TRANSVERSALITY, REGULARITY AND ERROR BOUNDS IN VARIATIONAL ANALYSIS AND OPTIMISATION

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DISSERTATION

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of the requirement for the degree of **DOCTOR OF PHILOSOPHY**

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Abstract

> An important feature of the new variational techniques is that they can handle nonsmooth functions, sets and multifunctions equally well.
>
> *Borwein and Zhu* [36]

Transversality properties of collections of sets, regularity properties of set-valued mappings, and error bounds of extended-real-valued functions lie at the core of variational analysis because of their importance for stability analysis, constraint qualifications, qualification conditions in coderivative and subdifferential calculus, and convergence analysis of numerical algorithms. The thesis is devoted to investigation of several research questions related to the aforementioned properties.

We develop a general framework for quantitative analysis of nonlinear transversality properties by establishing primal and dual characterizations of the properties in both convex and nonconvex settings. The Hölder case is given special attention. Quantitative relations between transversality properties and the corresponding regularity properties of set-valued mappings as well as nonlinear extensions of the new transversality properties of a set-valued mapping to a set in the range space are also discussed.

We study a new property so called *semitransversality of collections of set-valued mappings* on metric (in particular, normed) spaces. The property is a generalization of the semitransversality of collections of sets and the negation of the corresponding *stationarity*, a weaker property than the *extremality of collections of set-valued mappings*. Primal and dual characterizations of the property as well as quantitative relations between the property and semiregularity of set-valued mappings are formulated. As a consequence, we establish dual necessary and sufficient conditions for stationarity of collections of set-valued mappings as well as optimality conditions for efficient solutions with respect to variable ordering structures in multiobjective optimization.

We examine a comprehensive (i.e. not assuming the mapping to have any particular structure) view on the regularity theory of set-valued mappings and clarify the relationships between the existing primal and dual quantitative sufficient and necessary conditions including their hierarchy. The typical sequence of regularity assertions, often hidden in the proofs, and the roles of the assumptions involved in the assertions, in particular, on the underly-

ing space: general metric, normed, Banach or Asplund are exposed. As a consequence, we formulate primal and dual conditions for the stability properties of solution mappings to inclusions.

We propose a unifying general framework of quantitative primal and dual sufficient and necessary error bound conditions covering linear and nonlinear, local and global settings. The function is not assumed to possess any particular structure apart from the standard assumptions of lower semicontinuity in the case of sufficient conditions and (in some cases) convexity in the case of necessary conditions. We expose the roles of the assumptions involved in the error bound assertions, in particular, on the underlying space: general metric, normed, Banach or Asplund. As a consequence, the error bound theory is applied to characterize subregularity of set-valued mappings, and calmness of the solution mapping in convex semi-infinite optimization problems.

STATEMENT OF AUTHORSHIP

I certify that this work contains no material which has been accepted for the award of any other degree or diploma, in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. No other person's work has been relied upon or used without due acknowledgment in the main text and bibliography of the thesis. The content of this thesis, which is presented as a thesis incorporating published papers, consists of nine published or submitted papers.

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Chapter 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Differential calculus has been recognized as a very powerful tool in various models, in particular for solving optimization problems. However, the applicability of differential calculus is restricted due to the requirement of smoothness assumption of data, while nonsmooth structures frequently arise in numerous problems in mathematics and applied sciences. In fact, nonsmoothness naturally enters not only through initial data of optimization-related problems (particularly those with inequality and geometric constraints) but also via variational principles and other optimization, approximation, and perturbation techniques applied to problems with even smooth data. Besides, many fundamental objects frequently appearing in the framework of variational analysis (e.g., the distance/marginal/maximum functions, solution mappings, etc.) are inevitably nonsmooth and/or set-valued.

The necessity of new forms of analysis that involve generalized differentiation to work with nonsmooth objects leads to the development of modern variational analysis, which indicates a broad spectrum of mathematical theory including variational principles, nonsmooth and set-valued analysis, optimization, equilibrium, control, stability of linear and nonlinear systems, and sensitivity with respect to data perturbations. Variational analysis has been recognized as a fruitful and rapidly developing area, rich with applications in mathematics in the last few decades. One of the main ingredients of the area is the construction of generalized differentiations in terms of primal and dual objects for nondifferentiable functions (subderivatives and subdifferentials), sets with nonsmooth boundaries (tangent and normal cones), and set-valued mappings (derivatives and coderivatives), which has become a powerful tool in the study of sensitivity issues for various mathematical problems.

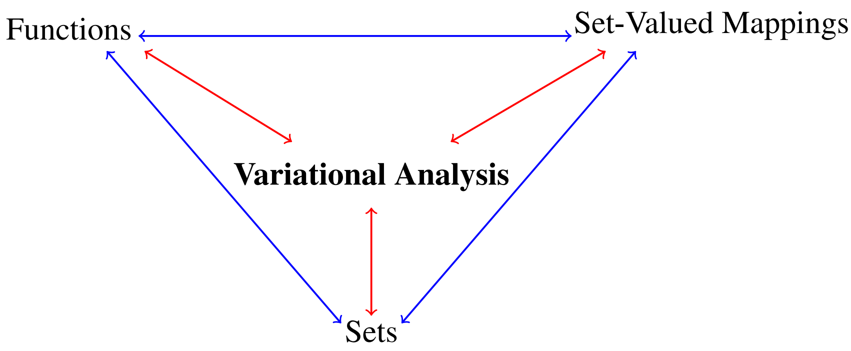

Figure 1. *Connections of sets, functions, and set-valued mappings*

The thesis is devoted to studying three fundamental elements of variational analysis: sets, functions, and set-valued mappings, all of which are generally nonconvex and have nonsmooth boundaries. We establish new and upgraded characterizations of transversality properties of collections of sets, regularity properties of set-valued mappings, error bounds of extended-real-valued functions, and apply the properties in studying optimization-related problems.

1.1.1 Transversality Properties

Transversality is about study of 'good arrangements' of collections of sets in normed vector spaces near a point in their intersection. The origins of the concept of regular arrangement of sets in space can be traced back to that of transversality in Differential Geometry which plays of course with smooth manifolds [100,104]. Given two smooth manifolds and a point in their intersection, a natural question is when the intersection is also a smooth manifold near the common point. It has been shown that one of the sufficient conditions is when the pair of two manifolds is *transversal* at this point in the sense that the sum of the tangent spaces to each manifold at the point equals the whole space, or equivalently, the intersection of two normal spaces (i.e., orthogonal complements to the tangent spaces) of the two manifolds at the common point equals vector 0.

Due to the wide variety of applications coming from different areas, some transversality properties together with the corresponding necessary and/or sufficient conditions have been rediscovered many times in different situations and often under different names. In the context of modern variational analysis the terms *transversality* was coined by Alexander Ioffe [114], and he then explained [116, p. 301] that '*Regularity is a property of a single object while transversality relates to the interaction of two or more independent objects*'. In the thesis we refer to transversality broadly as a group of 'good arrangement' properties, which includes *semitransversality*, *subtransversality*, *transversality* (a specific property) and some others.

Transversality properties are fundamental concepts in variational analysis in both theory and practice aspects. They are crucial for the validity of qualification conditions in optimization as well as subdifferential, normal cone and coderivative calculus, and convergence analysis of computational algorithms. Significant efforts have been invested into studying various transversality properties and establishing their primal and dual necessary and/or sufficient characterizations in various settings (convex and nonconvex, finite and infinite dimensional, finite and infinite collections of sets). Interested readers are refer to [9, 20, 25–27, 40, 70, 83, 116, 127–129, 134–136, 138–143, 148, 152, 176, 177, 183, 190, 194, 216, 218] for comprehensive discussions and interesting results.

Our working model in the thesis is a collection of $n \geq 2$ arbitrary subsets $\Omega_1, \ldots, \Omega_n$ of a normed vector space X, having a common point $\bar{x} \in \cap_{i=1}^n \Omega_i$. We are now going to discuss the three properties: subtransversality, transversality and semitransversality. Each property is presented in both geometric (how the sets intersect) and metric (how the sets are measured by distances) terms. It is worth mentioning that geometric and metric versions of the respective properties were not established simultaneously.

Definition 1 Let $\alpha > 0$, $\delta_1 > 0$ and $\delta_2 > 0$. The collection $\{\Omega_1, \ldots, \Omega_n\}$ is $\alpha-$subtransversal with δ_1 and δ_2 if one of the following two equivalent conditions are satisfied:

(i) for all $\rho \in]0, \delta_1[$ and $x \in B_{\delta_2}(\bar{x})$ with $\max_{1 \leq i \leq n} d(x, \Omega_i) < \alpha\rho$, it holds

$$\bigcap_{i=1}^{n} \Omega_i \cap B_\rho(x) \neq \emptyset; \tag{1.1}$$

(ii) for all $x \in B_{\delta_2}(\bar{x})$ with $\max_{1 \leq i \leq n} d(x, \Omega_i) < \alpha\delta_1$, it holds

$$\alpha d\left(x, \bigcap_{i=1}^{n} \Omega_i\right) \leq \max_{1 \leq i \leq n} d(x, \Omega_i). \tag{1.2}$$

The exact upper bound of all $\alpha \in]0, 1[$ such that condition (1.1) or condition (1.2) is satisfied for some $\delta_1 > 0$ and $\delta_2 > 0$ is denoted $\text{str}[\Omega_1, \ldots, \Omega_n](\bar{x})$ with the convention that the supremum of the empty subset of $\mathbb{R}_+$ equals 0.

The property in part (i) as well as the qualitative equivalence between (i) and (ii) are mentioned in [142], see also [38, Theorem 2] for the quantitative equivalence. Property (i) can be viewed as a local analogue of the global *uniform normal property* introduced in the convex setting in [20, Definition 3.1(4)] as a generalization of the *property (N)* of convex cones by Jameson [119]. The subtransversality of $\{\Omega_1, \ldots, \Omega_n\}$ is equivalent to the condition $\text{str}[\Omega_1, \ldots, \Omega_n](\bar{x}) > 0$, and the modulus provides a quantitative characterization of this property. It is worth mentioning that the constant is, in a sense, a local analogue of the *normality constant* in [20, Definition 4.2].

According to our understanding, property (ii) was first studied by Dolecki [77] under the name *separate decisively at a linear rate*, while its nonlinear version was discussed earlier [76, Lemma 4.2]. The property has been used (under different names) as a qualification condition for establishing subdifferential and normal cone calculus rules: [111, Section 5], *metric inequality* [183], *linear coherence* [194, Theorem 4.75]. Inequality (1.2) provides the upper bound of the distance from x to the intersection, which is very useful in computation algorithms (especially when dealing with feasibility problems).

Given a collection of sets, the feasibility problem consists of finding a point in their intersection, which is a very general model including, in particular, solving systems of all sorts of equations and inequalities (algebraic, differential, etc.). Bauschke and Borwein [24, 25] used the property under the name *linear regularity* as a sufficient condition for the linear convergence of alternating and cyclic projection algorithms for solving convex feasibility problems in Hilbert spaces. Very recently, Luke et al. makes the picture in the convex setting complete by showing that subtransversality is not only sufficient for the linear convergence of alternating projections, but it is also necessary [160, Theorem 8].

The next definition introduces *transversality* which is a stronger property than subtransversality.

Definition 2 Let $\alpha > 0$, δ_1 and $\delta_2 > 0$. The collection $\{\Omega_1, \ldots, \Omega_n\}$ is $\alpha-$transversal with δ_1 and δ_2 if one of the following two equivalent conditions are satisfied:

(i) for all $\rho \in]0, \delta_1[$, $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1, \ldots, n)$ with $\max_{1 \le i \le n} \|x_i\| < \alpha\rho$, it holds

$$\bigcap_{i=1}^{n} (\Omega_i - \omega_i - x_i) \cap (\rho\mathbb{B}) \neq \emptyset; \tag{1.3}$$

(ii) for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1, \ldots, n)$ with $\varphi(\max_{1 \le i \le n} \|x_i\|) < \delta_1$, it holds

$$\alpha d\left(0, \bigcap_{i=1}^{n} (\Omega_i - \omega_i - x_i)\right) \le \max_{1 \le i \le n} \|x_i\|. \tag{1.4}$$

The exact upper bound of all $\alpha \in]0, 1[$ such that condition (1.3) or condition (1.4) is satisfied for some $\delta_1 > 0$ and $\delta_2 > 0$ is denoted $\mathrm{tr}[\Omega_1, \ldots, \Omega_n](\bar{x})$ with the convention that the supremum of the empty subset of $\mathbb{R}_+$ equals 0.

Property (i) was originally studied by Kruger [127] under the name *regularity* as the negation of the *approximate stationarity property*, which is a weaker property than the (local) *extremality* [39,40]. The equivalence between (i) and (ii) is recently proved in [38, Theorem 2]. It is worth mentioning that transversality admits several equivalent metric characterizations (see, Chapter 2). The table below recalls several names of the property.

2005 [127]	2006 [128]	2009 [129]	2013 [140]	2017 [139]
Regularity	Strong Regularity	Property $(\mathrm{UR})_S$	Uniform Regularity	Transversality

We have known that subtransversality is not only sufficient but also necessary for the local linear convergence of the convex alternating projection. In the nonconvex case, the situation is much more complicated. In fact, it has been shown in [116, Example 9.25] that sequences generated by the algorithm fail to converge to some points in the intersection although the subtransversality holds. The need for a stronger property which ensures the linear convergence is essential, and this is exactly where transversality comes to play in practice for the first time. In general, the requirements of nonconvex optimization have caused an expansion of the range of transversality properties under consideration. Lewis and Malick [149] and Lewis et al. [148] used the property under the name *linear regular intersection* to prove the local linear convergence for alternating and averaged projections for nonconvex feasibility problems. It is worth mentioning that, unlike subtransversality, transversality is employed in computational areas under its equivalent dual characterizations in terms of normal cones instead of the original primal (geometric and metric) ones. The dual characterizations of transversality in finite dimensional spaces have been well known for about 30 years under the names *basic qualification condition* and *normal qualification condition*. We refer the reader to Kruger et al. [138, 139] for comprehensive discussions as well as useful results about the property.

Subtransversality and transversality have strong connections with the error bound theory which is going to be dicussed in detail in Section 1.1.3. In fact, the former is equivalent to the function $x \mapsto \max_{1\le i\le n} d(x,\Omega_i)$ having a local error bound at $\bar{x}$, while the latter can represent the existence of a local error bound at $\bar{x}$ uniformly near $(0,\dots,0)$ of the function $(x,(x_1,\dots,x_n)) \mapsto \max_{1\le i\le n} d(x,(\Omega_i - x_i))$; cf. [121].

The next definition introduces semitransversality which has not attracted much attention compared to its two siblings.

Definition 3 Let $\alpha > 0$ and $\delta > 0$. The collection $\{\Omega_1,\dots,\Omega_n\}$ is $\alpha-$semitransversal with δ if one of the following two equivalent conditions are satisfied:

(i) for all $\rho \in]0,\delta[$ and $x_i \in X$ $(i = 1,\dots,n)$ with $\max_{1\le i\le n} \|x_i\| < \alpha\rho$, it holds

$$\bigcap_{i=1}^{n} (\Omega_i - x_i) \cap B_\rho(\bar{x}) \neq \emptyset; \tag{1.5}$$

(ii) for all $x_i \in X$ $(i = 1,\dots,n)$ with $\max_{1\le i\le n} \|x_i\| < \alpha\delta$, it holds

$$\alpha d\left(\bar{x}, \bigcap_{i=1}^{n} (\Omega_i - x_i)\right) \le \max_{1\le i\le n} \|x_i\|. \tag{1.6}$$

The exact upper bound of all $\alpha > 0$ such that condition (1.5) or condition (1.6) is satisfied for some $\delta > 0$ is denoted $\mathrm{s_e tr}[\Omega_1,\dots,\Omega_n](\bar{x})$ with the convention that the supremum of the empty subset of $\mathbb{R}_+$ equals 0.

The property in part (i) was first studied by Kruger [127] as the negation of the *stationarity* property, while qualitative equivalence between (i) and (ii) was first proved in [140, Theorem 3.1(i)] (see also [38, Theorem 2] for the quantitative equivalence). The table below represents the involve of the terminology.

2005 [127]	2006 [128]	2009 [129]	2018 [142]	2020 [70]
Weak Regularity	Regularity	Property $(R)_S$	Semiregularity	Semitransversality

It is obvious that the property is weaker than transversality. We recently showed that semitransversality and transversality are equivalent in the convex setting [38, Proposition 8]. On the other hand, semitransversality and subtransversality are generally independent [141, 142]. Unlike the other two transversality properties, semitransversality does not have an exact counterpart within the conventional error bound theory. The three transversality properties, namely, semitransversality, subtransversality and transversality can be interpreted, respectively, as direct analogues of *semiregularity*, *subregularity* and *regularity* of set-valued mappings; cf. Section 1.1.2.

The maximum of the distances in Definitions 1, 2 and 3 and some other representations in the sequel corresponds to the maximum norm in $\mathbb{R}^n$ employed in all these definitions and assertions. It can be replaced everywhere by the sum norm [114, 116, 183] or any other equivalent norm. All the assertions above including the quantitative characterizations will remain valid (as long as the same norm is used everywhere), although the exact values of the moduli do depend on the chosen norm and some estimates can change.

The subtransversality is not sufficient for the local linear convergence of the nonconvex alternation projections, but it has an attractive property that transversality lacks: it is intrinsic in the sense that it is connected with the sets themselves and not with the ambient space. On the contrary, two sets that are transversal in a subspace of a bigger space are no longer transversal in the latter. So it is natural to look for a property that is connected with the sets as such, not with the space where they are defined, and at the same time strong enough to guarantee the convergence of alternating projections in small neighborhoods of points of intersection. This is where the *intrinsic transversality* comes to play. The property was introduced recently by Drusvyatskiy, Ioffe and Lewis [83] as a sufficient condition for local linear convergence of alternating projections for nonconvex feasibility problems in finite dimensional spaces. Several characterizations of this property involving limiting objects in infinite dimensional setting were established by Kruger [134]. It has been shown that intrinsic transversality and subtransversality coincide in the convex setting [116, Proposition 9.29]. Intrinsic transversality is originally defined on normal vectors in Euclidean spaces, and it has been proved recently that the property admits equivalent characterizations that do not involve normal vectors in the Hilbert setting [205].

Very recently, another important property called "tangential transversality" has come to life. The property is used to obtain necessary optimality conditions for optimization problems in terms of abstract Lagrange multipliers and to formulate intersection rules for tangent cones in Banach spaces. For more discussions about the property as well as connections with other transversality properties, we refer the reader to these recent papers [3, 4, 30, 31].

1.1.2 Regularity Properties

Let $F : X \rightrightarrows Y$ be a set-valued mapping between metric spaces, and consider the solvability of generalized equations, that is, the problem of

$$\text{finding} \quad x \in X \quad \text{such that} \quad F(x) \ni y, \tag{1.7}$$

where $y \in Y$ is given.

It is well-known that many important problems in variational analysis and optimization are covered by this generalized equation model. In fact, the relation $F(x) \ni y$ can stand for a variational inequality or more broadly it can express a mixture of inequality and equality conditions. The solution set is, of course, $F^{-1}(y)$ which is nonempty if and only if $y \in \operatorname{range} F$. The central issue in investigating problem (1.7) is to study the behavior of the solution set $F^{-1}(y)$ with respect to perturbations in y as a parameter, and one of the powerful tools is the concept of metric regularity.

The mapping F is regular at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if there exists an $\alpha > 0$, neighbourhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\alpha d(x, F^{-1}(y)) \le d(y, F(x)) \tag{1.8}$$

for all $x \in U$ and $y \in V$. The exact upper bound of all $\alpha > 0$ such that condition (1.8) is satisfied for some U and V is denoted $\operatorname{rg}[F](\bar{x},\bar{y})$ with the convention that the supremum of the empty subset of $\mathbb{R}_+$ equals 0.

The constant α measures sensitivity or conditioning of the generalized equation, and larger values of α correspond to more favorable behavior. The largest such constant α is denoted by $\operatorname{rg}[F](\bar{x},\bar{y})$ and called the modulus of regularity which allows us to check how large a perturbation can be before good behavior of a solution mapping breaks down.

Metric regularity establishes linear relationship between data change and solution error. Strictly speaking, inequality (1.8) provides an estimate for how far a point x is from being a solution to the general equation problem for F and the data y; the expression $d(y,F(x))$ measures the 'residua' when $y \notin F(x)$. It is clear that computing the residua is much more easier than finding a solution of the generalized equation. Such an estimate is of importance for numerous optimization problems, especially for numerical purposes. The following observation is by Dontchev and Rockafellar [81]. Let $\bar{x}$ be a solution of the inclusion $F(x) \ni \bar{y}$, and let F be metrically regular at $(\bar{x},\bar{y})$, and let (x_n, y_n) be sufficiently close to $(\bar{x},\bar{y})$. Then, the product of α and distance from x_n to the solution set of the inclusion $F(x) \ni y_n$ is bounded above by the residual $d(y_n, F(x_n))$. Metric regularity says that there exists a solution to the inclusion $F(x) \ni y_n$ at distance from x_n proportional to the residual. In particular, if we know the rate of convergence of the residual to zero, then we will obtain the rate of convergence of approximate solutions to an exact one.

The metric regularity is a basic quantitative property of mappings in variational analysis which is widely used in both theoretical and computational studies. In fact, it has been well recognized that this property plays a fundamental role in many aspects of nonlinear analysis, optimization, control theory, etc., especially for studying sensitivity and stability of solutions to generalized equations and various aspects of subdifferential calculus and optimization theory. Metric regularity is also one of the powerful tools for establishing convergence rates of various computational algorithms, for example, the proximal point method [52, 82], the Newton method [1, 5], Picard iterations [8, 103, 161].

The name *metric regularity* was coined by Borwein in the 1986 work [34], but the concept itself was established far earlier. Indeed, this concept can be traced back to the Banach open mapping theorem for linear operators obtained in the 1930s by Banach and Schauder [21, 203] in functional analysis. This theorem was extended for nonlinear operators: the tangent space theorem of Lyusternik [166] and the surjection theorem of Graves [99], which has been known as the cerebrated Graves-Lyusternik theorem. Specifically, the theorem says that if a single-valued function $F : X \to Y$ is continuously Fréchet differentiable near $\bar{x}$, then it is regular at $(\bar{x}, F(\bar{x}))$ if and only if the operator derivative $\nabla F(\bar{x})$ is surjective, i.e. $F'(\bar{x})X = Y$. Another fundamental result in variational analysis is the Robinson-Ursescu theorem (Robinson [199] and Ursescu [207]), an extension of the Banach open mapping principle to set-valued mappings, which says that for a set-valued mapping F with closed and convex graph, it is metrically regular at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if and only if $\bar{y}$ is an interior point of range F.

Metric regularity of a set-valued mapping F is known to be equivalent to two other properties: the *openness with a linear rate around* of F [81, Theorem 3E.9] and the *Aubin property* (also called pseudo-Lipschitz or Lipschitz-like) of the inverse F^{-1} [81, Theorem 3E.7], a property which in the single-valued case reduces to the Lipschitz continuity. Comprehensive characterizations of metric regularity and related well-posedness properties of set-valued mappings are available in the literature via both primal and dual constructions of generalized differentiation. For the state of the art of the regularity theory of set-valued mappings and its numerous applications we refer the reader to the books by Dontchev and Rockafellar [81] and Ioffe [116].

We would like to mention another equivalent property of metric regularity [116, Proposition 2.20]. The mapping F is graph-regular at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there exist an $\alpha > 0$, neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\alpha d(x, F^{-1}(y)) \leq d((x, y), \operatorname{gph} F) \tag{1.9}$$

for all $x \in U$ and $y \in V$.

The property was initially studied by Jourani and Thibault [122]. The main idea of graph regularity is that metric regularity is qualitatively unchanged when reasonable restrictions on x and y are added, for instance, $(x, y) \notin \operatorname{gph} F$. The function $(x, y) \mapsto d(y, F(x))$ in the definition of metric regularity has a drawback that it generally fails to be lower semicontinuous in infinite dimentional spaces. The function $(x, y) \mapsto d((x, y), \operatorname{gph} F)$, on the other hand, is Lipschitz continuous. Hence, necessary and sufficient conditions for the property can be easily obtained when applying the error bound theory for lower semicontinuous functions.

There are two basic ways of weakening the definition of metric regularity by fixing one of the points involved in the definition. The first resulting property is called metric subregularity and has attracted a lot of attention during the last decades.

We say that the mapping F is subregular at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if there exists an $\alpha > 0$ and a neighborhood U of $\bar{x}$ such that

$$\alpha d(x, F^{-1}(\bar{y})) \leq d(\bar{y}, F(x))$$

for all $x \in U$. The exact upper bound of all $\alpha > 0$ such that condition (1.1.2) is satisfied for some U is denoted $\operatorname{srg}[F](\bar{x},\bar{y})$ with the convention that the supremum of the empty subset of $\mathbb{R}_+$ equals 0.

Metric subregularity property was introduced in [108] for single-valued maps in the context of necessary optimality conditions. In [112] this property was called "regularity at a point" and the terminology "metric subregularity" was suggested in [80]. Metric subregularity lacks stability under small perturbations compared to metric regularity, i.e. the data input $\bar{y}$ is now fixed and not perturbed to a nearby y. Despite its instability, subregularity has attracted remarkable attention because of its great importance in studying optimization-related problems. It has been known that metric subregularity is equivalent to two properties: *pseudo-open at a linear rate* of F [58], *calmness* [61] of its inverse F^{-1}, a property which corresponds to the Aubin continuity with one of the variables fixed. The number of publications that study this property is huge, see [2, 37, 78, 81, 172, 213] and the references therein for many theoretical results as well as various applications of metric subregularity.

Another equivalent property of subregularity is of interest. The mapping F is graph subregular at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if there exist an $\alpha > 0$ and neighborhood U of $\bar{x}$ such that

$$\alpha d(x, F^{-1}(\bar{y})) \leq d((x,\bar{y}), \operatorname{gph} F) \tag{1.10}$$

for all $x \in U$.

The above four properties have direct counterparts in the error bound theory. Subregularity and graph subregularity are equivalent to the existence, respectively, of local error bounds of the functions: $x \mapsto d(\bar{y}, F(x))$ and $x \mapsto d((x,\bar{y}), \operatorname{gph} F)$, while regularity and graph regularity are equivalent to the existence, respectively, of local parametric error bounds of the functions: $(x,y) \mapsto d(y, F(x))$ and $(x,y) \mapsto d((x,y), \operatorname{gph} F)$. We refer the reader to [13, 116, 130, 131, 133, 182, 185, 188] and references therein for comprehensive discussions as well as connections between the regularity and error bound theory.

We say that F is semiregular at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if there exists an $\alpha > 0$ and a neighborhood V of $\bar{y}$ such that

$$\alpha d(\bar{x}, F^{-1}(y)) \leq d(\bar{y}, y) \tag{1.11}$$

for all $y \in V$. The exact upper bound of all $\alpha > 0$ such that condition (1.11) is satisfied for some U is denoted $\mathrm{s_erg}[F](\bar{x},\bar{y})$ with the convention that the supremum of the empty subset of $\mathbb{R}_+$ equals 0.

Being also weaker than metric regularity and general not comparable with subregularity, the property is first investigated by Kruger [129]. It has been shown very recently that semiregularity is equivalent to two properties: the *open with a linear rate at* (also known as *c-covering* [129], *regularity* [128], *controllability* [114]) of F [58, Proposition 2.1], and the *Lipschitz lower semicontinuity* (also known as *pseudo-calmness* [85], *linear recession* [114]) of the inverse F^{-1} [124].

In [7, 85, 206], semiregularity has been used under the name *hemiregularity*. Unlike its two well-established siblings, the semiregularity property has only started attracting attention of researchers thanks to its importance, e.g., in the convergence analysis of inexact Newton-type schemes for generalized equations [58, 129]. For primal and dual characterizations and comprehensive discussions about the property, we refer the reader to the recent paper by Cibulka, Fabian and Kruger [58].

1.1.3 Error Bounds

As in numerical analysis, an initial motivation to study error bounds in mathematical programming arose from practical considerations in the computer implementation of iterative methods for solving optimization and equilibrium programs. Iterative algorithms practically do not give us an exact solution even if there is one, for instance, the alternating projection algorithm for finding a point in the intersection of a pair of sets. Hence, when an algorithm stops at an iteration and returns us a point, it is essential to know how good the current point is in compassion with the ones in the solution set of the problem which is assumed nonempty but typically unknown. In other words, it would be helpful if we know how far is that point from the solution set. These observations become a primary motivation to investigate the existence of an error bound of a function.

Given an extended-real-valued function $f: X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ on a metric space X, and $\mu \in]0, +\infty]$, we denote

$$[f \le 0] := \{x \in X \mid f(x) \le 0\}, \quad [0 < f < \mu] := \{x \in X \mid 0 < f(x) < \mu\}.$$

The sets $[f > 0]$, $[f < \mu]$ and $[f \le \mu]$ are defined in a similar way.

Definition 4 Suppose X is a metric space, $f: X \to \mathbb{R}_\infty$, and $\tau > 0$. The function f admits a $\tau-$error bound at $\bar{x} \in X$ if there exist $\delta \in]0, +\infty]$ and $\mu \in]0, +\infty]$ such that

$$\tau d(x, [f \le 0]) \le f(x) \tag{1.12}$$

for all $x \in B_\delta(\bar{x}) \cap [0 < f < \mu]$, and either $\bar{x} \in [f \le 0]$ or $\delta = +\infty$.

The value of τ in Definition 4 obviously depends on the values of δ and μ. We will often say that f admits a $\tau-$error bound at $\bar{x}$ with δ and μ.

Definition 4 combines the cases of local and global error bounds that are usually treated separately. The conventional local error bound property corresponds to the case $\bar{x} \in [f \le 0]$, and δ being a (sufficiently small) finite number. In this case, we say that f admits a local τ−error bound (with δ and μ). When $\delta = +\infty$, we have $B_\delta(\bar{x}) = X$, i.e. the error bound property in Definition 4 is not related to any particular point, and we are in the setting of global error bounds. In this case, we simply say that f admits a global τ−error bound.

The (local) error bound modulus of f at $\bar{x}$ is defined as the exact upper bound of all $\tau > 0$ such that f admits a τ−error bound at $\bar{x}$, i.e.

$$\mathrm{Er}\, f(\bar{x}) := \liminf_{x \to \bar{x},\, f(x) > 0} \frac{f(x)}{d(x, [f \le 0])} = \liminf_{x \to \bar{x},\, f(x) \downarrow 0} \frac{f(x)}{d(x, [f \le 0])}. \tag{1.13}$$

In the above definition and throughout the thesis, we use the conventions $d(x, \emptyset_X) = +\infty$, $\inf \emptyset_{\mathbb{R}} = +\infty$ and $\frac{+\infty}{+\infty} = +\infty$. (The last convention is only needed to accommodate for the trivial case $f \equiv +\infty$.) The second equality in (1.13) is straightforward.

By definition (1.13), $\mathrm{Er}\, f(\bar{x}) \ge 0$. If $\mathrm{Er}\, f(\bar{x}) = 0$, then f does not admit a τ−error bound at $\bar{x}$ for any $\tau > 0$.

Depending of a specific structure of f, the solution set $[f \le 0]$ can be a polyhedron defined by finitely many linear inequalities, or more generally, the solution set of a nonlinear nonsmooth optimization problem. Error bounds provide a linear estimate of the distance from points and the solution set in terms residual values. In applications, this information is very essential since computing the upper bound is manageable, while finding the solution set of a general nonlinear inequality is considerably hard. Such an estimation has been intensively used in sensitivity analysis [121, 168, 170], and convergence analysis of computational algorithms [83, 116, 163–165, 169, 192]. The reader is refereed to excellent surveys by Pang [193], Lewis and Pang [150] for intensive discussions and diverse applications of error bounds.

Hoffman is believed to be the first person to study error bounds. In [105] he proved the existence of a global ($\delta = +\infty$) error bound for affine functions on finite dimensional spaces. The paper is now classic and has become the foundation for much of the subsequent work in this area. In fact, error bounds have played an important role in many areas in mathematical programming and variational analysis, and the number of publications invested in studying this property is huge. Most of the earlier results were dedicated to studying global error bounds for convex functions on both finite and infinite dimensional spaces by requiring additional conditions on the solution set such as the Slater condition, the boundedness, asymptotic constraint qualifications, the Abadie constraint qualification; cf. [10, 11, 74, 75, 150, 157, 162, 167, 197]. Recently, it has been shown that error bounds of convex functions can be examined by using conjugate counterparts [45, 63, 63, 220].

An alternative approach to study criteria for error bounds is to examine points outside the solution set by employing basic tools in variational analysis. The initial work in this area is by Ioffe [109]. Making use of the Ekeland variational principle and the sum rule for Clarke subdifferentials, he proved the existence of global error bounds for locally Lipschitz functions on Banach spaces under the condition that any Clarke subradient of the constraint function at each point outside the solution set be norm bounded away from zero. The technique has been used by many researchers to extend for lower semicontinuous functions and other kind of subdifferentials [12,121,131,178,179,208,209]. In the 2000 paper [112], Ioffe made another pioneering work by using a primal object called *slope* to formulate sufficient conditions for error bounds of lower semicontinuous function in metric spaces. Compared with subdifferential sufficient characterizations, slope counterparts are weaker and often easier to check. It is worth mentioning that the two conditions are equivalent in the convex setting. Characterizations of this kind have been used by many researchers in several cases, for instance, vector-valued functions [28, 29], set-valued mappings [101, 102, 181]. Interested readers are refereed to [12, 64, 92, 131–133, 180, 184, 186] and references therein for more discussions and results.

Error bounds has strong connections with two groups of important concepts: transversality of collections of sets and regularity of set-valued mappings. In the previous section, we have seen that metric regularity properties of set-valued mappings can be seen as the existence of a local error bound of appropriate functions. On the other hand, for any extended-real-valued function f the existence of a local error bound of f at $\bar{x}$ where $f(\bar{x}) = 0$ is equivalent to the subregularity at $(\bar{x}, 0)$ of the epigraphical mapping $\mathrm{Epi} f : X \rightrightarrows \mathbb{R}$ defined for any $x \in X$ by $\mathrm{Epi} f(x) = \{\alpha \in \mathbb{R} : \alpha \geq f(x)\}$ [116, 154]. Error bounds are also closely related to many other important properties in variational analysis, for instance, weak sharp minima [41–44,128,220,221], conditioning [63], Kurdyka–Łojasiewicz inequality [33,120].

