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Original article

Ternary shape-preserving subdivision schemes

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Abstract

We analyze the shape-preserving properties of ternary subdivision schemes generated by bell-shaped masks. We prove that any bell-shaped mask, satisfying the basic sum rules, gives rise to a convergent monotonicity preserving subdivision scheme, but convexity preservation is not guaranteed. We show that to reach convexity preservation the first order divided difference scheme needs to be bell-shaped, too. Finally, we show that ternary subdivision schemes associated with certain refinable functions with dilation 3 have shape-preserving properties of higher order.

Keywords: Subdivision scheme; Bell-shaped mask; Shape-preserving

1. Introduction

Shape-preserving approximations are used in the design of curves or surfaces to *predict or control their 'shape'* by the shape of the *control points*, i.e. the vertices of a given polygonal arc or polyhedral surface.

Efficient methods to construct shape-preserving approximations starting from an initial data sequence are *shape-preserving subdivision schemes*, i.e. schemes that preserve the *shape* – for instance, the monotonicity or the convexity – of the control points. There are several examples of binary monotonicity and convexity preserving subdivision schemes (see, for instance [1,7,9,17] and references therein) while the literature on ternary shape-preserving subdivision schemes is much smaller. Some first attempts in this direction are, for instance, the methods proposed in [2,3,13,16,19,20]. The use of dilation 3 gives more flexibility in the construction of subdivision schemes endowed with properties useful in applications. For instance, ternary subdivision schemes can improve the smoothness of the limit function while keeping a small support. Moreover, the convergence rate of ternary subdivision schemes is faster than the convergence rate of binary schemes since at each iteration the new sequence has three times as many points as the previous one so reducing the computational cost (cf. [13]).

Well-known examples of approximating shape-preserving subdivision schemes are the schemes generating spline curves [15]. Interestingly enough, for any arity they are associated with refinement masks that are *bell-shaped*.

Our goal is to analyze the shape-preserving properties of ternary subdivision schemes generated by bell-shaped refinement masks. Some results on monotonicity preservation of bell-shaped binary schemes can be found in [21]. Here, we want to analyze how the bell-shape property of a mask, as defined in Section 2, reflects on the shape-preserving

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properties of the associated subdivision scheme. In particular, the main result of the paper is that bell-shaped masks always generate monotonicity preserving subdivision schemes (see Section 3) while the bell-shape property is not sufficient to guarantee convexity preservation. In Section 4 we show which further assumptions on the masks are required to reach convexity preservation, too. Finally, in Section 5 we analyze a subdivision scheme family having shape-preserving properties of higher order and compare the behavior of these schemes with other ternary shape-preserving schemes from the literature.

2. Preliminaries

A (stationary) *ternary subdivision scheme* $\mathcal{S}_{\mathbf{a}}$ is described by the algorithm

$$\mathcal{S}_{\mathbf{a}}: \quad \begin{cases} \lambda^0 = \lambda = \{\lambda_\alpha\}_{\alpha\in\mathbb{Z}} \in \ell(\mathbb{Z}), \\ \lambda^{k+1} := S_{\mathbf{a}}\,\lambda^k, \quad k \geq 0, \end{cases} \tag{2.1}$$

where $S_{\mathbf{a}}: l(\mathbb{Z}) \to l(\mathbb{Z})$ is the *ternary subdivision operator* defined as

$$(S_{\mathbf{a}}\,\lambda)_\alpha = \sum_{\beta\in\mathbb{Z}} a_{\alpha-3\beta}\,\lambda_\beta, \quad \alpha \in \mathbb{Z}, \tag{2.2}$$

and the sequence $\mathbf{a} = \{a_\alpha \in \mathbb{R}\}_{\alpha\in\mathbb{Z}}$ is the *refinement mask.*

If the subdivision scheme is convergent, then there exists an uniformly continuous *limit function* f_λ, depending on the starting sequence λ, satisfying

$$\lim_{k\to\infty} \sup_{\alpha\in\mathbb{Z}} |\lambda_\alpha^k - f_\lambda(3^{-k}\alpha)| = 0, \tag{2.3}$$

with $f_\lambda \neq 0$ for at least a starting sequence. In the following, we will use the notation

$$f_\lambda = S_{\mathbf{a}}^\infty\,\lambda. \tag{2.4}$$

An equivalent definition of convergence requires the existence of the so-called *basic limit function* $\varphi_{\mathbf{a}}$ as the limit of the subdivision process when applied to the δ sequence, i.e.

$$\varphi_{\mathbf{a}} = S_{\mathbf{a}}^\infty\,\delta. \tag{2.5}$$

In fact, if the ternary subdivision scheme $\mathcal{S}_{\mathbf{a}}$ converges, the basic limit function is *refinable* with dilation 3, i.e. it satisfies the *refinement equation*

$$\varphi_{\mathbf{a}} = \sum_{\alpha\in\mathbb{Z}} a_\alpha\,\varphi_{\mathbf{a}}(3\cdot - \alpha), \tag{2.6}$$

and the limit function can be represented as

$$f_\lambda = \sum_{\alpha\in\mathbb{Z}} \lambda_\alpha\,\varphi_{\mathbf{a}}(\cdot - \alpha). \tag{2.7}$$

Most of the theory of the subdivision schemes consists in deriving the convergence conditions of the subdivision process and the properties of the basic limit function $\varphi_{\mathbf{a}}$ from the mask properties (see, for instance [5–7,12,15] and references therein).

