Layered fractal fibers and potentials

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Abstract

We study spectral asymptotic properties of conductive layered-thin-fibers of invasive fractal nature. The problem is formulated as a boundary value problem for singular elliptic operators with potentials in a quasi-filling geometry for the fibers. The methods are those of variational singular homogenization and M-convergence. We prove that the spectral measures of the differential problems converge to the spectral measure of a non-trivial self-adjoint operator with fractal terms.

Résumé

Nous étudions les propriétés asymptotiques spectrales de certains fibres conductrices minces stratifiées de nature fractale. Le problème est formulé comme un problème au bord pour des operateurs elliptiques singuliers avec un potentiel et une géométrie fractale des fibres, invasive de l'espace. Les méthodes sont ceux de l'homogénéisation singulière et de la M-convergence. Nous prouvons que les mesures spectrales des problèmes différentiels convergent vers la mesure spectrale d'un operateur auto-adjoint non banale avec des termes fractals.

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1. Introduction

Boundary value problems for second order elliptic and parabolic operators in Euclidean domains with irregular and possibly fractal boundaries have been studied in recent years from various points of view by several authors, among them [14], [31], [10], [12], [19], [21], [13], [1], [2], [4], [29], [22], [6], [30], [11], [8], [9]. A distinctive feature of the boundary value problems considered in [19], [21], [29], and [30] is the dynamical and fractal character of the boundary, or part of the boundary. In these problems, a Euclidean open domain is given which has a fractal boundary component. Moreover, in addition to a second order partial differential operator in the open domain, another operator, also of second order, is assigned on the fractal boundary. The interaction between the interior operator and the boundary operator, both of second order, gives rise to a second order transmission condition on the boundary, which captures the main global dynamical features of the problem at hand. The dimensional relationship that relies the domain to the boundary may take an unusual turn in the case the boundary displays a fractal geometry. For example, we may have an open two-dimensional Euclidean domain Ω in $\mathbb{R}^2$ with a boundary or a component of the boundary, $\mathcal{G}$, which has a Hausdorff dimension d_H that is any number $1 < d_H < 2$ close to 2 as we wish, and which fills a two-dimensional open subset of Ω up to an arbitrarily small set. Moreover, for boundaries which have their own dynamics, in addition to d_H also the so-called spectral dimension d_s of the fractal $\mathcal{G}$ plays a fundamental role. This splitting of the static and dynamical dimensions for fractal boundaries is new in comparison with the Euclidean boundaries even of irregular Lipschitz type, for which the two dimensions d_H and d_s do in fact coincide.

Our goal is to investigate such boundary value problems, with quasi-filling dynamical fractal boundaries, from the variational point of view of the singular homogenization theory for elliptic operators in Euclidean domains. We construct a sequence of second order elliptic operators $L^n u = -div(a^n \nabla u)$ in divergence form and a sequence of scalar potentials V^n in a bounded open domain Ω of the plane. The coefficient matrix $a^n Id$ of L^n and the potential V^n are adjusted to the geometry of a system of open thin fibers $\Sigma^n_{\varepsilon_n}$, which is moved in Ω as $n \to +\infty$ by the iterated action of the contractive self-similarities of a fractal inclusion $\mathcal{G}$ in Ω. The fibers have a two-layer structure, with an inner layer of increasingly high conductivity, and an external layer of increasingly low conductivity. Together with suitable boundary conditions, these operators define a sequence of self-adjoint operators $A^n u = L^n u - V^n u$ in the Lebesgue Hilbert space $L^2(\Omega)$. We take in consideration the spectral measures $P^n(d\lambda)$ of the operators A^n in $L^2(\Omega)$. Our objective is to prove that the spectral measures $P^n(d\lambda)$ do in fact converge to the spectral measure of a non trivial self-adjoint operator A in $L^2(\Omega)$. Under appropriate boundary and scaling assumptions, we prove in this paper that the limit operator A does in fact exist. The spectrum converges, despite the coefficients and the potential develop fractal singularities as $n \to +\infty$. From the energy point of view, the operator A incorporates energy from the surrounding domain, as well as energy conveyed into the fibers by the balancing asymptotic action of the insulating and the conductive layers. As a boundary value problem, A is given by the sum of the two-dimensional Laplace operator and a zero order term in $\Omega \setminus \mathcal{G}$ and on $\mathcal{G}$ by the sum of a fractal second order operator $L_{\mathcal{G}}$ and a zero order term. $L_{\mathcal{G}}$ is the self-adjoint operator defined by the intrinsic energy form on $\mathcal{G}$ in the space $L^2(\mathcal{G}, \mu_{\mathcal{G}})$, where $\mu_{\mathcal{G}}$ is the d_H-Hausdorff measure on $\mathcal{G}$. The equations in $\Omega \setminus \mathcal{G}$ and the equation in $\mathcal{G}$ are coupled by a second-order transmission condition on $\mathcal{G}$ and by a point-wise condition at the intrinsic boundary of $\mathcal{G}$.

Our setting and results apply to wide classes of so called nested fractals. However, as we wish to keep this paper constructive in nature, we provide details of our study only for two special families of fractals $\mathcal{G}$, namely a family of Koch curves $\mathcal{K}_\alpha$, $2 < \alpha \leq 4$, which are invasive as $\alpha \to 2$, and a family of Sierpiński gaskets $\mathcal{S}_\alpha$, α an integer ≥ 2, which are invasive as $\alpha \to +\infty$. In both cases, α^{-1} is the contraction factor of the similarity maps defining $\mathcal{G}$. The fractal $\mathcal{G}$ is the non-rectifiable component of the boundary of an open Euclidean domain $\Omega \setminus \mathcal{G}$, where Ω is a given bounded open domain of $\mathbb{R}^2$ that contains $\mathcal{G}$ in its interior. We point out that the fractals in consideration are dynamical in nature, as they support nontrivial intrinsic energy forms. Moreover, they display some of the most significant geometric features of fractal sets: the Koch curves, though topologically homeomorphic to a line, have no infinitesimal linear path and display sharp turns at every small scale; the Sierpiński sets totally disconnect the surrounding domain displaying ramifications at all small scales.

Spectral convergence is particularly hard to prove in the quasi-filling geometry of $\mathcal{G}$, when the surface and the boundary come closer in touch and the dynamical interaction between them is stronger. As mentioned before, our approach to these dynamical problems consists in assuming that the invasive fibers have a two-layer structure, with an increasingly conductive inner core and an increasingly insulating external layer. In the theory of composite media, two-layer coated fibers have a physical interest on their own. In the singular homogenization approach adopted in this paper, the fibers are the physical regions where the singularities and degeneracies of the elliptic operators occur as $n \to +\infty$. Our spectral asymptotic results provide then a physical foundation to the fractal boundary value problems of the kind described before. This may contribute to opening new perspectives to the study of small bodies with very rich dynamical boundary effects.

The geometry of the domains Ω and of the fractal boundary $\mathcal{G}$ is described in more detail in Section 2. The operators are introduced in Section 3 and the results are given in Section 4. Section 5 is dedicated to various preliminaries. The proofs are given in Sections 6 and 7.

2. The geometry

By Ω of $\mathbb{R}^2$ we denote the open triangle with vertices $D = (-3/2, -\sqrt{3}/2)$, $E = (5/2, -\sqrt{3}/2)$, $F = (1/2, 3\sqrt{3}/2)$. The domain Ω contains the triangle of vertices $A = (0,0)$, $B = (1,0)$, $C = (1/2, \sqrt{3}/2)$. In this triangle we construct a fractal inclusion, $\mathcal{G}$. The set $\mathcal{G}$ is the invariant (self-similar) compact set of $\mathbb{R}^2$ associated with a family $\Psi = \{\psi_1, \ldots, \psi_N\}$ of $N \geq 2$ similarities in $\mathbb{R}^2$, which are contraction maps in $\mathbb{R}^2$ with a common contraction factor α^{-1}, $\alpha > 1$, and which satisfy the so-called open set condition. The set of the essential fixed-points of these maps will be denoted by Γ. We recall that a fixed-point of a map of Ψ, $b_r \in \mathbb{R}^2$ is an essential fixed-point for the family Ψ if $\psi_i(b_r) = \psi_j(b_s)$ for some $i \in \{1, \ldots, N\}$, $j \neq i$, $j \in \{1, \ldots, N\}$ and b_s a fixed-point of a map of Ψ. The invariant (self-similar) regular Borel measure in $\mathbb{R}^2$ supported on $\mathcal{G}$, associated with Ψ, is given by

$$\mu = \mu_{\mathcal{G}} := \frac{1}{\mathcal{H}^d(\mathcal{G})} \mathbf{1}_{\mathcal{G}} \mathcal{H}^d \,, \tag{2.1}$$

where

$$d = d_H = \ln N / \ln \alpha \tag{2.2}$$

is the Hausdorff dimension of $\mathcal{G}$ and $\mathcal{H}^d$ is the Hausdorff measure of dimension d in $\mathbb{R}^2$. In particular,

$$\mu(\psi_{i|n}(\mathcal{G})) = \frac{1}{N^n} \mu(\mathcal{G}) = \frac{1}{N^n} \tag{2.3}$$

where $\psi_{i|n} = \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_n}$ if $n > 0$. For these properties, and for the related theory of so-called nested fractals associated with similarity maps as the one considered here, we refer to Hutchinson, [17], and Lindstrøm, [23].

As noted in the Introduction, our results refer specifically to a family of Koch curves, $\mathcal{K}_\alpha$, and a family of Sierpiński gasket, $\mathcal{S}_\alpha$. The Koch curves are obtained as the invariant set of the family $\Psi = \{\psi_1, \ldots, \psi_4\}$ of the following $N = 4$ similitudes, each one contractive with a factor $\alpha \in (2, 4]$:

$$\begin{cases} \psi_1(z) = \frac{z}{\alpha}, & \psi_2(z) = \frac{z}{\alpha}\, e^{i\vartheta} + \frac{1}{\alpha}, \\ \psi_3(z) = \frac{z}{\alpha}\, e^{-i\vartheta} + \frac{1}{2} + \frac{i\sin\vartheta}{\alpha}, & \psi_4(z) = \frac{z+\alpha-1}{\alpha}, \end{cases} \tag{2.4}$$

where

$$\vartheta = \arcsin\sqrt{\alpha - \frac{\alpha^2}{4}} \in [0, \frac{\pi}{2}) \tag{2.5}$$

and $z = x + iy \in \mathbb{C}$. The set of the essential fixed-points of this family is $\Gamma = \{A, B\}$. The Sierpiński fractals $\mathcal{S}_\alpha$, α integer ≥ 2, are produced by $N = \alpha(\alpha+1)/2$ similarities with contraction factor α^{-1}. We describe the set $\mathcal{S}_\alpha$, for simplicity, only in the case $\alpha = 2$. This is the standard gasket in $\mathbb{R}^2$, which we denote in the following by $\mathcal{S}$. However, since we keep general notation, the extension of our proofs to any other fractal of the family $\mathcal{S}_\alpha$ is straightforward. $\mathcal{S}$ is the invariant set of the family $\Psi = \{\psi_1, \psi_2, \psi_3\}$, where

$$\psi_1(z) = \frac{z}{2}, \quad \psi_2(z) = \frac{z}{2} + \frac{1}{2}, \quad \psi_3(z) = \frac{z}{2} + \frac{1}{4} + i\frac{\sqrt{3}}{4}. \tag{2.6}$$

These $N = 3$ maps have common contraction factor $\alpha = 2$. The set of essential fixed-points is now $\Gamma = \{A, B, C\}$.

In the domain Ω we introduce a reference two-layer fiber constructed as follows. This fiber is made of two co-axial thin hexagons

$$\Sigma^0_{0,\varepsilon} \subset \Sigma^0_{0,2\varepsilon}$$

of largest transversal size $\varepsilon > 0$ and 2ε, respectively. The common axis of the fibers is the segment connecting the points $A = (0, 0)$ and $B = (1, 0)$, and the middle point of the segment AB is denoted by $AB/2$. The fibers are symmetric with respect to the $x - axis$ and the vertical line $x = 1/2$. Therefore, it suffices to describe the geometry of the inner fiber $\Sigma^0_{0,\varepsilon}$ and of the outer fiber $\Sigma^0_{0,2\varepsilon}$ only in the region $y \geq 0$, $x \leq 1/2$. We consider the right triangle with vertices $A, AB/2, Q_0$ which makes the angle of amplitude ϑ^* at A, the value $\vartheta^* \in (0, \pi/6]$ will be chosen according to the fractal layer imbedded in the domain. Thus, $Q_0 = (1/2, \varepsilon_0)$, where $\varepsilon_0 = h_0/2$, $h_0 = \tan(\vartheta^*)$. For every $0 < \varepsilon \leq \varepsilon_0$, we consider the two points $Q_1(\varepsilon) = (\varepsilon/h_0, \varepsilon)$ and $Q_0(\varepsilon) = (1/2, \varepsilon)$ and the quadrilateral $A, AB/2, Q_0(\varepsilon), Q_1(\varepsilon)$. We then define the set $\Sigma^0_{0,2\varepsilon}$ to be the thin hexagon obtained by reflection of this quadrilateral across the $x - axis$, followed by a symmetry across the vertical axis $x = 1/2$. The vertices of $\Sigma^0_{0,2\varepsilon}$, listed clockwise, are the points $A, Q_1(\varepsilon), Q_2(\varepsilon), B, Q_3(\varepsilon), Q_4(\varepsilon)$, where now $Q_2(\varepsilon) = (1 - \varepsilon/h_0, \varepsilon)$, $Q_3(\varepsilon) = (1 - \varepsilon/h_0, -\varepsilon)$, $Q_4(\varepsilon) = (\varepsilon/h_0, -\varepsilon)$. The perimeter of the hexagon $\Sigma^0_{0,2\varepsilon}$ gives the external profile of our two-layer fiber. Inside the hexagon $\Sigma^0_{0,2\varepsilon}$, we now insert a smaller hexagon $\Sigma^0_{0,\varepsilon}$. The construction of this hexagon is similar to that of $\Sigma^0_{0,2\varepsilon}$, by replacing now the triangle $A, AB/2, Q_0$ with the smaller right triangle with vertices $A, AB/2, P_0$, where $P_0 = (1/2, \varepsilon_0/2)$. The angle of this triangle at A is $\arctan(h_0/2)$. The vertices of the hexagon $\Sigma^0_{0,\varepsilon}$, again listed clockwise, are the points $A, P_1(\varepsilon), P_2(\varepsilon), B, P_3(\varepsilon), P_4(\varepsilon)$, where now $P_1(\varepsilon) = (\varepsilon/h_0, \varepsilon/2)$, $P_2(\varepsilon) = (1 - \varepsilon/h_0, \varepsilon/2)$, $P_3(\varepsilon) = (1 - \varepsilon/h_0, -\varepsilon/2)$, $P_4(\varepsilon/h_0, -\varepsilon/2)$. Note that the two hexagon meet at the common vertices A and B and that set $\Sigma^0_{0,\varepsilon} \setminus \{A, B\}$ is contained in the interior of $\Sigma^0_{0,2\varepsilon}$. As observed before, for the Koch curves $\Gamma = \{A, B\}$, while for the Sierpiński curve $\Gamma = \{A, B, C\}$. In the construction of the initial fiber $\Sigma^0_{0,2\varepsilon}$ a possible choice for ϑ^* in both the previous examples is

