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POLYTOPES CLOSE TO BEING SIMPLE

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Abstract. It is known that polytopes with at most two nonsimple vertices are reconstructible from their graphs, and that d-polytopes with at most $d-2$ nonsimple vertices are reconstructible from their 2-skeletons. Here we close the gap between 2 and $d-2$, showing that certain polytopes with more than two nonsimple vertices are reconstructible from their graphs. In particular, we prove that reconstructibility from graphs also holds for d-polytopes with $d+k$ vertices and at most $d-k+3$ nonsimple vertices, provided $k \geqslant 5$. For $k \leqslant 4$, the same conclusion holds under a slightly stronger assumption.

Another measure of deviation from simplicity is the *excess degree* of a polytope, defined as $\xi(P) := 2f_1 - df_0$, where f_k denotes the number of k-dimensional faces of the polytope. Simple polytopes are those with excess zero. We prove that polytopes with excess at most $d-1$ are reconstructible from their graphs, and this is best possible. An interesting intermediate result is that d-polytopes with less than $2d$ vertices, and at most $d-1$ nonsimple vertices, are necessarily pyramids.

1. Introduction and summary

The k-dimensional *skeleton* of a polytope P is the set of all its faces of dimension $\leqslant k$. The 1-skeleton of P is the *graph* $G(P)$ of P. Reconstructing a polytope from its k-skeleton amounts to giving the combinatorial structure of the polytope (i.e the lattice of its faces, ordered by inclusion) solely by querying the k-skeleton. It however suffices to reconstruct the facets of P, since the combinatorial structure of a polytope is determined by the vertex-facet incidence graph, where a facet is adjacent to a vertex if and only if, it contains the vertex [7, Sec. 16.1.1]. Throughout the paper, we let d denote the dimension of P, deg denote the *degree* of a vertex, i.e. the number of edges incident to the vertex in the polytope P, and $V(P)$ and $E(P)$ denote the vertex and edge set of a polytope P, respectively.

Every d-polytope is reconstructible from its $(d-2)$-skeleton [9, Thm. 12.3.1], and there are combinatorially inequivalent d-polytopes with the same $(d-3)$-skeleton: take, for instance, a bipyramid over a $(d-1)$-simplex and a pyramid over a bipyramid over a $(d-2)$-simplex. For some classes of polytopes, the graph somewhat surprisingly determines the combinatorial structure of the polytope: polytopes with dimension at most three and simple polytopes [2, 11]. A d-polytope is called *simple*

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if every vertex is simple. A vertex is called *simple* if its degree is exactly d; otherwise it is called *nonsimple*. Equivalently, a vertex is simple if it is contained in exactly d facets, and nonsimple otherwise.

Define the *excess degree* $\xi(u)$ of a vertex u in a d-polytope as $\deg u - d$. Then the *excess* ξ of a d-polytope is defined as the sum of the excess of all its vertices; i.e.

$$\xi(P) := \sum_{u \in \operatorname{vert} P} (\deg u - d).$$

Simple polytopes have excess zero. Polytopes with small excess are a natural generalisation of simple polytopes. The excess degree is studied in detail in [13]. For the early sections of this paper, we will just need the following basic but surprisingly useful result [13, Lem. 2.4, 2.5 and 2.6(i)].

Lemma 1. *Let P be a d-polytope, F a facet, and let v be a vertex in F.*

(i) Suppose v is a simple vertex in F, but is not simple in P. Then there is facet J containing v whose intersection with F is not a ridge.

(ii) Let J be any facet which is distinct from F, such that $F \cap J$ is not a ridge. Then every vertex in $F \cap J$ is nonsimple in P.

(iii) Suppose v is nonsimple in P, and adjacent to a simple vertex w of P in $P \setminus F$. Then v must be adjacent to another vertex in $P \setminus F$, other than w.

It was shown in [5] that polytopes with at most two nonsimple vertices are reconstructible from their graphs. Since it will be used several times, we state this explicitly here.

Theorem 2. [5, Thms. 4.5 and 4.8] *Every polytope with only one or two nonsimple vertices can be reconstructed from its graph.*

On the other hand [5] also exhibited a pair of inequivalent 4-polytopes Q_4^1 and Q_4^2 with eight vertices, three of them nonsimple in each case, and with the same graph. In particular, they are not reconstructible from their common graph, and Theorem 2 does not extend to polytopes with more than two nonsimple vertices. Their construction is described in detail in [5, Sec. 2]; here we simply illustrate them (Fig. 1 (b-c)). There are eight facets in Q_4^1 but only seven in Q_4^2. One of the facets of Q_4^2 is a bipyramid over a simplex, namely 02467; its missing 2-face 246 is highlighted in Fig. 1(c). This bipyramid is split into two simplices to form Q_4^1 (Fig. 1(b)). In fact, this construction extends to higher dimensions; we summarise further information about it in Remark 3 and refer to [5, Prop. 2.2] for the details. Other reconstruction results can be found in [7, Sec. 20.5].

Remark 3. For $d \geqslant 4$, there are inequivalent d-polytopes Q_d^1 and Q_d^2, each with $2d$ vertices, exactly $d-1$ of them nonsimple, and the same $(d-3)$-skeleton. The polytope Q_d^1 contains exactly $2d$ facets, while the polytope Q_d^2 contains $2d-1$ facets. The polytope Q_d^2 is obtained from Q_d^1 by gluing two simplex facets of Q_d^1 along a common ridge R to create the bipyramid over R, which becomes a facet of Q_d^2.