We recall that a necessary condition for a ternary subdivision scheme to be convergent [12] is that its mask satisfies the *basic sum rules*

$$\sum_{\alpha\in\mathbb{Z}} a_{3\alpha+r} = 1, \quad r = 0, 1, 2. \tag{2.8}$$

The sum rules imply that the *mask symbol*, defined as the z-transform

$$a(z) = \sum_{\alpha\in\mathbb{Z}} a_\alpha\,z^\alpha, \tag{2.9}$$

can be factorized as $a(z) = (1 + z + z^2)\, d(z)$, where the *difference symbol*

$$d(z) = \sum_{\alpha\in\mathbb{Z}} d_\alpha\, z^\alpha \tag{2.10}$$

is a Laurent polynomial such that $d(1) = 1$. As in the case of binary subdivision schemes (see, for instance [7]), it can be shown that

$$S_{\mathbf{d}}\, \Delta\lambda = \Delta\,(S_{\mathbf{a}}\lambda), \tag{2.11}$$

where $S_{\mathbf{d}}$ is the subdivision operator associated with the difference mask $\mathbf{d} = \{d_\alpha\}_{\alpha\in\mathbb{Z}}$ and

$$\Delta\lambda = \{\Delta\lambda_\alpha\}_{\alpha\in\mathbb{Z}}, \quad \Delta\lambda_\alpha = \lambda_\alpha - \lambda_{\alpha-1}. \tag{2.12}$$

It is well known that the subdivision scheme $\mathcal{S}_{\mathbf{a}}$ is convergent if the subdivision scheme $\mathcal{S}_{\mathbf{d}}$ is contractive, i.e. $S_{\mathbf{d}}^\infty\lambda = 0$ for any initial sequence [7,12]. Moreover, if $a(z) = (1 + z + z^2)^{m+1}/3^m\, d(z)$, $m \geq 0$, with $d(1) = 1$, and $\mathcal{S}_{\mathbf{d}}$ is contractive, then $\varphi_{\mathbf{a}} \in C^m(\mathbb{R})$ and

$$\varphi_{\mathbf{a}}^{(\ell)} = S_{\mathbf{d}^\ell}^\infty\, \delta, \quad 1 \leq \ell \leq m, \tag{2.13}$$

where $S_{\mathbf{d}^\ell}^\infty$ is the subdivision operator associated with the *ℓ-order divided difference symbol*

$$d^\ell(z) = (1 + z + z^2)^{-\ell}\, 3^\ell\, a(z), \quad 1 \leq \ell \leq m. \tag{2.14}$$

In this paper, we will focus on ternary subdivision schemes generated by *bell-shaped masks*, i.e. compactly supported masks such that

$$\begin{cases} supp\,(\mathbf{a}) = [0, 2(N+1)], & \\ a_\alpha > 0, & 0 \leq \alpha \leq 2(N+1), \\ a_\alpha = a_{2(N+1)-\alpha}, & 0 \leq \alpha \leq 2(N+1), \\ a_\alpha < a_{\alpha+1}, & 0 \leq \alpha \leq N. \end{cases} \tag{2.15}$$

Since a subdivision scheme associated with a positive mask satisfying the basic sum rules converges [8,14], subdivision schemes associated with bell-shaped masks satisfying (2.8) are always convergent. Moreover, the positivity of their mask ensures that the subdivision scheme preserves the sign of the initial sequence so that refinable functions associated with bell-shaped masks are always positive.

Well known examples of bell-shaped masks are the B-spline masks, i.e. the masks associated with subdivision schemes having piecewise polynomials as limit functions. In this case the basic limit function is a cardinal B-spline, i.e. a piecewise polynomial on integer knots (see the monographs [15,18] for basic definitions and main properties of the B-splines and the associated subdivision schemes). The coefficients of the ternary refinement mask $\mathbf{b}_N = \{b_{\alpha,N}\}_{0\leq\alpha\leq 2(N+1)}$, corresponding to the cardinal B-spline of degree N, can be obtained by the equality

$$b_N(z) = \sum_{\alpha=0}^{2(N+1)} b_{\alpha,N}\, z^\alpha = \frac{1}{3^N}(1 + z + z^2)^{N+1}, \tag{2.16}$$

and have the explicit expression [11]

$$b_{\alpha,N} = \frac{1}{3^N} \sum_{\beta=[(\alpha+1)/2]}^{\alpha} \binom{N+1}{\beta} \binom{\beta}{\alpha-\beta}, \quad 0 \leq \alpha \leq 2(N+1). \tag{2.17}$$

For $N = 0$ the mask $\mathbf{b}_0 = \{1, 1, 1\}$, corresponding to the characteristic function of the interval [0, 1], is the unique mask of support [0, 2] satisfying the basic sum rules, so that there are no bell-shaped masks with support [0, 2] satisfying (2.8). For any $N \geq 1$, $\mathbf{b}_N$ is bell-shaped and the symbol $b_N(z)$ satisfies the basic sum rules.