$$\vartheta^* = \min\{\pi/2 - \vartheta, \vartheta/2\} \tag{2.7}$$

where ϑ is the rotation angle of the similarities generating the Koch curve $\mathcal{K}_\alpha$ (see (2.5)). We note that $\min\{\pi/2 - \vartheta, \vartheta/2\} \leq \pi/6$. In the Sierpiński case we can take $\vartheta^* = \pi/6$. In our theory we need to connect pair-wise all essential fixed points of the fractal under consideration. This requirement is fulfilled so far only in the case of the Koch curves, not for the Sierpiński curve. For the Sierpiński fractal the requirement can be easily met by just adding to $\Sigma^0_{0,2\varepsilon}$ two more co-axial fibers of the kind as $\Sigma^0_{0,\varepsilon} \subset \Sigma^0_{0,2\varepsilon}$, now connecting both A and B to C. To keep generality in our notation, for every pair $b_r \neq b_s$ in Γ, $r, s = 1, \ldots, \chi_\Gamma$ – where χ_Γ denotes the cardinality of Γ – we introduce the similitude

$$\Phi_{b_r,b_s}(z) = \frac{|b_r - b_s|}{|A - B|} e^{i\theta_{b_r,b_s}} z + b_r, \qquad (\theta_{b_r,b_s} = Arg\overrightarrow{b_r b_s})$$

that maps the vector $\overrightarrow{AB}$ to $\overrightarrow{b_r b_s}$ preserving the orientation. In our examples, the distance $|b_r - b_s|$ between any two fixed-points is the same, therefore the ratio $|b_r - b_s|/|A - B|$ equals 1 for every $b_r \neq b_s$ in Γ and Φ_{b_r,b_s} is a (Euclidean) isometry. This simplifies our future calculations. We now connect every pair b_r, b_s in Γ with the co-axial fibers

$$\Sigma^0_\varepsilon(b_r, b_s) \subset \Sigma^0_{2\varepsilon}(b_r, b_s)$$

where

$$\Sigma^0_{2\varepsilon}(b_r, b_s) = \Phi_{b_r,b_s}\Sigma^0_{0,2\varepsilon}$$

and

$$\Sigma^0_\varepsilon(b_r, b_s) = \Phi_{b_r,b_s}\Sigma^0_{0,\varepsilon}\,.$$

We then define the array of co-axial fibers

$$\Sigma^0_\varepsilon \subset \Sigma^0_{2\varepsilon}$$

connecting all points in Γ, by taking

$$\Sigma^0_\varepsilon = \bigcup_{b_r \neq b_s \in \Gamma} \Sigma^0_\varepsilon(b_r, b_s)\,, \qquad \Sigma^0_{2\varepsilon} = \bigcup_{b_r \neq b_s \in \Gamma} \Sigma^0_{2\varepsilon}(b_r, b_s)\,.$$

We now iteratively transform the arrays $\Sigma_\varepsilon^0 \subset \Sigma_{2\varepsilon}^0$ into finer and finer arrays, by the action, for each integer $n > 0$ of the maps

$$\psi_{i|n} = \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_n}$$

associated with arbitrary $n-$tuples of indices $i|n = (i_1, i_2, \dots, i_n) \in \{1, \dots, N\}^n$. If $n = 0$ we define $\psi_{i|n}$ to be the identity map in $\mathbb{R}^2$. For every set $\mathcal{O} \subseteq \mathbb{R}^2$, we define $\mathcal{O}^{i|n} = \psi_{i|n}(\mathcal{O})$, and, occasionally, we call $i|n$ the $n-$ address of the set $\mathcal{O}^{i|n}$. With this notation, for every ε and every $n \geq 0$, we then define the arrays of co-axial fibers

$$\Sigma_\varepsilon^n \subset \Sigma_{2\varepsilon}^n$$

by setting

$$\Sigma_{2\varepsilon}^n = \bigcup_{i|n} \Sigma_{2\varepsilon}^{i|n}, \quad \Sigma_{2\varepsilon}^{i|n} = \psi_{i|n}(\Sigma_{2\varepsilon}^0) = \bigcup_{b_r \neq b_s \in \Gamma} \Sigma_{2\varepsilon}^{i|n}(b_r, b_s)\,, \tag{2.8}$$

$$\Sigma_\varepsilon^n = \bigcup_{i|n} \Sigma_\varepsilon^{i|n}, \quad \Sigma_\varepsilon^{i|n} = \psi_{i|n}(\Sigma_\varepsilon^0) = \bigcup_{b_r \neq b_s \in \Gamma} \Sigma_\varepsilon^{i|n}(b_r, b_s)\,. \tag{2.9}$$

3. The partial differential operators

Our approach to spectral convergence is of variational nature and is based on the tools developed in [26]. However, before introducing the relevant energy functionals, we prefer to describe the partial differential operators and, though only formally, the boundary value problems that underly our theory.

The domain Ω of the Euclidean plane has been described in the previous section. In Ω we now introduce a sequence of linear elliptic operators in divergence form

$$A^n u = -div(a^n(x,y)\nabla u) - V^n(x,y)u\,, \tag{3.1}$$

where $(x, y) \in \Omega$, $a^n(x,y)Id$ is the 2×2 symmetric matrix of the coefficients and $V^n(x,y)$ is a potential function in Ω. We describe the structure of the operator A^n in more detail for each given n, and also the behavior as $n \to +\infty$. In this description, important material parameters are the positive real numbers $\varepsilon_n > 0$ and $\chi_n > 0$ that will be specified later on, both converging to zero as $n \to +\infty$.

For each given n, we introduce in Ω the two-layer fibers $\Sigma_\varepsilon^n \subset \Sigma_{2\varepsilon}^n$ described in Section 2, by choosing now $\varepsilon = \varepsilon_n$. We recall that the fibers are constructed by iteration of a family $\Psi = \{\psi_1, \dots, \psi_N\}$ of $\alpha^{-1}-$contractive similarities generating the fractal $\mathcal{G}$ in Ω, as described in Section 2. The coefficients $a^n(x,y)$ and the potential $V^n(x,y)$ present discontinuities across the fibers $\Sigma_{\varepsilon_n}^n$ and $\Sigma_{2\varepsilon_n}^n$. In particular, within the fibers, the internal layer $\Sigma_{\varepsilon_n}^n$ is a region of increasingly high conductivity as $n \to +\infty$, while the external layer $\Sigma_{2\varepsilon_n}^n \setminus \Sigma_{\varepsilon_n}^n$ is a region of increasingly low conductivity. More precisely, we assume that the coefficient matrix $a^n Id$ is defined at every $(x,y) \in \Omega$ by

$$a^n(x,y)Id = \begin{cases} \zeta_n \mathbf{1}_{\Omega \setminus \Sigma^n_{2\varepsilon_n}}(x,y)Id + \chi_n \mathbf{1}_{\Sigma^n_{2\varepsilon_n} \setminus \Sigma^n_{\varepsilon_n}}(x,y)Id + \\ \gamma_n \sigma_n w^n_{\varepsilon_n}(x,y) \mathbf{1}_{\Sigma^n_{\varepsilon_n}}(x,y)Id \end{cases} \tag{3.2}$$

and the potential V^n by

$$V^n(x,y) = -\lambda^*_n \mathbf{1}_{\Omega \setminus \Sigma^n_{\varepsilon_n}}(x,y) + \lambda_* \tau_n w^n_{\varepsilon_n} \mathbf{1}_{\Sigma^n_{\varepsilon_n}}(x,y). \tag{3.3}$$

In the expression of $a^n(x,y)$, $w^n_{\varepsilon_n} = w^n_{\varepsilon_n}(x,y)$ is equal – up to a numerical constant that will be specified in (6.2) of Section 6 – to the reciprocal of the length $\ell^n_{\varepsilon_n}(x,y)$ of the section of $\Sigma^n_{\varepsilon_n}$ perpendicular to the point (x,y) of its longitudinal axis, while in $V^n(x,y)$, the same $w^n_{\varepsilon_n}$, in a light abuse of notation, is taken to be exactly equal to the reciprocal of that length, that is $w^n_{\varepsilon_n}(x,y) = (\ell^n_{\varepsilon_n}(x,y))^{-1}$. Note that for each given n, $\ell^n_{\varepsilon_n}(x,y)$ decreases to zero as the point (x,y) – belonging to an open hexagonal component of $\Sigma^n_{\varepsilon_n}$ – approaches one of the two vertices that lie beside (x,y) on the longitudinal axis of the fiber. Therefore, for each n, $w^n_{\varepsilon_n}$ has a singularity at every longitudinal vertex of $\Sigma^n_{\varepsilon_n}$. Moreover, $w^n_{\varepsilon_n}$ increases to $+\infty$ everywhere as $n \to +\infty$, because then $\varepsilon_n \to 0$ and $\ell^n_{\varepsilon_n}(x,y) \to 0$. In particular, $w^n_{\varepsilon_n}$ is singular at every point P of the set Γ introduced in Section 2. The potential V^n is also singular on $\Sigma^n_{\varepsilon_n}$ – at the longitudinal vertices for each n and everywhere else as $n \to +\infty$ – again in consequence of the coefficient $w^n_{\varepsilon_n}$. In contrast to the internal layer $\Sigma^n_{\varepsilon_n}$ of the fiber, which is highly conductive as seen before, the external layer $\Sigma^n_{2\varepsilon_n} \setminus \Sigma^n_{\varepsilon_n}$ displays vanishing conductivity $\chi_n \to 0$ as $n \to +\infty$, as also anticipated before.

In addition to ε_n and χ_n, the constants ζ_n, γ_n and λ^*_n which also occur in the expressions of the coefficients and the potential, are positive constants that specify the material properties of Ω, while σ_n and τ_n are suitable scaling parameters that depend on the structural constants of the fractal $\mathcal{G}$. As to the constant λ_*, which will be specified in the next section, we point out that a proper choice of its value is crucial in assuring the convergence of the operators.

In order to realize the operators A^n as densely defined self-adjoint operators in $L^2(\Omega)$, we must prescribe suitable boundary conditions – for example, homogeneous Neumann or Dirichlet conditions – on the boundary $\partial\Omega$ of Ω and, as we shall see, also on the set Γ of the end-points of the fibers $\Sigma^n_{2\varepsilon_n}$. While $\partial\Omega$ does not play a special role in our problems, the fact that boundary conditions can be also pre-assigned on the set Γ is unusual, because Γ is a discrete set made of a finite number of points with zero capacity inside Ω. Here the singularities of the operators play the relevant role.

We begin with the Neumann conditions. As far as Ω is concerned, our boundary value problem can be formally stated, for a given function $f \in L^2(\Omega)$, as

$$\begin{cases} -div(a^n(x,y)\nabla u) - V^n(x,y)u \;=\; f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

By decomposing the terms in the definitions for a^n and V^n, this problem can be written, again formally but somehow more transparently, as

$$\begin{cases} -\zeta_n div(\nabla u) + \lambda_n^* u \;=\; f & \text{in } \Omega \setminus \Sigma^n_{2\varepsilon_n} \\ -\chi_n div(\nabla u) + \lambda_n^* u \;=\; f & \text{in } \Sigma^n_{2\varepsilon_n} \setminus \Sigma^n_{\varepsilon_n} \\ -\gamma_n \sigma_n div(w^n_{\varepsilon_n} \nabla u) - \lambda_* \tau_n w^n_{\varepsilon_n} u \;=\; f & \text{in } \Sigma^n_{\varepsilon_n} \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \\ \text{natural transmission conditions on } \partial(\Sigma^n_{2\varepsilon_n} \setminus \Sigma^n_{\varepsilon_n})\,. \end{cases}$$

As this strong formulation of the boundary value problem has in this paper only an illustrative purpose, we do not enter here into the regularity and geometrical details which would be needed for a rigorous formulation of all the preceding equations and boundary conditions. Instead, we proceed by giving the variational formulation of the problem at hand, and by defining the self-adjoint operator A^n. For every n we define the functional

$$\mathcal{F}^n : \; L^2(\Omega) \mapsto (-\infty, +\infty]$$

by

$$\mathcal{F}^n[u] = \begin{cases} \int_\Omega a^n(x,y)|\nabla u|^2 dxdy - \int_\Omega V^n(x,y) u^2 dxdy & \text{if} \quad u \in H^1(\Omega; a^n) \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus H^1(\Omega; a^n) \end{cases} \tag{3.4}$$

where $a^n Id$ is the coefficient matrix given in (3.2) and the domain $H^1(\Omega; a^n) \subset L^2(\Omega)$ is the completion of $C^1(\overline{\Omega})$ in the norm

$$\|u\|_{H^1(\Omega;a^n)} = \{\int_\Omega |u|^2 dxdy + \int_\Omega a^n |\nabla u|^2 dxdy\}^{\frac{1}{2}}\,. \tag{3.5}$$

We point out that ∇u is well defined since the weight $w^n_{\varepsilon_n}$ and its reciprocal $1/w^n_{\varepsilon_n}$ belong both to $L^1(\Omega)$. Then, the operator A^n, with Neumann conditions on $\partial\Omega$ and on Γ, is defined as the self-adjoint operator A^n with dense domain $D[A^n]$ in $L^2(\Omega)$ given by the identity

$$\mathcal{F}^n(u,v) = \int_\Omega (A^n u) v dxdy\,, \quad u \in D[A^n]\,, \quad v \in H^1(\Omega; a^n)\,,$$

where $\mathcal{F}^n(u,v)$ is the bilinear form on $H^1(\Omega; a^n)$ associated with the functional $\mathcal{F}^n$.

We now move to the case of Dirichlet conditions on Ω and Γ. Formally, the boundary value problem can be stated now as

$$\begin{cases} -\zeta_n div(\nabla u) + \lambda_n^* u \;=\; f & \text{in } \Omega \setminus \Sigma^n_{2\varepsilon_n} \\ -\chi_n div(\nabla u) + \lambda_n^* u \;=\; f & \text{in } \Sigma^n_{2\varepsilon_n} \setminus \Sigma^n_{\varepsilon_n} \\ -\gamma_n \sigma_n div(w^n_{\varepsilon_n} \nabla u) - \lambda_* \tau_n w^n_{\varepsilon_n} u \;=\; f & \text{in } \Sigma^n_{\varepsilon_n} \\ u = 0 & \text{on } \partial\Omega \\ u_{\Sigma^n_{\varepsilon_n}}(P) = 0 & \text{on } \Gamma \\ \text{natural transmission conditions on } \partial(\Sigma^n_{2\varepsilon_n} \setminus \Sigma^n_{\varepsilon_n}) \setminus \Gamma \,. \end{cases}$$

Here again $u_{\Sigma^n_{\varepsilon_n}}$ is the restriction of u to $\Sigma^n_{\varepsilon_n}$ and Γ the set of the end-points of $\Sigma^n_{\varepsilon_n}$.

