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On conjugacy of maximal abelian subalgebras and the outer automorphism group of the Cuntz algebra

Roberto Conti

Dipartimento di Scienze di Base e Applicate per l'Ingegneria,
Sezione di Matematica, Sapienza Università di Roma, Via A. Scarpa 16,
00161 Roma, Italy (roberto.conti@sbai.uniroma1.it)

Jeong Hee Hong

Department of Data Information,
Korea Maritime and Ocean University, Busan 606–791,
Republic of Korea (hongjh@kmou.ac.kr)

Wojciech Szymański

Department of Mathematics and Computer Science,
University of Southern Denmark, Campusvej 55,
5230 Odense M, Denmark (szymanski@imada.sdu.dk)

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We investigate the structure of the outer automorphism group of the Cuntz algebra and the closely related problem of conjugacy of maximal abelian subalgebras in \mathcal{O}_n . In particular, we exhibit an uncountable family of maximal abelian subalgebras, conjugate to the standard maximal abelian subalgebra \mathcal{D}_n via Bogolubov automorphisms, that are not inner conjugate to \mathcal{D}_n .

1. Introduction

The main motivation for the present paper comes from the desire to better understand the structure of the outer automorphism group of the Cuntz algebra \mathcal{O}_n [12, 14]. As the Cuntz algebras are among the most intensely investigated operator algebras, it is not surprising that both their single automorphisms and the structure of their automorphism groups have attracted a lot of interest. In addition to the obvious intrinsic value of this line of research, we would also like to point out its importance for the current efforts within Elliott's classification programme. In this context, for example, the question as to whether $\text{Aut}(\mathcal{O}_2)$ is a universal Polish group is raised in [25, question 9.1].

Our point of departure for the investigations of $\text{Out}(\mathcal{O}_n)$ is the recent progress in the understanding of $\text{Aut}(\mathcal{O}_n, \mathcal{D}_n)$, the group of those automorphisms of \mathcal{O}_n that globally preserve the standard maximal abelian subalgebra (MASA) \mathcal{D}_n [8, 11]. This group has the structure of a semi-direct product $\lambda(\mathcal{U}(\mathcal{D}_n)) \rtimes \lambda(\mathcal{S}_n)^{-1}$, where $\lambda(\mathcal{U}(\mathcal{D}_n))$ is a maximal abelian subgroup of $\text{Aut}(\mathcal{O}_n)$ of those automorphisms that

fix \mathcal{D}_n pointwise (see [13, proposition 1.3]) and $\lambda(\mathcal{S}_n)^{-1}$ is countable and discrete, the so-called Weyl group of \mathcal{O}_n . Of particular note here is the relation between the image of $\lambda(\mathcal{S}_n)^{-1}$ in $\text{Out}(\mathcal{O}_n)$ and the group of automorphisms of the full two-sided n -shift shown in [10].

The next logical step in the study of $\text{Out}(\mathcal{O}_n)$ would be to learn if every automorphism of \mathcal{O}_n has a representative in $\text{Out}(\mathcal{O}_n)$ coming from $\text{Aut}(\mathcal{O}_n, \mathcal{D}_n)$, and, if not, to classify MASAs of \mathcal{O}_n that are outer, but not inner, conjugate to \mathcal{D}_n (that is, one is mapped onto the other by an outer automorphism but no such inner automorphism exists). In fact, this question was raised a few years ago by Joachim Cuntz in a conversation with the third author. In the present paper, we show that Bogolubov automorphisms either globally preserve \mathcal{D}_n or move it to other, not inner conjugate, MASAs; see theorem 3.7 and corollary 3.8.

Naturally, any investigations of the structure of an outer automorphism group are significantly helped by classification of single automorphisms up to conjugacy. In the context of the Cuntz algebras, a great deal of progress has been achieved in this direction and we would like to specifically call the attention of the reader to [18, 19, 21]. By [21, theorem 9] and [24, theorem 3.6], any two aperiodic automorphisms of \mathcal{O}_n are outer conjugate, which is in nice analogy with the classification of aperiodic automorphisms of the hyperfinite II_1 factor due to Connes [4]. The classification of non-aperiodic automorphisms of \mathcal{O}_n is also related to the seminal work of Connes [5], although in this case the C^* -algebraic setting is much more intricate by comparison with the von Neumann algebraic one. Indeed, non-aperiodic automorphisms of the hyperfinite II_1 factor are completely classified by pairs (k, γ) , with k a positive integer (outer period) and γ a k th root of unity [5, theorem 1.5]. It is shown in [5, proposition 1.6] that each invariant (k, γ) may be realized in $\text{Aut}(\mathcal{O}_k)$ by an automorphism of the form $\lambda_d \lambda_w$ with d a unitary in the canonical MASA \mathcal{D}_n and λ_w a Bogolubov permutation. However, there exist automorphisms of \mathcal{O}_2 with $k = 2$ and $\gamma = 1$ that are not outer conjugate to each other [18, example 5.8]. In the present paper, classification results for single automorphisms of \mathcal{O}_n are used in the proofs of several structural properties of $\text{Out}(\mathcal{O}_n)$ collected in §4.

2. Notation and preliminaries

If n is an integer greater than 1, then the Cuntz algebra \mathcal{O}_n is a unital simple purely infinite C^* -algebra generated by n isometries S_1, \dots, S_n satisfying $\sum_{i=1}^n S_i S_i^* = 1$ [12]. We denote by W_n^k the set of k -tuples $\mu = (\mu_1, \dots, \mu_k)$ with $\mu_m \in \{1, \dots, n\}$, and by W_n the union $\bigcup_{k=0}^{\infty} W_n^k$, where $W_n^0 = \{0\}$. If $\mu \in W_n^k$, then $|\mu| = k$ is the length of μ . If $\mu = (\mu_1, \dots, \mu_k) \in W_n$ then $S_\mu = S_{\mu_1} \dots S_{\mu_k}$ ($S_0 = 1$ by convention) is an isometry with range projection $P_\mu = S_\mu S_\mu^*$. Every word in $\{S_i, S_i^* \mid i = 1, \dots, n\}$ can be uniquely expressed as $S_\mu S_\nu^*$ for $\mu, \nu \in W_n$ [12, lemma 1.3].

