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Current Density Equation in Turbulent Magnetized Plasmas

R. Gatto[*,1] and I. Chavdarovski[2]

[1]*Department of Mathematics, Natural Science and Computer Science, John Cabot University, 00165 Rome, Italy*

[2]*National Fusion Research Institute, Gwahangno 113, Yuseong-gu, Daejeon 305-333, Korea*

Abstract: A turbulent extension of Ohm's law, derived from the self-consistent action-angle transport theory, is presented. The equation describes the steady-state profile of the current density in axisymmetric magnetized plasmas in the presence of magnetic turbulence. The hyper-resistive, helicity-conserving contribution, usually derived in the framework of magneto-hydro-dynamics, is recovered, and the hyper-resistivity is defined. Additionally, the generalized Ohm's law contains an anomalous resistivity term, and a term proportional to the current density derivative. For given thermodynamic profiles, the numerical solution of the equation shows that turbulent contributions, besides regularizing the current density profile in the central region, lead to an increase of the total plasma current. This "turbulent bootstrap" effect provides a possible explanation to discrepancies recently observed between experimental current profiles and neoclassical predictions.

Keywords: Controlled thermonuclear fusion energy, tokamak, action-angle transport theory, turbulent Ohm's law, current density profile.

1. INTRODUCTION

Large part of the effort in the quest to controlled thermonuclear fusion energy focuses on the tokamak concept, a toroidal chamber in which a high temperature plasma is confined by a complex magnetic field topology. For stability reason, an important contribution to the latter is created by a toroidal plasma current which is induced by slowly increasing a current through an electromagnetic winding linked with the plasma torus. This is inherently a pulsed process, and a fusion reactor based on the original tokamak concept could only operate for short periods. An important advance in the path toward a tokamak reactor would be to achieve a steady-state scenario in which the toroidal plasma current is generated non-inductively. Unfortunately, all external current generation means envisioned so far, such as radio-frequency-wave or neutral beams injection, besides representing additional expensive components, are characterized by small efficiencies. A possible way out is to conceive new plasma regimes that combine favorable confinement properties with a large fraction of the so-called "bootstrap" current. The latter is a current amplification mechanism tied to intrinsic plasma diffusion in toroidal geometries [1], which therefore requires no external power. At best, a steady-state tokamak reactor would sustain all of its current *via* the bootstrap effect. An obstacle to this scenario is related to the fact that this current drive mechanism is ineffective around the plasma center where the pressure gradient vanishes [2]. The growing attention posed by the fusion community to the effects of turbulence in explaining anomalous transport, however, led in the early 90's to the realization that turbulence, always present at some level, actually helps to partly overcome this intrinsic limitation by diffusing the bootstrap current toward the center. In particular, since a tokamak with very large bootstrap current is inherently unstable to tearing modes, poloidal flux is generated spontaneously near the axis by the dynamo effect induced by this instability. This realization has stimulated various theoretical investigations in which the idea of a "fully bootstrapped tokamak" was put forward [3, 4], and experimental campaigns aimed at realizing, *via* a careful optimization of the plasma profiles, steady-state operation with a large bootstrap contribution [5]. The first results of this research effort are encouraging, in that several machines have now identified, and routinely realize, fully non-inductive, high-confinement steady-state regimes in which a minimal part of the current density depends on externally injected power [6, 7].

In this context, the ability to correctly predict the evolution of the current density profile - *including the effect of plasma turbulence* - becomes essential. To this end, theoretical studies aiming at deriving a turbulent extension of Ohm's law are advisable. Such equation, which governs the radial diffusion of the current density, must include, together with the neoclassical terms describing the effects of the finite orbit widths of trapped electrons (the bootstrap term), anomalous terms that model the effects induced by turbulence. Besides from being a key aid in investigating the feasibility of steady-state tokamak discharges with a large fraction of bootstrap current, in which neoclassical current amplification and turbulent current diffusion play synergistic roles, such an equation is relevant to study reversed-field pinches, in which turbulence is at the heart of the reversal of the toroidal field, as well as various phenomena in space and astrophysical plasmas, like solar wind, magnetosphere and

*Address correspondence to this author at the Department of Mathematics, Natural Science and Computer Science, John Cabot University, 00165 Rome, Italy; Tel: (+39) 066819121; Fax: (+39) 066832088;
E-mail: rgatto@johncabot.edu

accretion disks, in which plasma turbulence plays an important role.

At present, there is still an ongoing debate on which are the relevant turbulent terms that can explain the anomalous current diffusion observed in experiments, their detailed structure, and their physical meaning. It has been proposed that turbulence leads to three additional terms in the Ohm's law: (i) a viscous-like term which induces current diffusion [8], (ii) an anomalous resistive term which adds to the neoclassical resistivity [9], and (iii) a source-like term which generates current density to the expenses of the energy in the turbulence [10, 11]. Each one of these terms has been derived both with fluid and kinetic approaches, so that despite their structural similarities they often describe different physical phenomena. The main effect of the turbulent electron viscosity, or hyper-resistive, term is that of redistributing the current density. Since this is accomplished by the generation of poloidal field, this term is formally equivalent to the α-term in the dynamo theory [12, 13]. In a plasma with a stochastic magnetic field, whenever the collisional electron scattering length is long compared to the correlation length of a magnetic field line, as it is in weakly collisional plasmas, the electrons streaming along the stochastic magnetic field lines will execute a random-walk in the direction perpendicular to the equilibrium magnetic field. As shown by Rechester and Rosenbluth [14] using a *kinetic approach*, the ensuing particle transport is regulated by a particle diffusivity proportional to the field-aligned electron velocity. It is reasonable, however, that magnetic field braiding will induce an equally strong transport of electron parallel momentum across magnetic surfaces [15]. When electrons are scattered (by Coulomb collisions or particle-fluctuation resonances) over a sizable fraction of the minor radius in one mean-free time, they carry field-aligned momentum with them. This kinetic effect has been discussed by Jacobson and Moses [16], and shown to induce a relaxation of the plasma toward a Woltjer-Taylor state [17]. An alternative vision of anomalous current diffusion has been obtained following a *single-fluid, magneto-hydro-dynamic (MHD) approach*, i.e., by averaging the Ohm's law for a force-free mean field over small scale, small amplitude MHD fluctuations (mean-field theory). This approach has lead to the identification of the hyper-resistive term with the dynamo electric field, i.e., the perturbed $\langle V\times B\rangle_{\parallel}$ term [13, 18, 19]. Similar results have been found from reduced MHD models as well [9]. In the MHD approach, the essential element is the presence of fluctuations, and magnetic stochasticity is not required. For a force-free plasma, the hyper-resistive effect can be written as an added term in the Ohm's law having the form $B_0^{-2}B_0\nabla\cdot[\eta_h\nabla(j_{\parallel}/B_0)]$, and has the property of incorporating in a natural way the helicity conservation properties of the plasma whenever the helicity flux $\propto\eta_{\mathrm{h}}\nabla(j_{\parallel}/B_0)$ is zero at the outer plasma surface [19]. It has been shown that this functional form encompasses both the fluid and the kinetic framework [18]. What differs is the realization of the coefficient η_h, i.e., the underlying physical mechanism responsible for the magnetic fluctuations. The remaining two turbulent terms usually thought to affect the current density evolution in turbulent plasmas are the anomalous resistivity, and the source term. The first is a natural output of most turbulent theories, already well documented in the literature [9]. Since it represents a dissipative effect, it is formally equivalent to the β-term in the dynamo theory [12]. Finally, the source term could be the result of different physical processes, the end result of which is the local generation of current density due to the resonant transfer of linear momentum to the electrons.

We propose a turbulent extension of Ohm's law derived in the framework of the action-angle [20] self-consistent [21] transport theory. Self-consistency means considering a collision operator that includes both diffusion and drag in action-space, as opposed to the quasi-linear approach which retains only the diffusion part. Our result applies to collisionless plasmas where irreversibility is caused by the resonance between particle and stochastic magnetic field fluctuations, and thus it pertains to the class of kinetic dynamos. The turbulent Ohm's law is built upon results derived in a series of papers that present the momentum flux and source due to a background sea of magnetic turbulence [22, 23]. Here we do not report the details of the derivation, referring the interested reader to the literature; instead, we start from the momentum flux and source, and manipulate them so to obtain the Ohm's law. Our procedure leads to a version of the turbulent electric field that contains an hyper-resistive term, an anomalous resistivity term, and an additional term, referred to as the "cross"-resistivity term, that in the current diffusion equation is proportional to the derivative of the current density. The detailed structure of the corresponding three turbulent transport coefficients is presented, showing their dependence on the strength of the magnetic turbulence, and on the equilibrium plasma profiles. Both the anomalous and cross coefficients contain a piece that originates from the electron momentum source, and that, for certain plasma profiles, leads to current amplification. The effect of these three turbulent terms in shaping the current profile is put in evidence by a simple numerical study.

This paper is organized as follows. In Sec. 2 we present the main features of the action-angle approach to tokamak transport. The procedure to derive the electron momentum transport equation from the action-angle collision operator is outlined in Sec. 3. This equation contains a flux term and a source term, both proportional to transport coefficients that depend on the fluctuation spectrum of the underlying turbulence. An exact specification of the latter is a difficult task that is not attempted here. Instead, as described in Sec. 3.1, we adopt an ansatz spectrum that describes the special case of magnetic micro-turbulence with $\omega<\Omega_{\mathrm{g}}$ and $k_{\perp}\gg k_{\parallel}$. By incorporating this ansatz in the momentum equation we arrive to the final expressions for the momentum flux and source, which are presented in Sec. 3.2. The turbulent electric field, derived from the electron momentum equation, is presented in Sec. 4, while in Sec. 5 we introduce the turbulent electric field in the neoclassical Ohm's law, and derive the diffusion equation for the parallel current density. A power balance describing the energetics associated to the current density profile is also formulated. Sec. 6 presents the results of the numerical solution of the current equation which clarify the role of each turbulent term in shaping the current profile. The summary and the conclusions are presented in Sec. 7.

2. ACTION-ANGLE TRANSPORT THEORY IN TOROIDAL GEOMETRY

For completeness, in the present and in the next section we give a brief overview of the transport theory in action-space, adapted to describe transport phenomena in axisymmetric, magnetically-confined plasmas (as in tokamak machines). For a more detailed presentation of the theory, and for examples of its applications, we refer to Refs. [20-32]. The results that will be used from Sec. 4 on are Eqs. (20) and (21).

We consider an axisymmetric toroidal device which confines a plasma by a slowly varying magnetic field. In this context, the collision operators can be expressed in a particularly simple and general form [20, 24, 28] by (i) introducing action-angle variables to describe the particle motion, and (ii) expressing the fields as a sum over plasma normal modes. While the use of action-angle variables leads to a natural inclusion in the operator of toroidal effects (as particle trapping, drifts, etc.), the normal mode expansion permits, through a formal solution of Maxwell's equations, the inclusion in the transport coefficients of fluctuation spectra that are valid in fully inhomogeneous geometries. In the present section we summarize (following closely the exposition of Ref. [22]) the main ingredients of the action-angle approach. In particular, in Sec. 2.1 we define the actions and the mode decomposition, in Sec. 2.2 we present the self-consistent version of the operator (referred to as the "generalized Balescu-Lenard" (gBL) operator in the literature), and in Sec. 2.3 we specialize to the case of large aspect-ratio tokamaks with strong toroidal field.