The set Γ is the intrinsic boundary of the fractal $\mathcal{G}$ and, for every n, it is also the set of the end-points of the fibers $\Sigma^n_{\varepsilon_n}$ and $\Sigma^n_{2\varepsilon_n}$. As mentioned before, the weight $w^n_{\varepsilon_n}$ has a singularity at every $P \in \Gamma$. The effect of such a singularity is that for every function u of finite energy $\int_\Omega a^n(x,y)|\nabla u|^2 dxdy$ – as is the case in the variational formulation of the problem that we shall give below – the point-wise value

$$u(P) = \lim_{r \to 0} \frac{\int_{B_r(P) \cap \Sigma^n_{\varepsilon_n}} u dxdy}{|B_r(P) \cap \Sigma^n_{\varepsilon_n}|}$$

is well defined, as it can be seen by a direct calculation.

The functional $\mathcal{F}^n : \; L^2(\Omega) \mapsto (-\infty, +\infty]$ is now defined by

$$\mathcal{F}^n[u] = \begin{cases} \int_\Omega a^n(x,y)|\nabla u|^2 dxdy - \int_\Omega V^n(x,y) u^2 dxdy & \text{if} \quad u \in D_0[a_n] \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus D_0[a_n] \end{cases} \tag{3.6}$$

where $a^n Id$ is again the coefficient matrix defined in (3.2) and the domain $D_0[a_n] \subset L^2(\Omega)$ is now the completion of the set $\mathcal{C} = \{u \in C^1_0(\Omega) : \; u_{|\Gamma} = 0\}$ in the norm

$$\|u\|_{H^1(\Omega; a^n)} = \{\int_\Omega |u|^2 dxdy + \int_\Omega a^n |\nabla u|^2 dxdy\}^{\frac{1}{2}} \,. \tag{3.7}$$

The operator A^n, with Dirichlet conditions on $\partial\Omega$ and on Γ, is now defined as the self-adjoint operator A^n with dense domain $D[A^n]$ in $L^2(\Omega)$ given by the identity

$$\mathcal{F}^n(u,v) = \int_\Omega (A^n u) v dxdy \,, \quad u \in D[A^n] \,, \quad v \in D_0[a_n] \,,$$

where $\mathcal{F}^n(u,v)$ is the bilinear form on $D_0[a_n]$ defined by the new functional $\mathcal{F}^n$.

In both cases, the operators A^n, with the boundary conditions specified as before, have been defined as self-adjoint operators with a dense domain $D[A^n]$ in $L^2(\Omega)$. The object of our study are the spectral projectors $P^n((\lambda,\mu])$ and the spectral subspaces $X^n = P^n((\lambda,\mu])L^2(\Omega)$ in $L^2(\Omega)$ associated with these operators, in particular, the convergence of $P^n((\lambda,\mu])$ and

$X^n = P^n((\lambda,\mu])L^2(\Omega)$ as $n \to +\infty$, in a sense to be made precise. We point out that under the assumptions specified in the following, the spectrum of A^n is a discrete eigenvalue spectrum, each eigenvalue occurring with finite multiplicity, and the spectral measures are of pure jump type. Our results are described in the next section.

4. The results

Before stating our results, we recall some definitions and properties which refer to the fractal $\mathcal{G}$. In addition to the structural constants N and α of the fractal $\mathcal{G}$ already introduced before, we now consider the additional parameter $\delta = \delta_{\mathcal{G}} > 1$ defined by $\delta = \frac{d_H}{d_s}$ where d_H is the Hausdorff dimension of $\mathcal{G}$ and d_s is the so-called spectral dimension of $\mathcal{G}$; for the two main cases in consideration, $\delta_{\mathcal{K}_\alpha} = \frac{\ln 4}{\ln \alpha}$ and $\delta_{\mathcal{S}} = \frac{\lg 5}{\lg 4}$. The parameter δ determines the parameter ρ by the relation

$$\rho = \frac{\alpha^{2\delta}}{N}. \tag{4.1}$$

On $\mathcal{G}$ an energy bilinear form $\mathcal{E}_{\mathcal{G}}$ is also defined, with a dense domain $D[\mathcal{E}_{\mathcal{G}}]$ in $L^2(\mathcal{G},\mu)$. It is the Friedrichs extension of the form obtained in the limit as $n \to +\infty$ of the discrete energies

$$\frac{1}{2}\frac{\alpha^{2n\delta}}{N^n}\sum_{i|n}\sum_{b_r\neq b_s\in\Gamma}(u(\psi_{i|n}(b_r)) - u(\psi_{i|n}(b_s)))(v(\psi_{i|n}(b_r)) - v(\psi_{i|n}(b_s)))$$

for all $u, v \in C(\mathcal{G})$ such that $\mathcal{E}_{\mathcal{G}}[u] = \mathcal{E}_{\mathcal{G}}(u,u) < +\infty, \mathcal{E}_{\mathcal{G}}[v] = \mathcal{E}_{\mathcal{G}}(v,v) < +\infty$. The *Neumann* operator $L_{\mathcal{G}}$ on $\mathcal{G}$ is the self-adjoint operator $L_{\mathcal{G}}$ with dense domain $D[L_{\mathcal{G}}]$ in $L^2(\mathcal{G},\mu)$ defined by the identity

$$\mathcal{E}_{\mathcal{G}}(u,v) = \int_{\mathcal{G}} L_{\mathcal{G}}u\, v d\mu \quad \forall u \in D[L_{\mathcal{G}}],\, v \in D[\mathcal{E}_{\mathcal{G}}]\,.$$

The *Dirichlet* operator $L_{\mathcal{G}}$ on $\mathcal{G}$ is the self-adjoint operator $L_{\mathcal{G}}$ defined again by the identity above when we replace the domain $D[\mathcal{E}_{\mathcal{G}}]$ with the domain $D_0[\mathcal{E}_{\mathcal{G}}] = \{u \in D[\mathcal{E}_{\mathcal{G}}] : u(P) = 0 \;\; \forall P \in \Gamma\}$, also dense in $L^2(\mathcal{G},\mu)$.

Here are the assumptions on the operators A^n, which are common to both Neumann and Dirichlet cases described in the previous section. For every n we set

$$\sigma_n = (\frac{\rho}{\alpha})^n\,, \quad \tau_n = \frac{\alpha^n}{\tau_0 N^n} \tag{4.2}$$

where τ_0 denotes the number of the segments $b_r b_s$, $b_r \neq b_s \in \Gamma$. We assume that the material constants $\zeta_n, \gamma_n, \lambda_n^*$ satisfy the conditions

$$\zeta_n \to \zeta^*, \quad \gamma_n \to \gamma^*, \quad \lambda_n^* \to \lambda^* \tag{4.3}$$

as $n \to +\infty$, with $\zeta^*, \gamma^*, \lambda^* \in (0,+\infty)$. Moreover, we assume that

$$\varepsilon_n = (\frac{\rho}{N})^n \omega_n \text{ with } \omega_n \to 0 \quad \text{and } \chi_n \to 0\,, \tag{4.4}$$

such that

$$\frac{\chi_n}{\varepsilon_n} \to 0 \tag{4.5}$$

and

$$\frac{\chi_n}{(N^n \varepsilon_n \cdot \alpha^{-2n})^q} \geq C_0 \tag{4.6}$$

as $n \to +\infty$, with $C_0 > 0$ and $q \in (0, \frac{1}{2})$. We note that the expression in the denominator of the fraction in (4.6) represents the total area of the fibers $\Sigma^n_{\varepsilon_n}$ in fact $|\Sigma^n_{\varepsilon_n}|$ behaves as $N^n \varepsilon_n \cdot \alpha^{-2n}$ as n tends to $+\infty$.

For both Neumann and Dirichlet realizations of the operators A^n, as described in Section 3, the limit operators A are obtained from the same bilinear energy form $a(u, v)$, taken however with different domains for each of the versions, Neumann or Dirichlet, of the sequence A^n. The form $a(u, v)$ is given by

$$a(u, v) = \zeta^* \int_\Omega \nabla u \nabla v dx dy + \lambda^* \int_\Omega uv dx dy + \gamma^* \mathcal{E}_\mathcal{G}(u_\mathcal{G}, v_\mathcal{G}) - \lambda_* \int_\mathcal{G} uv d\mu \tag{4.7}$$

where $u, v \in H^1(\Omega)$, $u_\mathcal{G}, v_\mathcal{G}$ are the traces of u and v on $\mathcal{G}$ and $\mathcal{E}_\mathcal{G}$ is the energy form on the fractal $\mathcal{G}$ described before. In the Neumann case, the domain of the form a is

$$D[a] = \{u \in H^1(\Omega), u_\mathcal{G} \in D[\mathcal{E}_\mathcal{G}]\} ,$$

while in the Dirichlet case the domain is

$$D[a] = \{u \in H^1_0(\Omega), u_\mathcal{G} \in D[\mathcal{E}_\mathcal{G}], u_\mathcal{G} = 0 \text{ on } \Gamma\}$$

and in both cases the domain $D[a]$ is dense in $L^2(\Omega)$. Occasionally, in order to distinguish the two domains, we shall also denote the latter, that is the domain for Dirichlet boundary conditions, by $D_0[a]$. In both Neumann and Dirichlet cases, the limit operator A is the self-adjoint operator A with dense domain $D[A]$ in $L^2(\Omega)$ defined by the identity

$$a(u, v) = \int_\Omega Au \, v dx dy \quad \forall u \in D[A], \, v \in D[a] .$$

Our main result is in fact:

Theorem 4.1. *Let the conditions (4.2), (4.3), (4.4), (4.5) and (4.6) be satisfied. Let the operators A^n for $n \geq 1$ be the self-adjoint operators in $L^2(\Omega)$ constructed in Section 3, with domains $D[A^n]$ of Neumann, resp. Dirichlet type in $L^2(\Omega)$. Then, there exists a constant $\Lambda > 0$ such that, for every $\lambda_* < \Lambda$, the spectral projector operators $P^n((\lambda, \mu])$ of A^n, for every λ, μ not eigenvalues of A, converge strongly in $L^2(\Omega)$ as $n \to +\infty$ to the spectral projector $P((\lambda, \mu])$ of the self-adjoint operator A in $L^2(\Omega)$, taken with the domain $D[A]$ of Neumann, resp. Dirichlet type. Moreover, the spectral subspaces $X^n_{(\lambda,\mu]} = P^n((\lambda, \mu])L^2(\Omega)$ of A^n $M-$converge in $L^2(\Omega)$ to the spectral subspace $X_{(\lambda,\mu]} = P((\lambda, \mu])L^2(\Omega)$ of A. In the*

Dirichlet case, the restriction $\lambda^* > 0$ can be omitted.

We specify that the subspaces $X^n_{(\lambda,\mu]}$ $M-$converge to the subspace $X_{(\lambda,\mu]}$ in $L^2(\Omega)$ if both conditions below hold:

(i) for every $u \in X_{(\lambda,\mu]}$ there exists $u_n \in X^n_{(\lambda,\mu]}$ converging strongly to u in $L^2(\Omega)$

(ii) for every $v_{n_k} \in X^{n_k}_{(\lambda,\mu]}$ converging weakly to u in $L^2(\Omega)$, then $u \in X_{(\lambda,\mu]}$.

In the previous section we have stated, formally, the boundary value problems incorporated into the self-adjoint operators A^n, in both Neumann and Dirichlet cases for the boundary conditions on $\partial\Omega$ and Γ. We now outline, again only formally, the limit boundary value problems underlying the self-adjoint operator A.

In the Neumann case, the limit boundary value problem associated with A can be formally stated, for a given $f \in L^2(\Omega)$, as follows:

$$\begin{cases} -\zeta^*\Delta u + \lambda^* u = f & \text{on } \Omega \setminus \mathcal{G} \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \\ \gamma^* L_{\mathcal{G}} u_{\mathcal{G}} - \lambda_* u_{\mathcal{G}} = \left[\frac{\partial u}{\partial \nu}\right] & \text{on } \mathcal{G} \\ \nu_\Gamma u_{\mathcal{G}} = 0 & \text{on } \Gamma. \end{cases}$$

In this expression, $\frac{\partial u}{\partial \nu}$ is the exterior normal derivative of u on $\partial\Omega$ in the dual space of the fractional Sobolev space $H^{1/2}(\partial\Omega)$. The jump $\left[\frac{\partial u}{\partial \nu}\right]$ of the normal derivative of u across the fractal $\mathcal{G}$ belongs to the dual space of the domain $D[\mathcal{E}_{\mathcal{G}}]$, see [19]. The trace $u_{\mathcal{G}}$ of u on $\mathcal{G}$ is well defined and continuous on $\mathcal{G}$, see (5.9) below. $L_{\mathcal{G}}$ is the Neumann operator within the fractal $\mathcal{G}$, that is, the second order operator in $L^2(\mathcal{G}, \mu_{\mathcal{G}})$ defined by the form $\mathcal{E}_{\mathcal{G}}$ under the Neumann condition $\nu_\Gamma u_{\mathcal{G}} = 0$ on Γ, as described at the beginning of this section. The exterior normal derivative $\nu_\Gamma u_{\mathcal{G}}(P)$ is well defined at any point $P \in \Gamma$ for every $u \in L^2(\mathcal{G}, \mu)$ with $L_{\mathcal{G}} u \in L^2(\mathcal{G}, \mu)$, see [28], [32].

In the Dirichlet case, the boundary value problem for A is

$$\begin{cases} -\zeta^*\Delta u + \lambda^* u = f & \text{on } \Omega \setminus \mathcal{G} \\ u = 0 & \text{on } \partial\Omega \\ \gamma^* L_{\mathcal{G}} u_{\mathcal{G}} - \lambda_* u_{\mathcal{G}} = \left[\frac{\partial u}{\partial \nu}\right] & \text{on } \mathcal{G} \\ u_{\mathcal{G}} = 0 & \text{on } \Gamma. \end{cases}$$

As mentioned at the beginning of Section 3, we get the spectral results of Theorem 4.1 by a variational method, namely by applying the convergence theory for functionals developed in [25] and [26]. This approach leads to the spectral convergence results stated in Theorem 4.1, once we establish that the energy functionals $\mathcal{F}^n$ associated with the operators A^n $M-$converge in $L^2(\Omega)$ to the energy functional $\mathcal{F}$ of the operator A. The functionals $\mathcal{F}^n$ have already been defined in Section 3, both in the Neumann case, (3.4), and in the

Dirichlet case, (3.6). In order to state our variational result, we only need to introduce the limit functional $\mathcal{F}$ in each of the two cases.

We define the functional $\mathcal{F} : L^2(\Omega) \mapsto (-\infty, +\infty]$, associated with the operator A, in the Neumann case by

$$\mathcal{F}[u] = \begin{cases} a(u,u) & \text{if} \quad u \in D[a] \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus D[a] \end{cases} \tag{4.8}$$

where $D[a] = \{u \in H^1(\Omega), u|_{\mathcal{G}} \in D[\mathcal{E}_{\mathcal{G}}]\}$, and in the Dirichlet case by

$$\mathcal{F}[u] = \begin{cases} a(u,u) & \text{if} \quad u \in D_0[a] \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus D_0[a] \end{cases} \tag{4.9}$$

where $D_0[a] = \{u \in H^1_0(\Omega), u|_{\mathcal{G}} \in D[\mathcal{E}_{\mathcal{G}}], u|_{\mathcal{G}} = 0 \text{ on } \Gamma\}$.

