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Holder Error Bounds and Hölder Calmness with Applications to Convex Semi-Infinite Optimization

Alexander Y. Kruger · Marco A. Lopez · Xiaoqi Yang · Jiangxing Zhu

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Abstract Using techniques of variational analysis, necessary and sufficient subdifferential conditions for Holder error bounds are investigated and some new estimates for the corresponding modulus are obtained. As an application, we consider the setting of convex semi-infinite optimization and give a characterization of the Holder calmness of the argmin mapping in terms of the level set mapping (with respect to the objective function) and a special supremum function. We also estimate the Holder calmness modulus of the argmin mapping in the framework of linear programming.

Keywords Holder error bounds · Hölder calmness · Convex programming · Semi-infinite programming

Mathematics Subject Classification (2000) 49J53 · 90C25 · 90C31 · 90C34

1 Introduction

This paper mainly concerns the study and some applications of the notions of Holder error bounds and Hölder calmness.

Given an extended-real-valued function $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ on a metric space X , a point $\bar{x} \in [f \leq 0] := \{x \in X \mid f(x) \leq 0\}$ and a number $q > 0$, we say that f admits a q -order local error bound at \bar{x} , if there exist positive numbers τ and δ such that

$$\tau \alpha(x, [f \leq 0]) \leq [f(x)]_+^q, \quad \forall x \in B_\delta(\bar{x}), \quad (1.1)$$

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Alexander Y. Kruger (✉)
Centre for Informatics and Applied Optimization, Federation University Australia, Ballarat, Australia E-mail: a.kruger@federation.edu.au

Marco A. Lopez
Department of Mathematics, University of Alicante, Alicante, Spain and Centre for Informatics and Applied

 E-mail: marco.antonio@ua.es

Xiaoqi Yang

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong

E-mail: xiao.qi.yang@polyu.edu.hk

Jiangxing Zhu

Department of Mathematics, Yunnan University, Kunming 650091, China

E-mail: jiangxingzhu@yahoo.com

where $[f(x)]_+ := \max\{f(x), 0\}$.

The supremum of all $\tau > 0$ in (1.1) is called the *modulus (conditioning rate [40])* of q -order error bounds of f at \bar{x} and is denoted by $\text{Er}_q f(\bar{x}^-)$; explicitly,

$$\text{Er}_q f(\bar{x}^-) := \liminf_{x \rightarrow \bar{x}^-, f(x) > 0} \frac{f(x)^q}{d(x, [f \leq 0])} \quad (1.2)$$

It provides a quantitative characterization of error bounds. The absence of error bounds, i.e., the situation when (1.1) does not hold for any $\tau > 0$, is signaled by $\text{Er}_q f(\bar{x}^-) = 0$. When $q = 1$, we write simply $\text{Er} f(\bar{x}^-)$ instead of $\text{Er}_1 f(\bar{x}^-)$.

The case $q = 1$ corresponds to the conventional *linear error bounds*. Linear error bounds have been well studied, especially in the last 15 years, because of numerous applications; see, e.g., [1,2,13,14,18,23,24,29,31,34,37,45,46,48]. The study of Holder (q -order) and more general nonlinear error bounds started relatively recently thanks to more subtle applications, where conventional linear estimates do not hold; see [21,39,45,47,50].

Many authors have studied seemingly more general than error bounds, but in a sense equivalent to them properties of nonlinear subregularity and calmness of set-valued mappings, which are of great importance for optimization as well as subdifferential calculus, optimality conditions, stability and sensitivity issues, convergence of numerical methods, etc; interested readers are referred to [15,25,26,28,30,32,33,36–39,47,50] and the references therein. Sufficient conditions for (nonlinear) error bounds generate sufficient conditions for (nonlinear) subregularity and calmness; see, e.g., [28–30].

Given a set-valued mapping $S : Y \Rightarrow X$ between metric spaces Y and X , a point $(\bar{y}, \bar{x}^-) \in \text{gph}(S)$ and a number $q > 0$, we say that S is *q -order calm* (or possesses *q -order calmness property*) at (\bar{y}, \bar{x}^-) if there exist a number $\tau > 0$ and neighborhoods U of \bar{x} and V of \bar{y} such that

$$\tau d(x, S(\bar{y}^-)) \leq d(y, \bar{y}^-)^q, \quad \forall y \in V \text{ and } x \in S(y) \cap U. \quad (1.3)$$

If, additionally, \bar{x} is an isolated point in $S(\bar{y}^-)$, i.e., $S(\bar{y}^-) \cap U = \{\bar{x}^-\}$, then we say that S possesses *q -order isolated calmness property*

The supremum of all $\tau > 0$ in (1.3) is called the *q -order calmness modulus* of S at (\bar{y}, \bar{x}^-) and is denoted by $\text{clm}_q S(\bar{y}, \bar{x}^-)$; explicitly,

$$\text{clm}_q S(\bar{y}, \bar{x}^-) := \liminf_{\substack{y \rightarrow \bar{y}^- \\ x \rightarrow \bar{x}^-, x \in S(y)}} \frac{d(y, \bar{y}^-)^q}{d(x, S(\bar{y}^-))} = \liminf_{x \rightarrow \bar{x}^-} \frac{d(\bar{y}^-, S^{-1}(x))^q}{d(x, S(\bar{y}^-))}. \quad (1.4)$$

It provides a quantitative characterization of the calmness property. Following the lines of [9, Theorem 2.2], it is easy to verify that $\text{clm}_q S(\bar{y}, \bar{x}^-)$ coincides with the modulus of *q -order metric subregularity* of S^{-1} at (\bar{x}^-, \bar{y}^-) .

Using techniques of variational analysis, we continue the study of necessary and sufficient subdifferential conditions for Holder error bounds, particularly merging the conventional approach with the new advancements proposed recently in [47]. We formulate a general lemma collecting the main arguments used in the proofs of the

subdifferential sufficient error bound conditions and demonstrate that both linear and Hölder type conditions, both conventional and those in [47], can be obtained as direct consequences of this lemma.

Moreover, we clarify the relationship between the error bound characterizations in [47] and those obtained using the conventional approach. Some new estimates for the modulus of q -order error bounds are obtained. The main sufficient subdifferential conditions are combinations of two assertions: in Asplund spaces in terms of Frechet subdifferentials and for convex functions in general Banach spaces.

In [6], the authors compute or estimate the calmness modulus of the argmin mapping of linear semi-infinite optimization problems under canonical perturbations (see Section 4 for its explicit meaning). Motivated by this and as an application, the last goal of the paper is to study in detail Hölder calmness in convex semi-infinite programs. In this setting we clarify the relationship among the Hölder calmness of the solution mapping S , the (lower) level set mapping L , and the Hölder error bound of a special supremum function (see their definitions in Section 4). Moreover, we also estimate the Hölder calmness modulus of the argmin mapping in the framework of linear programming.

The rest of the paper is organized as follows: Section 2 summarizes some preliminary facts from variational analysis and generalized differentiation widely used in the formulations and proofs of our main results. In Section 3 we establish and discuss some necessary and sufficient subdifferential conditions for Hölder error bounds. The last Section 4

devoted to convex semi-infinite optimization, shows the equivalence among the Hölder calmness of the (lower) level set and argmin mappings and the Hölder error bounds of a special supremum function, and also provides an estimate of Hölder calmness of the argmin mapping under some particular conditions.

The paper is dedicated to our friend Professor Asen Dontchev on the occasion of his 70th birthday

2 Preliminaries

In this section, we summarize some fundamental tools of variational analysis and nonsmooth optimization.

Our basic terminology and notation are standard, see, e.g., [7, 8, 10, 21, 35, 43, 44]. Throughout the paper, X and Y are either metric or normed vector spaces. We use the standard notations $d(\cdot, \cdot)$ and $\|\cdot\|$ for the distance and the norm in any space. For $x \in X$ and $\delta > 0$, $B_\delta(x)$ denotes the open ball centered at x with radius δ . Given a set A and a point x in the same space, $d(x, A) := \inf_{a \in A} d(x, a)$ is the distance from x to A . In particular, $d(x, \emptyset) = +\infty$ for any x . If X is a normed vector space, its topological dual is denoted by X^* , while $\langle \cdot, \cdot \rangle$ denotes the bilinear form defining the pairing between the two spaces. We denote by B and B^* the open unit balls in a normed vector space and its dual, respectively.

Given an extended-real-valued function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$, we denote by $\text{dom } f$ its domain: $\text{dom } f := \{x \in X \mid f(x) < +\infty\}$. For a set-valued mapping $\Phi : X \rightrightarrows Y$, the graph and the domain of Φ are defined, respectively, by

$$\text{gph}(\Phi) := \{(x, y) \in X \times Y \mid x \in X, y \in \Phi(x)\} \quad \text{and} \quad \text{dom } \Phi := \{x \in X \mid \Phi(x) \neq \emptyset\}.$$

The inverse $F^{-1}: Y \rightrightarrows X$ of F (which always exists with possibly empty values at some y) is defined by

$$F^{-1}(y) := \{x \in X \mid y \in F(x)\}, \quad y \in Y.$$

Obviously, $\text{dom } F^{-1} = F(X)$.

Recall that the Frechet subdifferential of f at $x \in \text{dom } f$ is defined as

$$\partial f(x) := \left\{ x^* \in X^* \mid \liminf_{y \rightarrow x} \frac{f(y) - f(x) - \langle x^*, y - x \rangle}{\|y - x\|} \geq 0 \right\},$$

and $\partial f(x) := \emptyset$ if $x \notin \text{dom } f$. It is well-known that the set $\partial f(x)$ reduces to the classical subdifferential of convex analysis if f is convex.

The proofs of the main results rely on certain fundamental results of variational analysis: the *Ekeland variational principle* (Ekeland [11]; see also [10,21,35]) and several kinds of *subdifferential sum rules*. Below we provide these results for completeness.

Lemma 2.1 (Ekeland variational principle) *Suppose X is a complete metric space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and bounded from below, $\varepsilon > 0$ and $\lambda > 0$. If*

$$f(x^-) < \inf_X f + \varepsilon,$$

then there exists an $\hat{x} \in X$ such that

- (a) $d(\hat{x}, x^-) < \lambda$,
- (b) $f(\hat{x}) \leq f(x^-)$,
- (c) $f(x) + (\varepsilon/\lambda)d(x, \hat{x}) \geq f(\hat{x})$ for all $x \in X$.

Lemma 2.2 (Subdifferential sum rules) *Suppose X is a normed linear space, $f_1, f_2 : X \rightarrow \mathbb{R} \cup \{+\infty\}$, and $x^- \in \text{dom } f_1 \cap \text{dom } f_2$.*

(i) *Fuzzy sum rule. Suppose X is Asplund, f_1 is Lipschitz continuous and f_2 is lower semicontinuous in a neighbourhood of x . Then, for any $\varepsilon > 0$, there exist $x_1, x_2 \in X$ with $\|x_i - x^-\| < \varepsilon$, $|f_i(x_i) - f_i(x^-)| < \varepsilon$ ($i = 1, 2$), such that*

$$\partial(f_1 + f_2)(x^-) \subset \partial f_1(x_1) + \partial f_2(x_2) + \varepsilon B^*.$$

(ii) *Convex sum rule. Suppose f_1 and f_2 are convex and f_1 is continuous at a point in $\text{dom } f_2$. Then*

$$\partial(f_1 + f_2)(x^-) = \partial f_1(x^-) + \partial f_2(x^-).$$

The first sum rule in the lemma above is known as the *fuzzy* or *approximate* sum rule (Fabian [12]; cf., e.g., [27, Rule 2.2], [35, Theorem 2.33]) for Frechet subdifferentials in Asplund spaces. The other one is an example of an *exact* sum rule. It is valid in arbitrary normed spaces. For rule (ii) we refer the readers to [22, Theorem 0.3.3] and [49, Theorem 2.8.7].

Recall that a Banach space is *Asplund* if every continuous convex function on an open convex set is Frechet differentiable on a dense subset [41], or equivalently, if the dual of each its separable subspace is separable. We refer the reader to [3,35,41] for discussions about and characterizations of Asplund spaces. All reflexive, in particular, all finite dimensional Banach spaces are Asplund.

The following fact is an immediate consequence of the definition of the Frechet subdifferential (cf., e.g., [27, Propositions 1.10]).