We denote by \mathcal{F}_n^k the C^* -subalgebra of \mathcal{O}_n spanned by all words of the form $S_\mu S_\nu^*$, $\mu, \nu \in W_n^k$, which is isomorphic to the matrix algebra $M_{n^k}(\mathbb{C})$. The norm closure \mathcal{F}_n of $\bigcup_{k=0}^{\infty} \mathcal{F}_n^k$ is the uniformly hyperfinite algebra (UHF-algebra) of type n^∞ [16], called the core UHF-subalgebra of \mathcal{O}_n [12]. We denote by τ the unique normalized trace on \mathcal{F}_n . Subalgebra \mathcal{F}_n is the fixed-point algebra for the gauge action $\gamma: U(1) \rightarrow \text{Aut}(\mathcal{O}_n)$, such that $\gamma_z(S_j) = z S_j$ for $z \in U(1)$ and $j = 1, \dots, n$.

For an integer $m \in \mathbb{Z}$, we define

$$\mathcal{O}_n^{(m)} := \{x \in \mathcal{O}_n : \gamma_z(x) = z^m x \ \forall z \in U(1)\},$$

a spectral subspace for γ . Then $\mathcal{O}_n^{(0)} = \mathcal{F}_n$ and for each positive integer m and each $\alpha \in W_n^m$ we have $\mathcal{O}_n^{(m)} = \mathcal{F}_n S_\alpha$ and $\mathcal{O}_n^{(-m)} = S_\alpha^* \mathcal{F}_n$. Furthermore,

$$E_m(x) = \int_{z \in U(1)} z^{-m} \gamma_z(x) dz$$

is a completely contractive projection from \mathcal{O}_n onto $\mathcal{O}_n^{(m)}$, such that $E_m(xay) = xE_m(a)y$ for all $a \in \mathcal{O}_n, x, y \in \mathcal{F}_n$. In particular, $E := E_0$ is the faithful conditional expectation from \mathcal{O}_n onto \mathcal{F}_n given by averaging action γ over $U(1)$ with respect to the Haar measure.

The C^* -subalgebra of \mathcal{O}_n generated by projections $P_\mu, \mu \in W_n$, is a MASA in \mathcal{O}_n . We call it the *diagonal* and denote it by \mathcal{D}_n , also writing \mathcal{D}_n^k for $\mathcal{D}_n \cap \mathcal{F}_n^k$. The spectrum of \mathcal{D}_n is naturally identified with X_n , the full one-sided n -shift space (a Cantor set). Occasionally, we will view X_n as a metric space equipped with the metric $\text{dist}(x, y) = n^{-k}$, where $k = \min\{m \in \mathbb{N} \mid x_m \neq y_m\}$.

As shown by Cuntz in [13], there exists the following bijective correspondence between unitaries in \mathcal{O}_n (whose collection is denoted by $\mathcal{U}(\mathcal{O}_n)$) and unital $*$ -endomorphisms of \mathcal{O}_n (whose collection we denote by $\text{End}(\mathcal{O}_n)$), determined by

$$\lambda_u(S_i) = uS_i, \quad i = 1, \dots, n.$$

Composition of endomorphisms corresponds to the ‘convolution’ multiplication of unitaries: $\lambda_u \circ \lambda_w = \lambda_{\lambda_u(w)u}$. In the case $u, w \in \mathcal{U}(\mathcal{F}_n^1)$, this formula simplifies to $\lambda_u \circ \lambda_w = \lambda_{uw}$ and, in particular, there exists an imbedding $u \mapsto \lambda_u$ of $U(n) \cong \mathcal{U}(\mathcal{F}_n^1)$ into $\text{Aut}(\mathcal{O}_n)$ [15]. If A is either a unital C^* -subalgebra of \mathcal{O}_n or a subset of $\mathcal{U}(\mathcal{O}_n)$, then we denote $\lambda(A) = \{\lambda_u \in \text{End}(\mathcal{O}_n) : u \text{ unitary in } A\}$ and $\lambda(A)^{-1} = \{\lambda_u \in \text{Aut}(\mathcal{O}_n) : u \text{ unitary in } A\}$.

We denote by φ the canonical shift on the Cuntz algebra

$$\varphi(x) = \sum_{i=1}^n S_i x S_i^*, \quad x \in \mathcal{O}_n.$$

Clearly, $\varphi(\mathcal{F}_n) \subset \mathcal{F}_n$ and $\varphi(\mathcal{D}_n) \subset \mathcal{D}_n$. We denote by $\sigma : X_n \rightarrow X_n$ the shift on X_n . Then we have $\varphi(f)(x) = f(\sigma(x))$ for all $f \in C(X_n)$ and $x \in X_n$.

For all $u \in \mathcal{U}(\mathcal{O}_n)$, we have $\text{Ad}(u) = \lambda_{u\varphi(u^*)}$. If $u \in \mathcal{U}(\mathcal{O}_n)$, then for each positive integer k we define

$$u_k = u\varphi(u) \cdots \varphi^{k-1}(u). \tag{2.1}$$

Here, $\varphi^0 = \text{id}$, and we agree that u_k^* stands for $(u_k)^*$. If α and β are multi-indices of length k and m , respectively, then $\lambda_u(S_\alpha S_\beta^*) = u_k S_\alpha S_\beta^* u_m^*$. This is established through a repeated application of the identity $S_i x = \varphi(x) S_i$, valid for all $i = 1, \dots, n$ and $x \in \mathcal{O}_n$.

We often consider elements of \mathcal{O}_n of the form $w = \sum_{(\alpha, \beta) \in \mathcal{J}} c_{\alpha, \beta} S_\alpha S_\beta^*$, where \mathcal{J} is a finite collection of pairs (α, β) of words $\alpha, \beta \in W_n$ and $c_{\alpha, \beta} \in \mathbb{C}$. In particular, we consider the group \mathcal{S}_n of those unitaries in \mathcal{O}_n that can be written as finite sums of words, i.e. in the form $w = \sum_{(\alpha, \beta) \in \mathcal{J}} S_\alpha S_\beta^*$. Each $w \in \mathcal{S}_n$ normalizes \mathcal{D}_n ,

and hence $\lambda_w(\mathcal{D}_n) \subseteq \mathcal{D}_n$ [13, proposition 1.4]. We define $\mathcal{P}_n := \mathcal{S}_n \cap \mathcal{U}(\mathcal{F}_n)$ and $\mathcal{P}_n^k := \mathcal{S}_n \cap \mathcal{U}(\mathcal{F}_n^k)$. It is known that \mathcal{S}_2 is isomorphic to the Thompson group V [22, proposition 9.6].