A wide family of ternary bell-shaped masks was introduced in [11]. The corresponding basic limit functions have a behavior similar to that one of the B-splines – compact support, central symmetry, smoothness – but they depend on a shape parameter that gives more flexibility in applications (see Section 5 for some details on these masks).

As the B-spline masks, the masks in [11] generate not only monotonicity preserving but also convexity preserving approximations [11,16], so that they can be used in several applications.

3. Monotonicity preservation

Let $\Delta : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$ be the first difference operator defined as

$$\Delta\lambda = \{(\Delta\lambda)_\alpha\}_{\alpha\in\mathbb{Z}}, \quad (\Delta\lambda)_\alpha = \lambda_\alpha - \lambda_{\alpha-1}. \tag{3.1}$$

Definition 1. A subdivision scheme is said to be monotonicity preserving if it preserves the monotonicity of the starting sequence. This means that at any iteration $k \geq 0$ the difference sequence $\Delta\lambda^k$ is positive (resp. negative) whenever the sequence $\Delta\lambda^0$ is positive (resp. negative).

From (2.11) it follows that

$$\Delta\lambda^{k+1} = \Delta(S_{\mathbf{a}}\lambda^k) = S_{\mathbf{d}}(\Delta\lambda^k), \tag{3.2}$$

where $S_{\mathbf{d}}$ is the subdivision scheme associated with the difference symbol

$$d(z) = \sum_{\alpha\in\mathbb{Z}} d_\alpha z^\alpha = (1+z+z^2)^{-1} a(z). \tag{3.3}$$

As a consequence, $\mathcal{S}_{\mathbf{a}}$ is monotonicity preserving if the difference scheme $\mathcal{S}_{\mathbf{d}}$ is positive, i.e. the mask sequence $\mathbf{d} = \{d_\alpha\}_{\alpha\in\mathbb{Z}}$ is positive.

The following theorem shows that bell-shaped masks satisfying the basic sum rules are monotonicity preserving.

Theorem 1. *Any subdivision scheme associated with a bell-shaped mask satisfying the basic sum rules is monotonicity preserving.*

Proof. Since $\mathbf{a}$ is compactly supported and satisfies (2.8), $d(z)$ in (3.3) is a Laurent polynomial of degree $2N$. Thus, the mask sequence $\mathbf{d}$ is compactly supported with *supp* $\mathbf{d} = [0, 2N]$. From the equality $a(z) = (1+z+z^2)\, d(z)$ it follows

$$a_\alpha = d_\alpha + d_{\alpha-1} + d_{\alpha-2}, \quad 0 \leq \alpha \leq 2(N+1), \tag{3.4}$$

from which, by induction, we get

$$d_{3\alpha+r} = \sum_{\beta=0}^{\alpha} \Delta a_{3\beta+r}, \quad r = 0, 1, 2. \tag{3.5}$$

Now, since $\mathbf{a}$ is bell-shaped, $(\Delta a)_\alpha = a_\alpha - a_{\alpha-1} > 0$ for $0 \leq \alpha \leq N$, so that $d_\alpha > 0$ for $0 \leq \alpha \leq N$. Recalling the symmetry property of the mask coefficients, the equality (3.4) gives

$$a_\alpha = a_{2(N+1)-\alpha} = d_{2N-\alpha+2} + d_{2N-\alpha+1} + d_{2N-\alpha}, \quad 0 \leq \alpha \leq 2(N+1),$$

from which we get

$$d_{2N-\alpha} = a_\alpha - d_{2N-\alpha+2} - d_{2N-\alpha+1}, \quad 0 \leq \alpha \leq 2N.$$

Applying the relation above recursively, we find that for $\alpha = 0, 1, \ldots, N$ $d_{2N-\alpha} = d_\alpha$ and the claim follows. □

Just to give an idea of the behavior of subdivision schemes associated with bell-shaped masks, for $N = 3$ we consider the ternary mask family

$$\mathbf{a}(\gamma, \zeta, \eta) = \{\gamma, \zeta, \eta, 1-\gamma-\eta, 1-2\zeta, 1-\gamma-\eta, \eta, \zeta, \gamma\},$$

whose coefficients satisfy the basic sum rules (2.8). We assume

$$0 < \gamma < \zeta < \eta < 1-\gamma-\eta < 1-2\zeta$$

in order $\mathbf{a}(\gamma, \zeta, \eta)$ be bell-shaped and the subdivision scheme $\mathcal{S}_{\mathbf{a}(\gamma,\zeta,\eta)}$ be convergent and monotonicity preserving.

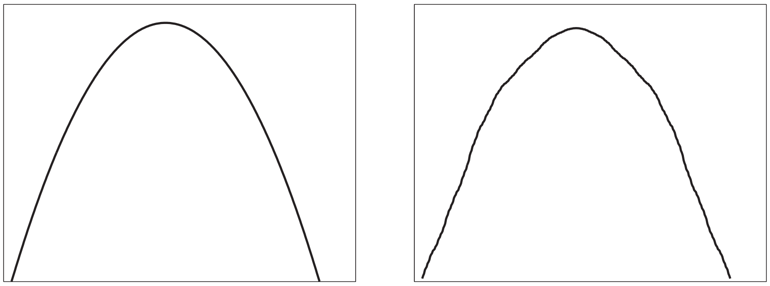

Fig. 1. On the left, the sequence λ^k after 4 iterations of $\mathcal{S}_{\mathbf{b}_3}$. On the right, the sequence λ^k after 4 iterations of $\mathcal{S}_{\tilde{\mathbf{a}}}$. The starting sequence is $\lambda = \{0, 5, 8, 9, 8, 5, 0\}$.