2.1. Action-Angle Variables and Normal Mode Decomposition

To specify the action-angle variables we adopt, together with the Cartesian coordinate set x, the flux coordinates $\xi = (\alpha, \theta, \zeta)$, where α is a minor radius-like variable constant on a flux surface, and θ and ζ are, respectively, the poloidal and the toroidal angle, both with periodicity 2π. To be specific, we will take α to be the toroidal flux function $\alpha \equiv \Psi_t / (2\pi)$, where Ψ_t is the toroidal flux enclosed by the flux surface. [From now on, the subscript $t(p)$ stands for toroidal (poloidal)]. To express some of the results in a more perspicuous way, we also introduce an "equivalent" cylindrical radial coordinate r, defined by the relation $\Psi_t(\alpha) = 2\pi\alpha = \int dS\, \mathbf{B}\cdot\hat{\zeta} \equiv \pi r^2 B_{0,t}$ (an over-hat indicates unit vectors), where the reference field $B_{0,t}$ is chosen to be the toroidal magnetic field on the magnetic axis. Thus, $r = r(\alpha) = [2\alpha / B_{0,t}]^{1/2}$. The actions parametrizing the phase-space particle point are taken to be [20, 26] $\boldsymbol{J} = (J_g, J_b, J_\zeta)$, $J_g = \mu_0 M^2 c / q$ being the gyro-action [where $\mu_0 \equiv v_\perp^2 / (2B_0)$ is the (lowest order) magnetic moment, B_0 the total equilibrium magnetic field, $v_\perp$ the component of the particle velocity perpendicular to the magnetic field, M and q the particle mass and charge, respectively, and c the speed of light], J_ζ the toroidal angular momentum,

$$J_\zeta = MR^2\dot{\zeta} - \frac{q}{c}\psi_p(\alpha) \tag{1}$$

where $\psi_p \equiv \Psi_p / 2\pi$ is the poloidal flux function and R the major radius, and the longitudinal invariant or bounce action J_b, equal to the toroidal flux enclosed by a drift orbit. Denoting the projection on the poloidal cross-section of the guiding center trajectory with $\alpha = \tilde{\alpha}(\theta)$, the bounce action can be written as $J_b = \oint (d\theta / 2\pi)(q/c)\tilde{\alpha}(\theta; H_0, J_g, J_\zeta)$, where the triplet (H_0, J_g, J_ζ) singles out one particular particle trajectory. The conjugate angles $\Theta = (\Theta_g, \Theta_b, \Theta_\zeta)$, which are cyclic coordinates (i.e., $\partial H_0 / \partial\Theta = 0$ where H_0 is the unperturbed Hamiltonian), represent respectively the orbit-averaged gyro-phase, the phase of the bounce motion and the bounce-averaged toroidal angle. As usual in Hamiltonian theory, they are obtained by derivatives of the appropriate generating function with respect to the conjugate new action. Because is constant in the absence of perturbations, the unperturbed motion is trivial: Θ develops linearly in time. The corresponding set of bounce-averaged gyration frequency, bounce frequency, and bounce-averaged toroidal (or banana) drift are defined in terms of the unperturbed quasi-static (i.e., slowly varying on the transit time scale) Hamiltonian H_0 which is a function of $\boldsymbol{J}$ only: $\Omega \equiv \Theta = \partial H_0(\boldsymbol{J}) / \partial \boldsymbol{J}$.

To linearize, the electrostatic and the magnetic potentials are decomposed into unperturbed and perturbed parts, $\Phi = \Phi_0 + \Phi_1$, $A = A_0 + A_1$. Because $\boldsymbol{J}$ is a constant of the unperturbed motion and Θ is a cyclic coordinate, we write $H(\boldsymbol{J},\Theta) = H_0(\boldsymbol{J}) + h(\boldsymbol{J},\Theta) + \cdots$. The unperturbed Hamiltonian, $H_0 = (1/2M)[p - (q/c)A_0]^2 + q\Phi_0 = Mv^2/2 + q\Phi_0$ (where p is the canonical angular momentum $p = Mv + qA_0/c$, and v, v^2 is the particle velocity) is allowed to change, due to variation of the background fields, but only very slowly (quasi-statically) on a transit time scale. The first order perturbing Hamiltonian $h = q\Phi_1 - (q/c)v \cdot A_1$ is expanded in a Fourier series in the ignorable and periodic angle coordinates, $h(\boldsymbol{J},\Theta;t) = \sum_l h(\boldsymbol{J},l;t)\exp(+il\cdot\Theta)$. The triplet of integers $\ell = (\ell_g, \ell_b, \ell_\zeta)$ singles out each one of the harmonics of the particle perturbing Hamiltonian, or, analogously, of the orbital motion. Note that in the general case these integers differ from the usual poloidal (m_a) and toroidal (n_a) mode numbers entering the Fourier decomposition of a field with respect of θ and ζ, and indicating its spatial dependence. The space-time Fourier transform of the perturbing Hamiltonian, $h(\boldsymbol{J},\ell;\omega)=\oint[d\Theta/(2\pi)^3]\int(dt/2\pi)\exp(-i\ell\cdot\Theta+i\omega t)\,h(\boldsymbol{J},\Theta;t)$, describes the energy exchange between waves and particles, and is therefore a crucial quantity of the theory.

2.2. Self-Consistent Action-Angle Collision Operator

The gBL transport theory [27] extends the validity of Kaufman's original quasilinear action-angle theory [20] in two ways, i.e., (i) is valid for steady-state fluctuation spectra, and (ii) is applicable, although through a simplifying approximation (the "pseudo-thermal ansatz" of Ref. [21]), to realistic plasma turbulence. The latter point will be discussed in Sec. 3.1. The former extension is achieved by the inclusion in the collision operator of the polarization drag felt by the scattered particle. This leads to a transfer of momentum and energy to the medium surrounding the particle, and therefore introduces self-consistency, since the fluctuations are produced by the surrounding medium itself (i.e., the remaining particles in the plasma). The distribution function of the scattered species 1 evolves according to [28].

$$\partial_t f_0(\boldsymbol{J}_1;t)=\partial\boldsymbol{J}_1\cdot\left[D(\boldsymbol{J}_1;t)\cdot\partial\boldsymbol{J}_1 f_0(\boldsymbol{J}_1;t)-F(\boldsymbol{J}_1;t)f_0(\boldsymbol{J}_1;t)\right],\quad(2)$$

where F is the friction vector which considers the polarization field. Indicating with 2 the scattering species and with a each of the normal modes constituting the fluctuation spectrum, we write [28] $D(\boldsymbol{J}_1)=\sum_2\sum_{\ell_1,\ell_2}\ell_1\,\ell_1\,D_0(\boldsymbol{J}_1,\ell_1,\ell_2)$ and $F(\boldsymbol{J}_1)=\sum_2\sum_{\ell_1,\ell_2}\ell_1\,\ell_2\cdot F_0(\boldsymbol{J}_1,\ell_1,\ell_2)$, where

$$\begin{bmatrix} D_0(\boldsymbol{J}_1;\ell_1,\ell_2) \\ F_0(\boldsymbol{J}_1;\ell_1,\ell_2) \end{bmatrix}=\left(\frac{2\pi}{M_2}\right)^3\int d\boldsymbol{J}_2\,Q(\ell_1,\boldsymbol{J}_1;\ell_2,\boldsymbol{J}_2)\begin{bmatrix} f_0(\boldsymbol{J}_2) \\ \partial\boldsymbol{J}_2 f_0(\boldsymbol{J}_2) \end{bmatrix},\quad(3)$$

$$Q(\ell_1,\boldsymbol{J}_1,\ell_2,\boldsymbol{J}_2)\equiv 2\pi\,\delta(\ell_1\cdot\Omega_1-\ell_2\cdot\Omega_2)\sum_a\left|C_a(\ell_1,\boldsymbol{J}_1,\ell_2,\boldsymbol{J}_2,\omega=\ell_2\cdot\Omega_2)\right|^2 \quad(4)$$

and $C_a(\ell_1,\boldsymbol{J}_1,\ell_2,\boldsymbol{J}_2,\omega)\equiv[4\pi\,h_a(\ell_1,\boldsymbol{J}_1,\omega)\,h_a^*(\ell_2,\boldsymbol{J}_2,\omega)]/N_a\Delta_a(\omega)$. In the latter expression, $\Delta_a(\omega)$ is the eigenvalue of the Maxwell operator, and N_a is a normalization factor. Eq. (2) describes the slow (compared to the characteristic particle frequencies) evolution of $f_0(\boldsymbol{J}_1;t)$ as a result of a random walk in action-space, induced by the normal modes a generated by species 2. The coefficient C_a, the "coupling coefficient", measures the effectiveness of mode a in coupling particles 1 and 2. The Hamiltonian $h_a=q\Phi_1^a-(q/c)v\cdot A_1^a$ to be used in C_a must be calculated using the expression of the particle velocity valid in the magnetic geometry of interest, toroidal in our case, and evaluating the fields Φ_1 and A_1 along the particle orbit.

It can be shown that the gBL operator is characterized by the following three properties [21, 28]: (i) interaction between particles of the same species do not produce any net particle transport; (ii) the particle fluxes of the two species are equal; (iii) the transport is independent of the radial electrostatic potential. Note that the presence of the friction vector F is essential for the operator to possess these properties. The self-consistent approach has already been employed to study various aspects of particle and energy transport driven by thermodynamic forces [21, 27].

2.3. Specialization to Large Aspect-Ratio Tokamaks

Following tokamak theory, we adopt the large aspect-ratio and the small gyro-radius orderings, $\varepsilon\equiv|(R-R_0)/R_0|\sim|r(\alpha)/R_0|,\delta\equiv\rho/L\ll 1$, where $\rho\equiv v_{th}/\Omega_g$ is the gyro-radius, L is a length characterizing the variation of the equilibrium quantities, $v_{th}=(2T/M)^{1/2}$ is the thermal velocity (T being the temperature in energy units), and R_0 is the major radius. In these orderings, the safety factor is well approximated by its cylindrical version, $q_{saf}\simeq rB_{0,t}/(R_0B_p)$. Since the toroidal field is predominant ($B_p/B_t\approx\varepsilon$), we can approximate the modulus of the unperturbed magnetic field as

$$B\approx B_{0,t}\left[1-\frac{r(\alpha)}{R_0}\cos\theta\right]=B_{0,t}\left[1-\frac{r(\alpha)}{R_0}\right]+B_{0,t}\frac{r(\alpha)}{R_0}2\sin^2\frac{\theta}{2}.\quad(5)$$

To the lowest order in the gyro-radius, particles in a large aspect-ratio tokamak simply stream along the field lines with a parallel velocity that can be expressed in terms of the Hamiltonian: $v_\parallel\equiv\sigma\,u=[(2/M)(H_0-q\Phi_0-Mv_\perp^2/2)]^{1/2}$, where $\sigma=\pm1$ (indicating the two possible direction of motion along the field lines), and u is the parallel flow speed. In general, because of the twisting of the magnetic field lines, the toroidal velocity differs from the parallel velocity. In view of the tokamak ordering $B_0\simeq B_{0,t}$, however, we approximately set $v_t\approx v_\parallel$, and from Eq. (1) we obtain

$$J_\zeta=MRv_t-\frac{q}{c}\psi_p(\alpha)\approx MR\sigma u-\frac{q}{c}\psi_p(\alpha).\quad(6)$$

Using expression (5) for the unperturbed magnetic field, we can recast the parallel flow speed as $u(\theta;\boldsymbol{J})=u_0(\mu_0)[\kappa^2(\boldsymbol{J})-\sin^2(\theta/2)]^{1/2}$, where we have defined the quantities $u_0(\mu_0)\equiv 2\left[\mu_0B_0\varepsilon\right]^{1/2}$ and $\kappa^2(\boldsymbol{J})\equiv\{H_0(\boldsymbol{J})-q\Phi_0[\alpha(\boldsymbol{J})]-\mu_0MB_0(1-\varepsilon)\}/(2\mu_0MB_0\varepsilon)$, and where any θ dependence of Φ_0 is neglected. The trapped region is identified by $\kappa=[0,1]$, while the untrapped region by $\kappa=(1,+\infty]$.