The sequence $\mathcal{F}^n : L^2(\Omega) \mapsto (-\infty, +\infty]$ $M-$converges in $L^2(\Omega)$ to the functional $\mathcal{F} : L^2(\Omega) \mapsto (-\infty, +\infty]$ if:

(a) for every $u \in L^2(\Omega)$ there exists u_n converging strongly to u in $L^2(\Omega)$, such that

$$\limsup \mathcal{F}^n[u_n] \leq \mathcal{F}[u], \quad as \quad n \to +\infty\,; \tag{4.10}$$

(b) for every v_n converging weakly to u in $L^2(\Omega)$,

$$\liminf \mathcal{F}^n[v_n] \geq \mathcal{F}[u] \quad as \quad n \to +\infty\,. \tag{4.11}$$

We prove the theorem

Theorem 4.2. *Let the conditions (4.2), (4.3), (4.4), (4.5) and (4.6) be satisfied. Let $\mathcal{F}^n$ for $n \geq 1$ be the functionals in $L^2(\Omega)$ defined in (3.4) (Neumann), resp. (3.6) (Dirichlet). Then, there exists a constant $\Lambda > 0$ such that for every $\lambda_* < \Lambda$, the sequence of functionals $\mathcal{F}^n$ M-converges in $L^2(\Omega)$ to the functional $\mathcal{F}$ defined in (4.8) (Neumann), resp. (4.9) (Dirichlet), as $n \to +\infty$. Moreover, in the Dirichlet case, the restriction $\lambda^* > 0$ can be omitted.*

The proof of Theorem 4.2 provides us also with an estimate of the constant Λ in terms of suitable imbedding and trace results. In the Neumann case, $\lambda^* > 0$ and Λ is obtained as

$$\Lambda = \min\big\{C_1^*, \frac{1}{C_2^*}\big\}, \tag{4.12}$$

with

$$C_1^* = \min\{\zeta^*, \lambda^*\} C_1^{-1}\,, \tag{4.13}$$

$$C_2^* = C_2 \cdot C_3 \cdot C_3^*\,, \tag{4.14}$$

where

$$C_3^* = 2^{3(1-p/2)} \max\Big\{\frac{(1+C_4(1+C_5))}{C_0^{p/2}}, \frac{(1+C_4(1+C_5))|\Omega|^{(1-p/2)}}{(\zeta^*)^{p/2}}, \frac{(C_4C_5)|\Omega|^{(1-p/2)}}{(\lambda^*)^{p/2}}\Big\}. \tag{4.15}$$

Here $p = \frac{2}{q+1}$; s is a fixed number in the interval $(1/2, 2-2/p)$; $C_1 = C_{Tr,1,\mathcal{G}}$ ($C_2 = C_{Tr,s}$) is the constant (independent of n) of the trace estimate from $H^1(\Omega)$ (resp., $H^s(\Omega)$) into the L^2 space of the fractal $\mathcal{G}$ (resp, of the pre-fractal approximations of $\mathcal{G}$), see Propositions 5.3, 5.2; $C_3 = C_{p,s}$ is the constant of the fractional Sobolev imbedding $W^{1,p}(\Omega) \subset H^s(\Omega)$, see Proposition 5.5; $C_4 = C_{P;p,p}$ is the Poincaré constant in $W^{1,p}(\Omega)$, see Proposition 5.6; $C_5 = C_{Tr,p}$ is the constant (independent of n) of the trace estimate in $L^p(\partial\Omega)$ for functions in $W^{1,p}(\Omega \setminus \Sigma^n_{\varepsilon_n})$, with Ω at the right hand side replaced by $\Omega \setminus \Sigma^n_{\varepsilon_n}$, see Proposition 5.4.

In the Dirichlet case, λ^* can be zero and the constant Λ is obtained as

$$\Lambda = \min\Big\{C^*_{1,D}, \frac{1}{C^*_{2,D}}\Big\}, \tag{4.16}$$

with

$$C^*_{1,D} = \max\Big\{\frac{\gamma^*}{C_7}, \frac{\max\{\min\{\lambda^*, \zeta^*\}, \zeta^*(1+C_8)\}}{C_1}\Big\}, \tag{4.17}$$

$$C^*_{2,D} = C_2 \cdot C_3 \cdot C^*_{3,D}\,, \tag{4.18}$$

where

$$C^*_{3,D} = 2^{2-p} \max\Big\{\frac{(1+C_4)}{C_0^{p/2}}, \frac{(1+C_4)|\Omega|^{(1-p/2)}}{(\zeta^*)^{p/2}}\Big\}. \tag{4.19}$$

Here $C_7 = C_{\mathcal{G}}$ is the Poincaré constant on the fractal (see 5.10), $C_8 = C_{P;2,2}$ is the Poincaré constant on the domain Ω (see Proposition 5.6), and $C_1 = C_{Tr,1,\mathcal{G}}$ is the trace constant on the fractal (see Proposition 5.3).

Remark 4.1. In the Koch case, sufficient conditions for the assumptions (4.4), (4.5), (4.6) to hold are:

$$\varepsilon = \varepsilon_n = N^{-s_1 n}, \quad \chi_n = N^{-s_2 n}\,, \tag{4.20}$$

where $1 - \ln\rho/\ln N < s_1 < s_2 \le s_3$ and $s_3 = (s_1 + 2\ln\alpha/\ln N - 1)q$ for some $q \in (0, \frac{1}{2})$. In the Sierpiński case, and in the case of the Koch curve with $\alpha = 3$, assumption (4.5) can be omitted due to the simpler geometry, as it will be clear from the proof in Section 6. The theory can be further developed by assuming different relative scalings for the parameters of the conductive and insulating layers of the fibers. Different asymptotic boundary conditions on $\mathcal{G}$ occur in the limit, which may be of first order type. This will be the object of another research.

5. Preliminaries

We collect a few results instrumental to our proofs. To facilitate their application, we state them in the geometry of our domains.

By $G^n, n \geq 0$ we denote the pre-fractal set:

$$G^n = \bigcup_{i|n} G^{i|n} = \bigcup_{i|n} \bigcup_{b_r \neq b_s \in \Gamma} G^{i|n}_{b_r b_s} \tag{5.1}$$

where $b_r\, b_s$ is the segment with end-points b_r and b_s and $G^{i|n}_{b_r b_s} = \psi_{i|n}(b_r\, b_s)$. We define the arc-length measure ds on G^n by $ds = \sum_{i|n} \sum_{b_r \neq b_s \in \Gamma} \mathcal{L}^1_{G^{i|n}_{b_r b_s}}$, where $\mathcal{L}^1_{G^{i|n}_{b_r b_s}} = ds$ is the 1−dimensional Lebesgue measure ds on the segment $G^{i|n}_{b_r b_s}$ of G^n. The sets G^n converge to the fractal set $\mathcal{G}$ as $n \to +\infty$ (in the Hausdorff metric) and also the renormalized measures $\tau_n ds$ on G^n weakly converge to the measure μ on $\mathcal{G}$ defined in (2.1).

Proposition 5.1. *Let μ be the measure defined in (2.1), and τ_n as in (4.2). Then, for every $\varphi \in C(\bar{\Omega})$,*

$$\tau_n \int_{G^n} \varphi \; ds \to \int_{\mathcal{G}} \varphi \; d\mu \tag{5.2}$$

as $n \to +\infty$.

The trace inequality below shows in particular how the estimate in question depends on the number of sides of G^n (see Theorem 5.3 in [6] and Theorem 3.5 in [7]):

Proposition 5.2. *Let G^n be as in (5.1), τ_n be as in (4.2), and let $s > 1/2$. For functions in the fractional Sobolev space $H^s(\Omega)$, the following inequality holds*

$$\tau_n ||v||^2_{L^2(G^n)} \leq C_{Tr,s} ||v||^2_{H^s(\Omega)} \tag{5.3}$$

where the constant $C_{Tr,s}$ depends on s but is independent of n.

Analogous results hold on the set $\mathcal{G}$ and on the boundary $\partial\Omega$:

Proposition 5.3. *Let d be the Hausdorff dimension of $\mathcal{G}$, and let $s > 1 - d/2$. Then for functions in the space $H^s(\Omega)$ the following inequality holds*

$$\int_{\mathcal{G}} |v|^2 d\mu \leq C_{Tr,s,\mathcal{G}} ||v||^2_{H^s(\Omega)} \tag{5.4}$$

For the proof we refer, for instance, to Theorem 1 of Chapter V in [16].
The fractal $\mathcal{G}$ is the closure in $\mathbb{R}^2$ of the set $\mathcal{V}^\infty = \bigcup_{n=0}^{+\infty} \mathcal{V}^n$ where for every $n \geq 0$

$$\mathcal{V}^n = \bigcup_{i|n} \psi_{i|n}(\Gamma). \tag{5.5}$$

The fractal energy $\mathcal{E}[u] = \mathcal{E}_{\mathcal{G}}[u]$ is the limit of the increasing sequence

$$\mathcal{E}[u] = \lim_{n \to +\infty} \mathcal{E}_n[u]\,, \tag{5.6}$$

with

$$\mathcal{E}_n[u] = \frac{1}{2}\,\frac{\alpha^{2n\delta}}{N^n}\sum_{i|n}\ \sum_{b_r\neq b_s\in\Gamma}\left(u(\psi_{i|n}(b_r)) - u(\psi_{i|n}(b_s))\right)^2, \tag{5.7}$$

on the domain

$$D[\mathcal{E}_{\mathcal{G}}] = \{u\in C(\mathcal{G})|\quad \sup_{n\geq 0}\mathcal{E}_n[u|_{\mathcal{V}^n}] < +\infty\}\,. \tag{5.8}$$

$D[\mathcal{E}_{\mathcal{G}}]\subset C^{\beta}(\mathcal{G})$ and the estimate

$$|u(P)-u(Q)|\leq C_H\sqrt{\mathcal{E}[u]}\,|P-Q|^{\beta} \tag{5.9}$$

holds for every $P,Q\in\mathcal{G}$. For these Hölder estimates we refer to Kozlov [18] (see also [27], where Kozlov's result is interpreted as an intrinsic Morrey's imbedding). From the previous inequality we get the Poincaré inequality for functions v in the domain $D_0[\mathcal{E}_{\mathcal{G}}]$

$$\int_{\mathcal{G}} v^2 d\mu \leq C_{P,\mathcal{G}}\,\mathcal{E}[v]\,. \tag{5.10}$$

The propositions that follow are standard trace and Poincaré inequalities for Sobolev spaces:

Proposition 5.4. *For functions in the Sobolev space $W^{1,p}(\Omega)$, $p>1$, the following inequality holds*

$$\int_{\partial\Omega}|v|^p dxdy \leq C_{Tr,p}\Big(\int_{\Omega}|\nabla v|^p dxdy + \int_{\Omega}|v|^p dxdy\Big)\,. \tag{5.11}$$

Proposition 5.5. *If $p>1$ and $s-2/2<1-2/p$, then the fractional Sobolev space $W^{1,p}(\Omega)$ is imbedded with compact inclusion in the Sobolev space $H^s(\Omega)$, moreover*

$$||v||^2_{H^s(\Omega)} \leq C_{p,s}\Big(\int_{\Omega}|\nabla v|^p dxdy + \int_{\Omega}|v|^p dxdy\Big)^{\frac{2}{p}} \tag{5.12}$$

For the proof we refer, for instance, to Theorem 2.19 in [5].

Proposition 5.6. *If $p>1$ and $1\leq r<\frac{2p}{2-p}$, for functions in $W^{1,p}(\Omega)$ the following inequality holds*

$$\int_{\Omega}|v|^r dxdy \leq C_{P;r,p}\Big(\int_{\Omega}|\nabla v|^p dxdy + \int_{\partial\Omega}|v|^p ds\Big)^{r/p} \tag{5.13}$$

For the proof we refer, for instance, to Lemma 3.1.1 in [24], and to Theorem 2.19 in [5].

6. Strong limit

In this section we start the proof of Theorem 4.2 in the Neumann case. Namely, we prove property (a) in (4.10) for the functionals $\mathcal{F}^n$ and $\mathcal{F}$ introduced in (3.4) and (4.8), respectively. In the Dirichlet case the proof is quite similar. Property (b) in (4.11) will be given in the next section: again the proof in the Neumann case, for the functionals $\mathcal{F}^n$ and

$\mathcal{F}$ introduced in (3.4) and (4.8), respectively, is very similar to the proof in Dirichet case, for the functionals $\mathcal{F}^n$ and $\mathcal{F}$ introduced in (3.6) and (4.9), respectively. We will specify the different constants in the estimates in Remark 7.1.

We note that in proving property (a) in (4.10) for a given $u \in L^2(\Omega)$ we can further assume without loss of generality that $u \in H^1(\Omega)$ and $u_{\mathcal{G}} \in D[\mathcal{E}_{\mathcal{G}}]$. In fact, if $u \in L^2(\Omega)$ does not satisfy both conditions, then $\mathcal{F}[u] = +\infty$ and property (a) is trivial. Our objective then is, given $u \in H^1(\Omega)$, to construct a sequence of functions $u_n \in L^2(\Omega)$, such that $u_n \to u$ in $L^2(\Omega)$ and

$$\limsup \mathcal{F}^n[u_n] \le \mathcal{F}[u] \tag{6.1}$$

as $n \to +\infty$. The proof will be achieved in two steps, first by assuming more regularity for u, then removing this additional assumption later. Moreover, since the geometry of the fibers is different in the two fractal cases under consideration, we shall split our proof in two parts, by proving the inequality (6.1) separately, first in the Sierpiński case, then in the Koch case. Before proceeding we specify the value of $w^n_{\varepsilon_n}$ in notation from Sections 2 and 3. In the expression of $a^n(x,y)$, for any fiber $\Sigma^{i|n}_{\varepsilon_n}(b_r,b_s)$ and for every fixed address $i|n$ and every pair $b_r \neq b_s \in \Gamma$, we define

$$w^n_{\varepsilon_n}(x,y) = \begin{cases} \frac{2+h_0^2}{4|P^*-P^\perp|} & \text{if} \quad (x,y) \in \mathcal{T}^{i|n}(b_r,b_s) \\ \frac{1}{2|P^*-P^\perp|} & \text{if} \quad (x,y) \in \mathcal{R}^{i|n}(b_r,b_s) \end{cases} \tag{6.2}$$

where $\mathcal{R}^{i|n}(b_r,b_s) = \psi_{i|n}(\mathcal{R}(b_r,b_s))$ with $\mathcal{R}(b_r,b_s) = \Phi_{b_r,b_s}\mathcal{R}$ and $\mathcal{T}^{i|n}(b_r,b_s) = \psi_{i|n}(\mathcal{T}(b_r,b_s))$ and with $\mathcal{T}(b_r,b_s) = \Phi_{b_r,b_s}\mathcal{T}$. Here $\mathcal{R}$ is the central rectangle in $\Sigma^0_{0,\varepsilon}$, which has vertices P_1, P_2, P_3, P_4, and $\mathcal{T}$ is the union of the two isosceles triangles A, P_1, P_4 and P_2, B, P_3. Moreover, for every point $P = (x,y) \in \Sigma^{i|n}_{\varepsilon_n}(b_r,b_s)$ we denote by $P^\perp$ the orthogonal projection of the point P on the longitudinal median axis of $\Sigma^{i|n}_{\varepsilon_n}(b_r,b_s)$ and we denote by P^* the intersection of the orthogonal line through $P^\perp$ with the boundary $\partial\Sigma^{i|n}_{\varepsilon_n}(b_r,b_s)$ of $\Sigma^{i|n}_{\varepsilon_n}(b_r,b_s)$ in the half fiber containing the point P. Here $|P^* - P^\perp|$ is the (Euclidean) distance between P^* and $P^\perp$ in $\mathbb{R}^2$. Instead, in the expression of $V^n(x,y)$, in every fiber $\Sigma^{i|n}_{\varepsilon_n}(b_r,b_s)$ we simply take

$$w^n_{\varepsilon_n}(x,y) = (\ell^n_{\varepsilon_n}(x,y))^{-1} \tag{6.3}$$

where $\ell^n_{\varepsilon_n}(x,y) = 2|P^* - P^\perp|$ is the transversal size of $\Sigma^{i|n}_{\varepsilon_n}(b_r,b_s)$ at (x,y).