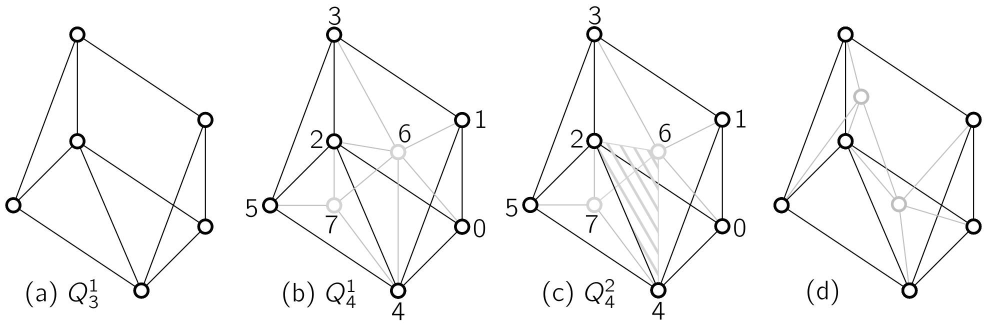

FIGURE 1. Examples of d-polytopes with $2d$ vertices, of which precisely $d-1$ are nonsimple. (a) The polytope Q_3^1. (b-c) The pair of 4-polytopes Q_4^1 and Q_4^2. The missing 2-face of the bipyramid face 02467 in Q_4^2 is highlighted. (d) Polytope obtained from a simplicial 4-prism by "pulling" a vertex (cf. [9, Sec. 5.2]) along one of the edges of a simplex facet.

For $d = 3$, the construction in [5] leads to two inequivalent 3-polytopes Q_3^1 and Q_3^2, each with six vertices, and hence the same 0-skeleton. The polytope Q_3^1 (Fig. 1(a)), sometimes called the tetragonal antiwedge, contains exactly six facets and two nonsimple vertices, but the polytope Q_3^2, the triangular prism, contains only five facets and is actually simple.

Here we keep studying the structure and reconstruction of polytopes which are "close" to being simple. We consider two approaches which generalise the concept of simplicity and guarantee reconstructibility from graphs.

Approach 1: Consider d-polytopes with "few" nonsimple vertices; this is the approach taken in [5].

Approach 2: Consider d-polytopes with small excess.

Concerning Approach 1, the polytopes Q_d^1 and Q_d^2 (cf. Remark 3) show that in general by "few", we must mean at most $d-2$ nonsimple vertices. With regard to Approach 2, the aforementioned pair of a bipyramid over a $(d-1)$-simplex and a pyramid over a bipyramid over a $(d-2)$-simplex have excess exactly d. So by small excess we must mean excess at most $d-1$. But then the excess theorem [13, Thm. 3.4] states that the smallest values of the excess of a d-polytope are 0 and $d-2$.

The main results of this paper are summarised next. We want to highlight that some of our results here came about after the authors tested a number of hypotheses on `polymake` [8].

In §3, we apply Approach 1. A polytope with only $d+1$ vertices is obviously a simplex. The structure of polytopes with $d+2$ vertices is well understood [9, Sec. 6.1]. They have either $d-2, d$ or $d+2$ nonsimple vertices. If such a polytope has only $d-2$ nonsimple vertices, it must be a $(d-2)$-fold pyramid over a quadrilateral, and so is reconstructible from its graph. We have already noted distinct examples with d nonsimple vertices but the same graph.

Facet	Polytope 1	Polytope 2	Polytope 3	Polytope 4
0:	{2 3 4 5 6}	{2 3 4 5 6}	{1 2 3 4 5 6}	{2 3 4 5 6}
1:	{1 3 4 5 6}	{1 3 4 5 6}	{0 3 4 5 6}	{1 3 4 5 6}
2:	{1 2 5 6}	{0 1 2 5 6}	{0 2 5 6}	{0 1 2 5 6}
3:	{1 2 4 6}	{1 2 4 6}	{0 2 4 6}	{0 1 2 3 4}
4:	{1 2 3 5}	{0 2 3 5}	{0 2 3 5}	{1 2 4 6}
5:	{0 2 3 4}	{0 1 3 5}	{0 1 3 4}	{0 2 3 5}
6:	{0 1 3 4}	{1 2 3 4}	{0 1 2 4}	{0 1 3 5}
7:	{0 1 2 4}	{0 1 2 3}	{0 1 2 3}	
8:	{0 1 2 3}			

Table 1. Vertex-facet incidences of all nonpyramidal 4-polytopes with seven vertices and four nonsimple vertices. They all have the same graph, and degree sequence $(4,4,4,5,5,6,6)$. They can be obtained from the catalogues in [6, 9].

For d-polytopes with $d+3$ vertices, we have a new result: if such a polytope has at most $d-1$ nonsimple vertices, then it is a $(d-3)$-fold pyramid over one of just three 3-dimensional examples, and consequently, the graph determines its entire combinatorial structure. This is best possible in the sense that there are nonpyramidal d-polytopes with $d+3$ vertices and exactly d nonsimple vertices which are not reconstructible from their graphs (cf. Table 1).

For a d-polytope P with $d+4$ vertices, the slightly stronger assumption that P has at most $d-2$ nonsimple vertices, is enough to ensure that the graph of P determines its entire combinatorial structure. Furthermore, in the case of P having $d-1$ nonsimple vertices, the polytope is still reconstructible from its 2-skeleton. Again these results are best possible, as shown by the examples Q_4^1 and Q_4^2.

