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On asymptotically harmonic manifolds of negative curvature

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Abstract We study asymptotically harmonic manifolds of negative curvature, without any cocompactness or homogeneity assumption. We show that asymptotic harmonicity provides a lot of information on the asymptotic geometry of these spaces: in particular, we determine the volume entropy, the spectrum and the relative densities of visual and harmonic measures on the ideal boundary. Then, we prove an asymptotic analogue of the classical mean value property of harmonic manifolds, and we characterize asymptotically harmonic manifolds, among Cartan–Hadamard spaces of strictly negative curvature, by the existence of an asymptotic equivalent $\tau(u)e^{Er}$ for the volume-density of geodesic spheres (with τ constant in case DR_M is bounded). Finally, we show the existence of a Margulis function, and explicitly compute it, for all asymptotically harmonic manifolds.

Keywords Asymptotically harmonic manifolds · Spectrum · Asymptotic geometry

Mathematics Subject Classification (2010) Primary 53C20

1 Introduction

Harmonic manifolds are those Riemannian manifolds whose geodesic spheres have constant mean curvature; equivalently, such that the volume density function, in normal coordinates at any point x, only depends on the distance $d(x, \cdot)$. Another equivalent condition is that the mean-value property

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$$f(x_0) = \frac{1}{\mathrm{vol}(S_{x_0}(R))} \int\limits_{S_{x_0}(R)} f(x) dv_{S_{x_0}(R)}$$

holds for all harmonic functions f on M (cf. [3]). It is not difficult to show that these are Einstein spaces, hence constant curvature spaces in dimensions 2 and 3.

In 1944, A. Lichnerowicz conjectured, and proved in dimension 4, that the rank one symmetric spaces (denoted ROSS, in the sequel) are the only harmonic manifolds. The conjecture was then proved by Z.I. Szabo for *compact simply connected manifolds* (cf. [25]), and for *negatively curved Cartan Hadamard manifolds admitting compact quotients* by G. Besson, G. Courtois and S. Gallot (also using work of P. Foulon, Y. Benoist and F. Labourie on the geodesic flow of manifolds having smooth horospherical distribution, cf. [4] section 9.C and [2,10]). The assumption of negative curvature was further relaxed by G. Knieper in [17], where he studied relations between volume growth, dynamical properties of the geodesic flow and Gromov hyperbolicity; in particular he proved that compact harmonic manifolds with a Gromov hyperbolic fundamental group are quotients of a ROSS. On the other hand, A. Ranjan and H. Shah proved in [21] that harmonic manifolds with polynomial volume growth are flat. However, the Lichnrowicz conjecture was proved to be false: E. Damek and F. Ricci constructed harmonic *homogeneous* manifolds which are not ROSS (cf. [8]). Since then, J. Heber proved that Damek-Ricci spaces and ROSS are the only homogeneous harmonic manifolds (cf. [11]).

In several of these works, an asymptotic version of harmonicity naturally appears (cf. [10,11]): a Cartan–Hadamard manifold M is *asymptotically harmonic* if its horospheres have constant mean curvature h. It is easy to show that, in dimension 2, the only asymptotically harmonic manifolds are $\mathbb{H}^2$ and $\mathbb{R}^2$. Recently (cf. [12]), it was also proved that, in dimension 3, the only asymptotically harmonic Cartan–Hadamard manifold of strictly negative curvature is the hyperbolic space ; this result was then generalized in [24], removing the curvature bounds and assuming that the mean curvature of the horopsheres is positive.

However, the notion of asymptotically harmonic manifolds was introduced by F. Ledrappier in [19], mainly to study the *cocompact* case (i.e. when the space admits compact quotients). F. Ledrappier proved that, within cocompact Cartan–Hadamard spaces, asymptotic harmonicity is equivalent to the condition $\inf \sigma(\Delta) = \frac{E^2}{4}$ (where $\sigma(\Delta)$ is the spectrum of the Laplacian of M, and E its volume-entropy) and to the coincidence of the families of harmonic and Bowen-Margulis measures; moreover, he showed that if M is asymptotically harmonic, then $E = nh$. The above recalled work of Besson, Courtois, Gallot, together with [10], settled the problem both for harmonic and for asymptotically harmonic manifolds, among cocompact, negatively curved Cartan–Hadamard spaces, showing that they are all ROSS. On the other hand, in [7] necessary and sufficient conditions are given in order that a homogeneous, negatively curved Cartan–Hadamard manifold is asymptotically harmonic; however, as far as the authors know, the problem whether any asymptotically harmonic manifold is ROSS is still open in this class.

The aim of this paper is to show that, for Cartan–Hadamard manifolds of strictly negative curvature of any dimension, even without any cocompactness or homogeneity assumption, asymptotic harmonicity provides a lot of information on the asymptotic geometry. In view of [19], we are naturally interested in the volume entropy, the spectrum and the relations between visual and harmonic measures on the ideal boundary of a general asymptotically harmonic manifold.

In particular, in Sect. 3, we show rigidity of Cartan–Hadamard asymptotically harmonic manifolds under suitable curvature bounds (Corollary 4.7), we determine the volume entropy and the spectrum (cf. Theorems 4.3 & 4.4) and, when the curvature is negatively pinched, we find sharp upper and lower bounds for the volume-growth of the horospheres (Theorem 4.8 and ff. Remarks 4.9 & 4.10). Moreover, we prove an asymptotic analogue of the classical mean-value property holding on harmonic manifolds (Theorem 4.11).

In Sect. 4, we characterize asymptotically harmonic manifolds as those manifolds whose volume density function is asymptotically equivalent to a function $\tau(u)\mathrm{e}^{Er}$, for some positive function τ on SM (Theorem 5.1); then, we show that the function τ is constant if DR_M (the derivative of the Riemann tensor) is bounded (Proposition 5.4(ii)). The existence of this function τ can also be deduced from the computations in [17], but we stress the fact that our result is stronger as we get some uniformity in the limit (Proposition 5.4 and Remark 5.3).

In Sect. 6 we prove the existence of a Margulis function (Proposition 6.2), we explicitly compute it for all asymptotically harmonic manifolds, and we find the relative densities of visual and harmonic measures on the ideal boundary (Proposition 6.1); we also show that they coincide when DR_M is bounded. This result is to compare to what is known in the cocompact and homogeneous cases, where coincidence of two of the three natural families of measures on the ideal boundary (visual, harmonic and Patterson-Sullivan measures) forces, respectively in the two cases, symmetry and asymptotic harmonicity of the manifold (cf. [19, 20, 28, 27, 7]); unfortunately, a similar characterization for general asymptotically harmonic Cartan–Hadamard manifolds is still missing.

The main tools we use are a comparison lemma for the second fundamental forms of two tangent spheres, which is proved in Sect. 2, and the Riccati equation. The first section is devoted to notations and preliminary results.

We thank professor S. Gallot for his suggestions and encouragement, and professor G. Knieper for explaining us the expression of the function τ in terms of Jacobi tensors.

2 Notations

Unless otherwise stated, throughout all the paper (M, g) will always be a *Cartan–Hadamard manifold* (CH-manifold, for short) of dimension $n+1$, i.e. a complete, simply connected Riemanniann manifold with nonpositive curvature.

The *ideal boundary* of M, denoted $\partial_\infty M$, is the set of equivalent classes of geodesic rays, γ and σ being equivalent if $\sup\{d(\gamma(t), \sigma(t)) \mid t \geq 0\} < \infty$ (cf. [5] definition II.8.1). For $\xi \in \partial_\infty M$, $\lim_{t\to+\infty} \gamma(t) = \xi$ will mean that ξ is the equivalence class defined by γ. The cone topology turn $M \cup \partial_\infty M$ into a compact manifold with boundary (cf [5] definition II.8.6).

For $\xi \in \partial_\infty M$ and $x \in M$, the *Busemann function* $b_{\xi,x}$, centered at ξ and vanishing at x, is defined by $b_{\xi,x}(y) = \lim_{t\to+\infty}(d(y, \gamma(t)) - t)$, where γ is the unique geodesic such that $\gamma(0) = x$ and $\lim_{t\to+\infty} \gamma(t) = \xi$. Two Busemann functions centered in the same point at infinity differ from a constant ; in many situations, we only need to know the Busemann functions up to a constant, and we shall note b_ξ some Busemann function centered in ξ. Busemann functions are Lipschitz and, on CH-manifolds, they are at least C^2, cf. [13].

The *horospheres* centered in $\xi \in \partial_\infty M$ are the level hypersurfaces of b_ξ: $H_\xi(t) = \{x \in M \mid b_\xi(x) = t\}$; we shall also use the convenient notation $H_\xi(x)$ for the horosphere centred at ξ and passing through x. As the Busemann functions are limit of distance functions, the horospheres centered in ξ are (locally) limit of spheres whose centers tends to ξ. Since $|\nabla b_\xi| = 1$ and the gradient lines of b_ξ are the geodesics γ such that $\lim_{t\to-\infty} \gamma(t) = \xi$, we can define the *inner* unit vector field of horospheres centred at ξ as $\nu = -\nabla b_\xi$ (i.e. ν points towards the center ξ of the horosphere).

For a general hypersurface N of M, $\vec{\mathcal{A}}^N$ denotes its *second* (vector valued) *fundamental form* ; that is, for $u, v \in T_x N$, $\vec{\mathcal{A}}^N(u, v)$ is the component of $D^M_u V$ normal to N, where D^M is the connection of M and V extends v in a neighborhood of x. Associated to the choice of a unit normal vector field ν to N we then have the *second* (scalar) *fundamental form* $\mathcal{A}^N = \langle \vec{\mathcal{A}}^N, \nu\rangle$ and the *shape operator* $A^N \in End(T_x N)$, defined by $\langle A^N u, v\rangle = \langle D^M_u \nu, v\rangle = -\langle \vec{\mathcal{A}}^N(u, v), \nu\rangle$. The *mean curvature vector* of N at x is $\vec{h}^N(x) = \frac{1}{n} Tr \vec{\mathcal{A}}^N(x)$, while the (scalar) mean curvature, associated with ν, is $h^N = \langle \vec{h}^N, \nu\rangle$.

A manifold M is called *asymptotically harmonic* if all its horospheres have constant mean curvature h. The curvature of M being nonpositive, the horospheres are convex and we have $h \geq 0$ when choosing ν pointing to the center of the horosphere.