In the expansion of the radial coordinate, $\alpha=\alpha_0+\alpha_1+\cdots$, the lowest order contribution α_0 represents the toroidal flux enclosed by the magnetic surface around which the particle motion evolves, and α_1 the excursion from the surface due to drifts. We can therefore Taylor expand the unperturbed poloidal flux function as $\psi_p(\alpha)=\psi_p(\alpha_0)+(\partial\psi_p/\partial\alpha)_{\alpha_0}\alpha_1=\psi_p(\alpha_0)+[1/q_{saf}(\alpha_0)]\alpha_1$ (where q_{saf} is the safety factor), obtaining from Eq. (6): $J_\zeta+(q/c)\psi_p(\alpha_0)+(q/c)[\alpha_1/q_{saf}(\alpha_0)]=M(R_0+\varepsilon\cos\theta)v_\parallel$. At zero and first order we have

$$J_\zeta+\frac{q}{c}\psi_p(\alpha_0)=0\,,\ \text{and}\quad \frac{q}{c}\frac{\alpha_1}{q_{saf}(\alpha_0)}=MR_0v_\parallel\,. \tag{7}$$

α_0 depends only on J_ζ, $\alpha=\alpha_0(J_\zeta)$, while α_1 depends on both $\boldsymbol{J}$ and θ because here $u=u(\theta;)$ does so. Solving the second relation in Eq. (7) we find for the α_1 correction: $\alpha_1=[cq_{saf}(\alpha_0)/q]MR_0v_\parallel(\theta;\boldsymbol{J})$.

3. MOMENTUM TRANSPORT EQUATION

In this section we continue the overview of the action-angle transport theory by presenting the general radial transport equation, under a drifting Maxwellian ansatz for the lowest order distribution function. Let $\chi(x,v;t)$ be some quantity (v = particle velocity) whose mean with respect to the distribution function (suppressing species label for the moment),

$$\overline{\chi}(x;t)\equiv\int dv\,\chi(x,v;t)\,f_0(x,v;t)\,, \tag{8}$$

is of physical interest. In particular, the moments $\chi=1$, $Mv_\parallel$ and $Mv^2/2$ (where $v_\parallel,v$ are the parallel and total particle speed, respectively) will lead to the density, parallel momentum and energy transport laws, respectively. We define the flux surface average of $\overline{\chi}$ as its normalized volume average restricted to a selected flux surface $\bar{\alpha}$,

$$\left\langle\overline{\chi}\right\rangle_{\bar{\alpha}}(\bar{\alpha},t)\equiv\frac{1}{V'}\int dx\,\delta[\alpha(x,t)-\bar{\alpha}]\,\overline{\chi}(x;t)\,, \tag{9}$$

where $V'\equiv dV(\bar{\alpha})/d\bar{\alpha}$, with $V(\bar{\alpha})$ the volume inside the flux surface $\bar{\alpha}$.

The derivation of the transport law begins by taking the time derivative of (9), after having made use of definition (8) and of the approximation $\partial V'/\partial t\simeq 0$ (stationary toroidal flux surfaces):

$$\partial\left\langle\overline{\chi}\right\rangle_{\bar{\alpha}}/\partial t=(1/V')\int dx\int dv\,(\partial/\partial t)$$
$$\left[\delta(\alpha-\bar{\alpha})\,f_0(x,v,t)\,\chi(x,v,t)\right],$$

or, in terms of action-angle variables ($M^3\int dx\,dv=\int d\ \ d\Theta$),

$$\frac{\partial}{\partial t}\left\langle\overline{\chi}\right\rangle_{\bar{\alpha}}=\frac{1}{V'M^3}\int d\boldsymbol{J}\int d\Theta\,\frac{\partial}{\partial t}\left\{\chi(\boldsymbol{J},\Theta,t)\,\delta[\alpha(\boldsymbol{J},\Theta,t)-\bar{\alpha}]\,f_0(\boldsymbol{J},\Theta,t)\right\}. \tag{10}$$

The time dependence of α arises from the quasi-static variation of the equilibrium. As an ansatz to accomplish closure, we assume a displaced, locally Maxwellian distribution function, so to be able to evaluate transport fluxes and sources including the presence of a toroidal current. Keeping only the first order term in $|V_\parallel/v_{th}|\ll 1$, we have $f_0(\boldsymbol{J},t)=f_M(\boldsymbol{J},t)\{1+V_\parallel[\alpha(\boldsymbol{J},t)]\,P[\alpha(\boldsymbol{J},t)]/T[\alpha(\boldsymbol{J},t)]\}$ where $V_\parallel$ is the parallel drift (flow) speed, $P\equiv Mv_\parallel$ is the parallel particle momentum, and

$$f_M(\boldsymbol{J},t)=N(\alpha)\,M^{3/2}/[\pi^{3/2}\,2^{3/2}\,T(\alpha)^{3/2}]$$
$$\times\exp\{-[H_0-q\Phi_0(\alpha)]/T(\alpha)\}.$$

$K_0\equiv H_0-q\Phi_0=Mv^2/2$ is the (unperturbed) kinetic energy when a particle is at α_0. Note that in general $\alpha=\alpha(\boldsymbol{J},\Theta;t)$; here however, $f_0(\boldsymbol{J})$ is the Θ-averaged lowest-order solution, so that $\alpha=\alpha(\boldsymbol{J};t)$. Since for far-untrapped particles the quantity α_1 is small, we evaluate all the quantities (like N, T, etc.) at $\alpha=\alpha_0(J_\zeta)$. Under the Maxwellian ansatz, the derivation of the transport law is straightforward, and it has been detailed in Ref. [23]. We present here only the final result. In terms of the factors

$$X^\chi(\boldsymbol{J},t,\ell;\bar{\alpha})\equiv\int\frac{d\Theta}{(2\pi)^3}\,\chi(\boldsymbol{J},\Theta,t)\,\ell\cdot\frac{\partial\alpha(\boldsymbol{J},\Theta,t)}{\partial\boldsymbol{J}}\,\delta[\alpha(\boldsymbol{J},\Theta,t)-\bar{\alpha}] \tag{11}$$

$$Y^\chi(\boldsymbol{J},t,\ell;\bar{\alpha})\equiv\int\frac{d\Theta}{(2\pi)^3}\,\ell\cdot\frac{\partial\chi(\boldsymbol{J},\Theta,t)}{\partial\boldsymbol{J}}\,\delta[\alpha(\boldsymbol{J},\Theta,t)-\bar{\alpha}]\,, \tag{12}$$

$$g\equiv\ell\cdot\nabla\boldsymbol{J}\alpha\,,\qquad G\equiv\ell\cdot\nabla\boldsymbol{J}P\,,\qquad K_0\equiv H_0-q\Phi_0=\frac{Mv^2}{2}\,, \tag{13}$$

as well as the thermodynamic forces (a prime indicating a derivative with respect to α)

$$A_N=\frac{N'}{N}+\frac{q}{T}\Phi_{0'}-\frac{3}{2}\frac{T'}{T}\,,\quad A_T=\frac{T'}{T}\,,\quad A_V=\frac{V_\parallel'}{V_\parallel}-\frac{T'}{T}\,, \tag{14}$$

the transport equation reads

$$\begin{aligned}&\frac{\partial}{\partial t}\left\langle\overline{\chi}_1\right\rangle_{\bar{\alpha}}-\left\langle\overline{\frac{\partial\chi_1}{\partial t}}\right\rangle_{\bar{\alpha}}+\frac{1}{V'}\frac{\partial}{\partial\bar{\alpha}}V'\left\langle\overline{\chi_1\frac{\partial\alpha(t)}{\partial t}}\right\rangle_{\bar{\alpha}}\\&+\frac{1}{V'}\frac{\partial}{\partial\bar{\alpha}}V'\Gamma_1(\bar{\alpha})=U_1(\bar{\alpha})\end{aligned} \tag{15}$$

where the flux Γ_1 and the source U_1 are given by

$$\begin{aligned}&\begin{bmatrix}\Gamma_1(\bar{\alpha})\\ U_1(\bar{\alpha})\end{bmatrix}=\sum_2\sum_{\ell_1,\ell_2}\frac{1}{V'}\left(\frac{2\pi}{M_1}\right)^2\int d\boldsymbol{J}_1\int d^6z_2\,Q(\ell_1,\boldsymbol{J}_1;\ell_2,\boldsymbol{J}_2)\\&f_M(\boldsymbol{J}_1)f_M(\boldsymbol{J}_2)\begin{bmatrix}X^\chi(\boldsymbol{J}_1,\ell_1;\bar{\alpha},t)\\ Y^\chi(\boldsymbol{J}_1,\ell_1;\bar{\alpha},t)\end{bmatrix}A(\boldsymbol{J}_1,\boldsymbol{J}_2,\ell_1,\ell_2)\,,.\end{aligned} \tag{16}$$

where to shorten the notation we have defined $z\equiv(\boldsymbol{J},\Theta)$ and

$$\begin{aligned}A(\boldsymbol{J}_1,\boldsymbol{J}_2,\ell_1,\ell_2)\equiv&\left(1+\frac{V_{\parallel,1}P_1}{T_1}\right)\left(1+\frac{V_{\parallel,2}P_2}{T_2}\right)\left(\frac{\ell_1\cdot\Omega_1}{T_1}-\frac{\ell_2\cdot\Omega_2}{T_2}\right)\\&-\left[\left(1+\frac{V_{\parallel,2}P_2}{T_2}\right)G_1\frac{V_{\parallel,1}}{T_1}-\left(1+\frac{V_{\parallel,1}P_1}{T_1}\right)G_2\frac{V_{\parallel,2}}{T_2}\right]\end{aligned}$$

$$-\left(1+\frac{V_{\parallel,1}P_1}{T_1}\right)\left(1+\frac{V_{\parallel,2}P_2}{T_2}\right)\left(g_1\,A_{N,1}-g_2\,A_{N,2}\right)$$

$$-\left[\frac{V_{\parallel,1}P_1}{T_1}\left(1+\frac{V_{\parallel,2}P_2}{T_2}\right)g_1\,A_{V,1}-\frac{V_{\parallel,2}P_2}{T_2}\left(1+\frac{V_{\parallel,1}P_1}{T_1}\right)g_2\,A_{V,2}\right]$$

$$-\left(1+\frac{V_{\parallel,1}P_1}{T_1}\right)\left(1+\frac{V_{\parallel,2}P_2}{T_2}\right)\left(g_1\,\frac{K_{0,1}}{T_1}A_{T,1}-g_2\,\frac{K_{0,2}}{T_2}A_{T,2}\right). \quad (17)$$

Note that the Θ integrations are present only in the X or Y factors, the remaining terms not depending on the angles. Eq. (15) with definitions (11)-(17) is the radial transport law which describes the time-evolution of any flux-surface averaged moment of the scattered particle distribution function. Note that in order to evaluate Eq. (16) we need to specify the term $\ell\cdot\Omega$ and the two factors X, Y, g and G defined in Eqs. (11), (12) and (13). The general expressions of these factors have been reported in Ref. [29]. Here we will make use only of their limiting expressions valid in the far-untrapped case.