For algebras $A \subseteq B$ we denote by $\mathcal{N}_B(A) = \{u \in \mathcal{U}(B) : uAu^* = A\}$ the normalizer of A in B and by $A' \cap B = \{b \in B : (\forall a \in A) ab = ba\}$ the relative commutant of A in B . We also denote by $\text{Aut}(B, A)$ the collection of all those automorphisms α of B such that $\alpha(A) = A$, and by $\text{Aut}_A(B)$ those automorphisms of B that fix A pointwise. Likewise, we denote by $\text{End}_A(B)$ the collection of those endomorphisms of B that fix pointwise subalgebra A .

3. Conjugacy of MASAs

If A_1 and A_2 are two MASAs in B , then we say they are *conjugate* if there exists an $\alpha \in \text{Aut}(B)$ such that $\alpha(A_1) = A_2$. We say A_1 and A_2 are *inner conjugate* if there exists a $u \in \mathcal{U}(B)$ such that $uA_1u^* = A_2$.

PROPOSITION 3.1. *Let $z \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_z \in \text{Aut}(\mathcal{O}_n)$. Then there exists a $u \in \mathcal{U}(\mathcal{O}_n)$ such that*

$$\lambda_z(\mathcal{D}_n) = \text{Ad}(u)(\mathcal{D}_n) \tag{3.1}$$

if and only if there exist $u \in \mathcal{U}(\mathcal{O}_n)$ and $w \in \mathcal{S}_n$ such that

$$\lambda_z(d) = \text{Ad}(u)\lambda_w(d) \quad \forall d \in \mathcal{D}_n. \tag{3.2}$$

Proof. Suppose that (3.1) holds. Then $\text{Ad}(u^*)\lambda_z \in \text{Aut}(\mathcal{O}_n, \mathcal{D}_n)$, and thus there exist $v \in \mathcal{U}(\mathcal{D}_n)$ and $w \in \mathcal{S}_n$ such that $\text{Ad}(u^*)\lambda_z = \lambda_v\lambda_w$ [8, theorem 2.1]. Since $\lambda_v|_{\mathcal{D}_n} = \text{id}$ and $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$, for each $d \in \mathcal{D}_n$ we have $\text{Ad}(u^*)\lambda_z(d) = \lambda_v\lambda_w(d) = \lambda_w(d)$, and identity (3.2) holds.

Conversely, suppose that (3.2) holds. Then $\lambda_z^{-1}\text{Ad}(u)\lambda_w \in \text{End}_{\mathcal{D}_n}(\mathcal{O}_n)$, and thus there exists a $v \in \mathcal{U}(\mathcal{D}_n)$ such that $\lambda_z^{-1}\text{Ad}(u)\lambda_w = \lambda_v$ (see [6, proposition 3.2]). Since $\lambda(\mathcal{D}_n) \subseteq \text{Aut}(\mathcal{O}_n)$ (see [13, proposition 1.3]), λ_w is an automorphism of \mathcal{O}_n . Since $\lambda_w(\mathcal{D}_n) \subseteq \mathcal{D}_n$ and \mathcal{D}_n is a MASA in \mathcal{O}_n , we may conclude that $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$, and identity (3.1) follows. \square

Before proving our main result, theorem 3.7, we need some preparation.

LEMMA 3.2. *If $x \in \mathcal{O}_n$, $x \geq 0$, and $x\mathcal{D}_n = \mathcal{D}_n x$, then $x \in \mathcal{D}_n$.*

Proof. We may assume that $0 \leq x \leq 1$. Let Φ be the faithful conditional expectation from \mathcal{O}_n onto \mathcal{D}_n . Since $0 \leq x^2 \leq x$, we have $0 \leq \Phi(x^2) \leq \Phi(x)$. Let $d \in \mathcal{D}_n$ be such that $d\Phi(x) = 0$. Then $0 \leq d\Phi(x^2)d^* \leq d\Phi(x)d^* = 0$, and hence $\Phi(dx^2d^*) = d\Phi(x^2)d^* = 0$. Consequently, $dx = 0$. Now, for an arbitrary $a \in \mathcal{D}_n$, let $b \in \mathcal{D}_n$ be such that $xa = bx$. Then $(a - b)\Phi(x) = 0$, and thus $(a - b)x = 0$. This shows that x is in the commutant of \mathcal{D}_n , and therefore $x \in \mathcal{D}_n$. \square

REMARK 3.3. The conclusion of lemma 3.2 remains valid if $\mathcal{D}_n \subseteq \mathcal{O}_n$ are replaced by any C^* -algebras $D \subseteq A$ such that D is a MASA in A and there exists a faithful conditional expectation from A onto D . However, it may fail if x is merely self-adjoint but not positive. For example, simply take $A = M_2(\mathbb{C})$, D the diagonal matrices and x the matrix with both off-diagonal entries equal to 1 and the diagonal ones both 0.

