THE INFLUENCE OF DIFFERENT GEOMETRIES OF MATRIX/SCAFFOLD ON THE RESPONSE OF A BONE AND RESORBABLE MATERIAL MIXTURE WITH VOIDS

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Outline

Objectives

- A 2-D dimensional sample made of natural bone tissue and artificial bioresorbable material is numerically investigated to study the influence of different geometries of the assemblage of matrix and scaffold.
- We consider a mixture made by bone tissue with evolving apparent mass density and a bio-resorbable material of the kind used in bone reconstruction;
- The linear theory of elastic materials with voids developed by Cowin and Nunziato (1) is employed for the mechanical behavior of such a mixture;
- the evolution model proposed in (2) and (3) is used to describe biological phenomena associated to the remodeling processes

References I

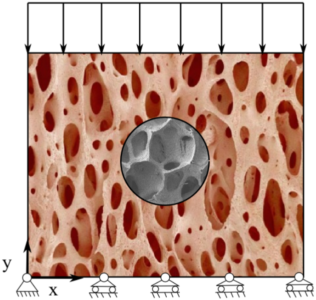
(1) S. C. Cowin, J. W. Nunziato (1983)
Linear elastic materials with voids.

(2) A. Madeo, T. Lekszycki, F. dell'Isola (2011)
A continuum model for the bio-mechanical interactions between living tissue and bio-resorbable graft after bone reconstructive surgery.

(3) A. Madeo, D. George, T. Lekszycki, M. Nierenberger, and Y. Rémond (2012)
A second gradient continuum model accounting for some effects of micro-structure on reconstructed bone remodeling.

Model and Materials

Solid mixture of bone tissue and bio-resorbable material



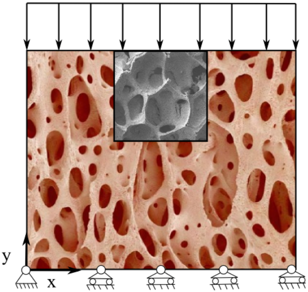


Figure : Initial configurations

The material under investigation is constituted by a mixture of bio-resorbable material (used in bone reconstructive surgery) and living bone-tissue, both porous materials.

Preliminary Assumptions I

Elastic Materials with Voids

1. The mathematical structure of an elastic material with voids is based on the fact that the apparent mass density ρ in the actual configuration $\mathscr{C}_t$ has the decomposition:

$$\rho(x,t) = \varsigma(x,t)\rho_{\text{Max}}(x,t) \tag{1}$$

   where ρ_{Max} is the density of matrix material (also called the true density in the theory of the mixtures) and ς is the volume fraction field; the same relation holds also for the ref. configuration $\mathscr{C}^*$

2. thus the independent variables are the usual placement field χ and, newly introduced, the scalar field ξ i.e. the change in matrix volume fraction from the reference configuration to the actual configuration:

$$\xi(\mathbf{X},t) = \varsigma(\chi(\mathbf{X},t),t) - \varsigma^*(\mathbf{X},t) \tag{2}$$

Preliminary Assumptions II

1. The mass density of the mixture can be expressed by

$$\rho = \rho_{\mathrm{b}} + \rho_{\mathrm{m}} \tag{3}$$

2. the volume fraction of mixture is

$$\varsigma = \left(\frac{\rho_{\mathrm{b}}}{\rho_{\mathrm{b\,Max}}} + \frac{\rho_{\mathrm{m}}}{\rho_{\mathrm{m\,Max}}} \right) = \varsigma_{\mathrm{b}} + \varsigma_{\mathrm{m}} \tag{4}$$