1.1.4 Structure of Thesis

Up until recently, mostly 'linear' transversality properties have been studied, although it has been observed that such properties often fail in very simple situations, for instance, when it comes to convergence analysis of computational algorithms. The first attempts to consider nonlinear extensions of the transversality properties have been made recently in [70,141,177,190]. Even in the linear setting, some dual sufficient conditions have been established only for subtransversality and transversality properties (cf. [127–129, 138–140, 142]). Motivated by these observations, we develop in Chapter 2 a comprehensive (primal and dual) quantitative analysis for transversality properties in the nonlinear setting. Apart from the conventional Hölder case, which is given a special attention, our general model covers also so called *Hölder-type* settings that have recently come into play in the closely related error bound theory due to their importance for applications. We would like to emphasize that our main objective is not formally extending earlier results in the literature from the Hölder to a more general nonlinear setting, but rather to develop a comprehensive theory of transversality. In fact, not many characterizations of semitransversality and transversality have been known even in the linear case. The nonlinearity is just a simple setting, which allows us to unify the existing results on the topic. In Sections 2.2 – 2.4, we establish primal (geometric, metric, and slope) necessary and sufficient conditions for transversality properties. Unlike earlier publications [141, 142], besides quantitative estimates for the rates/moduli of the corresponding properties, we establish here also estimates for the other parameters involved in the definitions, particularly the size of the neighbourhood where a property holds, which is important from the computational point of view. We show that subtransversality and transversality admit several geometric and metric characterizations. In the convex setting, geometric and metric characterizations of nonlinear semitransversality and transversality can be simplified significantly, and they are equivalent in the Hölder setting. Slope necessary conditions for the properties follow directly from definitions of the respective properties, while the sufficient counterparts stem from applying the Ekeland variational principle to appropriate functions; the proofs are rather straightforward. This type of conditions are often considered as just a first step on the way to producing more involved dual (subdifferential and normal cone) conditions, and the primal sufficient characterizations remain hidden in the proofs. We believe that primal conditions (being in a sense analogues of very popular slope characterizations of error bounds) can be of importance for applications since the slope is a derivative-like object. Moreover, subdividing the conventional regularity/transversality theory into primal and dual parts clarifies the roles of the main tools employed within the theory: the Ekeland variational principle used in the primal part and the subdifferential sum rules used in the dual part. Section 2.5 dedicates to dual (subdifferential and normal cone) characterizations of nonlinear transversality properties. These characterizations are obtained by employing corresponding slope necessary and sufficient conditions. Dual necessary conditions are formulated under convexity assumptions by applying the conventional subdifferential sum rule for convex functions in normed spaces. When formulating dual sufficient conditions, the underlying space is assumed Banach or Asplund, and the sets are assumed closed. In the setting of a general Banach space, the characterizations are formulated in terms of Clarke normals and subdifferentials. When the space is Asplund, Fréchet normals and subdifferentials are used in the statements. These characterizations can be easily reformulated in terms of more general abstract subdifferentials possessing some natural properties (in particular, appropriate sum rules) in a reference space (*trustworthy* subdifferentials [110]).

In Chapter 3, we study transversality and regularity properties of set-valued mappings. Section 3.1 is about quantitative relations between transversality properties of collections of sets and regularity properties of set-valued mappings in the nonlinear setting. In the Hölder (in particular, linear) cases, these results improve corresponding ones in [141, 142]. Primal and dual characterizations of nonlinear extensions of the new property *transversality of a mapping to a set in the range space* due to Ioffe [114, 116] are also formulated. We study in Section 3.2 primal and dual characterizations of a new property so called *semitransversality of collections of set-valued mappings* on metric (in particular, normed) spaces. The property is a natural extension of the semitransversality of collections of sets as well as the negation of the corresponding stationarity, a weaker property than the *extremality* of collections of set-valued mappings [175]. Several examples are provided, and connections between the property and semiregularity of set-valued mappings are also discussed. In Section 3.3, we establish new and upgraded primal (geometric and slope) and dual (subdifferential and normal cone) characterizations of uniform regularity properties of set-valued mappings. The approach follows the same pattern as that of the transversality theory. Specifically, slope sufficient conditions comes from the application of the Ekeland variational principle, while the necessary counterparts are obtained from definitions of the properties. Dual necessary and sufficient conditions for regularity are obtained as consequences of the corresponding slope conditions by applying appropriate subdifferential sum rules. They are formulated in terms of distances from the respective dual vectors to normal cones to the graph of the set-valued mapping. The conditions in terms of coderivatives [56, 57, 145, 189, 212] follow as consequences. As a consequence, we establish characterizations of the conventional metric regularity and subregularity of set-valued mappings as well as stability properties of solution mappings to generalized equations.

Chapter 4 dedicates to local and global error bounds of extended-real-valued functions in both linear and nonlinear settings. Conventional linear error bound conditions are discussed in Section 4.2. It contains a preliminary statement – Proposition 46 – treating the case when x in (1.12) is fixed and the general Theorem 32, the latter being an easy consequence of the first. Both statements contain a condition, which has not been used in this type of statements earlier. Conventional and alternative nonlinear error bound conditions are discussed in Sections 4.3 and 4.4, respectively. We demonstrate that the conventional nonlinear conditions are straightforward consequences of the corresponding linear ones, while the alternative conditions are consequences of the conventional ones. In Sections 4.5 and 4.6, we illustrate the sufficient and necessary conditions for nonlinear error bounds by applying them to characterizing nonlinear *subregularity* of general set-valued mappings and *calmness* of solution and level set mappings of canonically perturbed convex semi-infinite optimization problems, respectively.

1.2 Preliminaries

1.2.1 Notation and Definitions

Our basic notation is standard, see, e.g., [81, 172, 202]. The topological dual of a normed vector space X is denoted by X^*, while $\langle\cdot,\cdot\rangle$ denotes the bilinear form defining the pairing between the two spaces. The open unit balls in X and X^* are denoted by $\mathbb{B}$ and $\mathbb{B}^*$, respectively, and $B_\delta(x)$ stands for the open ball with center x and radius $\delta > 0$. If not explicitly stated otherwise, products of normed spaces are assumed equipped with the maximum norm $\|(x,y)\| := \max\{\|x\|,\|y\|\}$, $(x,y) \in X \times Y$. Symbols $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the real line (with the usual norm), the set of all nonnegative real number and the set of all positive integers, respectively.

For a set $\Omega \subset X$, its boundary and interior are denoted by $\operatorname{bd}\Omega$ and $\operatorname{int}\Omega$. The *indicator function* i_Ω of Ω is defined as follows: $i_\Omega(x) = 0$ if $x \in \Omega$ and $i_\Omega(x) = +\infty$ if $x \notin \Omega$. The distance from a point $x \in X$ to Ω is defined by $d(x,\Omega) := \inf_{u\in\Omega}\|u-x\|$, and we use the convention $d(x,\emptyset) = +\infty$.

For an extended-real-valued function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ on a metric space X, its domain and epigraph are defined, respectively, by $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x,\alpha) \in X \times \mathbb{R} \mid f(x) \le \alpha\}$. The inverse of f (if it exists) is denoted by f^{-1}. Note that f is allowed to take the value $-\infty$. This convention is needed only to accommodate for the general chain rule in Lemma 9. Throughout the thesis, we employ the conventional definitions of the lower and upper limits:

$$\liminf_{x\to\bar{x}} f(x) := \sup_{\varepsilon>0} \inf_{0<d(x,\bar{x})<\varepsilon} f(x) \quad \text{and} \quad \limsup_{x\to\bar{x}} f(x) := \inf_{\varepsilon>0} \sup_{0<d(x,\bar{x})<\varepsilon} f(x).$$

The function f is lower semicontinuous and upper semicontinuous at $\bar{x} \in \operatorname{dom} f$ if $\liminf_{x\to\bar{x}} f(x) = f(\bar{x})$ and $\limsup_{x\to\bar{x}} f(x) = f(\bar{x})$, respectively.

A set-valued mapping $F : X \rightrightarrows Y$ between two sets X and Y is a mapping, which assigns to every $x \in X$ a subset (possibly empty) $F(x)$ of Y. We use the notations $\operatorname{gph} F := \{(x,y) \in X \times Y \mid y \in F(x)\}$ and $\operatorname{dom} F := \{x \in X \mid F(x) \ne \emptyset\}$ for the graph and the domain of F, respectively, and $F^{-1} : Y \rightrightarrows X$ for the inverse of F. This inverse (which always exists with possibly empty values at some y) is defined by $F^{-1}(y) := \{x \in X \mid y \in F(x)\}$, $y \in Y$. Obviously $\operatorname{dom} F^{-1} = F(X)$.

Given a subset Ω of a normed space X and a point $\bar{x} \in \Omega$, the sets (cf. [62, 126])

$$N^F_\Omega(\bar{x}) := \left\{ x^* \in X^* \mid \limsup_{\Omega \ni x \to \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \le 0 \right\}, \tag{1.14}$$

$$N^C_\Omega(\bar{x}) := \left\{ x^* \in X^* \mid \langle x^*, z\rangle \le 0 \quad \text{for all} \quad z \in T^C_\Omega(\bar{x}) \right\} \tag{1.15}$$

are the *Fréchet* and *Clarke normal cones* to Ω at $\bar{x}$. In the last definition, $T^C_\Omega(\bar{x})$ stands for the *Clarke tangent cone* [62] to Ω at $\bar{x}$ that consists of all $v \in X$ such that, whenever one has sequences $t_k \downarrow 0$ and $x_k \to \bar{x}$ with $x_k \in \Omega$, there exist $v_k \to v$ with $x_k + t_k v_k \in \Omega$ for all k. The sets (1.14) and (1.15) are nonempty closed convex cones satisfying $N^F_\Omega(\bar{x}) \subset N^C_\Omega(\bar{x})$. If Ω is a convex set, they reduce to the normal cone in the sense of convex analysis:

$$N_\Omega(\bar{x}) := \{x^* \in X^* \mid \langle x^*, x-\bar{x}\rangle \le 0 \quad \text{for all} \quad x \in \Omega\}.$$

Given a function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ and a point $\bar{x} \in X$ with $|f(\bar{x})| < +\infty$, the *Fréchet* and *Clarke subdifferentials* of f at $\bar{x}$ are defined as (cf. [62, 126])

$$\partial^F f(\bar{x}) := \left\{ x^* \in X^* \mid \liminf_{x\to\bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \ge 0 \right\}, \tag{1.16}$$

$$\partial^C f(\bar{x}) := \{x^* \in X^* \mid \langle x^*, z\rangle \le f^\circ(\bar{x}, z) \quad \text{for all} \quad z \in X\}, \tag{1.17}$$

where $f^\circ(\bar{x}, z)$ is the *Clarke–Rockafellar directional derivative* [201] of f at $\bar{x}$ in the direction $z \in X$:

$$f^\circ(\bar{x}; z) := \lim_{\varepsilon\downarrow 0} \limsup_{\substack{(x,\alpha)\to(\bar{x},f(\bar{x}))\\ f(x)\le\alpha,\, t\downarrow 0}} \inf_{\|z'-z\|<\varepsilon} \frac{f(x+tz')-\alpha}{t}.$$

The last definition admits simplifications if f is lower semicontinuous at $\bar{x}$ and especially if it is Lipschitz continuous near $\bar{x}$; cf. [201]. The sets (1.16) and (1.17) are closed and convex, and satisfy $\partial^F f(\bar{x}) \subset \partial^C f(\bar{x})$. If f is convex, they reduce to the subdifferential in the sense of convex analysis (cf., e.g., [62, 126]):

$$\partial f(\bar{x}) := \{x^* \in X^* \mid f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle \geq 0 \quad \text{for all} \quad x \in X\}.$$

The relationships between the subdifferentials of f at $\bar{x} \in \operatorname{dom} f$ and the corresponding normal cones to its epigraph at $(\bar{x}, f(\bar{x}))$ are given by

$$\partial^F f(\bar{x}) = \left\{x^* \in X^* \mid (x^*, -1) \in N^F_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))\right\},$$
$$\partial^C f(\bar{x}) = \left\{x^* \in X^* \mid (x^*, -1) \in N^C_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))\right\}.$$

By convention, we set $N^F_\Omega(\bar{x}) = N^C_\Omega(\bar{x}) := \emptyset$ if $\bar{x} \notin \Omega$ and $\partial^F f(\bar{x}) = \partial^C f(\bar{x}) := \emptyset$ if $|f(\bar{x})| = +\infty$. It is easy to check that $N^F_\Omega(\bar{x}) = \partial^F i_\Omega(\bar{x})$ and $N^C_\Omega(\bar{x}) = \partial^C i_\Omega(\bar{x})$; cf., e.g., [62, 126].

We often use the generic notations N and ∂ for both Fréchet and Clarke objects, specifying wherever necessary that either $N := N^F$ and $\partial := \partial^F$, or $N := N^C$ and $\partial := \partial^C$.

Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces, the *coderivative* of F at $(x, y) \in \operatorname{gph} F$ is a set-valued mapping $D^*F(x, y) : Y^* \rightrightarrows X^*$ defined by

$$D^*F(x,y)(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N_{\operatorname{gph} F}(x,y)\}, \quad y^* \in Y^*. \tag{1.18}$$

Depending on the type of the normal cone, the above formula defines the Fréchet D^*_F or Clarke D^*_C coderivative.

1.2.2 Primal Tools

The fundamental tools for formulating primal characterizations of transversality, regularity and error bound properties are the celebrated Ekeland variational principle (cf. [89]), and the concept of slopes.

Lemma 1 (Ekeland variational principle) Suppose X is a complete metric space, $f : X \to \mathbb{R}_\infty$ is lower semicontinuous, $x \in X$, $\varepsilon > 0$ and $\lambda > 0$. If

$$f(x) < \inf_X f + \varepsilon,$$

then there exists an $\hat{x} \in X$ such that

(i) $d(\hat{x}, x) < \lambda$;

(ii) $f(\hat{x}) \leq f(x)$;

(iii) $f(u) + (\varepsilon/\lambda) d(u, \hat{x}) \geq f(\hat{x})$ for all $u \in X$.

For an extended real-valued function f on a metric space, its *slope* and *nonlocal slope* at $x \in \operatorname{dom} f$ are defined, respectively, by

$$|\nabla f|(x) := \limsup_{u \to x, u \neq x} \frac{[f(x) - f(u)]_+}{d(x,u)} \quad \text{and} \quad |\nabla f|^{\diamond}(x) := \sup_{u \neq x} \frac{[f(x) - f_+(u)]_+}{d(x,u)}, \tag{1.19}$$

where $\alpha_+ := \max\{0, \alpha\}$ for any $\alpha \in \mathbb{R}$ and $f_+(u) := [f(u)]_+$. If $x \notin \operatorname{dom} f$, we set $|\nabla f|(x) = |\nabla f|^{\diamond}(x) := +\infty$.

The slope was introduced in 1980 by DeGiorgi, Marino and Tosques [73], while the nonlocal slope was suggested later by Ngai and Théra [185] (see also, Kruger [131]).

Remark 1 (i) The quantity $|\nabla f|(x)$ provides the rate of steepest descent of f at x. If X is a normed space, and f is Fréchet differentiable at x, then $|\nabla f|(x) = \|f'(x)\|$. If f has a local minimum at x, then $|\nabla f|(x) = 0$.

(ii) Several extensions of the concept of slope for vector-valued functions as well as set-valued mapping can be found in [28, 29, 101, 102, 181].

Recall that a Banach space is *Asplund* if every continuous convex function on an open convex set is Fréchet differentiable on a dense subset [196], or equivalently, if the dual of each its separable subspace is separable. We refer the reader to [172, 196] for discussions about and characterizations of Asplund spaces. All reflexive, particularly, all finite dimensional Banach spaces are Asplund.

The next lemma collects several facts about slopes and their relationships with subdifferentials; cf. [12, 13, 15–17, 92, 93, 112, 116, 131].

Lemma 2 Let X be a metric space, $f : X \to \mathbb{R}_\infty$, and $x \in \operatorname{dom} f$.

(i) If f is not lower semicontinuous at x, then $|\nabla f|(x) = +\infty$.

(ii) If $f(x) > 0$, then $|\nabla f|(x) \le |\nabla f|^{\diamond}(x)$.

Suppose X is a normed space.

(iii) $|\nabla f|(x) \le |\partial^F f|(x)$.

(iv) If f is convex, then $|\nabla f|(x) = |\partial f|(x)$. Moreover, if $f(x) > 0$, then $|\nabla f|(x) = |\nabla f|^{\diamond}(x)$.

(v) If X is a Banach space and f is lower semicontinuous, then $|\nabla f|(x) \ge |\partial^C f|(x)$.

(vi) If X is an Asplund space and f is lower semicontinuous, then

$$|\nabla f|(x) \ge \liminf_{u \to x,\, f(u) \to f(x)} |\partial^F f|(u).$$

Remark 2 Assertions (i)–(iv) in Lemma 2 are straightforward. The more involved assertions (v) and (vi) are consequences of the sum rules in Lemma 4 for the respective subdifferentials. Observe that the fuzzy sum rule for Fréchet subdifferentials in Lemma 4(iii) naturally translates into the 'fuzzy' (due to the $\liminf$ operation) inequality in (vi). The latter estimate was first established (in a slightly more general setting of 'abstract' subdifferentials) in [16, Proposition 4.1] (see also [112, Proposition 3.1]). We have failed to find assertion (v) explicitly formulated in the literature. However, its proof only requires replacing the fuzzy sum rule in part (iii) of Lemma 4 with the 'exact' one in part (ii); cf. the reasoning provided in [17, Remark 6.1] to justify a similar fact involving the limiting subdifferentials.

Clarke subdifferentials in Lemma 2(v) and all the other statements in the thesis can be replaced with Ioffe's *approximate G-subdifferentials* [116] as they possess a sum rule similar to the one in Lemma 4(ii); see [116, Theorem 4.69]. Moreover, it is clear that instead of Clarke subdifferentials in general Banach spaces and Fréchet subdifferentials in Asplund spaces as in parts (v) and (vi) of Lemma 2, one can consider more general *subdifferential pairs* [19] (subdifferentials 'trusted' on a given space [116]) with subdifferentials possessing a sum rule either in the exact or fuzzy form, respectively. We use Clarke and Fréchet subdifferentials in this thesis to keep the presentation simple.

The next lemma provides chain rules for slopes.

Lemma 3 Let X be a metric space, $f: X \to \mathbb{R}_\infty$, $\varphi: \mathbb{R} \to \mathbb{R}_\infty$, $x \in \operatorname{dom} f$ and $f(x) \in \operatorname{dom} \varphi$. Suppose φ is nondecreasing on $\mathbb{R}$ and differentiable at $f(x)$ with $\varphi'(f(x)) > 0$.

(i) $|\nabla(\varphi \circ f)|(x) = \varphi'(f(x))|\nabla f|(x)$.

(ii) Let $f(x) > 0$, $\varphi(t) \le 0$ if $t \le 0$, $\varphi(t) > 0$ if $t > 0$, and φ is differentiable on $]0, f(x)[$, with φ' is nonincreasing (nondecreasing). Then $|\nabla(\varphi \circ f)|^\diamond(x) \ge \varphi'(f(x))|\nabla f|^\diamond(x)$ $(|\nabla(\varphi \circ f)|^\diamond(x) \le \varphi'(f(x))|\nabla f|^\diamond(x))$.

Proof

(i) If x is a local minimum of f, then, thanks to the monotonicity of φ, it is also a local minimum of $\varphi \circ f$, and consequently, $|\nabla(\varphi \circ f)|(x) = |\nabla f|(x) = 0$. Suppose x is not a local minimum of f. If f is not lower semicontinuous at x, i.e. $\alpha := \lim_{k\to+\infty} f(x_k) < f(x)$ for some sequence $x_k \to x$, then, in view of the assumption $\varphi'(f(x)) > 0$, φ is strictly increasing near $f(x)$, and consequently, $\liminf_{k\to+\infty} \varphi(f(x_k)) \le \varphi(\alpha) < \varphi(f(x))$ (with the convention that $\varphi(-\infty) = -\infty$), i.e. $\varphi \circ f$ is not lower semicontinuous at x; hence, in view of Lemma 2(i), $|\nabla(\varphi \circ f)|(x) = |\nabla f|(x) = +\infty$. Suppose f is lower semicontinuous at x, i.e. $\liminf_{u\to x, u\ne x} f(u) = f(x)$. Then, taking into account that x is not a local minimum of f,

$$\begin{aligned}
|\nabla(\varphi\circ f)|(x) &= \limsup_{u\to x,\, u\neq x} \frac{\varphi(f(x))-\varphi(f(u))}{d(u,x)}\\
&= \limsup_{\substack{u\to x,\, u\neq x\\ f(u)<f(x)}} \frac{\varphi(f(x))-\varphi(f(u))}{d(u,x)} = \limsup_{\substack{u\to x,\, u\neq x\\ f(u)\uparrow f(x)}} \frac{\varphi(f(x))-\varphi(f(u))}{d(u,x)}\\
&= \limsup_{\substack{u\to x,\, u\neq x\\ f(u)\uparrow f(x)}} \left(\frac{\varphi(f(x))-\varphi(f(u))}{f(x)-f(u)}\cdot\frac{f(x)-f(u)}{d(u,x)}\right)\\
&= \varphi'(f(x))\limsup_{u\to x,\, u\neq x} \frac{f(x)-f(u)}{d(u,x)} = \varphi'(f(x))|\nabla f|(x).
\end{aligned}$$

The proof is complete.

(ii) If f attains its minimum on X at x, then, thanks to the monotonicity of φ, we have $|\nabla(\varphi\circ f)|^\diamond(x) = |\nabla f|^\diamond(x) = 0$. Thanks to the assumptions on φ, we have $(\varphi\circ f)_+(u) = \varphi(f_+(u))$ and, by the mean value theorem,

$$\frac{\varphi(f(x))-\varphi(f_+(u))}{d(u,x)} = \varphi'(\theta)\frac{f(x)-f_+(u)}{d(u,x)}$$

for some $\theta \in]f_+(u), f(x)[$. The claimed inequalities follow from the respective monotonicity assumptions on φ'.

$\square$

Remark 3 (i) The slope chain rule in Lemma 3(i) is a local result. Instead of assuming that φ is defined on the whole real line, one can assume that φ is defined and finite on a closed interval $[\alpha,\beta]$ around the point $f(x)$: $\alpha < f(x) < \beta$. It is sufficient to define the composition $\varphi\circ f$ for x with $f(x)\notin[\alpha,\beta]$ as follows: $(\varphi\circ f)(x) := \varphi(\alpha)$ if $f(x)<\alpha$, and $(\varphi\circ f)(x) := \varphi(\beta)$ if $f(x)>\beta$. This does not affect the conclusion of the lemma.

(ii) Part (i) of Lemma 3 slightly improves [18, Lemma 4.1], where f and φ are assumed lower semicontinuous and continuously differentiable, respectively.

We are going to use in Chapter 4 the following dual counterparts of the slope $|\nabla f|(x)$:

$$|\partial^F f|(x) := d(0,\partial^F f(x)) \quad \text{and} \quad |\partial^C f|(x) := d(0,\partial^C f(x)). \tag{1.20}$$

Following [92, 204], we call them (respectively, Fréchet and Clarke) *subdifferential slopes* of f at x. When f is convex, we write simply $|\partial f|(x)$.

When formulating sufficient error bound conditions, we use special collections of slope operators. This allows us to combine several assertions into one. The collections are defined recursively as follows:

(i) $|\mathfrak{D} f|^\circ := \{|\nabla f|\}$, $|\mathfrak{D} f| = |\mathfrak{D} f|^\dagger := \{|\nabla f|^\diamond, |\nabla f|\}$;

(ii) if X is Banach, then $|\mathfrak{D}f|^\circ := |\mathfrak{D}f|^\circ \cup \{|\partial^C f|\}$, $|\mathfrak{D}f| = |\mathfrak{D}f|^\dagger := |\mathfrak{D}f| \cup \{|\partial^C f|\}$;

(iii) if X is Asplund, then $|\mathfrak{D}f|^\circ := |\mathfrak{D}f|^\circ \cup \{|\partial^F f|\}$, $|\mathfrak{D}f| := |\mathfrak{D}f| \cup \{|\partial^F f|\}$.

Thus, if X is an Asplund space, then $|\mathfrak{D}f| = \{|\nabla f|^\diamond, |\nabla f|, |\partial^C f|, |\partial^F f|\}$, and if X is a Banach space and f is convex, then $|\mathfrak{D}f| = |\mathfrak{D}f|^\dagger = \{|\nabla f|^\diamond, |\nabla f|, |\partial f|\}$. The 'full' set $|\mathfrak{D}f|$ is going to play the main role in the sufficient error bounds conditions below. In some conditions, we also use the 'truncated' sets $|\mathfrak{D}f|^\circ$ and $|\mathfrak{D}f|^\dagger$, excluding the operators $|\nabla f|^\diamond$ and $|\partial^F f|$, respectively.

1.2.3 Dual Tools

The main dual tools are various subdifferential sum rules and subdifferential-related properties.

Lemma 4 (Subdifferential sum rules) Suppose X is a normed vector space, $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$.

(i) **Convex sum rule**. If f_1 and f_2 are convex, and f_1 is continuous at a point in $\operatorname{dom} f_2$, then

$$\partial(f_1 + f_2)(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x}).$$

(ii) **Clarke–Rockafellar sum rule**. If f_1 is Lipschitz continuous, and f_2 is lower semicontinuous in a neighbourhood of $\bar{x}$, then

$$\partial^C(f_1 + f_2)(\bar{x}) \subset \partial^C f_1(\bar{x}) + \partial^C f_2(\bar{x}).$$

(iii) **Fuzzy sum rule**. If X is Asplund, f_1 is Lipschitz continuous, and f_2 is lower semicontinuous in a neighbourhood of $\bar{x}$, then, for any $\varepsilon > 0$, there exist $x_1, x_2 \in X$ with $\|x_i - \bar{x}\| < \varepsilon$, $|f_i(x_i) - f_i(\bar{x})| < \varepsilon$ $(i = 1, 2)$, such that

$$\partial^F(f_1 + f_2)(\bar{x}) \subset \partial^F f_1(x_1) + \partial^F f_2(x_2) + \varepsilon \mathbb{B}^*.$$

The first sum rule in the lemma above is the conventional subdifferential sum rule of convex analysis; see e.g. [117, Theorem 0.3.3] and [214, Theorem 2.8.7]. The second sum rule is formulated in terms of Clarke subdifferentials. It was established in Rockafellar [201, Theorem 2]. The expressions in (i) and (ii) are examples of *exact* sum rules. The third rule is known as the *fuzzy* or *approximate* sum rule (Fabian [91]) for Fréchet subdifferentials in Asplund spaces; cf., e.g., [126, Rule 2.2] and [172, Theorem 2.33]. Note that, unlike the sum rules in parts (i) and (ii) of the lemma, the subdifferentials in the right-hand side of the inclusion are computed not at the reference point, but at some points nearby. This explains the name. Similar to the previous one, it is valid generally only as inclusion.

The following two facts are immediate consequences of the definitions of the Fréchet and Clarke subdifferentials and normal cones (cf., e.g., [62, 126, 172]).

Lemma 5 Suppose X is a normed space and $f : X \to \mathbb{R} \cup \{+\infty\}$. If $\bar{x} \in \operatorname{dom} f$ is a point of local minimum of f, then $0 \in \partial^F f(\bar{x})$.

Lemma 6 Let Ω_1 and Ω_2 be subsets of a normed space X and $\omega_i \in \Omega_i$ $(i = 1,2)$. Then $N_{\Omega_1 \times \Omega_2}(\omega_1, \omega_2) = N_{\Omega_1}(\omega_1) \times N_{\Omega_2}(\omega_2)$, where in both parts of the equality N stands for either the Fréchet ($N := N^F$) or the Clarke ($N := N^C$) normal cone.

The following result presents a representation of the (convex) subdifferential of a norm function; cf. [214, Corollary 2.4.16] and [159, Example 3.2.7].

Lemma 7 Let $(Y, \|\cdot\|)$ be a normed space. Then

(i) $\partial\|\cdot\|(0) = \{y^* \in Y^* \mid \|y^*\| \le 1\}$;

(ii) $\partial\|\cdot\|(y) = \{y^* \in Y^* \mid \langle y^*, y\rangle = \|y\| \text{ and } \|y^*\| = 1\}, \; y \ne 0$.

We are going to use a representation of the subdifferential of a special convex function on X^{n+1} given in the next lemma; cf. [138, Lemma 3].

Lemma 8 Let X be a normed space and

$$\psi(u_1, \ldots, u_n, u) := \max_{1 \le i \le n} \|u_i - a_i - u\|, \quad u_1, \ldots, u_n, u \in X, \tag{1.21}$$

where $a_i \in X$ $(i = 1, \ldots, n)$. Let $x_1, \ldots, x_n, x \in X$ and $\max_{1 \le i \le n} \|x_i - a_i - x\| > 0$. Then

$$\begin{aligned} \partial\psi(x_1, \ldots, x_n, x) = \Big\{ (x_1^*, \ldots, x_n^*, x^*) \in (X^*)^{n+1} \mid x^* + \sum_{i=1}^n x_i^* = 0, \\ \sum_{i=1}^n \|x_i^*\| = 1, \; \sum_{i=1}^n \langle x_i^*, x_i - a_i - x\rangle = \max_{1 \le i \le n} \|x_i - a_i - x\| \Big\}. \end{aligned} \tag{1.22}$$

Proof The convex function ψ given by (1.21) is a composition of the continuous linear mapping $A : X^{n+1} \to X^n$: $(x_1, \ldots, x_n, x) \mapsto (x_1 - a_1 - x, \ldots, x_n - a_n - x)$ and the maximum norm on X^n: $g(x_1, \ldots, x_n) := \max_{1 \le i \le n} \|x_i\|$. The adjoint mapping A^* is in the form $(x_1^*, \ldots, x_n^*) \mapsto (x_1^*, \ldots, x_n^*, -\sum_{i=1}^n x_i^*)$, while (cf., e.g., [214])

$$\begin{aligned} \partial g(x_1, \ldots, x_n) = \Big\{ (x_1^*, \ldots, x_n^*) \in X^* \mid \sum_{i=1}^n \|x_i^*\| = 1, \\ \langle (x_1^*, \ldots, x_n^*), (x_1, \ldots, x_n)\rangle = \max_{1 \le i \le n} \|x_i\| \Big\}. \end{aligned}$$

The conclusion is a consequence of the convex chain rule (cf., e.g., [214]). □

Remark 4 (i) It is easy to notice that in the representation (1.22), for any $i = 1, \ldots, n$, either $\langle x_i^*, x_i - a_i - x\rangle = \max_{1 \le j \le n} \|x_j - a_j - x\|$ or $x_i^* = 0$.

(ii) The maximum norm on X^n used in (1.21) is a composition of the given norm on X and the maximum norm on $\mathbb{R}^n$. The corresponding dual norm produces the sum of the norms in (1.22). Any other norm on $\mathbb{R}^n$ can replace the maximum in (1.21) as long as the corresponding dual norm is used to replace the sum in (1.22).

We now formulate chain rules for Fréchet and Clarke subdifferentials, which are going to be used in the sequel. Such rules are extensively used when proving dual characterizations of nonlinear transversality, regularity and error bound properties. The next statement seems to present the chain rules under the weakest assumptions, compared to the existing assertions of this type, cf. [60, 62, 126, 186, 194, 211]. We provide an elementary direct proof based only on the definitions of the respective subdifferentials.

In the statement below, X is a general normed space, $\psi : X \to \mathbb{R}\cup\{+\infty\}$ and $\varphi : \mathbb{R} \to \mathbb{R}\cup\{\pm\infty\}$. The composition function $\varphi\circ\psi$ is defined in the usual way:

$$(\varphi\circ\psi)(x) := \begin{cases} \varphi(\psi(x)) & \text{if } x \in \operatorname{dom}\psi, \\ +\infty & \text{if } x \notin \operatorname{dom}\psi. \end{cases} \tag{1.23}$$

The outer function φ is assumed differentiable (or strictly differentiable) at the reference point, while the inner function ψ is arbitrary and does not have to be even (semi-)continuous. Note that $\varphi\circ\psi$ can take the value $-\infty$.

Recall that φ is *strictly differentiable* at $\bar t \in \mathbb{R}$ if it is finite near $\bar t$ and

$$\varphi'(\bar t) = \lim_{t',t''\to\bar t,\, t''\neq t'} \frac{\varphi(t'')-\varphi(t')}{t''-t'},$$

which automatically holds if φ is continuously differentiable at $\bar t$.

Lemma 9 Let X be a normed space, $\psi : X \to \mathbb{R}\cup\{+\infty\}$, $\varphi : \mathbb{R} \to \mathbb{R}\cup\{\pm\infty\}$, and $\bar x \in \operatorname{dom}\psi$. Suppose that φ is nondecreasing on $\mathbb{R}$, and finite and differentiable at $\psi(\bar x)$ with $\varphi'(\psi(\bar x)) > 0$. Then

$$\partial(\varphi\circ\psi)(\bar x) = \varphi'(\psi(\bar x))\partial\psi(\bar x), \tag{1.24}$$

where ∂ stands for the Fréchet subdifferential ($\partial := \partial^F$). If φ is strictly differentiable at $\psi(\bar x)$, then

$$(\varphi\circ\psi)^\circ(\bar x; z) = \varphi'(\psi(\bar x))\psi^\circ(\bar x; z) \quad \text{for all} \quad z \in X, \tag{1.25}$$

and equality (1.24) holds true with ∂ standing for the Clarke subdifferential ($\partial := \partial^C$).

Proof In view of the assumptions on φ, we have $\varphi(t) > \varphi(\psi(\bar x))$ for all $t > \psi(\bar x)$ and $\varphi(t) < \varphi(\psi(\bar x))$ for all $t < \psi(\bar x)$. Set $\gamma := \liminf_{u\to 0}\psi(\bar x+u)$. If $\gamma \neq \psi(\bar x)$, then

$$\begin{aligned} \liminf_{u\to 0}\frac{\psi(\bar x+u)-\psi(\bar x)}{\|u\|} &= \lim_{u\to 0}\frac{\gamma-\psi(\bar x)}{\|u\|}, \\ \liminf_{u\to 0}\frac{\varphi(\psi(\bar x+u))-\varphi(\psi(\bar x))}{\|u\|} &= \lim_{u\to 0}\frac{\varphi(\gamma)-\varphi(\psi(\bar x))}{\|u\|}. \end{aligned}$$

If $\gamma < \psi(\bar{x})$, then both limits above equal $-\infty$ and $\partial^F(\varphi\circ\psi)(\bar{x}) = \partial^F\psi(\bar{x}) = \emptyset$, while if $\gamma > \psi(\bar{x})$, then both limits equal $+\infty$ and $\partial^F(\varphi\circ\psi)(\bar{x}) = \partial^F\psi(\bar{x}) = X^*$.

Let $\gamma = \psi(\bar{x})$ and $x^* \in X^*$. Since $\varphi'(\psi(\bar{x})) > 0$, we have

$$\begin{aligned}
&\liminf_{u\to 0} \frac{\varphi(\psi(\bar{x}+u)) - \varphi(\psi(\bar{x})) - \varphi'(\psi(\bar{x}))\langle x^*, u\rangle}{\|u\|} \\
= &\liminf_{\substack{u\to 0\\ \psi(\bar{x}+u)\to\psi(\bar{x})}} \frac{\varphi(\psi(\bar{x}+u)) - \varphi(\psi(\bar{x})) - \varphi'(\psi(\bar{x}))\langle x^*, u\rangle}{\|u\|} \\
= &\varphi'(\psi(\bar{x})) \liminf_{u\to 0} \frac{\psi(\bar{x}+u) - \psi(\bar{x}) - \langle x^*, u\rangle}{\|u\|}.
\end{aligned}$$

It follows that $x^* \in \partial^F\psi(\bar{x})$ if and only if $\varphi'(\psi(\bar{x}))x^* \in \partial^F(\varphi\circ\psi)(\bar{x})$. Hence, by the definition (1.16), equality (1.24) holds true with ∂ standing for the Fréchet subdifferential ($\partial := \partial^F$).