LEMMA 3.4. *Let $z \in \mathcal{U}(\mathcal{F}_n^1) \setminus \mathcal{N}_{\mathcal{F}_n^1}(\mathcal{D}_n^1)$ and $x \in \mathcal{F}_n$. If $\lambda_z(\mathcal{D}_n)x = x\mathcal{D}_n$, then $x = 0$.*

Proof. Since $z \in \mathcal{U}(\mathcal{F}_n^1) \setminus \mathcal{N}_{\mathcal{F}_n^1}(\mathcal{D}_n^1)$, there exist a minimal projection $p \in \mathcal{D}_n^1$ and a $\delta > 0$ such that, for all $m \in \mathbb{N}$, all projections $e \in \mathcal{D}_n^m$ and all $h \in \mathcal{D}_n$, we have

$$\|e\lambda_z(\varphi^m(p)) - h\| \geq \delta. \tag{3.3}$$

Indeed, since $z \notin \mathcal{N}_{\mathcal{F}_n^1}(\mathcal{D}_n^1)$, there exists an $i \in W_n^1$ such that $\lambda_z(P_i) \notin \mathcal{D}_n$. By the Hahn–Banach theorem, there exists a functional ω of norm 1 on \mathcal{O}_n such that $\omega(\lambda_z(P_i)) = \delta > 0$ and $\omega|_{\mathcal{D}_n} = 0$. Now, take $m \in \mathbb{N}$, a projection $e \in \mathcal{D}_n^m$ and an $h \in \mathcal{D}_n$, and let $\alpha \in W_n^m$ be such that $eP_\alpha = P_\alpha$. Then

$$\begin{aligned} \|e\lambda_z(\varphi^m(P_i)) - h\| &= \|e\varphi^m(\lambda_z(P_i)) - h\| \\ &\geq \|P_\alpha\varphi^m(\lambda_z(P_i)) - P_\alpha h\| \\ &= \|S_\alpha(\lambda_z(P_i) - S_\alpha^*hS_\alpha)S_\alpha^*\| \\ &= \|\lambda_z(P_i) - S_\alpha^*hS_\alpha\| \\ &\geq |\omega(\lambda_z(P_i) - S_\alpha^*hS_\alpha)| \\ &= \delta. \end{aligned}$$

Now suppose there is an $x \neq 0$ in \mathcal{F}_n such that $\lambda_z(\mathcal{D}_n)x = x\mathcal{D}_n$. We may assume that $\|x\| = 1$. Since $x^*\lambda_z(\mathcal{D}_n) = \mathcal{D}_n x^*$ as well, we have $x^*x\mathcal{D}_n = \mathcal{D}_n x^*x$, and thus $x^*x \in \mathcal{D}_n$ by lemma 3.2.

Take a small $\varepsilon > 0$. For some $l \in \mathbb{N}$, there exists a $y_0 \in \mathcal{F}_n^l$ such that $\|x - y_0\| < \varepsilon$. Then $\|x^*x - y_0^*y_0\| < \varepsilon(2 + \varepsilon)$. For some $k \geq l$ there exists a $d \in \mathcal{D}_n^k$ such that $\|x^*x - d\| < \varepsilon$ and $d \geq 0$. Then $\|y_0^*y_0 - d\| < \varepsilon(3 + \varepsilon)$, and hence $\||y_0| - \sqrt{d}\| < \sqrt{\varepsilon(3 + \varepsilon)}$ due to operator monotonicity of the square root function. Indeed, since $y_0^*y_0 \leq d + \varepsilon(3 + \varepsilon)$, we have $|y_0| = \sqrt{y_0^*y_0} \leq \sqrt{d + \varepsilon(3 + \varepsilon)} \leq \sqrt{d} + \sqrt{\varepsilon(3 + \varepsilon)}$ and, likewise, $\sqrt{d} \leq |y_0| + \sqrt{\varepsilon(3 + \varepsilon)}$. Now, write $y_0 = w|y_0|$ with w a unitary in $\mathcal{F}_n^l \subseteq \mathcal{F}_n^k$. Setting $y := w\sqrt{d}$, we have $y \in \mathcal{F}_n^k$, $y^*y = d \in \mathcal{D}_n^k$ and

$$\|x - y\| \leq \|x - y_0\| + \|w|y_0| - w\sqrt{d}\| < \varepsilon + \sqrt{\varepsilon(3 + \varepsilon)} =: \varepsilon'.$$

Let p be the projection in \mathcal{D}_n^1 and let $\delta > 0$ be such that identity (3.3) holds. Let $g \in \mathcal{D}_n$ satisfy $\lambda_z(\varphi^k(p))x = xg$. Also, let q be the spectral projection of d corresponding to eigenvalue $\|d\|$. Then we have

$$\begin{aligned} 0 &= \|\lambda_z(\varphi^k(p))x - xg\| \\ &= \|(\lambda_z(\varphi^k(p))y - yg) + (\lambda_z(\varphi^k(p))(x - y) - (x - y)g)\| \\ &> \|y(\lambda_z(\varphi^k(p)) - g)\| - \varepsilon'(1 + \|g\|) \\ &\geq \frac{1}{1 + \varepsilon'} \|y^*y(\lambda_z(\varphi^k(p)) - g)\| - \varepsilon'(1 + \|g\|). \end{aligned}$$

We have $y^*y = d$, $dq = \|d\|q$ and $\|d\| \geq \|x^*x\| - \varepsilon = 1 - \varepsilon$. Thus,

$$\begin{aligned} &\frac{1}{1 + \varepsilon'} \|y^*y(\lambda_z(\varphi^k(p)) - g)\| - \varepsilon'(1 + \|g\|) \\ &\geq \frac{1 - \varepsilon}{1 + \varepsilon'} \|q(\lambda_z(\varphi^k(p)) - g)\| - \varepsilon'(1 + \|g\|) \\ &\geq \frac{1 - \varepsilon}{1 + \varepsilon'} \delta - \varepsilon'(1 + \|g\|) \end{aligned}$$

by (3.3). Since ε and ε' can be simultaneously arbitrarily small, this is a contradiction, which shows that $x = 0$. \square

REMARK 3.5. MASAs of the form $\lambda_z(\mathcal{D}_n)$, $z \in \mathcal{U}(\mathcal{F}_n^1)$, are called *standard* and are used to compute non-commutative topological entropy of certain endomorphisms [17, 26, 27]. They are abstractly characterized in [1].

For lemma 3.6, note that, given any partial isometry $S_\alpha S_\beta^*$ with $|\alpha|, |\beta| \geq 1$, there exists a $w \in \mathcal{S}_n$ of the form $w = S_\alpha S_\beta^* + \sum_{(\mu, \nu)} S_\mu S_\nu^*$. This is easily verified with help of the pigeonhole principle. Recall also that E denotes the faithful conditional expectation from \mathcal{O}_n onto \mathcal{F}_n .