When $\gamma = 1/27$, $\beta = 4/27$, $\eta = 10/27$, the mask

$$\mathbf{a}\left(\frac{1}{27}, \frac{4}{27}, \frac{10}{27}\right) = \left\{\frac{1}{27}, \frac{4}{27}, \frac{10}{27}, \frac{16}{27}, \frac{19}{27}, \frac{16}{27}, \frac{10}{27}, \frac{4}{27}, \frac{1}{27}\right\} \equiv \mathbf{b}_3$$

is the ternary mask of the cardinal B-spline of degree 3. When $\gamma = 1/8$, $\beta = 3/16$, $\eta = 3/8$, we get the mask

$$\mathbf{a}\left(\frac{1}{8}, \frac{3}{16}, \frac{3}{8}\right) = \left\{\frac{1}{8}, \frac{3}{16}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{1}{2}, \frac{3}{8}, \frac{3}{16}, \frac{1}{8}\right\} \equiv \tilde{\mathbf{a}}.$$

In Fig. 1 the sequences λ^k obtained after 4 iterations of the subdivision schemes $\mathcal{S}_{\mathbf{b}_3}$ and $\mathcal{S}_{\tilde{\mathbf{a}}}$, when applied to the convex starting sequence $\lambda = \{0, 5, 8, 9, 8, 5, 0\}$, are displayed.

4. Convexity preservation

Let $\Delta^2 : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$ be the second difference operator defined as

$$\Delta^2\lambda = \Delta(\Delta\lambda), \tag{4.1}$$

so that

$$\Delta^2\lambda = \{\Delta^2\lambda_\alpha\}_{\alpha\in\mathbb{Z}}, \quad \Delta^2\lambda_\alpha = \Delta(\Delta\lambda_\alpha) = \lambda_\alpha - 2\lambda_{\alpha-1} + \lambda_{\alpha-2}. \tag{4.2}$$

Definition 2. A subdivision scheme is said to be convexity preserving if it preserves the convexity of the starting sequence. This means that at any iteration $k \geq 0$ the difference sequence $\Delta^2\lambda^k$ is positive (resp. negative) whenever the sequence $\Delta^2\lambda^0$ is positive (resp. negative).

Applying the difference operator Δ to (3.2) we get

$$\Delta^2\lambda^{k+1} = \Delta^2(S_{\mathbf{a}}\lambda^k) = S_{\mathbf{c}}(\Delta^2\lambda^k), \tag{4.3}$$

where $S_{\mathbf{c}}$ is the subdivision scheme associated with the second order difference symbol

$$c(z) = (1 + z + z^2)^{-2}\, 3\, a(z). \tag{4.4}$$

Hence, the subdivision scheme $S_{\mathbf{a}}$ is convexity preserving if the second order difference scheme $\mathcal{S}_{\mathbf{c}}$ is positive, i.e. the mask sequence $\mathbf{c} = \{c_\alpha\}_{\alpha\in\mathbb{Z}}$ is positive.

Now, Fig. 1 shows that the B-spline subdivision scheme $\mathcal{S}_{\mathbf{b}_3}$ preserves the convexity while the scheme $\mathcal{S}_{\tilde{\mathbf{a}}}$ does not. This example shows that the assumptions on the mask in Theorem 1 are not sufficient to guarantee the convexity preservation and additional conditions are required. Actually, convexity preservation is ensured if also the divided difference mask $\mathbf{d}^1$ is bell-shaped, since in this case the same argument used in the proof of Theorem 1 can be used to show that the mask $\mathbf{c}$ is positive, too.

In fact, when $a(z)$ is bell-shaped and has a factor $(1 + z + z^2)^2$, then $d^1(z)$ is bell-shaped and satisfies the basic sum rules. Thus,

$$c(z) = (1 + z + z^2)^{-1}\, d^1(z) \tag{4.5}$$

Table 1
The coefficients of the first divided difference masks corresponding to $\mathbf{b}_3$ (first row) and $\widetilde{\mathbf{a}}$ (second row).

α	0	1	2	3	4	5	6
d^1_α for $\mathbf{b}_3$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$
d^1_α for $\widetilde{\mathbf{a}}$	$\frac{3}{8}$	$\frac{3}{16}$	$\frac{9}{16}$	$\frac{3}{4}$	$\frac{9}{16}$	$\frac{3}{16}$	$\frac{3}{8}$

is a Laurent polynomial of degree $2(N-1)$ and, reasoning as in the previous section, we have

$$d^1_\alpha = c_\alpha + c_{\alpha-1} + c_{\alpha-2}, \quad 0 \le \alpha \le 2N, \tag{4.6}$$

from which we get

$$c_{3\alpha+r} = \sum_{\beta=0}^{\alpha} \Delta d^1_{3\beta+r}, \quad r = 0, 1, 2. \tag{4.7}$$

Due to the bell-shape property of $\mathbf{d}^1$, the mask $\mathbf{c}$ is symmetric and positive and the following theorem holds.