3. Constitutive equation for the porosity is

$$\phi\left(x,t\right) = 1 - \theta \left(\frac{\rho_{\mathrm{b}}\left(x,t\right)}{\rho_{\mathrm{b\,Max}}} + \frac{\rho_{\mathrm{m}}\left(x,t\right)}{\rho_{\mathrm{m\,Max}}} \right) = 1 - \theta\left(\varsigma_{\mathrm{b}} + \varsigma_{\mathrm{m}}\right) \tag{5}$$

   being $\theta = 1$

4. $\Delta\phi = \phi - \phi^* = -\xi$

Modeling

◂ Go

Stored Energy Density

1. Assuming that the material possesses a center of symmetry and is isotropic, the stored energy density $\mathscr{E}$ associated with strain tensor E and void volume distortion from the reference configuration ξ is:

$$\mathscr{E} = \frac{1}{2}\frac{Y(\rho_{\mathrm{b}}^{*},\rho_{\mathrm{m}}^{*})\nu}{(1-2\nu)(1+\nu)}E_{\mathrm{ii}}^{2} + \frac{1}{2}\frac{Y(\rho_{\mathrm{b}}^{*},\rho_{\mathrm{m}}^{*})}{(1+\nu)}E_{\mathrm{ij}}E_{\mathrm{ij}} + \\ + \frac{1}{2}K_1(\rho_{\mathrm{b}}^{*},\rho_{\mathrm{m}}^{*})\xi^2 + \frac{1}{2}K_2\xi_{,\mathrm{i}}\xi_{,\mathrm{i}} + K_3(\rho_{\mathrm{b}}^{*},\rho_{\mathrm{m}}^{*})\,\xi\,E_{\mathrm{ii}}$$

   where Y, ν are Young's modulus and Poisson's ratio of the mixture.

2. We assume for interface conditions at the boundary between bone and material zones, $\partial\mathscr{B}_{i.f.}$ the continuity of placement χ and an energy density for the dual condition related to field ξ as

$$\mathscr{E}_{i.f.} = \frac{1}{2}K_4\left(\xi^{+} - \xi^{-}\right)^2 = \frac{1}{2}K_4\,[|\xi|]^2$$

The effect of the voids on the mixture I

A note on the principle of local equilibrium

Assuming K_2 to be zero in the stored energy density $\mathscr{E}$ yields:

$$\mathscr{E}(E,\xi) = \frac{1}{2}\mathbb{C}E \cdot E + \frac{1}{2}K_1\xi^2 + K_3\,\xi\,\mathrm{tr}(E)$$

Our aim is to find all the couples (u,ξ) corresponding to relative minima of the functional $\mathscr{E}(u,\xi)$ or in an equivalent form to find (u) such that

$$\min_{u,\xi}\left\{\int_{\mathscr{B}}\mathscr{E}(E,\xi)\mathrm{d}\mathscr{V}\right\} \equiv \min_{u}\left\{\int_{\mathscr{B}}\mathscr{E}(E,\hat{f}(E))\mathrm{d}\mathscr{V}\right\}$$

where $\hat{f}(E)$ is given by the following equation:

$$\left.\frac{\partial\mathscr{E}}{\partial\xi}\right|_{\xi=\hat{f}(E)} = K_1\,\hat{f}(E) + K_3\,\mathrm{tr}(E) = 0 \tag{6}$$

The effect of the voids on the mixture II

An equivalent Lamé's first parameter

Solving Eq. (6) in terms of $\hat{f}(E)$ gives

$$\hat{f}(E) = -\frac{K_3}{K_1}\operatorname{tr}(E) \tag{7}$$

Recalling the expression of the energy for linearly elastic isotropic materials and substituting Eq. (6), the stored energy density becomes:

$$\mathscr{E}(E,\xi) = \frac{1}{2}\{2\mu\, E\cdot E + \lambda\,[\operatorname{tr}(E)]^2\} + \frac{1}{2}K_1\left[-\frac{K_3}{K_1}\operatorname{tr}(E)\right]^2 + K_3\left[-\frac{K_3}{K_1}\operatorname{tr}(E)\right]\operatorname{tr}(E) =$$

$$= \frac{1}{2}\{2\mu\, E\cdot E + \left[\lambda - \frac{(K_3)^2}{K_1}\right][\operatorname{tr}(E)]^2\} = \frac{1}{2}\{2\mu\, E\cdot E + \lambda_{\text{eq}}\,[\operatorname{tr}(E)]^2\} \tag{8}$$

where for the material with voids an equivalent Lamé's parameter is

$$\lambda_{\text{eq}} = \left[\lambda - \frac{(K_3)^2}{K_1}\right] = \lambda - \lambda_v \tag{9}$$