LEMMA 3.6. *Let $a \in \mathcal{O}_n$. If $E(av) = 0$ for all $v \in \mathcal{S}_n$, then $a = 0$.*

Proof. For each projection $P_\beta \in \mathcal{D}_n$, $\beta \in W_n$, we have $0 = E(av)P_\beta = E(a(vP_\beta))$. Thus $E(aS_\alpha S_\beta^*) = 0$ for all α, β , with $|\alpha|, |\beta| \geq 1$. Since the linear span of such elements $S_\alpha S_\beta^*$ is dense in \mathcal{O}_n and E is faithful, we conclude that $a = 0$. \square

THEOREM 3.7. *If $z \in \mathcal{U}(\mathcal{F}_n^1) \setminus \mathcal{N}_{\mathcal{F}_n^1}(\mathcal{D}_n^1)$ and $a \in \mathcal{O}_n$ is such that $\lambda_z(\mathcal{D}_n)a = a\mathcal{D}_n$, then $a = 0$. In particular, there is no unitary $u \in \mathcal{O}_n$ such that $\lambda_z(\mathcal{D}_n) = \text{Ad}(u)(\mathcal{D}_n)$.*

Proof. Suppose, by way of contradiction, that $\lambda_z(\mathcal{D}_n)a = a\mathcal{D}_n$. Then, since unitaries from \mathcal{S}_n normalize \mathcal{D}_n , for any $v \in \mathcal{S}_n$ we have $\lambda_z(\mathcal{D}_n)av = av\mathcal{D}_n$. Since $\lambda_z(\mathcal{D}_n) \subseteq \mathcal{F}_n$, this implies that $\lambda_z(\mathcal{D}_n)E(av) = E(av)\mathcal{D}_n$. Therefore, $E(av) = 0$ by lemma 3.4, and consequently $a = 0$ by lemma 3.6. \square

An immediate consequence of theorem 3.7 is the existence of two MASAs of the Cuntz algebra \mathcal{O}_n that are outer, but not inner, conjugate. In fact, theorem 3.7 implies the following stronger fact.

COROLLARY 3.8. *There exists an uncountable family of MASAs in \mathcal{O}_n , indexed by the cosets $\mathcal{U}(\mathcal{F}_n^1)/\mathcal{N}_{\mathcal{F}_n^1}(\mathcal{D}_n^1)$, such that each of them is outer conjugate to \mathcal{D}_n but no two of them are inner conjugate.*

To the best of our knowledge, corollary 3.8 exhibits the very first example of two MASAs in a simple purely infinite C^* -algebra that are outer, but not inner, conjugate¹.

It was shown in [8] that

$$\begin{aligned} & \{v \in \mathcal{U}(\mathcal{O}_n) \mid \lambda_v \in \text{Aut}(\mathcal{O}_n, \mathcal{D}_n)\} \\ &= \{dw \mid d \in \mathcal{U}(\mathcal{D}_n), w \in \mathcal{S}_n \text{ such that } \lambda_w \in \text{Aut}(\mathcal{O}_n)\}. \end{aligned}$$

On the other hand, the set $\{u\varphi(u^*) \mid u \in \mathcal{U}(\mathcal{O}_n)\}$ is dense in $\mathcal{U}(\mathcal{O}_n)$ by [24, theorem 3.6]. Furthermore, $\mathcal{U}_a(\mathcal{O}_n) := \{v \in \mathcal{U}(\mathcal{O}_n) \mid \lambda_v \in \text{Aut}(\mathcal{O}_n)\}$ is a dense G_δ -subset of $\mathcal{U}(\mathcal{O}_n)$ such that $\mathcal{U}(\mathcal{O}_n) \setminus \mathcal{U}_a(\mathcal{O}_n)$ is also dense [2, proposition 7]. In this context, we mention the following corollary.

¹The authors are grateful to Mikael Rørdam for his comments on this point.

COROLLARY 3.9. *The following inclusion is proper:*

$$\{udw\varphi(u^*) \mid u \in \mathcal{U}(\mathcal{O}_n), d \in \mathcal{U}(\mathcal{D}_n), w \in \mathcal{S}_n \text{ such that } \lambda_w \in \text{Aut}(\mathcal{O}_n)\} \subset \mathcal{U}_a(\mathcal{O}_n).$$

Proof. Suppose this is not that case. Then, for each $\alpha \in \text{Aut}(\mathcal{O}_n)$, there exist $u \in \mathcal{U}(\mathcal{O}_n)$, $d \in \mathcal{U}(\mathcal{D}_n)$ and $w \in \mathcal{S}_n$ such that $\alpha = \lambda_{udw\varphi(u^*)} = \text{Ad}(u)\lambda_{dw}$. Since dw normalizes \mathcal{D}_n , we have $\lambda_{dw}(\mathcal{D}_n) = \mathcal{D}_n$ [13, proposition 1.4]. But then $\alpha(\mathcal{D}_n) = \text{Ad}(u)(\mathcal{D}_n)$, contradicting theorem 3.7. \square

We close this section by posing the following question. Suppose that $z \in \mathcal{U}(\mathcal{F}_n)$ is such that $\lambda_z \in \text{Aut}(\mathcal{O}_n)$ and that there exists a $u \in \mathcal{U}(\mathcal{O}_n)$ such that $\lambda_z(\mathcal{D}_n) = \text{Ad}(u)(\mathcal{D}_n)$. Does this imply the existence of a $v \in \mathcal{U}(\mathcal{F}_n)$ such that $\lambda_z(\mathcal{D}_n) = \text{Ad}(v)(\mathcal{D}_n)$?

4. The outer automorphism group of \mathcal{O}_n

In this section, we collect a few observations about the structure of the outer automorphism group of \mathcal{O}_n . We denote by $\pi: \text{Aut}(\mathcal{O}_n) \rightarrow \text{Out}(\mathcal{O}_n)$ the canonical surjection.