Step1,(a): Sierpiński. We assume that $u \in C^\beta(\bar\Omega) \cap H^1(\Omega)$ and $u_{\mathcal{G}} \in D[\mathcal{E}_{\mathcal{G}}]$, where $C^\beta(\bar\Omega)$ is the space of Hölder continuous functions on $\bar\Omega$ with Hölder exponent $\beta = \ln\rho/(2\ln\alpha)$. We note that β is the Hölder regularity parameter in the inequality (5.9) for functions of $D[\mathcal{E}_{\mathcal{G}}]$. As u is continuous on $\bar\Omega$ we may simply write u instead of $u_{\mathcal{G}}$ in an integral over $\mathcal{G}$. We decompose the functional $\mathcal{F}^n[v]$, for every $v \in L^2(\Omega)$, as follows:

$$\mathcal{F}^n[v] = B^n[v] - \int_\Omega V^n(x,y) v^2 dxdy$$

where we define

$$B^n[v] = \begin{cases} \int_\Omega a^n(x,y)|\nabla v|^2 dxdy & \text{if} \quad v \in H^1(\Omega; a^n) \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus v \in H^1(\Omega; a^n)\,. \end{cases} \tag{6.4}$$

The functionals B^n can be easily compared with the functionals $F^n_{\varepsilon_n}$ introduced in [30] in relation to the same Sierpiński-type system of fibers: we have

$$B^n[v] \leq F^n_{\varepsilon_n}[v]$$

for every $v \in H^1(\Omega; a^n)$. In fact, the coefficient a^n in the two functionals differs only in the external layer $\Sigma^n_{2\varepsilon} \setminus \Sigma^n_\varepsilon$ of the fiber. It is equal to 1 in $F^n_{\varepsilon_n}$, while in B^n we have $0 \leq a^n \leq 1$ as $n \to +\infty$, because of the insulating nature of $\Sigma^n_{2\varepsilon} \setminus \Sigma^n_\varepsilon$ in the case at hand.
In Proposition 4.3 of [30] it is proved that, given a function u as in Step 1, there exists a sequence of functions $u_n \in H^1(\Omega) \bigcap C^\beta(\bar{\Omega})$, such that $u_n \to u$ strongly in $L^2(\Omega)$ and

$$\limsup F^n_{\varepsilon_n}[u_n] \leq F[u]$$

as $n \to +\infty$. The functional F occurring in this inequality is defined by

$$F[u] = \begin{cases} \zeta^* \int_\Omega |\nabla u|^2 dxdy + \gamma^* \mathcal{E}_\mathcal{G}[u_\mathcal{G}] & \text{if} \quad u \in H^1(\Omega),\ u|_\mathcal{G} \in D[\mathcal{E}_\mathcal{G}] \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus \{u \in H^1(\Omega),\ u|_\mathcal{G} \in D[\mathcal{E}_\mathcal{G}]\}\,. \end{cases} \tag{6.5}$$

Therefore, for the same sequence u_n, we also get the inequality

$$\limsup B^n[u_n] \leq F[u] \tag{6.6}$$

as $n \to +\infty$.

Moreover as $u_n \to u$ strongly in $L^2(\Omega)$ we have

$$\limsup \int_{\Omega \setminus \Sigma^n_\varepsilon} -V^n\, u_n^2 dxdy \leq \lambda^* \int_\Omega u^2 dxdy. \tag{6.7}$$

The functions u_n in this inequality, that is, the functions u_n constructed in Proposition 4.3 of [30], have the following additional property. For each $(\bar{x}, \bar{y}) \in G^n$, $u_n(\bar{x}, y)$ is constant, and equal to $u_{I_n}(\bar{x}, \bar{y})$, on the transversal segment of the internal fiber $\Sigma^n_{\varepsilon_n}$ based at $\bar{x}$. More precisely, as the fiber $\Sigma^n_{\varepsilon_n}$ is the union over $i|n$ and $b_r \neq b_s \in \Gamma$ of all smaller fibers $\Sigma^{i|n}_\varepsilon(b_r, b_s)$, see (2.9), every $(\bar{x}, \bar{y}) \in G^n$ belongs to one of such $\Sigma^{i|n}_{\varepsilon_n}(b_r, b_s)$, and, in notation from the beginning of Section 6, the transversal segment mentioned before – which we shall denote simply by $S^{i|n}_{\varepsilon_n}(\bar{x}, \bar{y})$ – is the intersection with $\Sigma^{i|n}_{\varepsilon_n}(b_r, b_s)$ of the perpendicular line through $(\bar{x}, \bar{y})$ to the longitudinal axis $G^{i|n}_{b_r b_s}$ of $\Sigma^{i|n}_{\varepsilon_n}(b_r, b_s)$. For all $(\bar{x}, y) \in S^{i|n}_{\varepsilon_n}$, $u_n(\bar{x}, y) = u_{I_n}(\bar{x}, \bar{y})$. Here u_{I_n} is continuous on G^n and affine on each side of G^n and is obtained by interpolating the values of u at the vertices of G^n.

As we shall see in a moment, this property of the functions u_n allows us to prove that

$$\lim \int_{\Sigma^n_\varepsilon} V^n\, u_n^2 dxdy = \lambda_* \int_{\mathcal{G}} u^2 d\mu \tag{6.8}$$

as $n \to +\infty$, where μ is the Hausdorff measure on $\mathcal{G}$, see Section 2. Together with (6.6) and (6.7) , inequality (6.8) gives

$$\limsup\, [B^n[u_n] - \int_\Omega V^n\, u_n^2 dxdy] \le F[u] + \lambda^* \int_\Omega u^2 dxdy - \lambda_* \int_{\mathcal{G}} u_{\mathcal{G}}^2 d\mu\,. \tag{6.9}$$

Now we observe that if $\mathcal{F}[u] < +\infty$, then $u \in H^1(\Omega)$ and $u_{\mathcal{G}} \in D[\mathcal{E}_{\mathcal{G}}]$, hence

$$F[u] = \zeta^* \int_\Omega |\nabla u|^2 dxdy + \gamma^* \mathcal{E}_{\mathcal{G}}[u_{\mathcal{G}}]\,.$$

By replacing this expression in the previous inequality, we get

$$\limsup\, \mathcal{F}^n[u_n] = \limsup\, (B^n[u_n] - \int_\Omega V^n\, u_n^2 dxdy\,) \le \mathcal{F}[u]$$

and the proof of (a) in Step 1 will be completed.
Therefore, it only remains to prove the limit (6.8). By (2.9), we decompose

$$\int_{\Sigma^n_{\varepsilon_n}} V^n\, u_n^2 dxdy = \sum_{i|n} \sum_{b_r \neq b_s \in \Gamma} \int_{\Sigma^{i|n}_{\varepsilon_n}(b_r,b_s)} V^n\, u_n^2 dxdy\,.$$

In order to compute each integral in the sum at the right hand side, we consider the set $\Sigma^{i|n}_{\varepsilon_n}(b_r, b_s)$ as a normal domain based on its longitudinal axis $G^{i|n}_{b_r b_s}$ and use in $\Sigma^{i|n}_{\varepsilon_n}(b_r, b_s)$ the coordinates (s, t), with $s \in G^{i|n}_{b_r b_s}$ and $t \in S^{i|n}_{\varepsilon_n}$:

$$\int_{\Sigma^{i|n}_{\varepsilon_n}(b_r,b_s)} V^n\, u_n^2 dxdy = \int_{G^{i|n}_{b_r b_s}} ds \int_{S^{i|n}_{\varepsilon_n}} V^n(s,t)\, u_n^2(s,t) dt\,.$$

We now recall that $V^n(s,t) = \lambda_* \tau_n w^n_{\varepsilon_n}(s,t) = \lambda_* \tau_n \ell^n(s)^{-1}$, where $\ell^n(s) = |S^{i|n}_{\varepsilon_n}|$ is the length of the transversal segment $S^{i|n}_{\varepsilon_n} = S^{i|n}_{\varepsilon_n}(s,0)$ at the point $(s,0) \in G^{i|n}_{b_r b_s}$, and $u_n(s,t) = u_{I_n}(s,0)$ for all $t \in S^{i|n}_{\varepsilon_n}$, therefore

$$\int_{S^{i|n}_{\varepsilon_n}} V^n(s,t)\, u_n^2(s,t) dt = \lambda_* \tau_n u_{I_n}^2(s,0) \ell^n(s)^{-1} \int_{S^{i|n}_{\varepsilon_n}} dt = \lambda_* \tau_n u_{I_n}^2(s,0)$$

and we obtain

$$\int_{\Sigma^n_{\varepsilon_n}} V^n\, u_n^2 dxdy = \sum_{i|n} \sum_{b_r \neq b_s \in \Gamma} \lambda_* \tau_n \int_{G^{i|n}_{b_r b_s}} u_{I_n}^2(s,0) ds\,,$$

then

$$| \int_{\Sigma^n_{\varepsilon_n}} V^n\, u_n^2 dxdy - \int_{G^n} \lambda_* \tau_n u^2 ds| \leq \lambda_* C \alpha^{-\beta n} \tag{6.10}$$

where the constant C depends on the $C^\beta(\bar\Omega)$ norm of u but does not depend on n. By Proposition 5.1, we know that

$$\lim_{n\to+\infty} \lambda_* \tau_n \int_{G^n} u^2 ds = \lambda_* \int_{\mathcal{G}} u^2 d\mu\,, \tag{6.11}$$

This, with (6.10), proves our claim (6.8) and concludes the proof of property (a) in the Sierpiński case, under the additional assumption of Step 1.

Step 2,(a): Sierpiński. We remove the additional assumption made in Step 1, completing in this way the proof of (a) in the Sierpiński case. This is achieved by first approximating in $H^1(\Omega)$ any given function $u \in H^1(\Omega)$ that has the trace $u_{\mathcal{G}} \in D[\mathcal{E}_{\mathcal{G}}]$ with a sequence of functions $\check{u}_m \in C^\beta(\bar\Omega) \cap H^1(\Omega)$ which have same trace on $\mathcal{G}$ as u, then by a further approximation in $L^2(\Omega)$ of every such $\check{u}_m$, and finally by applying a diagonal argument to build up the required u_n. We refer to Propositions 4.4 and 4.5 in [30], for the details. However, we will describe Step 2 in more details in our proof of (a) in the Koch case, that comes next. Before, we point out that in the Sierpiński case treated so far the assumption (4.5) played no role in the proofs and it can be omitted in Theorems 4.1 and 4.2, as already noticed in Remark 4.1.

We now give the proof of (a) for the Koch curves $\mathcal{K}_\alpha$. When $\alpha \neq 3$, we cannot rely on the standard equilateral triangulations and interpolation operators in the whole domain Ω that are used in the Sierpiński case. The geometry of the Koch curves is more irregular and the interpolation technique is easier if restricted only to the pre-fractal curves G^n. New approximation tools from fractional Sobolev spaces come into play. This method is more general and can be applied to other cases, therefore we give the proof in some detail. As in the Sierpiński case dealt with before, the proof will be achieved again in two steps, first by assuming more regularity for u, then removing this additional assumption.

Step1,(a): Koch. We assume, in addition, that $u \in H^\varsigma(\Omega) \cap C^\beta(\bar\Omega)$, $\beta = \frac{\log\rho}{2\log\alpha}, \varsigma = \frac{\log\rho}{2\log\alpha} + 1 - \epsilon$. Here $H^\varsigma(\Omega)$ is the fractional Sobolev space.

For every pre-fractal polygonal curve G^n, we introduce the space $\mathbf{S}_n$ of the functions on G^n which are continuous on G^n and are affine on each side of G^n. We recall that the set of vertices of G^n is denoted by $\mathcal{V}^n$. For a given continuous function $g \in C(\bar\Omega)$, and for every n, we denote by g_{I_n} the function of $\mathbf{S}_n$ obtained by interpolating the values of g at the vertices of G^n, that is,

$$g_{I_n} \in \mathbf{S}_n \quad g_{I_n}(P) = u(P), \quad P \in \mathcal{V}^n. \tag{6.12}$$

We now introduce an auxiliary operator G_ε that acts on $C(\bar\Omega)$. This requires some preliminary notation. We set $\hat{C} = (1/2, h_1/2)$ with, $h_1 = \tan(\vartheta/2)$. The segment AB divides the set $\Sigma^0_{0,\varepsilon}$ (and the set $\Sigma^0_{0,2\varepsilon}$) in two parts: the one which lies inside the triangle of vertices $A, B, \hat{C}$ – denoted Σ^+_ε $(\Sigma^+_{2\varepsilon})$ – and the one which lies outside that triangle – denoted Σ^-_ε $(\Sigma^-_{2\varepsilon})$. For every point (x, y) in $\overline{\Sigma}^0_{0,2\varepsilon}$, by $(x^\top, y^\top)$ we denote the orthogonal projection of (x,y) on G^0; by $P_\pm = P_\pm(x,y) = (\widehat{x}_\pm, \widehat{y}_\pm) \in \partial\Sigma^0_{0,\varepsilon}$ the points where the straight line connecting (x,y) to $(x^\top, y^\top)$ intersects the boundary $\partial\Sigma^0_{0,\varepsilon}$ of $\Sigma^0_{0,\varepsilon}$ and, similarly, by $Q_\pm{=}Q_\pm(x,y) = (\widetilde{x}_\pm, \widetilde{y}_\pm) \in \partial\Sigma^0_{0,2\varepsilon}$ the points where the same line intersects the boundary $\partial\Sigma^0_{0,2\varepsilon}$ of $\Sigma^0_{0,2\varepsilon}$. In this notation, as before, the sign $+$ refers to points inside the triangle, the sign $-$ to points outside. We then define the operator $G_\varepsilon : C(\bar\Omega) \mapsto C(\bar\Omega)$, by putting $G_\varepsilon g = g_\varepsilon$ where for $(x,y) \in \bar\Omega$,

$$g_\varepsilon(x,y) = \begin{cases} g(x,y) & \text{if} \quad (x,y) \in \bar\Omega \setminus \overline{\Sigma}^0_{0,2\varepsilon} \\ g_{I_n}(x^\top, y^\top) & \text{if} \quad (x,y) \in \overline{\Sigma}^0_{0,\varepsilon} \\ g_{I_n}(x^\top, y^\top) t_\pm + g(Q_\pm)(1-t_\pm) & \text{if} \quad (x,y) \in \overline{\Sigma}^0_{0,2\varepsilon} \setminus \overline{\Sigma}^0_{0,\varepsilon}. \end{cases} \tag{6.13}$$

Here:

$$t_\pm = \frac{|\widetilde{y}_\pm - y| + |\widetilde{x}_\pm - x|}{|\widetilde{y}_\pm - \widehat{y}_\pm| + |\widetilde{x}_\pm - \widehat{x}_\pm|}\,.$$

Thus, g_ε is equal to g in $\Omega \setminus \Sigma^0_{0,2\varepsilon}$ and, on each segment J obtained as the intersection of $\overline{\Sigma}^0_{0,2\varepsilon}$ with the orthogonal line to AB at the point $(x^\top, y^\top)$, g_ε is the piecewise-affine function which is constant and equal to $g_{I_n}(x^\top, y^\top)$ on $J \cap \overline{\Sigma}^0_{0,\varepsilon}$ and equal to g at the intersection points $J \cap \partial\Sigma^0_{0,2\varepsilon}$ of J with $\partial\Sigma^0_{0,2\varepsilon}$.