Lemma 2.3 *Suppose X is a normed vector space and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$. If $x^- \in \text{dom } f$ is a point of local minimum of f , then $0 \in \partial f(x^-)$.*

The next subdifferential *chain rule* is a modification of [37, Lemma 1] and [27, Corollary 1.14.1]; see also [47, Proposition 2.1].

Lemma 2.4 *Suppose X is a normed linear space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in \text{dom } f$. Suppose also that $\psi : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ is differentiable at $f(x^-)$ with*

$$\psi'(f(x^-)) > 0 \text{ and nondecreasing on } [f(x^-), +\infty). \text{ Then}$$

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$$\partial(\psi \circ f)(x^-) = \psi^0(f(x^-))\partial f(x^-).$$

3 Characterizations of Holder error bounds

In this section, we establish and discuss some necessary and sufficient subdifferential conditions for Holder error bounds. We start with a slightly new look at the very well studied linear error bounds.

3.1 Linear error bounds

The next elementary lemma collects the main arguments used in the proofs of the subdifferential sufficient error bound conditions, the key tools being the Ekeland variational principle (Lemma 2.1) and subdifferential sum rules (Lemma 2.2). It establishes an error bound estimate for a fixed point $x \in / [f \leq 0]$ and actually combines two separate statements: for lower semicontinuous functions in Asplund spaces and for convex functions in general Banach spaces. All sufficient error bound conditions in this section are in a sense consequences of this lemma.

Lemma 3.1 *Suppose X is a Banach space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, $x \in X$ and $f(x) > 0$. Let $\tau > 0$ and $\alpha \in (0, 1]$. If either X is Asplund and, given an $M > f(x)$,*

$$d(0, \partial f(u)) \geq \tau \text{ for all } u \in X \text{ with } \|u - x\| < \alpha d(x, [f \leq 0]),$$

$$f(u) < M \text{ and } f(u) < \tau d(u, [f \leq 0]), \quad (3.1)$$

or f is convex and

$$d(0, \partial f(u)) \geq \tau \text{ for all } u \in X \text{ with } \|u - x\| < \alpha d(x, [f \leq 0]),$$

$$f(u) \leq f(x) \text{ and } f(u) < \tau d(u, [f \leq 0]), \quad (3.2)$$

then $[f \leq 0] \neq \emptyset$ and

$$\alpha \tau d(x, [f \leq 0]) \leq f(x). \quad (3.3)$$

Proof Suppose that condition (3.3) is not satisfied, i.e. $f(x) < \alpha \tau d(x, [f \leq 0])$ (this is the case, in particular, when $[f \leq 0] = \emptyset$). Choose a $\hat{\tau} \in (0, \tau)$ such that $f(x) < \hat{\tau} \alpha d(x, [f \leq 0])$. Then, by the Ekeland variational principle (Lemma 2.1) applied to the lower semicontinuous function f_+ , there exists a point $\hat{u} \in X$ such that

$$f_+(\hat{u}) \leq f(x), \quad \|\hat{u} - x\| < \alpha d(x, [f \leq 0]), \quad (3.4) \quad f_+(\hat{u}) \leq$$

$$f_+(u) + \hat{\tau} \|u - \hat{u}\| \quad \text{for all } u \in X. \quad (3.5)$$

Since $\alpha \in (0, 1]$, it follows from (3.4) that $\hat{u} \in / [f \leq 0]$, and consequently, $f_+(\hat{u}) = f(\hat{u}) > 0$. If $[f \leq 0] \neq \emptyset$, it follows from (3.5) that

$$f(\hat{u}) \leq \hat{\tau} d(\hat{u}, [f \leq 0]). \quad (3.6)$$

If $[f \leq 0] = \emptyset$, the last inequality is satisfied trivially. In view of the lower semicontinuity of f , we have $f_+(u) = f(u) > 0$ for all u near \hat{u} , and it follows from (3.5) and Lemma 2.3 that $0 \in \partial(f+g)(\hat{u})$ where $g(u) := \hat{\tau} \|u - \hat{u}\|$, $u \in X$.

(i) Suppose X is Asplund. Choose an $\varepsilon > 0$ such that

$$f(x) + \varepsilon < M, \quad \|\hat{u} - x\| + \varepsilon < \alpha d(x, [f \leq 0]), \quad \varepsilon < f(\hat{u}), \quad \varepsilon < \tau - \hat{\tau}$$

$$\text{and } \varepsilon(1 + \hat{\tau}) < (\tau - \hat{\tau}) d(\hat{u}, [f \leq 0]). \quad (3.7)$$

Applying the fuzzy sum rule (Lemma 2.2(i)), we find points $\hat{x}, x^{0*} \in B_\varepsilon(\hat{u})$ and subgradients $x^* \in \partial f(x^*)$, $x^{0*} \in \partial g(x^{0*})$ such that $|f(x^*) - f(\hat{u})| < \varepsilon$ and $\|x^* + x^{0*}\| < \varepsilon$.

By the definition of g , we have $kx^0 - k \leq \tau^*$. Using (3.7), (3.6), (3.4) and the obvious

inequality $d(x^*, [f \leq 0]) + kx^* - u^* - d(u^*, [f \leq 0]) \geq 0$, we obtain the following estimates:

$$kx^* - xk \leq kx^* - u^* + ku^* - xk < \varepsilon + ku^* - xk < \alpha d(x, [f \leq 0]),$$

$$\begin{aligned} (\hat{x}) &< f(\hat{u}) + \varepsilon \leq \hat{\tau} d(\hat{u}, [f \leq 0]) + \varepsilon \\ &\leq \hat{\tau} d(\hat{u}, [f \leq 0]) + \varepsilon + \tau (d(\hat{x}, [f \leq 0]) + \|\hat{x} - \hat{u}\| - d(\hat{u}, [f \leq 0])) \\ &< (\hat{\tau} - \tau) d(\hat{u}, [f \leq 0]) + \varepsilon(1 + \tau) + \tau d(\hat{x}, [f \leq 0]) < \tau d(\hat{x}, [f \leq 0]) \\ f(x^*) &> f(u^*) - \varepsilon > 0, \\ f \end{aligned}$$

$$f(x^*) < f(x) + \varepsilon < M,$$

$$d(0, \partial f(x^*)) \leq kx^* - k < \tau^* + \varepsilon < \tau.$$

This contradicts (3.1).

- (ii) Suppose f is convex. Since g is convex continuous, we can apply the convex sum rule (Lemma 2.2(ii)) to find a subgradient $x^* \in \partial f(u^*)$ such that $kx^* - k \leq \tau^*$. Thus, making use also of (3.6), we have $f(u^*) < \tau d(u^*, [f \leq 0])$ and $d(0, \partial f(u^*)) \leq kx^* - k \leq \tau^* < \tau$. This contradicts (3.2) and completes the proof. \square

In the setting of linear error bounds, the first part of Lemma 3.1 strengthens [37, Theorem 2], where a more general setting of Holder error bounds was studied. We are going to show in the next subsection that this seemingly more general setting can be treated within the conventional linear theory.

Dropping or weakening any or all of the conditions on u in (3.1) makes the sufficient condition in Lemma 3.1 stronger (while weakening the result). This way one can formulate simplified versions of Lemma 3.1. For instance, condition $f(u) < \tau d(u, [f \leq 0])$ in (3.1) does not seem practical when checking error bounds as it involves the unknown set $[f \leq 0]$, and basically says that only the points not satisfying the error bound property with constant τ should be checked. This condition is usually either dropped or replaced by the easier to check weaker condition $f(u) < \tau k u - x^* - k$, where x^* is some point in $[f \leq 0]$. Corollary 3.2 Suppose X is a Banach space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, $x \in X$ and $f(x) > 0$. Let $\tau > 0$ and $\alpha \in (0, 1]$. If either X is Asplund or f is convex, and $d(0, \partial f(u)) \geq \tau$ for all $u \in X$ with $ku - xk < \alpha d(x, [f \leq 0])$ and $f(u) < \tau d(u, [f \leq 0])$, then $[f \leq 0] \neq \emptyset$ and condition (3.3) holds true.

Remark 3.3 (i) The subdifferential characterizations in Lemma 3.1 are in fact consequences of the corresponding primal space characterizations in terms of slopes, some traces of which can be found in its proof; cf. [29,45]. We do not consider primal space characterizations in this paper.

- (ii) Elementary (primal or dual) error bound statements for a fixed point $x \in [f \leq 0]$, coming from the Ekeland variational principle and lying at the core of all sufficient error bound characterizations have been formulated by several authors; cf. [18,20,37,40].
- (iii) It is well understood by now that Frechet subdifferentials can be replaced in this type of results by other subdifferentials possessing reasonable sum rules in appropriate (trustworthy [19]) spaces. For instance, it is easy to establish analogues of Lemma 3.1 and the other statements in this section for lower semicontinuous functions in general Banach spaces in terms of Clarke subdifferentials. We do not do it in the

current paper to keep the presentation simple and avoid using several types of subdifferentials in one statement.

- (iv) The value of the parameter α in Lemma 3.1 determines a tradeoff between the strength of the error bound estimate (3.3) and the size of the neighbourhood of x involved in the sufficient conditions (3.1) and (3.2): increasing the value of α strengthens condition (3.3) at the expense of increasing the neighbourhood of x , all points from which have to be checked in conditions (3.1) and (3.2).

The next theorem is a slight generalization of the conventional linear error bound statement in the subdifferential form (which corresponds to taking $\alpha = 1$). It is an immediate consequence of Lemma 3.1.

Theorem 3.4 *Suppose X is a Banach space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in [f \leq 0]$. Let $\tau > 0$ and $\delta \in [0, \infty]$. If either X is Asplund or f is convex, and $d(0, \partial f(x)) \geq \tau$ for all $x \in B_\delta(x^-) \cap [f > 0]$ with $f(x) < \tau d(x, [f \leq 0])$, (3.8)*

then

$$\alpha \tau d(x, [f \leq 0]) \leq f_+(x) \quad \text{for all } \alpha \in (0, 1] \quad \text{and} \quad x \in B_{\frac{\delta}{1+\alpha}}(\bar{x}). \quad (3.9)$$

Proof Suppose that condition (3.9) is not satisfied, i.e. $f_+(x) < \alpha \tau d(x, [f \leq 0])$ for some $\alpha \in (0, 1]$ and some $x \in B_{\frac{\delta}{1+\alpha}}(\bar{x})$. Then $d(x, [f \leq 0]) > 0$, and consequently, $f_+(x) = f(x) > 0$. By Lemma 3.1, there exists a $u \in X$ with $\|ku - x\| < \alpha d(x, [f \leq 0])$ and $f(u) < \tau d(u, [f \leq 0])$ such that $d(0, \partial f(u)) < \tau$. This contradicts (3.8) because $\|ku - x^-\| \leq \|ku - x\| + \|x - x^-\| < (\alpha + 1)\|x - x^-\| < \delta$ and $f(u) > 0$. tu

The next statement is a simplified version of Theorem 3.4. with the the last inequality in (3.8) replaced by the easier to check weaker condition $f(u) < \tau \|ku - x^-\|$.

Corollary 3.5 *Suppose X is a Banach space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in [f \leq 0]$. Let $\tau > 0$ and $\delta \in [0, \infty]$. If either X is Asplund or f is convex, and $d(0, \partial f(x)) \geq \tau$ for all $x \in B_\delta(x^-) \cap [f > 0]$ with $f(x) < \tau \|x - x^-\|$,*

then condition (3.9) holds true.

Remark 3.6 (i) Theorem 3.4 and Corollary 3.5 allow for $\delta = \infty$, thus, covering also global error bounds.

- (ii) The value of the parameter α in Theorem 3.4 and Corollary 3.5 determines a tradeoff between the sharpness of the error bound estimate in (3.9) and the size of the neighbourhood of x , where this estimate holds. If the size of the neighbourhood is not important, one can take $\alpha = 1$, which insures the sharpest error bound estimate. Note that, unlike Lemma 3.1, in Theorem 3.4 and the subsequent statements in this section the parameter α is only present in the concluding part.

Thanks to Corollary 3.5, the limit $\underline{\text{Er}} f(x^-) := \liminf_{x \rightarrow x^-, f(x) \downarrow 0} d(0, \partial f(x))$

$$f(x)$$

$$x \rightarrow x^-, f(x) \downarrow 0$$

provides a lower estimate for the local error bound modulus $\text{Er} f(x^-)$ of f at x^- . Such estimates are often used in the literature.