PROPOSITION 4.1. *If $d \in \mathcal{U}(\mathcal{D}_n)$ and $u \in \mathcal{S}_n$, then $\lambda_d\lambda_u \in \text{Inn}(\mathcal{O}_n)$ if and only if there exist $v \in \mathcal{U}(\mathcal{D}_n)$ and $y \in \mathcal{S}_n$ such that $\lambda_d = \text{Ad}(v)$ and $\lambda_u = \text{Ad}(y)$.*

Proof. If $\lambda_d\lambda_u = \text{Ad}(w)$ for some $w \in \mathcal{U}(\mathcal{O}_n)$, then w normalizes \mathcal{D}_n , and thus $w = vy$, for some $v \in \mathcal{U}(\mathcal{D}_n)$ and $y \in \mathcal{S}_n$ [23, lemma 5.4]. Hence, $\text{Ad}(v^*)\lambda_d = \text{Ad}(y)\lambda_u^{-1}$ and, consequently, $\lambda_d = \text{Ad}(v)$ and $\lambda_u = \text{Ad}(y)$, since the intersection of $\lambda(\mathcal{U}(\mathcal{D}_n))$ and $\lambda(\mathcal{S}_n)^{-1}$ is trivial. \square

The semi-direct product decomposition in proposition 4.2 is a special case of [20, theorem 6.5] pertaining to a broader class of algebras. We include a short self-contained proof, different from Matsumoto’s argument.

Recall that an automorphism is *aperiodic* if its image in the outer automorphism group has infinite order. A Bogolubov automorphism λ_z of \mathcal{O}_n is aperiodic if and only if the corresponding unitary z has infinite order. Thus, in particular, a gauge automorphism γ_t is aperiodic if and only if t is not a root of unity.

PROPOSITION 4.2. *The subgroup $\pi(\text{Aut}(\mathcal{O}_n, \mathcal{D}_n))$ of $\text{Out}(\mathcal{O}_n)$ is not normal, and has the structure of a semi-direct product*

$$\pi(\text{Aut}(\mathcal{O}_n, \mathcal{D}_n)) = \pi(\lambda(\mathcal{U}(\mathcal{D}_n))) \rtimes \pi(\lambda(\mathcal{S}_n)^{-1}).$$

Proof. Since all aperiodic automorphisms of \mathcal{O}_n are outer conjugate (see [21, theorem 9] and [24, theorem 3.6]), a normal subgroup of $\text{Out}(\mathcal{O}_n)$ contains either none or all of them. Clearly, $\lambda(\mathcal{U}(\mathcal{F}_n^1))$ contains aperiodic automorphisms λ_z such that z does not normalize \mathcal{D}_n^1 . Thus, theorem 3.7 implies that $\pi(\text{Aut}(\mathcal{O}_n, \mathcal{D}_n))$ contains some, but not all, aperiodic automorphisms of \mathcal{O}_n . Consequently, it is not a normal subgroup of $\text{Out}(\mathcal{O}_n)$.

For the semi-direct product decomposition, it suffices to note that $\pi(\lambda(\mathcal{U}(\mathcal{D}_n)))$ and $\pi(\lambda(\mathcal{S}_n)^{-1})$ have trivial intersection. Indeed, suppose that $d \in \mathcal{U}(\mathcal{D}_n)$, $w \in \mathcal{S}_n$, $\lambda_w \in \text{Aut}(\mathcal{O}_n)$ and $u \in \mathcal{U}(\mathcal{O}_n)$ are such that $\lambda_d = \text{Ad}(u)\lambda_w$. Then, for each

$p \in \mathcal{D}_n$, we have $p = \lambda_d(p) = u\lambda_w(p)u^*$, and thus $u \in \mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n)$. Hence, there exist $d_1 \in \mathcal{U}(\mathcal{D}_n)$ and $w_1 \in \mathcal{S}_n$ such that $u = d_1w_1$ [23, lemma 5.4]. But then $\text{Ad}(d_1^*)\lambda_d = \text{Ad}(w_1)\lambda_w$, and equivalently $\lambda_{d_1^*d\varphi(d_1)} = \lambda_{w_1w\varphi(w_1^*)}$. This yields $d_1^*d\varphi(d_1) = 1 = w_1w\varphi(w_1^*)$, and thus both $\lambda_d = \text{Ad}(d_1)$ and $\lambda_w = \text{Ad}(w_1^*)$ are inner. \square

We have seen in proposition 4.2 that the subgroup $\pi(\text{Aut}(\mathcal{O}_n, \mathcal{D}_n))$ is not normal in $\text{Out}(\mathcal{O}_n)$. It follows from proposition 4.3 that the smallest normal subgroup of $\text{Out}(\mathcal{O}_n)$ containing $\pi(\text{Aut}(\mathcal{O}_n, \mathcal{D}_n))$ is quite large.

PROPOSITION 4.3. *If G is a normal subgroup of $\text{Out}(\mathcal{O}_n)$ containing at least one aperiodic element, then*

$$\pi(\mathcal{Z}(\text{Out}(\mathcal{O}_n)) \cup \lambda(\mathcal{U}(\mathcal{F}_n))^{-1} \cup \{\alpha \in \text{Aut}(\mathcal{O}_n) \mid \alpha \text{ aperiodic}\}) \subseteq G. \tag{4.1}$$

In particular, (4.1) holds with G , the commutator subgroup of $\text{Out}(\mathcal{O}_n)$.

Proof. Since all aperiodic automorphisms of \mathcal{O}_n are outer conjugate to one another (see [21, theorem 9] and [24, theorem 3.6]), group G contains classes of all of them. Now, let $\alpha \in \text{Aut}(\mathcal{O}_n)$ and suppose that there exists an aperiodic $\beta \in \text{Aut}(\mathcal{O}_n)$ such that $\alpha\beta$ is aperiodic. Then there exists a $\psi \in \text{Aut}(\mathcal{O}_n)$ such that $\pi(\alpha\beta) = \pi(\psi\beta\psi^{-1})$, and consequently $\pi(\alpha) = \pi(\alpha\beta\beta^{-1}) = \pi(\psi\beta\psi^{-1})\pi(\beta^{-1})$ belongs to G . Now, let $\alpha \in \text{Aut}(\mathcal{O}_n)$ be such that either $\pi(\alpha) \in \mathcal{Z}(\text{Out}(\mathcal{O}_n))$ or $\alpha \in \lambda(\mathcal{U}(\mathcal{F}_n))^{-1}$. Take an aperiodic gauge automorphism γ_t such that $\pi(\gamma_t^m) \neq \pi(\alpha^{-m})$ for all $m \in \mathbb{N}$. Since $\pi(\alpha)$ commutes with $\pi(\gamma_t)$, the product $\alpha\gamma_t$ is again aperiodic. This shows the first claim of the proposition. For the remaining one, simply note that if $\theta \in \text{Aut}(\mathcal{O}_n)$ is aperiodic, then θ^2 is outer conjugate to θ . Thus, $\pi(\theta^2) = \pi(\psi\theta\psi^{-1})$ for some $\psi \in \text{Aut}(\mathcal{O}_n)$, and hence $\pi(\theta) = \pi(\theta^2)\pi(\theta^{-1}) = \pi(\psi\theta\psi^{-1}\theta^{-1})$ belongs to the commutator subgroup $[\text{Out}(\mathcal{O}_n), \text{Out}(\mathcal{O}_n)]$. \square