For given n and for every ε_n as in the assumptions of the theorem we define

$$u_n(\xi,\eta) = \begin{cases} u(\xi,\eta) & \text{if} \quad (\xi,\eta) \in \Omega \setminus \Sigma^n_{2\varepsilon_n} \\ G_{\varepsilon_n}(u \circ \psi_{i|n}) \circ \psi^{-1}_{i|n}(\xi,\eta) & \text{if} \quad (\xi,\eta) \in \Sigma^{i|n}_{2\varepsilon_n}\,, \end{cases} \tag{6.14}$$

where G_{ε_n} is the operator G_ε defined before, here taken with $\varepsilon = \varepsilon_n$.

Lemma 6.1. *The functions u_n defined in (6.14) converge to u in $L^2(\Omega)$ and satisfy*

$$\limsup \mathcal{F}^n[u_n] = \mathcal{F}[u] \tag{6.15}$$

as $n \to +\infty$.

Proof. For every n, two distinct copies $G^{i|n}$, $G^{j|n}$ intersect each other at most at vertices belonging to the set $\mathcal{V}^n$. Moreover, for every $0 < \varepsilon \le h_0/2$, two distinct copies $\Sigma^{i|n}_{2\varepsilon}$, $\Sigma^{j|n}_{2\varepsilon}$ meet at most only in $\mathcal{V}^n$. Therefore, as the function u is regular in this Step 1, the functions u_n belong to $H^1(\Omega) \cap C^\beta(\overline{\Omega})$. Moreover,

$$\max_{\Sigma^n_{2\varepsilon}} |u_n| \le ||u||_{L^\infty(\Omega)} \tag{6.16}$$

and the sequence u_n converges strongly to u in $L^2(\Omega)$. We put

$$F^n[u_n] = \int_\Omega a^n(x,y) |\nabla u_n|^2 dxdy \tag{6.17}$$

and we decompose the integral as

$$F^n[u_n] = \zeta_n \int_{\Omega\setminus\Sigma^n_{2\varepsilon_n}} |\nabla u|^2 d\xi d\eta + \gamma_n\sigma_n \int_{\Sigma^n_{\varepsilon_n}} |\nabla u_n|^2 w^n d\xi d\eta + \chi_n \int_{\Sigma^n_{2\varepsilon_n}\setminus\Sigma^n_{\varepsilon_n}} |\nabla u_n|^2 d\xi d\eta. \quad (6.18)$$

Since the two-dimensional Lebesgue measure of $\Sigma^n_{2\varepsilon_n}$ goes to zero as $n \to +\infty$ and by assumption (4.3) we have

$$\lim_{n\to+\infty} \zeta_n \int_{\Omega\setminus\Sigma^n_{2\varepsilon_n}} |\nabla u|^2 d\xi d\eta = \zeta^* \int_\Omega |\nabla u|^2 d\xi d\eta. \quad (6.19)$$

Now we prove that:

$$\lim_{n\to+\infty} \chi_n \int_{\Sigma^n_{2\varepsilon_n}\setminus\Sigma^n_{\varepsilon_n}} |\nabla u_n|^2 d\xi d\eta = 0. \quad (6.20)$$

For every n, we have

$$\Sigma^n_{2\varepsilon_n\setminus\Sigma^n_{\varepsilon_n}} = \bigcup_{i|n} \Sigma^{i|n}_{2\varepsilon_n} \setminus \Sigma^{i|n}_{\varepsilon_n},$$

moreover, for a fixed n-address $i|n$, the set $\Sigma^{i|n}_{2\varepsilon_n} \setminus \Sigma^{i|n}_{\varepsilon_n}$ can be seen as the union of rectangles and triangles. We split the corresponding integrals according to this decomposition, namely, we write

$$\int_{\Sigma^{i|n}_{2\varepsilon_n}\setminus\Sigma^{i|n}_{\varepsilon_n}} |\nabla u_n|^2 d\xi d\eta \equiv R^+ + R^- + \sum_{j=3}^{6} X_j\,, \quad (6.21)$$

where

$$R^+ = \int_{\psi_{i|n}(\mathcal{R}^+_{\varepsilon_n})} |\nabla u_n|^2 d\xi d\eta, \quad R^- = \int_{\psi_{i|n}(\mathcal{R}^-_{\varepsilon_n})} |\nabla u_n|^2 d\xi d\eta$$

$$X_j = \int_{\psi_{i|n}(\mathcal{T}_{j,\varepsilon_n})} |\nabla u_n|^2 d\xi d\eta\,, \quad j = 3, 4, 5, 6.$$

Here $\mathcal{R}^+_{\varepsilon_n}$ is the rectangle of vertices P_1, P_2, Q_2, Q_1; $\mathcal{R}^-_{\varepsilon_n}$ is the rectangle of vertices Q_4, P_4, P_3, Q_3 and $\mathcal{T}_{j,\varepsilon_n}$ is the triangle of vertices A, P_h, Q_h if $j = 3, 6$, or the triangle P_h, Q_h, B if $j = 4, 5$, $h = j - 2$. It suffices to give the proof for R^+, as the other integrals can be evaluated similarly. By the change of coordinates $(\xi, \eta) = \psi_{i|n}(x, y)$, we define

$$g(x, y) = (u \circ \psi_{i|n})(x, y) \quad (6.22)$$

for all $(x, y) \in \mathcal{R}^+_{\varepsilon_n}$. In $\mathcal{R}^+_{\varepsilon_n}$ we have $\widehat{x}_+ = x$, $\widehat{y}_+ = \varepsilon_n/2$, $\widetilde{x}_+ = x$, $\widetilde{y}_+ = \varepsilon_n$, $Q_+ = (x, \varepsilon_n)$. Therefore, by applying (6.13) to the function g, we obtain

$$g_{\varepsilon_n}(x, y) = \{u(\psi_{i|n}(A))(1 - x) + u(\psi_{i|n}(B))x\} \frac{2(\varepsilon_n - y)}{\varepsilon_n} + u(\psi_{i|n}(x, \varepsilon_n)) \frac{2y - \varepsilon_n}{\varepsilon_n}\,.$$

The derivative with respect to the variable y leads to terms of the type

$$Z = \frac{(u(\psi_{i|n}(A)) - u(\psi_{i|n}(x,\varepsilon_n)))^2}{\varepsilon_n^2}$$

which after integration on $\mathcal{R}^+_{\varepsilon_n}$ gives

$$\chi_n \int_{\mathcal{R}^+_{\varepsilon_n}} Z dxdy \leq C \cdot \chi_n \, (1 - \frac{2\varepsilon_n}{h_0})\varepsilon_n^{-1} \cdot \alpha^{-2\beta n} \,. \tag{6.23}$$

Here and elsewhere by C we denote a constant which is independent of n. We note that in the previous inequality we have used the fact that u is Hölder continuous with exponent β that is

$$(u(\psi_{i|n}(A)) - u(\psi_{i|n}(x,\varepsilon_n)))^2 \leq C\alpha^{-2\beta n}.$$

The derivative with respect to the variable x leads to terms of the type $Y = u_x^2(\psi_{i|n}(x,\varepsilon_n))$, which after integration on $\mathcal{R}^+_{\varepsilon_n}$ gives

$$\chi_n \int_{\mathcal{R}^+_{\varepsilon_n}} Y dxdy$$

$$\leq C \cdot \chi_n \, \varepsilon_n/2 \int_{\frac{\varepsilon_n}{h_0}}^{1-\frac{\varepsilon_n}{h_0}} u_x^2(\psi_{i|n}(x,\varepsilon_n))dx \leq C\chi_n \cdot \alpha^{-n} \, \varepsilon_n/2 \int_{\psi_{i|n}([\frac{\epsilon}{h_0}, 1-\frac{\varepsilon_n}{h_0}], \varepsilon_n)} |\nabla u|^2(S) ds \tag{6.24}$$

where we denote by S any point in segment $\psi_{i|n}(x,\varepsilon_n)$ and by ds the arc-lenght measure along the polygonal curve $\psi_{i|n}(\partial\Sigma^+_{2\varepsilon_n})$. The last integral can be estimated taking into account the fact that $u \in H^{\varsigma}(\Omega)$ with $\varsigma = \frac{\ln \rho}{2 \ln \alpha} + 1 - \epsilon$ and this leads to estimate (6.28) below. We now evaluate the integral X_3. In $\mathcal{T}_{3,\varepsilon}$, we have $\widehat{x}_{0,+} = x$, $\widehat{y}_{0,+} = h_0 x/2$, $\widetilde{x}_{0,+} = x$, $\widetilde{y}_{0,+} = h_0 x$, $P_+ = (x, h_0 x/2)$, $Q_+ = (x, h_0 x)$ and

$$g_{\varepsilon}(x,y) = \{u(\psi_{i|n}(A))(1-x) + u(\psi_{i|n}(B))x\}\frac{2(h_0 x - y)}{h_0 x} + u(\psi_{i|n}(x,\varepsilon_n))\frac{2y - h_0 x}{h_0 x}. \tag{6.25}$$

As in the previous calculation the derivatives of the function in (6.25) lead to different terms that can be evaluated as the term

$$X = \frac{(u(\psi_{i|n}(A)) - u(\psi_{i|n}(x, h_0 x)))^2}{x^2};$$

then after integration on $\mathcal{T}_{3,\varepsilon_n}$ one obtain

$$\chi_n \int_{\mathcal{T}_{3,\varepsilon_n}} X dxdy \leq C \, \chi_n \, \varepsilon_n^{2\beta} \cdot \alpha^{-2n\beta}. \tag{6.26}$$

The integrals X_4, X_5 and X_6 can be dealt with in a similar way.

Since $u \in C^\beta(\overline{\Omega})$, by taking estimates (6.23) (6.24) into account we get from (6.21)

$$\chi_n \int_{\Sigma^n_{2\varepsilon} \setminus \Sigma^n_\varepsilon} |\nabla u_n|^2 d\xi d\eta =$$

$$\sum_{i|n} \chi_n \int_{\Sigma^{i|n}_{2\varepsilon} \setminus \Sigma^{i|n}_\varepsilon} |\nabla u_n|^2 d\xi d\eta \leq C \ \sum_{i|n} \{N^{-n} \frac{\chi_n}{\varepsilon_n} + \chi_n \varepsilon_n \alpha^{-n} \int_{\partial \Sigma^{i|n}_{2\varepsilon}} |\nabla u|^2 ds\} \tag{6.27}$$

where ds is the arc-lenght measure along the polygonal curve $\bigcup_{i|n} \partial \Sigma^{i|n}_{2\varepsilon_n}$. We note now that an estimates analogous to (5.3) holds on the boundary of $\Sigma^n_{2\varepsilon_n}$, see Proposition 5.2 with $s = \varsigma - 1$. We then obtain the estimate:

$$\tau_n \int_{\partial \Sigma^n_{2\varepsilon_n}} |\nabla u(S)|^2 ds \leq C ||\nabla u||^2_{H^{\varsigma-1}(\Omega)} \tag{6.28}$$

with a constant C independent of n. Claim (6.20) follows then from (6.27) and (6.28) by taking assumption (4.5) into account. We note that, for a given function $u \in H^1(\Omega) \cap C^\beta(\bar{\Omega}))$, the function u_n defined in (6.14) differs from the corresponding function u_n considered in [30] only in the set $\Omega \setminus \Sigma^n_\varepsilon$. In the fibers $\Sigma^n_{\varepsilon_n}$, both functions coincide, as they are constructed as the piece-wise affine continuous functions that interpolates the function u at the nodes $\mathcal{V}^n$ of G^n. Therefore, we can prove that $u_n \in H^1(\Omega; a^n)$ for every n and

$$\limsup_{n \to +\infty} \ \gamma_n \sigma_n \int_{\Sigma^n_{\varepsilon_n}} |\nabla u_n|^2 w^n_{\varepsilon_n} d\xi d\eta = \gamma^* \mathcal{E}[u]. \tag{6.29}$$

as in the proof of (4.22) in Proposition 4.3 of [30].
The limit

$$\lim \int_{\Sigma^n_{\varepsilon_n}} V^n_\varepsilon u^2_n dx dy = \lambda_* \int_{\mathcal{G}} u^2 d\mu. \tag{6.30}$$

can be evaluated as $n \to +\infty$ in the same way as the limit (6.8) and we omit the details. Finally, by (6.16), the functions u_n are uniformly bounded on the set $\Sigma^n_{2\varepsilon_n}$ and they are all equal to the function u on the set $\Omega \setminus \Sigma^n_{2\varepsilon_n}$. Therefore,

$$\lim \lambda^*_n \int_{\Omega \setminus \Sigma^n_{\varepsilon_n}} u^2_n dx dy = \lambda^* \int_\Omega u^2 dx dy \tag{6.31}$$

as $n \to +\infty$.

Putting all estimates (6.19), (6.20), (6.29), (6.31) and (6.30) together, we get (6.15) and conclude the proof of Lemma 6.1 and with it the proof of (a) in Step 1.

We now move to Step 2, by removing the additional assumption $u \in H^\varsigma(\Omega) \cap C^\beta(\overline{\Omega})$, made in Step 1.