Material Parameters I

The Young modulus of the mixture varies with reference mass densities

Material properties of the mixture are assumed inhomogeneous and instead of Lamé's parameters (λ and μ), we consider Young's modulus Y and Poisson's ratio ν.

$$Y = Y_{\mathrm{b\,Max}}({\varsigma_{\mathrm{b}}}^{*})^{\beta_{\mathrm{b}}} + Y_{\mathrm{m\,Max}}({\varsigma_{\mathrm{m}}}^{*})^{\beta_{\mathrm{m}}}$$

As a first approximation, we assume that the Poisson's ratio is constant ($\nu = 0.3$).

Material Parameters II

The material parameters K_1 and K_2

$$K_1 = K_{1\mathrm{b\,Max}}(\varsigma_{\mathrm{b}}{}^*)^{\alpha_{\mathrm{b}}} + K_{1\mathrm{m\,Max}}(\varsigma_{\mathrm{m}}{}^*)^{\alpha_{\mathrm{m}}}$$

In fact K_1 can be interpreted as a pore stiffness; while K_2 as a first approximation is assumed to be constant.

The material parameter K_3

It can be easily demonstrated that an equivalent Lamé parameter $\lambda_{eq} = \lambda - \lambda_v$ can be introduced to consider the presence of voids. We hypothesize that λ_v is a fraction of λ depending on the reference porosity and thus K_3 is expressed by

$$\lambda_v = K_3{}^2 / K_1 \propto \lambda \quad \Rightarrow \quad \lambda_{eq} = (1-\kappa)\lambda \quad \Rightarrow \quad K_3 = \sqrt{\kappa \lambda K_1}$$

where the parameter κ is non-negative and less than one.

Mechanical governing equations

Weak formulation

We require that the stored energy function $\mathscr{E}$ of a material with voids satisfies, for all variations in u_i and ς, the equality

$$\delta\mathscr{A} = \int_{\mathscr{B}} \left[T_{ij}\delta E_{ij} + K_1\,\xi\,\delta\xi + K_2\,\xi_{,i}\delta\xi_{,i} + K_3\left(E_{ii}\delta\xi + \xi\,\delta E_{ii}\right) - b_i\delta u_i\right] \mathrm{d}\mathscr{V} + \\ - \int_{\partial_\tau\mathscr{B}} \tau_i\delta u_i \mathrm{d}\mathscr{S} - \int_{\partial\mathscr{B}} \Xi\,\delta\xi\mathrm{d}\mathscr{S} + \int_{\partial\mathscr{B}_{i.f.}} K_4\,[|\xi|]\,\delta[|\xi|]\mathrm{d}\mathscr{S} = 0$$

- The test functions are defined by the 2-D displacement components δu_1, δu_2;
- components of the surface force density b_i are assumed to be zero;
- components of the edge force density are τ_i.

Evolution rules of growth and resorption

Governing equations for the mass densities of the two phases

$$\begin{cases} \dot{\rho}_{\mathrm{b}} & = A_{\mathrm{b}}(S)\, H(\phi) \\ \dot{\rho}_{\mathrm{m}} & = A_{\mathrm{m}}(S)\, H(\phi) \end{cases}$$

$$H = k\phi\,(1-\phi)$$

$$A_{\mathrm{b}}(S) = \begin{cases} s_{\mathrm{b}} S & \text{for } S \geq 0 \\ r_{\mathrm{b}} S & \text{for } S < 0 \end{cases}$$

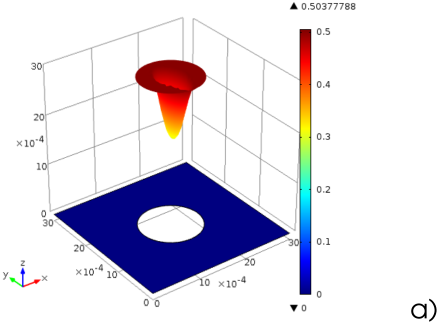
$$A_{\mathrm{m}}(S) = \begin{cases} 0 & \text{for } S \geq 0 \\ r_{\mathrm{m}} S & \text{for } S < 0 \end{cases}$$