3.2 Holder error bounds

The estimate (3.9) constitutes the *linear* error bound for the function f at \bar{x} with constant $\alpha\tau$. In many important situations such linear estimates do not hold, and this is where more subtle nonlinear (in particular, Holder type) models come into play. Surprisingly, such seemingly more general models can be treated within the conventional linear theory. The next theorem providing a characterization for the Holder error bounds is a consequence of Theorem 3.4.

Given a function $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$, a point $x \in X$ with $f(x) \geq 0$ and a number $q > 0$, $f^q(x)$ stands for $[f(x)]^q$. Thus, f^q is a function on $[f \geq 0]$. Note that the next theorem allows for $q > 1$.

Theorem 3.7 *Suppose X is a Banach space, $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in [f \leq 0]$. Let $\tau > 0$, $\delta \in (0, \infty]$ and $q > 0$. If either X is Asplund or f is convex, and $qf^{q-1}(x)d(0, \partial f(x)) \geq \tau$ for all $x \in B_\delta(x^-) \cap [f > 0]$ with $f^q(x) < \tau d(x, [f \leq 0])$, (3.10)*

then

$$\alpha\tau d(x, [f \leq 0]) \leq f_+^q(x) \quad \text{for all } \alpha \in (0, 1] \quad \text{and} \quad x \in B_{\frac{\delta}{1+\alpha}}(\bar{x}). \quad (3.11)$$

Proof Apply Theorem 3.4 with the lower semicontinuous function $x \mapsto f_+^q(x)$ in place of f . Observe that $[f_+^q \leq 0] = [f_+ = 0] = [f \leq 0]$ and, for any $x \in [f > 0]$, we have $f_+^q(x) = f^q(x)$ and $\partial f^q(x) = qf^{q-1}(x)\partial f(x)$ (by Lemma 2.4). \square

Theorem 3.7 strengthens [37, Corollary 2, parts (i) and (ii)]. When $q = 1$, Theorem 3.7 reduces to Theorem 3.4.

The next statement is a simplified version of Theorem 3.7, with the the last inequality in (3.10) replaced by the easier to check weaker condition $f^q(x) < \tau kx - x^-k$.

Corollary 3.8 *Suppose X is a Banach space, $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in [f \leq 0]$. Let $\tau > 0$, $\delta \in (0, \infty]$ and $q > 0$. If either X is Asplund or f is convex, and $qf^{q-1}(x)d(0, \partial f(x)) \geq \tau$ for all $x \in B_\delta(x^-) \cap [f > 0]$ with $f^q(x) < \tau kx - x^-k$, (3.12) then condition (3.11) holds true.*

In view of Corollary 3.8 and definition (1.2), the limit

$$\text{Er}_q f(\bar{x}) := q \liminf_{x \rightarrow \bar{x}, f(x) \downarrow 0} \frac{d(0, \partial f(x))}{f^{1-q}(x)} \quad (3.13)$$

provides a lower estimate for the modulus $\text{Er}_q f(x^-)$ of q -order error bounds of f at \bar{x} .

Example 3.9 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2$ if $x \geq 0$ and $f(x) = 0$ if $x < 0$. Then $[f \leq 0] = \mathbb{R}_-$ and, for any $x > 0$, we have $d(x, [f \leq 0]) = x$, $d(0, \partial f(x)) = f'(x) = 2x$, and, with $q = \frac{1}{2}$, $qf^{q-1}(x)d(0, \partial f(x)) = \frac{1}{2} \cdot \frac{1}{x} \cdot 2x = 1$. Hence, condition (3.12) is satisfied with $q = \frac{1}{2}$ and any $\tau \in (0, 1]$ and $\delta \in (0, \infty]$. With $q = \frac{1}{2}$, the inequality in (3.11) becomes $\alpha\tau x_+ \leq x_+$, where $x_+ := \max\{x, 0\}$. It is indeed satisfied for all $\tau \in (0, 1]$, $\alpha \in (0, 1]$ and $x \in \mathbb{R}$. \square

Example 3.10 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = \sqrt{x}$ if $x \geq 0$ and $f(x) = 0$ if $x < 0$. Then $[f \leq 0] = \mathbb{R}_-$ and, for any $x > 0$, we have $d(x, [f \leq 0]) = x$, $d(0, \partial f(x)) = f'(x) = \frac{1}{2\sqrt{x}}$, \square

and, with $q=2$, $qf^{q-1}(x)d(0,\partial f(x))=2\cdot x\cdot 2\sqrt{1-x}=1$. Hence, condition (3.12) is satisfied with $q=2$ and any $\tau \in (0,1]$ and $\delta \in [0,\infty]$. With $q=2$, the inequality in (3.11) becomes $\alpha\tau x \leq x$. It is indeed satisfied for all $\tau \in (0,1]$, $\alpha \in (0,1]$ and $x \in \mathbb{R}$. \square

It was observed in [47] that applying the Ekeland variational principle in the proof of results like Theorem 3.7 in a slightly different way, one can obtain a sufficient subdifferential condition for Hölder error bounds in a different form. Next we show that conditions of this type are also direct consequences of Lemma 3.1. The following statement is motivated by [47, Theorem 3.1]. Note that, unlike Theorem 3.7, it is restricted to the case $q \leq 1$.

Theorem 3.11 *Suppose X is a Banach space, $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in [f \leq 0]$. Let $\tau > 0$, $\delta \in [0,\infty]$ and $q \in (0,1]$. If either X is Asplund or f is convex, and*

$$d(x,[f \leq 0])^{q-1}d(0,\partial f(x))^q \geq \tau \text{ for all } x \in B_\delta(x^-) \cap [f > 0] \quad \text{with } f^q(x) < \tau d(x,[f \leq 0]), \quad (3.14)$$

then

$$(1-\alpha)^{1-q}\tau d(x,[f \leq 0]) \leq f_+^q(x) \quad \text{and} \quad \alpha \in (0,1) \in B_{\frac{\delta}{1+\alpha}}(\bar{x}). \quad (3.15)$$

for all α

Proof Suppose that condition (3.15) is not satisfied, i.e. $f_+^q(x) < \alpha^q(1-\alpha)^{1-q}\tau d(x,[f \leq 0])$ for some $\alpha \in (0,1)$ and $x \in B_{\frac{\delta}{1+\alpha}}(\bar{x})$. Then $d(x,[f \leq 0]) > 0$, and consequently, $f_+(x) = f(x) > 0$. Set $\tau' := \tau^{\frac{1}{q}}((1-\alpha)d(x,[f \leq 0]))^{\frac{1}{q}-1}$. Then $0 < f(x) < \alpha\tau' d(x,[f \leq 0])$. By Lemma 3.1, there exists a $u \in X$ with $\|ku-x\| < \alpha d(x,[f \leq 0])$ and $f(u) < \tau'^q d(u,[f \leq 0])$ such that $d(0,\partial f(u)) < \tau'$. Observe that $f(u) > 0$,

$$(\tau')^q = \tau((1-\alpha)d(x,[f \leq 0]))^{1-q} \stackrel{(3.17)}{<} \tau d(u,[f \leq 0])^{1-q} \quad (3.16)$$

$$\|ku-x\| \leq \|ku-x\| + \|kx-x^-\| < (\alpha+1)\|kx-x^-\| < \delta, \quad (3.17)$$

$$(1-\alpha)d(x,[f \leq 0]) < d(x,[f \leq 0]) - \|ku-x\| \leq d(u,[f \leq 0]), \quad (3.18)$$

$$f^q(u) < (\tau')^q d(u,[f \leq 0])^q \stackrel{(3.18)}{<} \tau d(u,[f \leq 0])^q, \quad (3.19)$$

$$d(u,[f \leq 0])^{q-1}d(0,\partial f(u))^q < d(u,[f \leq 0])^{q-1}(\tau')^q < \tau.$$

In view of (3.16) and (3.19), this contradicts (3.14) and completes the proof. \square

Just like Theorem 3.7, when $q=1$ Theorem 3.11 reduces to Theorem 3.4. Thus, both Theorems 3.7 and 3.11 generalize Theorem 3.4 to the Hölder setting.

The next statement is a simplified version of Theorem 3.11.

Corollary 3.12 *Suppose X is a Banach space, $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in [f \leq 0]$. Let $\tau > 0$, $\delta \in [0,\infty]$ and $q \in (0,1]$. If either X is Asplund or f is convex, and*

$$d(x,[f \leq 0])^{q-1}d(0,\partial f(x))^q \geq \tau \text{ for all } x \in B_\delta(x^-) \cap [f > 0] \quad \text{with } f^q(x) < \tau \|kx-x^-\|, \quad (3.20)$$

then condition (3.15) holds true.

As observed in Remark 3.6(ii) concerning Theorem 3.4 and Corollary 3.5, the value of

the parameter α in Theorem 3.11 and Corollary 3.12 determines a tradeoff between the sharpness of the error bound estimate in (3.15) and the size of the neighbourhood of \bar{x} , where this estimate holds. Thanks to the special form of the expression in the left-hand side of the inequality in (3.15), the range of values of α in (3.15) can be reduced, with the sharpest error bound estimate corresponding to taking $\alpha = q$.

Proposition 3.13 *Under the assumptions of Theorem 3.11, and adopting the convention $0^0 = 1$, condition (3.15) is equivalent to the following one:*

$$(1 - \alpha)^{1-q} \tau d(x, [f \leq 0]) \leq f_+^q(x) \quad \text{and} \quad q \in (0, q] \quad x \in B_{\frac{\delta}{1+\alpha}}(\bar{x}). \quad (3.21)$$

for all α

The latter condition implies

$$q^q (1 - q)^{1-q} \tau d(x, [f \leq 0]) \leq f_+^q(x) \quad \text{for all} \quad x \in B_{\frac{\delta}{1+q}}(\bar{x}). \quad (3.22)$$

Moreover, condition (3.22) is equivalent to (3.21) with the neighbourhood $B_{\frac{\delta}{1+\alpha}}(\bar{x})$ replaced by $B_{\frac{\delta}{1+q}}(\bar{x})$.

Proof The implication (3.21) \Rightarrow (3.22) is obvious, as well as the implication (3.15) \Rightarrow (3.21) when $q < 1$. It is easy to check that in the latter case the function $\alpha \mapsto \alpha^q(1-\alpha)^{1-q}$ is strictly increasing on $(0, q)$ and strictly decreasing on $(q, 1)$. Hence, when $\alpha > q$, one has $\alpha^q(1-\alpha)^{1-q} < q^q(1-q)^{1-q}$ and $B_{\frac{\delta}{1+\alpha}}(\bar{x}) \subset B_{\frac{\delta}{1+q}}(\bar{x})$, and consequently, (3.21) \Rightarrow (3.15).

When $q = 1$, the implication (3.21) \Rightarrow (3.15) is obvious. For the converse implication, only the case $\alpha = 1$ needs to be covered. Condition (3.15) implies

$$\alpha \tau d(x, [f \leq 0]) \leq f_+(x) \quad \text{for all} \quad \alpha \in (0, 1) \quad \text{and} \quad x \in B_{\frac{\delta}{2}}(\bar{x}).$$

Taking supremum over α in the left-hand side of the above inequality, we see that the inequality must hold also for $\alpha = 1$. The ‘moreover’ part is obvious since $\alpha^q(1-\alpha)^{1-q} \leq q^q(1-q)^{1-q}$ for all $\alpha \in (0, q]$. \square

Thanks to Proposition 3.13, the sufficient error bound condition in Corollary 3.12 can be simplified further.

Corollary 3.14 *Suppose X is a Banach space, $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $\bar{x} \in [f \leq 0]$. Let $\tau > 0$, $\delta \in [0, \infty]$ and $q \in (0, 1]$. If either X is Asplund or f is convex, and*

$$q^q(1-q)^{1-q} d(x, [f \leq 0])^{q-1} d(0, \partial f(x))^q \geq \tau \quad \text{for all} \quad x \in B_\delta(\bar{x}) \cap [f > 0]$$

with $q^q(1-q)^{1-q} f^q(x) < \tau \|x - \bar{x}\|$, (3.23)

then

$$\tau d(x, [f \leq 0]) \leq f_+^q(x) \quad \text{for all} \quad x \in B_{\frac{\delta}{1+q}}(\bar{x}). \quad (3.24)$$

In view of Corollary 3.8 and definition (1.2), the expression

$$\text{Er}_q f(x^-) := q(1-q)^{1-q} \liminf_{x \rightarrow 0} d(x, [f \leq 0])^{q-1} d(0, \partial f(x))^q \quad (3.25)$$

provides a lower estimate for the modulus $\text{Er}_q f(x^-)$ of q -order error bounds of f at x^- which complements (3.13).