Step 2,(a): Koch. We are given $u \in L^2(\Omega)$ and prove that there exists a sequence of functions $u_n \in L^2(\Omega)$, such that $u_n \to u$ in $L^2(\Omega)$ and

$$\limsup \mathcal{F}^n[u_n] \leq \mathcal{F}[u] \,. \tag{6.32}$$

As noticed at the beginning of this section, we can further assume that $u \in H^1(\Omega)$ and $u_{\mathcal{G}} \in D[\mathcal{E}_{\mathcal{G}}]$. We need the following lemma, that relies on trace, extension and density results for functions in Sobolev and Besov spaces on so-called $d-$sets. For these results we refer to Jonsson [15], Jonsson and Wallin [16].

Lemma 6.2. *Let $u \in H^1(\Omega)$ be such that $u|_{\mathcal{K}_\alpha} \in D[\mathcal{E}_{\mathcal{K}_\alpha}]$. Then, there exists a sequence of functions $\widehat{u}_m \in C^\beta(\overline{\Omega}) \cap H^\varsigma(\Omega)$, $\beta = \frac{\ln\rho}{2\ln\alpha}$, $\varsigma = \frac{\ln(\rho)}{2\ln\alpha} + 1 - \epsilon$, satisfying*

$$\hat{u}_m \equiv u_{|\mathcal{K}_\alpha} \text{ on } \mathcal{K}_\alpha \,,$$

which converges strongly to u in $H^1(\Omega)$ as $m \to +\infty$.

Proof. Let $u \in D[a]$. The trace $u_{|\mathcal{K}_\alpha}$ of u on $\mathcal{K}_\alpha$ belongs to the space $D[\mathcal{E}_{\mathcal{K}_\alpha}]$. This space coincides with the space $Lip_{\gamma,2,\infty}(\mathcal{K}_\alpha)$, $\gamma = \ln(N\rho)/2\ln\alpha$, introduced by Jonsson, see [15] and also [20]. As the space $Lip_{\gamma,2,\infty}(\mathcal{K}_\alpha)$ is a subspace of the Besov space $B^{2,\infty}_\gamma(\mathcal{K}_\alpha)$, $u_{|\mathcal{K}_\alpha}$ admits an extension $\check{u}$ to $\mathbb{R}^2$, such that $\check{u} \in B^{2,\infty}_{\gamma+1-d/2}(\mathbb{R}^2)$, where $B^{2,\infty}_{\gamma+1-d/2}(\mathbb{R}^2)$ is the fractional Besov space defined by Jonsson and Wallin, see [16]. We recall that the Besov space $B^{2,\infty}_{\gamma+1-d/2}(\mathbb{R}^2)$ is a subspace of the Besov space $B^{2,2}_{\gamma+1-d/2-\epsilon}(\mathbb{R}^2)$ for any positive ϵ. The latter space coincides with the fractional Sobolev space $H^{\gamma+1-d/2-\epsilon}(\mathbb{R}^2)$ (see [16]). By the imbedding properties of these Besov spaces (see [16]), we have

$$\check{u} \in C^\beta(\mathbb{R}^2) \,,$$

where $\beta = \gamma + 1 - d/2 - 2/2 = \frac{\ln\rho}{2\ln\alpha}$. We set $\hat{u} = \check{u}_{|\Omega}$ and we have $\hat{u} \in H^\varsigma(\Omega), \varsigma = \gamma + 1 - d/2 - \epsilon$. As $u \in H^1(\Omega)$, hence $\hat{u} - u \in H^1(\Omega)$ and the trace of $\hat{u} - u$ on $\mathcal{K}_\alpha$ is the function $\hat{u}|_{\mathcal{K}_\alpha} - u|_{\mathcal{K}_\alpha} \in C^\beta(\mathcal{K}_\alpha)$ and $\hat{u}|_{\mathcal{K}_\alpha} - u|_{\mathcal{K}_\alpha} \equiv 0$. Since $C^2(\bar{\Omega})$ is dense in $H^1(\Omega)$, there exists a sequence of functions $u^*_m \in C^2(\bar{\Omega})$, $u^*_m \equiv 0$ on $\mathcal{K}_\alpha$, that converges strongly to $u - \hat{u}$ in $H^1(\Omega)$ as $m \to +\infty$. For every m, we put

$$\hat{u}_m = u^*_m + \hat{u} \,. \tag{6.33}$$

The sequence $\{\hat{u}_m\}_m$ has the required properties, in particular $\hat{u}_m \in H^\varsigma(\Omega)$, it converges to u in $H^1(\Omega)$ and

$$\mathcal{E}[\hat{u}_m] = \mathcal{E}[u_{|\mathcal{K}_\alpha}] \tag{6.34}$$

for every m. This concludes the proof of the lemma.

We proceed with our proof of (a). By Lemma 6.2, given our function $u \in D[a]$, there exists a sequence of function $\hat{u}_m \in H^\varsigma(\Omega) \cap C^\beta(\overline{\Omega})$ such that

$$\hat{u}_m \to u \quad H^1(\Omega)$$

and $\hat{u}_m = u$ on $\mathcal{K}_\alpha$. By Lemma 6.1, for each function $\hat{u}_m$ there is a sequence of functions $\hat{u}_{m,n}$ such that

$$\lim_{n\to+\infty} \|\hat{u}_{m,n} - \hat{u}_m\|_{L^2(\Omega)} = 0 \quad \text{and} \quad \limsup_{n\to+\infty} \mathcal{F}^n[\hat{u}_{m,n}] = \mathcal{F}[\hat{u}_m]\,. \tag{6.35}$$

As $\hat{u}_m \to u$ in $H^1(\Omega)$, from (6.35), by taking (6.34) into account, we get

$$\begin{aligned}\mathcal{F}[u] &= \lim_{m\to+\infty} \Big\{\zeta^* \int_\Omega |\nabla \hat{u}_m|^2 dxdy + \lambda^* \int_\Omega \hat{u}_m^2 dxdy - \lambda_* \int_{\mathcal{G}} \hat{u}_m^2 d\mu + \gamma^* \mathcal{E}[\hat{u}_m]\Big\} \\ &= \lim_{m\to+\infty} \mathcal{F}[\hat{u}_m] = \lim_{m\to+\infty} \{\limsup_{n\to+\infty} \mathcal{F}^n[\hat{u}_{m,n}]\}\,,\end{aligned}$$

moreover,

$$\lim_{m\to+\infty} (\lim_{n\to+\infty} \|\hat{u}_{m,n} - u\|_{L^2(\Omega)}) = 0\,.$$

We now apply the diagonal formula of Corollary 1.16 in [3]. This gives a strictly increasing mapping $n \to m(n)$ with $\lim_{n\to+\infty} m(n) = +\infty$, such that, by denoting $u_n = u_{m(n),n}$:

$$\|u_n - u\|_{L^2(\Omega)} \to 0 \quad \text{and} \quad \limsup \mathcal{F}^n[u_n] \le \mathcal{F}[u],$$

as $n \to +\infty$.

7. Weak limit

In this Section we carry on the proof of condition (b) in (4.11). In the space $L^2(\Omega)$ we are given an arbitrary sequence v_n that converges weakly to a function u, and we must prove that

$$\liminf \mathcal{F}^n[v_n] \ge \mathcal{F}[u]\,. \tag{7.1}$$

By possibly extracting a subsequence of v_n, still denoted by v_n, it is not restrictive to assume that $v_n \in H^1(\Omega; a^n)$; that

$$v_n \to u \quad \text{weakly in } L^2(\Omega); \tag{7.2}$$

and that there exists a constant $\tilde{C}$ such that we have

$$\mathcal{F}^n[v_n] \le \tilde{C}\,, \tag{7.3}$$

with $\tilde{C}$ a constant independent of n, where $\mathcal{F}^n[\cdot]$ is the functional defined in (3.4).

Our first concern it to prove that $u \in H^1(\Omega)$. Against this property there are two features in the present setting: the loss of coercivity in the insulating layer of the fibers $\Sigma^n_{2\varepsilon_n} \setminus \Sigma^n_{\varepsilon_n}$, on the one hand, the action of the negative part of the potential V^n on the other hand. To overcome this difficulty we rely on suitable fractional Sobolev and trace inequalities. We begin by establishing a uniform estimate for the sequence v_n in the Sobolev space $W^{1,p}(\Omega)$ with $p<2$. More precisely, we prove

Lemma 7.1. *Let F^n be the functionals introduced in (6.17). Then for every $p>1$ the following inequalities hold:*

$$\int_\Omega |v|^p dxdy + \int_\Omega |\nabla v|^p dxdy \le C_3^* \Big\{ \lambda_n^* \int_{\Omega\setminus\Sigma^n_{\varepsilon_n}} v^2 dxdy + F^n[v] \Big\}^{p/2}, \tag{7.4}$$

where

$$C_3^* = 2^{3(1-p/2)} \max\Big\{ \frac{(1+C_4(1+C_5))}{C_0^{p/2}}, \frac{(1+C_4(1+C_5))|\Omega|^{(1-p/2)}}{(\zeta^*)^{p/2}}, \frac{(C_4C_5)|\Omega|^{(1-p/2)}}{(\lambda^*)^{p/2}} \Big\}$$

for every n and every $v\in H^1(\Omega;a^n)$.

Proof. The estimate of the term $\int_\Omega |v|^p dxdy$ follows from Propositions 5.6 and 5.4. We should only observe that inequality in Proposition 5.4 which involves the boundary $\partial\Omega$ also holds, with a constant that is independent of n, if we replace the set Ω with the set $\Omega\setminus\Sigma^n_{\varepsilon_n}$. In fact, $\partial\Omega$ can be taken to be part of the boundary of an open set $\tilde\Omega$ with $\tilde\Omega\subset\Omega\setminus\Sigma^n_{\varepsilon_n}$. The inequality obtained in this way is

$$||v||^p_{W^{1,p}(\Omega)} \le \Big\{ (1+C_{P;p,p}(1+C_{Tr,p})) \int_\Omega |\nabla v|^p dxdy + C_{P;p,p}C_{Tr,p} \int_{\Omega\setminus\Sigma^n_{\varepsilon_n}} |v|^p dxdy \Big\}. \tag{7.5}$$

As $\lambda_n^*\to\lambda^*$, the last term can be estimated for large n by

$$\int_{\Omega\setminus\Sigma^n_{\varepsilon_n}} |v|^p dxdy \le \{\lambda_n^* \int_{\Omega\setminus\Sigma^n_{\varepsilon_n}} v^2 dxdy \Big\}^{\frac{p}{2}} \cdot \frac{2^{(1-p/2)}|\Omega\setminus\Sigma^n_{\varepsilon_n}|^{1-\frac{p}{2}}}{(\lambda^*)^{\frac{p}{2}}}, \tag{7.6}$$

where by $|E|$, as usual, we denote the 2−dimensional Lebesgue measure of $E\subset\mathbb{R}^2$. Similarly,

$$\int_{\Sigma^n_{2\varepsilon_n}\setminus\Sigma^n_{\varepsilon_n}} |\nabla v|^p dxdy \le \Big\{ \chi_n \int_{\Sigma^n_{2\varepsilon_n}\setminus\Sigma^n_{\varepsilon_n}} |\nabla v|^2 dxdy \Big\}^{\frac{p}{2}} \cdot \frac{|\Sigma^n_{2\varepsilon_n}\setminus\Sigma^n_{\varepsilon_n}|^{1-\frac{p}{2}}}{\chi_n^{\frac{p}{2}}} \tag{7.7}$$

and by (4.6) with $q=2/p-1$

$$\frac{|\Sigma^n_{2\varepsilon}\setminus\Sigma^n_{\varepsilon}|^q}{\chi_n} \le \frac{2^q}{C_0}. \tag{7.8}$$

Estimates (7.7), (7.8) imply

$$\int_\Omega |\nabla v|^p dxdy = \int_{\Sigma^n_{2\varepsilon_n}\setminus\Sigma^n_{\varepsilon_n}} |\nabla v|^p dxdy + \int_{(\Omega\setminus\Sigma^n_{2\varepsilon_n})\bigcup\Sigma^n_{\varepsilon_n}} |\nabla v|^p dxdy \le$$

$$\leq 2^{(1-\frac{p}{2})} C_0^{-\frac{p}{2}} \Big\{ \chi_n \int_{\Sigma^n_{2\varepsilon_n} \setminus \Sigma^n_{\varepsilon_n}} |\nabla v|^2 dxdy \Big\}^{\frac{p}{2}} + \frac{2^{(1-\frac{p}{2})} |\Omega|^{1-\frac{p}{2}}}{(\zeta^*)^{p/2}} \Big\{ \zeta_n \int_{(\Omega \setminus \Sigma^n_{2\varepsilon_n}) \bigcup \Sigma^n_{\varepsilon_n}} |\nabla v|^2 dxdy \Big\}^{\frac{p}{2}}$$

for large n. Since for n large enough

$$\Big\{ \chi_n \int_{\Sigma^n_{2\varepsilon_n} \setminus \Sigma^n_{\varepsilon_n}} |\nabla v|^2 dxdy + \zeta_n \int_{(\Omega \setminus \Sigma^n_{2\varepsilon_n}) \bigcup \Sigma^n_{\varepsilon_n}} |\nabla v|^2 dxdy \Big\} \leq F^n[v]$$

our claim (7.4) follows and the lemma is proved.
Lemma 7.1 is applied with an appropriate choice of the exponent p in order to prove the lemma that follows, which is also instrumental to proving that $u \in H^1(\Omega)$. The choice of p makes it possible to apply the results of Propositions 5.5, 5.2. We also note that it is this special choice of p that leads to the requirement $q < 1/2$ in the assumptions of Theorem 4.1 and Theorem 4.2.

Lemma 7.2. *The functions v_n, satisfying conditions (7.2) and (7.3), have also the property:*

$$v_n \to u \quad \text{strongly in } H^s(\Omega) \tag{7.9}$$

for some $s > 1/2$, as $n \to +\infty$. Moreover,

$$\lim_{n \to +\infty} \lambda_n^* \int_{\Omega \setminus \Sigma^n_{\varepsilon}} v_n^2 dxdy = \lambda^* \int_\Omega u^2 dxdy \,. \tag{7.10}$$

Proof. We claim that the following inequality holds:

$$\int_\Omega a^n(x,y) |\nabla v_n|^2 dxdy + \lambda_n^* \int_{\Omega \setminus \Sigma^n_{\varepsilon_n}} v_n^2 dxdy \leq \tilde{C}^*. \tag{7.11}$$

with a constant $\tilde{C}^*$ that does not depend on n. By (7.4) of the previous lemma, this inequality implies that the sequence v_n is uniformly bounded in $W^{1,p}(\Omega)$. We can choose $p > \frac{4}{3}$. Then, by the Sobolev compact imbedding of Proposition 5.5, we get $v_n \to u$ strongly in $H^s(\Omega)$ with $s > 1/2$. Since this implies in particular that the integrals of v_n^2 over $\Sigma^n_{\varepsilon_n}$ vanish as $n \to +\infty$, and since $\lambda_n^* \to \lambda^*$, the inequality (7.10) also follows and the proof of the lemma will be achieved.
We now prove the claim. By Propositions 5.2, 5.5, we have

$$\tau_n \int_{G^n} v_n^2 ds \leq C_{Tr,s} ||v_n||^2_{H^s(\Omega)} \leq C_{Tr,s} \cdot C_{p,s} ||v_n||^2_{W^{1,p}(\Omega)},$$

and by (7.4) we get

$$\tau_n \int_{G^n} v_n^2 ds \leq C_2^* \big\{ F^n[v_n] + \lambda_n^* \int_{\Omega \setminus \Sigma^n_{\varepsilon_n}} v_n^2 dxdy \big\} \tag{7.12}$$

where

$$C_2^* = C_{Tr,s} \cdot C_{p,s} \cdot C_3^* \,.$$

This estimate allows us to reduce the estimate of the term $\int_{\Sigma^n_{\varepsilon_n}} V^n\, v_n^2 dxdy$ to the estimate of the term $\int_{\Sigma^n_{\varepsilon_n}} V^n v_n^2 dxdy - \lambda_* \tau_n \int_{G^n} v_n^2 ds$. This calculation can be carried out piece-wise on each fiber of $\Sigma^n_{\varepsilon_n}$.