Considering $\chi = Mv_\parallel \simeq Mv_\zeta$ and expressing the results in terms of r using $\langle\cdots\rangle_{\bar\alpha} = [1/(\bar r B_{0,t})]\langle\cdots\rangle_{\bar r}$, $\partial/\partial\bar\alpha = [1/(\bar r B_{0,t})]\partial/\partial\bar r$, Eq. (15) leads to the parallel momentum transport equation:

$$V_a\frac{\partial\left\langle MN(x;t)V_\parallel(x;t)\right\rangle_{\bar r}}{\partial t}-V_a\langle q_1 N E_t\rangle_{\bar r}+\frac{\partial}{\partial\bar r}\langle V_a V_f MNV_\parallel\rangle_{\bar r}\,S(1-\kappa)$$

$$+\frac{\partial}{\partial\bar r}V_a\Gamma^V(\bar r)=(\bar r B_{0,t})V_a U^V\ , \quad (18)$$

where $V_a = 4\pi^2 R_0 w\bar r$ is the toroidal shell (centered at $\bar r$ and of width w) over which the mode a is nonzero, and the flux and source terms are again given by (16). In the second term on the left-hand-side (LHS), E_t is the toroidal induction electric field $E_t = (1/cR)(\partial\psi_p/\partial t)$, where the time derivative is evaluated at a fixed position in space. While the poloidal magnetic field can remain constant (if the external fields remain so), the flux surface on which ψ_p has a particular value will move. In a tokamak, due to the smallness of B_p, the velocity of the flux surfaces is very large. As suggested by the presence of the velocity of a flux surface, $V_f \equiv cE_t/B_p$, the last term in the first raw of Eq. (18) is a collisionless version of the Ware-Galeev pinch, effective only for trapped particles [and this is the reason for the step function $S(1-\kappa)$ yielding 1 (0) for $\kappa<1$ ($\kappa>1$)]. The source term U^V accounts for the momentum generation due to fluctuations. Eq. (18) generalizes the quasilinear result of Ref. [30].

3.1. Fluctuation Spectrum for Magnetic Micro-Turbulence

As for the standard Balescu-Lenard operator, the spectrum contained in D and F of Eq. (2) is of the "thermal" type, and does not correctly model realistic turbulence [28]. This limitation has been overcome in Ref. [21], although in an approximate way, with the adoption of a supra-thermal spectrum (called the "pseudo-thermal" spectrum) which retains the structure of the original thermal spectrum (therefore maintaining the required properties of the collision operator), but replaces its form so as to better represent experimental features. Here we only outline the essence of this approximation, since it is well documented in the literature. It is assumed that the eigenvalue Δ_a is non-linearly modified from its thermal value in such a way that the turbulent vector potential driven by species 2 has the form
$|A^a|^2 \propto \tilde B^2(2)\exp\{-k_\perp^2/[2(\Delta k_\perp)^2]-k_\parallel^2/[2(\Delta k_\parallel)^2]\}/(\Delta k_\perp)^2$,
where $\Delta k_\perp \sim \rho_{gi}^{-1}$ and $\Delta k_\parallel \sim L_s^{-1}$ (with $L_s \sim q_{saf}R_0$ the shear length) are the spectral widths satisfying $\Delta k_\perp \gg \Delta k_\parallel$. The overall strength of the magnetic fluctuations induced by the entire species 2, indicated with $\tilde B^2(2) \equiv \langle B_r^2\rangle$, is assumed to be nonzero and approximately constant only within the volume of a narrow toroidal shell $V_a = 4\pi^2 r_a w_a R_0$. The "pseudo-thermal, magnetic micro-turbulence" version of Eq. (16) is

$$\begin{bmatrix}\Gamma_{12}(\bar\alpha)\\ U_{12}(\bar\alpha)\end{bmatrix}=\sum_{r_a}\sum_k\sum_{\ell_1,\ell_2}\frac{1}{V'(\bar r)}\left(\frac{2\pi}{M_1}\right)^3$$

$$\times\int_{V_a}d\boldsymbol{J}_1\,f_M(\boldsymbol{J}_1)\left(\frac{2\pi}{M_2}\right)^3\int_{V_a}d\boldsymbol{J}_2 f_M(\boldsymbol{J}_2)$$

$$\times\begin{bmatrix}X^{\chi_1}(\boldsymbol{J}_1,\ell_1)\\ Y^{\chi_1}(\boldsymbol{J}_1,\ell_1)\end{bmatrix}A(\boldsymbol{J}_1,\boldsymbol{J}_2,\ell_1,\ell_2)\,2\pi\delta[\ell_1\cdot\Omega_1(\boldsymbol{J}_1)-\ell_2\cdot\Omega_2(\boldsymbol{J}_2)]$$

$$\times|C_a|^2\,\delta(\ell_{\zeta,1}-n_a)\,\delta(\ell_{\zeta,2}-n_a)J^2_{\ell_{g1}}(z_{g1})\,J^2_{\ell_{b1}-m_a}(z_{b1})$$
$$\times J^2_{\ell_{g2}}(z_{g2})\,J^2_{\ell_{b2}-m_a}(z_{b2}) \quad (19)$$

where $|C_a|^2 = (q_1/c)^2\left|A_\parallel^a\right|^2\left|v_{\parallel 1}\right|^2\left|v_{\parallel 2}\right|^2/(V_a N_2\langle\left|v_{\parallel 2}\right|^2\rangle),\langle...\rangle$, indicates a thermal average, q is the particle charge, and A includes the terms of Eq. (17). Here, $z_g = k_\perp\rho_g$ where ρ_g is the gyration radius and $k_\perp$ the perpendicular wave-vector, and $z_b = [(k_r r_d)^2+(m\theta_d+n\zeta_d)^2]^{1/2}$ where (r_d,θ_d,ζ_d) are the amplitudes which measure the extent of the particle excursion from the field lines in the course of a transit period, and (k_r,m,n) are the radial wave-vector, the poloidal and the toroidal mode-number. The gyro- and bounce-related Bessel functions quantify the modification of the field-particle coupling due to drifts and the magnetic modulation of $v_\parallel$ by the magnetic well [31]. In the followings we assume $\omega_a < \Omega_g$ for both species, and therefore neglect effects from all gyro-harmonics except $\ell_g = 0$. It should be noted, however, that the pseudo-thermal ansatz limits the validity of the operator, and thus of Eq. (19), to strictly stable plasmas. A turbulent generalization of the operator would be required to remove this limitation, and obtain a truly self-consistent transport theory of strongly turbulent, inhomogeneous plasmas [32].

3.2. Momentum Flux and Source

Starting with Eq. (19), the momentum flux and the momentum source have been evaluated in Ref. [22] and Ref. [23], respectively, and therefore we report here only the final results. Introducing as expansion parameter the inverse scale length of the equilibrium quantities (density, temperature and safety factor) normalized to the electron (poloidal) gyro-radius, $\overline{\varepsilon} \equiv \rho_{e,p} A \ll 1$, where A can be $A_{N,j}, A_{V,j}, A_{T,j}$ (with j=e,i) or $A_{saf} \equiv (dq_{saf}/dr)/q_{saf}$ (note that we assume $A \sim 1$ and thus $\rho_{e,p} \sim \overline{\varepsilon}$), the $O(1)$ electron-ion momentum flux and source are, respectively,

$$\Gamma_{ei}^{V} = -3\left(\frac{L_{ei}}{N_e}\right)\left\{\frac{d(M_e N_e V_{\|e})}{dr}\right.$$
$$\left.+M_e N_e V_{\|e}\left[-2\left(\frac{T_i}{T_e}-\frac{3}{2}\right)\left(\frac{1}{q_{saf}}\frac{dq_{saf}}{dr}\right)+\left(\frac{1}{N_i}\frac{dN_i}{dr}+\frac{1}{T_i}\frac{dT_i}{dr}\right)\right]\right\} \quad (20)$$

$$(rB_{0t})U_{ei}^{V} = -3\left(\frac{L_{ei}}{N_e}\right)\frac{T_i}{T_e}\left(\frac{1}{q_{saf}}\frac{dq_{saf}}{dr}\right)\left\{\frac{d(M_e N_e V_{\|e})}{dr}\right.$$
$$\left.+M_e N_e V_{\|e}\left[5\left(\frac{1}{q_{saf}}\frac{dq_{saf}}{dr}\right)+\left(\frac{1}{N_i}\frac{dN_i}{dr}+\frac{2}{T_i}\frac{dT_i}{dr}\right)\right]\right\}. \quad (21)$$

The transport coefficient is defined by

$$L_{12} = \sum_{r_a} p^2 \pi N_1 b_t^2 D_{RR}(1,2),$$

with

$$D_{RR}(1,2) = v_{th,1}(v_{th,1}/\Omega_{b,M})[v_{th,2}^2/\langle | v_{\|2} |^2\rangle]$$
$$\times \tilde{b}^2(2)[J_0^2(z_{g,1})J_0^2(z_{g,2})/\pi z_{b,M}],$$

where $z_{b,M}$ is the maximum between $z_{b,e}$ and $z_{b,i}$. Using $v_{th,e}/\Omega_{b,M} \simeq q_{saf} R_0$ and $\langle | v_{\|2} |^2\rangle = v_{th,i}^2$, we see that $D_{RR}(e,i)$ is a generalized Rechester-Rosenbluth coefficient [14]. It is necessary to make a specification. The subscript "ei" attached to the flux and the source means "electron flux (or source) due to the fluctuation spectrum induced by the ion species". We are therefore neglecting the "ee" contributions, i.e., Γ_{ee}^{V} and U_{ee}^{V}, which in principle could be nonzero (only the volume-integrated "ee" source is zero). Because of the use of the pseudo-thermal ansatz for the fluctuation spectrum, however, the theory loses its level of refinement that would allow us to derive correct expressions for the "ee" contributions. We leave to future work the attempt to remove this limitation of the theory. In general, this could be done only through a self-consistent evaluation of the turbulent spectrum, a procedure that would lead to a truly turbulent version of the gBL collision operator.

Let's briefly discuss the various terms, starting with the flux, Eq. (20). The first contribution is a diagonal term which represents MHD effects. In the remaining off-diagonal terms, the dependence on the gradient of the electron density dropped out, leaving only the drives proportional to the ion thermodynamic forces, and the additional drive proportional to the magnetic profiles. These off-diagonal terms represent pure kinetic effects. The drive involving the safety factor contains a coefficient depending on the relative magnitude of the electron and ion temperatures, and is the only term which can give, for canonical N and T profiles, an inward contribution depending on the electron and ion temperatures and on the shape of q_{saf}. For example, in normal magnetic shear discharges (q_{saf} monotonically increasing with r), a momentum pinch is obtained for $T_i/T_e < 3/2$, a condition which is likely to be verified in most discharges. The existence of off-diagonal terms in the momentum transport matrix has been postulated theoretically and studied experimentally. For example, in Ref. [33], measurements of the toroidal rotation velocity profiles for discharges with neutral beam injection have led to the conclusion of the existence of a non-diffusive momentum transport tied to the gradient of the temperature. Similarly to that reference, our results exclude a non-diffusive drive due to the electron density gradient. Differently to that reference, however, we find the possibility of a momentum pinch driven by the magnetic configuration. This is similar to what suggested in the context of particle pinches, where turbulent equipartition or other approaches have put in evidence inward fluxes driven by the gradient of q_{saf}. Looking at the source, Eq. (21), we see that the only notable difference with respect to the flux is the overall proportionality to the safety factor gradient, so that a significant momentum source is possible only for large magnetic shear.

4. TURBULENT ELECTRIC FIELD

To derive the turbulent contribution to the electric field we consider the electron momentum balance Eq. (18), where the flux and the source are reported in Eqs. (20) and (21), respectively. For simplicity, in the followings we assume no ion drift velocity, $V_{\|i} = 0$, and approximate the electron velocity as $V_{\|e} \simeq -j_{\|}/(eN_e)$. From the passing-electron version of Eq. (17) we then obtain, after a division by $V_a e N_e$,

$$-\frac{M_e}{e^2 N_e}\frac{\partial\langle j_{\|}\rangle_r}{\partial t}+\frac{\langle N_e E_t\rangle_r}{N_e} = \eta_{neo} j_{\|} - E_{bs} + E_{\|}^{turb}. \quad (22)$$

On the right-hand-side (RHS) we have added the neoclassical resistive term and the neoclassical bootstrap term, $E_{bs} \propto -dp/dr$, p being the total equilibrium pressure.