Example 3.15 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as in Example 3.9: $f(x) = x^2$ if $x \geq 0$ and $f(x) = 0$ if $x < 0$. As computed in Example 3.9, for any $x > 0$, we have $d(x, [f \leq 0]) = x$ and $d(0, \partial f(x)) = f'(x) = 2x$. Now, with $q = \frac{1}{2}$, we have for any $x > 0$:

$$q^q(1-q)^{1-q} d(x, [f \leq 0])^{q-1} d(0, \partial f(x))^q = \frac{1}{2} \cdot x^{\frac{1}{2}} \cdot \sqrt{2x} = \frac{1}{2}.$$

Hence, condition (3.23) is satisfied with $q = \frac{1}{2}$ and any $\tau \in (0, \frac{1}{\sqrt{2}}]$ and $\delta \in [0, \infty)$. Thus, Corollary 3.14 gives in this example a global error bound estimate with constant up to $\frac{1}{\sqrt{2}}$, while we know from Example 3.9, which is a consequence of Theorem 3.7, that a global error bound estimate holds actually with any constant up to 1.

The next proposition shows that a slightly strengthened version of the sufficient error bound condition in Corollary 3.8 implies that in Corollary 3.12 (or 3.14). In view of the obvious similarity of the concluding conditions (3.11) (with $\alpha = 1$) in Corollary 3.8 and (3.24) in Corollary 3.14, below we compare their assumptions.

Proposition 3.16 Suppose X is a Banach space, $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in [f \leq 0]$. Let $\tau > 0$, $\delta \in (0, \infty]$ and $q \in (0, 1]$, and the convention $0^0 = 1$ be in force. Suppose also that either X is Asplund or f is convex. If $q f^{q-1}(x) d(0, \partial f(x)) \geq \tau$ for all $x \in B_\delta(x^-) \cap [f > 0]$ with $q^q f^q(x) < \tau k x - x^- k$, (3.26)

then

$$q^q d(x, [f \leq 0])^{q-1} d(0, \partial f(x))^q \geq \tau \text{ for all } x \in B_{\frac{\delta}{2}}(\bar{x}) \cap [f > 0] \text{ with } q^q f^q(x) < \tau k x - x^- k,$$

i.e. condition (3.23) is satisfied with $\tau^0 := (1-q)^{1-q} \tau$ and $\delta' := \frac{\delta}{2}$ in place of τ and δ , respectively.

Proof Suppose that condition (3.26) is satisfied. Then, condition (3.12) is satisfied too and, by Corollary 3.8, condition (3.11) holds true. In view of conditions (3.12) and (3.11) with $\alpha = 1$, we have for any $x \in B_{\frac{\delta}{2}}(\bar{x}) \cap [f > 0]$ with $q^q f^q(x) < \tau k x - x^- k$:

$$\begin{aligned} q^q d(x, [f \leq 0])^{q-1} d(0, \partial f(x))^q &= d(x, [f \leq 0])^{q-1} (q d(0, \partial f(x)))^q \\ &\geq d(x, [f \leq 0])^{q-1} (\tau f^{1-q}(x))^q = \tau \left((\tau d(x, [f \leq 0]))^{-1} f^q(x) \right)^{1-q} \geq \tau. \end{aligned}$$

This completes the proof. tu

Proposition 3.16 allows us to establish a relationship between the lower error bound estimates (3.13) and (3.25).

Corollary 3.17 Suppose X is a Banach space, $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in [f \leq 0]$. Let $q \in (0, 1]$ and the convention $0^0 = 1$ be in force. If X is Asplund or f is convex, then

$$(1-q)^{1-q} \text{Er}_q f(x^-) \leq \underline{\text{Er}}_q f(x^-). \quad (3.27)$$

Proof If $\text{Er}_q f(x^-) = 0$, the first inequality in (3.27) holds true trivially. Suppose that $0 < \tau < \text{Er}_q f(x^-)$. By definition (3.13), condition (3.26) is satisfied with some number $\delta > 0$ and, by Proposition 3.16, condition (3.23) is satisfied with $\tau^0 := (1-q)^{1-q}\tau$ and $\delta' := \frac{\delta}{2}$ in place of τ and δ , respectively. Hence, by definition (3.25), $\text{Er}_q^0 f(x^-) \geq (1-q)^{1-q}\tau$. Passing

to the limit as $\tau \uparrow \text{Er}_q f(x^-)$ proves (3.27). tu

Remark 3.18 In view of Corollary 3.17, the sufficient error bound condition $\text{Er}_q^0 f(x^-) > 0$ is in general weaker than $\text{Er}_q f(x^-) > 0$. At the same time, it also yields a weaker error

bound estimate – see (3.27). This is illustrated by Example 3.15, where $\text{Er}_q f(0) = 1$, $\text{Er}_q^0 f(0) = (1-q)^{1-q} = \frac{1}{\sqrt{2}}$, i.e. condition (3.27) holds as equality.

Inequality (3.27) relating the two lower estimates for the modulus $\text{Er}_q f(x^-)$ can be strict. Moreover, it can happen that $\text{Er}_q f(x^-) = 0$ while $\text{Er}_q^0 f(x^-) > 0$. In such cases, $\text{Er}_q^0 f(x^-)$

detects q -order error bounds while $\text{Er}_q f(x^-)$ fails.

Example 3.19 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ x^2 + \frac{1}{n} - \frac{1}{n^2} & \text{if } \frac{1}{n} < x \leq \frac{1}{n-1}, n = 3, 4, \dots, \\ x^2 + \frac{1}{4} & \text{if } x > \frac{1}{2}. \end{cases}$$

For any $n = 3, 4, \dots$ and $x \in (\frac{1}{n}, \frac{1}{n-1}]$, we have $\frac{1}{n} < f(x) \leq \frac{1}{(n-1)^2} + \frac{1}{n} - \frac{1}{n^2}$. Hence $[f \leq 0] = \emptyset$ and $f(x) \rightarrow 0$ as $x \downarrow 0$. At the points $x_n := \frac{1}{n-1}$, $n = 3, 4, \dots$, the function is continuous from the left. Moreover,

$$(x_n) - \lim_{x \downarrow x_n} f(x) = \frac{1}{(n-1)^2} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n^2} = \frac{n^2 - 3n + 1}{(n-1)^2 n^2} < 0.$$

Hence, f is lower semicontinuous. For any $x > 0$, we have $d(x, [f \leq 0]) = x$, $f^0(x) = 2x$ if $x \in (x_n, x_{n+1})$ and $\partial f(x_n) = [2x_n, +\infty)$, $n = 3, 4, \dots$, and consequently, $d(0, \partial f(x)) = 2x$ for all $x > 0$.

0. With $q = \frac{1}{2}$, we have d

$$\begin{aligned} (x_n) - d(0, \partial f(x_n)) &< \left(\frac{1}{n}\right)^{-\frac{1}{2}} \frac{2}{n-1} = \frac{2\sqrt{n}}{n-1} \rightarrow 0 \text{ as } n \rightarrow \infty \\ (x, [f \leq 0])^{q-1} d(0, \partial f(x))^q &= x^{-\frac{1}{2}} (2x)^{\frac{1}{2}} = \sqrt{2} > 0 \text{ for any } x > 0; \end{aligned}$$

hence, $\text{Er}_q^0 f(x^-) > 0$. At the same time, $f^q n \rightarrow \infty$; hence, $\text{Er}_q f(x^-) = 0$.

In some situations, it can be convenient to reformulate Theorem 3.11 in a slightly different form given in the next corollary.

Corollary 3.20 Suppose X is a Banach space, $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in [f \leq 0]$. Let $\tau > 0$, $\delta \in (0, \infty]$ and $p \geq 0$. If either X is Asplund or f is convex, and $d(0, \partial f(x)) \geq \tau d(x, [f \leq 0])^p$ for all $x \in B_\delta(x^-) \cap [f > 0]$

$$\text{with } f(x) < \tau d(x, [f \leq 0])^{p+1}, \quad (3.28)$$

$$\alpha(1-\alpha)^p \tau d(x, [f \leq 0])^{p+1} \leq f_+(x) \quad \text{for all } \alpha \in (0,1) \quad \text{and} \quad x \in B_{\frac{\delta}{1+\alpha}}(\bar{x}). \quad (3.29)$$

Proof Setting $q := \frac{1}{p+1}$ and replacing τ with τ^{p+1} in the statement of Theorem 3.11, reduces it to that of the above corollary. \square

Remark 3.21 Corollary 3.20 improves [47, Corollary 3.1], which claims a weaker conclusion under stronger assumptions. Condition (3.29) is referred to in [47] as $(p+1)$ -order error bound.

Combining Corollaries 3.8 and 3.14, and Proposition 3.16, we can formulate quantitative and qualitative sufficient subdifferential conditions for Holder error bounds.

Theorem 3.22 *Suppose X is a Banach space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in [f \leq 0]$. Let either X is Asplund or f is convex, $\tau > 0$, $q \in (0,1]$, and the convention $0^0 = 1$ be in force. Consider the following conditions:*

- (i) $\tau d(x, [f \leq 0]) \leq f_+^q(x)$ for all x near x^- ;
- (ii) $q f^{q-1}(x) d(0, \partial f(x)) \geq \tau$ for all $x \in [f > 0]$ near x^- ;
- (iii) $q^q (1-q)^{1-q} d(x, [f \leq 0])^{q-1} d(0, \partial f(x))^q \geq \tau$ for all $x \in [f > 0]$ near x^- .

Then (ii) \Rightarrow (i), (iii) \Rightarrow (i), and (ii) \Rightarrow (iii) with $(1-q)^{1-q} \tau$ in place of τ . If $q = 1$, then conditions (ii) and (iii) coincide.

Corollary 3.23 *Suppose X is a Banach space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, $x^- \in [f \leq 0]$ and $q \in (0,1]$. Let either X is Asplund or f is convex. f admits a q -order local error bound at x if one of the following conditions is satisfied:*

- (i) $\liminf_{x \rightarrow x^-, f(x) \downarrow 0} f^{q-1}(x) d(0, \partial f(x)) > 0$;
- (ii) $\liminf_{f(x) \downarrow 0} d(x, [f \leq 0])^{q-1} d(0, \partial f(x))^q > 0$.

The last inequality in (3.20) involving the q th power of the function f can sometimes be replaced by a similar inequality involving the function f itself.

Proposition 3.24 *Suppose X is a Banach space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in [f \leq 0]$. Let $\tau > 0$, $\delta \in (0, \infty]$, $\beta > 0$ and $q \in (0,1)$. If either X is Asplund or f is convex, and $d(x, [f \leq 0])^{q-1} d(0, \partial f(x))^q \geq \tau$ for all $x \in B_\delta(x^-) \cap [f$*

> 0]

$$\text{with } f(x) < \beta \|x - x^-\|, \quad (3.30)$$

then, with $r := \min \left\{ \delta, \beta^{\frac{q}{1-q}} \tau^{-\frac{1}{1-q}} \right\}$,

$$(1-\alpha)^{1-q} \tau d(x, [f \leq 0]) \leq f_+^q(x) \quad \text{for all } \alpha \in (0,1) \text{ and } x \in B_r(x^-). \quad (3.31)$$

Proof If $x \in B_r(x^-) \cap [f > 0]$, then $x \in B_\delta(x^-) \cap [f > 0]$. If, additionally, $f^q(x) < \tau \|x - x^-\|$, then

$$f(x) < \tau^{\frac{1}{q}} \|x - x^-\|^{\frac{1}{q}} < \tau^{\frac{1}{q}} r^{\frac{1}{q}-1} \|x - x^-\| \leq \beta \|x - x^-\|$$

Hence, condition (3.30) implies (3.20) with r in place of δ . The statement follows from Theorem 3.11. \square

Remark 3.25 The above proposition is formulated for the case $q < 1$. When $q = 1$, a similar assertion is trivially true with $\beta \geq \tau$ (as a consequence of Theorem 3.4), but, as the next

example shows, fails when $\theta < \tau$. This example shows also that [47, Proposition 3.1] fails

when $p = 0$.