Then for $k \geqslant 5$, any d-polytope with $d+k$ vertices and at most $d-k+3$ nonsimple vertices is determined by its graph of P. In the particular case $k=5$, the pair of polytopes Q_5^1 and Q_5^2 shows that this is best possible.

In view of these three results and the results of [5] we venture to conjecture the following.

Conjecture 4. *Let P be a d-polytope with at most $d-2$ nonsimple vertices. Then the graph of P determines its entire combinatorial structure.*

These results depend on some results about pyramids, which are presented first in §2. Recall a polytope is an *r-fold pyramid* if it is a pyramid whose *basis* is an $(r-1)$-fold pyramid, and any polytope is a 0-fold pyramid. If a vertex u is an apex of a pyramid P, we say that P is *pyramidal* at u. The main conclusion of §2, of interest in its own right, is that a d-polytope with at most $d+k$ vertices and at most $d-1$ nonsimple vertices, is necessarily a $(d-k)$-fold pyramid. This is only informative if $k<d$. For $k=d$, we have the following modification: a d-polytope with $2d$ vertices, and at most $d-2$ nonsimple vertices, is either a simplicial d-prism or a pyramid. Furthermore, this is best possible as there are 4-polytopes with eight vertices and three nonsimple vertices which are neither simplicial 4-prisms

nor pyramids (namely Q_4^1 and Q_4^2). Recall a *simplicial d-prism* is any prism whose base is a $(d-1)$-simplex.

In §4, we pay regard to polytopes with small excess, and completely settle the reconstruction problem for them by proving that for a d-polytope with excess at most $d-1$, the graph determines its entire combinatorial structure. This result is best possible in the sense that there are d-polytopes with excess d which are not reconstructible from their graphs.

2. Polytopes with a small number of nonsimple vertices are Pyramids

Here we show that knowing a polytope has strictly less than d nonsimple vertices gives us a lot of information about its structure. In particular, large classes of such polytopes must be pyramids. This is crucial for the reconstruction results in the next section. Let us call two vertices of a polytope *nonneighbours* if they are not adjacent. Note that every vertex is thus a nonneighbour of itself.

Theorem 5. *Let P be a d-polytope, which contains at most $d-k$ nonsimple vertices, where $k \leqslant d$. Then either P is a pyramid, or each nonsimple vertex has at least k simple nonneighbours.*

In case P is pyramidal, it is, for each j, reconstructible from its j-skeleton if and only if the basis is reconstructible from its j-skeleton.

Proof. The second alternative in the conclusion is trivially true if $k = 0$. If $k = d$, then P is simple, and the second alternative in the conclusion is vacuously true. Henceforth, we assume that $0 < k < d$.

Consider first the case that some nonsimple vertex u is nonadjacent to at most $k-1$ simple vertices.

Removing all the nonsimple vertices and all the simple vertices which are not adjacent to u cannot disconnect the graph, according to Balinski's theorem [1]. Therefore, the graph $G(S)$ induced by the set S of simple vertices which are neighbours of u is connected. Let x be one such simple vertex in S. Then, there is a facet F containing x but not u. Given any simple vertex in $V(F) \cap S$, all its neighbours other than u must also be in F. Since $G(S)$ is connected, F contains all the vertices in S. If some vertex $y \neq u$ is not in this facet, it cannot be a neighbour of any member of S. But outside S there are at most $d-k+k-1$ vertices, including y; this mean that y has degree at most $d-2$ in the polytope. This absurdity implies that every vertex of P is in $F \cup \{u\}$, i.e. P is a pyramid with basis F and apex u.

We only prove the reconstruction statement for graphs (1-skeletons), but the result extends to j-skeletons for $j \geqslant 2$.

So suppose now that P is pyramidal with basis F and apex u. If F is reconstructible from its graph, then we can obtain the vertex set of each $(d-2)$-face R of F, and from it, the corresponding facet of P, the one with vertex set $V(R) \cup \{u\}$. Thus, we can get the vertex-facet incidence graph of P. Otherwise P is not reconstructible. □

There are examples of nonsimplicial pyramidal and simplicial nonpyramidal d-polytopes with exactly d nonsimple vertices and the same graph; look no further than our old friends, the bipyramid over a $(d-1)$-simplex and the pyramid over a bipyramid over a $(d-2)$-simplex. Thus we get the following as corollary of Theorem 5.

Corollary 6. *A d-polytope having at most $d-1$ nonsimple vertices, and a vertex adjacent to every other vertex, must be a pyramid.*

Furthermore, this statement is best possible, as there are pyramidal and non-pyramidal d-polytopes with exactly d nonsimple vertices, at least one of which is adjacent to every other vertex; and in fact with the same graph.

Before proceeding with our results, we need two simple lemmas.

Lemma 7 ([14, Lem. 10(iii)])**.** *Up to combinatorial equivalence, the d-simplex and the simplicial d-prism are the only simple d-polytopes with no more than $2d$ vertices.*

Lemma 8. *Suppose P is a d-polytope with $2d$ or fewer vertices, and that some facet F of P contains only simple vertices. Then P is either a simplicial prism, or a pyramid over F.*

Proof. Recall that the *Minkowski sum* of two polytopes $Q+R$ is defined simply as $\{x+y : x \in Q, y \in R\}$, and that a polytope is called *(Minkowski) decomposable* if it can be written as the Minkowski sum of two polytopes, which are not similar to it. Actually, these definitions are not really important to us now; more important are the following two results about decomposability.