As in the previous section (see (6.10)) for any fiber $\Sigma^{i|n}_{\varepsilon_n}(b_r, b_s)$ of $\Sigma^n_{\varepsilon_n}$ we have

$$|\int_{\Sigma^{i|n}_{\varepsilon_n}(b_r,b_s)} V^n\, v_n^2 dxdy - \lambda_* \tau_n \int_{G^{i|n}_{b_r b_s}} v_n^2 ds| = \lambda_* \tau_n |\int_{G^{i|n}_{b_r b_s}} (\ell_{\varepsilon_n})^{-1}(s) \int_{S^{i|n}_{\varepsilon_n}} \Big(v_n^2(s,t) - v_n^2(s,0) \Big) dtds\, |$$

$$\le 2\lambda_* \tau_n \int_{G^{i|n}_{b_r b_s}} (\int_{S^{i|n}_{\varepsilon_n}} |\nabla v|_n^2 dt)^{1/2} (\int_{S^{i|n}_{\varepsilon_n}} v_n^2 dt)^{1/2} ds;$$

by summing over the fibers we get

$$|\int_{\Sigma^n_{\varepsilon_n}} V^n v_n^2 dxdy - \lambda_* \tau_n \int_{G^n} v_n^2 ds| \le \tau_n \lambda_* \Big\{ \int_{\Sigma^n_{\varepsilon_n}} |\nabla v_n|^2 dxdy + \int_{\Sigma^n_{\varepsilon_n}} v_n^2 dxdy \Big\}\,. \tag{7.13}$$

By Propositions 5.6 and 5.4, as in the proof of previous Lemma 7.1 (see in particular (7.5),(7.6),(7.7) and (7.8))

$$\int_{\Sigma^n_{\varepsilon_n}} V^n v_n^2 dxdy \le \lambda_* (C_2^* + \tau_n C_4^*)(F^n[v_n] + \lambda_n^* \int_{\Omega\setminus\Sigma^n_{\varepsilon_n}} v_n^2 dxdy) \tag{7.14}$$

where C_4^* is a positive constant that depends on the constants $C_{P;2,p}$, $C_{Tr,p}$, C_0, ζ^* and λ^* (see the proof of Lemma 7.1). Therefore

$$\mathcal{F}^n[v_n] \ge \Big(F^n[v_n] + \lambda_n^* \int_{\Omega\setminus\Sigma^n_{\varepsilon_n}} v_n^2 dxdy \Big)(1 - \lambda_* (C_2^* + \tau_n C_4^*))$$

The claim has been proved and with it the lemma.

Remark 7.1. In the Dirichlet case, Lemma 7.1 can be replaced by: let F^n be the functionals introduced in (6.17), then for every $p > 1$ the following estimate holds

$$\int_\Omega |v|^p dxdy + \int_\Omega |\nabla v|^p dxdy \le C^*_{3,D} \{F^n[v]\}^{p/2}\,, \tag{7.15}$$

where

$$C^*_{3,D} = 2^{2-p} \max\Big\{ \frac{1 + C_4}{C_0^{p/2}}, \frac{(1 + C_4)|\Omega|^{(1-p/2)}}{(\zeta^*)^{p/2}} \Big\}$$

for every n and every $v \in D_0[a_n]$. Consequently, in the Dirichlet case, inequality (7.14) in the proof of Lemma 7.2 can be replaced by:

$$\int_{\Sigma^n_{\varepsilon_n}} V^n v_n^2 dxdy \le \lambda_*(C^*_{2,D} + \tau_n C^*_4) F^n[v_n] \tag{7.16}$$

where

$$C^*_{2,D} = C_2 \cdot C_3 \cdot C^*_{3,D}$$

and C^*_4 is a positive constant that depends on the constant $C^*_{3,D}$. Therefore

$$\mathcal{F}^n[v_n] \ge F^n[v_n](1 - \lambda_*(C^*_{2,D} + \tau_n C^*_4)).$$

We are ready to prove that $u \in H^1(\Omega)$ and that additional properties of u hold.

Lemma 7.3. *The functions u has the property:*

$$u \in H^1(\Omega)\,. \tag{7.17}$$

Moreover, for the functions v_n satisfying the properties (7.2) and (7.3), the following conditions hold:

$$\zeta^* \int_\Omega |\nabla u|^2 dxdy \le \liminf\, \zeta_n \int_{\Omega\setminus\Sigma^n_{2\varepsilon_n}} |\nabla v_n|^2 dxdy\,; \tag{7.18}$$

for every $\varphi \in C(\bar\Omega)$,

$$\tau_n \int_{G^n} \varphi v_n ds \to \int_{\mathcal{G}} \varphi u\, d\mu\,, \tag{7.19}$$

as $n \to +\infty$.

Proof. We exploit suitable monotonicity properties of the integration domains and related Fatou's properties. We first consider the Sierpiński case. By T_0 we denote the open triangle of vertices A, B, C and we put $T^*_0 = \overline{T_0 \cup \Sigma^0_{2\varepsilon}}$, $\Omega_0 = \Omega \setminus T^*_0$, and for every $n \ge 1$

$$T^*_n = \overline{\cup_{i|n}\psi_{i|n}(T_0) \cup \Sigma^n_{2\varepsilon}}\,, \quad \Omega_n = \Omega \setminus T^*_n\,.$$

We note that the sequence of the open sets Ω_n is increasing. Moreover, Ω_n converges to the domain Ω, in the sense that the sequence of the two-dimensional Lebesgue measures of Ω_n tends to the two-dimensional Lebesgue measure of Ω and the sequence of the indicatrix functions $\mathbf{1}_{\Omega_n}$ tends to the indicatrix function $\mathbf{1}_\Omega$ a.e in Ω. In the case of the Koch curves $\mathcal{K}_\alpha$ we denote by $\hat T_0$ the open triangle of vertices $A = (0,0)$, $B = (1,0)$ and $\hat C = (1/2, h_1/2)$ where $h_1 = \tan(\frac{\vartheta}{2})$, ϑ the rotation angle (2.5). The triangle $\hat T_0$ satisfies the open set condition with the maps Ψ, that is, $\psi_{i|n}(\hat T_0) \subset \hat T_0$ for every $i|n$ and $\psi_{i|n}(\hat T_0) \cap \psi_{j/n}(\hat T_0) = \emptyset$ for every $i|n \ne j/n$. For every n, we define the (open) polygonal fiber $\hat T^n$

$$\hat T^n = \bigcup_{i|n} \hat T^{i|n} \quad \text{where} \quad \hat T^{i|n} = \psi_{i|n}(\hat T_0), \tag{7.20}$$

and we set

$$\Omega_n = \Omega \setminus \overline{(\hat{T}^n \cup \Sigma^n_{2\varepsilon_n})}\,.$$

By the properties of $\hat{T}_0$ and of the geometry of the fibers, it follows that – as in the previous case of the Sierpiński set – the sequence of open sets Ω_n is increasing and Ω_n converges to Ω as specified before. Therefore, in both cases, Koch and Sierpiński, for every $n \geq m$ we have

$$\Omega_m \subset \Omega_n \subset \Omega \setminus \Sigma^n_{2\varepsilon_n}\,.$$

This property implies, by the estimate (7.11), that v_n converge weakly in $H^1(\Omega_m)$ to u, moreover,

$$\int_{\Omega_m} |\nabla u|^2 dxdy \leq \liminf_{n\to+\infty} \int_{\Omega\setminus\Sigma^n_{2\varepsilon_n}} |\nabla v_n|^2 dxdy \leq \tilde{C}^{**} \tag{7.21}$$

where the constant $\tilde{C}^{**}$ depends on the constant $\tilde{C}^{*}$ in (7.11) and on the material constants ζ^*, λ^*, but it is independent of n and m. By applying Fatou's Lemma to the integrals at the left-hand side of the previous inequality, we obtain that both (7.17) and (7.18) hold. We now prove the last property in the lemma, (7.19). By (5.3) of Proposition 5.2,

$$\tau_n \int_{G^n} (\varphi v_n - \varphi u) ds \to 0 \tag{7.22}$$

as $n \to +\infty$. By the density of $C(\bar{\Omega})$ in $H^1(\Omega)$ and (5.2), we have

$$\tau_n \int_{G^n} \varphi u ds \to \int_{\mathcal{G}} \varphi u \, d\mu \tag{7.23}$$

as $n \to +\infty$. This conclude the proof of (7.19) and of the lemma.

We now proceed to the proof of condition (b), that is, of inequality (7.1). We proceed in two steps.

Step1. We assume, in addition, that

$$v_n \in C^1(\bar{\Omega})$$

for every n. The inequality to prove, (7.1), is

$$\zeta^* \int_\Omega |\nabla u|^2 dxdy + \lambda^* \int_\Omega u^2 dxdy + \gamma^* \mathcal{E}_{\mathcal{G}}[u_{\mathcal{G}}] - \lambda_* \int_{\mathcal{G}} u^2 d\mu \tag{7.24}$$

$$\leq \liminf \left(F^n[v_n] - \int_\Omega V^n(x,y) v_n^2 dxdy\right) \tag{7.25}$$

as $n \to +\infty$, where

$$F^n[v_n] = \zeta_n \int_{\Omega\setminus\Sigma^n_{2\varepsilon_n}} |\nabla v_n|^2 dxdy + \chi_n \int_{\Sigma^n_{2\varepsilon_n}\setminus\Sigma^n_{\varepsilon_n}} |\nabla v_n|^2 dxdy + \gamma_n\sigma_n \int_{\Sigma^n_{\varepsilon_n}} w^n_{\varepsilon_n} |\nabla v_n|^2 dxdy$$

and

$$\int_\Omega V^n(x,y) v_n^2 dxdy = -\lambda_n^* \int_{\Omega\setminus\Sigma^n_{\varepsilon_n}} v_n^2 dxdy + \lambda_*\tau_n \int_{\Sigma^n_{\varepsilon_n}} w^n_{\varepsilon_n} v_n^2 dxdy\,.$$

The inequalities (7.10) and (7.18) of Lemma 7.2 and Lemma 7.3, together, give

$$\zeta^* \int_\Omega |\nabla u|^2 dxdy + \lambda^* \int_\Omega u^2 dxdy \le \liminf\,(\zeta_n \int_{\Omega\setminus\Sigma^n_{2\varepsilon_n}} |\nabla v_n|^2 dxdy + \lambda_n^* \int_{\Omega\setminus\Sigma^n_{\varepsilon_n}} v_n^2 dxdy) \tag{7.26}$$

as $n \to +\infty$. Next, we claim that the following inequality holds:

$$\gamma^* \mathcal{E}_{\mathcal{G}}[u_{\mathcal{G}}] \le \liminf \gamma_n\sigma_n \int_{\Sigma^n_{\varepsilon_n}} w^n_{\varepsilon_n} |\nabla v_n|^2 dxdy\,. \tag{7.27}$$

We note that this inequality does not involve the insulating external fibers $\Sigma^n_{2\varepsilon_n} \setminus \Sigma^n_{\varepsilon_n}$ nor the potentials V^n, which are the peculiarities of the present setting. The inequality has in fact been obtained in [30], see (5.51) of that paper. The proof is rather technical, and it will not be reproduced here.

The term at the right hand side of inequality (7.27) is bounded, because all integrals are part of $F^n[v_n]$ which, by (7.11), are uniformly bounded in n. This implies that $\mathcal{E}_{\mathcal{G}}[u_{\mathcal{G}}] < +\infty$, what implies in particular that $u_{\mathcal{G}}$ is continuous on $\mathcal{G}$.
Finally, we shall prove in a moment that

$$\lim \int_{\Sigma^n_{\varepsilon_n}} V^n v_n^2 dxdy = \lambda_* \int_{\mathcal{G}} u^2 d\mu. \tag{7.28}$$

By adding (7.26), (7.27), (7.28) together, we get the inequality (7.1) and bring Step 1 to its end.

To prove (7.28), we write

$$\begin{aligned} & \int_{\Sigma^n_{\varepsilon_n}} V^n v_n^2 dxdy - \lambda_* \int_{\mathcal{G}} u^2 d\mu = \\ & (\lambda_*\tau_n \int_{\Sigma^n_{\varepsilon_n}} w^n_{\varepsilon_n} v_n^2 dxdy - \lambda_*\tau_n \int_{G^n} v_n^2 ds) + \lambda_*\tau_n(\int_{G^n} v_n^2 ds - \int_{G^n} u^2 ds) + \lambda_*(\tau_n \int_{G^n} u^2 ds - \int_{\mathcal{G}} u^2 d\mu) \\ = \; & J_1 + J_2 + J_3\,. \end{aligned}$$

By (7.13),

$$|J_1| = \lambda_*\tau_n|\Big(\int_{\Sigma^n_{\varepsilon_n}} v_n^2 w^n_{\varepsilon_n} dxdy - \int_{G^n} v_n^2 ds\Big)| \le \tau_n\lambda_* C_4^* \Big\{F^n[v_n] + \lambda_n^* \int_{\Omega\setminus\Sigma^n_{\varepsilon_n}} v_n^2 dxdy\Big\}$$

with C_4^* a positive constant that depends only on the constants $C_{P;2,p}$, $C_{Tr,p}$, C_0, ζ^* and λ^* (see the proof of Lemma 7.2) occurring in (7.4) and (7.14). The terms in brackets are uniformly bounded in n and $\tau_n \to 0$. Therefore, $J_1 \to 0$ as $n \to +\infty$.

By the estimate (5.3) of Proposition 5.2

$$|J_2| = \lambda_* \tau_n \Big|\Big(\int_{G^n} v_n^2 ds - \int_{G^n} u^2 ds \Big)\Big| \leq \lambda_* C_{Tr,s} ||v_n - u||_{H^s(\Omega)} ||v_n + u||_{H^s(\Omega)} \to 0.$$

Finally, by property (5.2) of Proposition 5.1

$$J_3 = \tau_n \int_{G^n} u^2 ds - \int_{\mathcal{G}} u^2 d\mu \to 0\,.$$

This concludes the proof.

Step 2. We remove the assumption that $v_n \in C^1(\bar{\Omega})$. This is done by a density argument based on the fact that the space $C^1(\bar{\Omega})$ is dense in the weighted space $H^1(\Omega; a^n)$ (see Proposition 5.4 in [30] for more details).

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