Before giving the expression for the anomalous contribution $E_{\|}^{turb}$, it is convenient to express Eqs. (20) and (21) more succinctly as

$$\Gamma_{ei}^{V} == -L_{ei}\hat{\Gamma}_1^{ei}\frac{M_e V_{\|e}}{\rho_{e,p}} - L_{ei}\hat{\Gamma}_2^{ei} M_e \frac{dV_{\|e}}{dr}$$

and

$$(rB_{0t})U_{ei}^{V} = -L_{ei}\hat{U}_1^{ei}\frac{M_e V_{\|e}}{\rho_{e,p}^2} - L_{ei}\hat{U}_2^{ei}\frac{M_e}{\rho_{e,p}}\frac{dV_{\|e}}{dr},$$

where we have introduced the following non-dimensional radial functions

$$\hat{\Gamma}_1^{ei} = 3\rho_{e,p}\left[\frac{1}{p_e}\frac{dp_e}{dr} + \frac{1}{p_i}\frac{dp_i}{dr} - 2\left(\frac{T_i}{T_e} - \frac{3}{2}\right)\left(\frac{1}{q_{saf}}\frac{dq_{saf}}{dr}\right)\right], \quad (23)$$

$$\hat{\Gamma}_2^{ei} = 3,$$

$$\hat{U}_1^{ei} = 15\rho_{e,p}^2\left(\frac{1}{q_{saf}}\frac{dq_{saf}}{dr}\right)^2\frac{T_i}{T_e} + 3\rho_{e,p}^2\left(\frac{1}{q_{saf}}\frac{dq_{saf}}{dr}\right) \times \frac{T_i}{T_e}\left(\frac{1}{N_e}\frac{dN_e}{dr} + \frac{1}{N_i}\frac{dN_i}{dr} + \frac{2}{T_i}\frac{dT_i}{dr}\right), \quad (24$$

$$\hat{U}_2^{ei} = 3\rho_{e,p}\left(\frac{1}{q_{saf}}\frac{dq_{saf}}{dr}\right)\frac{T_i}{T_e}, \quad (25)$$

with the electron poloidal gyro-radius and gyro-frequency given by, respectively, $\rho_{e,p} = v_{th,e}/\Omega_{e,p}$ and $\Omega_{e,p} = eB_{0,\theta}/(cM_e)$. In terms of the following additional functions, all proportional to the transport coefficients $\eta_{ei} = M_e L_{ei}/(e^2 N_e)$ and $L_{ei}/N_e = \pi v_{th,e} q_s R_0 \tilde{b}^2$,

$$\Lambda_1^{\Gamma} = \frac{\eta_{ei}}{\rho_{e,p}}\left(\hat{\Gamma}_2^{ei}\frac{\rho_{e,p}}{L_{N_e}} - \hat{\Gamma}_1^{ei}\right), \quad (26)$$

$$\Lambda_1^{U} = \frac{1}{N_e\rho_{e,p}}\frac{\eta_{ei}}{\rho_{e,p}}\left(\hat{U}_2^{ei}\frac{\rho_{e,p}}{L_{N_e}} - \hat{U}_1^{ei}\right), \quad (27)$$

$$\Lambda_2^{\Gamma} = \eta_{ei}\hat{\Gamma}_2^{ei}, \quad \Lambda_2^{U} = \frac{1}{N_e\rho_{e,p}}\eta_{ei}\hat{U}_2^{ei}, \quad (28)$$

where $L_{N_e}^{-1} \equiv (1/N_e)(dN_e/dr)$, we finally express the turbulent electric field as

$$E_{\|}^{turb} = \frac{1}{N_e}\frac{1}{r}\frac{\partial}{\partial r}r\left(\Lambda_1^{\Gamma} j_{\|} - \Lambda_2^{\Gamma}\frac{dj_{\|}}{dr}\right) + \left(\Lambda_2^{U}\frac{dj_{\|}}{dr} - \Lambda_1^{U} j_{\|}\right). \quad (29)$$

As the superscripts Γ and U indicate, the first term originates from the momentum flux, while the second contribution originates from the momentum source. The latter is a result of the self-consistency of the theory, and would be missing in the more conventional quasi-linear approach. It is not a true source term, since it is proportional to either the current density or its derivative. In this respect it is similar to the neoclassical bootstrap electric field, $E_{bs} \propto \eta_{neo}(T/B_p)(dn/dr)$ where the poloidal magnetic field is $B_p \propto (\int drrj)/r$.

To make contact with results obtained in the framework of mean-field MHD [18, 19], we manipulate the expression (29) for the turbulent electric field so to isolate the helicity-conserving hyper-resistivity term, obtaining

$$E_{\|}^{turb} = \eta_{an} j_{\|} + \eta_{\times}\frac{dj_{\|}}{dr} - \frac{1}{B_0 r}\frac{d}{dr}\left[r\eta_h\frac{d}{dr}\left(\frac{j_{\|}}{B_0}\right)\right], \quad (30)$$

where B_0 is the total equilibrium magnetic field, and where the three transport coefficients are defined by

$$\eta_{an} \equiv -\frac{1}{B_0 r}\frac{d}{dr}\left(\frac{r\Lambda_2^{\Gamma}}{N_e}\frac{dB_0}{dr}\right) + \frac{\Lambda_1^{\Gamma} + r(d\Lambda_1^{\Gamma}/dr)}{N_e r} - \Lambda_1^{U}, \quad (31)$$

$$\eta_{\times} \equiv -\frac{\Lambda_2^{\Gamma}}{N_e L_{N_e}} + \frac{\Lambda_1^{\Gamma}}{N_e} + \Lambda_2^{U}, \quad (32)$$

and

$$\eta_h \equiv \Lambda_2^{\Gamma} B_0^2 / N_e. \quad (33)$$

The last term in Eq. (30), proportional to the anomalous viscosity, or hyper-resistivity, η_h, originates from the momentum flux, and accounts for the tendency of the turbulence to smooth out the perpendicular gradient of the parallel current density. Since the magnetic turbulence (and hence η_h) vanishes at the boundary, this term is helicity conserving. Note that in a Taylor state [17], $j_{\|}/B_0 = constant$, so that the hyper-resistive term is driven by departures from a Taylor state. This term has been shown to be necessary to fit experimental current profiles in the central region of tokamak plasmas, profiles that despite the presence of bootstrap current (and/or edge current drive) remain non-hollow [34]. In pinches, the anomalous viscous term leads to toroidal field reversal at the plasma edge as observed [18]. The first term on the RHS of Eq. (30) is an anomalous resistive electric field. From the expression of the anomalous resistivity η_{an}, Eq. (31), and definitions (26)-(28), we see that both the momentum flux and source contribute to η_{an}. Finally, the middle term in Eq. (30), proportional to the coefficient $\eta_{\times}$ and to the first derivative of the current density, is also due to both flux and source, as Eq. (32) shows. Contrary to the other two turbulent terms, it is non-dissipative in form. We will refer to this term as the "cross"-term, or the "cross"-resistivity contribution.

We conclude this section by showing the radial dependence of the three turbulent transport coefficients (31)-(33). In order to do that, we need to specify the geometry, the magnetic field and the thermodynamic plasma profiles. We choose $a = 71$ cm and $R_0 = 240$ cm, giving an aspect ratio of $\varepsilon \equiv a/R_0 = 0.295$. The electron and ion temperature profiles are modeled by $T_e = T_i = (T_{e,0} - T_{e,a})(1 - x^2)^{\gamma_T} + T_{e,a}$ where $x = r/a$, $\gamma_T = 2.0$, and where the boundary values are $T_{e,a} = 0.1$ keV, $T_{e,0} = 4.8$ keV. The electron (and ion) density profile is given by $N_e = N_i = (N_{e,0} - N_{e,a})(1 - x^2)^{\gamma_N} + N_{e,a}$ where $\gamma_N = 2.5$, $N_{e,0} = 2\times10^{13}$ $1/\mathrm{cm}^3$, and $N_{e,a} = 0.01\times N_{e,0} = 2\times10^{11}$

1/cm^3. The total (electron plus ion) pressure is $p = p_e + p_i = 2T_e N_e$. For the normalized (i.e., divided by the central value of the toroidal field $B_{z,0}$) perturbed magnetic field we adopt the model $\tilde{b}^2 = \tilde{b}_0^2 \times (1-x^2)^{2.5}$ where $\tilde{b}_0^2 = 1\times 10^{-8}$. Such a perturbation, being zero at the plasma edge, leads to an helicity-conserving hyper-resistive term. For all turbulent coefficients we use as a normalizing parameter $\overline{\eta}_{cl}$, the cylindrical cross-section average of the classical resistivity $\eta_{cl} = 4.77 e^2 Z_{eff} \ln\Lambda / [M_e v_{th,e}^3]$, where the ion charge and the Coulomb logarithm are assumed to be, respectively, $Z_{eff} = 1$ and $\ln\Lambda = 17$. In a tokamak plasma, however, the classical resistivity must be modified to take into account toroidal effects. The result is the neoclassical resistivity, $\eta_{neo} = \eta_{cl} / (1 - 1.95\sqrt{r/R_0})$. For our parameters, this function is singular inside the plasma; to avoid complications not essential to the purpose of our work, in all numerical calculations we use the classical resistivity. To specify the equilibrium magnetic profiles we use the safety factor that is solution of the ohmic case presented in Sec. 6.1, and showed in Fig. (**6**) with a line with boxes. The non-dimensional hyper-resistive coefficient $\hat{\eta}_{\mathrm{h}} = \eta_{\mathrm{h}} / (a^2 B_{z,0}^2 \overline{\eta}_{cl})$, the non-dimensional cross coefficient $\hat{\eta}_{\mathrm{X}} = \eta_{\mathrm{X}} / (a\overline{\eta}_{\mathrm{cl}})$, and the non-dimensional anomalous coefficient $\hat{\eta}_{an} = \eta_{an} / \overline{\eta}_{\mathrm{cl}}$ are plotted in Figs. (**1-3**), respectively. All coefficients are zero at the outer boundary of the plasma, in virtue of the adopted shape of the magnetic perturbation. We see that, while both the hyper- and the cross-resistive contribution are positive, the anomalous resistivity is positive near the central region of the plasma, and negative in the outer region. Moreover, the absolute value of $\hat{\eta}_{an}$ is on average larger then the other two coefficients, even though it remains smaller than that of the classical resistivity. As noted before, while the hyper-resistive coefficient $\hat{\eta}_{\mathrm{h}}$ originates from the momentum flux Γ, the remaining two coefficients contain both a momentum flux and a momentum source contribution, a result of the self-consistency of our theory. To clarify the origin of the negative values of the anomalous resistivity, we compare the parts of $\hat{\eta}_{an}$ proportional to the flux with the one proportional to the source:

$$\eta_{an,1} \equiv -\frac{1}{B_0 r}\frac{d}{dr}\left(\frac{r\Lambda_2^\Gamma}{N_e}\frac{dB_0}{dr}\right),$$

$$\eta_{an,2} \equiv \frac{\Lambda_1^\Gamma + r(d\Lambda_1^\Gamma / dr)}{N_e r}, \quad \eta_{an,3} \equiv -\Lambda_1^U .$$

In Fig. (**4**) we plot $\eta_{an,2}$ (dashed-line) and $\eta_{an,3}$ (solid-line). The function $\eta_{an,1}$ has a much smaller amplitude and therefore is not shown. We find that the major contribution is given by $\eta_{an,2}$ and therefore by the momentum flux. $\eta_{an,2}$

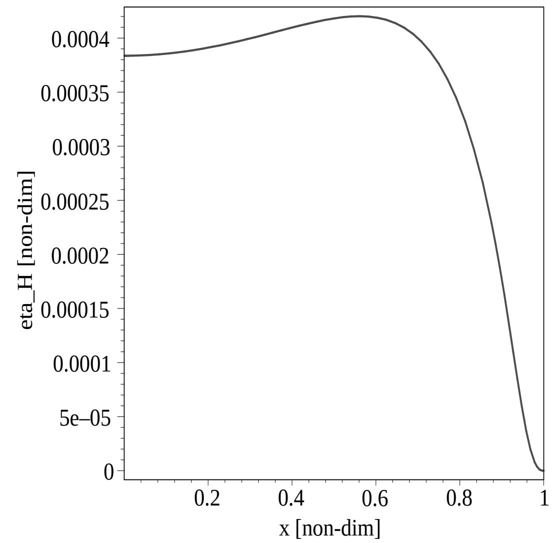

Fig. (1). Radial profile of the hyper-resistivity, $\hat{\eta}_{\mathrm{h}}$.