Theorem 2. *Any subdivision scheme associated with a bell-shaped mask* $\mathbf{a}$ *such that its symbol* $a(z)$ *has a factor* $(1+z+z^2)^2$ *is convexity preserving.*

We notice that in this case the limit curve is at least C^2 (see (2.13)).

Theorem 2 allows us to conclude that the subdivision scheme $\mathcal{S}_{\mathbf{b}_3}$ is convexity preserving while the subdivision scheme $\mathcal{S}_{\widetilde{\mathbf{a}}}$ is not. In fact, the mask corresponding to the first order divided difference symbol $(1+z+z^2)^{-1}\, 3\, b_3(z)$ has bell-shaped coefficients (cf. (2.16)), while the mask of the first order divided difference symbol $(1+z+z^2)^{-1}\, 3\, \tilde{a}(z)$, where $\tilde{a}(z)$ is the symbol associated with the mask $\tilde{\mathbf{a}}$, neither is bell-shaped nor satisfies (2.8). The values of both the difference masks are listed in Table 1.

5. Higher order shape-preserving subdivision schemes

As a first example of ternary shape preserving subdivision schemes we consider the approximating schemes introduced in [19,20]. The scheme in [20] generates C^2 limit functions and is associated with the mask

$$\tilde{\mathbf{a}}_2 = \{\tilde{a}_\alpha\}_{0\le\alpha\le 8} = \left\{ \frac{1}{72}, \frac{1}{8}, \frac{25}{72}, \frac{23}{36}, \frac{3}{4}, \frac{23}{36}, \frac{25}{72}, \frac{1}{8}, \frac{1}{72} \right\}. \tag{5.1}$$

The scheme in [19] generates C^4 limit functions and is associated with the mask

$$\tilde{\mathbf{a}}_4 = \{\tilde{a}_\alpha\}_{0\le\alpha\le 14} = \left\{ \frac{1}{31104}, \frac{1}{384}, \frac{625}{31104}, \frac{599}{7776}, \frac{19}{96}, \frac{2879}{7776}, \frac{2761}{5184}, \frac{115}{192}, \frac{2761}{5184}, \frac{2879}{7776}, \frac{19}{96}, \frac{599}{7776}, \frac{625}{31104}, \frac{1}{384}, \frac{1}{31104} \right\}. \tag{5.2}$$

Both masks are bell-shaped but $\tilde{\mathbf{a}}_2$ has just bell-shaped first and second order divided difference masks, while $\tilde{\mathbf{a}}_4$ has bell-shaped difference masks till order forth. As a consequence, $S_{\tilde{\mathbf{a}}_2}$ is monotonicity and convexity preserving, while $S_{\tilde{\mathbf{a}}_4}$ preserves differences till order forth.

A large class of shape-preserving subdivision schemes of higher order are the schemes generated by the class of refinement masks introduced in [11] (see also [16]). They are bell-shaped compactly supported masks whose coefficients are given by

$$\begin{aligned} &\mathbf{a}_{\gamma,N} = \{a_{\alpha,\gamma,N}\}_{0\le\alpha\le 2(N+1)}, \\ &a_{\alpha,N,\gamma} = \frac{\gamma}{3}\, b_{\alpha,N-1} + \frac{1}{3}(3-2\gamma) b_{\alpha-1,N-1} + \frac{\gamma}{3}\, b_{\alpha-2,N-1}, \end{aligned} \tag{5.3}$$

where $0 \le \gamma \le 3/2$ is a shape parameter.

For any admissible γ the subdivision scheme $S_{\mathbf{a}_{\gamma,N}}$ converges and generates C^{N-1} limit function (see [11] for details).

Since the symbol $a_{\gamma,N}(z) = \sum_{\alpha=0}^{2(N+1)} a_{\alpha,N,\gamma}\, z^\alpha$ is given by

$$a_{\gamma,N}(z) = \frac{1}{3^N}\,(1+z+z^2)^N\,(\gamma z^2 + (3-2\gamma)z + \gamma), \tag{5.4}$$