2.1 Hessian and Laplacian of Busemann functions

The second fundamental form naturally appears when restricting a function to a submanifold:

Proposition 2.1 *Let $i : N \to M$ be an isometric immersion, let $F : M \to \mathbb{R}$ be a smooth function and let $f = F_{|N}$ be its restriction to N.*

For all $x \in N$ and all $u, v \in T_x N$ we have

$$(\mathrm{Hess}^N f)(u, v) = (\mathrm{Hess}^M F)(u, v) + \langle \nabla^M F, \vec{\mathcal{A}}^N(u, v)\rangle$$

Proof The proof is standard. □

As a consequence, the Hessian of the Busemann function is given by the second scalar fundamental form of its horospheres, with respect to the inner normal vector field ν; taking the trace we get $\Delta b_\xi(y) = -Tr(\mathrm{Hess}_y b_\xi) = -nh_\xi(y)$, where $h_\xi(y)$ is the mean curvature at y of the horosphere centered in ξ passing through y, with respect to ν. (Similarly, the second fundamental form of spheres is the Hessian of the distance function to the center and the Laplacian of the distance from a point x gives the mean curvature of the spheres centered in x).

It follows, by the regularity theory of solutions of elliptic equations, that for asymptotically harmonic manifolds Busemann functions and horospheres are at least as regular as the metric (whereas they are known to be real analytic on harmonic manifolds, cf. [22]). Moreover, it is then straightforward to check that, for any asymptotically harmonic manifold M with horospheres of mean curvature h, the function $f(y) = \mathrm{e}^{-nhb_\xi(y)}$ is harmonic.

2.2 The Riccati equation

Let $\xi \in \partial_\infty M$ and γ be a geodesic such that $\lim_{t\to-\infty} \gamma(t) = \xi$. For each t, let $A_\xi(t)$ be the shape operator of the horosphere centered in ξ passing through $\gamma(t)$, with respect to the inner unit vector field $\nu = -\nabla b_\xi = -\gamma'(t)$; this family of operators satisfies the Riccati equation (cf. [14] §1.3):

$$A'_\xi(t) + A^2_\xi(t) + R_M(\dot\gamma(t), .)\dot\gamma(t) = 0 \tag{2.1}$$

where R_M is the Riemann tensor of M.

3 Comparison of spheres on CH-manifolds

In the sequel, we note $\mathbb{M}^n(-a^2)$ the simply connected Riemannian manifold with constant sectional curvature $-a^2$, and we shall note C_a and $\cot_a$ the functions defined by:

$$C_a(s) = \begin{cases} \frac{1}{a^2}(\cosh(as) - 1) & \text{if } a > 0 \\ \frac{s^2}{2} & \text{if } a = 0 \end{cases} \quad \text{and} \quad \cot_a(s) = \begin{cases} a\coth(as) & \text{if } a > 0 \\ \frac{1}{s} & \text{if } a = 0 \end{cases}$$

3.1 Comparison of triangles

When assuming a sectional curvature upper bound $K_M \le -a^2$ for M, the classical Toponogov theorem (cf. [14]) implies that, given two edges of a triangle in M with angle α at the common vertex, then the third edge is larger than the one of a triangle in the model space $\mathbb{M}^2(-a^2)$ with the same lengths for the first two edges and the same angle at the common vertex. The following lemma is a slight modification of this result, where we compare the ratio of (some function of) the lengths of the third edge and of an "intermediate edge".

Lemma 3.1 (Triangle comparison with curvature upper bound) *Let M be a CH-manifold with $K_M \le -a^2 \le 0$. Let (xyz) and $(\tilde{x}\tilde{y}\tilde{z})$ be triangles in M and $\mathbb{M}^2(-a^2)$ respectively, such that $r_1 = d(x, y) = d(\tilde{x}, \tilde{y})$, $r_2 = d(x, z) = d(\tilde{x}, \tilde{z})$ and $\alpha = \angle_x(y, z) = \angle_{\tilde{x}}(\tilde{y}, \tilde{z})$. Moreover, for $\theta \in]0, 1[$ let p, q and $\tilde{p}$, $\tilde{q}$ be respectively the points on the geodesic segments xy, xz and $\tilde{x}\tilde{y}$, $\tilde{x}\tilde{z}$ such that $d(x, p) = d(\tilde{x}, \tilde{p}) = \theta r_1$ and $d(x, q) = d(\tilde{x}, \tilde{q}) = \theta r_2$ (cf. Fig. 1). Then:*

$$\frac{C_a(d(y,z))}{C_a(d(p,q))} \ge \frac{C_a(d(\tilde{y},\tilde{z}))}{C_a(d(\tilde{p},\tilde{q}))} = F_a(r_1, r_2, \alpha, \theta).$$

Remark 3.2 By the cosine formula in $\mathbb{M}^2(-a^2)$ (cf. [5] proposition I.2.7) we know that the right-hand side of the above inequality only depends on the lengths r_1, r_2, α and θ, whence the existence of the function F_a.

When $a = 0$ we have $F_0 = \frac{1}{\theta^2}$ and lemma 3.1 is a direct consequence of the convexity of the distance function in $CAT(0)$-spaces (cf. [5] proposition II.2.2).

When $a > 0$ we find:

$$F_a(r_1, r_2, \alpha, \theta) = \frac{\cosh(ar_1)\cosh(ar_2) - \sinh(ar_1)\sinh(ar_2)\cos(\alpha) - 1}{\cosh(a\theta r_1)\cosh(a\theta r_2) - \sinh(a\theta r_1)\sinh(a\theta r_2)\cos(\alpha) - 1}.$$

An important point in the proof of lemma 3.1 is that, whenever $\theta \le 1$, the function F_a is nondecreasing with respect to α.

Proof of lemma (2.1) First consider a comparison triangle $(\bar{x}\bar{y}\bar{z})$ in $\mathbb{M}^2(-a^2)$, that is such that $d(\bar{x}, \bar{y}) = r_1$, $d(\bar{x}, \bar{z}) = r_2$, and $d(\bar{y}, \bar{z}) = d(y, z)$. Define $\bar{p}$, $\bar{q}$ to be the points on the geodesic segments $\bar{x}\bar{y}$ and $\bar{x}\bar{z}$ respectively, such that $d(\bar{x}, \bar{p}) = \theta r_1$, $d(\bar{x}, \bar{q}) = \theta r_2$, and let $\bar{\alpha} = \angle_{\bar{x}}(\bar{y}, \bar{z})$. By Toponogov theorem, we have $d(\bar{p}, \bar{q}) \ge d(p, q)$ and $\bar{\alpha} \ge \alpha$. Using these inequalities and remark 3.2 we have

$$\frac{C_a(d(y,z))}{C_a(d(p,q))} \ge \frac{C_a(d(y,z))}{C_a(d(\bar{p},\bar{q}))} = F_a(r_1, r_2, \bar{\alpha}, \theta) \ge F_a(r_1, r_2, \alpha, \theta) = \frac{C_a(d(\tilde{y},\tilde{z}))}{C_a(d(\tilde{p},\tilde{q}))}$$

□

A similar inequality holds for CH-manifolds with curvature lower bound:

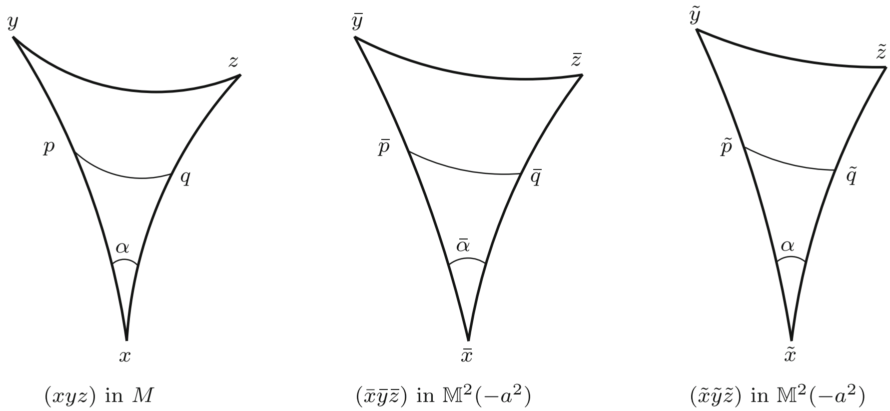

Fig. 1 Comparison triangles

Lemma 3.3 (Triangle comparison with curvature lower bound) *Let M be a CH-manifold with $K_M \geq -b^2$. Let (xyz) and $(\tilde{x}\tilde{y}\tilde{z})$ be triangles in M and $\mathbb{M}^2(-b^2)$ respectively, such that $r_1 = d(x, y) = d(\tilde{x}, \tilde{y})$, $r_2 = d(x, z) = d(\tilde{x}, \tilde{z})$ and $\alpha = \angle_x(y, z) = \angle_{\tilde{x}}(\tilde{y}, \tilde{z})$. Moreover, for $\theta \in]0, 1[$ let p, q and $\tilde{p}$, $\tilde{q}$ be respectively the points on the geodesic segments xy, xz and $\tilde{x}\tilde{y}$, $\tilde{x}\tilde{z}$ such that $d(x, p) = d(\tilde{x}, \tilde{p}) = \theta r_1$ and $d(x, q) = d(\tilde{x}, \tilde{q}) = \theta r_2$. Then:*

$$\frac{C_b(d(y, z))}{C_b(d(p, q))} \leq \frac{C_b(d(\tilde{y}, \tilde{z}))}{C_b(d(\tilde{p}, \tilde{q}))} = F_b(r_1, r_2, \alpha, \theta).$$

Proof The proof is similar to that of lemma 3.1. Toponogov theorem gives $d(\bar{p}, \bar{q}) \leq d(p, q)$ and $\bar{\alpha} \leq \alpha$, and by the monotonicity of the function F_b we get

$$\frac{C_b(d(y, z))}{C_b(d(p, q))} \leq \frac{C_b(d(y, z))}{C_b(d(\bar{p}, \bar{q}))} = F_b(r_1, r_2, \bar{\alpha}, \theta) \leq F_b(r_1, r_2, \alpha, \theta) = \frac{C_b(d(\tilde{y}, \tilde{z}))}{C_b(d(\tilde{p}, \tilde{q}))}$$

□

3.2 Comparison of spheres

Let $S_x(r)$ and $S_y(R)$ be two geodesic spheres in M, with $r < R$, tangent at some point z, with $S_x(r)$ internal to $S_y(R)$. Let $\vec{\mathcal{A}}_x$ and $\vec{\mathcal{A}}_y$ (resp. $\mathcal{A}_x$, $\mathcal{A}_y$) be the second, vector-valued (resp. scalar) fundamental forms of $S_x(r)$ and $S_y(R)$, and let ν be the common inner unit normal vector at z. We will now compare the two second fundamental forms $\mathcal{A}_x$ and $\mathcal{A}_y$.