Example 3.26 Let $f(x) = |x|$ ($x \in \mathbb{R}$), $\bar{x} = 0$, $\tau = 2$, $\beta = \frac{1}{2}$ and $q = 1$. Then there are no $x \in [f > 0]$ with $f(x) < \theta|x|$, i.e. condition (3.30) is trivially satisfied with any $\delta > 0$. Similarly, $\tau|x| > f(x)$ for any $x \neq 0$, i.e. condition (3.31) fails with any $r > 0$.

Remark 3.27 (i) One can easily formulate a statement similar to Proposition 3.13 for the error bounds statements in Proposition 3.24 and Corollary 3.20. In the latter case, the sharpest error bound estimate in (3.29) corresponds to taking $\alpha = \frac{1}{p+1}$, where the maximum of $\alpha(1-\alpha)^p$ over $\alpha \in (0,1)$ is attained.

- (ii) The neighbourhood $B_{\frac{\delta}{1+\alpha}}(\bar{x})$ in (3.15), (3.21) and (3.29), and the $\frac{\delta}{1+q}$ neighbourhood $B(x^-)$ in (3.22) and (3.24) can always be replaced by the $\frac{\delta}{2}$ smaller neighbourhood $B(x^-)$, independent of α or q . A similar simplification is possible also in (3.31).
- (iii) Conditions (3.14) in Theorem 3.11, (3.20) in Corollary 3.12, (3.28) in Corollary 3.20, (iii) in Theorem 3.22, (ii) in Corollary 3.23 and (3.30) in Proposition 3.24, although sufficient for the corresponding Holder error bound estimates, do not seem practical as they involve the unknown distance $d(x, [f \leq 0])$, which error bounds are supposed to estimate. This remark also applies to the next more general theorem. Nevertheless, such conditions are in use in the literature; see [47,51,52].

The next theorem combines the sufficient Holder error bound conditions from Theorems 3.7 and 3.11 in a single statement. It is still a consequence of Lemma 3.1.

Theorem 3.28 Suppose X is a Banach space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^- \in [f \leq 0]$. Let $\tau > 0$, $\delta \in (0, \infty]$, $\lambda \in [0,1]$ and $q \in (0,1]$. If either X is Asplund or f is convex, and

$$\left(\frac{\lambda}{d(x, [f \leq 0])^{\frac{1}{q}-1}} + q(1-\lambda)f^{q-1}(x) \right) d(0, \partial f(x)) \geq \tau \text{ for all } x \in B_\delta(x^-) \cap [f > 0]$$

$$\text{with } \lambda f(x)^{\frac{1}{q}-1} + (1-\lambda)f^q(x) < \tau d(x, [f \leq 0]), \quad (3.32)$$

$$d(x, [f \leq 0])$$

then

$$\tau_\alpha d(x, [f \leq 0]) \leq f_+^q(x) \quad \text{for all } \alpha \in (0,1) \quad \text{and} \quad x \in B_{\frac{\delta}{1+\alpha}}(\bar{x}), \quad (3.33)$$

where $\tau_\alpha > 0$ is the unique solution for the equation

$$\frac{\lambda \tau_\alpha^{\frac{1}{q}}}{(1-\alpha)^{\frac{1}{q}-1}} + (1-\lambda)\tau_\alpha = \alpha \tau \quad (3.34)$$

Proof Observe that the function $t \mapsto \phi(t) := \lambda(1-\alpha)^{1-\frac{1}{q}}t^{\frac{1}{q}} + (1-\lambda)t$ is continuous and strictly increasing on \mathbb{R}_+ and satisfies $\phi(0) = 0$ and $\lim_{t \rightarrow \infty} \phi(t) = \infty$. Hence, the equation (3.34) has a solution for any $\alpha > 0$ and $\tau > 0$, which is unique. Suppose that condition (3.33) is not satisfied, i.e.

$$f_+(x) < \tau_\alpha d(x, [f \leq 0]) \quad (3.35)$$

for some $\alpha \in (0,1)$ and $x \in B_{\frac{\delta}{1+\alpha}}(\bar{x})$. Then $d(x, [f \leq 0]) > 0$, and consequently, $f_+(x) =$

$\alpha f(x) > 0$. Consider a

function $g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$g(u) := \frac{\lambda f_+(u)}{((1-\alpha)d(x, [f \leq 0]))^{\frac{1}{q}-1}} + (1-\lambda)f_+^q(u), \quad u \in X. \quad (3.36)$$

It is obviously lower semicontinuous, $g(u) \geq 0$ for all $u \in X$, $[g \leq 0] = [g = 0] = [f \leq 0]$ and $g(x) > 0$. Observe that $g = \psi \circ f_+$, where $\psi(t) := \lambda((1-\alpha)d(x, [f \leq 0]))^{1-\frac{1}{q}}t + (1-\lambda)t^q$, and $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly increasing and continuously differentiable on $(0, \infty)$. Hence, by (3.35),

$$\begin{aligned} g(x) &= \psi(f(x)) < \psi(\tau_\alpha d(x, [f \leq 0]))^{\frac{1}{q}} \\ &= \left(\frac{\lambda \tau_\alpha^{\frac{1}{q}}}{(1-\alpha)^{\frac{1}{q}-1}} + (1-\lambda)\tau_\alpha \right) d(x, [f \leq 0]) = \alpha \tau d(x, [g \leq 0]) \end{aligned}$$

g

Thus, $0 < g(x) < \alpha \tau d(x, [g \leq 0])$. By Lemma 3.1, there exists a $u \in X$ such that $\|ku - x\| <$

$$\alpha d(x, [f \leq 0]), \quad g(u) < \tau d(u, [f \leq 0]) \text{ and } d(0, \partial g(u)) < \tau. \quad (3.37)$$

Hence, $f(u) > 0$. The first inequality in (3.37) immediately yields estimates (3.16) and (3.17). By (3.36) and the second inequality in (3.37), we have

$$\frac{\lambda f(u)}{(u, [f \leq 0])^{\frac{1}{q}-1}} + (1-\lambda)f^q(u) \leq g(u) < \tau d(u, [f \leq 0]) \quad (3.38)$$

Applying Lemma 2.4, we get

$$\partial g(u) = \left(\frac{\lambda}{((1-\alpha)d(x, [f \leq 0]))^{\frac{1}{q}-1}} + q(1-\lambda) \right) f^{q-1}(u) \partial f(u),$$

and consequently, by (3.17) and the third inequality in (3.37),

$$\left(\frac{\lambda}{d(u, [f \leq 0])^{\frac{1}{q}-1}} + q(1-\lambda)f^{q-1}(u) \right) d(0, \partial f(u)) < d(0, \partial g(u)) < \tau.$$

In view of (3.16) and (3.38), this contradicts (3.32) and completes the proof. \square

Remark 3.29 When $\lambda = 0$, Theorem 3.28 reduces to Theorem 3.7 except for the case $\alpha = 1$ in (3.11). When $\lambda = 1$, Theorem 3.28 reduces to Theorem 3.11. When $q = 1$, Theorem 3.28 reduces to Theorem 3.4 except for the case $\alpha = 1$ in (3.9). The case $\alpha = 1$ in (3.11) when $\lambda = 0$ and in (3.9) when $q = 1$ is an immediate consequence of the case $\alpha \in (0, 1)$; see the argument in the proof of Proposition 3.13.

The next statement is a simplified version of Theorem 3.28.

Corollary 3.30 *Suppose X is a Banach space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x^* \in [f \leq 0]$. Let $\tau > 0$, $\delta \in [0, \infty)$, $\lambda \in [0, 1]$ and $q \in [0, 1]$. If either X is Asplund or f is convex, and*

$$\left(\frac{\lambda}{d(x, [f \leq 0])^{\frac{1}{q}-1}} + q(1-\lambda)f^{q-1}(x) \right) d(0, \partial f(x)) \geq \tau \text{ for all } x \in B_\delta(x^-) \cap [f > 0]$$

with $\frac{\lambda f(x)}{\|x-x^-\|} + (1-\lambda)f(x) < \tau$

$$d(x, [f \leq 0])^q$$

then condition (3.33) holds true.

3.3 Convex case

In this subsection X is a normed vector space and the function $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is assumed convex. The statement of Lemma 3.1 can be partially reversed (at the reference point).

Lemma 3.31 Suppose X is a normed vector space, $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex, $x \in X$, $f(x) > 0$ and $\tau > 0$. If

$$\tau d(x, [f \leq 0]) \leq f(x), \quad (3.39)$$

then $d(0, \partial f(x)) \geq \tau$.

Proof Let condition (3.39) be satisfied and $x^* \in \partial f(x)$. Then, for any $u \in [f \leq 0]$, we have

$$\|x^*\| \|u - x\| \geq -\langle x^*, u - x \rangle \geq f(x) - f(u) \geq f(x) \geq \tau d(x, [f \leq 0]).$$

Taking the infimum in the left-hand side over all $u \in [f \leq 0]$, we get $\|x^*\| \geq \tau$, which concludes the proof. \square

Combining Theorem 3.4 and Lemma 3.31, we can formulate the standard subdifferential linear error bound criterion for convex functions.

Theorem 3.32 Suppose X is a Banach space, $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex lower semicontinuous, $x^- \in [f \leq 0]$ and $\tau > 0$. The following conditions are equivalent:

- (i) $\tau d(x, [f \leq 0]) \leq f_+(x)$ for all x near x^-
- (ii) $d(0, \partial f(x)) \geq \tau$ for all $x \in [f > 0]$ near x^- .

The convex case ‘reverse’ linear error bound statement in Lemma 3.31 can also be easily adjusted to the Holder setting both in the ‘conventional’ form as in Theorem 3.7 and its modification as in Theorem 3.11. It is easy to see that the conclusion of the next lemma is actually a combination of two different conditions.

Lemma 3.33 Suppose X is a normed vector space, $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex, $x \in X$ and $f(x) > 0$. Let $\tau > 0$ and $q \in (0, 1]$. If

$$\tau d(x, [f \leq 0]) \leq f^q(x), \quad (3.40)$$

then $d(0, \partial f(x)) \geq \max \left\{ \tau f^{1-q}(x), \tau^{\frac{1}{q}} d(x, [f \leq 0])^{\frac{1}{q}-1} \right\}$ or equivalently, $\min \left\{ f^{q-1}(x) d(0, \partial f(x)), d(x, [f \leq 0])^{q-1} d(0, \partial f(x))^q \right\} \geq \tau$.

Proof Condition (3.40) can be rewritten as

$$\tau f^{1-q}(x) d(x, [f \leq 0]) \leq f(x).$$

Applying Lemma 3.31 with $\tau^0 := \tau f^{1-q}(x)$ in place of τ , we get $d(0, \partial f(x)) \geq \tau f^{1-q}(x)$. Similarly, rewriting condition (3.40) as

$$\tau^{\frac{1}{q}} d(x, [f \leq 0])^{\frac{1}{q}-1} d(x, [f \leq 0]) \leq f(x),$$

and applying Lemma 3.31 with $\tau' := \tau^{\frac{1}{q}} d(x, [f \leq 0])^{\frac{1}{q}-1}$ in place of τ , we get $d(0, \partial f(x)) \geq \tau^{\frac{1}{q}} d(x, [f \leq 0])^{\frac{1}{q}-1}$. tu

Combining Theorem 3.22 and Lemma 3.33, we can formulate quantitative and qualitative subdifferential characterizations of Hölder error bounds for convex functions.

Theorem 3.34 *Suppose X is a Banach space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex lower semicontinuous and $x^- \in [f \leq 0]$. Let $\tau > 0$, $q \in (0, 1]$, and the convention $0^0 = 1$ be in force. Consider conditions (i), (ii) and (iii) in Theorem 3.22. Then*

- (a) (ii) \Rightarrow (i) and (i) \Rightarrow (ii) with $q\tau$ in place of τ ;
- (b) (iii) \Rightarrow (i) and (i) \Rightarrow (iii) with $q^q(1-q)^{1-q}\tau$ in place of τ .

If $q = 1$, then all the conditions are equivalent.

Corollary 3.35 *Suppose X is a Banach space, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex lower semicontinuous, $x^- \in [f \leq 0]$ and $q \in (0, 1]$. The following conditions are equivalent:*

- (i) $\tau d(x, [f \leq 0]) \leq f_+^q(x)$ for some $\tau > 0$ and all x near x^- ;
- (ii) $\liminf_{x \rightarrow x^-, f(x) > 0} f^{q-1}(x) d(0, \partial f(x)) > 0$;
- (iii) $\liminf_{x \rightarrow x^-, f(x) > 0} d(x, [f \leq 0])^{q-1} d(0, \partial f(x))^q > 0$.