Now suppose φ is strictly differentiable at $\psi(\bar{x})$. Then φ is strictly increasing and invertible in a neighbourhood of $\psi(\bar{x})$. If $(x_k, \alpha_k) \to (\bar{x}, \psi(\bar{x}))$ and $\psi(x_k) \le \alpha_k$ $(k = 1, 2, \ldots)$, then $\beta_k := \varphi(\alpha_k) \to \varphi(\psi(\bar{x}))$ and $\varphi(\psi(x_k)) \le \beta_k$ $(k = 1, 2, \ldots)$. Conversely, if $(x_k, \beta_k) \to (\bar{x}, \varphi(\psi(\bar{x})))$ and $\varphi(\psi(x_k)) \le \beta_k$ $(k = 1, 2, \ldots)$, then $\alpha_k := \varphi^{-1}(\beta_k) \to \psi(\bar{x})$ and $\psi(x_k) \le \alpha_k$ for all sufficiently large k.

Let $z \in X$. If for some $\varepsilon > 0$ and sequences $(x_k, \alpha_k) \to (\bar{x}, \psi(\bar{x}))$ and $t_k \downarrow 0$, it holds $\psi(x_k) \le \alpha_k$ $(k = 1, 2, \ldots)$ and

$$\lim_{k\to+\infty} \inf_{\|z'-z\|<\varepsilon} (\psi(x_k + t_k z') - \alpha_k) \ne 0,$$

then $|\varphi(\alpha_k)| < +\infty$ for all sufficiently large k,

$$\liminf_{k\to+\infty,\, \|z'-z\|<\varepsilon} (\varphi(\psi(x_k + t_k z')) - \varphi(\alpha_k)) \ne 0,$$

and the above limits are either both positive or both negative. Set

$$\gamma := \lim_{\varepsilon\downarrow 0} \limsup_{\substack{(x,\alpha)\to(\bar{x},\psi(\bar{x}))\\ \psi(x)\le\alpha,\, t\downarrow 0}} \inf_{\|z'-z\|<\varepsilon} (\psi(x + tz') - \alpha).$$

If $\gamma < 0$, then, in view of the above observation, $(\varphi\circ\psi)^\circ(\bar{x}; z) = \psi^\circ(\bar{x}; z) = -\infty$. Similarly, if $\gamma > 0$, then $(\varphi\circ\psi)^\circ(\bar{x}; z) = \psi^\circ(\bar{x}; z) = +\infty$. If $\gamma = 0$, then

$$\begin{aligned}
(\varphi\circ\psi)^\circ(\bar{x}; z) &= \lim_{\varepsilon\downarrow 0} \limsup_{\substack{(x,\alpha)\to(\bar{x},\psi(\bar{x}))\\ \psi(x)\le\alpha,\, t\downarrow 0}} \inf_{\substack{\|z'-z\|<\varepsilon\\ |\psi(x+tz')-\alpha|<\varepsilon}} \frac{\varphi(\psi(x+tz')) - \varphi(\alpha)}{t} \\
&= \varphi'(\psi(\bar{x})) \lim_{\varepsilon\downarrow 0} \limsup_{\substack{(x,\alpha)\to(\bar{x},\psi(\bar{x}))\\ \psi(x)\le\alpha,\, t\downarrow 0}} \inf_{\substack{\|z'-z\|<\varepsilon\\ |\psi(x+tz')-\alpha|<\varepsilon}} \frac{\psi(x+tz') - \alpha}{t} = \varphi'(\psi(\bar{x}))\psi^\circ(\bar{x}; z).
\end{aligned}$$

This proves (1.25). In view of the definition (1.17), equality (1.24) with ∂ standing for the Clarke subdifferential ($\partial := \partial^C$) is a consequence of (1.25). $\square$

Remark 5 (i) The chain rules in Lemma 9 are local results. Instead of assuming that φ is defined on the whole real line with possibly infinite values, one can assume that φ is defined and finite on a closed interval $[\alpha,\beta]$ around the point $\psi(\bar{x})$: $\alpha<\psi(\bar{x})<\beta$. The proof above remains valid if the definition (1.23) of the composition function is slightly modified. The 'if' condition in the first line should be changed to $x\in\operatorname{dom}\psi$ and $\psi(x)\in\operatorname{dom}\varphi$, and another two lines should be added: $(\varphi\circ\psi)(x):=\varphi(\alpha)$ if $\psi(x)<\alpha$, and $(\varphi\circ\psi)(x):=\varphi(\beta)$ if $\beta<\psi(x)<+\infty$. This change does not affect the conclusion of the proposition.

(ii) If, additionally, ψ is assumed lower semicontinuous at $\bar{x}$, then the proof of the proposition can be shortened as the cases $\gamma<\psi(\bar{x})$ in the first part of the proof and $\gamma<0$ in the second part cannot happen. In the lower semicontinuous setting, it is sufficient to assume that φ is nondecreasing only on $[\psi(\bar{x}),+\infty[$ and there is no need to allow φ to take the value $-\infty$.

The next lemma presents several elementary relations between groups of vectors in a normed space, which are frequently used when deducing dual characterizations of extremality, stationarity and transversality properties of collections of sets, although often hidden in numerous proofs; cf. [40, Lemma 2.4].

Lemma 10 Let $K_1,\ldots,K_n$ be cones in a normed space, $\varepsilon>0$, $\rho>0$, and $\mu>0$. Suppose that vectors $z_1,\ldots,z_n$ satisfy

$$\rho\left\|\sum_{i=1}^{n} z_i\right\|+\mu\sum_{i=1}^{n} d(z_i,K_i)<\varepsilon,\quad \sum_{i=1}^{n}\|z_i\|=1. \tag{1.26}$$

(i) If $\varepsilon+\rho\le\mu$, then there exist vectors $\hat{z}_i$ $(i=1,\ldots,n)$ such that

$$\hat{z}_i\in K_i\ (i=1,\ldots,n),\quad \sum_{i=1}^{n}\|\hat{z}_i\|=1,\quad \left\|\sum_{i=1}^{n}\hat{z}_i\right\|<\frac{\varepsilon}{\rho}. \tag{1.27}$$

(ii) If $\varepsilon+\mu\le\rho$, then there exist vectors $\hat{z}_i$ $(i=1,\ldots,n)$ such that

$$\sum_{i=1}^{n}\hat{z}_i=0,\quad \sum_{i=1}^{n}\|\hat{z}_i\|=1,\quad \sum_{i=1}^{n} d(\hat{z}_i,K_i)<\frac{\varepsilon}{\mu}.$$

(iii) Moreover, if the underlying space is dual to a normed space, and

$$\sum_{i=1}^{n}\langle z_i,x_i\rangle\ge\tau\max_{1\le i\le n}\|x_i\| \tag{1.28}$$

for some vectors x_i $(i=1,\ldots,n)$, not all zero, and a number $\tau\in]0,1]$, then the vectors $\hat{z}_i$ $(i=1,\ldots,n)$ in parts (i) or (ii) satisfy

$$\sum_{i=1}^{n}\langle\hat{z}_i,x_i\rangle>\hat{\tau}\max_{1\le i\le n}\|x_i\|, \tag{1.29}$$

where $\hat{\tau}:=\frac{\tau\mu-\varepsilon}{\mu+\varepsilon}$ under the assumptions in part (i), and $\hat{\tau}:=\frac{\tau\rho-\varepsilon}{\rho+\varepsilon}$ under the assumptions in part (ii).

Chapter 2

Transversality Properties of Collections of Sets

The content of this chapter is based on the publications [38, 65, 67, 68, 70].

2.1 Definitions and Basic Relationships

Recall that our working model is a collection of $n \geq 2$ arbitrary subsets $\Omega_1, \ldots, \Omega_n$ of a normed vector space X, having a common point $\bar{x} \in \cap_{i=1}^n \Omega_i$.

The next definition introduces three most common Hölder transversality properties. It is a modification of [141, Definition 1].

Definition 5 Let $\alpha > 0$ and $q > 0$. The collection $\{\Omega_1, \ldots, \Omega_n\}$ is

(i) $\alpha-$semitransversal of order q at $\bar{x}$ if there exists a $\delta > 0$ such that condition (1.5) is satisfied for all $\rho \in]0, \delta[$ and $x_i \in X$ $(i = 1, \ldots, n)$ with $\max_{1 \leq i \leq n} \|x_i\|^q < \alpha\rho$;

(ii) $\alpha-$subtransversal of order q at $\bar{x}$ if there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that condition (1.1) is satisfied for all $\rho \in]0, \delta_1[$ and $x \in B_{\delta_2}(\bar{x})$ with $\max_{1 \leq i \leq n} d^q(x, \Omega_i) < \alpha\rho$;

(iii) $\alpha-$transversal of order q at $\bar{x}$ if there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that condition (1.3) is satisfied for all $\rho \in]0, \delta_1[$, $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1, \ldots, n)$ with $\max_{1 \leq i \leq n} \|x_i\|^q < \alpha\rho$.

The three properties in the above definition were referred to in [141] as $[q]-$semiregularity, $[q]-$subregularity and $[q]-$regularity, respectively. Property (ii) was defined in [141] in a slightly different but equivalent way, under an additional assumption that $q \leq 1$. When $\cap_{i=1}^n \Omega_i$ is closed and $\bar{x} \in \text{bd} \cap_{i=1}^n \Omega_i$, the condition $q \leq 1$ is indeed necessary for the $\alpha-$subtransversality and $\alpha-$transversality properties; see Remark 7. At the same time, as observed in [141], the property of $\alpha-$semitransversality can be meaningful with any positive q (and any positive α); see Example 1. With $q = 1$ (linear case), the three properties reduce to the conventional transversality properties, respectively.

If a collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\alpha-$semitransversal (respectively, $\alpha-$subtransversal or $\alpha-$transversal) of order q at $\bar{x}$ with some $\alpha>0$ and $\delta>0$ (or $\delta_1>0$ and $\delta_2>0$), we often simply say that $\{\Omega_1,\ldots,\Omega_n\}$ is *semitransversal* (respectively, *subtransversal* or *transversal*) of order q at $\bar{x}$. The number α characterizes the corresponding property quantitatively. The exact upper bound of all $\alpha>0$ such that the property holds with some $\delta>0$ (or $\delta_1>0$ and $\delta_2>0$) is called the *modulus* of this property. We use the notations $\mathrm{s_etr}_q[\Omega_1,\ldots,\Omega_n](\bar{x})$, $\mathrm{str}_q[\Omega_1,\ldots,\Omega_n](\bar{x})$ and $\mathrm{tr}_q[\Omega_1,\ldots,\Omega_n](\bar{x})$ for the moduli of the respective properties. If the property does not hold, then by convention the respective modulus equals 0.

If $q<1$, the Hölder transversality properties in Definition 5 are obviously weaker than the corresponding conventional linear properties and can be satisfied for collections of sets when the conventional ones fail. This can happen in many natural situations (see examples in [141, Section 2.3]), which explains the growing interest of researchers to studying the more subtle nonlinear transversality properties.

We now introduce general nonlinear versions of the properties in Definition 5. The nonlinearity in the definitions of the properties is determined by a continuous strictly increasing function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ satisfying $\varphi(0)=0$ and $\lim_{t\to+\infty}\varphi(t)=+\infty$. The collection of all such functions is denoted by $\mathscr{C}$. We denote by $\mathscr{C}^1$ the subfamily of functions from $\mathscr{C}$ which are differentiable on $]0,+\infty[$ with $\varphi'(t)>0$ for all $t>0$. Obviously, if $\varphi\in\mathscr{C}$ ($\varphi\in\mathscr{C}^1$), then $\varphi^{-1}\in\mathscr{C}$ ($\varphi^{-1}\in\mathscr{C}^1$). If not explicitly stated otherwise, we assume from now on that $\varphi\in\mathscr{C}$. It is worth mentioning that for the purposes of our research it suffices to assume that functions $\varphi\in\mathscr{C}$ are defined and invertible near 0.

If not explicitly stated otherwise, we assume from now on that $\varphi\in\mathscr{C}$.

Definition 6 The collection $\{\Omega_1,\ldots,\Omega_n\}$ is

(i) $\varphi-$semitransversal at $\bar{x}$ if there exists a $\delta>0$ such that condition (1.5) is satisfied for all $\rho\in]0,\delta[$ and $x_i\in X$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|)<\rho$;

(ii) $\varphi-$subtransversal at $\bar{x}$ if there exist $\delta_1>0$ and $\delta_2>0$ such that condition (1.1) is satisfied for all $\rho\in]0,\delta_1[$ and $x\in B_{\delta_2}(\bar{x})$ with $\varphi(\max_{1\le i\le n}d(x,\Omega_i))<\rho$;

(iii) $\varphi-$transversal at $\bar{x}$ if there exist $\delta_1>0$ and $\delta_2>0$ such that condition (1.3) is satisfied for all $\rho\in]0,\delta_1[$, $\omega_i\in\Omega_i\cap B_{\delta_2}(\bar{x})$ and $x_i\in X$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|)<\rho$.

Observe that conditions (1.5) and (1.3) are trivially satisfied when $x_i=0$ $(i=1,\ldots,n)$. Hence, in parts (i) and (iii) of Definition 6 (as well as Definition 5) one can additionally assume that $\max_{1\le i\le n}\|x_i\|>0$. Similarly, in part (ii) of Definition 6 (as well as Definition 5) one can assume that $x\notin\cap_{i=1}^n\Omega_i$.

Each of the properties in Definition 6 is determined by a function $\varphi\in\mathscr{C}$, and a number $\delta>0$ in item (i) or numbers $\delta_1>0$ and $\delta_2>0$ in items (ii) and (iii). The function plays the role of a kind of rate or modulus of the respective property, while the role of the δ's is more technical: they control the size of the interval for the values of ρ and, in the case of

$\varphi-$subtransversality and $\varphi-$transversality in parts (ii) and (iii), the size of the neighbourhoods of $\bar{x}$ involved in the respective definitions. Of course, if a property is satisfied with some $\delta_1 > 0$ and $\delta_2 > 0$, it is satisfied also with the single $\delta := \min\{\delta_1, \delta_2\}$ in place of both δ_1 and δ_2. Unlike our previous publications on (linear and Hölder) transversality properties, we use in the current chapter two different parameters to emphasise their different roles in the definitions and the corresponding characterizations. Moreover, we are going to provide quantitative estimates for the values of these parameters.

Given a $\delta > 0$ in item (i) ($\delta_1 > 0$ and $\delta_2 > 0$ in items (ii) and (iii)), if a property is satisfied for some function $\varphi \in \mathscr{C}$, it is obviously satisfied for any function $\hat{\varphi} \in \mathscr{C}$ such that $\hat{\varphi}^{-1}(t) \le \varphi^{-1}(t)$ for all $t \in]0, \delta[$ ($t \in]0, \delta_1[$), or equivalently, $\hat{\varphi}(t) \ge \varphi(t)$ for all $t \in]0, \varphi^{-1}(\delta)[$ ($t \in]0, \varphi^{-1}(\delta_1)[$). Thus, it makes sense looking for the smallest function in $\mathscr{C}$ (if it exists) ensuring the corresponding property for the given sets. Observe also that taking a smaller $\delta > 0$ (smaller $\delta_1 > 0$ and $\delta_2 > 0$) may allow each of the properties to be satisfied with a smaller φ. When the exact value of δ (δ_1 and δ_2) in the definition of the respective property is not important, it makes sense to look for the smallest function ensuring the corresponding property for some $\delta > 0$ (δ_1 and δ_2).

The most important realization of the three properties in Definition 6 corresponds to the Hölder setting, i.e. φ being a power function, given for all $t \ge 0$ by $\varphi(t) := \alpha^{-1}t^q$ with some $\alpha > 0$ and $q > 0$. In this case, Definition 6 reduces to a (slight modification of) [141, Definition 1], and we refer to the respective properties as $\alpha-$*semitransversality*, $\alpha-$*subtransversality* and $\alpha-$*transversality* of order q at $\bar{x}$. With $q = 1$ (linear case), the properties were studied in [128, 129, 142].

Another important for applications class of functions is given by the so called *Hölder-type* [33, 153] ones, i.e. functions of the form $t \mapsto \alpha^{-1}(t^q + t)$, frequently used in the error bound theory, or more generally, functions $t \mapsto \alpha^{-1}(t^q + \beta t)$ with some $\alpha > 0$, $\beta > 0$ and $q > 0$. Depending on the value of q, transversality properties determined by such functions can be approximated by Hölder (if $q < 1$) or even linear (if $q \ge 1$) ones.

Proposition 1 Let $\varphi(t) := \alpha^{-1}(t^q + \beta t)$ with some $\alpha > 0$, $\beta > 0$ and $q > 0$. If the collection $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$(semi-/sub-)transversal at $\bar{x}$, then it is $\alpha'-$(semi-/sub-) transversal of order q' at $\bar{x}$, where:

(i) if $q < 1$, then $q' = q$ and α' is any number in $]0, \alpha[$;

(ii) if $q = 1$, then $q' = 1$ and $\alpha' := \alpha(1+\beta)^{-1}$;

(iii) if $q > 1$, then $q' = 1$ and α' is any number in $]0, \alpha\beta^{-1}[$.

Proof The assertions follow from Definition 5 in view of the following observations:

(i) if $q < 1$ and $\alpha' \in]0, \alpha[$, then, for all sufficiently small $t > 0$, it holds $\alpha'(1 + \beta t^{1-q}) < \alpha$, and consequently, $\varphi(t) = \alpha^{-1}(1 + \beta t^{1-q})t^q < (\alpha')^{-1}t^q$;

(ii) if $q = 1$ and $\alpha' = \alpha(1+\beta)^{-1}$, then $\varphi(t) = \alpha^{-1}(1+\beta)t = (\alpha')^{-1}t$;

(iii) if $q > 1$ and $\alpha' \in]0, \alpha\beta^{-1}[$, then, for all sufficiently small $t > 0$, it holds $\alpha'(\beta^{-1}t^{q-1}+1) < \alpha\beta^{-1}$, and consequently, $\varphi(t) = \alpha^{-1}\beta(\beta^{-1}t^{q-1}+1)t < (\alpha')^{-1}t$.

□

The next two propositions collect some simple facts about the properties in Definition 6 and clarify relationships between them.

Proposition 2 (i) If $\Omega_1 = \ldots = \Omega_n$, and there exists a $\delta_1 > 0$ such that $\varphi(t) \geq t$ for all $t \in]0, \delta_1[$, then $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and any $\delta_2 > 0$.

(ii) If $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then it is $\varphi-$semitransversal at $\bar{x}$ with δ_1 and $\varphi-$subtransversal at $\bar{x}$ with any $\delta_1' \in]0, \delta_1]$ and $\delta_2' > 0$ such that $\varphi^{-1}(\delta_1') + \delta_2' \leq \delta_2$.

(iii) If $\bar{x} \in \operatorname{int} \cap_{i=1}^n \Omega_i$, then $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$.

Proof

(i) Let $\Omega := \Omega_1 = \ldots = \Omega_n$. Then condition (1.1) becomes $\Omega \cap B_\rho(x) \neq \emptyset$. This inclusion is trivially satisfied if $\varphi(d(x, \Omega)) < \rho$ and $\varphi(\rho) \geq \rho$.

(ii) Let $\{\Omega_1, \ldots, \Omega_n\}$ be $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$. Since condition (1.5) is a particular case of condition (1.3) with $\omega_i = \bar{x}$ $(i = 1, \ldots, n)$, we can conclude that $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ_1. Let $\delta_1' \in]0, \delta_1]$ and $\delta_2' > 0$ be such that $\varphi^{-1}(\delta_1') + \delta_2' \leq \delta_2$, and let $\rho \in]0, \delta_1'[$ and $x \in B_{\delta_2'}(\bar{x})$ with $\varphi(\max_{1 \leq i \leq n} d(x, \Omega_i)) < \rho$. Choose $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ such that $\varphi(\max_{1 \leq i \leq n} \|x - \omega_i\|) < \rho$. Then, for any $i = 1, \ldots, n$,

$$\|\omega_i - \bar{x}\| \leq \|x - \omega_i\| + \|x\| < \varphi^{-1}(\rho) + \delta_2' < \delta_2.$$

Set $x_i := x - \omega_i$ $(i = 1, \ldots, n)$. We have $\rho \in]0, \delta_1[$, $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1, \ldots, n)$ and $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \rho$. By Definition 6(iii), condition (1.3) is satisfied. This is equivalent to condition (1.1). In view of Definition 6(ii), $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1' and δ_2'.

(iii) Let $\bar{x} \in \operatorname{int} \cap_{i=1}^n \Omega_i$. Choose numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that, with $\delta := \varphi^{-1}(\delta_1) + \delta_2$, it holds $B_\delta(\bar{x}) \subset \cap_{i=1}^n \Omega_i$. Then, for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1, \ldots, n)$ with $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \delta_1$, it holds $0 \in \cap_{i=1}^n (\Omega_i - \omega_i - x_i)$, and consequently, condition (1.3) is satisfied with any $\rho > 0$. Hence, $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2.

□

Remark 6 (i) The inequality $\varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$ in Proposition 2(ii) and some statements below can obviously be replaced by the equality $\varphi^{-1}(\delta_1') + \delta_2' = \delta_2$ providing in a sense the best estimate for the values of the parameters δ_1' and δ_2'.

(ii) In the Hölder setting, parts (i) and (iii) of Proposition 2 recapture [141, Remarks 4 and 3], respectively, while part (ii) improves [141, Remark 1].

(iii) The nonlinear semitransversality and subtransversality properties are in general independent; see examples in [141, Section 2.3] and [142, Section 3.2].

Proposition 3 Let $\cap_{i=1}^n \Omega_i$ be closed and $\bar{x} \in \operatorname{bd} \cap_{i=1}^n \Omega_i$. If $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$subtransversal (or $\varphi-$transversal) at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then there exists a $\bar{t} \in]0, \min\{\delta_2, \varphi^{-1}(\delta_1)\}[$ such that $\varphi(t) \ge t$ for all $t \in]0, \bar{t}]$.

Proof Let $\{\Omega_1, \ldots, \Omega_n\}$ be $\varphi-$subtransversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$. Choose a point $\hat{x} \notin \cap_{i=1}^n \Omega_i$ such that $\|\hat{x} - \bar{x}\| < \min\{\varphi^{-1}(\delta_1), \delta_2\}$ and set $\bar{t} := d(\hat{x}, \cap_{i=1}^n \Omega_i)$. Then $\bar{t} < \min\{\varphi^{-1}(\delta_1), \delta_2\}$. Besides, $\bar{t} > 0$ since $\cap_{i=1}^n \Omega_i$ is closed. Thanks to the continuity of the function $d(\cdot, \cap_{i=1}^n \Omega_i)$, for any $t \in]0, \bar{t}]$ there is an $x \in]\bar{x}, \hat{x}]$ such that $d(x, \cap_{i=1}^n \Omega_i) = t$. We have $\|x - \bar{x}\| \le \|\hat{x} - \bar{x}\| < \delta_2$ and $\varphi(t) \le \varphi(\bar{t}) < \delta_1$. Take a $\rho \in]\varphi(t), \delta_1[$. Then $\varphi(\max_{1 \le i \le n} d(x, \Omega_i)) \le \varphi(t) < \rho$. By Definition 6(ii), $t = d(x, \cap_{i=1}^n \Omega_i) < \rho$, and letting $\rho \downarrow \varphi(t)$, we arrive at $t \le \varphi(t)$. If $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$transversal at $\bar{x}$, the conclusion follows in view of Proposition 2(ii). $\square$

Remark 7 The conditions on φ in Proposition 3 in the Hölder setting can only be satisfied if either $q < 1$, or $q = 1$ and $\alpha \le 1$. This reflects the well known fact that the Hölder subtransversality and transversality properties are only meaningful when $q \le 1$ and, moreover, the linear case ($q = 1$) is only meaningful when $\alpha \le 1$; cf. [138, p. 705], [134, p. 118]. The extreme case $q = \alpha = 1$ is in a sense singular for subtransversality as in this case Definition 5(ii) yields $d(x, \cap_{i=1}^n \Omega_i) = \max_{1 \le i \le n} d(x, \Omega_i)$ for all x near $\bar{x}$.

In accordance with Proposition 3, the $\varphi-$subtransversality and $\varphi-$transversality properties impose serious restrictions on the function φ. This is not the case with the $\varphi-$semitransversality property: φ can be, e.g., any power function.

Example 1 Let $\mathbb{R}^2$ be equipped with the maximum norm, and let $q > 0$, $\gamma > 0$, $\Omega_1 := \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \gamma^{\frac{1}{q}} \xi_2 + |\xi_1|^{\frac{1}{q}} \ge 0\right\}$, $\Omega_2 := \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \gamma^{\frac{1}{q}} \xi_2 - |\xi_1|^{\frac{1}{q}} \le 0\right\}$ and $\bar{x} := (0, 0)$. Note that, when $q > 1$, the sets Ω_1 and Ω_2 are nonconvex. We claim that the pair $\{\Omega_1, \Omega_2\}$ is $\varphi-$semitransversal at $\bar{x}$ with $\varphi(t) := \gamma t^q$ ($t \ge 0$).

Proof Given an $r > 0$, set $x_1 := (0, -r)$ and $x_2 := (0, r)$. Then $\|x_1\| = \|x_2\| = r$ and $(\pm\gamma r^q, 0) \in (\Omega_1 - x_1) \cap (\Omega_2 - x_2)$. Moreover, it is easy to notice that either $(\gamma r^q, 0)$ or $(-\gamma r^q, 0)$ belongs to $(\Omega_1 - x_1) \cap (\Omega_2 - x_2)$ for any choice of vectors $x_1, x_2 \in \mathbb{R}^2$ with $\max\{\|x_1\|, \|x_2\|\} \le r$. Hence, $(\Omega_1 - x_1) \cap (\Omega_2 - x_2) \cap B_\rho(\bar{x}) \ne \emptyset$ for all such vectors $x_1, x_2 \in \mathbb{R}^2$ as long as $\rho > \gamma r^q$, and consequently, $\{\Omega_1, \Omega_2\}$ is $\varphi-$semitransversal at $\bar{x}$. $\square$

2.2 Geometric Characterizations

The next proposition provides alternative geometric representations of $\varphi-$transversality. They differ from those in Definition 6(iii) by values of the parameters δ_1 and δ_2. Note also the relations between the values of the parameters in the two groups of representations and observe the similarity with those in Proposition 2(ii). One of the advantages of the alternative representations of $\varphi-$transversality given below is their direct relations with those in the definition of $\varphi-$subtransversality. Some other advantages will be exposed later.

Proposition 4 Let $\delta_1 > 0$ and $\delta_2 > 0$. The following properties are equivalent:

(i) condition (1.3) holds for all $\rho \in]0,\delta_1[$, $\omega_i \in \Omega_i$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\omega_i + x_i \in B_{\delta_2}(\bar{x})$ and $\varphi(\max_{1\le i\le n}\|x_i\|) < \rho$;

(ii) condition (1.5) holds for all $\rho \in]0,\delta_1[$ and $x_i \in \delta_2\mathbb{B}$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n} d(\bar{x},\Omega_i - x_i)) < \rho$;

(iii) for all $\rho \in]0,\delta_1[$ and $x,x_i \in X$ with $x + x_i \in B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$ and $\varphi(\max_{1\le i\le n} d(x,\Omega_i - x_i)) < \rho$, it holds

$$\bigcap_{i=1}^{n}(\Omega_i - x_i)\cap B_\rho(x) \neq \emptyset. \tag{2.1}$$

Moreover, if $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then properties (i)–(iii) hold with any $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ satisfying $\varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$ in place of δ_1 and δ_2.

If properties (i)–(iii) hold with some $\delta_1 > 0$ and $\delta_2 > 0$, then $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2, and $\varphi-$transversal at $\bar{x}$ with any $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ satisfying $\varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$.

Proof We first prove the equivalence of the properties (i)–(iii).

(i) $\Rightarrow$ (ii). Let $\rho \in]0,\delta_1[$ and $x_i \in \delta_2\mathbb{B}$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n} d(\bar{x},\Omega_i - x_i)) < \rho$. Choose $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ such that $\varphi(\max_{1\le i\le n}\|\bar{x} + x_i - \omega_i\|) < \rho$. Set $x_i' := \bar{x} + x_i - \omega_i$ $(i = 1,\ldots,n)$. Then $\omega_i + x_i' \in B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i'\|) < \rho$. By (i), condition (1.3) is satisfied with x_i' in place of x_i $(i = 1,\ldots,n)$. This is equivalent to condition (1.5).

(ii) $\Rightarrow$ (iii). Let $\rho \in]0,\delta_1[$ and $x,x_i \in X$ with $x + x_i \in B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$ and $\varphi(\max_{1\le i\le n} d(x,\Omega_i - x_i)) < \rho$. Set $x_i' := x + x_i - \bar{x}$ $(i = 1,\ldots,n)$. Then $x_i' \in \delta_2\mathbb{B}$ $(i = 1,\ldots,n)$ and $\varphi(\max_{1\le i\le n} d(\bar{x},\Omega_i - x_i')) < \rho$. By (ii), condition (1.5) is satisfied with x_i' in place of x_i $(i = 1,\ldots,n)$. This is equivalent to condition (2.1).

(iii) $\Rightarrow$ (i). Let $\rho \in]0,\delta_1[$, $\omega_i \in \Omega_i$ and $x_i \in X$ with $\omega_i + x_i \in B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i\|) < \rho$. Set $x_i' := \omega_i + x_i - \bar{x}$ $(i = 1,\ldots,n)$. Then, $\bar{x} + x_i' \in B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$

and $\varphi(d(\bar{x},\Omega_i-x_i'))\le\varphi(\max_{1\le i\le n}\|x_i\|)<\rho$. By (iii), condition (2.1) is satisfied with $\bar{x}$ and x_i' in place of x and x_i $(i=1,\ldots,n)$, respectively. This is equivalent to condition (1.3).

Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, and let $\delta_1'\in]0,\delta_1]$ and $\delta_2'>0$ be such that $\varphi^{-1}(\delta_1')+\delta_2'\le\delta_2$. Then, for all $\rho\in]0,\delta_1'[$, $\omega_i\in\Omega_i$ and $x_i\in X$ with $\omega_i+x_i\in B_{\delta_2'}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i\|)<\rho$, we have $\|\omega_i-\bar{x}\|\le\|x_i\|+\|\omega_i+x_i-\bar{x}\|<\varphi^{-1}(\delta_1')+\delta_2'\le\delta_2$ $(i=1,\ldots,n)$. By Definition 6(iii), condition (1.3) is satisfied, and consequently, property (i) holds with δ_1' and δ_2'.

Suppose property (iii) holds with some $\delta_1>0$ and $\delta_2>0$. Setting $x_i:=0$ $(i=1,\ldots,n)$, we obtain the property in Definition 6(ii), i.e. $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2.

Suppose property (i) holds with some $\delta_1>0$ and $\delta_2>0$, and let $\delta_1'\in]0,\delta_1]$ and $\delta_2'>0$ be such that $\varphi^{-1}(\delta_1')+\delta_2'\le\delta_2$. Then, for all $\rho\in]0,\delta_1'[$, $\omega_i\in\Omega_i\cap B_{\delta_2'}(\bar{x})$ and $x_i\in X$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|)<\rho$, we have $\|\omega_i+x_i-\bar{x}\|\le\|\omega_i-\bar{x}\|+\|x_i\|<\varphi^{-1}(\delta_1')+\delta_2'\le\delta_2$ $(i=1,\ldots,n)$. By (i), condition (1.3) is satisfied, and consequently, $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1' and δ_2'. $\square$

Corollary 1 The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ if and only if there exist $\delta_1>0$ and $\delta_2>0$ such that any of the properties (i)–(iii) in Proposition 4 holds.

When the sets are convex, the definitions and characterizations of the $\varphi-$semitransversality and $\varphi-$transversality admit simplifications. We are unsure about possible meaningful simplifications of the $\varphi-$subtransversality.

Given a $\delta>0$, we denote by $\widehat{\mathscr{C}}_\delta$ the subfamily of functions from $\mathscr{C}$ satisfying the following property:

$$\frac{\varphi^{-1}(\rho)}{\rho}\le\frac{\varphi^{-1}(\delta)}{\delta}\quad\text{for all}\quad\rho\in]0,\delta[.\tag{2.2}$$

Observe that any $\varphi\in\mathscr{C}$ such that the function $t\mapsto\frac{\varphi^{-1}(t)}{t}$ is nondecreasing on $]0,\delta]$ satisfies this property. This is true (for all $\delta>0$), in particular, in the Hölder setting, i.e. when $\varphi(t):=\alpha^{-1}t^q$ $(t\ge0)$ for some $\alpha>0$ and $q\in]0,1]$.

In the convex case, the requirements that the relations in parts (i) and (iii) of Definition 6 hold for all small $\rho>0$ can be significantly relaxed.

Proposition 5 Suppose $\Omega_1,\ldots,\Omega_n$ are convex, $\delta>0$, and $\varphi\in\widehat{\mathscr{C}}_\delta$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is

(i) $\varphi-$semitransversal at $\bar{x}$ with δ if and only if

$$\bigcap_{i=1}^{n}(\Omega_i-x_i)\cap B_\delta(\bar{x})\neq\emptyset\tag{2.3}$$

for all $x_i\in X$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|)<\delta$;

(ii) $\varphi-$transversal at $\bar{x}$ with $\delta_1 := \delta$ and some $\delta_2 > 0$ if and only if

$$\bigcap_{i=1}^{n}(\Omega_i - \omega_i - x_i) \cap (\delta_1 \mathbb{B}) \neq \emptyset \tag{2.4}$$

for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1, \ldots, n)$ with $\varphi(\max_{1\le i\le n} \|x_i\|) < \delta_1$.

Proof

(i) If $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ, then, by Definition 6(i), for any $x_i \in X$ $(i = 1, \ldots, n)$ with $\varphi(\max_{1\le i\le n} \|x_i\|) < \delta$, and any number ρ satisfying $\varphi(\max_{1\le i\le n} \|x_i\|) < \rho < \delta$, condition (1.5) holds. The latter condition obviously implies (2.3).

Conversely, suppose condition (2.3) is satisfied for all $x_i \in X$ $(i = 1, \ldots, n)$ with $\varphi(\max_{1\le i\le n} \|x_i\|) < \delta$. Let ρ be an arbitrary number in $]0, \delta[$ and let $x_i \in X$ $(i = 1, \ldots, n)$ with $\varphi(\max_{1\le i\le n} \|x_i\|) < \rho$. Set $t := \varphi^{-1}(\rho)/\varphi^{-1}(\delta)$ and $x_i' := x_i/t$ $(i = 1, \ldots, n)$. Then $0 < t < 1$ and $\|x_i'\| = \|x_i\|/t < \varphi^{-1}(\rho)/t = \varphi^{-1}(\delta)$ $(i = 1, \ldots, n)$, and consequently, there exists an $x' \in \cap_{i=1}^{n}(\Omega_i - x_i') \cap B_\delta(\bar{x})$, i.e. $x' \in B_\delta(\bar{x})$ and $x' = \omega_i - x_i'$ for some $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$, or equivalently, $x_i = t(\omega_i - x')$ $(i = 1, \ldots, n)$. In view of the convexity of the sets, we have $t\omega_i + (1-t)\bar{x} \in \Omega_i$ $(i = 1, \ldots, n)$. Set $x := \bar{x} + t(x' - \bar{x})$. We have $x = t\omega_i + (1-t)\bar{x} - t(\omega_i - x') \in \Omega_i - x_i$ $(i = 1, \ldots, n)$. Moreover, in view of (2.2), $\|x - \bar{x}\| = t\|x' - \bar{x}\| < \varphi^{-1}(\rho)\delta/\varphi^{-1}(\delta) \le \rho$. Hence, condition (1.5) is satisfied. By Definition 6(i), $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ.