Let $u \in T_z S_x(r)$ be a unitary vector, and let $c_u(s)$ be the geodesic of $S_x(r)$ with initial tangent vector u. Denote by r_x and r_y the distance functions to x and y respectively, and let $r_y(s) = r_y(c_u(s))$ be the restriction of the function r_y to the curve c_u. Applying Proposition 2.1 to c and r_y we find

$$r_y''(0) = \left(\mathrm{Hess}^M r_y\right)(u, u) + \langle \nabla r_y, \vec{\mathcal{A}}_x(u, u)\rangle,$$

and, since $\mathrm{Hess}^M r_y$ gives the second fundamental form of $S_y(R)$ w.r. to ν,

$$r_y''(0) = \mathcal{A}_y(u, u) - \mathcal{A}_x(u, u) \tag{3.1}$$

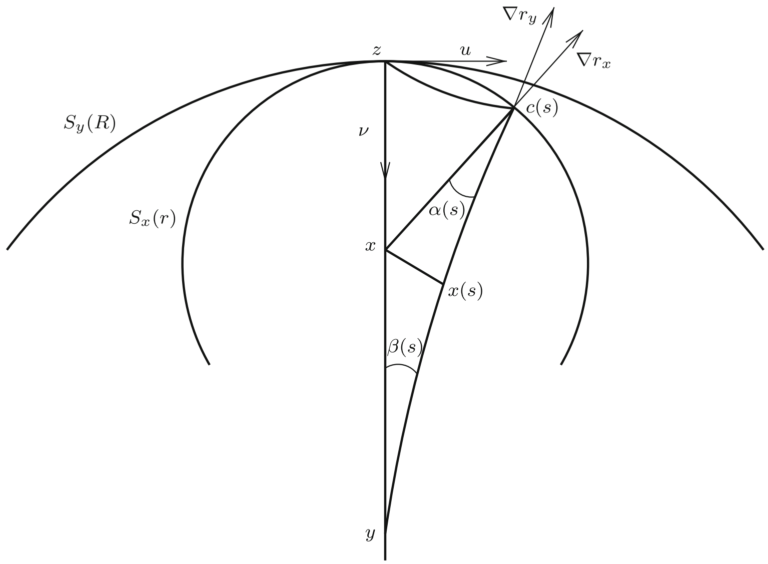

Fig. 2 Comparing the second fundamental forms of tangent spheres

But $r_y''(0) \leq 0$ as z is the maximum of r_y on $S_x(r)$, thus at the point z we have $\mathcal{A}_y \leq \mathcal{A}_x$ which means that $S_x(r)$ is "more curved" than $S_y(R)$. Using the above comparison lemmas for triangles, we get sharper comparison estimates for the tangent spheres:

Lemma 3.4 *Let (M, g) be a CH-manifold with $K_M \leq -a^2$. With the above notations, the second fundamental forms of $S_x(r)$, $S_y(R)$ at the tangent point z satisfy:*

$$0 \leq \mathcal{A}_x - \mathcal{A}_y \leq (\cot_a r - \cot_a R)\, g$$

Moreover, if we assume $-b^2 \leq K_M$ then at the tangent point z we also have:

$$(\cot_b r - \cot_b R)\, g \leq \mathcal{A}_x - \mathcal{A}_y$$

Remark 3.5 These estimates are optimal, since they are equalities when M has, respectively, constant curvature $-a^2$ or $-b^2$.

Proof We only consider the case $a > 0$; when $a = 0$, the proof is similar (just replace the hyperbolic laws by the Euclidean ones) and is left to the reader.

As before, let $u \in T_z S_x(r)$ be a unitary vector, let $c(s)$ be the geodesic of $S_x(r)$ with initial tangent vector u and let $r_y(s)$ be the restriction of the function r_y to the curve c. For $s > 0$ we consider (cf. Fig. 2):

- the angle $\alpha(s)$ between ∇r_y and ∇r_x at $c(s)$;
- the angle $\beta(s)$ between the geodesic lines from y to z and from y to $c(s)$;
- $\theta = \frac{R-r}{R}$ and the point $x(s)$ of the geodesic from y to $c(s)$ such that $d(y, x(s)) = \theta d(y, c(s))$;

so $r_y'(s) = \langle \nabla r_y, \dot{c}(s) \rangle = -\sin(\alpha(s))$. Using Toponogov theorem for the triangle $(c(s)x(s)x)$ and the law of cosine in $\mathbb{M}^2(-a^2)$ we get

$$\begin{aligned}\cosh(ad(x, x(s))) &\geq \cosh(ar)\cosh(a(1-\theta)r_y(s)) \\ &\quad - \sinh(ar)\sinh(a(1-\theta)r_y(s))\cos(\alpha(s)) \\ &\geq 1 + \sinh(ar)\sinh(a(1-\theta)r_y(s))(1-\cos(\alpha(s)))\end{aligned} \tag{3.2}$$

On the other hand, Lemma 3.1 applied to the triangle $(yzc(s))$ implies that

$$\cosh(ad(x, x(s))) - 1 \leq \frac{\cosh(ad(z, c(s))) - 1}{F_a(R, r_y(s), \beta(s), \theta)} \tag{3.3}$$

which, plugged in (3.2), yields:

$$1 - \cos(\alpha(s)) \leq \frac{\cosh(ad(z, c(s))) - 1}{\sinh(ar)\sinh(a(1-\theta)r_y(s))F_a(R, r_y(s), \beta(s), \theta)} \tag{3.4}$$

We divide by s^2 and pass to the limit for $s \to 0$ in (3.4): as $r_y'(s)^2 = \sin^2\alpha(s)$ and $r_y'(0) = 0$, we have $\lim_{s\to 0} \frac{1-\cos\alpha(s)}{s^2} = \frac{1}{2}\lim_{s\to 0}\left(\frac{r_y'(s)}{s}\right)^2 = \frac{1}{2}r_y''(0)^2$; then, notice that $\frac{d(z,c_u(s))}{s} \to 1$ and that, as $r_y(s) - R = O(s^2)$ and $\beta(s) = O(s)$, we have $\lim_{s\to 0} F_a(R, r_y(s), \beta(s), \theta) = \frac{\sinh^2(aR)}{\sinh^2(a(R-r))}$. So from (3.4) we get

$$\left|r_y''(0)\right| \leq \frac{a\sinh(a(R-r))}{\sinh(ar)\sinh(aR)} = a(\coth(ar) - \coth(aR))$$

By (3.1), as $r_y''(0) \leq 0$ we deduce $\mathcal{A}_x(u,u) - \mathcal{A}_y(u,u) \leq \cot_a r - \cot_a R$.

Consider now the curvature lower bound $-b^2 \leq K_M$. By Toponogov theorem and the law of cosine, Eq. (3.2) becomes

$$1 - \cos(\alpha(s)) \geq \frac{\cosh(bd(x, x(s))) - 1}{\sinh(br)\sinh(b(1-\theta)r_y(s))} + \frac{1 - \cosh(br(1 - \frac{r_y(s)}{R}))}{\sinh(br)\sinh(b(1-\theta)r_y(s))} \tag{3.5}$$

while Lemma 3.3 implies

$$\cosh(bd(x, x(s))) - 1 \geq \frac{\cosh(bd(z, c(s))) - 1}{F_b(R, r_y(s), \beta(s), \theta)}, \tag{3.6}$$

which plugged in (3.5) yields

$$\begin{aligned}1 - \cos(\alpha(s)) \geq\ & \frac{\cosh(bd(z, c(s))) - 1}{\sinh(br)\sinh(b(1-\theta)r_y(s))F_b(R, r_y(s), \beta(s), \theta)} \\ &+ \frac{1 - \cosh(br(1 - \frac{r_y(s)}{R}))}{\sinh(br)\sinh(b(1-\theta)r_y(s))}\end{aligned} \tag{3.7}$$

Dividing by s^2 and letting $s \to 0$ as before, we get $(\cot_b r - \cot_b R)g \leq \mathcal{A}_x - \mathcal{A}_y$. □

In the sequel, we will be mainly interested in the second fundamental form of horospheres. We will use a result similar to Lemma 3.4, where the sphere $S_y(R)$ is replaced by a horosphere:

Lemma 3.6 *Let (M, g) be a CH-manifold with $K_M \leq -a^2$. Let $S_x(r)$ and $H_\xi(z)$ be respectively a sphere and a horosphere tangent at a point z, with $S_x(r)$ internal to the horosphere.*

Let $\mathcal{A}_x$, $\mathcal{A}_\xi$ be the second fundamental forms of $S_x(r)$, $H_\xi(z)$ with respect to be the common inner unit normal vector at z. Then, at the tangent point z we have:

$$0 \leq \mathcal{A}_x - \mathcal{A}_\xi \leq (\cot_a r - a)\, g \tag{3.8}$$

Moreover, if we assume $-b^2 \leq K_M$, then at the tangent point z we also have:

$$(\cot_b r - b)\, g \leq \mathcal{A}_x - \mathcal{A}_\xi$$

This result can be obtained in two different ways: taking limits, in the inequalities of Lemma 3.4, for y tending to ξ along the geodesic $\exp_z(t\nu)$, or following the same proof with the Busemann function b_ξ in place of r_y. The proof is left to the reader.

Remark 3.7 In lemma 2.3 of [17], the author gives an expression of $\mathcal{A}_x - \mathcal{A}_y$ and $\mathcal{A}_x - \mathcal{A}_\xi$ in term of integrals along the geodesic $t \mapsto \exp_z(t\nu)$ involving operators satisfying a Jacobi equation. Using this expression and comparison results for solutions of Jacobi equations under suitable bounds on the curvature, one might also give proofs of our Lemmas 3.4 and 3.6 purely in terms of Jacobi tensors.

4 Asymptotically harmonic CH-manifolds

In this section, M will always be an asymptotically harmonic CH-manifold with horospheres of constant mean curvature h.

4.1 The entropy and the spectrum

We are interested here in two invariants of the manifold M: the volume entropy and the spectrum. The entropy is determined by the behaviour of the volume of balls whose second derivative (with respect to the radius) is given, in turns, by the mean curvature of the spheres. On the other hand, the spectrum can be determined by using special functions whose Laplacian has a nice behaviour; in our case, the distance function, whose Laplacian is again given by the mean curvature of spheres (see discussion in Sect. 2).

For points x and y in M, let $\vec{h}_x(y)$ be the mean curvature vector at y of the sphere $S_x(d(x, y))$, and $h_x(y) = -\langle \vec{h}_x(y), \nabla r_x \rangle$. Notice that, as $K_M \leq 0$, balls and horoballs are convex, so both h and $h_x(y)$ are non-negative.