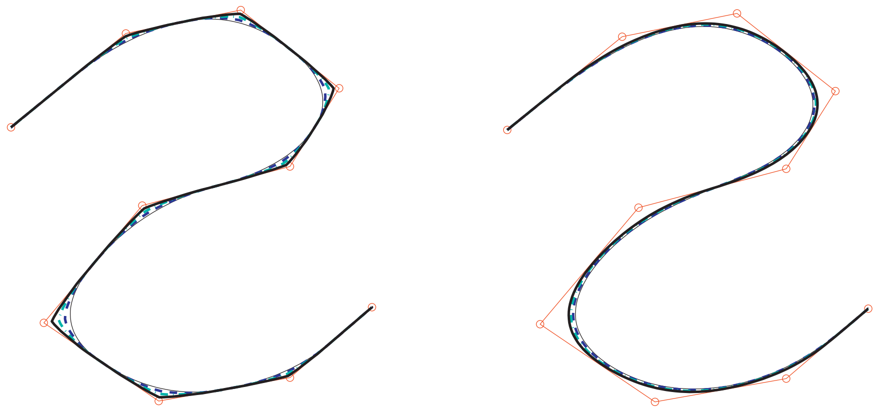

Fig. 2. The C^1 limit curves obtained by $\mathcal{S}_{\mathbf{a}_{\gamma,2}}$ (left) and the C^2 limit curves obtained by $\mathcal{S}_{\mathbf{a}_{\gamma,3}}$ (right) for $\gamma = 0.75$ (blue dashed line), $\gamma = 0.5$ (cyan dash-dotted line) and $\gamma = 0.25$ (black solid line). The B-spline curves generated by $\mathcal{S}_{\mathbf{b}_2}$ and $\mathcal{S}_{\mathbf{b}_3}$ are also displayed (black thin line). The control polygon (red thin line) and the control points (red circles) are also displayed. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

the divided difference masks having symbol $d^{\ell}_{\gamma,N}(z) = (1 + z + z^2)^{-\ell}\, 3^{\ell}\, a_{\gamma,N}(z)$, $1 \leq \ell \leq N-1$, are all bell-shaped. Thus, the subdivision scheme $S_{\mathbf{a}_{\gamma,N}}$ has shape-preserving properties of order $N-1$, i.e. $\mathcal{S}_{\mathbf{a}_{\gamma,N}}$ preserves the sign of the ℓ-order difference $\Delta^{\ell}\lambda$ for $1 \leq \ell \leq N-1$, where $\Delta^{\ell}\lambda = \Delta(\Delta^{\ell-1}\lambda)$. The subdivision scheme $\mathcal{S}_{\mathbf{a}_{\gamma,N}}$ has *optimal shape-preserving properties* in the sense that it preserves the highest – w.r.t. to the mask support – possible order differences.

We notice that the mask family (5.3) contains both $\mathbf{b}_N$ and $\tilde{\mathbf{a}}_2$. In fact, it easy to show that for any N $\mathbf{b}_N \equiv \mathbf{a}_{1,N}$ while the mask (5.1) coincides with $\mathbf{a}_{\gamma,3}$ when $\gamma = 3/2^3$. Interestingly, the mask (5.2) has approximatively the same values of $\mathbf{a}_{\gamma,6}$ when $\gamma = 3/2^7$, being the difference in the order of 3%. However, the mask $\mathbf{a}_{\gamma,6}$ generates smoother curves, since the corresponding subdivision schemes is C^5, and preserves the sign of the differences till order fifth.

To show how the shape parameter affects the shape of the limit curve, in Fig. 2 the limit curves obtained by $\mathcal{S}_{\mathbf{a}_{\gamma,2}}$ and by $\mathcal{S}_{\mathbf{a}_{\gamma,3}}$, when the starting sequence is an open polygon, are displayed for different values of the parameter γ. For comparison the spline curves of degree 2 and 3 are also displayed. A more realistic example is shown in Fig. 3 (see also the details in Fig. 4). The pictures show that acting both on the smoothness and on the shape parameter we can generate either limit curves very close to the control polygon or limit curves cutting the edges smoothly.

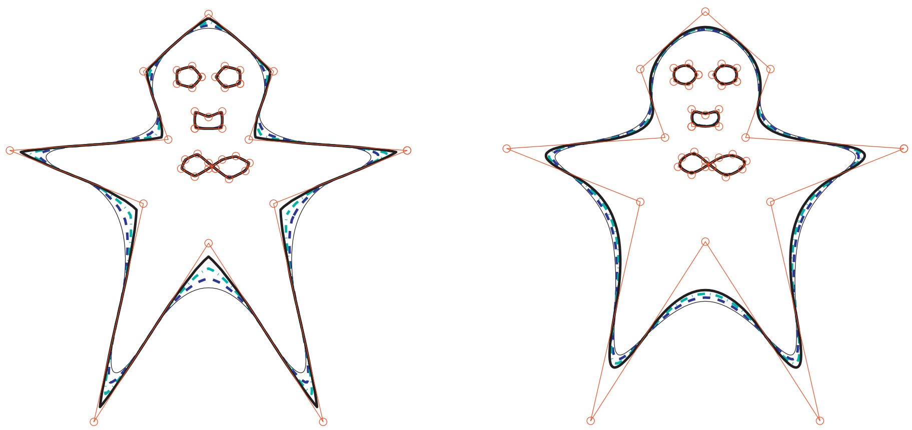

Fig. 3. The C^1 limit curves obtained by $\mathcal{S}_{\mathbf{a}_{\gamma,2}}$ (left) and the C^2 limit curves obtained by $\mathcal{S}_{\mathbf{a}_{\gamma,3}}$ (right) for $\gamma = 0.75$ (blue dashed line), $\gamma = 0.5$ (cyan dash-dotted line) and $\gamma = 0.25$ (black solid line). The B-spline curve generated by $\mathcal{S}_{\mathbf{b}_2}$ (thin line), the control polygon (red thin line) and the control points (red circles) are also displayed. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