REMARK 4.4. No class of non-trivial gauge automorphism of \mathcal{O}_n belongs to the centre of $\text{Out}(\mathcal{O}_n)$. Indeed, consider the unitary $u = S_{11}S_{121}^* + S_{121}S_{11}^* + P_{122} + P_2$ in \mathcal{S}_2 , discussed in [11, theorem 5.2]. Then λ_u is an automorphism of \mathcal{O}_2 such that $\lambda_u^2 = \text{id}$. For $1 \neq t \in U(1)$ we have $\lambda_u\gamma_t\lambda_u^{-1}\gamma_t^{-1} = \lambda_w$, with $w = t\lambda_u(P_{121}) + t\lambda_u(P_{11}) + P_{122} + P_2$. The automorphism λ_w of \mathcal{O}_2 is outer, for otherwise there exists a $d \in \mathcal{U}(\mathcal{D}_2)$ such that $w = d\varphi(d^*)$. But then w , viewed as a function on X_2 , would take value 1 at the infinite word $111\dots$ (fixed by the shift on X_2). However, this is not the case. A similar argument applies to all n , with a suitably modified $u \in \mathcal{S}_n$ (cf. [11, theorem 5.2]).

This is in stark contrast with what happens for the weak closure M of \mathcal{O}_n in the Gelfand–Naimark–Segal representation of the canonical Kubo–Martin–Schwinger state $\omega = \tau \circ E$, which is the approximately finite-dimensional factor of type $III_{1/n}$. Indeed, gauge automorphisms of \mathcal{O}_n extend to $M = \pi_\omega(\mathcal{O}_n)''$ [7], thereby providing the $(2\pi/\log(n)$ -periodic) modular automorphisms (with respect to the normal extension of ω), which then lie in the centre of $\text{Out}(M)$ by the Connes–Radon–Nikodým theorem [3, theorem 1.2.8]. It also follows from this argument that the automorphism λ_u above does not extend to M (i.e. it is not normal).

REMARK 4.5. Those classes under inner equivalence of all automorphisms of \mathcal{O}_2 known to us at the moment belong to the commutator subgroup of $\text{Out}(\mathcal{O}_2)$. For example, consider the unitary $u \in \mathcal{S}_2$ discussed in remark 4.4. Then, for an aperiodic

gauge automorphism γ_t , the automorphism $\lambda_u\gamma_t$ is aperiodic as well. Indeed, since λ_u has order 2, it suffices to show that all even powers of $\lambda_u\gamma_t$ are outer. But we have $\lambda_u\gamma_t\lambda_u\gamma_t = \lambda_v$, with $v = t^3\lambda_u(P_{121}) + t\lambda_u(P_{11}) + P_{122} + P_2$, and λ_v is aperiodic for the same reason as given in remark 4.4. Now, the same argument as in proposition 4.3 gives the conclusion that λ_u is a commutator modulo an inner automorphism of \mathcal{O}_2 .

We close this paper with a few simple, albeit potentially useful, observations about the outer automorphism group of \mathcal{O}_2 . First of all, it is worth noting that by combining some of the results from [21] and [8] one easily obtains the following.

PROPOSITION 4.6. *Every element of infinite order in $\text{Out}(\mathcal{O}_2)$ is a product of two involutions in the commutator subgroup. In particular, both $[\text{Out}(\mathcal{O}_2), \text{Out}(\mathcal{O}_2)]$ and $\text{Out}(\mathcal{O}_2)$ are generated by elements of finite order.*

Proof. We consider the inner equivalence classes of the automorphisms λ_A and λ_F defined in [8, § 5.3]. As shown therein, the commutator $\lambda_F\lambda_A\lambda_F\lambda_A^{-1}$ has infinite order in $\text{Out}(\mathcal{O}_2)$ and is a product of two involutions. The conclusion follows immediately, as every aperiodic automorphism is a conjugate of such commutator. \square

We believe that the same result holds true for $\text{Out}(\mathcal{O}_n)$ for all $n \geq 2$, with a suitable modification of the lengthy computations in [8]. As this falls outside the scope of the present work, we leave the task to the interested reader.

COROLLARY 4.7. *The normal subgroup of $\text{Out}(\mathcal{O}_2)$ generated by $\lambda(\mathcal{S}_2)^{-1}$ is generated by elements of finite order.*

Proof. There are two possible cases.

(1) If $\pi(\lambda(\mathcal{S}_2)^{-1})$ is not contained in the commutator subgroup of $\text{Out}(\mathcal{O}_2)$, then there is an element in $\pi(\lambda(\mathcal{S}_2)^{-1})$ but not in the commutator subgroup, say g , necessarily of finite order. Moreover, gh must have finite order for any h of infinite order in $\pi(\lambda(\mathcal{S}_2)^{-1})$. Accordingly, any h of infinite order in $\pi(\lambda(\mathcal{S}_2)^{-1})$ can be written as $g^{-1}(gh)$, a product of finite order elements.