(ii) If $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with $\delta_1 := \delta$ and some $\delta_2 > 0$, then, by Definition 6(iii), for any $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1, \ldots, n)$ with $\varphi(\max_{1\le i\le n} \|x_i\|) < \delta_1$, and any number ρ satisfying $\varphi(\max_{1\le i\le n} \|x_i\|) < \rho < \delta_1$, condition (1.3) holds. The latter condition obviously implies (2.4).

Conversely, suppose condition (2.4) is satisfied for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1, \ldots, n)$ with $\varphi(\max_{1\le i\le n} \|x_i\|) < \delta_1$. Then the collection of convex sets $\Omega_i - \omega_i$ $(i = 1, \ldots, n)$, considered near their common point 0, satisfies the conditions in part (i) and is consequently $\varphi-$semitransversal at 0 with δ_1 uniformly over $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1, \ldots, n)$. This means that $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2.

$\square$

Remark 8 (i) The 'linear' version of Proposition 5 was established recently in [38].

(ii) Conditions (2.3) and (2.4) are equivalent to the metric estimates $d(\bar{x}, \cap_{i=1}^{n}(\Omega_i - x_i)) < \delta$ and $d(0, \cap_{i=1}^{n}(\Omega_i - \omega_i - x_i)) < \delta_1$, respectively.

(iii) The convexity assumption as well as condition $\varphi \in \widehat{\mathscr{C}}_\delta$ in Proposition 5 are only needed in the sufficiency parts.

Employing the same arguments as in the proof of Proposition 5, it is easy to establish simplified convex case versions of the alternative representations of $\varphi-$transversality in Proposition 4, and the 'restricted' two-set versions of $\varphi-$semitransversality and $\varphi-$transversality in Proposition 10.

Proposition 6 Suppose $\Omega_1,\ldots,\Omega_n$ are convex, $\delta_1>0$, $\delta_2>0$, and $\varphi\in\widehat{\mathscr{C}}_{\delta_1}$. Properties (i)–(iii) in Proposition 4 hold if and only if the following equivalent properties hold true:

(i) condition (2.4) holds for all $\omega_i\in\Omega_i$, $x_i\in X$ with $\omega_i+x_i\in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i\|)<\delta_1$;

(ii) condition (2.3) holds with δ_1 in place of δ for all $x_i\in\delta_2\mathbb{B}$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}d(\bar{x},\Omega_i-x_i))<\delta_1$;

(iii) $\cap_{i=1}^n(\Omega_i-x_i)\cap B_{\delta_1}(x)\neq\emptyset$ for all $x,x_i\in X$ with $x+x_i\in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}d(x,\Omega_i-x_i))<\delta_1$.

Proposition 7 Suppose Ω_1 and Ω_2 are convex, $\bar{x}\in\Omega_1\cap\Omega_2$, $\alpha>0$ and $\alpha':=(1+2\alpha)^{-1}$.

(i) If $\{\Omega_1,\Omega_2\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta>0$, then

$$(\Omega_1-x)\cap\Omega_2\cap B_\delta(\bar{x})\neq\emptyset \tag{2.5}$$

for all $x\in X$ with $\varphi(\|x\|)<\delta$.

Suppose $\delta>0$, $\bar{t}:=\varphi^{-1}(\delta)$, $\varphi\in\widehat{\mathscr{C}}_\delta$, and $\varphi(t)\le\alpha t$ for all $t\in]0,\bar{t}]$. If condition (2.5) holds for all $x\in\bar{t}\mathbb{B}$, then $\{\Omega_1,\Omega_2\}$ is $\alpha'-$semitransversal at $\bar{x}$ with $\delta':=\left(\alpha+\frac{1}{2}\right)\bar{t}$.

(ii) If $\{\Omega_1,\Omega_2\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then

$$(\Omega_1-\omega_1-x)\cap(\Omega_2-\omega_2)\cap(\delta_1\mathbb{B})\neq\emptyset \tag{2.6}$$

for all $\omega_i\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,2)$ and $x\in X$ with $\varphi(\|x\|)<\delta_1$.

Suppose $\delta_1>0$, $\delta_2>0$, $\bar{t}:=\varphi^{-1}(\delta_1)$, $\varphi\in\widehat{\mathscr{C}}_{\delta_1}$, and $\varphi(t)\le\alpha t$ for all $t\in]0,\bar{t}]$. If condition (2.6) holds for all $\omega_i\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,2)$ and $x\in\bar{t}\mathbb{B}$, then $\{\Omega_1,\Omega_2\}$ is $\alpha'-$transversal at $\bar{x}$ with $\delta_1':=\left(\alpha+\frac{1}{2}\right)\bar{t}$ and δ_2.

The next statement clarifies the relationship between the nonlinear semitransversality and transversality in the convex setting.

Proposition 8 Suppose $\Omega_1,\ldots,\Omega_n$ are convex.

(i) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta>0$, then it is $\psi-$transversal at $\bar{x}$ with any $\psi\in\widehat{\mathscr{C}}_\delta$, $\delta_1:=\delta$ and any $\delta_2>0$ such that $\delta_2+\psi^{-1}(\delta)\le\varphi^{-1}(\delta)$.

(ii) Suppose $\alpha>0$, $\delta>0$, $\varphi\in\widehat{\mathscr{C}}_\delta$, and $\varphi^{-1}(\rho)/\rho\ge\alpha$ for all $\rho\in]0,\delta[$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ, then, for any $\varepsilon\in]0,\alpha[$, it is $\psi-$transversal at $\bar{x}$ with $\psi\in\mathscr{C}$ such that $\psi^{-1}(t)=\varphi^{-1}(t)-\varepsilon t$ if $t\in[0,\alpha]$, $\delta_1:=\delta$ and $\delta_2:=\varepsilon\delta$.

Proof

(i) Let $\{\Omega_1,\dots,\Omega_n\}$ be $\varphi-$semitransversal at $\bar{x}$ with some $\delta>0$, and let $\psi\in\widehat{\mathscr{C}}_\delta$, $\delta_1:=\delta$ and $\delta_2>0$ be such that $\delta_2+\psi^{-1}(\delta)\le\varphi^{-1}(\delta)$. Let $\omega_i\in\Omega_i\cap B_{\delta_2}(\bar{x})$ and $x_i\in X$ $(i=1,\dots,n)$ with $\psi(\max_{1\le i\le n}\|x_i\|)<\delta$. Set $x_i':=\omega_i+x_i-\bar{x}$ $(i=1,\dots,n)$. Then $\|x_i'\|\le\|\omega_i-\bar{x}\|+\|x_i\|<\delta_2+\psi^{-1}(\delta)\le\varphi^{-1}(\delta)$ $(i=1,\dots,n)$, and by Proposition 5(i), $\cap_{i=1}^n(\Omega_i-x_i')\cap B_\delta(\bar{x})\neq\emptyset$, which is equivalent to condition (2.4). In view of Proposition 5(ii), $\{\Omega_1,\dots,\Omega_n\}$ is $\psi-$transversal at $\bar{x}$ with δ_1 and δ_2.

(ii) Observe that $\psi^{-1}\in\mathscr{C}$, hence $\psi\in\mathscr{C}$; $\delta_2+\psi^{-1}(\delta)=\varphi^{-1}(\delta)$, and $\frac{\psi^{-1}(\rho)}{\rho}=\frac{\varphi^{-1}(\rho)}{\rho}-\varepsilon\le\frac{\psi^{-1}(\delta)}{\delta}$ for all $\rho\in]0,\delta[$.

□

In the Hölder setting, the above corollary yields the following assertion.

Corollary 2 Suppose $\Omega_1,\dots,\Omega_n$ are convex. Let $\alpha>0$ and $q\in]0,1]$. If $\{\Omega_1,\dots,\Omega_n\}$ is $\alpha-$semitransversal of order q at $\bar{x}$, then it is $\alpha'-$transversal of order q at $\bar{x}$ with any $\alpha'\in]0,\alpha[$. As a consequence, $\{\Omega_1,\dots,\Omega_n\}$ is semitransversal of order q at $\bar{x}$ if and only if it is transversal of order q at $\bar{x}$.

Remark 9 In the linear case ($q=1$), the second part of Corollary 2 recaptures [127, Proposition 13(iv)].

2.3 Metric Characterizations

The three transversality properties are defined in Definition 6 geometrically. We now show that they can be characterized in metric terms. These metric characterizations can be used as equivalent definitions of the respective properties.

Theorem 1 The collection $\{\Omega_1,\dots,\Omega_n\}$ is

(i) $\varphi-$semitransversal at $\bar{x}$ with some $\delta>0$ if and only if

$$d\left(\bar{x},\bigcap_{i=1}^n(\Omega_i-x_i)\right)\le\varphi\left(\max_{1\le i\le n}\|x_i\|\right)\tag{2.7}$$

for all $x_i\in X$ $(i=1,\dots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|)<\delta$;

(ii) $\varphi-$subtransversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$ if and only if the following equivalent conditions hold:

(a) for all $x\in B_{\delta_2}(\bar{x})$ with $\varphi\left(\max_{1\le i\le n}d(x,\Omega_i)\right)<\delta_1$, it holds

$$d\left(x,\bigcap_{i=1}^n\Omega_i\right)\le\varphi\left(\max_{1\le i\le n}d(x,\Omega_i)\right);\tag{2.8}$$

(b) for all $x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1$ and $\omega_1 + x_1 = \ldots = \omega_n + x_n \in B_{\delta_2}(\bar{x})$, it holds

$$d\left(0, \bigcap_{i=1}^{n}(\Omega_i - \omega_i - x_i)\right) \le \varphi\left(\max_{1\le i\le n}\|x_i\|\right); \tag{2.9}$$

(iii) $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$ if and only if inequality (2.9) holds for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1$.

Proof

(i) Let $\{\Omega_1,\ldots,\Omega_n\}$ be $\varphi-$semitransversal at $\bar{x}$ with some $\delta > 0$, and let $x_i \in X$ $(i = 1,\ldots,n)$ with $\rho_0 := \varphi(\max_{1\le i\le n}\|x_i\|) < \delta$. Choose a $\rho \in]\rho_0,\delta[$. By (1.5), $d\left(\bar{x}, \cap_{i=1}^{n}(\Omega_i - x_i)\right) < \rho$. Letting $\rho \downarrow \rho_0$, we arrive at inequality (2.7).

Conversely, let $\delta > 0$ and inequality (2.7) hold for all $x_i \in X$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta$. For all $\rho \in]0,\delta[$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \rho$, we have $d\left(\bar{x}, \cap_{i=1}^{n}(\Omega_i - x_i)\right) < \rho$, which implies condition (1.5). By Definition 6(i), $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ.

(ii) We first prove the equivalence between (a) and (b).

Suppose condition (a) is satisfied. Let $\omega_i \in \Omega_i$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1$ and $x := \omega_1 + x_1 = \ldots = \omega_n + x_n \in B_{\delta_2}(\bar{x})$. Then

$$\varphi\left(d(x,\Omega_i)\right) = \varphi\left(d(\omega_i + x_i,\Omega_i)\right) \le \varphi(\|x_i\|) < \delta_1 \quad (i = 1,\ldots,n),$$

and consequently, inequality (2.8) is satisfied. Hence,

$$d\Big(0, \bigcap_{i=1}^{n}(\Omega_i - \omega_i - x_i)\Big) = d\Big(x, \bigcap_{i=1}^{n}\Omega_i\Big) \le \varphi\left(\max_{1\le i\le n} d(x,\Omega_i)\right) \le \varphi\Big(\max_{1\le i\le n}\|x_i\|\Big).$$

Suppose condition (b) is satisfied. Let $x \in B_{\delta_2}(\bar{x})$ with $\varphi\left(\max_{1\le i\le n} d(x,\Omega_i)\right) < \delta_1$. Choose $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ such that $\varphi(\max_{1\le i\le n}\|x - \omega_i\|) < \delta_1$ and set $x_i' := x - \omega_i$ $(i = 1,\ldots,n)$. Then $x = x_i' + \omega_i \in B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i'\|) < \delta_1$. In view of inequality (2.9) with x_i' in place of x_i $(i = 1,\ldots,n)$, we obtain

$$d\Big(x, \bigcap_{i=1}^{n}\Omega_i\Big) \le \varphi\Big(\max_{1\le i\le n}\|x - \omega_i\|\Big).$$

Taking infimum in the right-hand side over $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$, we arrive at inequality (2.8).

We show that $\varphi-$subtransversality is equivalent to condition (i). Let $\{\Omega_1,\ldots,\Omega_n\}$ be $\varphi-$subtransversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, and let $x \in B_{\delta_2}(\bar{x})$ with $\rho_0 := \varphi\left(\max_{1\le i\le n} d(x,\Omega_i)\right) < \delta_1$. Choose a $\rho \in]\rho_0,\delta_1[$. By Definition 6(ii), $\cap_{i=1}^{n}\Omega_i \cap$

$B_\rho(x) \neq \emptyset$, and consequently, $d\left(x, \cap_{i=1}^n \Omega_i\right) < \rho$. Letting $\rho \downarrow \rho_0$, we arrive at inequality (2.8).

Conversely, let $\delta_1 > 0$ and $\delta_2 > 0$, and inequality (2.8) hold for all $x \in B_{\delta_2}(\bar{x})$ with $\varphi(\max_{1\le i\le n} d(x,\Omega_i)) < \delta_1$. For any $\rho \in]0,\delta_1[$ and $x \in B_{\delta_2}(\bar{x})$ with $\varphi(\max_{1\le i\le n} d(x,\Omega_i)) < \rho$, we have $d\left(x, \cap_{i=1}^n \Omega_i\right) < \rho$, which implies condition (1.1). By Definition 6(ii), $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2.

(iii) Let $\{\Omega_1,\ldots,\Omega_n\}$ be $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, and let $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\rho_0 := \varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1$. Choose a $\rho \in]\rho_0,\delta_1[$. By (1.3), $d\left(0, \cap_{i=1}^n(\Omega_i - \omega_i - x_i)\right) < \rho$. Letting $\rho \downarrow \rho_0$, we arrive at inequality (2.9).

Conversely, let $\delta_1 > 0$ and $\delta_2 > 0$, and inequality (2.9) hold for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1$. For any $\rho \in]0,\delta_1[$, $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \rho$, we have $d\left(0, \cap_{i=1}^n(\Omega_i - \omega_i - x_i)\right) < \rho$, which is equivalent to condition (1.3). By Definition 6(iii), $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2.

□

Example 2 Let $\mathbb{R}^2$ be equipped with the maximum norm, and let $\Omega_1 := \{(\xi_1,\xi_2) \in \mathbb{R}^2 \mid \xi_2 \ge 0\}$, $\Omega_2 := \{(\xi_1,\xi_2) \in \mathbb{R}^2 \mid \xi_2 \le \xi_1^2\}$ and $\bar{x} := (0,0)$. Thus, $\Omega_1 \cap \Omega_2 = \{(\xi_1,\xi_2) \in \mathbb{R}^2 \mid 0 \le \xi_2 \le \xi_1^2\}$, and no shift of the sets can make their intersection empty. We claim that the pair $\{\Omega_1,\Omega_2\}$ is $\varphi-$semitransversal at $\bar{x}$ with $\varphi(t) := \sqrt{2t}$ $(t \ge 0)$ and $\delta := 2$.

Proof Observe that, given any $\varepsilon \ge 0$, the vertical shifts of the sets determined by $x_{1\varepsilon} := (0,-\varepsilon)$ and $x_{2\varepsilon} := (0,\varepsilon)$ produce the largest 'gap' between them compared to all possible shifts x_1 and x_2 with $\max\{\|x_1\|,\|x_2\|\} \le \varepsilon$. Indeed,

$$
\begin{aligned}
(\Omega_1 - x_{1\varepsilon}) \cap (\Omega_2 - x_{2\varepsilon}) &= \{(\xi_1,\xi_2) \in \mathbb{R}^2 \mid \varepsilon \le \xi_2 \le \xi_1^2 - \varepsilon\} \\
&\subset (\Omega_1 - x_1) \cap (\Omega_2 - x_2),
\end{aligned}
$$

as long as $\max\{\|x_1\|,\|x_2\|\} \le \varepsilon$. Observe also that $(\sqrt{2\varepsilon},\varepsilon) \in (\Omega_1 - x_{1\varepsilon}) \cap (\Omega_2 - x_{2\varepsilon})$. Hence, for any $x_1, x_2 \in \mathbb{R}^2$ with $\varepsilon := \max\{\|x_1\|,\|x_2\|\} < \varphi^{-1}(\delta) = 2$, we have

$$
d(\bar{x}, (\Omega_1 - x_1) \cap (\Omega_2 - x_2)) \le \|(\sqrt{2\varepsilon},\varepsilon)\| = \sqrt{2\varepsilon} = \varphi\left(\max\{\|x_1\|,\|x_2\|\}\right).
$$

In view of Theorem 1(i), $\{\Omega_1,\Omega_2\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ. □

Example 3 Let $\mathbb{R}^2$ be equipped with the maximum norm, and let $\Omega_1 := \{(\xi_1,\xi_2) \in \mathbb{R}^2 \mid \xi_2 = \xi_1^2\}$, $\Omega_2 := \{(\xi_1,\xi_2) \in \mathbb{R}^2 \mid \xi_2 = -\xi_1^2\}$ and $\bar{x} := (0,0)$. Thus, $\Omega_1 \cap \Omega_2 = \{\bar{x}\}$. We claim that, for any $\gamma > 1$, the pair $\{\Omega_1,\Omega_2\}$ is $\varphi-$subtransversal at $\bar{x}$ with $\varphi(t) := \gamma\sqrt{t}$ $(t \ge 0)$ and any $\delta_1 > 0$ and $\delta_2 > 0$ satisfying $\delta_2 + \frac{1}{2} + \sqrt{\delta_2 + \frac{1}{4}} < \gamma^2$.

Proof Observe that, $d(x,\Omega_1\cap\Omega_2)=\|x\|$ for all $x\in\mathbb{R}^2$ and, given any $\varepsilon\geq 0$ and the corresponding point $x_\varepsilon:=(0,\varepsilon)$, one has

$$\min_{\|x\|=\varepsilon}\max\{d(x,\Omega_1),d(x,\Omega_2)\}=d(x_\varepsilon,\Omega_1)=d(x_\varepsilon,\Omega_2)=\min_{t\geq 0}\max\{\varepsilon-t,t^2\}.$$

It is easy to see that the minimum in the rightmost minimization problem is attained at $t:=\sqrt{\varepsilon+\frac{1}{4}}-\frac{1}{2}$ satisfying $\varepsilon-t=t^2$. Thus,

$$\min_{\|x\|=\varepsilon}\max\{d(x,\Omega_1),d(x,\Omega_2)\}=\varepsilon+\frac{1}{2}-\sqrt{\varepsilon+\frac{1}{4}}=\frac{\varepsilon^2}{\varepsilon+\frac{1}{2}+\sqrt{\varepsilon+\frac{1}{4}}}.$$

Hence, for any $x\in\mathbb{R}^2$ with $\|x\|<\delta_2$, we have

$$d(x,\Omega_1\cap\Omega_2)=\|x\|\leq\frac{\gamma\|x\|}{\sqrt{\|x\|+\frac{1}{2}+\sqrt{\|x\|+\frac{1}{4}}}}\leq\varphi(\max\{d(x,\Omega_1),d(x,\Omega_2)\}).$$

In view of Theorem 1(ii), $\{\Omega_1,\Omega_2\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2. □

The next statement provides alternative metric characterizations of $\varphi-$transversality. These characterizations differ from the one in Theorem 1(iii) by values of the parameters δ_1 and δ_2 and have certain advantages, e.g., when establishing connections with metric regularity of set-valued mappings. The relations between the values of the parameters in the two groups of metric characterizations can be estimated.

Theorem 2 Let $\delta_1>0$ and $\delta_2>0$. The following conditions are equivalent:

(i) inequality (2.9) is satisfied for all $x_i\in X$ and $\omega_i\in\Omega_i$ with $\omega_i+x_i\in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\leq i\leq n}\|x_i\|)<\delta_1$;

(ii) for all $x_i\in\delta_2\mathbb{B}$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\leq i\leq n}d(\bar{x},\Omega_i-x_i))<\delta_1$, it holds

$$d\Big(\bar{x},\bigcap_{i=1}^{n}(\Omega_i-x_i)\Big)\leq\varphi\Big(\max_{1\leq i\leq n}d(\bar{x},\Omega_i-x_i)\Big);\tag{2.10}$$

(iii) for all $x,x_i\in X$ with $x+x_i\in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\leq i\leq n}d(x,\Omega_i-x_i))<\delta_1$, it holds

$$d\Big(x,\bigcap_{i=1}^{n}(\Omega_i-x_i)\Big)\leq\varphi\Big(\max_{1\leq i\leq n}d(x,\Omega_i-x_i)\Big).\tag{2.11}$$

Moreover, if $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then conditions (i)–(iii) hold with any $\delta_1'\in]0,\delta_1]$ and $\delta_2'>0$ satisfying $\varphi^{-1}(\delta_1')+\delta_2'\leq\delta_2$ in place of δ_1 and δ_2.

Conversely, if conditions (i)–(iii) hold with some $\delta_1>0$ and $\delta_2>0$, then $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with any $\delta_1'\in]0,\delta_1]$ and $\delta_2'>0$ satisfying $\varphi^{-1}(\delta_1')+\delta_2'\leq\delta_2$.

Proof We first prove the equivalence of conditions (i)–(iii).

(i) $\Rightarrow$ (ii). Let $x_i \in \delta_2\mathbb{B}$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n} d(\bar{x},\Omega_i - x_i)) < \delta_1$. Choose $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$ such that $\varphi(\max_{1\le i\le n}\|\bar{x}+x_i-\omega_i\|) < \delta_1$. Set $x_i' := \bar{x}+x_i-\omega_i$ $(i=1,\ldots,n)$. Then $\omega_i + x_i' \in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i'\|) < \delta_1$. By (i), inequality (2.9) is satisfied with x_i' in place of x_i $(i=1,\ldots,n)$, i.e.

$$d\Big(\bar{x},\bigcap_{i=1}^{n}(\Omega_i - x_i)\Big) \le \varphi\Big(\max_{1\le i\le n}\|\bar{x}+x_i-\omega_i\|\Big).$$

Taking the infimum in the righ-hand side of the above inequality over $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$, we arrive at inequality (2.10).

(ii) $\Rightarrow$ (iii). Let $x, x_i \in X$ with $x + x_i \in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n} d(x,\Omega_i - x_i)) < \delta_1$. Set $x_i' := x + x_i - \bar{x}$ $(i=1,\ldots,n)$. Then $x_i' \in \delta_2\mathbb{B}$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n} d(\bar{x},\Omega_i - x_i')) < \delta_1$. By (ii), inequality (2.10) is satisfied with x_i' in place of x_i $(i=1,\ldots,n)$. This is equivalent to inequality (2.11).

(iii) $\Rightarrow$ (i). Let $x_i \in X$ and $\omega_i \in \Omega_i$ with $\omega_i + x_i \in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1$. Set $x_i' := \omega_i + x_i - \bar{x}$ $(i=1,\ldots,n)$. Then, $\bar{x} + x_i' \in B_{\delta_2}(\bar{x})$ and $\varphi(\max_{1\le i\le n} d(\bar{x},\Omega_i - x_i')) \le \varphi(\|x_i\|) < \delta_1$. By (iii), inequality (2.11) is satisfied with $\bar{x}$ and x_i' in place of x and x_i $(i=1,\ldots,n)$, respectively, i.e.

$$d\Big(0,\bigcap_{i=1}^{n}(\Omega_i - \omega_i - x_i)\Big) \le \varphi\Big(\max_{1\le i\le n} d(0,\Omega_i - \omega_i - x_i)\Big).$$

Since $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$, inequality (2.9) is satisfied.

Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, and let $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ be such that $\varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$. Then, for all $x_i \in X$ and $\omega_i \in \Omega_i$ with $\omega_i + x_i \in B_{\delta_2'}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1'$, we have $\|\omega_i - \bar{x}\| \le \|x_i\| + \|\omega_i + x_i - \bar{x}\| < \varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$ $(i=1,\ldots,n)$. By Theorem 1(iii), inequality (2.9) is satisfied, and consequently, condition (i) (as well as conditions (ii) and (iii)) holds with δ_1' and δ_2'.

Conversely, suppose conditions (i)–(iii) hold with some $\delta_1 > 0$ and $\delta_2 > 0$, and let $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ be such that $\varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$. Then, for all $\omega_i \in \Omega_i \cap B_{\delta_2'}(\bar{x})$ and $x_i \in X$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1'$, we have $\|\omega_i + x_i - \bar{x}\| \le \|x_i\| + \|\omega_i - \bar{x}\| < \varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$ $(i=1,\ldots,n)$. By (i), inequality (2.9) is satisfied, and consequently, $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1' and δ_2' according to Theorem 1(iii). $\square$

Remark 10 (i) In the Hölder case, i.e. when $\varphi(t) := \alpha^{-1}t^q$ $(t \ge 0)$ for some $\alpha > 0$ and $q \in]0,1]$, condition (2.11) served as the main metric characterization of transversality; cf. [141, 142]. In the linear case, condition (2.10) has been picked up recently in [38, 40]. This condition seems an important advancement as it replaces an arbitrary point x in (2.11) with the given reference point $\bar{x}$. Condition (2.9) in part (i) seems new. In view of Theorem 1(iii), it is the most straightforward metric counterpart of the original geometric property (1.3).

(ii) The metric characterizations of the three $\varphi-$transversality properties in the above theorems look similar: each of them provides an upper error bound type estimate for the distance from a point to the intersection of sets, which can be useful from the computational point of view. For the account of nonlinear error bounds theory, we refer the reader to [17, 18, 64, 211].

The next corollary provides qualitative metric characterizations of the three nonlinear transversality properties. They are direct consequences of Theorems 1 and 2.

Corollary 3 The collection $\{\Omega_1, \ldots, \Omega_n\}$ is

(i) $\varphi-$semitransversal at $\bar{x}$ if and only if there exists a $\delta > 0$ such that inequality (2.7) holds for all $x_i \in \delta\mathbb{B}$ $(i = 1, \ldots, n)$;

(ii) $\varphi-$subtransversal at $\bar{x}$ if and only if the following equivalent conditions hold:

 (a) there exists a $\delta > 0$ such that inequality (2.8) holds for all $x \in B_\delta(\bar{x})$;

 (b) there exists a $\delta > 0$ such that inequality (2.9) holds for all $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$ and $x_i \in \delta\mathbb{B}$ $(i = 1, \ldots, n)$ with $\omega_1 + x_1 = \ldots = \omega_n + x_n$;

(iii) $\varphi-$transversal at $\bar{x}$ if and only if the following equivalent conditions hold:

 (a) there exists a $\delta > 0$ such that inequality (2.9) holds for all $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$ and $x_i \in \delta\mathbb{B}$ $(i = 1, \ldots, n)$;

 (b) there exists a $\delta > 0$ such that inequality (2.10) holds for all $x_i \in \delta\mathbb{B}$ $(i = 1, \ldots, n)$;

 (c) there exists a $\delta > 0$ such that inequality (2.11) holds for all $x \in B_\delta(\bar{x})$ and $x_i \in \delta\mathbb{B}$ $(i = 1, \ldots, n)$.

Remark 11 In the Hölder setting, i.e. when $\varphi(t) := \alpha^{-1}t^q$ $(t \geq 0)$ with some $\alpha > 0$ and $q > 0$, the above corollary improves [141, Theorem 1]. In the linear case, the equivalence of the three characterizations of transversality in Corollary 3(iii) has been established in [38]. We refer the readers to [134, 138, 139] for more discussions and historical comments.

The next two propositions identify important situations when 'restricted' versions of the metric characterizations of nonlinear transversality properties in Theorem 1 can be used: with all but one sets being translated in the cases of $\varphi-$semitransversality and $\varphi-$transversality, and with the point x restricted to one of the sets in the case of $\varphi-$subtransversality. The latter restricted version is of importance, for instance, when dealing with alternating (or cyclic) projections. The first proposition formulates simplified necessary characterizations of the transversality properties which are direct consequences of the respective statements, while the second one gives conditions under which these characterizations become sufficient in the case of two sets.

Proposition 9 (i) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta>0$, then

$$d\Big(\bar{x},\bigcap_{i=1}^{n-1}(\Omega_i-x_i)\cap\Omega_n\Big)\le\varphi\Big(\max_{1\le i\le n-1}\|x_i\|\Big)$$

for all $x_i\in X$ $(i=1,\ldots,n-1)$ with $\varphi(\max_{1\le i\le n-1}\|x_i\|)<\delta$.

(ii) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then

$$d\Big(x,\bigcap_{i=1}^{n}\Omega_i\Big)\le\varphi\Big(\max_{1\le i\le n-1}d(x,\Omega_i)\Big)$$

for all $x\in\Omega_n\cap B_{\delta_2}(\bar{x})$ with $\varphi(\max_{1\le i\le n-1}d(x,\Omega_i))<\delta_1$.

(iii) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then

$$d\Big(0,\bigcap_{i=1}^{n-1}(\Omega_i-\omega_i-x_i)\cap(\Omega_n-\omega_n)\Big)\le\varphi\Big(\max_{1\le i\le n-1}\|x_i\|\Big)$$

for all $\omega_i\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $x_i\in X$ $(i=1,\ldots,n-1)$ with $\varphi(\max_{1\le i\le n-1}\|x_i\|)<\delta_1$.

Proposition 10 Let Ω_1,Ω_2 be subsets of a normed space X, and $\bar{x}\in\Omega_1\cap\Omega_2$. Let $\alpha>0$, $\bar{t}>0$, $\varphi(t)\le\alpha t$ for all $t\in]0,\bar{t}]$, and $\alpha':=(1+2\alpha)^{-1}$.

(i) If for all $x\in\bar{t}\mathbb{B}$,

$$d\left(\bar{x},(\Omega_1-x)\cap\Omega_2\right)\le\varphi(\|x\|),\tag{2.12}$$

then $\{\Omega_1,\Omega_2\}$ is $\alpha'-$semitransversal at $\bar{x}$ with $\delta:=\left(\alpha+\frac{1}{2}\right)\bar{t}$.

(ii) If there exists a $\delta_2>0$ such that, for all $x\in\Omega_2\cap B_{2\delta_2}(\bar{x})$ with $d(x,\Omega_1)<\bar{t}$,

$$d\left(x,\Omega_1\cap\Omega_2\right)\le\varphi(d(x,\Omega_1)),\tag{2.13}$$

then $\{\Omega_1,\Omega_2\}$ is $\alpha'-$subtransversal at $\bar{x}$ with $\delta_1:=\left(\alpha+\frac{1}{2}\right)\bar{t}$ and δ_2.

(iii) If there exists a $\delta_2>0$ such that, for all $\omega_i\in\Omega_i\cap B_{\delta_2}(\bar{x})(i=1,2)$ and $x\in\bar{t}\mathbb{B}$,

$$d\left(0,(\Omega_1-\omega_1-x)\cap(\Omega_2-\omega_2)\right)\le\varphi(\|x\|)$$

then $\{\Omega_1,\Omega_2\}$ is $\alpha'-$transversal at $\bar{x}$ with $\delta_1:=\left(\alpha+\frac{1}{2}\right)\bar{t}$ and δ_2.

Proof

(i) Let $\delta := \left(\alpha+\frac{1}{2}\right)\bar{t}$, and inequality (2.12) be satisfied for all $x \in \bar{t}\mathbb{B}$. Let $\rho \in]0,\delta[$ and $x_1, x_2 \in X$ with $\max\{\|x_1\|,\|x_2\|\} < \alpha'\rho$. Set $x' := x_1 - x_2$. Thus, $\|x'\| \le 2\max\{\|x_1\|,\|x_2\|\} < 2\alpha'\delta = \bar{t}$. Hence, by (2.12) with x' in place of x,

$$\begin{aligned} d(\bar{x},(\Omega_1-x_1)\cap(\Omega_2-x_2)) &\le \|x_2\| + d(\bar{x}-x_2,(\Omega_1-x_1)\cap(\Omega_2-x_2)) \\ &= \|x_2\| + d(\bar{x},(\Omega_1-x')\cap\Omega_2) \\ &\le \|x_2\| + \varphi(\|x'\|) \le \|x_2\| + \alpha\|x'\| \\ &\le (1+2\alpha)\max\{\|x_1\|,\|x_2\|\} < \rho. \end{aligned}$$

Hence, $(\Omega_1 - x_1)\cap(\Omega_2 - x_2)\cap B_\rho(\bar{x}) \neq \emptyset$ and, by Definition 5(i), $\{\Omega_1,\Omega_2\}$ is $\alpha'-$semitransversal at $\bar{x}$ with δ.

(ii) Let $\delta_1 := \left(\alpha+\frac{1}{2}\right)\bar{t}$, $\delta_2 > 0$, and inequality (2.13) be satisfied for all $x \in \Omega_2 \cap B_{2\delta_2}(\bar{x})$ with $d(x,\Omega_1) < \bar{t}$. Let $\rho \in]0,\delta_1[$ and $x \in B_{\delta_2}(\bar{x})$ with $\max\{d(x,\Omega_1),d(x,\Omega_2)\} < \alpha'\rho$. Choose a number $\gamma > 1$ such that

$$\|x-\bar{x}\| < \gamma^{-1}\delta_2 \quad \text{and} \quad \max\{d(x,\Omega_1),d(x,\Omega_2)\} < \gamma^{-1}\alpha'\rho,$$

and a point $x' \in \Omega_2$ such that $\|x - x'\| \le \gamma d(x,\Omega_2)$. Then

$$\begin{aligned} \|x'-\bar{x}\| &\le \|x-x'\| + \|x-\bar{x}\| \le \gamma d(x,\Omega_2) + \|x-\bar{x}\| \\ &\le (\gamma+1)\|x-\bar{x}\| < (1+\gamma^{-1})\delta_2 < 2\delta_2, \\ d(x',\Omega_1) &\le \|x-x'\| + d(x,\Omega_1) \le (\gamma+1)\max\{d(x,\Omega_1),d(x,\Omega_2)\} \\ &< (1+\gamma^{-1})\alpha'\delta_1 < 2\alpha'\delta_1 = \bar{t}. \end{aligned}$$

Hence, by (2.13) with x' in place of x,

$$\begin{aligned} d(x,\Omega_1\cap\Omega_2) &\le \|x-x'\| + d(x',\Omega_1\cap\Omega_2) \le \|x-x'\| + \varphi(d(x',\Omega_1)) \\ &\le \|x-x'\| + \alpha d(x',\Omega_1) \le (1+\alpha)\|x-x'\| + \alpha d(x,\Omega_1) \\ &\le (1+\alpha)\gamma d(x,\Omega_2) + \alpha d(x,\Omega_1) \\ &\le ((1+\alpha)\gamma + \alpha)\max\{d(x,\Omega_1),d(x,\Omega_2)\}. \end{aligned}$$

Letting $\gamma \downarrow 1$, we arrive at

$$d(x,\Omega_1\cap\Omega_2) \le (1+2\alpha)\max\{d(x,\Omega_1),d(x,\Omega_2)\} < \rho.$$

Hence, $\Omega_1\cap\Omega_2\cap B_\rho(x) \neq \emptyset$ and, by Definition 5(ii), $\{\Omega_1,\Omega_2\}$ is $\alpha'-$subtransversal at $\bar{x}$ with δ_1 and δ_2.