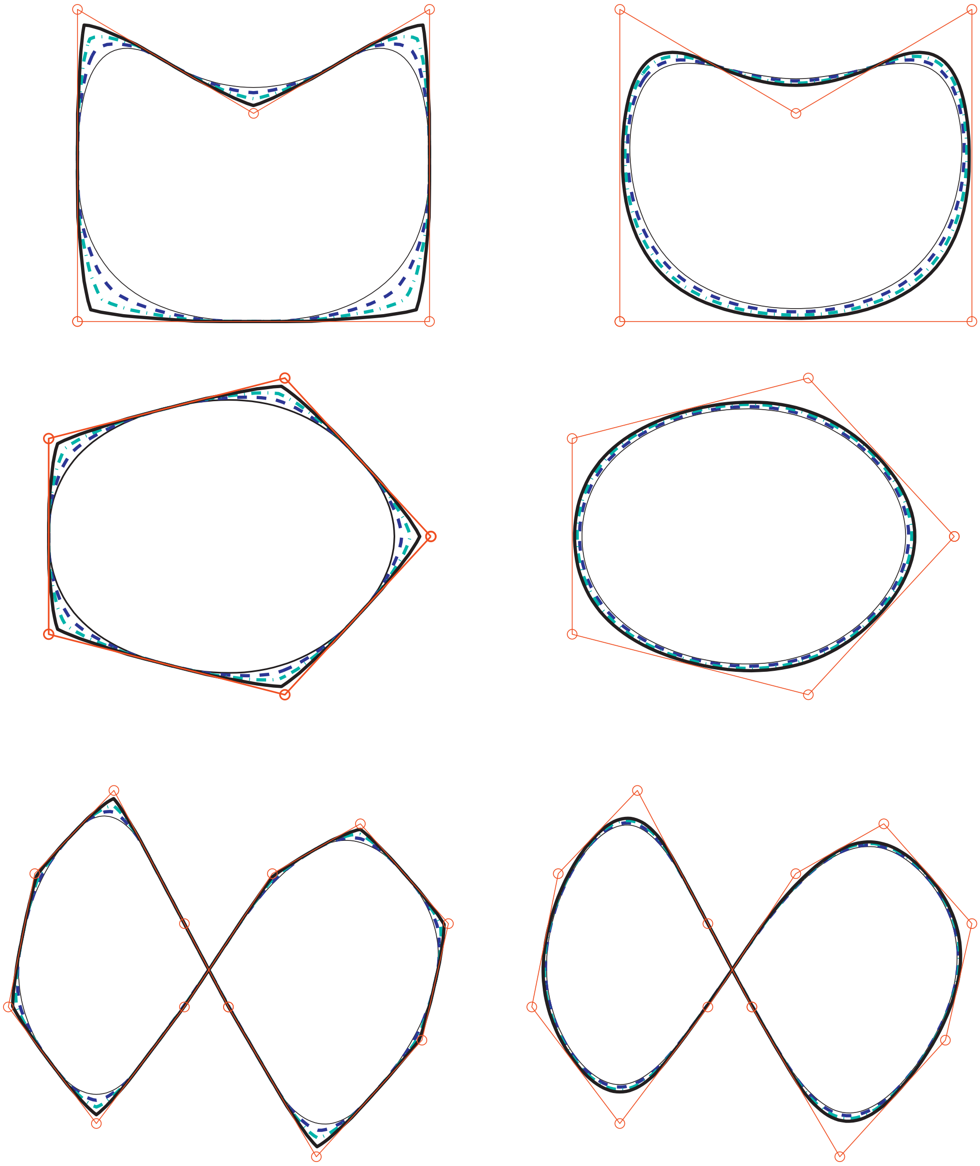

Fig. 4. Details of Fig. 3: the C^1 limit curves obtained by $\mathcal{S}_{\mathbf{a}_{\gamma,2}}$ (left) and the C^2 limit curves obtained by $\mathcal{S}_{\mathbf{a}_{\gamma,3}}$ (right) for $\gamma = 0.75$ (blue dashed line), $\gamma = 0.5$ (cyan dash-dotted line) and $\gamma = 0.25$ (black solid line). The B-spline curve generated by $\mathcal{S}_{\mathbf{b}_2}$ (thin line), the control polygon (red thin line) and the control points (red circles) are also displayed. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

Finally, we consider the ternary 4-point interpolatory schemes introduced in [13], i.e. the linear scheme

$$\begin{cases} \lambda_{3\alpha}^{k+1} = \lambda_{\alpha}^{k}, \\ \lambda_{3\alpha+1}^{k+1} = \dfrac{-1-3\mu}{18}\,\lambda_{\alpha-1}^{k} + \dfrac{13+9\mu}{18}\,\lambda_{\alpha}^{k} + \dfrac{7-9\mu}{18}\,\lambda_{\alpha+1}^{k} + \dfrac{-1+3\mu}{18}\,\lambda_{\alpha+2}^{k}, \\ \lambda_{3\alpha+2}^{k+1} = \dfrac{-1+3\mu}{18}\,\lambda_{\alpha-1}^{k} + \dfrac{7-9\mu}{18}\,\lambda_{\alpha}^{k} + \dfrac{13+9\mu}{18}\,\lambda_{\alpha+1}^{k} + \dfrac{-1-3\mu}{18}\,\lambda_{\alpha+2}^{k}. \end{cases} \tag{5.5}$$