(2) On the other hand, if $\pi(\lambda(\mathcal{S}_2)^{-1})$ is in $[\text{Out}(\mathcal{O}_2), \text{Out}(\mathcal{O}_2)]$, then any element of infinite order in $\pi(\lambda(\mathcal{S}_2)^{-1})$ is a product of two conjugates of involutions in $\pi(\lambda(\mathcal{P}_2)^{-1})$, by proposition 4.6. \square

To the best of our knowledge, the following result provides the first non-trivial structural result about the rather mysterious group $\lambda(\mathcal{S}_2)^{-1}$. Recall that a group G is called *almost simple* if there exists a non-abelian simple group H such that $H \subseteq G \subseteq \text{Aut}(H)$.

PROPOSITION 4.8. *The group $\lambda(\mathcal{S}_2)^{-1}|_{\mathcal{S}_2}$ is almost simple.*

Proof. Clearly, any automorphism of the form λ_w , with $w \in \mathcal{S}_2$, restricts to an automorphism of \mathcal{S}_2 . Also, \mathcal{S}_2 , being isomorphic to the Thompson group V , is simple [22, proposition 9.6] and, in particular, its centre is trivial. Thus, one has inclusions

$$\mathcal{S}_2 \simeq \text{Inn}(\mathcal{S}_2) \subseteq \lambda(\mathcal{S}_2)^{-1}|_{\mathcal{S}_2} \subseteq \text{Aut}(\mathcal{S}_2),$$

and the conclusion follows. \square

Of course, one might wonder whether the kernel of the restriction map $\lambda(\mathcal{S}_2)^{-1} \rightarrow \lambda(\mathcal{S}_2)^{-1}|_{\mathcal{S}_2}$ is trivial. The analogous statement for \mathcal{P}_n has been discussed in [9] (see the paragraph following proposition 3.6 therein).

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References

- 1 E. J. Beggs and P. Goldstein. Maximal abelian subalgebras of \mathcal{O}_n . *C. R. Acad. Sci. Paris Sér. I* **24** (2002), 26–32.
- 2 O. Bratteli and A. Kishimoto. Homogeneity of the pure state space of the Cuntz algebra. *J. Funct. Analysis* **171** (2000), 331–345.
- 3 A. Connes. Une classification des facteurs de type III. *Annales Scient. Éc. Norm. Sup.* **6** (1973), 133–252.
- 4 A. Connes. Outer conjugacy classes of automorphisms of factors. *Annales Scient. Éc. Norm. Sup.* **8** (1975), 383–419.
- 5 A. Connes. Periodic automorphisms of the hyperfinite factor of type II_1 . *Acta Sci. Math. (Szeged)* **39** (1977), 39–66.
- 6 R. Conti. Automorphisms of the UHF algebra that do not extend to the Cuntz algebra. *J. Austral. Math. Soc.* **89** (2010), 309–315.
- 7 R. Conti and C. Pinzari. Remarks on the index of endomorphisms of Cuntz algebras. *J. Funct. Analysis* **142** (1996), 369–405.
- 8 R. Conti and W. Szymański. Labeled trees and localized automorphisms of the Cuntz algebras. *Trans. Am. Math. Soc.* **363** (2011), 5847–5870.
- 9 R. Conti and W. Szymański. Automorphisms of the Cuntz algebras. In *Progress in operator algebras, noncommutative geometry, and their applications: proceedings 4th annual meeting of the European noncommutative geometry network, Bucharest, 2011*, pp. 1–15 (Bucharest: Theta, 2012).
- 10 R. Conti, J. H. Hong and W. Szymański. The restricted Weyl group of the Cuntz algebra and shift endomorphisms. *J. Reine Angew. Math.* **667** (2012), 177–191.
- 11 R. Conti, J. H. Hong and W. Szymański. The Weyl group of the Cuntz algebra. *Adv. Math.* **231** (2012), 3147–3161.
- 12 J. Cuntz. Simple C^* -algebras generated by isometries. *Commun. Math. Phys.* **57** (1977), 173–185.
- 13 J. Cuntz. Automorphisms of certain simple C^* -algebras. In *Quantum fields: algebras, processes* (ed. L. Streit), pp. 187–196 (Springer, 1980).
- 14 J. Cuntz. K -theory for certain C^* -algebras. *Annals Math. (2)* **113** (1981), 181–197.
- 15 M. Enomoto, H. Takehana and Y. Watatani. Automorphisms on Cuntz algebras. *Math. Japon.* **24** (1979), 231–234.
- 16 J. Glimm. On a certain class of operator algebras. *Trans. Am. Math. Soc.* **95** (1960), 318–340.
- 17 J. H. Hong, A. Skalski and W. Szymański. On invariant MASAs for endomorphisms of the Cuntz algebras. *Indiana Univ. Math. J.* **59** (2010), 1873–1892.
- 18 M. Izumi. Finite group actions on C^* -algebras with the Rohlin property. I. *Duke Math. J.* **122** (2004), 233–280.

- 19 A. Kishimoto. The Rohlin property for shifts on UHF algebras and automorphisms of Cuntz algebras. *J. Funct. Analysis* **140** (1996), 100–123.
- 20 K. Matsumoto. Orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras. *Pac. J. Math.* **246** (2010), 199–225.
- 21 H. Nakamura. Aperiodic automorphisms of nuclear purely infinite simple C^* -algebras. *Ergod. Theory Dynam. Syst.* **20** (2000), 1749–1765.
- 22 V. Nekrashevych. Cuntz–Pimsner algebras of group actions. *J. Operat. Theory* **52** (2004), 223–249.
- 23 S. C. Power. Homology for operator algebras. III. Partial isometry homotopy and triangular algebras. *New York J. Math.* **4** (1998), 35–56.
- 24 M. Rørdam. Classification of inductive limits of Cuntz algebras. *J. Reine Angew. Math.* **440** (1993), 175–200.
- 25 M. Sabok. Completeness of the isomorphism problem for separable C^* -algebras. Preprint, 2013. (Available at <http://arxiv.org/abs/1306.1049>.)
- 26 A. Skalski. Noncommutative topological entropy of endomorphisms of Cuntz algebras. II. *Publ. RIMS Kyoto* **47** (2011), 887–896.
- 27 A. Skalski and J. Zacharias. Noncommutative topological entropy of endomorphisms of Cuntz algebras. *Lett. Math. Phys.* **86** (2008), 115–134.

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