Lemma 4.1 *Let M^{n+1} be an asymptotically harmonic CH-manifold. For all $x \in M$ and $r > 0$, the sphere $S_x(r)$ satisfies*

$$\forall\, y \in S_x(r) \quad h \leq h_x(y) \leq h + \frac{1}{r}.$$

Proof From Lemma 3.6 we have

$$\mathcal{A}_\xi(u, u) \leq \mathcal{A}_x(u, u) \leq \mathcal{A}_\xi(u, u) + \frac{1}{r}|u|^2$$

where $\mathcal{A}_x$ and $\mathcal{A}_\xi$ are, respectively, the second fundamental forms of $S_x(r)$ and of the horosphere $H_\xi(y)$, tangent to $S_x(r)$ at y. Taking the trace on an orthonormal basis gives the result. ◻

We fix $x \in M$. For $r > 0$, let $B_x(r)$ be the ball of radius r centered in x, and $V(r) = \mathrm{Vol}(B_x(r))$ the growth function. The *entropy* of M is defined by

$$E = \limsup_{r\to\infty} \frac{1}{r} \log V(r).$$

A first consequence of asymptotic harmonicity is the following linear isoperimetric inequality:

Proposition 4.2 *Let M^{n+1} be an asymptotically harmonic CH-manifold. For any domain $\Omega \subset M$ with smooth boundary $\partial\Omega$ we have $nh\mathrm{Vol}(\Omega) \leq \mathrm{vol}(\partial\Omega)$.*

Proof Fix some $\xi \in \partial_\infty M$. Since $-\Delta b_\xi = nh$, integrating by parts on Ω the function $-\Delta b_\xi$ gives the result. □

Theorem 4.3 *Let M^{n+1} be an asymptotically harmonic CH-manifold. The entropy of M is $E = nh$.*

Proof By the co-area formula we have $V'(r) = \mathrm{vol}(S_x(r))$, and by Proposition 4.2 we get $nhV(r) \leq V'(r)$. Integrating this inequality we get $V(r) \geq A\mathrm{e}^{nhr}$ for some constant A, so that the entropy is bounded below by nh.

Now, the second derivative of V is given by $V''(r) = n \int_{S_x(r)} h_x(y) dv_r(y)$ where dv_r is the volume form of $S_x(r)$. Choose $\varepsilon > 0$ and let $r_0 = \frac{1}{\varepsilon}$. By Lemma 4.1, we have $V''(r) \leq n(h+\varepsilon)V'(r)$ for any $r \geq r_0$. Integrating this inequality between r_0 and r, yields $V'(r) \leq A\mathrm{e}^{n(h+\varepsilon)r}$ for some constant A. Integrating once again between r_0 and r, we get $V(r) \leq B + C\mathrm{e}^{n(h+\varepsilon)r}$, which implies that $E \leq n(h+\varepsilon)$. Since ε is arbitrarily small, this concludes the proof. □

Theorem 4.4 *Let M^{n+1} be an asymptotically harmonic CH-manifold. The spectrum of the Laplacian of M is $\sigma(\Delta) = [\frac{n^2h^2}{4}, +\infty)$*

Proof By Proposition 4.2 and Cheeger's inequality, we have $\sigma(\Delta) \subset [\frac{n^2h^2}{4}, +\infty)$.

Conversely, we choose $x \in M$ and consider the distance function r_x to x. Since the Laplacian of r_x is given by the mean curvature of spheres, we have

$$\sup_{y\in M\setminus B_x(R)} \{\, |\Delta r_x(y) - nh| \,\} \;\leq\; \frac{n}{R} \tag{4.1}$$

Using (4.1) and the fact that $|\nabla r_x| = 1$, we can follow the method initiated by H. Donnelly to determine the essential spectrum (cf. [9]): for each $\lambda > \frac{n^2h^2}{4}$ we use radial functions to construct sequences satisfying Weyl's criterion for λ (cf. [23] theorem VII.12 p. 237). See for example [18] theorem 1.2 for a general result, whose hypotheses are satisfied by the function r_x. □

Remark 4.5 From Theorems 4.3 and 4.4 we deduce $\inf\{\sigma(\Delta)\} = \frac{E^2}{4}$.

For cocompact negatively curved manifolds, this equality is equivalent to the asymptotic harmonicity (cf. [19] theorem 1). But, in the general case, it is easy to construct manifolds satisfying this inequality, which are not asymptotically harmonic. For example, the conclusions of Theorems 4.3 and 4.4 hold true for any Cartan–Hadamard manifold with curvature less than $-h^2$ and tending to $-h^2$ at infinity.

4.2 Rigidity

Consider the second fundamental form $\mathcal{A}_\xi$ of a horosphere H_ξ, and let $\lambda_1, \ldots, \lambda_n$ be the principal curvatures of H_ξ at some point x, with respect to the inner unit normal of H_ξ. If M satisfies the curvature upper bound $K_M \leq -a^2$, then it is well known that $\lambda_i \geq a$ (cf [14]). Therefore we get

$$n^2h^2 = \left(\sum_i \lambda_i\right)^2 = \sum_i \lambda_i^2 + 2\sum_{i<j} \lambda_i\lambda_j \geq |\mathcal{A}_\xi|^2 + n(n-1)a^2,$$

and

$$|\mathcal{A}_\xi|^2 \leq n^2h^2 - n(n-1)a^2. \tag{4.2}$$

When assuming a curvature lower bound $K_M \geq -b^2$, a similar argument gives

$$|\mathcal{A}_\xi|^2 \geq n^2h^2 - n(n-1)b^2. \tag{4.3}$$

Now, as the mean curvature is the same for all horospheres, taking the trace of Riccati Eq. (2.1) gives $|\mathcal{A}_\xi|^2 + \mathrm{Ric}_M(u,u) = 0$ for any $u \in SM$, for the second fundamental form $\mathcal{A}_\xi$ of a horosphere tangent to $u^\perp$. Therefore we get:

Proposition 4.6 *Let M^{n+1} be an asymptotically harmonic CH-manifold. For any $u \in SM$ we have*

(i). *if M satisfies $K_M \leq -a^2$, then* $\mathrm{Ric}_M(u,u) \geq -n^2h^2 + n(n-1)a^2$*;*
(ii). *if M satisfies $K_M \geq -b^2$, then* $\mathrm{Ric}_M(u,u) \leq -n^2h^2 + n(n-1)b^2$.

As a consequence, we have the following characterization of constant curvature spaces:

Corollary 4.7 *Let M^{n+1} be an asymptotically harmonic CH-manifold.*

(i). *if M satisfies $K_M \leq -a^2$ then $h \geq a$, and $h = a$ if and only if $M = \mathbb{M}^{n+1}(-a^2)$;*
(ii). *if M satisfies $K_M \geq -b^2$ then $h \leq b$, and $h = b$ if and only if $M = \mathbb{M}^{n+1}(-b^2)$.*

Proof The curvature upper bound $K_M \leq -a^2$ implies $h \geq a$. If $h = a$, then Proposition 4.6 gives $\mathrm{Ric}_M \geq -na^2$, and since the Ricci curvature is a sum of n sectional curvatures which are not greater then $-a^2$, this implies that all the sectional curvatures are equal to $-a^2$. The proof is the same when assuming a curvature lower bound. □

4.3 Growth of horospheres

It is well known that, on CH-manifolds with *pinched* curvature, horospheres have polynomial volume growth, whose degree depend on the bounds on the curvature (cf. [15]). We will now see that, under the asymptotic harmonicity assumption, an upper bound $K_M \leq -a^2 < 0$ is enough to estimate from above the polynomial growth of horospheres.

Let H_ξ be a horosphere centered in some point at infinity ξ, let b_ξ be the Busemann function vanishing on H_ξ, and let g_0 be the Riemannian metric induced on H_ξ. For each $t \in \mathbb{R}$, there is a natural diffeomorphism $\varphi_t : H_\xi \to H_\xi(t)$ defined by $\varphi_t(x) = \exp_x(t\nabla b_\xi)$, which in turns induces a diffeomorphism

$$\Phi \begin{cases} \mathbb{R} \times H_\xi(0) \to M \\ (t,x) \mapsto \varphi_t(x) \end{cases}$$

In these "horospherical" coordinates (t, x), the metric of M reads $g = dt^2 + g_t$, where $g_t = \varphi_t^* g_{H_\xi(t)}$ and $g_{H_\xi(t)}$ is the induced Riemannian metric of $H_\xi(t)$.

When assuming a sectional curvature upper bound $K \leq -a^2$, the map φ_t increases the distance for $t > 0$ and decreases the distance if $t < 0$. In fact, as a consequence of comparison theorem for Jacobi fields, we have that all the eigenvalues of $d\varphi_t$ are greater than or equal to e^{at} if $t > 0$, and less than or equal to e^{at} if $t < 0$ (cf. [13]).

Now, it is a standard fact that the mean curvature gives the derivative of the volume form of a submanifold under a deformation. In our setting, if $dv_t = J_t(x)dv_0$ is the volume form of the metric g_t and $J_t(x)$ is the density of dv_t with respect to dv_0, we have $J_t' = nh_t J_t$, where h_t is the mean curvature of $H_\xi(t)$. By asymptotic harmonicity, we deduce that $dv_t = \mathrm{e}^{nht}dv_0$ for all t; therefore, in horocyclic coordinates the volume form of M reads $dv_M = \mathrm{e}^{nht}dtdv_0$.

On the other hand, by Theorem 4.3, the volume entropy of M is nh: heuristically, this means that the exponential rate of the volume growth of M comes from the behaviour of the volume form in the $\mathbb{R}$ direction, and that the volume growth of the slices $H_\xi(t)$ should be subexponential. Namely:

Theorem 4.8 *Let M^{n+1} be an asymptotically harmonic CH-manifold with sectional curvature upper bound $K_M \leq -a^2 < 0$. Then, there exists a constant C (depending only on n, a and h) such that, for any horosphere H of M, the balls of H satisfy* $\mathrm{vol}(B_x^H(r)) \leq Cr^{\frac{nh}{a}}$ *for all $r > 0$.*

Proof Let $H = H_\xi$ be a horosphere centered in ξ. For any two orthonormal vectors $u, v \in TH$, Gauss equation implies that $K_H(u, v) = K_M(u, v) + \mathcal{A}_\xi(u, u)\mathcal{A}_\xi(v, v) - \mathcal{A}_\xi(u, v)^2$, where K_H and K_M are the sectional curvatures of H and M respectively, and $\mathcal{A}_\xi$ is the second fundamental form of H. Taking the trace (with respect to v) on an orthonormal frame $(e_i)_{i=1..n}$, we get

$$\mathrm{Ric}_H(u, u) = \mathrm{Ric}_M(u, u) - K_M(u, \nu) + nh\mathcal{A}_\xi(u, u) - \sum_{i=1}^{n} \mathcal{A}_\xi(u, e_i)^2$$

Since the curvature of M is negative, the second fundamental form $\mathcal{A}_\xi$ is positive, and by definition of its norm we have $\sum_{i=1}^n \mathcal{A}_\xi(u, e_i)^2 \leq |\mathcal{A}_\xi|^2$. Therefore we get

$$\mathrm{Ric}_H(u, u) \geq \mathrm{Ric}_M(u, u) - |\mathcal{A}_\xi|^2 \geq -2n^2h^2 + 2n(n-1)a^2$$

where the last inequality comes from (4.2) and Proposition 4.6. Therefore, by Bishop's comparison theorem, there exists a constant C (depending only on n, a and h) such that, for any x in H we have $\mathrm{Vol}(B_x^H(1)) \leq C$.