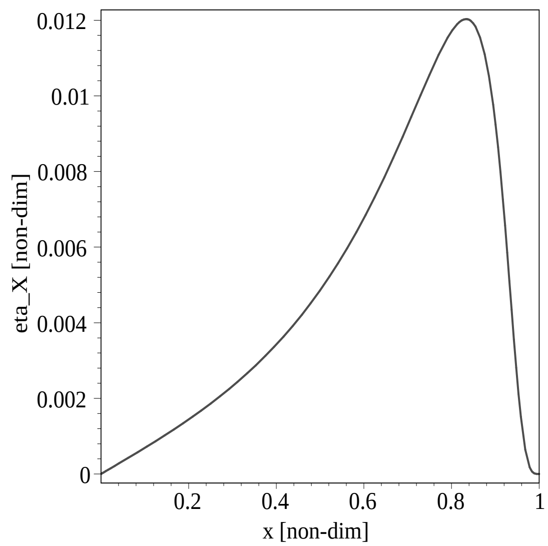

Fig. (2). Radial profile of the cross-resistivity, $\hat{\eta}_{\times}$.

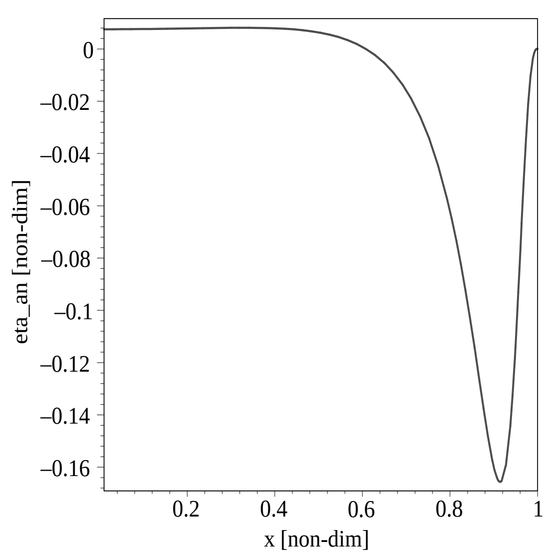

Fig. (3). Radial profile of the anomalous resistivity, $\hat{\eta}_{an}$.

has also the feature of changing sign at around $x \simeq 0.6$, while $\eta_{an,3}$, which originates from the momentum source, is always positive. The possibility of negative anomalous resistivity has been pointed out in several works adopting the MHD approach [9]. In our model, the sign of it depends on the details of the equilibrium thermodynamic (N,T) and magnetic (q_{saf}) profiles, as can be inferred from Eqs. (23)-(25).

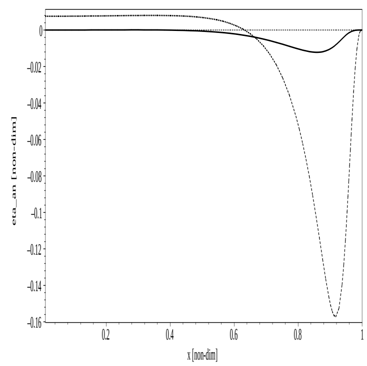

Fig. (4). $\hat{\eta}_{an,2}$ (dashed-line) and $\hat{\eta}_{an,3}$ (solid-line) anomalous resistivity profiles. The horizontal line at zero is for reference.

5. TURBULENT OHM'S LAW

To investigate the effects of the turbulent contributions we carry out a numerical solution of Ohm's law assuming the presence of densely-packed micro-tearing modes acting over the entire plasma cross section, with (normalized) perturbation profile of the form $\tilde{b}^2 = \tilde{b}_0^2 \times (1-x^2)^{2.5}$ where $\tilde{b}_0^2 = 1\times 10^{-8}$. The model to solve is given by the equation $E_{\parallel}^0 = \eta_{neo}\left(j_{\parallel} - j_{bs}\right) + E_{\parallel}^{turb}$ or, using Eq. (30),

$$E_{\parallel}^0 = -\eta_{neo} j_{bs} + (\eta_{neo} + \eta_{an}) j_{\parallel} + \eta_{\times}\frac{dj_{\parallel}}{dr} - \frac{1}{B_0 r}\frac{d}{dr}\left[r\eta_h \frac{d}{dr}\left(\frac{j_{\parallel}}{B_0}\right)\right], \quad (34)$$

together with the following expression for the neoclassical bootstrap current [35-39]:

$$j_{bs} = -cF_{13}\left(\frac{r}{R}\right)^{1/2}\frac{n_e(T_e+T_i)}{B_\theta}\left(\frac{1}{n_e}\frac{dn_e}{dr}\right) - c\left(\frac{r}{R}\right)^{1/2}\frac{n_e T_e}{B_\theta}$$
$$\times\left[-\left(\frac{3}{2}F_{13} - F_{23}\right)\left(\frac{1}{T_e}\frac{dT_e}{dr}\right) - \left(\frac{3}{2} - y\right)F_{13}\frac{T_i}{T_e}\left(\frac{1}{T_i}\frac{dT_i}{dr}\right)\right], \quad (35)$$

where

$$F_{m3} = \frac{K_{m3}}{[1 + a_{m3}\nu_{e*}^{1/2} + b_{m3}\nu_{e*}][1 + c_{m3}\nu_{e*}(r/R)^{3/2}]}, \quad m = 1,2,$$

$$y = \frac{1.31(1 + 1.65\nu_{i*}^{1/2})}{1 + 0.862\nu_{i*}^{1/2}},$$

$$\nu_{j*} = \frac{B_z R^{3/2}}{\tau_j r^{1/2} B_\theta (T_j/m_j)^{1/2}}, \quad j = e,i,$$

the classical e-i and i-i collision times are

$\tau_e = 3m_e^{1/2}T_e^{3/2}/[4(2\pi)^{1/2}\ln\Lambda_e n_e e^4 Z_{eff}]$,

$\tau_i = 3m_i^{1/2}T_i^{3/2}/[4\pi^{1/2}\ln\Lambda_i n_i e^4]$, and where for $Z_{eff} = 1$ the K's and the a,b,c's are $K_{13} = 2.30$, $K_{23} = 4.19$, $a_{13} = 1.02$, $a_{23} = 0.57$, $b_{13} = 0.75$, $b_{23} = 0.38$, $c_{13} = 1.07$ and $c_{23} = 0.61$ [37]. The frequency ν_* represents the ratio of the effective collision frequency to the bounce frequency, and is a measure of the collisionality of the plasma. There are three main collisionality regimes: banana (collisionless), plateau (transition) and Pfrisch-Schlüter (collisional), approximately delineated by the relations: $\nu_* \ll 1$ (banana), $1 \le \nu_* \le 1/\varepsilon^{3/2}$ (plateau), and $\nu_* \gg 1$ (collisional). The bootstrap current is diminished as the plasma becomes more collisional, because the banana orbits are more and more impeded. In our case we obtain an average value of ν_{e*} of about 0.05, therefore indicating a banana regime. Eq. (34) is nonlinear, because the parallel current density is related to the poloidal component of the magnetic field *via* Ampere's law, $(4\pi/c)j_{\parallel} \simeq (1/r)(\partial/\partial r)rB_\theta$, and B_θ appears in the expression for the bootstrap current (35). In the numerical solution, the bootstrap term is therefore treated using an iterative procedure.

Ohm's law Eq. (34) represents a steady-state balance between momentum gain and momentum loss on the part of the charge carriers, and its numerical solution provides the parallel current density profile in the finite-pressure, relaxed state of the system. Its study is especially relevant in connection with the idea of achieving steady-state operation, in which all or a large fraction of the current is due to the bootstrap effect [3, 4, 39]. The question is, whether or not the turbulent contributions in Ohm's law can provide the necessary current diffusion and profile re-adjustment so to generate the needed current seed on axis. To our knowledge, all studies carried out so far have considered only the hyper-resistive term. On the contrary, our Ohm's law Eq. (34) contains, besides the usual hyper-resistive term, the other two terms proportional to η_{an} and $\eta_{\times}$. According to our theory, all these terms are effective in the presence of magnetic turbulence, and thus must be retained. The transport coefficients present in these three terms depend strongly on the plasma profiles, as definitions (31)-(33) show. An appropriate study of the influence of their corresponding terms on the evolution of plasma discharges should be done by considering a transport code which couples the turbulent Ohm's law to the evolution equations for the density and energy. In the present work, however, we limit ourselves to a much simpler application of the theory, i.e, we use Eq. (34) to calculate the steady-state current profile in a (cylindrical) tokamak for fixed pressure equilibrium profiles, and for a fixed level of magnetic turbulence.

To shed light on the physical meaning of the various terms in Eq. (34) we derive an energy balance by multiplying the equation by $r\, j_{\parallel}$ and then integrating it from zero to a, following the procedure adopted in Ref. [4]. We obtain the following power balance

$$\int_0^a dr\, r\, j_\parallel E_\parallel^0 + a\left\{\eta_h \frac{j_\parallel}{B_0}\frac{d}{dr}\left[\frac{j_\parallel}{B_0}\right] - \frac{1}{2}\eta_\times j_\parallel^2\right\}_{r=a}$$

$$= \int_0^a dr\,[\eta_{neo} + \eta_{an}] r\, j_\parallel^2 + \int_0^a dr\, r\, \eta_h \left[\frac{d}{dr}\left(\frac{j_\parallel}{B_0(r)}\right)\right]^2$$

$$-\int_0^a dr\, r\, j_\parallel \left[\eta_{neo} j_{bs} + \frac{1}{2r}\frac{d[r\eta_\times]}{dr} j_\parallel\right], \qquad (36)$$

where each term represents power per unit length of the torus. As seen before, for the chosen equilibrium profiles, all the η transport coefficients are positive everywhere in the plasma, except for η_{an} which is positive in the inner and central region, and negative near the outer edge. This fact leads to the following considerations on the various contributions to the power balance. The two terms on the LHS represent externally injected power. The first contribution is positive and represents the work done by the electric field driven by the induced loop voltage, and its strength is controlled by $E_\parallel^0$. The second term on the LHS is the power injected from the boundary surface ($r = a$) of the plasma (or, analogously, the injection of helicity), with an hyper-resistive contribution proportional to the derivative of the current density - so that it can be positive or negative, in the latter case meaning recharging - and an additional term, always negative, proportional to the squared current density. In all our simulation $j_\parallel(a) \simeq 0$, and therefore this boundary term is negligible. The first term on the RHS represents internal power dissipation (or generation) due to neoclassical and anomalous effects. Anomalous current generation would be in principle possible if the equilibrium profiles are such to make $\eta_{neo} + \eta_{an} < 0$. From Fig. (**3**) we see that the anomalous resistivity is negative on the outer region of the plasma. However, the current density is small on the same region, so that overall the anomalous term dissipates current, and $\eta_{neo} + \eta_{an} > 0$ everywhere. The second term on the RHS, which is positive definite, represents the additional power dissipation associated with hyper-resistive current diffusion. The last term on the RHS is the work done by the diffusion-driven (bootstrap) electromotive force, incremented by a turbulent contribution that is also negative over most of the plasma (see Sec. 6.4). Means to alleviate the requirement of a loop voltage reside on the boundary term on the LHS, and on the last integral term on the RHS containing the bootstrap and the cross contribution. In principle, these two terms could balance the remaining dissipative terms on the RHS, removing any need of externally supplied power and thus realizing steady-state operation.