(iii) The proof follows that of assertion (i) with the sets $\Omega_1 - \omega_1$ and $\Omega_2 - \omega_2$ in place of Ω_1 and Ω_2, respectively.

□

Remark 12 (i) In the linear case, Proposition 10(ii) recaptures [139, Theorem 1(iii)], while parts (i) and (iii) seem new.

(ii) Restricted versions of the metric conditions in Theorem 2 can be produced in a similar way.

Checking the metric estimates of the $\varphi-$subtransversality and $\varphi-$transversality can be simplified as illustrated by the following proposition referring to condition (2.8) in Theorem 1(ii). Equivalent versions of conditions (2.10) and (2.11) in Theorem 2 look similar.

Proposition 11 The following conditions are equivalent:

(i) inequality (2.8) holds true;

(ii) for all $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$, it holds

$$d\Big(x, \bigcap_{i=1}^{n} \Omega_i\Big) \le \varphi\Big(\max_{1\le i\le n} \|x-\omega_i\|\Big); \tag{2.14}$$

(iii) inequality (2.14) holds true for all $\omega_i \in \Omega_i$ with $\|\omega_i - \bar{x}\| < \|x-\bar{x}\| + \varphi^{-1}(\|x-\bar{x}\|)$ $(i = 1,\ldots,n)$;

(iv) inequality (2.14) holds true for all $\omega_i \in \Omega_i$ with $\varphi(\|\omega_i - x\|) < \|x-\bar{x}\|$ $(i = 1,\ldots,n)$.

Proof The equivalence (i) $\Leftrightarrow$ (ii) and implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are straightforward. We next show that (iv) $\Rightarrow$ (ii). Let condition (iv) hold true, $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$, and $\varphi(\|\omega_i - x\|) \ge \|x-\bar{x}\|$ for some i. Then

$$d\Big(x, \bigcap_{i=1}^{n} \Omega_i\Big) \le \|x-\bar{x}\| \le \varphi\Big(\max_{1\le i\le n} \|x-\omega_i\|\Big),$$

i.e. inequality (2.14) is satisfied, and consequently condition (ii) holds true. □

2.4 Slope Characterizations

2.4.1 Slope Sufficient Conditions

In this section, we formulate slope sufficient conditions for the properties in Definition 6. The conditions are straightforward consequences of the Ekeland variational principle (Lemma 1) applied to appropriate lower semicontinuous functions. Throughout this section, X is a Banach space, the sets $\Omega_1,\ldots,\Omega_n$ are closed; δ, δ_1 and δ_2 are given positive numbers. These are exactly the assumptions which ensure that the Ekeland variational principle is applicable. In view of Proposition 2(iii), it suffices to assume that $\bar{x} \in \mathrm{bd} \cap_{i=1}^{n} \Omega_i$.

The sufficient conditions for the three properties follow the same pattern. We first establish nonlocal slope sufficient conditions arising from the Ekeland variational principle.

These nonlocal conditions are largely of theoretical interest (unless the sets are convex): they encapsulate the application of the Ekeland variational principle and serve as a source of more practical local (infinitesimal) conditions. The corresponding local slope sufficient conditions, their Hölder as well as simplified (δ-free) versions are formulated as corollaries. This way we expose the hierarchy of this type of conditions. These results make the foundation for the dual sufficient conditions for the respective properties in Section 2.5.

Along with the standard maximum norm on X^{n+1}, we are going to use also the following norm depending on a parameter $\gamma > 0$:

$$\|(x_1,\ldots,x_n,x)\|_\gamma := \max\left\{\|x\|, \gamma \max_{1\le i\le n}\|x_i\|\right\}, \quad x_1,\ldots,x_n,x \in X. \tag{2.15}$$

Semitransversality

Theorem 3 The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ if, for some $\gamma > 0$ and any $x_i \in X$ $(i = 1,\ldots,n)$ satisfying

$$0 < \max_{1\le i\le n}\|x_i\| < \varphi^{-1}(\delta), \tag{2.16}$$

there exists a $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|), \delta[$ such that

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\ne(\omega_1,\ldots,\omega_n,x)}} \frac{\varphi\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right) - \varphi\left(\max_{1\le i\le n}\|u_i - x_i - u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma} \ge 1 \tag{2.17}$$

for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying

$$\|x-\bar{x}\| < \lambda, \quad \max_{1\le i\le n}\|\omega_i - \bar{x}\| < \frac{\lambda}{\gamma}, \tag{2.18}$$

$$0 < \max_{1\le i\le n}\|\omega_i - x_i - x\| \le \max_{1\le i\le n}\|x_i\|. \tag{2.19}$$

The proof below employs two closely related nonnegative functions on X^{n+1} determined by the given function $\varphi \in \mathscr{C}$ and vectors $x_1,\ldots,x_n \in X$:

$$f(u_1,\ldots,u_n,u) := \varphi\left(\max_{1\le i\le n}\|u_i - x_i - u\|\right), \quad u_1,\ldots,u_n,u \in X, \tag{2.20}$$

$$\widehat{f} := f + i_{\Omega_1\times\ldots\times\Omega_n}. \tag{2.21}$$

Proof Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is not $\varphi-$semitransversal at $\bar{x}$ with δ, and let $\gamma > 0$ be given. By Definition 6(i), there exist a $\rho \in]0,\delta[$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \rho$ such that $\cap_{i=1}^n(\Omega_i - x_i)\cap B_\rho(\bar{x}) = \emptyset$. Thus, $\max_{1\le i\le n}\|x_i\| > 0$. Let $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|), \delta[$ and $\lambda' := \min\{\lambda,\rho\}$. Then $\lambda' > \varphi(\max_{1\le i\le n}\|x_i\|)$, $\cap_{i=1}^n(\Omega_i - x_i)\cap B_{\lambda'}(\bar{x}) = \emptyset$, and consequently,

$$\max_{1\le i\le n}\|u_i - x_i - u\| > 0 \quad \text{for all} \quad u_i \in \Omega_i\ (i = 1,\ldots,n),\ u \in B_{\lambda'}(\bar{x}). \tag{2.22}$$

Let f and $\widehat{f}$ be defined by (2.20) and (2.21), respectively, while X^{n+1} be equipped with the metric induced by the norm (2.15). We have $\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x}) = \varphi\left(\max_{1\le i\le n}\|x_i\|\right) < \lambda'$. Choose a number ε such that $\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x}) < \varepsilon < \lambda'$. Applying the Ekeland variational principle, we can find points $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$ and $x \in X$ such that

$$\|(\omega_1,\ldots,\omega_n,x)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma < \lambda' \le \lambda, \quad f(\omega_1,\ldots,\omega_n,x) \le f(\bar{x},\ldots,\bar{x},\bar{x}), \tag{2.23}$$

$$f(\omega_1,\ldots,\omega_n,x) - f(u_1,\ldots,u_n,u) \le \frac{\varepsilon}{\lambda'}\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma \tag{2.24}$$

for all $(u_1,\ldots,u_n,u) \in \Omega_1 \times \ldots \times \Omega_n \times X$. In view of (2.22) and the definitions of λ' and f, conditions (2.23) yield (2.18) and (2.19). Since $\varepsilon/\lambda' < 1$, condition (2.24) contradicts (2.17). □

Remark 13 The expression in the left-hand side of (2.17) is the nonlocal γ-slope [131, p. 60] at $(\omega_1,\ldots,\omega_n,x)$ of the function (2.21).

The next statement is a localized version of Theorem 3.

Corollary 4 (i) The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ if, for some $\gamma > 0$ and any $x_i \in X$ $(i=1,\ldots,n)$ satisfying (2.16), there exists a $\lambda \in]\varphi\left(\max_{1\le i\le n}\|x_i\|\right),\delta[$ such that

$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i\ (i=1,\ldots,n),\ u\to x \\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,x)}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x_i-x\|\right) - \varphi\left(\max\limits_{1\le i\le n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma} \ge 1 \tag{2.25}$$

for all $x \in X$ and $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$ satisfying (2.18) and (2.19).

(ii) If $\varphi \in \mathscr{C}^1$, then inequality (2.25) in part (i) can be replaced by

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-x\|\right)\times$$
$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i\ (i=1,\ldots,n),\ u\to x \\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,x)}} \frac{\max\limits_{1\le i\le n}\|\omega_i-x_i-x\| - \max\limits_{1\le i\le n}\|u_i-x_i-u\|}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma} \ge 1. \tag{2.26}$$

Proof The expression in the left-hand side of (2.25) is the γ-slope [131, p. 61] of the function (2.21) at $(\omega_1,\ldots,\omega_n,x)$. The first assertion follows from Theorem 3 in view of Lemma 2(ii), while the second one is a consequence of Lemma 3 in view of Remark 3(i). □

In the Hölder setting, Theorem 3 and Corollary 4 yield the following statement.

Corollary 5 Let $\alpha > 0$ and $q > 0$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\alpha-$semitransversal of order q at $\bar{x}$ with δ if, for some $\gamma > 0$ and any $x_i \in X$ $(i=1,\ldots,n)$ with $0 < \max_{1\le i\le n}\|x_i\| < (\alpha\delta)^{\frac{1}{q}}$, there exists a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n}\|x_i\|)^q,\delta[$ such that

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X \\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,x)}} \frac{\left(\max\limits_{1\le i\le n}\|\omega_i-x_i-x\|\right)^q - \left(\max\limits_{1\le i\le n}\|u_i-x_i-u\|\right)^q}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma} \ge \alpha \tag{2.27}$$

for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.18) and (2.19), or all the more, such that

$$q\Big(\max_{1\le i\le n}\|\omega_i - x_i - x\|\Big)^{q-1} \times \limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i\ (i=1,\ldots,n),\ u\to x \\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,x)}} \frac{\max\limits_{1\le i\le n}\|\omega_i - x_i - x\| - \max\limits_{1\le i\le n}\|u_i - x_i - u\|}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma} \ge \alpha. \quad (2.28)$$

Proof The statement is a direct consequence of Theorem 3 and Corollary 4 with $\varphi(t) := \alpha^{-1}t^q$ for all $t \ge 0$. Observe that $\varphi^{-1}(t) = (\alpha t)^{\frac{1}{q}}$. $\square$

Remark 14 (i) On top of the explicitly given restriction $\|\omega_i - \bar{x}\| < \lambda/\gamma$ in Theorem 3 (and similar conditions in its corollaries) on the choice of the points $\omega_i \in \Omega_i$, which involves γ, the other conditions implicitly impose another one:

$$\|\omega_i - \bar{x}\| \le \|x - \bar{x}\| + \|\omega_i - x_i - x\| + \|x_i\| \le \|x - \bar{x}\| + 2\max_{1\le i\le n}\|x_i\|,$$

and consequently, $\|\omega_i - \bar{x}\| < \lambda + 2\varphi^{-1}(\delta)$. This alternative restriction can be of importance when γ is small.

(ii) The statements of Theorem 3 and its corollaries can be simplified (and weakened!) by dropping condition (2.19).

(iii) Inequalities (2.17), (2.25)–(2.28), which are crucial for checking nonlinear semitransversality, involve two groups of parameters: on one hand, sufficiently small vectors $x_i \in X$, not all zero, and on the other hand, points $x \in X$ and $\omega_i \in \Omega_i$ near $\bar{x}$. Note an important difference between these two groups. The magnitudes of x_i are directly controlled by the value of δ in the definition of $\varphi-$semitransversality: $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta$. At the same time, taking into account that λ can be made arbitrarily close to $\varphi(\max_{1\le i\le n}\|x_i\|)$, the magnitudes of $x - \bar{x}$ and $\omega_i - \bar{x}$ (as well as $\omega_i - x_i - x$) are determined by δ indirectly; they are controlled by $\max_{1\le i\le n}\|x_i\|$: cf. conditions (2.18) and (2.19).

(iv) In view of the definition of the parametric norm (2.15), if any of the inequalities (2.17), (2.25)–(2.28) holds true for some $\gamma > 0$, then it also holds for any $\gamma' \in]0,\gamma[$.

(v) Even in the linear setting, the characterizations in Corollary 5 are new.

The next corollary provides a simplified (and weaker!) version of Theorem 3. The simplification comes at the expense of eliminating the difference between the two groups of parameters highlighted in Remark 14(iii).

Corollary 6 The collection $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ if, for some $\gamma > 0$ and any $x_i \in X$ $(i = 1, \ldots, n)$ satisfying (2.16), inequality (2.17) holds for all $x \in B_\delta(\bar{x})$ and $\omega_i \in \Omega_i \cap B_{\delta/\gamma}(\bar{x})$ $(i = 1, \ldots, n)$ satisfying (2.19).

Sacrificing the estimates for δ in Theorem 3, and Corollaries 4 and 6, we arrive at the following 'δ-free' statement.

Corollary 7 The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ if, for γ and all $x_i \in X$ $(i=1,\ldots,n)$ near 0 with $\max_{1\le i\le n}\|x_i\| > 0$, $x \in X$ and $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$ near $\bar{x}$ satisfying (2.19), inequality (2.17) holds true. Moreover, inequality (2.17) can be replaced by its localized version (2.25), or by (2.26) if $\varphi \in \mathscr{C}^1$.

Subtransversality

Theorem 4 The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma > 0$ and any $x' \in X$ satisfying

$$\|x'-\bar{x}\| < \delta_2, \quad 0 < \max_{1\le i\le n} d(x',\Omega_i) < \varphi^{-1}(\delta_1), \tag{2.29}$$

there exists a $\lambda \in]\varphi\left(\max_{1\le i\le n} d(x',\Omega_i)\right), \delta_1[$ such that

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\ne(\omega_1,\ldots,\omega_n,x)}} \frac{\varphi\left(\max_{1\le i\le n}\|\omega_i - x\|\right) - \varphi\left(\max_{1\le i\le n}\|u_i - u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma} \ge 1 \tag{2.30}$$

for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i=1,\ldots,n)$ satisfying

$$\|x-x'\| < \lambda, \quad \max_{1\le i\le n}\|\omega_i - \omega_i'\| < \frac{\lambda}{\gamma}, \tag{2.31}$$

$$0 < \max_{1\le i\le n}\|\omega_i - x\| \le \max_{1\le i\le n}\|\omega_i' - x'\| < \varphi^{-1}(\lambda). \tag{2.32}$$

The proof below follows the pattern of that of Theorem 3. It employs a continuous real-valued function $f : X^{n+1} \to \mathbb{R}_+$ determined by the given function $\varphi \in \mathscr{C}$:

$$f(u_1,\ldots,u_n,u) := \varphi\left(\max_{1\le i\le n}\|u_i - u\|\right), \quad u_1,\ldots,u_n,u \in X, \tag{2.33}$$

and its restriction to $\Omega_1 \times \ldots \times \Omega_n \times X$ given by (2.21). Note that the function (2.33) is a particular case of (2.20) corresponding to setting $x_i := 0$ $(i=1,\ldots,n)$. We provide here the proof of Theorem 4 for completeness and to expose the differences in handling the two transversality properties, but we skip the proofs of most of its corollaries.

Proof Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is not $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2, and let $\gamma > 0$ be given. By Definition 6(ii), there exist a number $\rho \in]0,\delta_1[$ and a point $x' \in B_{\delta_2}(\bar{x})$ such that $\varphi\left(\max_{1\le i\le n} d(x',\Omega_i)\right) < \rho$ and $\cap_{i=1}^n \Omega_i \cap B_\rho(x') = \emptyset$. Hence, $x' \notin \cap_{i=1}^n \Omega_i$ and

$$0 < \varphi\left(\max_{1\le i\le n} d(x',\Omega_i)\right) < \rho \le d\left(x', \bigcap_{i=1}^n \Omega_i\right).$$

Let $\lambda \in]\varphi\left(\max_{1\le i\le n} d(x',\Omega_i)\right), \delta_1[$. Choose numbers ε and λ' such that

$$\varphi\left(\max_{1\le i\le n} d(x',\Omega_i)\right) < \varepsilon < \lambda' < \min\{\lambda,\rho\},$$

and points $\omega_i' \in \Omega_i$ $(i=1,\ldots,n)$ such that $\varphi\left(\max_{1\le i\le n}\|\omega_i'-x'\|\right)<\varepsilon$. Let f and $\widehat{f}$ be defined by (2.33) and (2.21), respectively, while X^{n+1} be equipped with the metric induced by the norm (2.15). We have $\widehat{f}(\omega_1',\ldots,\omega_n',x')<\varepsilon$. Applying the Ekeland variational principle, we can find points $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ and $x\in X$ such that

$$\|(\omega_1,\ldots,\omega_n,x)-(\omega_1',\ldots,\omega_n',x')\|_\gamma<\lambda',\ f(\omega_1,\ldots,\omega_n,x)\le f(\omega_1',\ldots,\omega_n',x'), \tag{2.34}$$

$$f(\omega_1,\ldots,\omega_n,x)-f(u_1,\ldots,u_n,u)\le\frac{\varepsilon}{\lambda'}\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma \tag{2.35}$$

for all $(u_1,\ldots,u_n,u)\in\Omega_1\times\ldots\times\Omega_n\times X$. Thanks to (2.34), we have $\|x-x'\|<\lambda'$, and consequently,

$$d\Big(x,\bigcap_{i=1}^n\Omega_i\Big)\ge d\Big(x',\bigcap_{i=1}^n\Omega_i\Big)-\|x-x'\|>d\Big(x',\bigcap_{i=1}^n\Omega_i\Big)-\lambda'>0.$$

Hence, $x\notin\cap_{i=1}^n\Omega_i$, and $\max_{1\le i\le n}\|\omega_i-x\|>0$. In view of the definitions of λ' and f, conditions (2.34) together with the last inequality yield (2.31) and (2.32). Since $\varepsilon/\lambda'<1$, condition (2.35) contradicts (2.30). □

The next statement is a localized version of Theorem 4.

Corollary 8 (i) The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma>0$ and any $x'\in X$ satisfying (2.29), there exists a $\lambda\in\left]\varphi\left(\max_{1\le i\le n}d(x',\Omega_i)\right),\delta_1\right[$ such that

$$\limsup_{\substack{u_i\xrightarrow{\Omega_i}\omega_i\ (i=1,\ldots,n),\ u\to x\\ (u_1,\ldots,u_n,u)\ne(\omega_1,\ldots,\omega_n,x)}}\frac{\varphi\Big(\max\limits_{1\le i\le n}\|\omega_i-x\|\Big)-\varphi\Big(\max\limits_{1\le i\le n}\|u_i-u\|\Big)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma}\ge 1 \tag{2.36}$$

for all $x\in X$ and $\omega_i,\omega_i'\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.31) and (2.32).

(ii) If $\varphi\in\mathscr{C}^1$, then inequality (2.36) in part (i) can be replaced by

$$\varphi'\Big(\max_{1\le i\le n}\|\omega_i-x\|\Big)\times\limsup_{\substack{u_i\xrightarrow{\Omega_i}\omega_i\ (i=1,\ldots,n),\ u\to x\\ (u_1,\ldots,u_n,u)\ne(\omega_1,\ldots,\omega_n,x)}}\frac{\max\limits_{1\le i\le n}\|\omega_i-x\|-\max\limits_{1\le i\le n}\|u_i-u\|}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma}\ge 1. \tag{2.37}$$

In the Hölder setting, Theorem 4 and Corollary 8 yield the following statement. In view of Remark 7, we assume that $q\le 1$.

Corollary 9 Let $\alpha>0$ and $q\in]0,1]$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\alpha-$subtransversal of order q at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma>0$ and any $x'\in B_{\delta_2}(\bar{x})$ with $0<\max\limits_{1\le i\le n}d(x',\Omega_i)<(\alpha\delta_1)^{\frac{1}{q}}$, there exists a $\lambda\in\left]\alpha^{-1}\Big(\max\limits_{1\le i\le n}d(x',\Omega_i)\Big)^q,\delta_1\right[$ such that

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\ne(\omega_1,\ldots,\omega_n,x)}}\frac{\Big(\max\limits_{1\le i\le n}\|\omega_i-x\|\Big)^q-\Big(\max\limits_{1\le i\le n}\|u_i-u\|\Big)^q}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma}\ge\alpha, \tag{2.38}$$

for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.31) and

$$0 < \max_{1 \le i \le n} \|\omega_i - x\| \le \max_{1 \le i \le n} \|\omega_i' - x'\| < (\alpha\lambda)^{\frac{1}{q}},$$

or all the more, such that

$$q\left(\max_{1 \le i \le n} \|\omega_i - x\|\right)^{q-1} \times \limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i \ (i=1,\ldots,n),\ u \to x \\ (u_1,\ldots,u_n,u) \ne (\omega_1,\ldots,\omega_n,x)}} \frac{\max_{1 \le i \le n} \|\omega_i - x\| - \max_{1 \le i \le n} \|u_i - u\|}{\|(u_1, \ldots, u_n, u) - (\omega_1, \ldots, \omega_n, x)\|_\gamma} \ge \alpha. \quad (2.39)$$

Remark 15 (i) The expressions in the left-hand sides of (2.30) and (2.36) are, respectively, the nonlocal γ-slope and the γ-slope at $(\omega_1, \ldots, \omega_n, x)$ of the function (2.21).

(ii) Under the conditions of Theorem 4, there are two ways for estimating $\|\omega_i - \bar{x}\|$:

$$\|\omega_i - \bar{x}\| \le \|x' - \bar{x}\| + \|\omega_i - \omega_i'\| + \|\omega_i' - x'\| < \delta_2 + \lambda/\gamma + \varphi^{-1}(\lambda), \quad \text{and}$$
$$\|\omega_i - \bar{x}\| \le \|x - \bar{x}\| + \|\omega_i - x\|$$
$$\le \|x' - \bar{x}\| + \|x - x'\| + \max_{1 \le i \le n} \|\omega_i' - x'\| < \delta_2 + \lambda + \varphi^{-1}(\lambda).$$

The second estimate does not involve γ and is better than the first one when $\gamma < 1$. A similar observation can be made about Corollary 10.

(iii) It can be observed from the proof of Theorem 4 that the sufficient conditions for $\varphi-$subtransversality can be strengthened by adding another restriction on the choice of x': $\varphi\left(\max_{1 \le i \le n} d(x', \Omega_i)\right) < d\left(x', \cap_{i=1}^n \Omega_i\right)$.

(iv) The statement of Theorem 4 and its corollaries can be simplified by dropping condition (2.32).

(v) Inequalities (2.30), (2.36)–(2.39), which are crucial for checking nonlinear subtransversality, involve points $x \in X$ and $\omega_i \in \Omega_i$ near $\bar{x}$. Their distance from $\bar{x}$ is determined in Theorem 4 via other points: $x' \notin \cap_{i=1}^n \Omega_i$ and $\omega_i' \in \Omega_i$; cf. conditions (2.31) and (2.32). Only the distance from x' to $\bar{x}$ and to the sets Ω_i is directly controlled by the values of δ_1 and δ_2 in the definition of $\varphi-$subtransversality: $x' \in B_{\delta_2}(\bar{x})$ and $\varphi\left(\max_{1 \le i \le n} d(x', \Omega_i)\right) < \delta_1$. All the other distances are controlled by λ, which can be made arbitrarily close to $\varphi\left(\max_{1 \le i \le n} d(x', \Omega_i)\right)$.

(vi) In view of the definition of the parametric norm (2.15), if any of the inequalities (2.30), (2.36)–(2.39) holds true for some $\gamma > 0$, then it also holds for any $\gamma' \in]0, \gamma[$.

(vii) Corollary 9 strengthens [141, Proposition 6]. In the linear case, it improves [138, Proposition 10].

The next corollary provides a simplified (and weaker!) version of Theorem 4; cf. Remark 15(v).

Corollary 10 The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma>0$, inequality (2.30) holds for all $x\in B_{\delta_1+\delta_2}(\bar{x})$ and $\omega_i\in\Omega_i\cap B_{\delta_2+\delta_1/\gamma+\varphi^{-1}(\delta_1)}(\bar{x})$ $(i=1,\ldots,n)$ satisfying $0<\max_{1\le i\le n}\|\omega_i-x\|<\varphi^{-1}(\delta_1)$.

Proof Let $\delta_1>0$ and $\delta_2>0$, $x'\in B_{\delta_2}(\bar{x})\setminus\cap_{i=1}^n\Omega_i$, $\lambda\in]\varphi\left(\max_{1\le i\le n}d(x',\Omega_i)\right),\delta_1[$, and points $x\in X$ and $\omega_i,\omega_i'\in\Omega_i$ $(i=1,\ldots,n)$ satisfy conditions (2.31) and (2.32). Then

$$\begin{aligned}
\|x-\bar{x}\| &\le \|x-x'\|+\|x'-\bar{x}\|<\lambda+\delta_2<\delta_1+\delta_2,\\
\|\omega_i-\bar{x}\| &\le \|x'-\bar{x}\|+\|\omega_i-\omega_i'\|+\|\omega_i'-x'\|\\
&\quad<\delta_2+\lambda/\gamma+\varphi^{-1}(\lambda)<\delta_2+\delta_1/\gamma+\varphi^{-1}(\delta_1),\\
\|\omega_i-x\| &< \varphi^{-1}(\lambda)<\varphi^{-1}(\delta_1),
\end{aligned}$$

i.e. points $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfy all the conditions in the corollary. Hence, inequality (2.30) holds. It follows from Theorem 4 that $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2. $\square$

Sacrificing the estimates for δ_1 and δ_2 in Theorem 4, and Corollaries 8 and 10, we can formulate the following 'δ-free' statement.

Corollary 11 The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ if inequality (2.30) holds true for some $\gamma>0$ and all $x\in X$ near $\bar{x}$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ near $\bar{x}$ satisfying $\max_{1\le i\le n}\|\omega_i-x\|>0$. Moreover, inequality (2.30) can be replaced by its localized version (2.36), or by (2.37) if $\varphi\in\mathscr{C}^1$.

Transversality

Since $\varphi-$transversality is in a sense an overarching property covering both $\varphi-$semitransversality and $\varphi-$subtransversality (see Proposition 2(iii)), the next theorem contains some elements of both Theorems 3 and 4, and its proof goes along the same lines. Similar to the proof of Theorem 3, it employs functions (2.20) and (2.21).

Theorem 5 The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma>0$ and any $\omega_i'\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\xi\in]0,\varphi^{-1}(\delta_1)[$, there exists a $\lambda\in]\varphi(\xi),\delta_1[$ such that inequality (2.17) holds for all $x,x_i\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying

$$\|x-\bar{x}\|<\lambda,\quad \max_{1\le i\le n}\|\omega_i-\omega_i'\|<\frac{\lambda}{\gamma},\tag{2.40}$$

$$0<\max_{1\le i\le n}\|\omega_i-x_i-x\|\le\max_{1\le i\le n}\|\omega_i'-x_i-\bar{x}\|=\xi.\tag{2.41}$$

Proof Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is not $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2, and let $\gamma>0$ be given. By Definition 6(iii), there exist a number $\rho\in]0,\delta_1[$ and points $\omega_i'\in\Omega_i\cap B_{\delta_2}(\bar{x})$ and $x_i'\in X$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i'\|)<\rho$ such that $\cap_{i=1}^n(\Omega_i-\omega_i'-x_i')\cap(\rho\mathbb{B})=\emptyset$. Thus, $\xi:=\max_{1\le i\le n}\|x_i'\|>0$ and $\xi<\varphi^{-1}(\rho)<\varphi^{-1}(\delta_1)$. Set $x_i:=\omega_i'+x_i'-\bar{x}$ $(i=1,\ldots,n)$. Then

$$\max_{1\le i\le n}\|\omega_i'-x_i-\bar{x}\|=\max_{1\le i\le n}\|x_i'\|=\xi.$$

Let $\lambda\in]\varphi(\xi),\delta_1[$ and $\lambda':=\min\{\lambda,\rho\}$. Then $\cap_{i=1}^n(\Omega_i-x_i)\cap B_{\lambda'}(\bar{x})=\emptyset$, and consequently, condition (2.22) holds true. Let f and $\widehat{f}$ be defined by (2.20) and (2.21), respectively, while X^{n+1} be equipped with the metric induced by the norm (2.15). We have $\widehat{f}(\omega_1',\ldots,\omega_n',\bar{x})=\varphi\left(\max_{1\le i\le n}\|x_i'\|\right)=\varphi(\xi)<\lambda'$. Choose a number ε such that $\widehat{f}(\omega_1',\ldots,\omega_n',\bar{x})<\varepsilon<\lambda'$. Applying the Ekeland variational principle, we can find points $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ and $x\in X$ such that

$$\|(\omega_1,\ldots,\omega_n,x)-(\omega_1',\ldots,\omega_n',\bar{x})\|_\gamma<\lambda',\;\; f(\omega_1,\ldots,\omega_n,x)\le f(\omega_1',\ldots,\omega_n',\bar{x}), \tag{2.42}$$

and condition (2.24) holds for all $u\in X$ and $u_i\in\Omega_i$ $(i=1,\ldots,n)$. In view of (2.22) and the definitions of λ' and f, conditions (2.42) yield (2.40) and (2.41). Since $\varepsilon/\lambda'<1$, condition (2.24) contradicts (2.17). □

The next statement is a localized version of Theorem 5.

Corollary 12 (i) The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma>0$ and any $\omega_i'\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\xi\in]0,\varphi^{-1}(\delta_1)[$, there exists a $\lambda\in]\varphi(\xi),\delta_1[$ such that inequality (2.25) holds for all $x,x_i\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.40) and (2.41).

(ii) If $\varphi\in\mathscr{C}^1$, then inequality (2.25) in part (i) can be replaced by (2.26).

In the Hölder setting, Theorem 5 and Corollary 12 yield the following statement. In view of Remark 7, we assume that $q\le 1$.

Corollary 13 Let $\alpha>0$ and $q\in]0,1]$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\alpha-$transversal of order q at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma>0$ and any $\omega_i'\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\xi\in]0,(\alpha\delta_1)^{\frac{1}{q}}[$, there exists a $\lambda\in]\alpha^{-1}\xi^q,\delta_1[$ such that inequality (2.27) holds true for all $x,x_i\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.40) and (2.41), or all the more, such that inequality (2.28) holds true.

Remark 16 (i) On top of the explicitly given restriction $\|\omega_i-\omega_i'\|<\lambda/\gamma$ in Theorem 5 (and similar conditions in its corollaries), which involves γ, the other conditions implicitly impose another one:

$$\begin{aligned}\|\omega_i-\omega_i'\|&\le\|x-\bar{x}\|+\|\omega_i-x_i-x\|+\|\omega_i'-x_i-\bar{x}\|\\&\le\|x-\bar{x}\|+2\xi<\lambda+2\varphi^{-1}(\delta_1).\end{aligned}$$

This alternative restriction can be of importance when γ is small.

(ii) It can be observed from the proof of Theorem 5 that the sufficient conditions for $\varphi-$transversality can be strengthened by adding another restriction on the choice of ξ and x_i: $\varphi(\xi) < d(\bar{x}, \cap_{i=1}^n(\Omega_i - x_i))$.

(iii) The sufficient conditions for $\varphi-$semitransversality and $\varphi-$subtransversality in Theorems 3 and 4 are particular cases of those in Theorem 5, corresponding to setting $\omega_i' := \bar{x}$ and $x_1 = \ldots = x_n$, respectively.

(iv) The statement of Theorem 5 and its corollaries can be simplified by dropping condition (2.41).

(v) Inequalities (2.17), (2.25)–(2.28), which are crucial for checking nonlinear transversality, involve a collection of parameters: $x, x_i \in X$ and $\omega_i \in \Omega_i$, which are related to another collection: a small number $\xi > 0$ and points $\omega_i' \in \Omega_i$ near $\bar{x}$. The value of ξ and magnitudes of $\omega_i' - \bar{x}$ are directly controlled by the values of δ_1 and δ_2 in the definition of $\varphi-$transversality: $\varphi(\xi) < \delta_1$ and $\omega_i' \in B_{\delta_2}(\bar{x})$. At the same time, taking into account that λ can be made arbitrarily close to $\varphi(\xi)$, the magnitudes of $x - \bar{x}$, $\omega_i - \omega_i'$ and x_i are determined by δ_1 and δ_2 indirectly; they are controlled by ξ: cf. conditions (2.40) and (2.41). Thus, the derived parameters $x, x_i \in X$ and $\omega_i \in \Omega_i$ involved in (2.17) possess the natural properties: when δ_1 and δ_2 are small, the points x and ω_i are near $\bar{x}$ and the vectors x_i are small.

(vi) In view of the definition of the parametric norm (2.15), if any of the inequalities (2.17), (2.25)–(2.28) holds true for some $\gamma > 0$, then it also holds for any $\gamma' \in]0, \gamma[$.

(vii) Even in the linear setting, the characterizations in Corollary 13 are new.

The next corollary provides a simplified (and weaker!) version of Theorem 5; cf. Remark 16(v).

Corollary 14 The collection $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma > 0$, inequality (2.17) holds for all $x \in B_{\delta_1}(\bar{x})$, $x_i \in X$ and $\omega_i \in \Omega_i \cap B_{\delta_2+\delta_1/\gamma}(\bar{x})$ $(i = 1, \ldots, n)$ satisfying $\varphi\left(\max_{1 \le i \le n} d(x_i + \bar{x}, \Omega_i)\right) < \delta_1$ and $0 < \max_{1 \le i \le n} \|\omega_i - x_i - x\| < \varphi^{-1}(\delta_1)$.

Proof Let $\omega_i' \in \Omega_i \cap B_{\delta_2}(\bar{x})$, $\xi \in]0, \varphi^{-1}(\delta_1)[$, $\lambda \in]\varphi(\xi), \delta_1[$, and points $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfy conditions (2.40) and (2.41). Then

$$\|x - \bar{x}\| < \lambda < \delta_1, \quad \|\omega_i - \bar{x}\| \le \|\omega_i' - \bar{x}\| + \|\omega_i - \omega_i'\| < \delta_2 + \lambda/\gamma < \delta_2 + \delta_1/\gamma,$$
$$d(x_i + \bar{x}, \Omega_i) \le \|x_i + \bar{x} - \omega_i'\| \le \xi < \varphi^{-1}(\delta_1),$$
$$0 < \max_{1 \le i \le n} \|\omega_i - x_i - x\| \le \xi < \varphi^{-1}(\delta_1),$$

i.e. points $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfy all the conditions in the corollary. Hence, inequality (2.17) holds. It follows from Theorem 5 that $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2. $\square$

Sacrificing the estimates for δ_1 and δ_2 in Theorem 5, and Corollaries 12 and 14, we can formulate the following 'δ-free' statement.