Let now $x \in H$ and consider the map $\varphi_{-t} : H \to H_\xi(-t)$ defined above, for $t > 0$. As $K_M \leq -a^2$, we have $\varphi_{-t}(B_x^H(r)) \subset B_{\varphi_{-t}(x)}^{H_\xi(-t)}(\mathrm{e}^{-at}r)$. Moreover, as $dv_{-t} = \mathrm{e}^{-nht}dv_0$, we have $\mathrm{vol}(\varphi_{-t}(B_x^H(r))) = \mathrm{e}^{-nht}\mathrm{vol}(B_x^H(r))$; so, choosing $t = \frac{\ln r}{a}$ we obtain

$$\mathrm{vol}(B_x^H(r))) \leq \mathrm{e}^{nh\frac{\ln r}{a}}\mathrm{vol}(B_{\varphi_{-t}(x)}^{H_\xi(-t)}(1)) \leq Cr^{\frac{nh}{a}}$$

□

Remark 4.9 This theorem proves that the degree of the polynomial volume growth of the horospheres is bounded above by $\frac{nh}{a}$. This upper bound is sharp, as it is the degree of the volume growth of the horospheres in the hyperbolic space (the horospheres being Euclidean in that case). Note that the upper bound is also sharp for the rank one symmetric spaces.

Remark 4.10 Using a similar proof, it is easy to see that the lower bound $-b^2 \leq K_M \leq 0$ gives a lower bound on the volume growth of the horospheres, namely $\mathrm{vol}(B_x^H(r)) \geq Cr^{\frac{nh}{b}}$. The proof is left to the reader.

4.4 The mean value property

Harmonic manifolds are characterized by the fact that the harmonic functions have the mean value property: for any harmonic function F and any $R > 0$

$$F(x_0) = \frac{1}{\mathrm{vol}(S_{x_0}(R))} \int\limits_{S_{x_0}(R)} F(x) dv_{S_{x_0}(R)}$$

This can be proved by taking the derivative of the right-hand side of the above equality, and by observing that it vanishes for any harmonic function F if and only if the spheres have constant mean curvature.

In the following theorem we prove that harmonic functions on an asymptotically harmonic manifold satisfy a mean value property, where, naturally, the mean is taken on horospheres. As the horospheres are non-compact, the mean on an horosphere is obtained as the limit of the means on an exhaustion. The computations of these horospherical means are inspired by those in [15].

Theorem 4.11 *Let M^{n+1} be an asymptotically harmonic manifold with sectional curvature upper bound $K_M \leq -a^2 < 0$, and let F be a function which is continuous on $M \cup \partial_\infty M$ and harmonic on M.*

For any $\xi \in \partial_\infty M$, any horosphere H_ξ centered in ξ, and any $x \in H_\xi$, there exists a sequence $(r_j)_{j\in\mathbb{N}}$ tending to $+\infty$ such that

$$\lim_{j\to\infty} \frac{1}{\mathrm{Vol}(B_x^{H_\xi}(r_j))} \int\limits_{B_x^{H_\xi}(r_j)} F dv_{H_\xi} = F(\xi)$$

where $B_x^{H_\xi}(R)$ denote the ball in H_ξ centered in x of radius R.

Proof Let H_ξ be a horosphere centered in some point at infinity ξ, and let $\varphi_t : H_\xi \to H_\xi(t)$ be the diffeomorphism defined in §3.3.

Choose $x \in H_\xi$. Because H_ξ has polynomial volume growth, there exists a sequence $(r_j)_{j\in\mathbb{N}}$ tending to $+\infty$ such that

$$\lim_{j\to\infty} \frac{\mathrm{vol}(\partial B_x^{H_\xi}(r_j))}{\mathrm{Vol}(B_x^{H_\xi}(r_j))} = 0.$$

For $t \in \mathbb{R}$ and $j \in \mathbb{N}$, let $\Omega_{j,t} = \varphi_t(B_x^{H_\xi}(r_j))$. As pointed out in Sect. 4.3, we have $\mathrm{Vol}(\Omega_{j,t}) = \mathrm{e}^{nht}\mathrm{Vol}(B_x^{H_\xi}(r_j))$. Moreover, the boundary of $\Omega_{j,t}$ satisfy

$$\frac{d}{dt}\mathrm{vol}(\partial\Omega_{j,t}) = -(n-1) \int\limits_{\partial\Omega_{j,t}} \left\langle \vec{k}_{j,t}, \frac{\partial}{\partial t} \right\rangle$$

where $\vec{k}_{j,t}$ is the mean curvature vector of $\partial\Omega_{j,t}$ (seen as a submanifold of M). Taking an orthonormal basis $(e_1, \ldots, e_{n-1})$ of $T\partial\Omega_{j,t}$ and $\eta_{j,t}$ its exterior unit normal in $H_\xi(t)$ we have

$$-(n-1)\langle\vec{k}_{j,t},\frac{\partial}{\partial t}\rangle=\sum_{i=1}^{n-1}\langle D^M_{e_i}\frac{\partial}{\partial t},e_i\rangle=nh-\left\langle D^M_{\eta_{j,t}}\frac{\partial}{\partial t},\eta_{j,t}\right\rangle\le nh-a$$

where the last inequality comes from the curvature upper-bound on M. Therefore we have $\frac{d}{dt}\mathrm{vol}(\partial\Omega_{j,t})\le(nh-a)\mathrm{vol}(\partial\Omega_{j,t})$, and integrating this inequality we get $\mathrm{vol}(\partial\Omega_{j,t})\le \mathrm{e}^{(nh-a)t}\mathrm{vol}(\partial\Omega_{j,0})$ and

$$\frac{\mathrm{vol}(\partial\Omega_{j,t})}{\mathrm{Vol}(\Omega_{j,t})}\le \mathrm{e}^{-at}\frac{\mathrm{vol}(\partial B_x^{H_\xi}(r_j))}{\mathrm{Vol}(B_x^{H_\xi}(r_j))}. \tag{4.4}$$

Consider now

$$g_j(t)=\frac{1}{\mathrm{Vol}(\Omega_{j,t})}\int_{\Omega_{j,t}}F\,dv_t \tag{4.5}$$

where dv_t is the volume form of $H_\xi(t)$ and F a function which is continuous on $M\cup\partial_\infty M$ and harmonic on M. In particular, F is bounded. Using the fact that horospheres have constant mean curvature, we have

$$g_j'(t)=\frac{1}{\mathrm{Vol}(\Omega_{j,t})}\int_{\Omega_{j,t}}\langle\nabla F,\frac{\partial}{\partial t}\rangle dv_t \tag{4.6}$$

and

$$g_j''(t)=\frac{1}{\mathrm{Vol}(\Omega_{j,t})}\int_{\Omega_{j,t}}(\mathrm{Hess}^M F)(\frac{\partial}{\partial t},\frac{\partial}{\partial t})dv_t \tag{4.7}$$

Using proposition 2.1 and the fact that F is harmonic in M we get

$$(\mathrm{Hess}^M F)(\frac{\partial}{\partial t},\frac{\partial}{\partial t})=-\mathrm{tr}((\mathrm{Hess}^{H_\xi(t)}F)_{|TH_\xi(t)})=\Delta^{H_\xi(t)}f+nh\langle\nabla F,\frac{\partial}{\partial t}\rangle$$

where f is the restriction of F to $H_\xi(t)$. Equation (4.7) gives

$$\begin{aligned}g_j''(t)-nhg_j'(t)&=\frac{1}{\mathrm{Vol}(\Omega_{j,t})}\int_{\Omega_{j,t}}\Delta^{H_\xi(t)}f\,dv_t\\&=-\frac{1}{\mathrm{Vol}(\Omega_{j,t})}\int_{\partial\Omega_{j,t}}\langle\nabla F,\eta_{j,t}\rangle dv_t.\end{aligned} \tag{4.8}$$

As Ric_M is bounded from below and F is bounded on M, using Yau's gradient estimate for harmonic functions [26], there exists a constant C (depending on n, a, h and $||F||_\infty$) such that $|\nabla F|\le C$ on M. Therefore, using (4.4), the right-hand side of (4.8) satisfies

$$\left|\frac{1}{\mathrm{Vol}(\Omega_{j,t})}\int_{\partial\Omega_{j,t}}\langle\nabla F,\eta_{j,t}\rangle dv_t\right|\le C\mathrm{e}^{-at}\frac{\mathrm{vol}(\partial B_x^{H_\xi}(r_j))}{\mathrm{Vol}(B_x^{H_\xi}(r_j))}$$

and tends uniformly to zero on bounded intervals when j tends to $+\infty$. In particular, it implies that, on bounded intervals, the C^0 norms of the functions g_j'' are uniformly bounded. The fact that F is bounded and Yau's gradient estimate also imply that the C^0 norms of the functions g_j and g'_j are uniformly bounded, and, using Arzela-Ascoli convergence theorem, we have that, up to a subsequence, $(g_j)_{j\in\mathbb{N}}$ tends in C^1 topology to a function g.

Moreover, multiplying (4.8) by a test function, integrating by part and letting j tend to $+\infty$ we find that, in the sense of distributions, g is a solution of

$$g''(t) - nhg'(t) = 0.$$

Therefore, by classical regularity theory, g is smooth and $g'(t) = g'(0)\mathrm{e}^{nht}$. Since g' is bounded on $\mathbb{R}$ we must have $g' \equiv 0$ and g is constant.

For any neighbourhood U of ξ (for the cone topology) there exist t such that the horosphere $H_\xi(t)$ is contained in U. By continuity of F on $M \cup \partial_\infty M$ and by the definition of g_j, the value $g_j(t)$ can be made arbitrary close to $F(\xi)$ (for any j). Therefore we have $g(t) = F(\xi)$ for any $t \in \mathbb{R}$, and $g(0) = F(\xi)$ gives the result. □

Remark 4.12 It would be better to have a similar result without taking a sequence of radii tending to infinity, that is to have

$$\lim_{r\to\infty} \frac{1}{\mathrm{Vol}(B_x^{H_\xi}(r))} \int_{B_x^{H_\xi}(r)} F dv_{H_\xi} = F(\xi).$$

For the proof to work in that case, one need to have $\lim_{r\to\infty} \frac{\mathrm{vol}(\partial B_x^{H_\xi}(r))}{\mathrm{Vol}(B_x^{H_\xi}(r))} = 0$. However, from the polynomial volume growth of horospheres one only get $\liminf_{r\to\infty} \frac{\mathrm{vol}(\partial B_x^{H_\xi}(r))}{\mathrm{Vol}(B_x^{H_\xi}(r))} = 0$.