4 Applications to convex semi-infinite optimization

In this section, we mainly consider the following convex optimization problem

$$\begin{aligned} P(c, b) : \quad & \text{minimize} \quad f(x) + \langle c, x \rangle \\ & \text{subject to} \quad g_t(x) \leq b_t, \quad t \in T, \end{aligned} \tag{4.1}$$

where $c, x \in \mathbb{R}^n$, T is a compact set in a metric space Z such that $T \times Z, f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g_t : \mathbb{R}^n \rightarrow \mathbb{R}, t \in T$, are given convex functions such that $(t, x) \mapsto g_t(x)$ is continuous on $T \times \mathbb{R}^n$, and $b \in C(T, \mathbb{R})$, i.e., $T \ni t \mapsto b_t \in \mathbb{R}$ is continuous on T . In this setting, the pair $(c, b) \in \mathbb{R}^n \times C(T, \mathbb{R})$ is regarded as the parameter to be perturbed. The parameter space $\mathbb{R}^n \times C(T, \mathbb{R})$ is endowed with the norm

$$\|(c, b)\| := \max\{\|c\|, \|b\|_{k_\infty}\}, \tag{4.2}$$

where \mathbb{R}^n is equipped with the Euclidean norm $\|\cdot\|$ and $\|b\|_{k_\infty} := \max_{t \in T} |b_t|$.

Our aim here is to analyze the *solution mapping* (also called *argmin mapping*) of problem (4.1):

$$S : (c, b) \mapsto \{x \in \mathbb{R}^n \mid x \text{ solves } P(c, b)\} \text{ with } (c, b) \in \mathbb{R}^n \times C(T, \mathbb{R}).$$

In the special case that c is fixed, S reduces to the partial solution mapping $S_c : C(T, \mathbb{R}) \Rightarrow \mathbb{R}^n$ given by

$$S_c(b) = S(c,b).$$

Associated with the parameterized problem $P(c,b)$, we denote by F the feasible set mapping, which is given by

$$F(b) := \{x \in \mathbb{R}^n \mid g_t(x) \leq b_t, t \in T\}.$$

The set of *active indices* at $x \in F(b)$ is the set $T_b(x)$ defined by

$$T_b(x) := \{t \in T \mid g_t(x) = b_t\}.$$

We say that the problem $P(c,b)$ satisfies the *Slater constraint qualification* (hereinafter called the *Slater condition*) if there exists $\hat{x} \in \mathbb{R}^n$ such that $g_t(\hat{x}) < b_t$ for all $t \in T$. The following well-known result (see [16, Theorems 7.8 and 7.9]) plays a key role in our analysis.

Proposition 4.1 *Let $(c^-, b^-) \in \mathbb{R}^n \times C(T, \mathbb{R})$ and assume that $P(c^-, b^-)$ satisfies the Slater condition. Then $x^- \in S(c^-, b^-)$ if and only if the Karush-Kuhn-Tucker (KKT) conditions hold, i.e.,*

$$\begin{aligned} & \square \\ & x^- \in F(b^-) \quad \text{and} \quad -(\partial f(x^-) + c^-) \setminus \left(\text{cone} \left(\bigcup_{t \in T_b^-(x^-)} \partial g_t(x^-) \right) \right) \neq \emptyset \end{aligned}$$

Here $\text{cone}(X)$ represents the conical convex hull of X , and we assume that $\text{cone}(X)$ always contain the zero-vector 0_n , in particular $\text{cone}(0) = \{0_n\}$.

In this section we provide a characterization for Holder calmness of S at $((c^-, b^-), x^-)$. To this aim, we use the following *level set mapping* $L : \mathbb{R} \times C(T, \mathbb{R}) \Rightarrow \mathbb{R}^n$ given by

$$L(\alpha, b) := \{x \in \mathbb{R}^n \mid f(x) + hc^-, x - x^- \leq \alpha; g_t(x) \leq b_t, t \in T\}$$

and the *supremum function* $f^- : \mathbb{R}^n \rightarrow \mathbb{R}$ defined as

$$\begin{aligned} f^-(x) &:= \sup \{f(x) - f(x^-) + hc^-, x - x^- \leq i; g_t(x) - b_t^-, t \in T\} \\ &= \sup \{f(x) + hc^-, x - i - (f(x^-) + hc^-, x^- \leq i); g_t(x) - b_t^-, t \in T\}. \end{aligned} \quad (4.3)$$

(See [6, (11) and (12)] for the linear counterparts of L and f^- .)

For a given $t_0 \in \overline{T} \setminus T$, we define

$$\overline{T} := T \cup \{t_0\}, \quad g_{t_0}(x) := f(x) + hc^-, x - i \quad \text{and} \quad b_{t_0}^- := f(x^-) + hc^-, x^- \leq i.$$

As T is a compact set (and t_0 is an isolated point in \overline{T}), the function $(t, x) \rightarrow g_t(x)$ is continuous on $\overline{T} \times \mathbb{R}^n$, $b \in C(\overline{T}, \mathbb{R})$ and, obviously,

$$f^-(x) = \sup \{g_t(x) - b_t^-, t \in \overline{T}\}.$$

For any $x \in \mathbb{R}^n$, we consider the extended active set

$$\overline{T}(x) := \{t \in \overline{T} : g_t(x) - b_t^- = f^-(x)\}.$$

The following well-known result is useful for us (e.g. [17, VI, Theorem 4.4.2]).

$$\partial f^-(x) = \text{co} \left(\bigcup_{t \in T(x)} \partial g_t(x) \right). \quad (4.4)$$

Observe that

$$\partial g_{t_0}(x) = \partial f(x) + c^-.$$

Since $((c^-, b^-), x^-) \in \text{gph}(S)$,

$$S(\bar{c}, \bar{b}) = [\bar{f} = 0] = [\bar{f} \leq 0] = \mathcal{L}(f(\bar{x}) + \langle \bar{c}, \bar{x} \rangle, \bar{b}). \quad (4.5)$$

Observe that $t_0 \in T(x^-)$. Consequently $0_n \in \partial \bar{f}(\bar{x})$, and by (4.4)

$$0_n = \sum_{i=1}^p \lambda_i u^i,$$

with $u^i \in \partial g_{t_i}(\bar{x})$, $\{t_i, i = 1, 2, \dots, p\} \subset \bar{T}(\bar{x})$, $\lambda_i > 0$ and $\sum_{i=1}^p \lambda_i = 1$.

If $P(c^-, b^-)$ satisfies the Slater condition, t_0 must be one of the indices involved in the sum above, and we shall write

$$0_n = \mu_0(u^0 + c) + \sum_{i=1}^q \mu_i u^i, \quad (4.6)$$

with $u^0 \in \partial f(x)$, $u^i \in \partial g_{t_i}(x)$, $\{t_i, i = 1, 2, \dots, q\} \subset T_b(x)$, $\mu_0 > 0$, $\mu_i \geq 0$, $i = 1, 2, \dots, q$, and $\sum_{i=0}^q \mu_i = 1$.

Otherwise $0_n \in \text{co} \left(\bigcup_{t \in T_b(\bar{x})} \partial g_t(\bar{x}) \right)$ and x would be a global minimum of the function $\phi(\cdot) := \sup\{g_t(\cdot) - b^-, t \in T\}$,

giving rise to the contradiction $\phi(x) < 0 = \phi(\bar{x})$. Observe that it may happen that $\mu_0 = 1$ and the sum in (4.6) vanishes (this is the case if $T_b(x) = \emptyset$).

The following lemma provides a uniform boundedness result which is needed later. It constitutes a convex counterpart of [5, Lemma 3.2].

Lemma 4.2 *Let $((c^-, b^-), x^-) \in \text{gph}(S)$ be given and assume that $P(c^-, b^-)$ satisfies the Slater condition. Then there exist $M > 0$ and neighborhoods U of x and V of*

(c^-, b^-) such that, for all $(c, b) \in V$ and all $x \in S(c, b) \cap U$, there exists $u \in \partial f(x)$ satisfying

$$-(c + u) \in [0, M] \text{co} \left(\bigcup_{t \in T_b(x)} \partial g_t(x) \right). \quad (4.7)$$

Proof The result follows arguing by contradiction. By continuity we can assume that $P(c, b)$ satisfies the Slater condition at any $(c, b) \in V$. Then, thanks to the KKT conditions, together with Caratheodory Theorem, there would exist a sequence $\text{gph}(S) \ni ((c^r, b^r), x^r) \rightarrow ((c^-, b^-), x^-)$ such that

$$-(c^r + u^r) = \sum_{i=1}^n \lambda_i^r u_i^r \quad (4.8)$$

for some $\{t_1^r, \dots, t_n^r\} \subset T_{b^r}(x^r)$, $\lambda_i^r \geq 0$, and some $u^r \in \partial f(x^r)$, $u_i^r \in \partial g_{t_i^r}(x^r)$, $r \in \mathbb{N}$, verifying $\sigma_r := \sum_{i=1}^n \lambda_i^r \rightarrow +\infty$ as r tends to $+\infty$.

If we apply a filtering process as in [4, Lemma 3.1], based on the compactness of T and [42, Theorem 24.5], we get the existence of points $\{t_1, \dots, t_n\} \subset T_b(x)$ such that $t_i^r \rightarrow t_i$, $i = 1, 2, \dots, n$, and $u_i^r \in \partial g_{t_i^r}(x^r)$, $i = 1, 2, \dots, n$, such that $u_i^r \rightarrow u_i$, $i = 1, 2, \dots, n$. Then, dividing both terms in (4.8) by σ_r and taking limits as $r \rightarrow \infty$ (after filtering again with respect to the bounded coefficients λ_i^r / σ_r , $i = 1, 2, \dots, n$), we reach the same contradiction with the Slater condition. \square

The following proposition gives a characterization of the q -order calmness property

for the level set mapping L in terms of the supremum function defined by (4.3). It constitutes a Holder convex counterpart of [5, Proposition 3.1] (see also [6, Theorem 4]). Recall that $q \in (0,1]$.

Proposition 4.3 *Let $((c^-, b^-), x^-) \in \text{gph}(S)$. Then the following are equivalent: (i)*

L is q -order calm at $((f(x^-) + hc^-, x^-, i, b^-), x^-) \in \text{gph}(L)$;

(ii) $\liminf_{x \rightarrow x^-, f(x) \downarrow 0} f(x)^{q-1} d(0, \partial f(x)) > 0$.

Proof This result is a direct consequence of the equivalence between the q -order calmness of L at $((f(x^-) + hc^-, x^-, i, b^-), x^-)$ and the existence of a q -order error bound of f at x^- , together with Corollary 3.35, (4.5), and the following inequalities:

$$\begin{aligned} [\bar{f}(x)]_+ &= [\sup\{f(x) - f(\bar{x}) + \langle \bar{c}, x - \bar{x} \rangle; g_t(x) - \bar{b}_t, t \in T\}]_+ \\ &= \sup\{[f(x) - f(\bar{x}) + \langle \bar{c}, x - \bar{x} \rangle]_+; [g_t(x) - \bar{b}_t]_+, t \in T\} \\ &= d((f(x^-) + hc^-, x^-, i, b^-), L^{-1}(x)) \text{ for all } x \in \mathbb{R}^n. \end{aligned} \quad (4.9)$$

The proof is complete. tu

Proposition 4.4 *Assume that L is not q -order calm at $((f(x^-) + hc^-, x^-, i, b^-), x^-) \in \text{gph}(L)$. Then, there will exist sequences $\{x^r\}_{r \in \mathbb{N}}$ converging to x^- with $f(x^r) \downarrow 0$, and $\{v^r\}_{r \in \mathbb{N}} \subset \partial \bar{f}(x^r) \setminus \{0_n\}$, such that v^r converges to 0_n , and*

$$(x^r, \mathcal{S}(\bar{c}, \bar{b})) \geq \frac{\bar{f}(x^r)}{\|v^r\|} \quad (4.10) \quad \text{d}$$

Proof Certainly, if

$$\liminf_{x \rightarrow x^-, f(x) \downarrow 0} f(x)^{q-1} d(0, \partial f(x)) = 0,$$

there must exist sequences $x^r \rightarrow x^-$, with $f(x^r) \downarrow 0$, and $v^r \in \partial \bar{f}(x^r)$, such that

$$\lim_{r \rightarrow +\infty} f(x^r)^{q-1} v^r = 0_n \quad (4.11)$$

entailing $v^r \rightarrow 0_n$ as

$$\lim_{r \rightarrow +\infty} \bar{f}(x^r)^{q-1} = \begin{cases} +\infty, & \text{if } q < 1 \\ 1, & \text{if } q = 1 \end{cases}$$

First, we observe that $v^r \neq 0_n$ for $r = 1, 2, \dots$. Otherwise, i.e. if $v^r = 0_n$ for some r , then

$0_n \in \partial \bar{f}(x^r)$ and x^r is a (global) minimum of the convex function f , entailing $f(x^r) \leq$

$f(x) = 0$, but this contradicts $f(x^r) > 0$.