The linear 4-point scheme is C^2 and convexity preserving for suitable values of the parameter μ [4]. In Fig. 5 we compare the behavior of the C^1 scheme $\mathcal{S}_{\mathbf{a}_{\gamma,2}}$ with $\gamma = 0.25$, the C^2 scheme $\mathcal{S}_{\mathbf{a}_{\gamma,3}}$ with $\gamma = 0.25$ and the schemes (5.5)

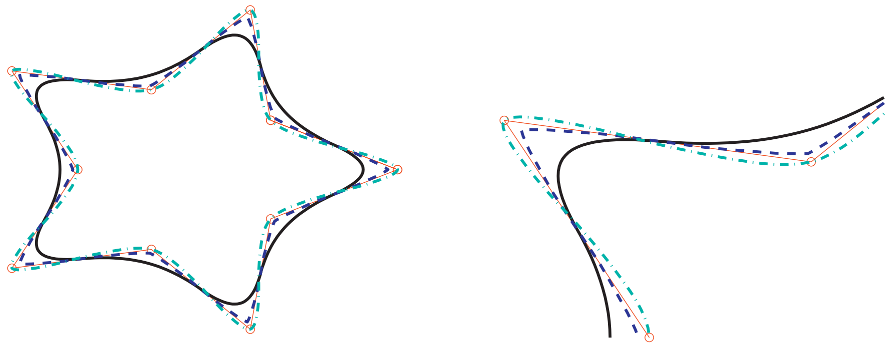

Fig. 5. The C^2 limit curve obtained by $\mathcal{S}_{\mathbf{a}_{\gamma,3}}$ (black solid line) and the C^1 limit curve obtained by $\mathcal{S}_{\mathbf{a}_{\gamma,2}}$ (blue dashed line) for $\gamma = 0.25$ in comparison with the ternary interpolatory 4-point scheme (cyan dash-dotted line). A detail of the curves is displayed on the right. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

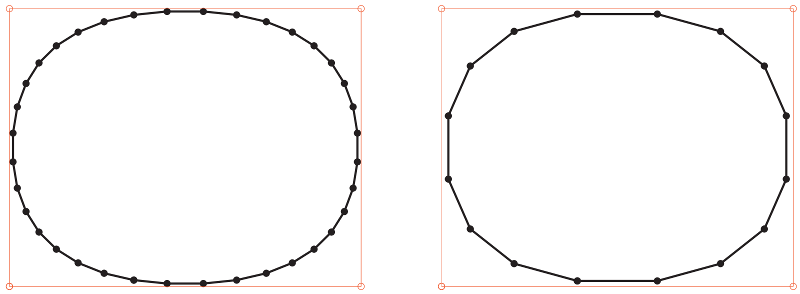

Fig. 6. On the left, the sequence λ^2 (black points) obtained after two iterations of the ternary scheme $\mathcal{S}_{\mathbf{a}_{\gamma,3}}$ with $\gamma = 0.25$. On the right, the sequence λ^2 (black points) obtained after two iterations of the binary analogue (cf. [10]). The control polygon (thin red line) and the control points (red circle) are also displayed. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

with $\mu = 7/81$. The picture shows that, even if the scheme $\mathcal{S}_{\mathbf{a}_{\gamma,2}}$ is not interpolatory, it generates limit curve closer to the control polygon than the limit curve generates by the 4-point interpolatory scheme.

6. Conclusions

We studied the shape-preserving properties of ternary approximating subdivision schemes associated with bell-shaped masks. We showed that bell-shaped masks always preserve the monotonicity of the starting sequence while to preserve convexity the first order divided difference mask have to be bell-shaped, too. Higher order shape-preserving properties can be achieved when the difference masks of higher order are also bell-shaped.

The examples in Section 5 show that there are many advantages in the use of ternary subdivisions. First of all it is possible to construct schemes with higher smoothness with respect to their binary counterpart. For instance, bell-shaped masks obtained by linear combinations of binary B-spline masks of different degree cannot give rise to subdivision schemes with the same smoothness as the B-spline schemes (see [10,11] for a discussion). Instead, the ternary schemes $\mathcal{S}_{\mathbf{a}_{\gamma,N}}$ have the same smoothness as the B-spline schemes and have optimal shape-preserving properties. Also in the case of the interpolatory 4-point scheme, the ternary 4-point scheme (5.5) is C^2 while the well-known binary 4-point scheme is C^1 (cf. [7]).

Moreover, ternary subdivision has a fast convergence rate since at any step the number of points in the sequence λ^k is multiplied by three. The effect of the ternary process can be seen in Fig. 6 where two iterations of the subdivision scheme $\mathcal{S}_{\mathbf{a}_{\gamma,3}}$ for $\gamma = 0.25$ are compared with two iterations of an analogous binary subdivision scheme, i.e. a binary 3-point approximating scheme generating C^2 limit functions (cf. [10] for details). The figure shows that after just two iterations the ternary scheme generates a curve that is pleasant to see so reducing the computational cost of the overall algorithm (cf. [13]). Thus, ternary subdivisions can have interesting applications in generating real world images that usually have a high number of control points in the starting sequence.

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