5 Asymptotic behaviour of the volume form

In the previous section, in order to compute the entropy, we integrated the inequalities of Lemma 3.6 on spheres. But since these inequalities hold pointwise, we can try to determine the asymptotic behaviour of the volume form at least in a fixed direction. Actually, let $\theta_x(u, r)$ be the density of the volume form of M in normal coordinates centered in some point x; so the volume form reads $dv_M = \theta_x(u, r) dv_{S_x M} dr$, where $dv_{S_x M}$ is the volume form of $S_x M$.

Harmonic manifolds are characterized by the fact that $\theta_x(u, r)$ only depends on r. In this section we give a characterization of asymptotically harmonic manifolds in term of the asymptotic behaviour of $\theta_x(u, r)$:

Theorem 5.1 *Let M be a CH-manifold with $K_M \le -a^2 < 0$ and entropy E. M is asymptotically harmonic if and only if there exists a positive function $\tau : SM \to \mathbb{R}_+$ such that $\theta_x(u, r)$ is uniformly equivalent to $\tau(u)\mathrm{e}^{Er}$ for $r \to \infty$.*

"Uniformly equivalent" here means that the quotient of $\theta_x(u, r)$ by $\tau(u)\mathrm{e}^{Er}$ converges to 1 for $r \to \infty$, uniformly with respect to $u \in SM$. For the sake of clarity, we will split this result in two Propositions 5.4 and 5.7 which will be proved in the two following subsections.

Remark 5.2 It is a natural question whether Theorem 5.1 remains true if we assume $K \le 0$ and $E > 0$.

(i). For the "only if" part (Proposition 5.4) to hold true, a sufficient condition for the existence and uniformity of the limit of $\theta(v, t)\mathrm{e}^{-nht}$ would be that $\det(U'_v(0) - S'_{v,t}(0))$ is bounded from below by a positive constant which does not depend on v (where U_v and $S_{v,t}$ are defined at the beginning of §4.1).

(ii). On the other hand, the "if" part (Proposition 5.7) still holds if we assume $K \leq 0$ and $E > 0$: it is enough to replace in the proof of Proposition 5.7 the upper bound in inequalities (5.7) by $n \ln(\frac{R+s}{r+s})$ which also tends to 0 as $s \to \infty$.

Remark 5.3 There is some overlap between the results of this section and the work of G. Knieper in [17]. In the setting of harmonic manifolds he uses Jacobi tensors to express the quotient $\frac{\theta_x(u,r)}{\mathrm{e}^{Er}}$ and its limit as $r \to \infty$ (when it exists). In particular, in corollary 2.5 he proves that $\frac{\theta_x(u,r)}{\mathrm{e}^{Er}}$ is a monotonically increasing function of r and therefore has a (possibly infinite) limit; he then gives sufficient conditions for this limit to be finite (such as an Anosov geodesic flow, or a rank equal to one). Once we know that the volume entropy of M is $E = nh$ (cf. Theorem 4.3), it is easy to see that the computations in [17] can be carried out in our setting. We will use this approach to prove the existence of the function τ in Theorem 5.1. Notice however that our comparison lemma also proves the *uniform* convergence of $\theta_x(u,r)\mathrm{e}^{-Er}$ to $\tau(u)$, and that this uniformity is crucial for proving the above characterization of Theorem 5.1.

5.1 The asymptotic volume-density function τ

The function θ_x is related to the mean curvature h_x of spheres centered in x of radius r by the formula

$$\frac{\theta_x'(u,r)}{\theta_x(u,r)} = nh_x(\exp_x(ru)) \tag{5.1}$$

where θ_x' denotes the derivative of θ_x with respect to r.

In what follows, we shall often write for short the point $\exp_x(ru)$ as (u,r) to avoid cumbersome notations; moreover, we will regard $\theta_x(u,r)$ as a function on $SM \times \mathbb{R}$, so we can drop the index x.

The quantity $\theta(u,r)\mathrm{e}^{-nhr}$ can be expressed in terms of Jacobi tensors. This was used to study the asymptotic behaviour of the volume on harmonic manifolds (cf. [7,12,17]). As we will use this approach to prove the first proposition of this section, let us recall some basic fact on Jacobi tensors.

A Jacobi tensor along a geodesic γ is a smooth family $J(t)$ of endomorphisms of $\dot\gamma(t)^\perp$ satisfying the Jacobi equation $J''(t)+R(t)J(t) = 0$, where $R(t)$ is defined from the Riemann tensor by $R(t)u = R(\dot\gamma(t),u)\dot\gamma(t)$. Then, applying J to any parallel vector field $V(t)$ along γ gives a Jacobi vector field $J(t)V(t)$.

Let $v \in S_xM$ and $\gamma(t) = \exp_x(tv)$, and consider the Jacobi tensor J_v along γ defined by $J_v(0) = 0$ and $J_v'(0) = \mathrm{Id}$. It is well known that $J_v'(r)J_v^{-1}(r)$ gives the shape operator $A_x(v,r)$ of the sphere $S_x(r)$ at $\exp_x(rv)$ (with respect to the inner normal to the sphere), and that $\theta(v,r) = \det(J_v(r))$.

For $r > 0$, let $U_{v,r}$, $S_{v,r}$ be the Jacobi tensors on γ defined by $U_{v,r}(-r) = 0$, $U_{v,r}(0) = S_{v,r}(0) = \mathrm{Id}$ and $S_{v,r}(r) = 0$. The unstable and stable Jacobi tensors at v are defined by $U_v = \lim_{r\to\infty} U_{v,r}$ and $S_v = \lim_{r\to\infty} S_{v,r}$. As $U_{v,r}'(0) = J_{\dot\gamma(-r)}'(r)J_{\dot\gamma(-r)}^{-1}(r)$ is the shape operator of the sphere $S_{\gamma(-r)}(r)$ at x, it follows (for example by Lemma 3.6) that $U_v'(0)$ is shape operator at x of the horosphere centered in $\xi_- = \lim_{r\to\infty}\gamma(-r)$. In a similar way, we have that $-S_v'(0)$ is the shape operator at x of the horosphere centered in $\xi_+ = \lim_{r\to\infty}\gamma(r)$.

Using Jacobi tensors and Lemma 3.6 we get the following result:

Proposition 5.4 *Let M be a CH-manifold with curvature $K_M \leq -a^2 < 0$. If M is asymptotically harmonic, then there exists a bounded, positive function $\tau : SM \to \mathbb{R}_+$ such that*

$$\forall u \in SM \quad \left|\frac{\theta(u,t)}{\tau(u)\mathrm{e}^{nht}} - 1\right| \leq \varepsilon(t)$$

where the function $\varepsilon(t)$ *only depends on a and n and satisfies* $\lim_{t\to\infty}\varepsilon(t)=0$.

Moreover, the function τ *has the following properties:*

(i). $\tau : SM \to \mathbb{R}_+$ *is invariant by the geodesic flow and flip invariant, i.e.:*

- $\tau(\dot\gamma(t))$ *is constant for any geodesic* γ*;*
- $\tau(v)=\tau(-v)$ *for all* $v\in SM$.

(ii). $\tau\geq\frac{1}{(2h)^n}$, *with equality if and only if the curvature is constant.*

Proof Since M is asymptotically harmonic, we have $\mathrm{tr}(U_v'(0))=nh$; following the proof of Corollary 2.5 of [17] we get

$$\theta(v,t)\mathrm{e}^{-nht}=\frac{1}{\det(U_v'(0)-S_{v,t}'(0))},$$

and

$$\lim_{t\to\infty}\theta(v,t)\mathrm{e}^{-nht}=\begin{cases}+\infty & \text{if } \det(U_v'(0)-S_v'(0))=0\\ \frac{1}{\det(U_v'(0)-S_v'(0))} & \text{if } \det(U_v'(0)-S_v'(0))>0\end{cases}$$

Since $U_v'(0)$ and $-S_v'(0))$ are the shape operators of horospheres and $K\leq -a^2$, we have, by standard comparison theorem, $U_v'(0)-S_v'(0))\geq 2aId$ and $\det(U_v'(0)-S_v'(0))\geq(2a)^n>0$. Therefore the limit is finite and we define

$$\tau(u)=\lim_{t\to\infty}\theta(u,t)\mathrm{e}^{-nht}=\frac{1}{\det(U_v'(0)-S_v'(0))} \tag{5.2}$$

which is a function on SM bounded above by $\frac{1}{(2a)^n}$.

The Jacobi tensor $-S_{v,t}'(0)$ is the shape operator of the sphere $S_{\gamma(t)}(t)$. Using Lemma 3.6 we have that $-S_{v,t}'(0)$ tends uniformly to $-S_v'(0)$, and $\det(U_v'(0)-S_{v,t}'(0))$ tends uniformly to $\det(U_v'(0)-S_v'(0))$. Moreover, we still have $\det(U_v'(0)-S_{v,t}'(0))\geq(2a)^n>0$ and therefore the function τ is the uniform limit of $\theta(u,t)\mathrm{e}^{-nht}$. Since the function τ is bounded above, we also have that $\frac{\theta(u,t)}{\tau(u)\mathrm{e}^{nht}}$ tends uniformly to 1, which proves the first part of Proposition 5.4.

The properties of τ also rely on its expression in term of Jacobi tensors. First, as $U_v'(0)$ and $-S_v'(0)$ are the shape operators of the horospheres centered in ξ_- and ξ_+, relative to their respective inner normals, it is clear that τ is flip invariant. The invariance by the geodesic flow is just lemma 2.2 in [12].

The second property is similar to Corollary 2.6 in [17]. As $U_v'(0)-S_v'(0)$ is a positive symmetric matrix, the arithmetic-geometric inequality gives

$$\det(U_v'(0)-S_v'(0))^{\frac{1}{n}}\leq\frac{1}{n}\mathrm{tr}(U_v'(0)-S_v'(0))=2h$$

and the inequality follows. The case of equality follows, as in the proof of Corollary 2.6 in [17], from the fact that $s\mapsto U_{\dot\gamma(s)}'(0)$ and $s\mapsto S_{\dot\gamma(s)}'(0)$ satisfy the Riccati equation, and because $U_{\dot\gamma(s)}'(0)-S_{\dot\gamma(s)}'(0)=\frac{2h}{n-1}Id$. □

A Riemannian manifold is harmonic if and only if the density function only depends on r. As an asymptotic analogue, one would expect that for asymptotically harmonic manifolds $\lim_{r\to\infty}\frac{\theta(u,r)}{\mathrm{e}^{Er}}$ does not depend on u, and thus that $\tau(u)$ be constant on SM. In the following proposition we prove that it is true under the additional assumption that DR_M is bounded.