Finally, (4.10) follows from the obvious fact

$$S(c^-, b^-) = \{x \in \mathbb{R}^n \mid f(x) = 0\} = \{x \in \mathbb{R}^n \mid 0_n \in \partial f(x)\}.$$

Since $v^r \in \partial \bar{f}(x^r)$, we have

$$x \in S(c^-, b^-) \Rightarrow 0 = f(x) \geq f(x^r) + hv^r, x - x^r,$$

and we conclude

$$S(\bar{c}, \bar{b}) \subset \{x \in \mathbb{R}^n \mid \langle v^r, x \rangle \leq \langle v^r, x^r \rangle - f^-(x^r)\}.$$

Applying the well-known Ascoli formula for the distance to a hyperplane we get

$$d(x^r, S(\bar{c}, \bar{b})) \geq \frac{[\bar{f}(x^r)]_+}{\|v^r\|} = \frac{\bar{f}(x^r)}{\|v^r\|}.$$

tu

The following proposition provides a necessary condition in the case that L is not q -order calm. The proof updates some arguments in [5, Theorem 3.1] to the convex q -Holder setting.

Proposition 4.5 *Let $((c^-, b^-), x^-) \in \text{gph}(S)$ and assume that $P(c^-, b^-)$ satisfies the Slater condition. Suppose that L is not q -order calm at $((f(x^-) + \langle hc^-, x^- \rangle, b^-), x^-) \in \text{gph}(L)$. Then there exist sequences $\{x^r\}_{r \in \mathbb{N}}$ converging to x^- and $\{b^r\}_{r \in \mathbb{N}} \subset C(T, \mathbb{R})$ converging to b^- such that*

$$x^r \in F(b^r), \lim_{r \rightarrow +\infty} \frac{d(x^r, S_c(b^r))}{d(x^r, S_c(b^-))} = 0, \quad (4.12)$$

as well as a finite set $T_0 \subset T_{b^r}(x^r)$ satisfying

$$-(c^- + u) \in \sum_{t \in T_0} \gamma_t u_t \quad (4.13)$$

$$t \in T_{b^r}(x^r)$$

with $\gamma_t > 0$, $u_t \in \partial g_t(x^-)$, $t \in T_0$, and $u \in \partial f(x^-)$.

Proof We have established that there exist sequences $\{x^r\}_{r \in \mathbb{N}}$ converging to x^- with $f^-(x^r) \downarrow 0$, and $\{v^r\}_{r \in \mathbb{N}}$, $v^r \in \partial \bar{f}(x^r) \setminus \{0_n\}$, such that $v^r \rightarrow 0$ and (4.10) and (4.11) hold.

Applying Proposition 5.1 (remember that $P(c^-, b^-)$ satisfies the Slater condition), we know that, associated with $x^- \in S(c^-, b^-)$, there is a finite subset $T_0 \subset T_{b^-}(x^-)$ such that

$$-(c^- + u) = \sum_{t \in T_0} \gamma_t u_t \quad (4.14)$$

for some $\gamma_t > 0$, $u_t \in \partial g_t(x^-)$, $t \in T_0$, and $u \in \partial f(x^-)$. Now we proceed by showing the existence of $N > 0$ such that

$$g_t(x^r) - b_t^- \geq -N f^-(x^r) \quad \forall t \in T_0 \text{ and } r \in \mathbb{N}. \quad (4.15)$$

We have that (4.14) gives rise to

$$\begin{aligned} - \sum_{t \in T_0} \gamma_t (g_t(x^r) - b_t^-) &= - \sum_{t \in T_0} \gamma_t (g_t(x^r) - g_t(x^-)) \leq - \sum_{t \in T_0} \gamma_t \langle u_t, x^r - x^- \rangle \\ &= \langle hc^- + u, x^r - x^- \rangle \leq \langle hc^-, x^r - x^- \rangle + f(x^r) - f(x^-) \\ &= f^-(x^r). \end{aligned} \quad (4.16)$$

$$|b_t^r - \bar{b}_t| \leq (N+1)\bar{f}(x^r), \quad (4.19)$$

In addition, (4.15) yields $T_0 \subset T_{b^r}(x^r)$ (as $\phi_r(t) = 0$ if $t \in T_0$). This, together with (4.14), leads us to (4.13).

Finally, appealing to (4.10), (4.19) and (4.11), we prove (4.12) as follows:

$$\begin{aligned} \|b^r - b^-\|_{\infty} &\leq \lim_{r \rightarrow +\infty} \frac{\|d(x^r, S(\bar{c}, \bar{b}))\|^q}{\|\bar{f}(x^r)\|^q} \\ &= \lim_{r \rightarrow +\infty} (N+1)^q \frac{\|\bar{f}(x^r)^{q-1} \|v^r\|}{\|v^r\|^q} = 0 \\ &\leq \lim_{r \rightarrow +\infty} (N+1) \bar{f}(x) \end{aligned} \quad (4.20)$$

The proof is complete. tu

Remark 4.6 The following example shows that, in the convex setting, the condition $x^r \in F(b^r)$ cannot be strengthened to $x^r \in S_c(b^r)$ for the sequence $\{x^r\}_{r \in \mathbb{N}}$ in Proposition 4.5 as it happens in the linear case (see the proof of [5, Theorem 3.1]). Consider the convex problem in \mathbb{R} :

$$\begin{aligned} &\text{minimize } x^2 \text{ subject} \\ &\text{to } x \leq 0. \end{aligned}$$

Given $q \in (\frac{1}{2}, 1]$, take $\bar{c} = 0$, $b^- = 0$, and $\bar{b} = 0$. Then $S_c(b^-) = \{0\}$ and the supremum function $f^-(x) = \sup\{x^2, x\}$. Clearly, $f^-(x) = x^2$ for $x \in (-\infty, 0]$. Then it is easy to verify that $\liminf_{x \rightarrow \bar{x}, \bar{f}(x) \downarrow 0} \bar{f}(x)^{q-1} d(0, \partial \bar{f}(x)) = 0$ and, thus, by Proposition 4.3, the level set mapping L is not q -order calm at $((0,0), 0) \in \text{gph}(L)$. Moreover, there exist sequences $x^r := -2^{-r}$ and $b^r := 2^{-2r}$ such that

$$x^r = -2^{-r} \in F(2^{-2r}) = F(b^r) \quad \text{and} \quad \lim_{r \rightarrow +\infty} \frac{\|b^r - \bar{b}\|_{\infty}^q}{d(x^r, S_c(\bar{b}))} = \lim_{r \rightarrow +\infty} \frac{1}{2^{(2q-1)r}} = 0. \quad (4.21)$$

Recalling that $\text{cone}(0) = \{0\}$, (4.13) also holds in this setting. Obviously, $x^r \in F(b^r)$ but $x^r \notin S_c(b^r) = \{0\}$ for any $r \in \mathbb{N}$. On the other hand, we have

$$S_c(b) = \begin{cases} \{0\} & \forall b \in [0, 1), \\ \{b\} & \forall b \in (-1, 0) \end{cases} \quad (4.22)$$

Noting that $\|b\| \leq \|b\|^{\frac{2}{3}}$ for all $b \in (-1, 1)$, it readily follows from (4.22) that

$$d(x, S_c(\bar{b})) \leq \|b - \bar{b}\|^{\frac{2}{3}} \quad \forall x \in S_c(b) \cap (-1, 1) \text{ and } b \in (-1, 1), \quad (4.23)$$

which guarantees that S_c is $\frac{2}{3}$ -order calm at $(0,0)$.

The above example reveals the fact that the q -order calmness of S at $((c^-, b^-), x^-)$ may not imply the validity of the q -order calmness of L at $((f(x^-) + hc^-, x^-, i, b^-), x^-)$. The following theorem constitutes a Hölder convex counterpart of [5, Theorem 3.1] for the linear case.

Theorem 4.7 Let $x^- \in S(c^-, b^-)$ and assume that $P(c^-, b^-)$ satisfies the Slater condition.

Consider the following statements: (i) S is q -order calm at $((c^-, b^-), x^-)$;

(ii) S_c^- is q -order calm at (b^-, x^-) ;

(iii) L is q -order calm at $((f(x^-) + hc^-, x^-, i, b^-), x^-)$;

(iv) f has a q -order local error bound at x^- .

Then (iii) \Leftrightarrow (iv) \Rightarrow (i) \Rightarrow (ii) hold. In addition, if f and g_t are linear, then (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv).

Proof (iii) \Leftrightarrow (iv) is Proposition 4.3, while (i) \Rightarrow (ii) is obvious. Now, we proceed by proving that (iv) \Rightarrow (i). According to (iv), there exist $\tau, \delta \in (0, +\infty)$ such that

$$\tau d(x, [\bar{f} \leq 0]) \leq \bar{f}_+(x)^q \quad \forall x \in B_\delta(\bar{x}).$$

According to Lemma 4.2, we may suppose that (4.7) holds for $U = B_\delta(x^-)$, together with a certain neighborhood V of (c^-, b^-) and a certain $M > 0$. Then, for all $(c, b) \in V$ and all $x \in S(c, b) \cap U \cap [f^- > 0]$, it follows from (4.5) that $\tau d(x, S(c^-, b^-)) = \tau d(x, [f^- \leq 0]) \leq f^-(x)^q$

$$\begin{aligned} &= \left[\sup \{ f(x) - f(\bar{x}) + \langle \bar{c}, x - \bar{x} \rangle; g_t(x) - \bar{b}_t, t \in T \} \right]^q \\ &\leq \left[\sup \{ f(x) - f(\bar{x}) + \langle \bar{c}, x - \bar{x} \rangle; b_t - \bar{b}_t, t \in T \} \right]^q, \end{aligned} \quad (4.24)$$

where we have used $x \in F(b)$. Let us take

$$-(u+c) = \sum_{t \in T_0} \eta_t u_t$$

for some finite subset $T_0 \subset T_b(x)$, $u \in \partial f(x)$, $u_t \in \partial g_t(x)$, and some $\eta_t > 0$, $t \in T_0$, satisfying $\sum_{t \in T_0} \eta_t \leq M$. Then we have

$$\begin{aligned} -\langle hu+c, x-x^- \rangle &= \sum_{t \in T_0} \eta_t \langle u_t, x-x^- \rangle = \sum_{t \in T_0} \eta_t \langle hu_t, x-x^- \rangle \\ &\geq \sum_{t \in T_0} \eta_t (g_t(x) - g_t(x^-)) \geq \sum_{t \in T_0} \eta_t (b_t - \bar{b}_t) \\ &\geq -M \|b - b^-\|_{k_\infty}, \end{aligned} \quad (4.25)$$

which implies, from $u \in \partial f(x)$, that $f(x) - f(x^-) + hc^-, x-x^- \leq hu+c^-, x-x^- =$

$$\begin{aligned} &hu+c^-, x-x^- - hc^-, x-x^- \\ &\leq M \|b - b^-\|_{k_\infty} + \|c - c^-\|_{k_c} \|x - x^-\|_{k_c}. \end{aligned} \quad (4.26)$$

Recalling that $\|x - x^-\|_{k_c} \leq \delta$ for all $x \in U$, (4.2) and (4.26) imply

$$f(x) - f(x^-) + hc^-, x-x^- \leq (M + \delta) \|c, b - c^-, b^-\|_{k_c},$$

and therefore (4.24) yields

$$\tau d(x, S(c^-, b^-)) \leq \max\{(M + \delta)^q, 1\} \|c, b - c^-, b^-\|_{k_c}^q, \quad (4.27)$$

whenever $x \in S(c, b) \cap U \cap [f^- > 0]$. Observe that (4.27) is trivial for $x \in [f^- \leq 0] = S(c^-, b^-)$ and, hence, we have established (i).