Shephard [15, Thm. (15)] proved that if some facet F of a polytope P contains only simple vertices, then P is either decomposable, or a pyramid over F; see [14, Prop. 5] for another proof. And according to [14, Thm. 9], the only decomposable polytope with $2d$ or fewer vertices is the prism. The lemma follows from combining these two results. □

By an application of Lemma 7, P must actually be a triplex as defined in [12]. We do not need this stronger conclusion here.

Additional assumptions about the total number of vertices now allow us to draw a stronger conclusion.

Theorem 9. *Let P be a d-polytope with fewer than $2d$ vertices, of which at most $d-1$ are nonsimple. Then P is a pyramid.*

Proof. We proceed by induction on d. The base case $d=2$ is easily proved: indeed, P must be a triangle. Now assume that the claim is true for dimensions $2,\dots,d-1$.

If P has a facet with $2d-2$ vertices, there will only be one vertex outside that facet, which ensures that P is a pyramid. So assume that every facet has at most $2d-3$ vertices.

The case of P having a facet in which every vertex is simple is settled by Lemma 8. Henceforth assume also that every facet has at least one nonsimple vertex.

Amongst all the facets of P which omit at least one nonsimple vertex, choose one, say F_1, with a maximum number of vertices. The induction hypothesis ensures that F_1 is a pyramid over some ridge R, say with apex u_1. If R is not a simplex, then u_1 is nonsimple in F_1. If R is a simplex, then so is F_1, and we may choose any nonsimple vertex to be its apex u_1. Then R contains at most $d-3$ nonsimple vertices, because there are at most $d-2$ nonsimple vertices in F_1. In particular, there are simple vertices in R.

Let F_2 be the other facet containing R. Clearly F_2 omits the nonsimple vertex u_1. By the induction hypothesis and the maximality of F_1, F_2 is also a pyramid over R. Let u_2 denote the apex of F_2. Any simple vertex in R has all of its neighbours in $F_1 \cup F_2$. Suppose that there is a vertex z outside $F_1 \cup F_2$. Then removing u_1, u_2 and the nonsimple vertices in R, at most $d-1$ vertices altogether, would disconnect z from the simple vertices in R, violating Balinski's theorem. This ensures that every vertex of P lies in $F_1 \cup F_2$.

Since there are only two vertices outside the ridge R, P is a 2-fold pyramid over R. □

Repeated application gives us the following corollary. The case $k = 2$ is essentially known, following from the characterisation of d-polytopes with $d+2$ vertices [9, Sec. 6.1].

Proposition 10. *Suppose $1 \leqslant k \leqslant d$, and that P is a d-polytope with $d+k$ vertices, of which at most $d-1$ are nonsimple. Then P is a $(d-k)$-fold pyramid over a k-polytope with $2k$ vertices.*

This begs the question of d-polytopes with $2d$ vertices. The next result covers that case, and is in the same spirit as Proposition 10.

Proposition 11. *Let P be a d-polytope with $2d$ vertices and at most $d-2$ nonsimple vertices. Then P is either a simplicial d-prism or a pyramid.*

Furthermore, this is best possible as there are 4-polytopes with eight vertices and three nonsimple vertices which are neither simplicial 4-prisms nor pyramids (namely Polytopes Q_4^1 and Q_4^2).

Proof. The idea used for Theorem 9 also proves this proposition, but we give the full details. As in the proof of Theorem 9, we proceed by induction on d. In the base case $d = 2$ we have that P is a quadrilateral, which is a simplicial prism in two dimensions. Now assume that the claim is true for dimensions $2, 3, \dots, d-1$.

If a facet of P has $2d-1$ vertices, then P is clearly a pyramid. If P is a simple polytope, then it is a simplicial d-prism by Lemma 7. We may henceforth assume that every facet has at most $2(d-1)$ vertices, and that some vertices are not simple.

The case when some facet contains only simple vertices is again taken care of by Lemma 8. This leaves us with the case that every facet has at least one nonsimple vertex.

There are facets which omit at least one nonsimple vertex. Amongst all such facets, choose one, say F_1, with the maximal number of vertices. Then F_1 has at most $2(d-1)$ vertices, of which at most $(d-2)-1$ are nonsimple. Then Theorem 9 and the induction hypothesis together ensure that F_1 is either a prism or a pyramid.

Suppose F_1 is a prism, and denote by v a vertex in F_1 which is not simple in P. Clearly every vertex in F_1 is simple in F_1. Then Lemma 1(i) ensures that there is another facet containing v, say J, which does not intersect F_1 in a ridge. There are only two vertices outside F_1, so J must intersect F_1 in a subridge. But then every vertex in $F_1 \cap J$ will be nonsimple in P, and so $F_1 \cap J$ would contain at least $d-2$ nonsimple vertices. In particular, F_1 contains every nonsimple vertex in P. This being contrary to the hypothesis that F_1 omits a nonsimple vertex, we conclude that F_1 is a pyramid.

We claim that the apex u_1 of this pyramid is, or can be chosen to be, nonsimple in P. If F_1 is a simplex, we choose u_1 to be a nonsimple vertex of P in F_1, and define R as the convex hull of the other vertices. Then R is a ridge, and F_1 is a pyramid over R, whose apex u_1 is nonsimple. If F_1 is not a simplex, we recall that it is a pyramid over some base R, necessarily a ridge, say with apex u_1. Since R will not be a simplex in this case, u_1 is automatically nonsimple in F_1, and thus in P. Let F_2 denote the other facet containing R.