Corollary 15 The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ if, for some $\gamma>0$ and all $x\in X$ near $\bar{x}$, $x_i\in X$ $(i=1,\ldots,n)$ near 0 and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ near $\bar{x}$ satisfying $\max_{1\le i\le n}\|\omega_i-x_i-x\|>0$, inequality (2.17) holds true. Moreover, inequality (2.17) can be replaced by its localized version (2.25), or by (2.26) if $\varphi\in\mathscr{C}^1$.

Remark 17 The sufficient conditions for $\varphi-$semitransversality and $\varphi-$transversality in Theorems 3 and 5 and their corollaries use the same (slope) inequalities (2.17), (2.25) and (2.26). Nevertheless, the sufficient conditions in Theorem 5 and Corollary 15 are stronger than the corresponding ones in Theorem 3 and Corollary 7, respectively, as they require the inequalities to be satisfied on a larger set of points. This is natural as $\varphi-$transversality is a stronger property than $\varphi-$semitransversality. At the same time, the 'δ-free' versions in Corollaries 7 and 15 are almost identical: the only difference is the additional condition

$$\max_{1\le i\le n}\|\omega_i-x_i-x\|\le\max_{1\le i\le n}\|x_i\|$$

in Corollary 7. The sufficient condition in Corollary 15 is still acceptable for characterizing $\varphi-$transversality, but the one in Corollary 7 seems a little too strong for $\varphi-$semitransversality. That is why we prefer not to oversimplify these sufficient conditions.

2.4.2 Slope Necessary Conditions

In this section, we formulate slope necessary conditions for the properties in Definition 6. They all follow the same pattern. We first establish nonlocal slope necessary conditions arising from the definitions of the respective properties. The corresponding local slope necessary conditions, their Hölder as well as simplified (δ-free) versions are formulated as corollaries. This way we expose the hierarchy of this type of conditions. These results make the foundation for the dual necessary conditions for the respective properties in Section 2.5.

The next theorem establishes nonlocal slope necessary conditions for the three transversality properties.

Theorem 6 Suppose there exist an $\alpha>0$ and a $\delta>0$ such that $\varphi(t)\ge\alpha t$ for all $t\in\,]0,\varphi^{-1}(\delta)[$, and $\gamma:=(\alpha^{-1}+1)^{-1}$.

(i) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ, then

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\ne(\bar{x},\ldots,\bar{x},\bar{x})}}\frac{\varphi\left(\max_{1\le i\le n}\|x_i\|\right)-\varphi\left(\max_{1\le i\le n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma}\ge 1 \tag{2.43}$$

for all $x_i\in X$ $(i=1,\ldots,n)$ satisfying

$$0<\max_{1\le i\le n}\|x_i\|<\varphi^{-1}(\delta). \tag{2.44}$$

(ii) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with $\delta_1 := \delta$ and some $\delta_2 > 0$, then

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,x)}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma} \ge 1 \tag{2.45}$$

for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying

$$\|x-\bar{x}\|<\delta_2,\quad 0<\max_{1\le i\le n}\|\omega_i-x\|<\varphi^{-1}(\delta_1). \tag{2.46}$$

(iii) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with $\delta_1 := \delta$ and some $\delta_2 > 0$, then

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,\bar{x})}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma} \ge 1 \tag{2.47}$$

for all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying

$$\max_{1\le i\le n}\|\omega_i-\bar{x}\|<\delta_2,\quad 0<\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|<\varphi^{-1}(\delta_1). \tag{2.48}$$

Proof

(i) Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ. Let $\gamma := (\alpha^{-1}+1)^{-1}$, $x_i\in X$ $(i=1,\ldots,n)$ satisfy (2.16). Denote $M := \varphi(\max_{1\le i\le n}\|x_i\|) < \delta$. Then $M \ge \alpha\max_{1\le i\le n}\|x_i\|$. Let $\eta\in]0,1[$, and choose a number $\gamma'\in]\eta\gamma,\gamma[$. Then $(\gamma')^{-1}-\alpha^{-1}>1$. Choose a $\xi>1$ such that $\xi\le\eta^{-1}$, $\xi\le(\gamma')^{-1}-\alpha^{-1}$ and $\xi M<\delta$. By Definition 6(i), we have $\cap_{i=1}^n(\Omega_i-x_i)\cap B_{\xi M}(\bar{x})\neq\emptyset$, and consequently, there exist $\hat{x}\in X$ and $\hat{\omega}_i\in\Omega_i$ $(i=1,\ldots,n)$, with $\hat{\omega}_1-x_1=\ldots=\hat{\omega}_n-x_n=\hat{x}$ such that $\|\bar{x}-\hat{x}\|<\xi M$. Since $\max_{1\le i\le n}\|\hat{\omega}_i-x_i-\hat{x}\|=0$ while $\max_{1\le i\le n}\|x_i\|>0$, we have $(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})\neq(\bar{x},\ldots,\bar{x},\bar{x})$. Moreover, for all $i=1,\ldots,n$,

$$\begin{aligned}\|\hat{\omega}_i-\bar{x}\| &\le \|\hat{\omega}_i-x_i-\bar{x}\|+\|x_i\|\\ &= \|\hat{x}-\bar{x}\|+\|x_i\| < \xi M+\alpha^{-1}M \le M(\gamma')^{-1} < M(\eta\gamma)^{-1},\end{aligned}$$

and consequently,

$$\begin{aligned}\|(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma &= \max\left\{\|\hat{x}-\bar{x}\|,\gamma\max_{1\le i\le n}\|\hat{\omega}_i-\bar{x}\|\right\}\\ &< M\max\left\{\xi,\eta^{-1}\right\} = M\eta^{-1}.\end{aligned}$$

Hence, $\varphi(\max_{1\le i\le n}\|x_i\|)=M>\eta\|(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma$, and consequently,

$$\begin{aligned}&\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\neq(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{\varphi(\max_{1\le i\le n}\|x_i\|)-\varphi(\max_{1\le i\le n}\|u_i-x_i-u\|)}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma}\\ &\qquad\qquad \ge \frac{\varphi(\max_{1\le i\le n}\|x_i\|)}{\|(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma} > \eta.\end{aligned}$$

Letting $\eta\uparrow 1$, we arrive at inequality (2.43).

(ii) Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and some $\delta_2 > 0$. Let $\gamma := (\alpha^{-1}+1)^{-1}$, $x \in B_{\delta_2}(\bar{x})$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ satisfy (2.46). Denote $M := \varphi(\max_{1\le i\le n}\|\omega_i - x\|) < \delta_1$. Then $M \ge \alpha\max_{1\le i\le n}\|\omega_i - x\|$. Let $\eta \in]0,1[$, and choose a number $\gamma' \in]\eta\gamma,\gamma[$. Then $(\gamma')^{-1} - \alpha^{-1} > 1$. Choose a $\xi > 1$ such that $\xi \le \eta^{-1}$, $\xi \le (\gamma')^{-1} - \alpha^{-1}$ and $\xi M < \delta_1$. By Definition 6(ii), there exists an $\omega \in \cap_{i=1}^n \Omega_i$ such that $\|\omega - x\| < \xi M$. Since $\max_{1\le i\le n}\|\omega_i - x\| > 0$, we have $(\omega,\ldots,\omega,\omega) \ne (\omega_1,\ldots,\omega_n,x)$. Moreover, for all $i = 1,\ldots,n$,

$$\|\omega - \omega_i\| \le \|\omega - x\| + \|\omega_i - x\| < \xi M + \alpha^{-1}M \le M(\gamma')^{-1} < M(\eta\gamma)^{-1},$$

and consequently,

$$\begin{aligned}\|(\omega,\ldots,\omega,\omega) - (\omega_1,\ldots,\omega_n,x)\|_\gamma &= \max\left\{\|\omega - x\|, \gamma\max_{1\le i\le n}\|\omega - \omega_i\|\right\}\\ &< M\max\left\{\xi,\eta^{-1}\right\} = M\eta^{-1}.\end{aligned}$$

Thus, $\varphi(\max_{1\le i\le n}\|\omega_i - x\|) = M > \eta\|(\omega,\ldots,\omega,\omega) - (\omega_1,\ldots,\omega_n,x)\|_\gamma$. Since $\eta \in]0,1[$ is arbitrary, we obtain

$$\begin{aligned}&\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\ne(\omega_1,\ldots,\omega_n,x)}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i - x\|\right) - \varphi\left(\max\limits_{1\le i\le n}\|u_i - u\|\right)}{\|(u_1,\ldots,u_n,u) - (\omega_1,\ldots,\omega_n,x)\|_\gamma}\\ &\qquad\qquad \ge \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i - x\|\right)}{\|(\omega,\ldots,\omega,\omega) - (\omega_1,\ldots,\omega_n,x)\|_\gamma} > \eta.\end{aligned}$$

Letting $\eta \uparrow 1$, we arrive at (2.45).

(iii) Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$. Let $\gamma := (\alpha^{-1}+1)^{-1}$, $\omega_i \in \Omega_i$ and $x_i \in X$ $(i = 1,\ldots,n)$ satisfy (2.48). Denote $M := \varphi(\max_{1\le i\le n}\|\omega_i - x_i - \bar{x}\|) < \delta_1$. Then $M \ge \alpha\max_{1\le i\le n}\|\omega_i - x_i - \bar{x}\|$. Let $\eta \in]0,1[$, and choose a number $\gamma' \in]\eta\gamma,\gamma[$. Then $(\gamma')^{-1} - \alpha^{-1} > 1$. Choose a $\xi > 1$ such that $\xi \le \eta^{-1}$, $\xi \le (\gamma')^{-1} - \alpha^{-1}$ and $\xi M < \delta_1$. By Definition 6(iii), $\cap_{i=1}^n(\Omega_i - \omega_i - x_i') \cap (\xi M)\mathbb{B} \ne \emptyset$, where $x_i' := \bar{x} + x_i - \omega_i$ $(i = 1,\ldots,n)$, or equivalently, $\cap_{i=1}^n(\Omega_i - x_i) \cap B_{\xi M}(\bar{x}) \ne \emptyset$. Thus, there exist $\hat{\omega}_i \in \Omega_i$ $(i = 1,\ldots,n)$ and $\hat{x} \in X$ such that $\hat{\omega}_1 - x_1 = \ldots = \hat{\omega}_n - x_n = \hat{x}$ such that $\|\bar{x} - \hat{x}\| < \xi M$. Since $\max_{1\le i\le n}\|\hat{\omega}_i - x_i - \hat{x}\| = 0$ while $\max_{1\le i\le n}\|\omega_i - x_i - \bar{x}\| > 0$, we have $(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x}) \ne (\omega_1,\ldots,\omega_n,\bar{x})$. Moreover, for all $i = 1,\ldots,n$,

$$\begin{aligned}\|\hat{\omega}_i - \omega_i\| &\le \|\hat{\omega}_i - x_i - \bar{x}\| + \|x_i + \bar{x} - \omega_i\|\\ &= \|\hat{x} - \bar{x}\| + \|x_i + \bar{x} - \omega_i\| < \xi M + \alpha^{-1}M \le M(\gamma')^{-1} < M(\eta\gamma)^{-1},\end{aligned}$$

and consequently,

$$\begin{aligned}\|(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x}) - (\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma &= \max\left\{\|\hat{x} - \bar{x}\|, \gamma\max_{1\le i\le n}\|\hat{\omega}_i - \omega_i\|\right\}\\ &< M\max\left\{\xi,\eta^{-1}\right\} = M\eta^{-1}.\end{aligned}$$

Hence, $\varphi\left(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)=M>\eta\|(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})-(\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma$, and consequently,

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\ne(\omega_1,\ldots,\omega_n,\bar{x})}} \frac{\varphi(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|)-\varphi(\max_{1\le i\le n}\|u_i-x_i-u\|)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma}$$
$$\ge\frac{\varphi\left(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)}{\|(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})-(\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma}>\eta.$$

Letting $\eta\uparrow 1$, we arrive at (2.47).

□

Remark 18 (i) The expressions in the left-hand sides of (2.43), (2.45) and (2.47) are the nonlocal γ-slopes [131, p. 60] computed at respective points of the extended-real-valued function $\widehat{f}$ (2.21) where $f: X^{n+1}\to\mathbb{R}_+$ is given by (2.20) in the case of (2.43) and (2.47), and by (2.33) in the case of (2.45).

(ii) In view of the definition of the parametric norm (2.15), if inequalities (2.43), (2.45) and (2.47) hold with the given γ, they also hold with any $\gamma'\in]0,\gamma[$. This observation is applicable to all slope inequalities in this section.

In the Hölder setting, Theorem 6 yields the following statement.

Corollary 16 Let $\alpha>0$ and $q\in]0,1]$.

(i) Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\alpha-$semitransversal of order q at $\bar{x}$ with some $\delta>0$. Set $\gamma:=(\alpha^{\frac{1}{q}}\delta^{\frac{1}{q}-1}+1)^{-1}$. Then

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\ne(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{\max\limits_{1\le i\le n}\|x_i\|^q-\max\limits_{1\le i\le n}\|u_i-x_i-u\|^q}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma}\ge\alpha$$

for all $x_i\in X$ $(i=1,\ldots,n)$ with $0<\max_{1\le i\le n}\|x_i\|<(\alpha\delta)^{\frac{1}{q}}$.

(ii) Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\alpha-$subtransversal of order q at $\bar{x}$. Set $\gamma:=(\alpha^{\frac{1}{q}}\delta_1^{\frac{1}{q}-1}+1)^{-1}$. Then

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\ne(\omega_1,\ldots,\omega_n,x)}} \frac{\max\limits_{1\le i\le n}\|\omega_i-x\|^q-\max\limits_{1\le i\le n}\|u_i-u\|^q}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma}\ge\alpha$$

for all $x\in B_{\delta_2}(\bar{x})$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ with $0<\max_{1\le i\le n}\|\omega_i-x\|<(\alpha\delta_1)^{\frac{1}{q}}$.

(iii) Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\alpha-$transversal of order q at $\bar{x}$. Set $\gamma:=(\alpha^{\frac{1}{q}}\delta_1^{\frac{1}{q}-1}+1)^{-1}$. Then

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\ne(\omega_1,\ldots,\omega_n,\bar{x})}} \frac{\max\limits_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|^q-\max\limits_{1\le i\le n}\|u_i-x_i-u\|^q}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma}\ge\alpha$$

for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1, \ldots, n)$ with $0 < \max_{1\le i\le n} \|\omega_i - x_i - \bar{x}\| < (\alpha\delta_1)^{\frac{1}{q}}$.

Proof The assertion in part (i) is a consequence of Theorem 6(i) with $\varphi(t) := \alpha^{-1}t^q$ for all $t \ge 0$; then of course, $\varphi^{-1}(t) = (\alpha t)^{\frac{1}{q}}$. To prove the statement, given an α and a δ, we need to compute a lower bound $\bar{\alpha}$ for $\varphi(t)/t$ on $]0, \varphi^{-1}(\delta)[$. The function $t \mapsto \varphi(t)/t = \alpha^{-1}t^{q-1}$ is nonincreasing on $]0, +\infty[$; hence, its value at $\varphi^{-1}(\delta) = (\alpha\delta)^{\frac{1}{q}}$ provides the exact lower bound. Thus, we can take $\bar{\alpha} := \alpha^{-1}(\alpha\delta)^{\frac{q-1}{q}} = \alpha^{-\frac{1}{q}}\delta^{1-\frac{1}{q}}$. Then $\gamma := (\bar{\alpha}^{-1} + 1)^{-1} = (\alpha^{\frac{1}{q}}\delta^{\frac{1}{q}-1} + 1)^{-1}$. The rest of the proof is straightforward. The proofs for parts (ii) and (iii) are similar. □

Remark 19 (i) When $q = 1$, we have $\gamma := (\alpha + 1)^{-1}$ in Corollary 16, and this value does not depend on δ. When $q < 1$, by choosing a sufficiently small δ, the value of γ can be made arbitrarily close to 1.

(ii) Part (ii) of Corollary 16 strengthens [138, Proposition 10], while parts (i) and (iii) are new even in the linear setting.

The next statement presents a localized version of Theorem 6 in the convex setting.

Corollary 17 Suppose $\Omega_1, \ldots, \Omega_n$ and φ are convex, $\varphi'_+(0) > 0$, and $\gamma := ((\varphi'_+(0))^{-1} + 1)^{-1}$.

(i) If $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta > 0$, then

$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \bar{x}\ (i=1,\ldots,n),\ u\to\bar{x} \\ (u_1,\ldots,u_n,u)\neq(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{\varphi\left(\max\limits_{1\le i\le n} \|x_i\|\right) - \varphi\left(\max\limits_{1\le i\le n} \|u_i - x_i - u\|\right)}{\|(u_1, \ldots, u_n, u) - (\bar{x}, \ldots, \bar{x}, \bar{x})\|_\gamma} \ge 1 \tag{2.49}$$

for all $x_i \in X$ $(i = 1, \ldots, n)$ satisfying (2.16).

(ii) If $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then

$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i\ (i=1,\ldots,n),\ u\to x \\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,x)}} \frac{\varphi\left(\max\limits_{1\le i\le n} \|\omega_i - x\|\right) - \varphi\left(\max\limits_{1\le i\le n} \|u_i - u\|\right)}{\|(u_1, \ldots, u_n, u) - (\omega_1, \ldots, \omega_n, x)\|_\gamma} \ge 1 \tag{2.50}$$

for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.46).

(iii) If $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then

$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i\ (i=1,\ldots,n),\ u\to\bar{x} \\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,\bar{x})}} \frac{\varphi\left(\max\limits_{1\le i\le n} \|\omega_i - x_i - \bar{x}\|\right) - \varphi\left(\max\limits_{1\le i\le n} \|u_i - x_i - u\|\right)}{\|(u_1, \ldots, u_n, u) - (\omega_1, \ldots, \omega_n, \bar{x})\|_\gamma} \ge 1 \tag{2.51}$$

for all $\omega_i \in \Omega_i$ and $x_i \in X$ $(i = 1, \ldots, n)$ satisfying (2.48).

Moreover, if $\varphi \in \mathscr{C}^1$, then inequalities (2.49), (2.50) and (2.51) in parts (i)–(iii) can be replaced, respectively, by

$$\varphi'\left(\max_{1\le i\le n}\|x_i\|\right) \limsup_{\substack{u_i \xrightarrow{\Omega_i} \bar{x}\ (i=1,\ldots,n),\ u\to\bar{x}\\ (u_1,\ldots,u_n,u)\ne(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{\max\limits_{1\le i\le n}\|x_i\| - \max\limits_{1\le i\le n}\|u_i-x_i-u\|}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma} \ge 1, \tag{2.52}$$

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x\|\right) \times \limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i\ (i=1,\ldots,n),\ u\to x\\ (u_1,\ldots,u_n,u)\ne(\omega_1,\ldots,\omega_n,x)}} \frac{\max\limits_{1\le i\le n}\|\omega_i-x\| - \max\limits_{1\le i\le n}\|u_i-u\|}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma} \ge 1, \tag{2.53}$$

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right) \times \limsup_{\substack{u_i \xrightarrow{\Omega_i} \bar{x}\ (i=1,\ldots,n),\ u\to\bar{x}\\ (u_1,\ldots,u_n,u)\ne(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{\max\limits_{1\le i\le n}\|\omega_i-x_i-\bar{x}\| - \max\limits_{1\le i\le n}\|u_i-x_i-u\|}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma} \ge 1. \tag{2.54}$$

Proof In view of the convexity of φ, it holds $\varphi(t) \ge \varphi'_+(0)t$ for all $t \ge 0$. Moreover, functions (2.20), (2.33) and (2.21) are convex. By Lemma 2(iv), the left-hand sides of inequalities (2.43), (2.45) and (2.47) are equal to the left-hand sides of inequalities (2.49), (2.50) and (2.51), respectively. If $\varphi \in \mathscr{C}^1$, then, thanks to Lemma 3, the left-hand sides of inequalities (2.49), (2.50) and (2.51) are equal, respectively, to the left-hand sides of inequalities (2.52), (2.53) and (2.54). □

Remark 20 (i) The expressions in the left-hand sides of the inequalities (2.49), (2.50) and (2.51) are the γ-slopes [131, p. 61] computed at respective points of the extended-real-valued function (2.21), where f is defined by either (2.20) or (2.33).

(ii) The slope necessary conditions for $\varphi-$semitransversality and $\varphi-$subtransversality in parts (i) and (ii) of Corollary 17 are particular cases of the slope condition of $\varphi-$transversality in part (iii) of this corollary, corresponding to setting $\omega'_i := \bar{x}$ $(i = 1,\ldots,n)$ and $x_1 = \ldots = x_n$, respectively.

Sacrificing the estimates for the δ's in Theorem 6 and Corollary 17, we can formulate 'δ-free' versions of these statements.

Corollary 18 Suppose $\varphi(t) \ge \alpha t$ for some $\alpha > 0$ and all $t > 0$ near 0, and $\gamma := (\alpha^{-1}+1)^{-1}$.

(i) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$, then inequality (2.43) holds for all $x_i \in X$ $(i = 1,\ldots,n)$ near 0 with $\max_{1\le i\le n}\|x_i\| > 0$.

(ii) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$, then inequality (2.45) holds for all $x \in X$ near $\bar{x}$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ near $\bar{x}$ with $\max_{1\le i\le n}\|\omega_i - x\| > 0$.

(iii) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$, then inequality (2.47) holds for all $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$ near $\bar{x}$ and $x_i \in X$ $(i=1,\ldots,n)$ near 0 with $\max_{1\le i\le n}\|\omega_i - x_i - \bar{x}\| > 0$.

Corollary 19 Suppose $\Omega_1,\ldots,\Omega_n$ and φ are convex, $\varphi'_+(0) > 0$, and $\gamma := ((\varphi'_+(0))^{-1}+1)^{-1}$.

(i) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$, then inequality (2.49) holds for all $x_i \in X$ $(i=1,\ldots,n)$ near 0 with $\max_{1\le i\le n}\|x_i\| > 0$.

(ii) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$, then inequality (2.50) holds for all $x \in X$ near $\bar{x}$ and $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$ near $\bar{x}$ with $\max_{1\le i\le n}\|\omega_i - x\| > 0$.

(iii) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$, then inequality (2.51) holds for all $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$ near $\bar{x}$ and $x_i \in X$ $(i=1,\ldots,n)$ near 0 with $\max_{1\le i\le n}\|\omega_i - x_i - \bar{x}\| > 0$.

Moreover, if $\varphi \in \mathscr{C}^1$, then inequalities (2.49), (2.50) and (2.51) in parts (i)–(iii) can be replaced by (2.52), (2.53) and (2.54), respectively.

Remark 21 If $\cap_{i=1}^n \Omega_i$ is closed and $\bar{x} \in \mathrm{bd}\cap_{i=1}^n \Omega_i$, then condition $\varphi'_+(0) > 0$ in parts (ii) and (iii) of Corollaries 17 and 19 can be dropped, as in this case Proposition 3 implies that $\varphi'_+(0) \ge 1$. Also in view of this proposition, one can suppose in parts (ii) and (iii) of Theorem 6 and Corollary 18 that $\alpha \ge 1$.

The next sufficient condition for $\varphi-$subtransversality was established in [70].

Proposition 12 Suppose X is Banach, and $\Omega_1,\ldots,\Omega_n$ are closed. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$ if, for some $\gamma > 0$ and any $x' \in X$ satisfying $\|x' - \bar{x}\| < \delta_2$ and $0 < \max_{1\le i\le n} d(x',\Omega_i) < \varphi^{-1}(\delta_1)$, there exists a $\lambda \in]\varphi(\max_{1\le i\le n} d(x',\Omega_i)),\delta_1[$ such that inequality (2.50) holds for all $x \in X$ and $\omega_i, \omega'_i \in \Omega_i$ $(i=1,\ldots,n)$ satisfying $\|x - x'\| < \lambda$, $\max_{1\le i\le n}\|\omega_i - \omega'_i\| < \lambda\gamma^{-1}$ and $0 < \max_{1\le i\le n}\|\omega_i - x\| \le \max_{1\le i\le n}\|\omega'_i - x'\| < \varphi^{-1}(\lambda)$.

From Proposition 12 and Corollary 19(ii), we obtain a complete slope characterization of $\varphi-$subtransversality in the convex case.

Corollary 20 Suppose X is Banach, $\Omega_1,\ldots,\Omega_n$ are closed and convex, φ is convex, $\varphi'_+(0) > 0$, and $\gamma := ((\varphi'_+(0))^{-1}+1)^{-1}$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ if and only if inequality (2.50) holds for all $x \in X$ near $\bar{x}$ and $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$ near $\bar{x}$ with $\max_{1\le i\le n}\|\omega_i - x\| > 0$.

Remark 22 Combining sufficient conditions from the previous section for the other two nonlinear transversality properties with the corresponding necessary conditions from Corollary 19 does not lead to their complete slope characterizations.

2.5 Dual Characterizations

2.5.1 Dual Sufficient Conditions

Semitransversality

If not explicitly stated otherwise, in this and the subsequent sections the space X is assumed to be Banach, the sets $\Omega_1,\ldots,\Omega_n$ are assumed closed; δ, δ_1 and δ_2 are given positive numbers.

The dual norm on $(X^*)^{n+1}$ corresponding to (2.15) has the following form:

$$\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma = \|x^*\| + \frac{1}{\gamma}\sum_{i=1}^{n}\|x_i^*\|, \quad x_1^*,\ldots,x_n^*,x^* \in X^*. \tag{2.55}$$

We will denote by d_γ the distance in $(X^*)^{n+1}$ determined by (2.55).

In this section, we use the function $\widehat{f}$ given by (2.21) with $f : X^{n+1} \to \mathbb{R}_+$ defined by (2.20). The next statement provides dual characterizations of $\varphi-$semitransversality in terms of subdifferentials of $\widehat{f}$.

Proposition 13 (i) The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ if, for some $\gamma > 0$ and any $x_i \in X$ $(i = 1,\ldots,n)$ satisfying (2.16), there exists a $\lambda \in]\varphi\left(\max_{1\le i\le n}\|x_i\|\right),\delta[$ such that

$$d_\gamma\left(0,\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)\right) \ge 1 \tag{2.56}$$

for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.18) and (2.19), where ∂ in (2.56) stands for the Clarke subdifferential ($\partial := \partial^C$).

(ii) If X is Asplund, then the above assertion is valid with ∂ in (2.56) standing for the Fréchet subdifferential ($\partial := \partial^F$), and condition (2.19) replaced by

$$0 < \max_{1\le i\le n}\|\omega_i - x_i - x\| < \varphi^{-1}(\lambda). \tag{2.57}$$

Proof

(i) Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is not $\varphi-$semitransversal at $\bar{x}$ with δ, and let $\gamma > 0$ be given. By Theorem 3, there exist points $x_i \in X$ $(i = 1,\ldots,n)$ satisfying (2.16) such that, for any $\lambda \in]\varphi\left(\max_{1\le i\le n}\|x_i\|\right),\delta[$, there exist $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.18) and (2.19), and a number $\tau \in]0,1[$ such that

$$\varphi\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right) - \varphi\left(\max_{1\le i\le n}\|u_i - x_i - u\|\right) \le \tau\|(u_1,\ldots,u_n,u) - (\omega_1,\ldots,\omega_n,x)\|_\gamma \tag{2.58}$$

for all $u_i \in \Omega_i$ near ω_i $(i=1,\ldots,n)$ and all u near x. In other words, $(\omega_1,\ldots,\omega_n,x)$ is a local minimizer of the function

$$(u_1,\ldots,u_n,u) \mapsto \widehat{f}(u_1,\ldots,u_n,u)+\tau\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma. \tag{2.59}$$

By Lemma 5, its Fréchet and, as a consequence, Clarke subdifferential at this point contains 0. Observe that (2.59) is the sum of the function $\widehat{f}$ and the Lipschitz continuous convex function $(u_1,\ldots,u_n,u) \mapsto \tau\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma$, and at any point all subgradients $(x_1^*,\ldots,x_n^*,x^*)$ of the latter function satisfy $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma \le \tau$. By the Clarke–Rockafellar sum rule (Lemma 4(i)), there exists a subgradient $(x_1^*,\ldots,x_n^*,x^*) \in \partial^C\widehat{f}(\omega_1,\ldots,\omega_n,x)$ such that $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma \le \tau < 1$. The last inequality contradicts (2.56).

(ii) If X is Asplund, then one can employ the fuzzy sum rule (Lemma 4(ii)): for any $\varepsilon>0$, there exist points $x' \in B_\varepsilon(x)$, $\omega_i' \in \Omega_i \cap B_\varepsilon(\omega_i)$ $(i=1,\ldots,n)$, and a subgradient $(x_1^*,\ldots,x_n^*,x^*) \in \partial^F\widehat{f}(\omega_1',\ldots,\omega_n',x')$ such that $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma < \tau+\varepsilon$. The number ε can be chosen small enough so that $\|x'-\bar{x}\| < \lambda$, $\max_{1\le i\le n}\|\omega_i'-\bar{x}\| < \lambda/\gamma$, $0 < \max_{1\le i\le n}\|\omega_i'-x_i-x'\| < \varphi^{-1}(\lambda)$, and $\tau+\varepsilon<1$. The last inequality again yields $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma < 1$, which contradicts (2.56).

□

The key condition (2.56) in Proposition 13 involves subdifferentials of the function $\widehat{f}$ given by (2.21) with $f: X^{n+1} \to \mathbb{R}_+$ defined by (2.20). Subgradients of $\widehat{f}$ have $n+1$ component vectors $x_1^*,\ldots,x_n^*,x^*$. As it can be seen from the representation (2.55) of the dual norm, the contribution of the vectors $x_1^*,\ldots,x_n^*$ on one hand and x^* on the other hand to condition (2.56) is different. The next corollary exposes this difference.

Corollary 21 (i) The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ if, for some $\gamma>0$ and any $x_i \in X$ $(i=1,\ldots,n)$ satisfying (2.16), there exists a $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ such that $\|x^*\| \ge 1$ for all $x \in X$ and $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$ satisfying (2.18) and (2.19), and all $(x_1^*,\ldots,x_n^*,x^*) \in \partial\widehat{f}(\omega_1,\ldots,\omega_n,x)$ with $\sum_{i=1}^n\|x_i^*\| < \gamma$, where ∂ stands for the Clarke subdifferential ($\partial := \partial^C$).

(ii) If X is Asplund, then the above assertion is valid with ∂ standing for the Fréchet subdifferential ($\partial := \partial^F$), and condition (2.19) replaced by (2.57).

Proof Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is not $\varphi-$semitransversal at $\bar{x}$ with δ. Let $\gamma>0$. By Proposition 13, there exist $x_i \in X$ $(i=1,\ldots,n)$ satisfying (2.16) such that, for any $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$, there exist $x \in X$ and $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$ satisfying (2.18) and (2.19) ((2.57) if X is Asplund), and a subgradient $(x_1^*,\ldots,x_n^*,x^*) \in \partial^C\widehat{f}(\omega_1,\ldots,\omega_n,x)$ ($(x_1^*,\ldots,x_n^*,x^*) \in \partial^F\widehat{f}(\omega_1,\ldots,\omega_n,x)$ if X is Asplund) such that $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma < 1$. By the definition (2.55) of the dual norm, this implies $\sum_{i=1}^n\|x_i^*\| < \gamma$ and $\|x^*\| < 1$. The latter inequality contradicts the assumption. □

The next 'δ-free' statement is a direct consequence of Corollary 21.

Corollary 22 (i) The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ if

$$\lim_{\gamma\downarrow 0}\ \liminf_{\substack{t\downarrow 0\\ t=\max_{1\le i\le n}\|x_i\|>0}}\ \lim_{\lambda\downarrow\varphi(t)}\ \inf_{\substack{\|x-\bar{x}\|<\lambda,\ \|\omega_i-\bar{x}\|<\lambda/\gamma,\ \omega_i\in\Omega_i\ (i=1,\ldots,n)\\ 0<\max_{1\le i\le n}\|\omega_i-x_i-x\|\le t\\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,x),\ \sum_{i=1}^n\|x_i^*\|<\gamma}}\|x^*\|>1, \tag{2.60}$$

or, in particular, if

$$\liminf_{\substack{x\to\bar{x},\ \omega_i\xrightarrow{\Omega_i}\bar{x},\ x_i\to 0,\ x_i^*\to 0\ (i=1,\ldots,n)\\ \max\limits_{1\le i\le n}\|\omega_i-x_i-x\|>0,\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)}}\|x^*\|>1, \tag{2.61}$$

where ∂ stands for the Clarke subdifferential ($\partial := \partial^C$).

(ii) If X is Asplund, then the above assertion is valid with ∂ standing for the Fréchet subdifferential ($\partial := \partial^F$).

Remark 23 Condition (2.61) is obviously stronger than (2.60). Moreover, it is in fact sufficient for the stronger $\varphi-$transversality property of $\Omega_1,\ldots,\Omega_n$; cf. Remark 28.

The proof of Proposition 13 utilizes the sum rules in Lemma 4 to obtain representations of subgradients of the sum function (2.59) in terms of subgradients of the summand functions. Note that one of the summands – the function $\widehat{f}$ – is itself a sum of functions; see (2.21). Next we apply these sum rules again to obtain characterizations of the nonlinear semitransversality in terms of normals to the given individual sets. This time the difference between the exact sum rule in Lemma 4(i) and the approximate sum rule in Lemma 4(ii) becomes explicit in the conclusions of the next theorem: compare the exact condition (2.65) and the corresponding approximate condition (2.66).

Theorem 7 Let $\varphi\in\mathscr{C}^1$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ if, for some $\mu>0$ and any $x_i\in X$ $(i=1,\ldots,n)$ satisfying (2.16), one of the following conditions is satisfied:

(i) there exists a $\lambda\in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ such that

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-x\|\right)\left(\left\|\sum_{i=1}^n x_i^*\right\|+\mu\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\omega_i)\right)\right)\ge 1 \tag{2.62}$$

for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.19) and

$$\|x-\bar{x}\|<\lambda,\quad \max_{1\le i\le n}\|\omega_i-\bar{x}\|<\mu\lambda, \tag{2.63}$$

and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying

$$\sum_{i=1}^n\|x_i^*\|=1, \tag{2.64}$$

$$\sum_{i=1}^n\langle x_i^*,x+x_i-\omega_i\rangle=\max_{1\le i\le n}\|x+x_i-\omega_i\|, \tag{2.65}$$

where N in (2.62) stands for the Clarke normal cone ($N := N^C$);

(ii) X is Asplund, and there exist a $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|), \delta[$ and a $\tau \in]0,1[$ such that inequality (2.62) holds with N standing for the Fréchet normal cone ($N := N^F$) for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.63) and (2.57), and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying condition (2.64) and

$$\sum_{i=1}^{n} \langle x_i^*, x + x_i - \omega_i\rangle > \tau \max_{1\le i\le n} \|x + x_i - \omega_i\|. \tag{2.66}$$

Proof Suppose $\{\Omega_1,\dots,\Omega_n\}$ is not $\varphi-$semitransversal at $\bar{x}$ with δ, and let $\mu > 0$ be given. Set $\gamma := \mu^{-1}$. By Proposition 13, there exist points $x_i \in X$ $(i = 1,\dots,n)$ satisfying (2.16) such that, for any $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|), \delta[$, there exist $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.19) ((2.57) if X is Asplund) and (2.63), and a subgradient $(\hat{x}_1^*,\dots,\hat{x}_n^*,\hat{x}^*) \in \partial\widehat{f}(\omega_1,\dots,\omega_n,x)$ such that

$$\|(\hat{x}_1^*,\dots,\hat{x}_n^*,\hat{x}^*)\|_\gamma < 1, \tag{2.67}$$

where ∂ stands for either the Clarke subdifferential (if X is a general Banach space) or the Fréchet subdifferential (if X is Asplund). Recall from (2.21) that $\widehat{f}$ is a sum of two functions: the function f given by (2.20) and the indicator function of the set $\Omega_1 \times \ldots \times \Omega_n$. Since $\max_{1\le i\le n}\|\omega_i - x_i - x\| > 0$, f is locally Lipschitz continuous near $(\omega_1,\dots,\omega_n,x)$.