As stated before, in all numerical studies presented in the next section the boundary term in Eq. (36) is equal to zero. This contribution can however be made nonzero in experiments, for example by producing a current sheath at the surface of the plasma using an external dc low-energy electron beam [40]. Due to the anomalous diffusion of the current, this edge current would diffuse toward the plasma center (and possibly amplified), and contribute to the sustenance of the plasma current against resistive dissipation. The advantage of this approach consists on the avoidance of more expensive, low efficiency current drive mechanisms such as radio-frequency.

6. NUMERICAL SOLUTION OF THE CURRENT EQUATION

To proceed with the numerical solution of Eq. (34), we introduce $x \equiv r/a$ and non-dimensionalize the relevant quantities as follows (an over-hat means non-dimensional): $\hat{j}_\parallel = 4\pi a j_\parallel/(cB_{z,0})$, $\hat{E}_\parallel^0 = 4\pi a E_\parallel^0/(\bar{\eta}_{cl} c B_{z,0})$, $\hat{P} = P/B_{z,0}^2$, $\hat{B} = B/B_{z,0}$, $\hat{\eta}_{neo} = \eta_{neo}/\bar{\eta}_{cl}$, $\hat{\eta}_h = \eta_h/(a^2 B_{z,0}^2 \bar{\eta}_{cl})$, $\hat{\eta}_\times = \eta_\times/(a\bar{\eta}_{cl})$, $\hat{\eta}_{an} = \eta_{an}/\bar{\eta}_{cl}$, where the cross-section averaged classical resistivity is $\bar{\eta}_{cl} = 0.1423796 \times 10^{-16}$ s, and $B_{z,0}$ is the toroidal field (that we assume constant and equal to 4×10^4 G). The equation to be solved for the parallel current density is then

$$\hat{D}_2(x)\frac{d^2\hat{j}_\parallel(x)}{dx^2} + \hat{D}_1(x)\frac{d\hat{j}_\parallel(x)}{dx} + \hat{D}_0(x)\hat{j}_\parallel(x) = \frac{\hat{D}_{-1}(x)}{\int dx'\, x'\, \hat{j}_\parallel(x')} + \hat{E}_\parallel^0(x), \qquad (37)$$

with coefficients

$$\hat{D}_2(x;\eta_h) = -\frac{\hat{\eta}_h}{\hat{B}_0^2}, \qquad (38)$$

$$\hat{D}_1(x;\eta_h,\eta_\times) = \hat{\eta}_\times + \frac{1}{\hat{B}_0^2}\left[-\frac{1}{x}\frac{d(x\hat{\eta}_h)}{dx} + \frac{2\hat{\eta}_h}{\hat{B}_0}\frac{d\hat{B}_0}{dx}\right],$$

$$\hat{D}_0(x;\eta_{neo},\eta_{an}) = \hat{\eta}_{neo} + \hat{\eta}_{an} + \frac{1}{\hat{B}_0^3}\left\{\begin{array}{l}\dfrac{1}{x}\dfrac{d\hat{B}_0}{dx}\dfrac{d(x\hat{\eta}_h)}{dx} + \hat{\eta}_h \\ \left[-\dfrac{2}{\hat{B}_0}\left(\dfrac{d\hat{B}_0}{dx}\right)^2 + \dfrac{d^2\hat{B}_0}{dx^2}\right]\end{array}\right\},$$

$$\hat{D}_{-1}(x;\eta_{neo}) = -4\pi\hat{\eta}_{neo}\left(\frac{a}{R_0}\right)^{1/2} x^{3/2}$$

$$\times\left\{F_{13}\frac{n_e[T_e+T_i]}{B_{z,0}^2}\left(\frac{1}{n_e}\frac{dn_e}{dr}\right) + \frac{n_e T_e}{B_{z,0}^2}\right.$$

$$\left.\times\left[-\left(\frac{3}{2}F_{13} - F_{23}\right)\left(\frac{1}{T_e}\frac{dT_e}{dr}\right) - \left(\frac{3}{2} - y\right)F_{13}\frac{T_i}{T_e}\left(\frac{1}{T_i}\frac{dT_i}{dr}\right)\right]\right\}. \qquad (39)$$

The contributions to $\hat{D}_1$ and $\hat{D}_0$ that are proportional to the radial derivative of the equilibrium magnetic field can be expanded in terms of the small parameter $\varepsilon \sim B_{0,p}/B_{0,z} \sim a/R_0$. We obtain

$$\hat{D}_1 = \hat{\eta}_\times - \frac{1}{\hat{B}_{0,z}^2}\frac{1}{x}\frac{d(x\hat{\eta}_h)}{dx} + \frac{1}{\hat{B}_{0,z}^2}\left(\frac{2\hat{\eta}_h\varepsilon^2}{\hat{L}_{p,1}}\right) - \frac{1}{\hat{B}_{0,z}^2}\frac{1}{x}\frac{d(x\hat{\eta}_h)}{dx}\varepsilon^2 + O(\varepsilon^4),$$

$$\hat{D}_0 = \hat{\eta}_{neo} + \hat{\eta}_{an} + \frac{1}{\hat{B}_{0,z}^2}\frac{1}{x}\frac{d(x\hat{\eta}_h)}{dx}\frac{\varepsilon^2}{\hat{L}_{p,1}} + \frac{1}{\hat{B}_{0,z}^2}\frac{\hat{\eta}_h\varepsilon^2}{\hat{L}_{p,2}} + O(\varepsilon^4).$$

Here, $1/\hat{L}_{p,1} \equiv (1/\hat{B}_{0,p})(d\hat{B}_{0,p}/dx)$ and $1/\hat{L}_{p,2} \equiv (1/\hat{B}_{0,p})(d^2\hat{B}_{0,p}/dx^2)$ are lengths characterizing the radial variation of the poloidal field. To lowest order in ε, the model to be solved is given by Eq. (37) and the coefficients (38), (39),

$$\hat{D}_1(x;\eta_h,\eta_\times) = \hat{\eta}_\times - \frac{1}{\hat{B}_{0,z}^2}\frac{1}{x}\frac{d(x\hat{\eta}_h)}{dx}, \qquad (40)$$

and

$$\hat{D}_0(x;\eta_{neo},\eta_{an}) = \hat{\eta}_{neo} + \hat{\eta}_{an}. \qquad (41)$$

The nonlinear character of Eq. (37) is manifest in the integral term on the RHS, due to the dependence of the bootstrap current on the poloidal magnetic field. This term requires an iterative procedure for the numerical solution. Since the magnetic profiles are present in the D coefficients as well, we had to include them in the iteration procedure to obtain a self-consistent result for the current density profile. The only equilibrium profiles that we keep constant during the numerical solution are the density and temperature profiles.

6.1. Ohmic and Bootstrap Profiles

Setting $\hat{\eta}_h = \hat{\eta}_\times = \hat{\eta}_{an} = 0$ (no turbulence), assuming $\hat{E}_\parallel^0 = 8.215\times10^{-3}$, and approximating $\hat{\eta}_{neo} \simeq \hat{\eta}_{cl}$ in Eq. (37) (an approximation that will be used in all numerical calculations), the current diffusion equation reduces to $\hat{\eta}_{cl}\hat{j}_\parallel = \hat{D}_{-1}/\int x'\hat{j}_\parallel(x') + \hat{E}_\parallel^0$, and its solution comprises ohmic and bootstrap contributions. The profiles of the ohmic (boxes), bootstrap (crosses) and ohmic + bootstrap (triangles) current density profiles are shown in Fig. (**5**), while in Fig. (**6**) we plot the safety factor profile corresponding to the ohmic (boxes) and ohmic + bootstrap (triangles) cases. Integrating over the plasma cross section we obtain the following total currents: $I_{ohm} = 723\,kA$, $I_{bs} = 95\,kA$, and $I_{tot} = I_{ohm} + I_{bs} = 818\,kA$. The bootstrap current is about $12\,\%$ of the total current.

6.2. Add the Hyper-Resistive Contribution

The effect of the hyper-resistive term is well known. It induces a diffusion of the current density toward the center, and it is therefore fundamental in mitigating the hollowness induced by the bootstrap effect in advanced scenarios. All hypothesis of non-inductive, steady-state tokamak operation rely on this term. To study this hyper-resistive contribution, we solve Eq. (37) with $\hat{\eta}_\times = \hat{\eta}_{an} = 0$. We assume the boundary conditions $\hat{j}_\parallel(x=1)=0$ and $\hat{j}_\parallel'(x=0)=0$, therefore ignoring non-inductive helicity injection at the plasma edge. We initialize all terms depending on the magnetic profiles using the ohmic profiles found in Sec. 6.1.

The new current density profile is shown with circles in Fig. (**5**). In all our numerical studies we have decided not to keep constant the total current, so to put in evidence turbulent dissipation and/or generation of current density. We obtain $I_{tot} = I_{ohm} + I_{bs} + I_h = 768\,kA$. The reduction in current with respect to the case of Sec. 6.1 is 6.1%. The effect of the hyper-resistive term is therefore twofold: a smoothing out of the central current density profile, and a dissipative reduction of the overall current.

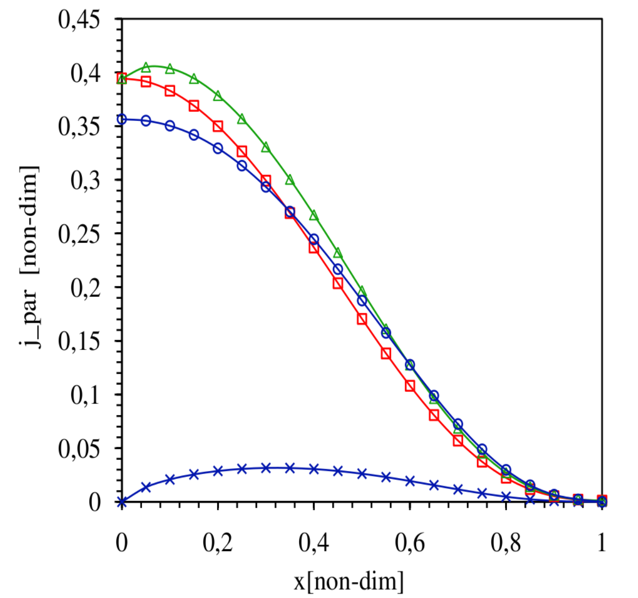

Fig. (5). Ohmic (boxes), bootstrap (crosses), ohmic + bootstrap (triangles), and ohmic + bootstrap + hyper-resistivity (circles) current density profiles.