Proposition 5.5 *Let M be a CH-manifold with curvature $K_M \leq -a^2 < 0$. If M is asymptotically harmonic, and if the derivative of the Riemann tensor DR_M is bounded on M, then τ is constant on SM.*

Proof Let us first show that $\tau(u) = \tau(v)$ when $u, v \in SM$ point towards the same boundary point $\xi \in \partial_\infty M$, i.e. $\lim_{s\to+\infty} \gamma_u(s) = \lim_{s\to+\infty} \gamma_v(s)$. By the invariance of τ under the geodesic flow, we may as well assume that u and v are normal to the same horosphere, so $d(\gamma_u(t), \gamma_v(t)) \leq c_1 \mathrm{e}^{-at}$ for all $t > 0$. For any $r, t > 0$ we have

$$|\tau(u) - \tau(v)| \leq |\tau(u) - \theta(\dot\gamma_u(t), r)\mathrm{e}^{-nhr}| + |\theta(\dot\gamma_u(t), r) - \theta(\dot\gamma_v(t), r)|\mathrm{e}^{-nhr}$$
$$+|\tau(v) - \theta(\dot\gamma_v(t), r)\mathrm{e}^{-nhr}|,$$

and using the invariance of τ by the geodesic flow and Proposition 5.4 we get

$$|\tau(u) - \tau(v)| \leq (\tau(u) + \tau(v))\varepsilon(r) + |\theta(\dot\gamma_u(t), r) - \theta(\dot\gamma_v(t), r)|\mathrm{e}^{-nhr}. \tag{5.3}$$

For $s \in]0, r]$, let $h_{u,t}(s)$ (resp. $h_{v,t}(s)$) be the mean curvature, at the point $\gamma_u(t+s)$ (resp. at $\gamma_v(t+s)$), of the sphere of radius s centered in $\gamma_u(t)$ (resp. $\gamma_v(t)$). Following the Lemma 2.3 in [12], we will use comparison theory for Riccati equation to estimate $|h_{u,t}(s) - h_{v,t}(s)|$. We choose orthonormal parallel basis $e_{u,i}(s)$ of $\dot\gamma_u(t+s)^\perp$ and $e_{v,i}(s)$ of $\dot\gamma_v(t+s)^\perp$) such that, for any i, $d(e_{u,i}(s), e_{v,i}(s)) \leq c_2\mathrm{e}^{-a(t+s)}$ in SM, for some constant c_2 (cf. [6] for the existence of such frame fields). Let $A_{u,t}(s)$ and $A_{v,t}(s)$ be the matrices of the second fundamental forms of the spheres of radius s centered in $\gamma_u(t), \gamma_v(t)$ in these basis. They satisfy the Riccati equations $A'_{u,t}(s) + A^2_{u,t}(s) + R_{u,t}(s) = 0$ and $A'_{v,t}(s) + A^2_{v,t}(s) + R_{v,t}(s) = 0$, where $R_{u,t}(s)$ is the matrix of the endomorphism $R(\dot\gamma_u(t+s), .)\dot\gamma_u(t+s)$, and analogously for $R_{v,t}(s)$. Because of the assumption on DR_M, we have that the tensor $r(s) = R_{u,t}(s) - R_{v,t}(s)$ satisfies

$$|r(s)| \leq C_3\mathrm{e}^{-a(t+s)}. \tag{5.4}$$

Consider now $B(s) = A_{u,t}(s) - A_{v,t}(s)$ and $Q(s) = \frac{1}{2}(A_{u,t}(s) + A_{v,t}(s))$. From the Riccati equations we have that B is solution of

$$B'(s) + B(s)Q(s) + Q(s)B(s) + r(s) = 0.$$

A direct computation shows that for any $0 < \varepsilon < s$ we have the formula

$$B(s) = {}^tC(s)\left[{}^tC(\varepsilon)^{-1}B(\varepsilon)C(\varepsilon)^{-1} - \int_\varepsilon^s {}^tC(\zeta)^{-1}r(\zeta)C(\zeta)^{-1}d\zeta\right]C(s) \tag{5.5}$$

where $C(s)$ is a solution of $C'(s) = -C(s)Q(s)$. In particular, because of the curvature upper bound we have $Q(s) \geq a\mathrm{Id}$ hence, for any $0 < \varepsilon < s$, $|C(\varepsilon)^{-1}C(s)| \leq \mathrm{e}^{-a(s-\varepsilon)}$. Plugging this estimate and (5.4) in the formula (5.5) we get

$$|B(s)| \leq |B(\varepsilon)|\mathrm{e}^{-2a(s-\varepsilon)} + c_4\mathrm{e}^{-a(t+s)}$$

Since both $A_{u,t}(s)$ and $A_{v,t}(s)$ behave, for $s \to 0$, as $\frac{1}{s}\mathrm{Id} + o(1)$ we have $\lim_{\varepsilon\to 0} B(\varepsilon) = 0$; therefore we deduce that $|B(s)| \leq c_4\mathrm{e}^{-a(t+s)}$ and, taking the trace,

$$|h_{u,t}(s) - h_{v,t}(s)| \leq c_5\mathrm{e}^{-a(t+s)}$$

for some constant c_5. By the expression (5.1) for h_x, integrating on $[0, r]$ yields

$$-c_6(1 - \mathrm{e}^{-ar})\mathrm{e}^{-at} \leq \ln\frac{\theta(\dot\gamma_u(t), r)}{\theta(\dot\gamma_v(t), r)} \leq c_6(1 - \mathrm{e}^{-ar})\mathrm{e}^{-at}$$

With these inequalities we can bound the last term of (5.3):

$$\begin{aligned}\theta(\dot\gamma_v(t), r)\mathrm{e}^{-nhr}&\left[\exp(-c_6(1-\mathrm{e}^{-ar})\mathrm{e}^{-at})-1\right]\\ &\le |\theta(\dot\gamma_u(t), r)-\theta(\dot\gamma_v(t), r)|\mathrm{e}^{-nhr}\\ &\quad\le \theta(\dot\gamma_v(t), r)\mathrm{e}^{-nhr}\left[\exp(c_6(1-\mathrm{e}^{-ar})\mathrm{e}^{-at})-1\right]\end{aligned}\tag{5.6}$$

Choosing r large enough, the term $\varepsilon(r)$ in (5.3) can be made arbitrary small; for this value of r, $\theta(\dot\gamma_v(t), r)\mathrm{e}^{-nhr}$ stays close to $\tau(v)$ for all t by Proposition 5.4, and the above estimate (5.6) implies that we can choose t large enough to make also the last term of (5.3) arbitrary small. Therefore $\tau(u)=\tau(v)$.

Consider now any vector $u, v \in SM$, and let σ be a geodesic such that $\lim_{s\to-\infty}\sigma(s)=\lim_{s\to+\infty}\gamma_u(s)$ and $\lim_{s\to+\infty}\sigma(s)=\lim_{s\to+\infty}\gamma_v(s)$. From the above computations we must have $\tau(v)=\tau(\dot\sigma(0))$ and $\tau(u)=\tau(-\dot\sigma(0))$, so by flip invariance we get $\tau(u)=\tau(v)$. Therefore τ is constant. □

Remark 5.6 This Proposition is very close to Lemma 2.3 and Corollary 2.1 of [12], but, in our proof, we don't need any lower bound on the curvature and rather use Proposition 5.4.

5.2 Characterization of asymptotic harmonicity

Propositions 5.4 and 5.5 say that the volume form of M has purely exponential growth, with isotropic exponential rate (and is asymptotically perfectly isotropic when DR_M is bounded). In fact this is a characterization of asymptotic harmonicity:

Proposition 5.7 *Let M be a CH-manifold with $K_M \le -a^2 < 0$ and entropy E. If there exists a positive function $\tau : SM \to \mathbb{R}$ such that $\theta(u, r)$ is uniformly equivalent to $\tau(u)\mathrm{e}^{Er}$ for $r\to\infty$, then M is asymptotically harmonic.*

Remark 5.8 Notice that, together with Benoist-Foulon-Labourie and Besson-Courtois-Gallot characterization of cocompact asymptotically harmonic spaces, Proposition 5.7 shows that *if a CH-manifold with compact quotients has volume form which is (uniformly) equivalent to a function $\tau(u)\mathrm{e}^{ER}$, then it is a ROSS.*

Proof Let $\gamma(t)$ be a geodesic of M with $\lim_{t\to-\infty}\gamma(t)=\xi\in\partial_\infty M$, and let $h(t)$ be the mean curvature at $\gamma(t)$ of the horosphere $H_\xi(t)$ centered in ξ and passing through $\gamma(t)$. We shall prove that the function $h(t)$ is constant.

Let $r<R$ be two real numbers, and choose $s>-r$. For any $t\in[r, R]$, we use Lemma 3.6 to compare the second fundamental forms of $H_\xi(t)$ and $S_{\gamma(-s)}(t+s)$ at $\gamma(t)$. Taking the trace in (3.8), we have

$$0\le\frac{\theta'(\dot\gamma(-s), t+s)}{\theta(\dot\gamma(-s), t+s)}-nh(t)\le na\left(\coth(a(t+s))-1\right)$$

and integrating on $[r, R]$ with respect to t we get

$$0\le\ln\frac{\theta(\dot\gamma(-s), R+s)}{\theta(\dot\gamma(-s), r+s)}-n\int_r^R h(t)dt\le\ln\Big(\frac{\sinh^n(a(R+s))}{\sinh^n(a(r+s))}\mathrm{e}^{-na(R-r)}\Big)\tag{5.7}$$

The right-hand side tends to 0 when s tends to infinity. Moreover, by hypothesis we have $|\frac{\theta(\dot\gamma(-s), R+s)}{\tau(\dot\gamma(-s))\mathrm{e}^{E(R+s)}}-1|\le\varepsilon(R+s)$ with $\lim_{s\to\infty}\varepsilon(R+s)=0$, and we get

$$\lim_{s\to\infty}\frac{\theta(\dot\gamma(-s), R+s)}{\tau(\dot\gamma(-s))\mathrm{e}^{Es}}=\mathrm{e}^{ER}.$$

Analogously, we find $\lim_{s\to\infty} \frac{\theta(\dot\gamma(-s),r+s)}{\tau(\dot\gamma(-s))e^{Es}} = e^{Er}$, so letting s tend to infinity in (5.7) we obtain

$$E(R-r) - n\int_r^R h(t)dt = 0.$$

Therefore $\int_r^R (E - nh(t))dt = 0$ for all $r < R$, from which we deduce that $h(t) = \frac{E}{n}$ for all $t \in \mathbb{R}$, and M is asymptotically harmonic. □

6 Margulis function and measures at infinity

In this last section, we assume that M is a asymptotically harmonic CH-manifold with pinched curvature $-b^2 \le K_M \le -a^2 < 0$, and h is always the mean curvature of the horospheres.