(i) X is a general Banach space, and $(\hat{x}_1^*,\dots,\hat{x}_n^*,\hat{x}^*)$ in (2.67) is a Clarke subgradient. By the Clarke–Rockafellar sum rule (Lemma 4(i)), there exist vectors $(\tilde{x}_1^*,\dots,\tilde{x}_n^*,\tilde{x}^*) \in \partial^C f(\omega_1,\dots,\omega_n,x)$ and $(u_1^*,\dots,u_n^*) \in N^C_{\Omega_1\times\ldots\times\Omega_n}(\omega_1,\dots,\omega_n)$ such that

$$(\hat{x}_1^*,\dots,\hat{x}_n^*,\hat{x}^*) = (\tilde{x}_1^*,\dots,\tilde{x}_n^*,\tilde{x}^*) + (u_1^*,\dots,u_n^*,0). \tag{2.68}$$

By Lemmas 6 and 9, Remark 5,

$$u_i^* \in N^C_{\Omega_i}(\omega_i) \quad (i = 1,\dots,n), \tag{2.69}$$

$$(\tilde{x}_1^*,\dots,\tilde{x}_n^*,\tilde{x}^*) = \varphi'(\psi(\omega_1,\dots,\omega_n,x))(-x_1^*,\dots,-x_n^*,x^*), \tag{2.70}$$

and $(-x_1^*,\dots,-x_n^*,x^*) \in \partial\psi(\omega_1,\dots,\omega_n,x)$, where

$$\psi(u_1,\dots,u_n,u) := \max_{1\le i\le n}\|u_i - x_i - u\|, \quad u_1,\dots,u_n,u \in X. \tag{2.71}$$

By Lemma 8, conditions (2.64) and (2.65) are satisfied, and

$$x^* = \sum_{i=1}^{n} x_i^*. \tag{2.72}$$

Combining (2.67), (2.68), (2.69), (2.70), (2.72) and (2.55), we obtain

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)\left(\left\|\sum_{i=1}^{n} x_i^*\right\| + \mu\sum_{i=1}^{n} d\left(x_i^*, N^C_{\Omega_i}(\omega_i)\right)\right) < 1.$$

This contradicts (2.62).

(ii) Let X be Asplund and a number $\tau \in]0,1[$ be given. Instead of the Clarke–Rockafellar sum rule, one can employ the fuzzy sum rule (Lemma 4(ii)): for any $\varepsilon > 0$, there exist points $x' \in B_\varepsilon(x)$, $y_i \in B_\varepsilon(\omega_i)$, $\omega_i' \in \Omega_i \cap B_\varepsilon(\omega_i)$ $(i = 1,\ldots,n)$, and vectors $(\tilde{x}_1^*,\ldots,\tilde{x}_n^*,\tilde{x}^*) \in \partial^F f(y_1,\ldots,y_n,x')$ and $(u_1^*,\ldots,u_n^*) \in N^F_{\Omega_1\times\ldots\times\Omega_n}(\omega_1',\ldots,\omega_n')$ such that

$$\|(\tilde{x}_1^*,\ldots,\tilde{x}_n^*,\tilde{x}^*) + (u_1^*,\ldots,u_n^*,0) - (\hat{x}_1^*,\ldots,\hat{x}_n^*,\hat{x}^*)\|_\gamma < \varepsilon. \tag{2.73}$$

Denote

$$\beta := \left|\varphi'\left(\max_{1\le i\le n}\|\omega_i' - x_i - x'\|\right) - \varphi'\left(\max_{1\le i\le n}\|y_i - x_i - x'\|\right)\right|, \tag{2.74}$$

and observe that $\beta \to 0$ as $\varepsilon \downarrow 0$. Recall that $0 < \max_{1\le i\le n}\|\omega_i - x_i - x\| \le \max_{1\le i\le n}\|x_i\| < \varphi^{-1}(\lambda)$. The number ε can be chosen small enough so that

$$\|x' - \bar{x}\| < \lambda, \quad \max_{1\le i\le n}\|\omega_i' - \bar{x}\| < \mu\lambda, \quad 0 < \max_{1\le i\le n}\|\omega_i' - x_i - x'\| < \varphi^{-1}(\lambda),$$
$$\|(\hat{x}_1^*,\ldots,\hat{x}_n^*,\hat{x}^*)\|_\gamma + (1+\mu)\beta + \varepsilon < 1, \tag{2.75}$$
$$\max_{1\le i\le n}\|y_i - \omega_i'\| < \frac{1-\tau}{2}\max_{1\le i\le n}\|x' - x_i - \omega_i'\|. \tag{2.76}$$

By Lemmas 6 and 9, Remark 5 and Lemma 8,

$$u_i^* \in N^F_{\Omega_i}(\omega_i') \quad (i = 1,\ldots,n), \tag{2.77}$$
$$(\tilde{x}_1^*,\ldots,\tilde{x}_n^*,\tilde{x}^*) = \varphi'\left(\max_{1\le i\le n}\|y_i - x_i - x'\|\right)(-x_1^*,\ldots,-x_n^*,x^*), \tag{2.78}$$

where vectors $x_1^*,\ldots,x_n^*,x^* \in X^*$ satisfy (2.64), (2.72) and

$$\sum_{i=1}^n \langle x_i^*, x' + x_i - y_i\rangle = \max_{1\le i\le n}\|x' + x_i - y_i\|. \tag{2.79}$$

It follows from (2.64), (2.76) and (2.79) that

$$\begin{aligned}\sum_{i=1}^n \langle x_i^*, x' + x_i - \omega_i'\rangle &\ge \sum_{i=1}^n \langle x_i^*, x' + x_i - y_i\rangle - \max_{1\le i\le n}\|y_i - \omega_i'\| \\ &= \max_{1\le i\le n}\|x' + x_i - y_i\| - \max_{1\le i\le n}\|y_i - \omega_i'\| \\ &\ge \max_{1\le i\le n}\|x' + x_i - \omega_i'\| - 2\max_{1\le i\le n}\|y_i - \omega_i'\| \\ &> \tau \max_{1\le i\le n}\|x' + x_i - \omega_i'\|.\end{aligned}$$

Combining (2.72), (2.73), (2.77), (2.78) and (2.55), we obtain

$$\begin{aligned}&\varphi'\left(\max_{1\le i\le n}\|y_i - x_i - x'\|\right)\left(\left\|\sum_{i=1}^n x_i^*\right\| + \mu\sum_{i=1}^n d\left(x_i^*, N^F_{\Omega_i}(\omega_i')\right)\right) \\ &\quad\overset{(2.72),(2.78)}{=} \|\tilde{x}^*\| + \mu\sum_{i=1}^n d\left(-\tilde{x}_i^*, N^F_{\Omega_i}(\omega_i')\right) \overset{(2.77)}{\le} \|\tilde{x}^*\| + \mu\sum_{i=1}^n \|\tilde{x}_i^* + u_i^*\| \\ &\quad\overset{(2.55)}{=} \|(\tilde{x}_1^*,\ldots,\tilde{x}_n^*,\tilde{x}^*) + (u_1^*,\ldots,u_n^*,0)\|_\gamma \overset{(2.73)}{<} \|(\hat{x}_1^*,\ldots,\hat{x}_n^*,\hat{x}^*)\|_\gamma + \varepsilon.\end{aligned}$$

Hence, thanks to (2.74), (2.75) and (2.79),

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i'-x_i-x'\|\right)\left(\left\|\sum_{i=1}^n x_i^*\right\|+\mu\sum_{i=1}^n d\left(x_i^*,N^F_{\Omega_i}(\omega_i')\right)\right)$$
$$<\|(\hat{x}_1^*,\ldots,\hat{x}_n^*,\hat{x}^*)\|_\gamma+(1+\mu)\beta+\varepsilon<1.$$

This contradicts (2.62).

□

The key dual transversality condition (2.62) in Theorem 7 combines two conditions on the dual vectors $x_i^*\in X^*$ $(i=1,\ldots,n)$: either their sum must be sufficiently far from 0, or the vectors themselves must be sufficiently far from the corresponding normal cones. From the point of view of applications, it can be convenient to have these conditions separated. The next corollary shows that it can be easily done. It collects four separate sufficient conditions for $\varphi-$semitransversality.

Corollary 23 Let $\varphi\in\mathscr{C}^1$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ if, for some $\gamma>0$ and any $x_i\in X$ $(i=1,\ldots,n)$ satisfying (2.16), one of the following conditions is satisfied:

(i) there exists a $\lambda\in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ such that

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-x\|\right)\left\|\sum_{i=1}^n x_i^*\right\|\ge 1 \tag{2.80}$$

for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.18) and (2.19), and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64), (2.65) and

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-x\|\right)\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\omega_i)\right)<\gamma, \tag{2.81}$$

where N stands for the Clarke normal cone ($N:=N^C$);

(ii) X is Asplund, and there exist a $\lambda\in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ and a $\tau\in]0,1[$ such that inequality (2.80) holds for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.18) and (2.57), and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64), (2.66) and (2.81), where N stands for the Fréchet normal cone ($N:=N^F$);

(iii) there exists a $\lambda\in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ such that

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-x\|\right)\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\omega_i)\right)\ge\gamma \tag{2.82}$$

for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.18) and (2.19), and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64), (2.65) and

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-x\|\right)\left\|\sum_{i=1}^n x_i^*\right\|<1, \tag{2.83}$$

where N in (2.82) stands for the Clarke normal cone ($N:=N^C$);

(iv) X is Asplund, and there exist a $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ and a $\tau \in]0,1[$ such that inequality (2.82) holds for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.18) and (2.57), and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64), (2.66) and (2.83), where N in (2.82) stands for the Fréchet normal cone ($N:=N^F$).

Proof It is sufficient to notice that the assumptions in parts (i)–(iv) of the above corollary imply those in the respective parts of Theorem 7 with $\mu:=\gamma^{-1}$. Indeed, the corollary replaces condition (2.62) by a stronger condition: either (2.80) in parts (i) and (ii) or (2.82) in parts (iii) and (iv). These conditions only need to be satisfied by $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.81) in the case of (2.80) or (2.83) in the case of (2.82). If any of the conditions (2.81) and (2.83) is violated, then condition (2.62) is automatically satisfied. □

Corollary 23 'separating' the two dual transversality conditions hidden in the key combined condition (2.62) in Theorem 7 still has a drawback. It does not allow for the two 'exact' cases: $x_i^*\in N_{\Omega_i}(\omega_i)$ $(i=1,\ldots,n)$ (with appropriate normal cones) or $\sum_{i=1}^n x_i^*=0$, very important for the transversality (as well as extremality and stationarity) theory; cf. conditions (2.81) and (2.83). Employing Lemma 10, we can accommodate for these important cases.

Corollary 24 Let $\varphi\in\mathscr{C}^1$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ if, for some $\mu>0$ and any $x_i\in X$ $(i=1,\ldots,n)$ satisfying (2.16), one of the following conditions is satisfied:

(i) there exists a $\lambda\in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ such that

$$\left[\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-x\|\right)\right]^{-1}+1\le\mu, \tag{2.84}$$

and condition (2.80) holds true for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.19) and (2.63), and all $x_i^*\in N^C_{\Omega_i}(\omega_i)$ $(i=1,\ldots,n)$ satisfying (2.64) and (2.66) with

$$\tau:=\frac{\mu\varphi'(\max_{1\le i\le n}\|\omega_i-x_i-x\|)-1}{\mu\varphi'(\max_{1\le i\le n}\|\omega_i-x_i-x\|)+1}; \tag{2.85}$$

(ii) X is Asplund, and there exist a $\lambda\in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ and a $\hat{\tau}\in]0,1[$ such that condition (2.80) hold true and

$$\left[\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-x\|\right)\right]^{-1}+1\le\hat{\tau}\mu \tag{2.86}$$

for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.57) and (2.63), and all $x_i^*\in N^F_{\Omega_i}(\omega_i)$ $(i=1,\ldots,n)$ satisfying (2.64) and (2.66) with

$$\tau:=\frac{\hat{\tau}\mu\varphi'(\max_{1\le i\le n}\|\omega_i-x_i-x\|)-1}{\mu\varphi'(\max_{1\le i\le n}\|\omega_i-x_i-x\|)+1}; \tag{2.87}$$

(iii) there exists a $\lambda\in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ such that

$$\left[\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-x\|\right)\right]^{-1}+\mu\le 1, \tag{2.88}$$

and condition (2.82) holds true with $\gamma := \mu^{-1}$ and N standing for the Clarke normal cone ($N := N^C$) for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.19) and (2.63), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying $\sum_{i=1}^n x_i^* = 0$, and conditions (2.64) and (2.66) with

$$\tau := \frac{\varphi'\left(\max_{1 \le i \le n} \|\omega_i - x_i - x\|\right) - 1}{\varphi'\left(\max_{1 \le i \le n} \|\omega_i - x_i - x\|\right) + 1}; \tag{2.89}$$

(iv) X is Asplund, and there exist a $\lambda \in]\varphi\left(\max_{1 \le i \le n} \|x_i\|\right), \delta[$ and a $\hat{\tau} \in]0,1[$ such that condition (2.82) hold true with $\gamma := \mu^{-1}$ and N standing for the Fréchet normal cone ($N := N^F$), and

$$\left[\varphi'\left(\max_{1 \le i \le n} \|\omega_i - x_i - x\|\right)\right]^{-1} + \mu \le \hat{\tau} \tag{2.90}$$

for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.57) and (2.63), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying $\sum_{i=1}^n x_i^* = 0$, and conditions (2.64) and (2.66) with

$$\tau := \frac{\hat{\tau}\varphi'\left(\max_{1 \le i \le n} \|\omega_i - x_i - x\|\right) - 1}{\varphi'\left(\max_{1 \le i \le n} \|\omega_i - x_i - x\|\right) + 1}. \tag{2.91}$$

Proof

(i) and (ii). Suppose $\{\Omega_1, \ldots, \Omega_n\}$ is not $\varphi-$semitransversal at $\bar{x}$ with δ, and let a $\mu > 0$ be given. Set $\gamma := \mu^{-1}$. By Theorem 7, there exist points $x_i \in X$ $(i = 1, \ldots, n)$ satisfying (2.16) such that both conditions (i) and (ii) in the theorem are not satisfied. Hence, for any $\lambda \in]\varphi\left(\max_{1 \le i \le n} \|x_i\|\right), \delta[$ (as well as $\tau \in]0,1[$ in case (ii)), there exist $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.19) (or (2.57) in case (ii)) and (2.63), and $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64) and (2.65) (or (2.66) in case (ii)) such that inequality (2.62) is violated with N standing for the Clarke (Fréchet in case (ii)) normal cone. Thus, conditions (1.26) hold true with $\varepsilon := \left[\varphi'\left(\max_{1 \le i \le n} \|\omega_i - x_i - x\|\right)\right]^{-1}$, $\rho := 1$, $z_i := x_i^*$ and K_i standing for $N^C_{\Omega_i}(\omega_i)$ (or $N^F_{\Omega_i}(\omega_i)$ in case (ii)) $(i = 1, \ldots, n)$. Besides, if inequality (2.84) is satisfied, we have $\varepsilon + \rho \le \mu$ and, by Lemma 10(i), there exist vectors $\hat{x}_i^* := \hat{z}_i \in X^*$ $(i = 1, \ldots, n)$ such that conditions (1.27) hold true, i.e. $\hat{x}_i^* \in N^C_{\Omega_i}(\omega_i)$ (or $\hat{x}_i^* \in N^F_{\Omega_i}(\omega_i)$ in case (ii)) $(i = 1, \ldots, n)$, $\sum_{i=1}^n \|\hat{x}_i^*\| = 1$, while condition (2.80), with $\hat{x}_i^*$ in place of x_i^* $(i = 1, \ldots, n)$, is violated.

Moreover, setting $\tau := 1$ in case (i), we have inequality (1.28) satisfied in both cases and, by Lemma 10(iii), inequality (1.29) holds with $\hat{\tau} := \frac{\tau\mu - \varepsilon}{\mu + \varepsilon}$ in place of τ. Observe that the above definition of $\hat{\tau}$ is exactly the definition of τ in (2.85) in case (i) or in (2.87) in case (ii). Since condition (2.80) is violated, both conditions (i) and (ii) are not satisfied.

(iii) and (iv). The proof proceeds as above, replacing the application of part (i) of Lemma 10 with that of its part (ii).

$\square$

Remark 24 Since τ defined by any of the formulas (2.85), (2.87), (2.89) and (2.91) is not in general a constant, but depends on x, ω_i and x_i $(i=1,\ldots,n)$, checking condition (2.66) with such a τ does not seem practical. Such a check becomes meaningful in the linear setting, i.e. when φ', and consequently, also τ is constant; cf. Remark 31.

Dropping condition (2.66) in all parts of Corollary 24, we can formulate its simplified and easier to use version.

Corollary 25 Let $\varphi \in \mathscr{C}^1$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ if, for some $\mu>0$ and any $x_i \in X$ $(i=1,\ldots,n)$ satisfying (2.16), one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ such that conditions (2.84) and (2.80) hold true for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.19) and (2.63), and all $x_i^*\in N^C_{\Omega_i}(\omega_i)$ $(i=1,\ldots,n)$ satisfying (2.64);

(ii) X is Asplund, and there exist a $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ and a $\hat{\tau}\in]0,1[$ such that conditions (2.86) and (2.80) hold true for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.57) and (2.63), and all $x_i^*\in N^F_{\Omega_i}(\omega_i)$ $(i=1,\ldots,n)$ satisfying (2.64);

(iii) there exists a $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ such that conditions (2.88) and (2.82) hold true with $\gamma:=\mu^{-1}$ and N standing for the Clarke normal cone ($N:=N^C$) for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.19) and (2.63), and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying $\sum_{i=1}^n x_i^*=0$ and condition (2.64);

(iv) X is Asplund, and there exist a $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|),\delta[$ and a $\hat{\tau}\in]0,1[$ such that conditions (2.90) and (2.82) hold true with $\gamma:=\mu^{-1}$ and N standing for the Fréchet normal cone ($N:=N^F$), for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.57) and (2.63), and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying $\sum_{i=1}^n x_i^*=0$ and condition (2.64).

Subtransversality

The dual characterizations in this section follow the pattern of those in the previous section with appropriate adjustments in the proofs. We use the function $\widehat{f}$ given by (2.21) with $f: X^{n+1}\to\mathbb{R}_+$ defined by (2.33). The next statement provides dual characterizations of $\varphi-$subtransversality in terms of subdifferentials of $\widehat{f}$.

Proposition 14 (i) The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma>0$ and any $x'\in X$ satisfying (2.29), there exists a $\lambda\in]\varphi(\max_{1\le i\le n} d(x',\Omega_i)),\delta_1[$ such that condition (2.56) is satisfied for all $x\in X$ and $\omega_i,\omega_i'\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.31) and (2.32), where ∂ in (2.56) stands for the Clarke subdifferential ($\partial:=\partial^C$).

(ii) If X is Asplund, then the above assertion is valid with ∂ in (2.56) standing for the Fréchet subdifferential ($\partial := \partial^F$), and condition (2.32) replaced by

$$0 < \max_{1\le i\le n} \|\omega_i - x\| < \varphi^{-1}(\lambda). \tag{2.92}$$

Proof

(i) Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is not $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2, and let $\gamma > 0$ be given. By Theorem 4, there exists a point $x' \in X$ satisfying (2.29) such that, for any $\lambda \in]\varphi(\max_{1\le i\le n} d(x',\Omega_i)), \delta_1[$, there exist points $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.31) and (2.32), and a number $\tau \in]0,1[$ such that

$$\varphi\left(\max_{1\le i\le n}\|\omega_i - x\|\right) - \varphi\left(\max_{1\le i\le n}\|u_i - u\|\right) \le \tau\|(u_1,\ldots,u_n,u) - (\omega_1,\ldots,\omega_n,x)\|_\gamma$$

for all $u_i \in \Omega_i$ near ω_i $(i = 1,\ldots,n)$ and all u near x. In other words, $(\omega_1,\ldots,\omega_n,x)$ is a local minimizer of the function (2.59). By Lemma 5, its Fréchet and, as a consequence, Clarke subdifferential at this point contains 0. Observe that (2.59) is the sum of the function $\widehat{f}$ and the Lipschitz continuous convex function $(u_1,\ldots,u_n,u) \mapsto \tau\|(u_1,\ldots,u_n,u) - (\omega_1,\ldots,\omega_n,x)\|_\gamma$. At any point, all subgradients $(x_1^*,\ldots,x_n^*,x^*)$ of the latter function satisfy $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma \le \tau$. By the Clarke–Rockafellar sum rule (Lemma 4(i)), there exists a subgradient $(x_1^*,\ldots,x_n^*,x^*) \in \partial^C\widehat{f}(\omega_1,\ldots,\omega_n,x)$ such that $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma \le \tau$. The last inequality contradicts (2.56).

(ii) If X is Asplund, then one can employ the fuzzy sum rule (Lemma 4(ii)): for any $\varepsilon > 0$, there exist points $y \in B_\varepsilon(x)$, $y_i \in \Omega_i \cap B_\varepsilon(\omega_i)$ $(i = 1,\ldots,n)$, and a subgradient $(x_1^*,\ldots,x_n^*,x^*) \in \partial^F\widehat{f}(y_1,\ldots,y_n,y)$ such that $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma < \tau + \varepsilon$. The number ε can be chosen small enough so that $\|y - x'\| < \lambda$, $\max_{1\le i\le n}\|y_i - \omega_i'\| < \lambda/\gamma$, $0 < \max_{1\le i\le n}\|y_i - y\| < \varphi^{-1}(\lambda)$, and $\tau + \varepsilon < 1$. The last inequality again yields $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma < 1$, which contradicts (2.56).

□

Similar to the case of nonlinear semitransversality, the difference in the contribution of components of subgradients of $\widehat{f}$ to the key dual condition (2.56) in Proposition 14 can be exposed.

Corollary 26 (i) The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma > 0$ and any $x' \in X$ satisfying (2.29), there exists a $\lambda \in]\varphi(\max_{1\le i\le n} d(x',\Omega_i)), \delta_1[$ such that $\|x^*\| \ge 1$ for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.31) and (2.32), and all $(x_1^*,\ldots,x_n^*,x^*) \in \partial\widehat{f}(\omega_1,\ldots,\omega_n,x)$ with $\sum_{i=1}^n \|x_i^*\| < \gamma$, where ∂ stands for the Clarke subdifferential ($\partial := \partial^C$).

(ii) If X is Asplund, then the above assertion is valid with ∂ standing for the Fréchet subdifferential ($\partial := \partial^F$), and condition (2.32) replaced by (2.92).

Proof Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is not $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2, and let $\gamma>0$ be given. By Proposition 14, there exists a point $x'\in X$ satisfying (2.29) such that, for any $\lambda\in\,]\varphi\left(\max_{1\le i\le n} d(x',\Omega_i)\right),\delta_1[$, there exist points $x\in X$ and $\omega_i,\omega_i'\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.31) and (2.32), and a subgradient $(x_1^*,\ldots,x_n^*,x^*)\in\partial^C\widehat{f}(\omega_1,\ldots,\omega_n,x)$ $((x_1^*,\ldots,x_n^*,x^*)\in\partial^F\widehat{f}(\omega_1,\ldots,\omega_n,x)$ if X is Asplund) such that $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma<1$. By the representation (2.55) of the dual norm, this implies $\sum_{i=1}^n\|x_i^*\|<\gamma$ and $\|x^*\|<1$. The latter inequality contradicts the assumption. $\square$

Remark 25 In the Hölder setting, i.e. when $\varphi(t)=\alpha^{-1}t^q$ with $\alpha>0$ and $q>0$, Corollary 26 improves [141, Proposition 7].

The next 'δ-free' statement is a direct consequence of Corollary 26.

Corollary 27 (i) The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ if

$$\lim_{\gamma\downarrow 0}\ \liminf_{\substack{x'\to\bar{x},\ t\downarrow 0\\ t=\max_{1\le i\le n} d(x',\Omega_i)>0}}\ \lim_{\lambda\downarrow\varphi(t)}\ \inf_{\substack{\|x-x'\|<\lambda,\ \|\omega_i-\omega_i'\|<\lambda/\gamma,\ \omega_i,\omega_i'\in\Omega_i\ (i=1,\ldots,n)\\ 0<\max_{1\le i\le n}\|\omega_i-x\|\le\max_{1\le i\le n}\|\omega_i'-x'\|<\varphi^{-1}(\lambda)\\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,x),\ \sum_{i=1}^n\|x_i^*\|<\gamma}}\|x^*\|>1, \tag{2.93}$$

or, in particular, if

$$\liminf_{\substack{x\to\bar{x},\ \omega_i\xrightarrow{\Omega_i}\bar{x},\ x_i^*\to 0\ (i=1,\ldots,n)\\ \max\limits_{1\le i\le n}\|\omega_i-x\|>0,\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)}}\|x^*\|>1, \tag{2.94}$$

where ∂ stands for the Clarke subdifferential ($\partial:=\partial^C$).

(ii) If X is Asplund, then the above assertion is valid with ∂ being the Fréchet subdifferential ($\partial:=\partial^F$).

Remark 26 Condition (2.94) is obviously stronger than (2.93).

Next, similar to the case of nonlinear semitransversality, we apply the sum rules again to the function $\widehat{f}$ to obtain characterizations of nonlinear subtransversality in terms of normals to the given individual sets.

Theorem 8 Let $\varphi\in\mathscr{C}^1$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\mu>0$ and any $x'\in X$ satisfying (2.29), one of the following conditions is satisfied:

(i) there exists a $\lambda\in\,]\varphi\left(\max_{1\le i\le n} d(x',\Omega_i)\right),\delta_1[$ such that

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x\|\right)\left(\left\|\sum_{i=1}^n x_i^*\right\|+\mu\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\omega_i)\right)\right)\ge 1 \tag{2.95}$$

for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.32) and

$$\|x - x'\| < \lambda, \quad \max_{1 \le i \le n} \|\omega_i - \omega_i'\| < \mu\lambda, \tag{2.96}$$

and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64) and

$$\sum_{i=1}^{n} \langle x_i^*, x - \omega_i \rangle = \max_{1 \le i \le n} \|x - \omega_i\|, \tag{2.97}$$

where N in (2.95) stands for the Clarke normal cone ($N := N^C$);

(ii) X is Asplund, and there exist a $\lambda \in]\varphi(\max_{1 \le i \le n} d(x', \Omega_i)), \delta_1[$ and a $\tau \in]0, 1[$ such that inequality (2.95) holds with N standing for the Fréchet normal cone ($N := N^F$) for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.92) and (2.96), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64) and

$$\sum_{i=1}^{n} \langle x_i^*, x - \omega_i \rangle > \tau \max_{1 \le i \le n} \|x - \omega_i\|. \tag{2.98}$$

Proof Suppose $\{\Omega_1, \ldots, \Omega_n\}$ is not $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2, and let $\mu > 0$ be given. By Proposition 14, there exists a point $x' \in X$ satisfying (2.29) such that, for any $\lambda \in]\varphi(\max_{1 \le i \le n} d(x', \Omega_i)), \delta_1[$, there exist points $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.31) and (2.32) ((2.92) if X is Asplund), and a subgradient $(\hat{x}_1^*, \ldots, \hat{x}_n^*, \hat{x}^*) \in \partial \widehat{f}(\omega_1, \ldots, \omega_n, x)$ such that condition (2.67) holds, where $\gamma := \mu^{-1}$ and ∂ stands for either the Clarke subdifferential (if X is a general Banach space) or the Fréchet subdifferential (if X is Asplund). Recall from (2.21) that $\widehat{f}$ is a sum of two functions: the function f given by (2.33) and the indicator function of the set $\Omega_1 \times \ldots \times \Omega_n$. Since $\max_{1 \le i \le n} \|\omega_i - x\| > 0$, f is locally Lipschitz continuous near $(\omega_1, \ldots, \omega_n, x)$.

(i) <u>X is a general Banach space</u>, and $(\hat{x}_1^*, \ldots, \hat{x}_n^*, \hat{x}^*)$ in (2.67) is a Clarke subgradient. By the Clarke–Rockafellar sum rule (Lemma 4(i)), there exist vectors $(\tilde{x}_1^*, \ldots, \tilde{x}_n^*, \tilde{x}^*) \in \partial^C f(\omega_1, \ldots, \omega_n, x)$ and $(u_1^*, \ldots, u_n^*) \in N^C_{\Omega_1 \times \ldots \times \Omega_n}(\omega_1, \ldots, \omega_n)$ such that equality (2.68) holds, where f is given by (2.33). By Lemmas 6 and 9 and Remark 5, these vectors satisfy (2.69), (2.70) and $(-x_1^*, \ldots, -x_n^*, x^*) \in \partial \psi(\omega_1, \ldots, \omega_n, x)$, where $\psi(u_1, \ldots, u_n, u) := \max_{1 \le i \le n} \|u_i - u\|$ $(u_1, \ldots, u_n, u \in X)$. By Lemma 8, conditions (2.64), (2.72) and (2.97) are satisfied. Combining (2.67), (2.68), (2.69), (2.70), (2.72) and (2.55), we obtain

$$\varphi'\left(\max_{1 \le i \le n} \|\omega_i - x\|\right) \left(\left\| \sum_{i=1}^{n} x_i^* \right\| + \mu \sum_{i=1}^{n} d\left(x_i^*, N^C_{\Omega_i}(\omega_i)\right) \right) < 1.$$

This contradicts (2.95).

(ii) <u>Let X be Asplund</u> and a number $\tau \in]0, 1[$ be given. Instead of the Clarke–Rockafellar sum rule, one can employ the fuzzy sum rule (Lemma 4(ii)): for any $\varepsilon > 0$, there exist points $y \in B_\varepsilon(x)$, $y_i \in B_\varepsilon(\omega_i)$, $\omega_i'' \in \Omega_i \cap B_\varepsilon(\omega_i)$ $(i=1,\ldots,n)$, and vectors $(\tilde{x}_1^*,\ldots,\tilde{x}_n^*,\tilde{x}^*) \in \partial^F f(y_1,\ldots,y_n,y)$ and $(u_1^*,\ldots,u_n^*) \in N^F_{\Omega_1\times\ldots\times\Omega_n}(\omega_1'',\ldots,\omega_n'')$ satisfying (2.73). Denote

$$\beta := \left|\varphi'\left(\max_{1\le i\le n}\|\omega_i''-y\|\right) - \varphi'\left(\max_{1\le i\le n}\|y_i-y\|\right)\right|, \tag{2.99}$$

and observe that $\beta \to 0$ as $\varepsilon \downarrow 0$. The number ε can be chosen small enough so that

$$\|y-x'\| < \lambda, \quad \max_{1\le i\le n}\|y_i'-\omega_i''\| < \frac{\lambda}{\gamma}, \quad 0 < \max_{1\le i\le n}\|y_i'-y\| < \varphi^{-1}(\lambda),$$

condition (2.75) is satisfied, and

$$\max_{1\le i\le n}\|y_i-\omega_i''\| < \frac{1-\tau}{2}\max_{1\le i\le n}\|y-\omega_i''\|. \tag{2.100}$$

By Lemmas 6 and 9, Remark 5 and Lemma 8, we have (2.77) and

$$(\tilde{x}_1^*,\ldots,\tilde{x}_n^*,\tilde{x}^*) = \varphi'\left(\max_{1\le i\le n}\|y_i-y\|\right)(-x_1^*,\ldots,-x_n^*,x^*), \tag{2.101}$$

where vectors $x_1^*,\ldots,x_n^*,x^* \in X^*$ satisfy (2.64), (2.72) and

$$\sum_{i=1}^n \langle x_i^*, y-y_i\rangle = \max_{1\le i\le n}\|y-y_i\|. \tag{2.102}$$

It follows from (2.100) and (2.102) that

$$\begin{aligned}\sum_{i=1}^n \langle x_i^*, y-\omega_i''\rangle &\ge \sum_{i=1}^n \langle x_i^*, y-y_i\rangle - \max_{1\le i\le n}\|y_i-\omega_i''\| \\ &= \max_{1\le i\le n}\|y-y_i\| - \max_{1\le i\le n}\|y_i-\omega_i''\| \\ &\ge \max_{1\le i\le n}\|y-\omega_i''\| - 2\max_{1\le i\le n}\|y_i-\omega_i''\| > \tau\max_{1\le i\le n}\|y-\omega_i''\|.\end{aligned}$$

Combining (2.72), (2.73), (2.77), (2.101) and (2.55), we obtain

$$\begin{aligned}&\varphi'\left(\max_{1\le i\le n}\|y_i-y\|\right)\left(\left\|\sum_{i=1}^n x_i^*\right\| + \mu\sum_{i=1}^n d\left(x_i^*, N^F_{\Omega_i}(\omega_i'')\right)\right) \\ &\quad\overset{(2.72),(2.101)}{=} \|\tilde{x}^*\| + \mu\sum_{i=1}^n d\left(-\tilde{x}_i^*, N^F_{\Omega_i}(\omega_i'')\right) \overset{(2.77)}{\le} \|\tilde{x}^*\| + \mu\sum_{i=1}^n \|\tilde{x}_i^*+u_i^*\| \\ &\quad\overset{(2.55)}{=} \|(\tilde{x}_1^*,\ldots,\tilde{x}_n^*,\tilde{x}^*) + (u_1^*,\ldots,u_n^*,0)\|_\gamma \overset{(2.73)}{<} \|(\hat{x}_1^*,\ldots,\hat{x}_n^*,\hat{x}^*)\|_\gamma + \varepsilon.\end{aligned}$$

Hence, thanks to (2.99), (2.75) and (2.102),

$$\begin{aligned}&\varphi'\left(\max_{1\le i\le n}\|\omega_i''-y\|\right)\left(\left\|\sum_{i=1}^n x_i^*\right\| + \mu\sum_{i=1}^n d\left(x_i^*, N^F_{\Omega_i}(\omega_i'')\right)\right) \\ &\quad < \|(\hat{x}_1^*,\ldots,\hat{x}_n^*,\hat{x}^*)\|_\gamma + (1+\mu)\beta + \varepsilon < 1.\end{aligned}$$

This contradicts (2.95).