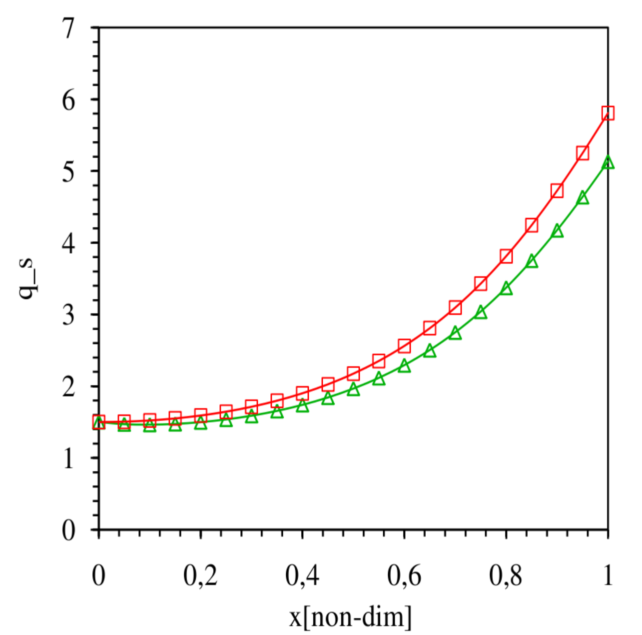

Fig. (6). Ohmic (boxes) and ohmic + bootstrap (triangles) safety factor profiles.

6.3. Add the Remaining Anomalous Contributions

We proceed by adding to the run of Sec. 6.2 (ohmic+bootstrap+hyper-resistivity) the remaining two turbulent contributions. We begin with the anomalous resistivity by setting to zero only $\hat{\eta}_\times$ in Eq. (37). The resulting current profile is presented with boxes in Fig. (**7**). The total current is reduced to $I_{tot} = I_{ohm} + I_{bs} + I_h + I_{an} = 696\,kA$, a $9.3\,\%$ reduction. The current is reduced mainly in the central part of the plasma, but there is also a small increase in the outer region due to the negative value of $\hat{\eta}_{an}$, as shown in Fig. (**4**).

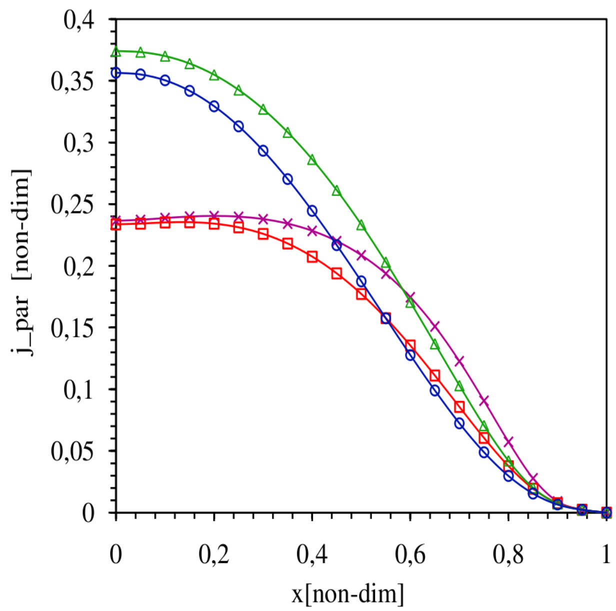

Fig. (7). Ohmic + bootstrap + hyper-resistivity (circles), ohmic + bootstrap + hyper-resistivity + anomalous resistivity (boxes), ohmic + bootstrap + hyper-resistivity + cross-resistivity (triangles), and ohmic + bootstrap + hyper-resistivity + anomalous resistivity + cross-resistivity (crosses) current density profiles.

The effect of the turbulent cross-resistivity term is studied next by running the code setting to zero only $\hat{\eta}_{an}$. The resulting current profile is shown with triangles in Fig. (**7**), leading to a total current of $I_{tot} = I_{ohm} + I_{bs} + I_h + I_\times = 931\,kA$. We thus find that the addition of the cross turbulent term leads to a $21.2\,\%$ increase in the total current, so that this term represents turbulent current amplification. It is interesting to note that recent bootstrap current validation experiments carried out in the TCV tokamak concluded that the magnitude of the bootstrap current exceeds purely neoclassical prediction [6]. The TCV team attempted an explanation of this discrepancy by referring to effective diffusion due to potato orbits, and/or to turbulence. Our results show that the latter effect could indeed be relevant.

As a final step, we solve the complete equation (37). The resulting current density profile is shown with crosses in Fig. (**7**), while in Fig. (**8**) we present the corresponding safety factor profile (circles) (for comparison, we report with boxes also the safety factor profiles of the ohmic + bootstrap + hyper-resistive case). The total current for this final run is $I_{tot} = I_{ohm} + I_{bs} + I_h + I_{an} + I_\times = 832\,kA$, a $1.7\,\%$ increase with respect to the neoclassical result of Sec. 6.1, and a $8.4\,\%$ increase with respect to the hyper-resistive case of Sec. 6.2.

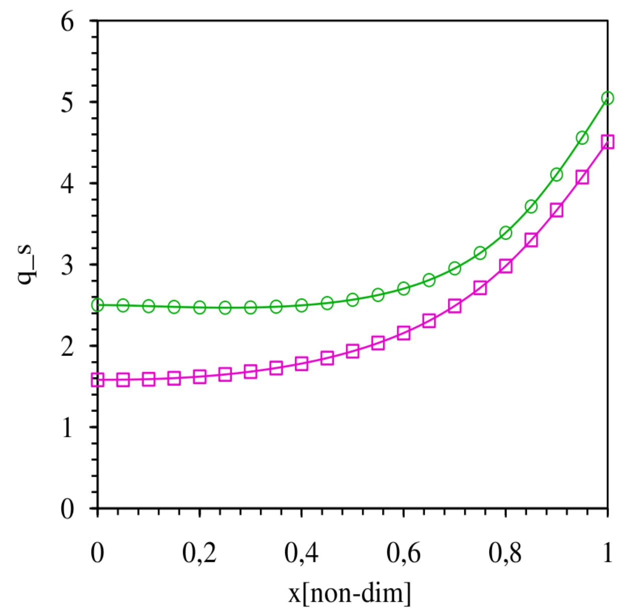

Fig. (8). Ohmic + bootstrap + hyper-resistivity + cross-resistivity + anomalous resistivity (circles) and ohmic + bootstrap + hyper-resistivity (boxes) safety factor profiles.

6.4. Energetics

We analyze further the results presented in Sec. 6.3, by evaluating the various terms of the power balance (36):

$$T_E \equiv +\int_0^a dr\, r\, E_\parallel^0\, j_\parallel\,,$$

$$T_a \equiv a\left[\eta_h \frac{j_\parallel}{B_0}\frac{d}{dr}\left(\frac{j_\parallel}{B_0}\right) - \frac{1}{2}\eta_\times j_\parallel^2\right]_{r=a}\,,$$

$$T_{neo} \equiv +\int_0^a dr\, \eta_{neo}\, r\, j_\parallel^2\,,$$

$$T_{an} \equiv +\int_0^a dr\, \eta_{an}\, r\, j_\parallel^2\,,$$

$$T_h \equiv +\int_0^a dr\, r\, \eta_h \left[\frac{d}{dr}\left(\frac{j_\parallel}{B_0}\right)\right]^2\,,$$

$$T_{bs} \equiv -\int_0^a dr\, r\, \eta_{neo} j_{bs}\, j_\parallel\,,$$

$$T_\times \equiv -\frac{1}{2}\int_0^a dr\, \frac{d(r\eta_\times)}{dr}\, j_\parallel^2\,.$$

Adopting the units W/cm, for the LHS of Eq. (36) we obtain $T_E = 6.253$ and $T_a = 0$. The first three contributions on the RHS are all positive, representing dissipation: $T_{neo} = 8.615$, $T_{an} = 0.168$ and $T_h = 0.360$. The hyper-resistive contribution $T_{\rm h}$, is responsible of the reduction of the hollowness of the current density at the plasma center when bootstrap current is relevant. The last two terms, one proportional to $\eta_{neo} j_{bs}$ and the other to $\eta_{\times}$, are negative and of comparable magnitude (-1.165 and -1.726, respectively), representing internally generated power. The second one, due to the presence of turbulence, represents therefore an important contribution to non-inductive current generation. Since an interesting output of our analysis is given by this "turbulent bootstrap" term, we have looked at the integrand of $T_{\times}$, finding out that it is negative everywhere in the plasma, except for a very small region near the edge.

7. SUMMARY AND CONCLUSIONS

The control of the current density profile is a key task in tokamak plasma experiments aimed at defining advanced regimes, in which the synergistic combination of improved confinement and of a large fraction of bootstrap current minimizes the need for external current drive. For this reason, the derivation of generalized versions of Ohm's law that include both neoclassical and turbulent effects becomes essential. Using the self-consistent action-angle transport theory, we have derived a turbulent version of Ohm's law which is valid in magnetized axisymmetric plasma with a background of magnetic turbulence, such that induced by micro-tearing modes. We have performed numerical studies to understand the role of turbulence in shaping the steady-state current profile. Although the results presented in this paper have been derived using a particular model of turbulent transport, many turbulent theories lead to results that are formally equivalent, and thus we believe that our results can be given a qualitative, if not a quantitative, significance.

The extended Ohm's law (34) includes the well-known hyper-resistive term, the diffusive effect of which ultimately allows for the radial diffusion of the bootstrap current toward the plasma center. Besides the hyper-resistive term, Eq. (34) contains two other turbulent contributions, the anomalous resistivity term, and a "cross"-resistivity term. The anomalous resistivity contribution leads to a significant reduction of the current density in the central part of the plasma, and to a small increase in the outer region. This is due to a radial profile of the anomalous resistivity that becomes negative for $x \geq 0.6$. Our results confirm the possibility of negative turbulent anomalous resistivity, a fact that according to our transport model is due to the combined effect of the thermodynamic and magnetic equilibrium profiles. The remaining turbulent term, the cross-resistivity term, leads to a significant increase of $j_{\|}$ everywhere in the plasma. It is therefore a turbulent bootstrap contribution, and could be a candidate for the explanation of observed discrepancies between the theoretically predicted bootstrap current and the measured one. In our simulation, the increase in total current when going from the ohmic + bootstrap + hyper-resistive case to the ohmic + bootstrap + hyper-resistive + cross-resistive case is about 20%. Most of this current increase is however compensated by the effect of the anomalous resistivity term, so that the increase in total current when adding all the turbulent contributions to the neoclassical case (ohmic + bootstrap only) is reduced to only 1.7%.

In conclusion, the main indication that can be drawn from our work is that anytime the hyper-resistivity is included in the modeling of the current density, additional turbulent terms must be included as well. These additional turbulent contributions could play a determinant role in explaining experimental current density profiles, and can be of help in addressing the many questions raised by the study of advanced tokamak regimes with a high bootstrap current fraction. The numerical solution of the particular turbulent transport model presented in this paper has shown, for example, that one of the turbulent terms (the cross-resistivity term) is analogous to a turbulent bootstrap effect, as it leads to an increase of the total plasma current.

Our work could be expanded in several directions, with the general goal of removing some of the limitations of our approach. First, we have adopted an arbitrary profile for the magnetic perturbation. Additional studies should be carried out by coupling the Ohm's law with a stability code, and modeling the magnetic perturbation profile according to the output of the stability analysis. Approaching the problem from another direction, it should be interesting to look for the magnetic perturbation profile that leads to the best fit of current density experimental data. Second, it is well known that the current density profile itself plays an important role in the global stability properties of the plasma. When a large fraction of the current is due to bootstrap, i.e., owns its existence to the pressure gradient, then the mutual interaction between current and thermodynamic profiles must be taken into account self-consistently. This has not been done here, and should be addressed by coupling the Ohm's law to a transport code evolving the equilibrium profiles. Finally, our code could be used to investigate the effects associated to the boundary term in the power balance, Eq. (36). In all computations we have set $j_{\|} = 0$ at the plasma outer boundary. Thanks to turbulent diffusion, however, a nonzero current at the plasma edge could potentially sustain a significant fraction of the plasma current.

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