To finish the proof, we are establishing (ii) \Rightarrow (iii) in the linear setting. Suppose to the

contrary that L is not q -order calm at $((c^-, b^-), x^-)$. To reach a contradiction, by Proposition 4.5 it suffices to show that the sequence $x^r \in F(b^r)$ in Proposition 4.5 is also contained in $S_c(b^r)$, which readily follows from the KKT conditions (4.13) in the linear setting (by continuity, it is not restrictive to assume that $P(c^-, b^r)$ satisfies the Slater condition). \square

Next we recall the so-called Extended Nurnberger Condition (ENC) [4, Definition 2.1], which plays a crucial role in the present paper.

Definition 4.8 We say that ENC is satisfied at $((c^-, b^-), x^-) \in \text{gph}(S)$ when

$$P(c^-, b^-) \text{ satisfies the Slater condition} \quad \text{and there is no} \\ D \subset T_{b^-}(x^-) \quad \left(\text{cone} \left(\bigcup_{t \in D} \partial g_t(x) \right) \cap \{0\} \neq \emptyset \right) \\ \text{with } |D| < n \text{ such that } -(\partial f(x^-) + c^-) \in \text{cone} \left(\bigcup_{t \in D} \partial g_t(x) \right) \quad (4.28)$$

The following lemma is also crucial in our analysis; interested readers are referred to [4, Theorem 2.1 and Lemma 3.1] for more details.

Lemma 4.9 *Assume that ENC is satisfied at $((c^-, b^-), x^-) \in \text{gph}(S)$. Then the following conditions hold:*

- (i) *S is single valued and Lipschitz continuous in a neighborhood of (c^-, b^-) .*
- (ii) *If a sequence $\{((c^r, b^r), x^r)\}_{r \in \mathbb{N}} \subset \text{gph}(S)$ converges to $((c^-, b^-), x^-)$, then $(b^r, x^r) \in \text{gph}(S_{c^-})$ for r large enough.*

Thanks to Lemma 4.9, we will arrive at the following theorem, which shows that the parameter c can be considered fixed in our analysis provided that ENC is fulfilled at $((c^-, b^-), x^-) \in \text{gph}(S)$.

Theorem 4.10 *Let $((c^-, b^-), x^-) \in \text{gph}(S)$ and suppose that ENC is satisfied at $((c^-, b^-), x^-)$. Then*

$$\text{clm}_q S((c^-, b^-), x^-) = \text{clm}_q S_{c^-}(b^-, x^-).$$

Proof According to Lemma 4.9(i), we have

$$k(c^r, ((\bar{c}, \bar{b}), \bar{x})) = \lim_{r \rightarrow +\infty} \frac{k(c^r, ((\bar{c}, \bar{b}), \bar{x}))}{d(x^r, \mathcal{S}(\bar{c}, \bar{b}))} \frac{\|b^r - (c^-, b^-)\|^q}{\text{clm}_q S} \quad (4.29)$$

for certain sequences $(c^r, b^r) \rightarrow (c^-, b^-)$ and $\{x^r\} = S(c^r, b^r)$ with $x^r \neq x^-$ and $x^r \rightarrow x^-$. By Lemma 4.9(ii), we have

$$\{x^r\} = S_{c^-}(b^r) \quad \text{for } r \text{ large enough.}$$

Therefore, (4.29) and the obvious consequence of (4.2)

$$k(c^r, b^r) - (c^-, b^-) \geq k b^r - b^- k_{\infty},$$

ensure

$$\frac{\text{clm}_q S((c^-, b^-), x^-) \geq \liminf_{r \rightarrow +\infty} d(\text{KKT}_{r, S}((c^-, b^-), x^-)) \geq \text{clm}_q S_c((b^-, x^-))}{\infty}$$

Since it readily follows from the definitions that $\text{clm}_q S_c((b^-, x^-)) \geq \text{clm}_q S((c^-, b^-), x^-)$, the previous lower limit must be an ordinary limit and we can obtain indeed

$$\text{clm}_q S((c^-, b^-, x^-)) = \text{clm}_q S_c((b^-, x^-)) = \lim_{r \rightarrow +\infty} d(\text{KKT}_{r, S}((c^-, b^-), x^-)).$$

The proof is complete. tu

In what follows, particularly in Theorem 4.11, we consider a rather weaker condition than ENC and then provide an upper estimate for $\text{clm}_q[S((c^-, b^-), x^-)]$. To this aim, we associate with $(b, x) \in \text{gph}(S_c)$ the family of KKT subsets of T given by

$$K_b(x) := \{D \subset T_b(x) \mid |D| \leq n \text{ and } -(u+c^-) \in \text{cone}\{\partial g_t(x), t \in D\} \text{ for some } u \in \partial f(x)\}.$$

For any $D \in K_b(x^-)$, we consider the supremum function $f_D: \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$\begin{aligned} f_D(x) &:= \sup\{g_t(x) - b_t^-, t \in T; -g_t(x) + b_t^-, t \in D\} \\ &= \sup\{g_t(x) - b_t^-, t \in T \setminus D; |g_t(x) - b_t^-|, t \in D\}. \end{aligned} \quad (4.30)$$

$K_b(x)$ and $f_D(x)$ are convex counterparts of the corresponding concepts in [6, Section 3] for the linear model.

Theorem 4.11 *Let $S(c^-, b^-) = \{x^-\}$ and assume that $P(c^-, b^-)$ satisfies the Slater condition. Then the following estimate holds*

$$\text{clm}_q S_c((b^-, x^-)) \leq \inf \liminf f_D(x)^{q-1} d(0, \partial f_D(x)).$$

$$\rightarrow_{\substack{b \rightarrow \bar{b} \\ \in \mathcal{F}(\cdot)}} \liminf_{\substack{x \rightarrow x^-, x \in \mathcal{F}(b)}} d(x, \mathcal{S}_c(\bar{b})) \quad D \in \mathcal{K}_b(\bar{x}) \quad \substack{x \rightarrow \bar{x} \\ f_D(x) > 0}$$

Proof The proof is based on similar arguments to those used in the proof of [6, Theorem 6]. Picking a fixed $D \in K_b(x^-)$, let us show that

$$(4.31) \quad \liminf_{\substack{b \rightarrow b^- \\ x \rightarrow x^-, x \in \mathcal{F}(b)}} d(x, \mathcal{S}_c(\bar{b})) \leq \liminf_{\substack{x \rightarrow x^- \\ f_D(x) > 0}} f_D(x)^{q-1} d(0, \partial f_D(x)) \quad \frac{\text{clm}_q S_c((b^-, x^-))}{f_D(x)}.$$

We have

$\liminf f_D(x)^{q-1} d(0, \partial f_D(x)) = \lim_{x \rightarrow x^-} f_D(x')^{q-1} d(0, \partial f_D(x'))$ $\xrightarrow{r \rightarrow +\infty} f_D(x) > 0$ for a certain sequence $\{x^r\}_{r \in \mathbb{N}}$ such that $\lim_{r \rightarrow +\infty} x^r = x^-$ and $f_D(x^r) > 0$ for all $r \in \mathbb{N}$. Obviously, $x^r \in S_c(b^-)$ since $S_c(b^-) = \{x^-\}$ and $f_D(x^-) = 0$. Note that $d(0, \partial f_D(x^r)) > 0$ since $x^r \in \text{argmin}_{x \in \mathbb{R}^n} f_D$.

To prove (4.31), we need to build a new sequence of parameters $\{b^r\}_{r \in \mathbb{N}} \subset C(T, \mathbb{R}^n)$

converging to b^- such that

$$(4.32) \quad x^r \in F(b^r) \quad \text{and} \quad \frac{\|b^r - \bar{b}\|_\infty^q}{\|x^r - \bar{x}\|} \leq \left(1 + \frac{1}{r}\right) f_D(x^r)^{q-1} d(0, \partial f_D(x^r)).$$

First we give a lower bound for $\|x^r - \bar{x}\|$. If $u^r \in \partial f_D(x^r)$, $u^r \cdot \bar{x} = 0$, and

$$\|x^r - \bar{x}\| \|u^r\| \geq \langle u^r, x^r - \bar{x} \rangle \geq f_D(x^r) - f_D(\bar{x}) = f_D(x^r),$$

and so

$$\|x^r - \bar{x}\| \geq \frac{f_D(x^r)}{\|u^r\|} \geq \frac{f_D(x^r)}{d(0, \partial f_D(x^r))}. \quad (4.33)$$

The next step consists of the construction of the desired sequence $\{b^r\}$ such that (4.32) holds. Once again we apply Urysohn's Lemma which guarantees the existence of a certain function $\phi_r \in C(T, [0, 1])$ such that

$$\phi_r(t) = \begin{cases} 0, & \text{if } t \in \{t \in T : g_t(x^r) - \bar{b}_t \leq -(1 + \frac{1}{r})f_D(x^r)\} \\ 1, & \text{if } t \in \{t \in T : g_t(x^r) - \bar{b}_t \geq f_D(x^r)\} \end{cases} \quad (4.34)$$

Recalling the definition of $f_D(x^r)$ and the fact that $f_D(x^r) > 0$, D and $\{t \in T : g_t(x^r) - \bar{b}_t \leq -(1 + \frac{1}{r})f_D(x^r)\}$ are disjoint closed sets in T . Certainly, if t belongs to both sets we reach the following contradiction:

$$f_D(x^r) \geq |g_t(x^r) - \bar{b}_t| \geq (1 + \frac{1}{r})f_D(x^r)$$

If the set $\{t \in T : g_t(x^r) - \bar{b}_t \leq -(1 + \frac{1}{r})f_D(x^r)\}$ is empty we take $\phi_r(t) = 0$ for all $t \in T$.

Now, let us define, for each $t \in T$,

$$b_t^r := (1 - \phi_r(t))g_t(x^r) + \phi_r(t)(\bar{b}_t + f_D(x^r)).$$

For each r , the definition of b^r and (4.30) clearly imply that

$$b_t^r - g_t(x^r) = \phi_r(t)(f_D(x^r) + \bar{b}_t - g_t(x^r)) \geq 0$$

and thus $x^r \in F(b^r)$. Finally, let us observe that $b_t^r - \bar{b}_t = f_D(x^r)$ when $\phi_r(t) = 1$, and

$$b_t^r - \bar{b}_t \leq f_D(x^r) < (1 + \frac{1}{r})f_D(x^r) < g_t(x^r) - \bar{b}_t$$

when $\phi_r(t) < 1$. Accordingly,

$$\|b^r - \bar{b}\|_\infty \leq (1 + \frac{1}{r})f_D(x^r)$$

which, together with (4.33), entails

$$\frac{\|b^r - \bar{b}\|_\infty^q}{\|x^r - \bar{x}\|} \leq \frac{d(0, \partial f_D(x^r))}{f_D(x^r)} \|b^r - \bar{b}\|_\infty^q \leq \left(1 + \frac{1}{r}\right) f_D(x^r)^{q-1} d(0, \partial f_D(x^r)),$$

which ensures (4.32). Finally, we will consider the linear counterpart of $P(c, b)$; namely, we will always assume that $f = 0$ and $g_t(x) = \langle a_t, x \rangle$ for all $t \in T$ therein, where $t \mapsto a_t \in \mathbb{R}^n$ is continuous on T .

Corollary 4.12 Let $S(c^-, b^-) = \{x^-\}$ and assume that $P(c^-, b^-)$ satisfies the Slater condition.

Then the following estimates hold $\text{clm}_q S((c^-, b^-), x^-) \leq \text{clm}_q S_c^-(b^-, x^-) \leq \inf \liminf$

$$f_D(x)^{q-1} d(0, \partial f_D(x)). \quad (4.35)$$

$$D \in K_{b^-}(x^-) f_{Dx}(\rightarrow x) x^- > 0$$

Proof The first inequality follows straightforwardly from (1.4). To prove (4.35), by Theorem 4.11 it suffices to show that the sequence $x^r \in F(b^r)$ produced in Theorem 4.11 is also contained in $S_c^-(b^r)$. Taking a fixed D as in Theorem 4.11, since for all $t \in D$, by (4.34) we could have $\phi_r(t) = 0$, which follows from the definition of b_t^r that $b_t^r = g_t(x^r)$ and then implies that $D \subset T_{b^r}(x^r)$. Noting that f and g_t are linear functions, we obtain $D \in K_{b^r}(x^r)$. Recalling that $x^r \in F(b^r)$, this certainly yields that $x^r \in S_c^-(b^r)$. \square

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