6.1 Visual and harmonic measures

There are two families of measures naturally defined on the ideal boundary of Cartan–Hadamard manifolds: the visual and harmonic measures.

To define the visual measures, consider the homeomorphism given by the "projection on $\partial_\infty M$ from x":

$$\phi_x : \begin{cases} S_x M \to \partial_\infty M \\ \quad u \mapsto \phi_x(u) = \lim_{t\to\infty} \exp_x(tu) \end{cases}$$

The measure λ_x is the push-forward on $\partial_\infty M$ of the (normalized) Riemannian measure of $S_x M$.

On the other hand, the family of harmonic measures comes from the uniqueness of the solution to the Dirichlet problem at infinity (cf. [1]): given a continuous function f on $\partial_\infty M$, there exists a unique bounded harmonic function F on M such that $\lim_{x\to\xi} F(x) = \xi$. Then, it is a consequence of Riesz representation theorem that there exists a unique family of measures $\mu_x,\ x \in M$, such that $F(x) = \int_{\partial_\infty M} f(\xi)d\mu_x(\xi)$.

Proposition 6.1 *Let M be an asymptotically harmonic CH-manifold with pinched curvature $-b^2 \le K_M \le -a^2 < 0$. For any $x, y \in M$ we have*

$$\frac{d\lambda_x}{d\lambda_y}(\xi) = \frac{\tau(\phi_y^{-1}(\xi))}{\tau(\phi_x^{-1}(\xi))} e^{-nh(b_\xi(x)-b_\xi(y))} \quad \textit{and} \quad \frac{d\mu_x}{d\mu_y}(\xi) = e^{-nh(b_\xi(x)-b_\xi(y))}.$$

Proof Consider the distance functions r_x and r_y to $x, y \in M$ respectively, and the sphere $S_x(t)$ centered in x of radius t. For t great enough, each geodesic ray from y intersect $S_x(t)$ at a unique point; for $v \in S_y M$, let $F_t(v)$ be the intersection point of the geodesic $s \mapsto \exp_y(sv)$ and $S_x(t)$.

The map $F_t : S_y M \to S_x(t)$ so defined is a diffeomorphism whose Jacobian can be expressed in the following way. Fix some $v \in S_y M$, let $R = r_y(F_t(v))$ and $z = F_t(v) = \exp_y(Rv)$. Consider the sphere $S_y(R)$ and the "projection" $P : S_x(t) \to S_y(R)$, where $P(m)$ is the intersection of $S_y(R)$ with the half-geodesic from y passing through m. Obviously we have $P \circ F_t(u) = \exp_y(Ru)$ for all $u \in S_y M$, and from the very definition of the density function we get

$$\mathrm{Jac}_v(P \circ F_t) = \theta(v, R). \tag{6.1}$$

On the other-hand, the tangent map $T_z P : T_z S_x(t) \to T_z S_y(R)$ is the orthogonal projection in $T_z M$ whose Jacobian is given by the scalar product of the unit normals of the hyperplanes $T_z S_x(t)$ and $T_z S_y(R)$; therefore we have

$$\mathrm{Jac}_z(P) = \langle \nabla r_y(F_t(v)), \nabla r_x(F_t(v))\rangle \tag{6.2}$$

From Eqs. (6.1) and (6.2) we get

$$\mathrm{Jac}_v F_t = \frac{\theta(v, r_y(F_t(v)))}{\langle \nabla r_y(F_t(v)), \nabla r_x(F_t(v))\rangle} \tag{6.3}$$

Now, let $U \subset \partial_\infty M$ be a measurable set with negligible boundary, and let $U_t = \{\exp_x(tu) \mid u \in \phi_x^{-1}(U)\}$ be the projection of U on $S_x(t)$ from x.

By definition of λ_x we have

$$\lambda_x(U) = \int_{\phi_x^{-1}(U)} d\sigma_x = \frac{1}{\mathrm{vol}(S^n)} \int_{U_t} \frac{1}{\theta(P_t^{-1}(z), t)} dv_{S_x(t)}(z)$$

where $P_t(u) = \exp_x(tu)$ for $u \in S_x M$, $d\sigma_x$ is the normalized measure of $S_x M$, and $dv_{S_x(t)}$ the volume forms of $S_x(t)$.

By (6.3), we get

$$\lambda_x(U) = \int_{F_t^{-1}(U_t)} \frac{\theta(v, r_y(F_t(v)))}{\theta(P_t^{-1} \circ F_t(v), t)} \langle \nabla r_y(F_t(v)), \nabla r_x(F_t(v))\rangle^{-1} d\sigma_y(v) \tag{6.4}$$

where $d\sigma_y$ is the normalized measure on $S_y M$.

Now we observe that, letting t tend to infinity, we have

- $\lim_{t\to\infty} P_t^{-1} \circ F_t(v) = \phi_x^{-1} \circ \phi_y(v)$;
- $\lim_{t\to\infty} \chi_{F_t^{-1}(U_t)} = \chi_{\phi_y^{-1}(U)}$ almost everywhere;
- $\lim_{t\to\infty} \langle \nabla r_y(F_t(v)), \nabla r_x(F_t(v))\rangle = 1$

Moreover, from Proposition 5.4 we know that

$$\begin{aligned}\frac{1-\varepsilon(r_y(F_t(v)))}{1+\varepsilon(t)} \frac{\tau(v)}{\tau(P_t^{-1} \circ F_t(v))} \mathrm{e}^{nh(r_y(F_t(v))-t)} &\le \frac{\theta_y(v, r_y(F_t(v)))}{\theta(P_t^{-1} \circ F_t(v), t)} \\ &\le \frac{1+\varepsilon(r_y(F_t(v)))}{1-\varepsilon(t)} \frac{\tau(v)}{\tau(P_t^{-1} \circ F_t(v))} \mathrm{e}^{nh(r_y(F_t(v))-t)}\end{aligned}$$

By definition of Busemann function we have that $r_y(F_t(v))-t$ converges, uniformly on $S_y M$, to $b_{\phi_y(v)}(y) - b_{\phi_y(v)}(x)$; so, as τ is continuous and bounded, by dominated convergence (6.4) yields

$$\begin{aligned}\lambda_x(U) &= \int_{\phi_y^{-1}(U)} \frac{\tau(v)}{\tau(\phi_x^{-1} \circ \phi_y(v))} \mathrm{e}^{-nh(b_{\phi_y(v)}(x)-b_{\phi_y(v)}(y))} dv_{S_y M}(v) \\ &= \int_U \frac{\tau(\phi_y^{-1}(\xi))}{\tau(\phi_x^{-1}(\xi))} \mathrm{e}^{-nh(b_\xi(x)-b_\xi(y))} d\lambda_y(\xi)\end{aligned}$$

which proves the first equality of the proposition.

The second equality follows from [1]: the relative densities of harmonic measures are given by the Poisson kernel, and, as $\Delta b_\xi = -nh$, by unicity of the Poisson kernel we have $\frac{d\mu_x}{d\mu_y}(\xi) = \mathrm{e}^{-nh(b_\xi(x)-b_\xi(y))}$. □

As a consequence of Theorem 5.4, we have that, *when the derivative of the Riemann tensor is bounded, the visual and harmonic measures class have the same relative densities*.

6.2 The Margulis function

For cocompact CH-manifolds Margulis introduced the function

$$m(x) = \lim_{r\to\infty} \mathrm{vol}(S_x(r))\mathrm{e}^{-Er}.$$

where E is the volume entropy of M. The main conjecture concerning this function is that it is constant if and only if M is a symmetric space, cf. [28,16] for some related results. Theorem 5.4 allows us to define the Margulis function for asymptotically harmonic manifolds (even noncocompact):

Proposition 6.2 *Let M be an asymptotically harmonic CH-manifold with $-b^2 \leq K \leq -a^2 < 0$. There exists a function $m : M \to \mathbb{R}_+$ such that*

$$\lim_{r\to\infty} \mathrm{vol}(S_x(r))\mathrm{e}^{-nhr} = m(x) \ \ \textit{and} \ \ \lim_{r\to\infty} \mathrm{vol}(B_x(r))\mathrm{e}^{-nhr} = \frac{m(x)}{nh}$$

for any $x \in M$. Moreover, the function m is harmonic.

Proof Let $V_x(r) = \mathrm{Vol}(B_x(r))$ and $v_x(r) = \mathrm{vol}(S_x(r))$, so $V_x'(r) = v_x(r)$. Since $v_x(r) = \int_{S_x M} \theta(u,r)du$, integrating (5.2) on $S_x M$, by monotone convergence we get the first equality with $m(x) = \int_{S_x M} \tau(u)du$.

Then, by Proposition 4.2, we have $V_x'(r) - nhV_x(r) \geq 0$, so $V_x(r)\mathrm{e}^{-nhr}$ is increasing. As $V_x(r) = \int_0^r \int_{S_x M} \theta(u,s)duds$, Theorem 5.4 implies, for any $r \geq 1$,

$$V_x(r) \leq V_x(1) + \frac{m(x)}{nh}(\mathrm{e}^{nhr} - \mathrm{e}^{nh}) + \mathrm{vol}(S^n)\int_1^r \varepsilon(s)\mathrm{e}^{nhs}ds$$

from which we deduce that $V_x(r)\mathrm{e}^{-nhr}$ is bounded; hence, it converges to some limit $l(x)$. As $V_x(r)\mathrm{e}^{-nhr}$ is increasing and converging, there exists a sequence $r_k \to \infty$ such that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \frac{d}{dr}_{|r=r_k} \left(V_x(r)\mathrm{e}^{-nhr}\right) \\ &= \lim_{k\to\infty} (v_x(r_k)\mathrm{e}^{-nhr_k} - nhV_x(r_k)\mathrm{e}^{-nhr_k}) \\ &= m(x) - nhl(x). \end{aligned}$$

Finally, to show that the Margulis function is harmonic, we write it using the visual measures:

$$m(x) = \int_{S_x M} \tau(u)du = \int_{\partial_\infty M} \tau(\phi_x^{-1}(\xi))d\lambda_x(\xi)$$

Choosing a fixed point $x_0 \in M$, we get

$$m(x) = \int_{\partial_\infty M} \tau(\phi_{x_0}^{-1}(\xi))\mathrm{e}^{-nh(b_\xi(x)-b_\xi(x_0))}d\lambda_{x_0}(\xi)$$

and we are done, because $\mathrm{e}^{-nh(b_\xi(x)}$ is harmonic. □

Remark 6.3 Notice that the existence of a Margulis function is a consequence of the existence of the function τ, and therefore it can also be deduced from [17] in a more general setting (cf. remark 5.3). However, to prove the harmonicity of the Margulis function, we need Proposition 6.1 whose proof uses our uniform estimate of Proposition 5.4.

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