Then F_2 omits the nonsimple vertex u_1, so by maximality, it is also a pyramid over R, say with apex u_2. Consequently, if there were a vertex outside $F_1 \cup F_2$, then removing the vertices u_1, u_2 and the nonsimple vertices in R, at most $d-2$ vertices altogether, would disconnect the graph of P, contradicting Balinski's theorem. Hence there are no vertices outside $F_1 \cup F_2$ and again P is a 2-fold pyramid over R. □

Proposition 11 gives a new proof of Grünbaum's result that there is no 4-polytope with eight vertices and 17 edges; see [9, Thm. 10.4.2, p. 193]. Indeed, such a 4-polytope must have at most two nonsimple vertices, in which case the polytope would be a pyramid. But this is impossible, as the base would have seven vertices and only ten edges.

It seems unlikely that there is any extension of these results to more than $2d$ vertices. The following question might seem to be natural:

> Must every d-polytope with $2d+1$ vertices, of which at most $d-2$ are nonsimple, be either a pentasm or pyramid?

A d-dimensional *pentasm* is the Minkowski sum of a d-simplex and a line segment which is parallel to one triangular face, but not parallel to any edge, of the simplex; or any polytope combinatorially equivalent to it. Pentasms were first defined in

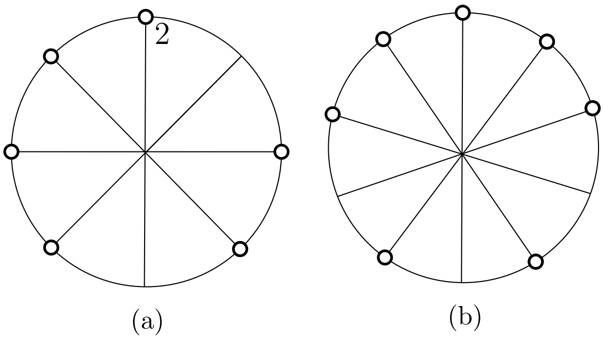

FIGURE 2. Gale diagrams of two nonpyramidal 4-polytopes whose duals have nine vertices, at most two nonsimple vertices and are neither a pentasm or a pyramid.

[12] and studied further in [13]; the graph in Fig. 3(b) is that of a 3-dimensional pentasm.

However even this modest question has a negative answer. A counterexample when $d = 3$ is given by the graph in Fig. 3(a). Two counterexamples in four dimensions are the duals of the polytopes whose Gale diagrams are 10^{th} and 17^{th} in the list [9, Fig. 6.3.4], which are depicted in Fig. 2 in that order.

3. Reconstruction: Polytopes with a small number of vertices

Now we apply the preceding work to obtain structural and reconstruction results for polytopes with less than $2d$ vertices, most of which are simple.

Theorem 12. *Let $k \geqslant 5$ and let P be a d-polytope with $d + k$ vertices, of which at most $d - k + 3$ are nonsimple. Then the graph of P determines the entire combinatorial structure of P. For the particular case of $k = 5$, this conclusion is best possible as shown by the pair of polytopes Q_5^1 and Q_5^2.*

Proof. If $d < k$ then the polytope is reconstructible from its graph by Theorem 2, and the results in [2, 11]. In the case of $d = k$ and P being nonsimple, since $d - k + 3 \leqslant d - 2$, Proposition 11 gives that P is a pyramid, and the reconstruction follows from Theorem 5 and Theorem 2 since the basis of the pyramid would have at most two nonsimple vertices. So assume $d \geqslant k+1$. From Proposition 10 it ensues that P is a $(d - k)$-fold pyramid over some k-polytope Q with $2k$ vertices. Since there are at most $d - k + 3$ nonsimple vertices in P, Q has at most three nonsimple vertices. By Theorem 5 the reconstruction statement now reduces to proving that Q is reconstructible from its graph. By Proposition 11, Q is either a simplicial k-prism, which is clearly reconstructible, or else a pyramid over a $(k - 1)$-polytope with $2k - 1$ vertices, at most two of which are nonsimple, in which case Q is also reconstructible by combining Theorem 5 and Theorem 2. □

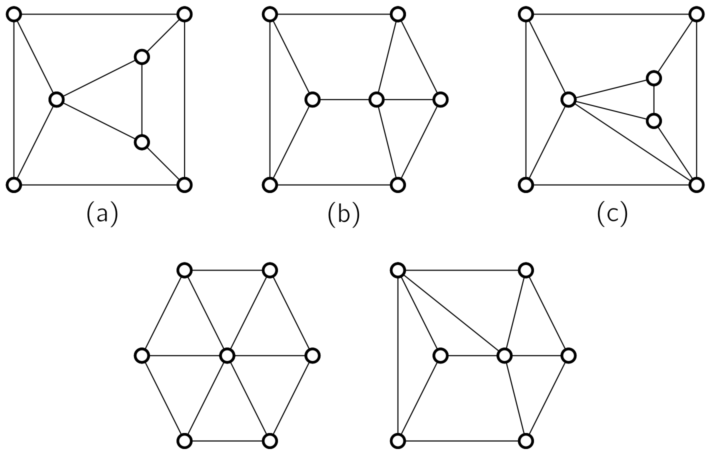

FIGURE 3. Graphs of the 3-polytopes with seven vertices and at most two nonsimple vertices (cf. [3, Fig. 4]).

The preceding theorem is not valid for polytopes with $d + 4$ or fewer vertices. However reconstructibility holds under a slightly stronger hypothesis about the number of nonsimple vertices.