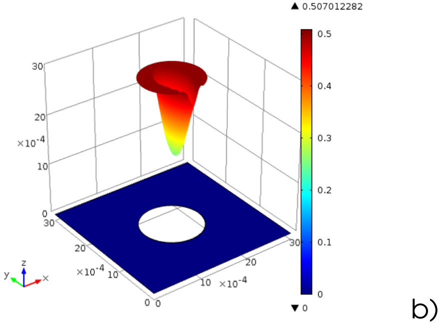
Stimulus

$$S(X,t) = \int_{\Omega} \mathscr{E}(X_0,t)\, d(\rho_{\mathrm{b}}(X_0,t))\; e^{-f(X-X_0)} dX_0 - P_{\mathrm{ref}} = P(X,t) - P_{\mathrm{ref}}$$

Results – Circular graft

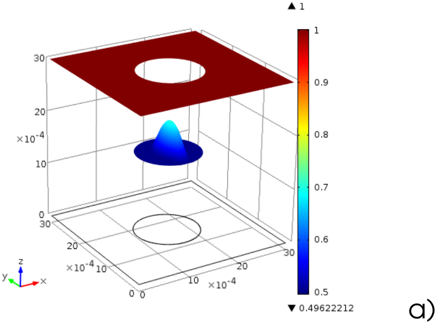
Final mass densities of bio-material with ξ **a**) and without ξ **b**)

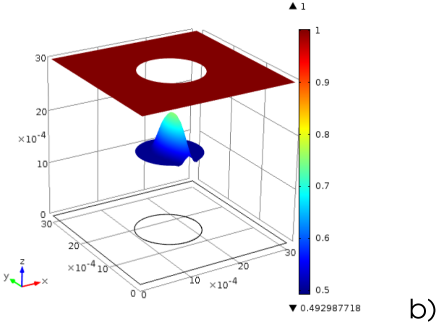




Results – Circular graft

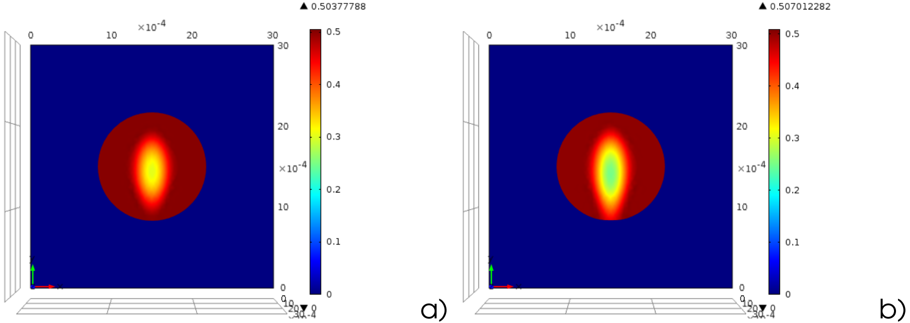
Final mass densities of bone with ξ (**a**) and without ξ (**b**)





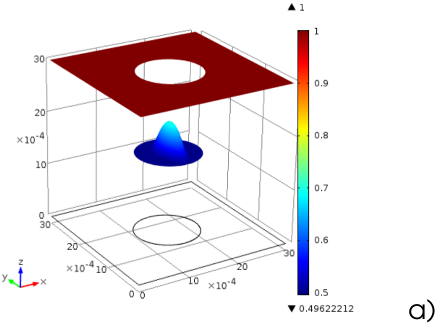
Results – Circular graft

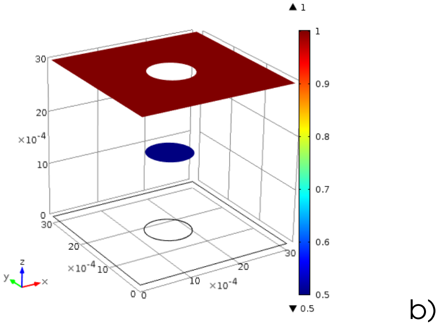
Final mass densities of bio-material with ξ (**a**) and without ξ (**b**)