Theorem 13. *Let P be a d-polytope with $d + 4$ vertices, of which at most $d - 1$ are nonsimple. If P has at most $d - 2$ nonsimple vertices, then the graph of P determines its entire combinatorial structure. Furthermore, in the case of P having $d - 1$ nonsimple vertices, the polytope is still reconstructible from its 2-skeleton.*

More precisely, for $d = 3$ there are exactly five such polytopes (see Fig. 3). For every $d \geqslant 4$ there are exactly nine such polytopes, namely a $(d - 4)$-fold pyramid over either a simplicial 4-prism one of the three polytopes in Fig. 1 (b-d), or a $(d - 3)$-fold pyramid over one of the five 3-polytopes in Fig. 3.

These results are best possible, in the sense that there are nonpyramidal 4-polytopes with eight vertices and three nonsimple vertices which are not reconstructible from their graphs, namely the polytopes Q_4^1 and Q_4^2 (cf. Remark 3).

Proof. If $d \leqslant 3$, P is of course reconstructible from its graph. For $d = 3$ the 3-polytopes with seven vertices and at most two nonsimple vertices can be found in [3, Fig. 4] or Fig. 3.

Graph reconstructibility also holds for simple polytopes. So suppose P is not a simple polytope and $d \geqslant 4$. Then by Proposition 10, P is a $(d-4)$-fold pyramid over a 4-polytope Q with eight vertices. By Theorem 5 the reconstruction statement now reduces to proving that Q is reconstructible from its graph, and this follows from Theorem 2 and [2, 11], in case P has at most $d - 2$ nonsimple vertices.

For $d \geqslant 4$, once we have that the polytope is a $(d - 4)$-fold pyramid over a 4-polytope with eight vertices and at most three nonsimple vertices, we first look at the catalogues [6] of 4-polytopes with eight vertices to find those with exactly three nonsimple vertices, all of which are depicted in Fig. 1(b-d). Thanks to Proposition 11, a 4-polytope with eight vertices and at most two nonsimple vertices is either a simplicial 4-prism or a pyramid over a 3-polytope with seven vertices and at most two nonsimple vertices (see Fig. 3); and in all these cases it is reconstructible from its graph.

Finally, note that reconstructing from the 2-skeleton reduces to showing that a 4-polytope with eight vertices and exactly three nonsimple vertices is reconstructible from its 2-skeleton, which is a very special case of [9, Thm. 12.3.1]. □

Finally, we come to the case of $d + 3$ vertices. Our original proof of the next result used Gale diagrams, but the following argument seems to be neater.

Theorem 14. *Let P be a d-polytope with $d + 3$ vertices, of which at most $d - 1$ are nonsimple. Then the polytope is either a $(d - 3)$-fold pyramid over a simplicial 3-prism, a $(d-3)$-fold pyramid over Q_3^1 (Fig. 1(a)), or a $(d-2)$-fold pyramid over a pentagon. As a consequence, the graph of P determines its entire combinatorial structure.*

These results are best possible in the sense that for any $d \geqslant 4$, there are nonpyramidal d-polytopes with $d+3$ vertices and exactly d nonsimple vertices which are not reconstructible from their graphs.

Proof. Let P be a d-polytope with $d + 3$ vertices and at most $d - 1$ nonsimple vertices. If P is simple, then either $d = 3$ and P is a simplicial prism, or $d = 2$ and P is a pentagon. So suppose P is not a simple polytope. Proposition 10 gives that P is a $(d - 3)$-fold pyramid over a 3-polytope, which must be reconstructible from its graph. Hence P is reconstructible by repeated application of Theorem 5.

Once we know that P is a $(d - 3)$-fold pyramid, we can obtain all such polytopes simply by looking for 3-polytopes with six vertices, at most two of which are nonsimple. They are the simplicial 3-prism, Q_3^1 (Fig. 1(a)), and the pyramid over a pentagon, see [3, Fig. 3] or [9, Fig. 6.3.1].

For examples of nonpyramidal 4-polytopes with $d + 3$ vertices and d nonsimple, see Table 1. Constructing multifold pyramids over these gives higher dimensional examples. □

In fact, there are $3d - 8$ distinct combinatorial types of d-polytopes with d nonsimple vertices and three simple vertices, and they have the same graph, namely the complete graph on $d+3$ vertices with a path of length four removed. However, for any d, there is a d-polytope with $d+1$ nonsimple vertices and two simple vertices which is reconstructible from its graph. We will study this in more detail elsewhere.

4. Polytopes with small excess

Recall that the excess ξ of a d-polytope P is $\xi(P) = \sum_{u\in \text{vert}\, P}(\deg u - d)$. Polytopes with small excess $\xi \leqslant d - 1$ were first studied in [13], where the excess theorem was established.

Theorem 15 (Excess theorem, [13, Thm. 3.3])**.** *The smallest values of the excess of a d-polytope are* 0 *and* $d-2$*.*

In this section we show that, like simple polytopes (those with excess zero) [2], all polytopes with small excess are reconstructible from their graphs. It is known that a polytope with dimension at most three is reconstructible from its graph. And there are pairs of d-polytopes with excess d and isomorphic $(d-3)$-skeleta: a bipyramid over a $(d-1)$-simplex and a pyramid over a bipyramid over a $(d-2)$-simplex. So by virtue of the excess theorem, we concentrate on polytopes with excess $d-2$ and $d-1$, for $d \geqslant 4$.

Our capstone result, Theorem 23, asserts that any d-polytope with excess less than d is reconstructible from its graph. Before delving into its proof, we recall some definitions and results from [13, 5, 10].