Results – Circular graft

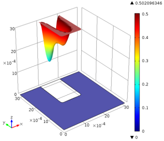
Final mass densities of bone for two different size grafts: "big" (**a**) and "small" (**b**)



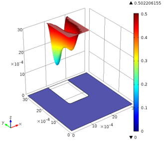


Results – Rectangular graft

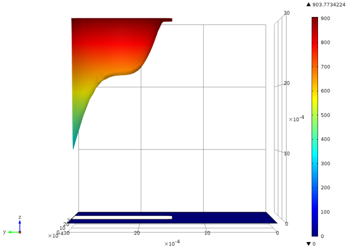
Final mass densities of bio-material with ξ (**a**) and without ξ (**b**)



a)



b)



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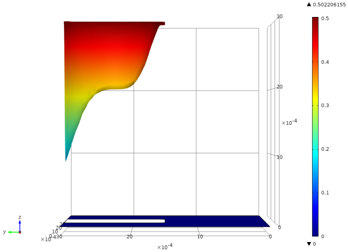
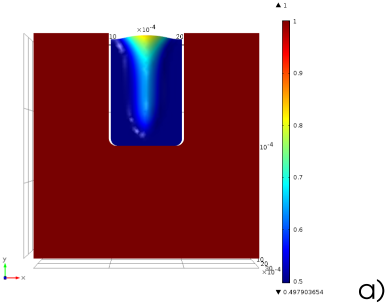


b)

Results – Rectangular graft

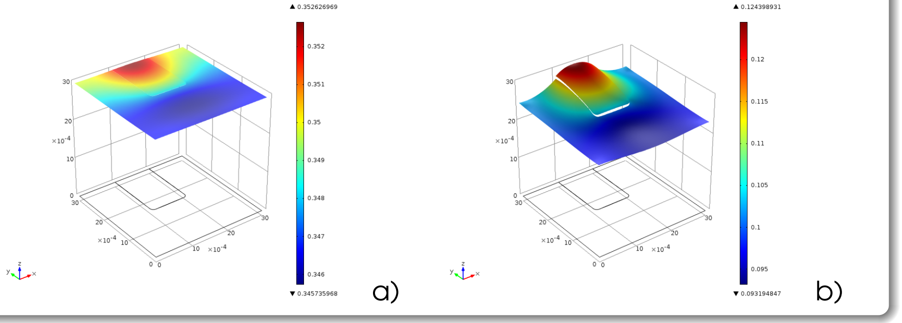
Final mass densities of bone with ξ (**a**) and without ξ (**b**)



Results – Rectangular graft

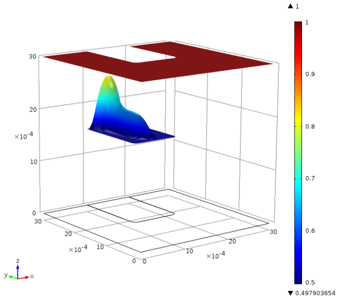
◂ Return

Initial ξ (**a**) and final ξ (**b**)

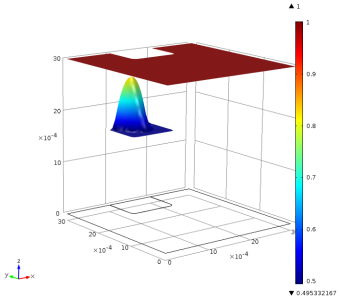


Results – Rectangular graft

Final mass densities of bone for two different size grafts: "big" (**a**) and "small" (**b**)



a)



b)

Conclutions

1. Effects of the newly introduced variable ξ, the change in matrix volume fraction from the reference configuration to the actual configuration;
2. Effects of different size of grafts;
3. Anisotropy in the remodeling process due to the application of load.

Thanks

THANK YOU VERY MUCH FOR YOUR KIND ATTENTION