Lemma 16 (Structure of d-polytopes with excess $d-2$, [13, Lem. 4.8, Thm. 4.10])**.** *Every d-polytope P with excess exactly $d-2$ has either*

(i) a unique nonsimple vertex; or
(ii) exactly $d-2$ nonsimple vertices, each of degree $d+1$ in P, which form a simplex $(d-3)$-face K.
(iii) In the latter case, every facet in P intersecting K, but not containing it, misses exactly one vertex of K and every vertex of K in the facet has degree d.

Lemma 17 (Structure of d-polytopes with excess $d-1$, [13, Thm. 4.18])**.** *Let P be d-polytope with excess degree $d-1$, where $d > 3$. Then $d = 5$ and either*

(i) there is a single vertex with degree nine; or
(ii) there are two vertices with degree seven; or
(iii) there are four vertices each with degree six, which form a quadrilateral 2-face Q which is the intersection of two facets. Furthermore, every facet in P intersecting Q but not containing it intersects Q at an edge, and every vertex of Q in such a facet has degree five.

Proof. Items (i), (ii), and the first sentence of (iii) are restatements of [13, Thm. 4.18]. Here we prove the second part of (iii).

Recall from Lemma 1(ii) that if two facets of P do not intersect in a ridge, then every vertex in their intersection is nonsimple; and if this intersection is either a vertex or an edge, then every vertex in their intersection has degree at least seven. So in case (iii), every pair of facets intersects in either a ridge, Q, or the empty set.

Let F_1 and F_2 be two facets whose intersection is Q. Fix a vertex $u \in Q$. Let F be a facet containing u but missing some neighbour v of u in Q. The intersection of F and F_i must be a ridge for each i. Thus, all the neighbours of u in P except v must be in F: there are two such facets.

Furthermore, any facet containing u and its two neighbours in Q must contain Q. This completes the proof of the lemma. □

Our methodology to establish the reconstruction of polytopes with small excess relies on a result of Joswig [10], which in turn builds on Kalai's idea to prove the reconstructibility of simple polytopes; see [11].

Define a *k-frame* as a subgraph of $G(P)$ isomorphic to the star $K_{1,k}$, where the vertex of degree k is called the *root* of the frame. If the root of a frame is a simple vertex, we say that the frame is *simple*. We say that a k-frame with root x is *valid* if there is a facet containing x and all the edges of the frame. If x is a simple vertex, each of its $(d-1)$-frames is valid.

Lemma 18 ([10, Thm. 2.3])**.** *A polytope can be reconstructed from its graph if the valid frames of each vertex are known.*

Call an acyclic orientation of the graph $G(P)$ of a polytope P *good* if for every nonempty face F of P the graph $G(F)$ of F has a unique sink. (A *sink* as usual means a vertex with no directed edges going out.) As in [5, Sec. 4], we only need that the acyclic orientation has a unique sink in every facet, so for us this possibly larger set represents the good orientations. An acyclic orientation of $G(P)$ induces a partial ordering of the vertices of $G(P)$.

Define an *initial* set of a graph $G(P)$ with respect to some orientation as a set such that no edge is directed from a vertex not in the set to a vertex in the set. Similarly, a *final* set with respect to some orientation is a set such that no edge is directed from a vertex in the set to a vertex not in the set. A *source* is a vertex with no directed edges coming into it.

The paper [5] established the existence of good orientations with some special properties, but first we need an important remark, also from [5].

Remark 19 ([5, Rem. 4.2])*.* Let P be a d-polytope, let F be a face of P and let O be a good orientation of $G(P)$ in which $V(F)$ is initial. Further, denote by $O|_F$ the good orientation of $G(F)$ induced by O. If O'_F is a good orientation of $G(F)$ other than $O|_F$, then the orientation O' of $G(P)$ obtained from O by redirecting the edges of $G(F)$ according to O'_F is also a good orientation.

Lemma 20 ([5, Lem. 4.3])**.** *Let P be a polytope. For every two disjoint faces F_i and F_j of P, there is a good orientation of $G(P)$ such that*

(i) the vertices in F_i are initial,
(ii) the vertices in F_j are final, and
(iii) within the face F_i, any two vertices (if they exist) can be chosen to be the (local) sink and the (global) source.

The following corollary follows from Lemma 20.

Corollary 21. *Let F be a facet of a polytope P. Then*

(i) For any two vertices $u, v \in F$, there exists a good orientation O of $G(P)$ such that u is the source of O, the vertices of F are an initial set, and v is the sink of $O|_F$.

(ii) For any face R in F, there is a good orientation O of $G(P)$ such that the vertices of R are an initial set in $O|_F$, and some vertex in $F \setminus R$ is the sink in $O|_F$.

Proof. Let F be a facet of a polytope. For the proof of (i), apply Lemma 20 to $F_i = F$ (and disregard F_j).

For the proof of (ii), we apply Lemma 20 twice. First apply it to $F = F_i$, disregarding F_j, to obtain that $V(F)$ is initial with respect to some good orientation O. Secondly, apply it to the polytope F in aff F (disregarding P), where the face R plays the role of F_i and a vertex u of F not in R plays the role of F_j; in this way, we obtain that, within F, the vertex set of R is initial and u is a sink with respect to some good orientation O'_F of $G(F)$. From Remark 19 it then follows that the orientation O' obtained from O by directing the edges of $G(F)$ according to O'_F is the desired good orientation. ☐

A *feasible* subgraph is any induced $(d-1)$-connected subgraph H of G in which the simple vertices of P in H each have degree $d-1$ in H. In this case, each nonsimple vertex of P in H has degree $\geqslant d-1$ in H.

Lemma 22 ([5, Lem. 4.4])**.** *Let P be a d-polytope, and let H be a feasible subgraph of $G(P)$ containing at most $d-2$ nonsimple vertices. If the graph $G(F)$ of some facet F is contained in H, then $H = G(F)$.*

We are now ready to state and prove the main result of the section.

Theorem 23. *Let P be a d-polytope with excess at most $d-1$. Then the graph of P determines the entire combinatorial structure of P.*

This result is best possible in the sense that there are d-polytopes with excess d which are not reconstructible from their graphs.

Proof. Let $d \geqslant 4$. By Lemma 18 graph reconstruction follows from determining the valid frames of each vertex. As a result, it suffices to determine the valid frames of nonsimple vertices.

We first consider the case of $\xi = d-2$. In view of Lemma 16, a d-polytope with excess $d-2$ has either a unique nonsimple vertex or has $d-2$ vertices of excess degree one, which form a $(d-3)$-simplex R. The reconstruction of the former case follows from Theorem 2. Hence we only deal with the latter case.

The facets containing a nonsimple vertex in R (and in P) fall into two classes: those touching but not containing R and those containing R. By Lemma 16(iii), for each nonsimple vertex u in P the facet containing u and missing some vertex v in R is given by the d-frame rooted at u which misses v: for each such vertex u there are exactly $d-3$ such facets.

We now deal with the facets containing R; here more work is required to get the valid frames of a nonsimple vertex.

Denote by $\mathcal{H}_R$ the set of feasible subgraphs which contain the complete graph $G(R)$ on the vertices of R, and by $\mathcal{A}_R$ the set of all acyclic orientations of $G(P)$ in which for some subgraph H_R in $\mathcal{H}_R$, (1) H_R is initial and (2) H_R contains $G(R)$ as an initial subgraph. Any such H_R has a sink which is a simple vertex. Observe that there is a facet F_R in P which contains R. The graph of F_R is in $\mathcal{H}_R$.

Claim 1. A feasible subgraph H_R of $G(P)$ is the graph of a facet containing R if and only if (1) H_R contains $G(R)$, (2) H_R is initial with respect to a good orientation O in $\mathcal{A}_R$, and (3) H_R has a unique sink which is a simple vertex.

Proof. We reason as in the proof of Claim 1 of [5, Thm. 4.8].

First consider a facet F_R containing R. Corollary 21(ii) ensures the existence of a good orientation of $G(P)$ in which the vertices of F_R are initial, that the vertices of R are initial within F_R, and that a simple vertex is a sink in the facet. This proves the "only if" part of the claim.

Let $O \in \mathcal{A}_R$ and let h_k^O denote the number of simple vertices of G with indegree k. Define

$$f_R^O := h_{d-1}^O + dh_d^O.$$

The function f_R^O counts the number of pairs (F, w), where F is a facet of P and w is a simple sink in F of the orientation O in $\mathcal{A}_R$. Since the orientation is acyclic, every facet has a sink.

Let H_R be a feasible subgraph in $\mathcal{H}_R$, and let x be the simple sink in H_R with respect to O. Suppose H_R does not represent the facet F_R containing x and the $d-1$ edges in H_R incident to x. Then, in view of Lemma 22, there are vertices of F_R outside H_R. Since H_R is initial with respect to O, the facet F_R would contain two sinks, one of them being x.

Consequently, given that there is a good orientation in $\mathcal{A}_R$ and a subgraph H_R representing a facet, we have that

$$\min_{O \in \mathcal{A}_R} f_R^O = f_{d-1},$$

where f_{d-1} denotes the number of facets in P. Observe that all the nonsimple vertices of P are in R, and thus, in H_R. Also, an orientation of $\mathcal{A}_R$ minimising f_R^O must be a good orientation.

Let x be the simple sink in H_R with respect to O, then x defines a unique facet F_R of P. Therefore, all the other vertices of F_R are smaller than x with respect to the ordering induced by O. Since H_R is an initial set in O and since there is directed path in $G(F_R)$ from any other vertex of $G(F_R)$ to x, we must have $V(F_R) \subseteq V(H_R)$ and we are home by Lemma 22. □

Thanks to Claim 1, running through all the good orientations in $\mathcal{A}_R$, we can recognise all the graphs of facets containing R; say that its set is $\mathcal{F}_R$. Consequently, for each nonsimple vertex in R we also have the valid frames in each of these facets.

We now know all the valid frames of each nonsimple vertex in P, and of course, of each simple vertex. Thus, the case now follows from Lemma 18.

For the case of $\xi = d-1$, thanks to Lemma 17 and Theorem 2, we can assume that the polytope has dimension five and the four vertices of degree six are contained in a quadrilateral 2-face R.

The facets containing a nonsimple vertex in R (and in P) fall into two classes: those intersecting but not containing R and those containing R. By virtue of Lemma 17(iii), for each nonsimple vertex u in P the facet containing u and missing some vertex v in R is given by the 5-frame rooted at u which misses v: for each such vertex u there are exactly two such facets.

To recognise the facets containing R we proceed mutatis mutandis as in the case of $\xi = d-2$, just replacing the $(d-3)$-simplex with the 2-face. In this way, we recognise all the valid frames of each nonsimple vertex in P. Thus, the case again follows from Lemma 18.

A bipyramid over a $(d-1)$-simplex and a pyramid over a bipyramid over a $(d-2)$-simplex give a pair of nonreconstructible d-polytopes with excess d, and exactly d vertices of degree $d+1$, for $d \geqslant 4$. Hence the theorem is tight, as claimed. □

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