$\square$

The next corollary 'separates' the two conditions on the dual vectors $x_i^* \in X^*$ $(i = 1, \ldots, n)$ combined in the key dual transversality condition (2.95) in Theorem 8.

Corollary 28 Let $\varphi \in \mathscr{C}^1$. The collection $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma > 0$ and any $x' \in X$ satisfying (2.29), one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\varphi(\max_{1\le i\le n} d(x', \Omega_i)), \delta_1[$ such that

$$\varphi'\left(\max_{1\le i\le n} \|\omega_i - x\|\right) \left\|\sum_{i=1}^{n} x_i^*\right\| \ge 1 \tag{2.103}$$

for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.31) and (2.32), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.97) and

$$\varphi'\left(\max_{1\le i\le n} \|\omega_i - x\|\right) \sum_{i=1}^{n} d\left(x_i^*, N_{\Omega_i}(\omega_i)\right) < \gamma, \tag{2.104}$$

where N stands for the Clarke normal cone ($N := N^C$);

(ii) X is Asplund, and there exist a $\lambda \in]\varphi(\max_{1\le i\le n} d(x', \Omega_i)), \delta_1[$ and a $\tau \in]0, 1[$ such that inequality (2.103) holds for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.31) and (2.92), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.98) and (2.104), where N stands for the Fréchet normal cone ($N := N^F$);

(iii) there exists a $\lambda \in]\varphi(\max_{1\le i\le n} d(x', \Omega_i)), \delta_1[$ such that

$$\varphi'\left(\max_{1\le i\le n} \|\omega_i - x\|\right) \sum_{i=1}^{n} d\left(x_i^*, N_{\Omega_i}(\omega_i)\right) \ge \gamma \tag{2.105}$$

for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.31) and (2.32), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.97) and

$$\varphi'\left(\max_{1\le i\le n} \|\omega_i - x\|\right) \left\|\sum_{i=1}^{n} x_i^*\right\| < 1, \tag{2.106}$$

where N in (2.105) stands for the Clarke normal cone ($N := N^C$);

(iv) X is Asplund, and there exist a $\lambda \in]\varphi(\max_{1\le i\le n} d(x', \Omega_i)), \delta_1[$ and a $\tau \in]0, 1[$ such that inequality (2.105) holds for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.31) and (2.92), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.98) and (2.106), where N in (2.105) stands for the Fréchet normal cone ($N := N^F$).

Proof It is sufficient to notice that the assumptions in parts (i)–(iv) imply those in the respective parts of Theorem 8. □

The next corollary complements Corollary 28 and accommodates for the two 'exact' cases $x_i^* \in N_{\Omega_i}(\omega_i)$ $(i = 1,\ldots,n)$ and $\sum_{i=1}^n x_i^* = 0$. It is a consequence of Theorem 8 and Lemma 10; cf. the proof of Corollary 24.

Corollary 29 Let $\varphi \in \mathscr{C}^1$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\mu > 0$ and any $x' \in X$ satisfying (2.29), one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\varphi(\max_{1\le i\le n} d(x',\Omega_i)), \delta_1[$ such that

$$\left[\varphi'\left(\max_{1\le i\le n} \|\omega_i - x\|\right)\right]^{-1} + 1 \le \mu, \tag{2.107}$$

and condition (2.103) holds true for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.32) and (2.96), and all $x_i^* \in N^C_{\Omega_i}(\omega_i)$ $(i = 1,\ldots,n)$ satisfying (2.64) and (2.98) with

$$\tau := \frac{\mu\varphi'(\max_{1\le i\le n} \|\omega_i - x\|) - 1}{\mu\varphi'(\max_{1\le i\le n} \|\omega_i - x\|) + 1}; \tag{2.108}$$

(ii) X is Asplund, and there exist a $\lambda \in]\varphi(\max_{1\le i\le n} d(x',\Omega_i)), \delta_1[$ and a $\hat{\tau} \in]0,1[$ such that condition (2.103) hold true and

$$\left[\varphi'\left(\max_{1\le i\le n} \|\omega_i - x\|\right)\right]^{-1} + 1 \le \hat{\tau}\mu \tag{2.109}$$

for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.92) and (2.96), and all $x_i^* \in N^F_{\Omega_i}(\omega_i)$ $(i = 1,\ldots,n)$ satisfying (2.64) and (2.98) with

$$\tau := \frac{\hat{\tau}\mu\varphi'(\max_{1\le i\le n} \|\omega_i - x\|) - 1}{\mu\varphi'(\max_{1\le i\le n} \|\omega_i - x\|) + 1}; \tag{2.110}$$

(iii) there exists a $\lambda \in]\varphi(\max_{1\le i\le n} d(x',\Omega_i)), \delta_1[$ such that

$$\left[\varphi'\left(\max_{1\le i\le n} \|\omega_i - x\|\right)\right]^{-1} + \mu \le 1, \tag{2.111}$$

and condition (2.105) holds true with $\gamma := \mu^{-1}$ and N standing for the Clarke normal cone ($N := N^C$) for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.32) and (2.96), and all $x_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying $\sum_{i=1}^n x_i^* = 0$, and conditions (2.64) and (2.98) with

$$\tau := \frac{\varphi'(\max_{1\le i\le n} \|\omega_i - x\|) - 1}{\varphi'(\max_{1\le i\le n} \|\omega_i - x\|) + 1}; \tag{2.112}$$

(iv) X is Asplund, and there exist a $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|), \delta[$ and a $\hat{\tau} \in]0,1[$ such that condition (2.105) hold true with $\gamma := \mu^{-1}$ and N standing for the Fréchet normal cone ($N := N^F$), and

$$\left[\varphi'\left(\max_{1\le i\le n}\|\omega_i - x\|\right)\right]^{-1} + \mu \le \hat{\tau} \tag{2.113}$$

for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.92) and (2.96), and all $x_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying $\sum_{i=1}^n x_i^* = 0$, and conditions (2.64) and (2.98) with

$$\tau := \frac{\hat{\tau}\varphi'(\max_{1\le i\le n}\|\omega_i - x\|) - 1}{\varphi'(\max_{1\le i\le n}\|\omega_i - x\|) + 1}. \tag{2.114}$$

Remark 27 Since τ defined by any of the formulas (2.108), (2.110), (2.112) and (2.114) is not in general a constant, but depends on x and ω_i $(i = 1,\ldots,n)$, checking condition (2.98) with such a τ does not seem practical. Such a check becomes meaningful in the linear setting, i.e. when φ', and consequently, also τ is constant; cf. Remark 31.

Dropping condition (2.98) in all parts of Corollary 29, we can formulate its simplified and easier to use version.

Corollary 30 Let $\varphi \in \mathscr{C}^1$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\mu > 0$ and any $x' \in X$ satisfying (2.29), one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\varphi(\max_{1\le i\le n} d(x',\Omega_i)), \delta_1[$ such that conditions (2.103) and (2.107) hold true for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.32) and (2.96), and all $x_i^* \in N^C_{\Omega_i}(\omega_i)$ $(i = 1,\ldots,n)$ satisfying (2.64);

(ii) X is Asplund, and there exists a $\lambda \in]\varphi(\max_{1\le i\le n} d(x',\Omega_i)), \delta_1[$ such that conditions (2.103) and (2.109) hold true for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.92) and (2.96), and all $x_i^* \in N^F_{\Omega_i}(\omega_i)$ $(i = 1,\ldots,n)$ satisfying (2.64);

(iii) there exists a $\lambda \in]\varphi(\max_{1\le i\le n} d(x',\Omega_i)), \delta_1[$ such that conditions (2.105) and (2.111) hold true with $\gamma := \mu^{-1}$ and N standing for the Clarke normal cone ($N := N^C$) for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.32) and (2.96), and all $x_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying $\sum_{i=1}^n x_i^* = 0$ and condition (2.64);

(iv) X is Asplund, and there exists a $\lambda \in]\varphi(\max_{1\le i\le n}\|x_i\|), \delta[$ such that conditions (2.113) and (2.105) hold true, the latter with $\gamma := \mu^{-1}$ and N standing for the Fréchet normal cone ($N := N^F$), for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.92) and (2.96), and all $x_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying $\sum_{i=1}^n x_i^* = 0$ and condition (2.64).

Transversality

We use the function $\widehat{f}$ given by (2.21) with $f : X^{n+1} \to \mathbb{R}_+$ defined by (2.20). The next statement provides dual characterizations of $\varphi-$transversality in terms of subdifferentials of $\widehat{f}$.

Proposition 15 (i) The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma > 0$ and any $\omega_i' \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$ and $\xi \in]0,\varphi^{-1}(\delta_1)[$, there exists a $\lambda \in]\varphi(\xi),\delta_1[$ such that condition (2.56) is satisfied for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.40) and (2.41), where ∂ in (2.56) stands for the Clarke subdifferential ($\partial := \partial^C$).

(ii) If X is Asplund, then the above assertion is valid with ∂ in (2.56) standing for the Fréchet subdifferential ($\partial := \partial^F$), and condition (2.41) replaced by (2.57).

Proof

(i) Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is not $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2, and let $\gamma > 0$ be given. By Theorem 5, there exist points $\omega_i' \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$ and a number $\xi \in]0,\varphi^{-1}(\delta_1)[$ such that, for any $\lambda \in]\varphi(\xi),\delta_1[$, there exist points $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$, and a number $\tau \in]0,1[$ such that conditions (2.40) and (2.41) are satisfied, and inequality (2.58) holds for all $u_i \in \Omega_i$ near ω_i $(i = 1,\ldots,n)$ and all u near x. In other words, $(\omega_1,\ldots,\omega_n,x)$ is a local minimizer of the function (2.59). By Lemma 5, its Fréchet and, as a consequence, Clarke subdifferential at this point contains 0. Observe that (2.59) is the sum of the function $\widehat{f}$ and the Lipschitz continuous convex function $(u_1,\ldots,u_n,u) \mapsto \tau\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma$, and at any point all subgradients $(x_1^*,\ldots,x_n^*,x^*)$ of the latter function satisfy $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma \le \tau$. By the Clarke–Rockafellar sum rule (Lemma 4(i)), there exists a subgradient $(x_1^*,\ldots,x_n^*,x^*) \in \partial^C\widehat{f}(\omega_1,\ldots,\omega_n,x)$ such that $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma \le \tau < 1$. The last inequality contradicts (2.56).

(ii) If X is Asplund, then one can employ the fuzzy sum rule (Lemma 4(ii)): for any $\varepsilon > 0$, there exist points $x' \in B_\varepsilon(x)$ and $y_i \in \Omega_i \cap B_\varepsilon(\omega_i)$ $(i = 1,\ldots,n)$, and a subgradient $(x_1^*,\ldots,x_n^*,x^*) \in \partial^F\widehat{f}(y_1,\ldots,y_n,x')$ such that $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma < \tau+\varepsilon$. The number ε can be chosen small enough so that $\|x'-\bar{x}\| < \lambda$, $\max_{1\le i\le n}\|y_i-\omega_i'\| < \lambda/\gamma$, $0 < \max_{1\le i\le n}\|y_i-x_i-x'\| < \varphi^{-1}(\lambda)$, and $\tau+\varepsilon < 1$. The last inequality again yields $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma < 1$, which contradicts (2.56).

$\square$

Similar to the cases of the nonlinear semitransversality and subtransversality, the difference in the contribution of components of subgradients of $\widehat{f}$ to the key dual condition (2.56) in Proposition 15 can be exposed.

Corollary 31 (i) The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma>0$ and any $\omega_i'\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\xi\in]0,\varphi^{-1}(\delta_1)[$, there exists a $\lambda\in]\varphi(\xi),\delta_1[$ such that $\|x^*\|\geq 1$ for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.40) and (2.41), and all $(x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)$ with $\sum_{i=1}^n\|x_i^*\|<\gamma$, where ∂ stands for the Clarke subdifferential ($\partial:=\partial^C$).

(ii) If X is Asplund, then the above assertion is valid with ∂ standing for the Fréchet subdifferential ($\partial:=\partial^F$), and condition (2.41) replaced by (2.57).

Proof Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is not $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2, and let $\gamma>0$ be given. By Proposition 15, there exist points $\omega_i'\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and a number $\xi\in]0,\varphi^{-1}(\delta_1)[$ such that, for any $\lambda\in]\varphi(\xi),\delta_1[$, there exist $x,x_i\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.40) and (2.41) ((2.57) if X is Asplund), and a subgradient $(x_1^*,\ldots,x_n^*,x^*)\in\partial^C\widehat{f}(\omega_1,\ldots,\omega_n,x)$ ($(x_1^*,\ldots,x_n^*,x^*)\in\partial^F\widehat{f}(\omega_1,\ldots,\omega_n,x)$ if X is Asplund) such that $\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma<1$. By the representation (2.55) of the dual norm, this implies $\sum_{i=1}^n\|x_i^*\|<\gamma$ and $\|x^*\|<1$. The latter inequality contradicts the assumption. $\square$

The next 'δ-free' statement is a direct consequence of Corollary 31.

Corollary 32 (i) The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ if

$$\lim_{\gamma\downarrow 0}\ \liminf_{\substack{\omega_i'\xrightarrow{\Omega_i}\bar{x}\ (i=1,\ldots,n)\\ \xi\downarrow 0}}\ \lim_{\lambda\downarrow\varphi(\xi)}\ \inf_{\substack{\|x-\bar{x}\|<\lambda,\ \|\omega_i-\omega_i'\|<\lambda/\gamma,\ \omega_i\in\Omega_i\ (i=1,\ldots,n)\\ 0<\max_{1\leq i\leq n}\|\omega_i-x_i-x\|\leq\xi\\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,x),\ \sum_{i=1}^n\|x_i^*\|<\gamma}}\|x^*\|>1, \tag{2.115}$$

or, in particular, if condition (2.61) is satisfied, where ∂ in both conditions stands for the Clarke subdifferential ($\partial:=\partial^C$).

(ii) If X is Asplund, then the above assertion is valid with ∂ standing for the Fréchet subdifferential ($\partial:=\partial^F$).

Remark 28 Condition (2.61) is obviously stronger than (2.115).

Next, similar to the cases of the nonlinear semitransversality and subtransversality, we apply the sum rules again to the function $\widehat{f}$ to obtain characterizations of the nonlinear transversality in terms of normals to the given individual sets.

Theorem 9 Let $\varphi\in\mathscr{C}^1$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\mu>0$ and any $\omega_i'\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\xi\in]0,\varphi^{-1}(\delta_1)[$, one of the following conditions is satisfied:

(i) there exists a $\lambda\in]\varphi(\xi),\delta_1[$ such that inequality (2.62) holds for all $x,x_i\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.41) and

$$\|x-\bar{x}\|<\lambda,\quad \max_{1\leq i\leq n}\left\|\omega_i-\omega_i'\right\|<\mu\lambda, \tag{2.116}$$

and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64) and (2.65), where N stands for the Clarke normal cone ($N:=N^C$);

(ii) X is Asplund, and there exist a $\lambda \in]\varphi(\xi), \delta_1[$ and a $\tau \in]0,1[$ such that inequality (2.62) holds for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.57) and (2.116), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64) and (2.66), where N stands for the Fréchet normal cone ($N := N^F$).

Proof Suppose $\{\Omega_1, \ldots, \Omega_n\}$ is not $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2, and let $\mu > 0$ be given. Set $\gamma := \mu^{-1}$. By Proposition 15, there exist points $\omega_i' \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1, \ldots, n)$ and a $\xi \in]0, \varphi^{-1}(\delta_1)[$ such that, for any $\lambda \in]\varphi(\xi), \delta_1[$, there exist points $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying conditions (2.41) ((2.57) if X is Asplund) and (2.116), and a subgradient $(\hat{x}_1^*, \ldots, \hat{x}_n^*, \hat{x}^*) \in \partial \widehat{f}(\omega_1, \ldots, \omega_n, x)$ such that condition (2.67) holds, where ∂ stands for either the Clarke subdifferential (if X is a general Banach space) or the Fréchet subdifferential (if X is Asplund). The rest of the proof follows that of Theorem 7. $\square$

Remark 29 The sufficient conditions in Theorems 7 and 8 correspond to setting, respectively, $\omega_i' := \bar{x}$ $(i = 1, \ldots, n)$ and $x_1 = \ldots = x_n$ in the sufficient conditions in Theorem 9.

The next corollary 'separates' the two conditions on the dual vectors $x_i^* \in X^*$ $(i = 1, \ldots, n)$ combined in the key dual transversality condition (2.62) in Theorem 9.

Corollary 33 Let $\varphi \in \mathscr{C}^1$. The collection $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma > 0$ and any $\omega_i' \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1, \ldots, n)$ and $\xi \in]0, \varphi^{-1}(\delta_1)[$, one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\varphi(\xi), \delta_1[$ such that inequality (2.80) holds for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.40) and (2.41), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.65) and (2.81), where N stands for the Clarke normal cone ($N := N^C$);

(ii) X is Asplund, and there exist a $\lambda \in]\varphi(\xi), \delta_1[$ and a $\tau \in]0,1[$ such that inequality (2.80) holds for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.40) and (2.57), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.66) and (2.81), where N stands for the Fréchet normal cone ($N := N^F$);

(iii) there exists a $\lambda \in]\varphi(\xi), \delta_1[$ such that inequality (2.82) holds for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.40) and (2.41), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.65) and (2.83), where N stands for the Clarke normal cone ($N := N^C$);

(iv) X is Asplund, and there exist a $\lambda \in]\varphi(\xi), \delta_1[$ and a $\tau \in]0,1[$ such that inequality (2.82) holds for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.40) and (2.57), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.66) and (2.83), where N stands for the Fréchet normal cone ($N := N^F$).

Proof It is sufficient to notice that the assumptions in parts (i)–(iv) imply those in the respective parts of Theorem 9; cf. the proof of Corollary 23. $\square$

The next corollary complements Corollary 33 and accommodates for the two 'exact' cases $x_i^* \in N_{\Omega_i}(\omega_i)$ $(i = 1,\dots,n)$ and $\sum_{i=1}^n x_i^* = 0$. They are consequences of Theorem 9 and Lemma 10; cf. the proof of Corollary 24.

Corollary 34 Let $\varphi \in \mathscr{C}^1$. The collection $\{\Omega_1,\dots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2 if, for some $\mu > 0$ and any $\omega_i' \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1,\dots,n)$ and $\xi \in]0, \varphi^{-1}(\delta_1)[$, one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\varphi(\xi), \delta_1[$ such that conditions (2.84) and (2.80) hold true for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.41) and (2.116), and all $x_i^* \in N^C_{\Omega_i}(\omega_i)$ $(i = 1,\dots,n)$ satisfying (2.64) and (2.66) with τ defined by (2.85);

(ii) X is Asplund, and there exist a $\lambda \in]\varphi(\xi), \delta_1[$ and a $\hat{\tau} \in]0,1[$ such that conditions (2.86) and (2.80) hold true for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.57) and (2.116), and all $x_i^* \in N^F_{\Omega_i}(\omega_i)$ $(i = 1,\dots,n)$ satisfying (2.64) and (2.66) with τ defined by (2.87);

(iii) there exists a $\lambda \in]\varphi(\xi), \delta_1[$ such that conditions (2.88) and (2.82) hold true with $\gamma := \mu^{-1}$ and N standing for the Clarke normal cone ($N := N^C$) for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.41) and (2.116), and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying $\sum_{i=1}^n x_i^* = 0$, and conditions (2.64) and (2.66) with τ defined by (2.89);

(iv) X is Asplund, and there exist a $\lambda \in]\varphi(\xi), \delta_1[$ and a $\hat{\tau} \in]0,1[$ such that conditions (2.90) and (2.82) hold true with $\gamma := \mu^{-1}$ and N standing for the Fréchet normal cone ($N := N^F$) for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.57) and (2.116), and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying $\sum_{i=1}^n x_i^* = 0$, and conditions (2.64) and (2.66) with τ defined by (2.91).

Remark 30 In all parts of Corollary 34, checking condition (2.66) (and the mentioning of $\hat{\tau} \in]0,1[$ in parts (ii) and (iv)) can be dropped.

Hölder Transversality Properties

In this section, we consider the most important realizations of the three nonlinear transversality properties corresponding to the Hölder setting, i.e. φ being a power function, given for all $t \ge 0$ by $\varphi(t) := \alpha^{-1} t^q$ with $\alpha > 0$ and $q > 0$ ($q \in]0,1]$ in the case of Hölder subtransversality and Hölder transversality). If not explicitly stated otherwise, the space X is assumed to be Banach, and the sets $\Omega_1,\dots,\Omega_n$ are assumed closed.

The next three statements are direct consequences of Theorem 7, and Corollaries 23 and 24, respectively.

Corollary 35 Let $\alpha > 0$ and $q > 0$. The collection $\{\Omega_1, \ldots, \Omega_n\}$ is $\alpha-$semitransversal of order q at $\bar{x}$ with δ if, for some $\mu > 0$ and any $x_i \in X$ $(i = 1, \ldots, n)$ satisfying

$$0 < \max_{1 \le i \le n} \|x_i\| < (\alpha\delta)^{\frac{1}{q}}, \tag{2.117}$$

one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n}\|x_i\|)^q, \delta[$ such that

$$q\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{q-1}\left(\left\|\sum_{i=1}^{n} x_i^*\right\| + \mu\sum_{i=1}^{n} d\left(x_i^*, N_{\Omega_i}(\omega_i)\right)\right) \ge \alpha \tag{2.118}$$

for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.19) and (2.63), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64) and (2.65), where N in (2.118) stands for the Clarke normal cone $(N := N^C)$;

(ii) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n}\|x_i\|)^q, \delta[$ and a $\tau \in]0, 1[$ such that inequality (2.118) holds with N standing for the Fréchet normal cone $(N := N^F)$ for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.63) and

$$0 < \max_{1\le i\le n}\|\omega_i - x_i - x\| < (\alpha\lambda)^{\frac{1}{q}}, \tag{2.119}$$

and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64) and (2.66).

Corollary 36 Let $\alpha > 0$ and $q > 0$. The collection $\{\Omega_1, \ldots, \Omega_n\}$ is $\alpha-$semitransversal at $\bar{x}$ of order q with δ if, for some $\gamma > 0$ and any $x_i \in X$ $(i = 1, \ldots, n)$ satisfying (2.117), one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n}\|x_i\|)^q, \delta[$ such that

$$q\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{q-1}\left\|\sum_{i=1}^{n} x_i^*\right\| \ge \alpha \tag{2.120}$$

for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.18) and (2.19), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.65) and

$$q\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{q-1}\sum_{i=1}^{n} d\left(x_i^*, N_{\Omega_i}(\omega_i)\right) < \alpha\gamma, \tag{2.121}$$

where N stands for the Clarke normal cone $(N := N^C)$;

(ii) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n}\|x_i\|)^q, \delta[$ and a $\tau \in]0, 1[$ such that inequality (2.120) holds for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.18) and (2.119), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.66) and (2.121), where N stands for the Fréchet normal cone $(N := N^F)$;

(iii) there exists a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n}\|x_i\|)^q,\delta[$ such that

$$q\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{q-1}\sum_{i=1}^n d\left(x_i^*, N_{\Omega_i}(\omega_i)\right) \ge \alpha\gamma \tag{2.122}$$

for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.18) and (2.19), and all $x_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying (2.64), (2.65) and

$$q\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{q-1}\left\|\sum_{i=1}^n x_i^*\right\| < \alpha, \tag{2.123}$$

where N in (2.122) stands for the Clarke normal cone ($N := N^C$);

(iv) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n}\|x_i\|)^q,\delta[$ and a $\tau \in]0,1[$ such that inequality (2.122) holds for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.18) and (2.119), and all $x_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying (2.64), (2.66) and (2.123), where N in (2.122) stands for the Fréchet normal cone ($N := N^F$).

Corollary 37 Let $\alpha > 0$ and $q > 0$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\alpha-$semitransversal of order q at $\bar{x}$ with δ if, for some $\mu > 0$ and any $x_i \in X$ $(i = 1,\ldots,n)$ satisfying (2.117), one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n}\|x_i\|)^q,\delta[$ such that

$$\alpha q^{-1}\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{1-q} + 1 \le \mu, \tag{2.124}$$

and condition (2.120) holds true for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.19) and (2.63), and all $x_i^* \in N^C_{\Omega_i}(\omega_i)$ $(i = 1,\ldots,n)$ satisfying conditions (2.64) and (2.66) with

$$\tau := \frac{\mu q\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{q-1} - \alpha}{\mu q\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{q-1} + \alpha}; \tag{2.125}$$

(ii) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n}\|x_i\|)^q,\delta[$ and a $\hat{\tau} \in]0,1[$ such that condition (2.120) holds true and

$$\alpha q^{-1}\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{1-q} + 1 \le \hat{\tau}\mu \tag{2.126}$$

for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.63) and (2.119), and all $x_i^* \in N^F_{\Omega_i}(\omega_i)$ $(i = 1,\ldots,n)$ satisfying conditions (2.64) and (2.66) with

$$\tau := \frac{\hat{\tau}\mu q\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{q-1} - \alpha}{\mu q\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{q-1} + \alpha}; \tag{2.127}$$

(iii) there exists a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n}\|x_i\|)^q, \delta[$ such that

$$\alpha q^{-1}\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{1-q} + \mu \le 1, \tag{2.128}$$

and condition (2.122) holds true with $\gamma := \mu^{-1}$ and N standing for the Clarke normal cone ($N := N^C$) for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.19) and (2.63), and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying $\sum_{i=1}^n x_i^* = 0$, and conditions (2.64) and (2.66) with

$$\tau := \frac{q(\max_{1\le i\le n}\|\omega_i - x_i - x\|)^{q-1} - \alpha}{q(\max_{1\le i\le n}\|\omega_i - x_i - x\|)^{q-1} + \alpha}; \tag{2.129}$$

(iv) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n}\|x_i\|)^q, \delta[$ and a $\hat{\tau} \in]0,1[$ such that condition (2.122) holds true with $\gamma := \mu^{-1}$ and N standing for the Fréchet normal cone ($N := N^F$), and

$$\alpha q^{-1}\left(\max_{1\le i\le n}\|\omega_i - x_i - x\|\right)^{1-q} + \mu \le \hat{\tau} \tag{2.130}$$

for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.63) and (2.119), and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying $\sum_{i=1}^n x_i^* = 0$, and conditions (2.64) and (2.66) with

$$\tau := \frac{\hat{\tau} q(\max_{1\le i\le n}\|\omega_i - x_i - x\|)^{q-1} - \alpha}{q(\max_{1\le i\le n}\|\omega_i - x_i - x\|)^{q-1} + \alpha}. \tag{2.131}$$

Remark 31 Since τ defined by any of the formulas (2.125), (2.127), (2.129) and (2.131) is not in general a constant, but depends on x, ω_i and x_i $(i = 1,\dots,n)$, checking condition (2.66) with such a τ does not seem practical unless $q = 1$. In the latter case, formulas (2.125), (2.127), (2.129) and (2.131) reduce, respectively, to

$$\tau := \frac{\mu - \alpha}{\mu + \alpha}, \quad \tau := \frac{\hat{\tau}\mu - \alpha}{\mu + \alpha}, \quad \tau := \frac{1 - \alpha}{1 + \alpha} \quad \text{and} \quad \tau := \frac{\hat{\tau} - \alpha}{1 + \alpha}. \tag{2.132}$$

In the general nonlinear case, condition (2.66) can be dropped, leading to more concise but stronger sufficient conditions requiring a larger number of candidates to be checked.

The next three statements are direct consequences of Theorem 8, and Corollaries 28 and 29, respectively.

Corollary 38 Let $\alpha > 0$ and $q \in]0,1]$. The collection $\{\Omega_1,\dots,\Omega_n\}$ is $\alpha-$subtransversal of order q at $\bar{x}$ with δ_1 and δ_2 if, for some $\mu > 0$ and any $x' \in X$ satisfying

$$\|x' - \bar{x}\| < \delta_2, \quad 0 < \max_{1\le i\le n} d(x', \Omega_i) < (\alpha\delta_1)^{\frac{1}{q}}, \tag{2.133}$$

one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n} d(x',\Omega_i))^q, \delta_1[$ such that

$$q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{q-1}\left(\left\|\sum_{i=1}^n x_i^*\right\| + \mu\sum_{i=1}^n d\left(x_i^*, N_{\Omega_i}(\omega_i)\right)\right) \ge \alpha \tag{2.134}$$

for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.96) and

$$0 < \max_{1\le i\le n}\|\omega_i - x\| \le \max_{1\le i\le n}\|\omega_i' - x'\| < (\alpha\lambda)^{\frac{1}{q}}, \tag{2.135}$$

and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying (2.64) and (2.97), where N in (2.134) stands for the Clarke normal cone ($N := N^C$);

(ii) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n} d(x',\Omega_i))^q, \delta_1[$ and a $\tau \in]0,1[$ such that inequality (2.134) holds with N standing for the Fréchet normal cone ($N := N^F$) for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.96) and

$$0 < \max_{1\le i\le n}\|\omega_i - x\| < (\alpha\lambda)^{\frac{1}{q}}, \tag{2.136}$$

and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying (2.64) and (2.98).

Corollary 39 Let $\alpha > 0$ and $q \in]0,1]$. The collection $\{\Omega_1,\dots,\Omega_n\}$ is $\alpha-$subtransversal of order q at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma > 0$ and any $x' \in X$ satisfying (2.133), one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n} d(x',\Omega_i))^q, \delta_1[$ such that

$$q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{q-1}\left\|\sum_{i=1}^n x_i^*\right\| \ge \alpha \tag{2.137}$$

for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.31) and (2.135), and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying (2.64), (2.97) and

$$q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{q-1}\sum_{i=1}^n d\left(x_i^*, N_{\Omega_i}(\omega_i)\right) < \alpha\gamma, \tag{2.138}$$

where N stands for the Clarke normal cone ($N := N^C$);

(ii) there exists a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n} d(x',\Omega_i))^q, \delta_1[$ such that

$$q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{q-1}\sum_{i=1}^n d\left(x_i^*, N_{\Omega_i}(\omega_i)\right) \ge \alpha\gamma \tag{2.139}$$

for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.31) and (2.135), and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying (2.64), (2.97) and

$$q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{q-1}\left\|\sum_{i=1}^n x_i^*\right\| < \alpha, \tag{2.140}$$

where N in (2.139) stands for the Clarke normal cone ($N := N^C$);

(iii) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}\left(\max_{1\le i\le n} d(x',\Omega_i)\right)^q, \delta_1[$ and a $\tau \in]0,1[$ such that inequality (2.137) holds for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.31) and (2.136), and all $x_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying (2.64), (2.98) and (2.138), where N stands for the Fréchet normal cone $(N := N^F)$;

(iv) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}\left(\max_{1\le i\le n} d(x',\Omega_i)\right)^q, \delta_1[$ and a $\tau \in]0,1[$ such that inequality (2.139) holds for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.31) and (2.136), and all $x_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying (2.64), (2.98) and (2.140), where N in (2.139) stands for the Fréchet normal cone $(N := N^F)$.

Remark 32 The sufficient condition in part (i) of Corollary 39 improves [141, Theorem 2].

Corollary 40 Let $\alpha > 0$ and $q \in]0,1]$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\alpha-$subtransversal of order q at $\bar{x}$ with δ_1 and δ_2 if, for some $\mu > 0$ and any $x' \in X$ satisfying (2.133), one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\alpha^{-1}\left(\max_{1\le i\le n} d(x',\Omega_i)\right)^q, \delta_1[$ such that

$$\alpha q^{-1}\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{1-q} + 1 \le \mu, \tag{2.141}$$

and condition (2.137) holds true for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.96) and (2.135), and all $x_i^* \in N^C_{\Omega_i}(\omega_i)$ $(i = 1,\ldots,n)$ satisfying (2.64) and (2.98) with

$$\tau := \frac{\mu q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^q - \alpha}{\mu q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{q-1} + \alpha}; \tag{2.142}$$

(ii) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}\left(\max_{1\le i\le n} d(x',\Omega_i)\right)^q, \delta_1[$ and a $\hat{\tau} \in]0,1[$ such that condition (2.137) holds true and

$$\alpha q^{-1}\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{1-q} + 1 \le \hat{\tau}\mu \tag{2.143}$$

for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.96) and (2.136), and all $x_i^* \in N^F_{\Omega_i}(\omega_i)$ $(i = 1,\ldots,n)$ satisfying (2.64) and (2.98) with

$$\tau := \frac{\hat{\tau}\mu q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^q - \alpha}{\mu q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{q-1} + \alpha}. \tag{2.144}$$

(iii) there exists a $\lambda \in]\alpha^{-1}\left(\max_{1\le i\le n} d(x',\Omega_i)\right)^q, \delta_1[$ such that

$$\alpha q^{-1}\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{1-q} + \mu \le 1, \tag{2.145}$$

and condition (2.139) holds true with $\gamma := \mu^{-1}$ and N standing for the Clarke normal cone $(N := N^C)$ for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying (2.96) and (2.135), and all $x_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying $\sum_{i=1}^n x_i^* = 0$, and conditions (2.64) and (2.98) with

$$\tau := \frac{q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{q-1} - \alpha}{q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{q-1} + \alpha}; \tag{2.146}$$

(iv) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}(\max_{1\le i\le n}\|x_i\|)^q, \delta[$ and a $\hat{\tau} \in]0,1[$ such that condition (2.139) holds true with $\gamma := \mu^{-1}$ and N standing for the Fréchet normal cone ($N := N^F$), and

$$\alpha q^{-1}\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{1-q} + \mu \le \hat{\tau} \tag{2.147}$$

for all $x \in X$ and $\omega_i, \omega_i' \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.96) and (2.136), and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying $\sum_{i=1}^n x_i^* = 0$, and conditions (2.64) and (2.98) with

$$\tau := \frac{\hat{\tau} q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{q-1} - \alpha}{q\left(\max_{1\le i\le n}\|\omega_i - x\|\right)^{q-1} + \alpha}. \tag{2.148}$$

Remark 33 Since τ defined by any of the formulas (2.142), (2.144), (2.146) and (2.148) is not in general a constant, but depends on x and ω_i $(i = 1,\dots,n)$, checking condition (2.98) with such a τ does not seem practical and can be dropped (at the expense of weakening the assertions). Such a check becomes meaningful in the linear setting, i.e. when $q = 1$. In this case, (2.142), (2.144), (2.146) and (2.148) reduce to the corresponding expressions in (2.132).

The next three statements are direct consequences of Theorem 9, and Corollaries 33 and 34, respectively.

Corollary 41 Let $\alpha > 0$ and $q \in]0,1]$. The collection $\{\Omega_1,\dots,\Omega_n\}$ is $\alpha-$transversal of order q at $\bar{x}$ with δ_1 and δ_2 if, for some $\mu > 0$ and any $\omega_i' \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1,\dots,n)$ and $\xi \in]0,(\alpha\delta_1)^{\frac{1}{q}}[$, one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\alpha^{-1}\xi^q, \delta_1[$ such that inequality (2.118) holds for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.41) and (2.116), and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying (2.64) and (2.65), where N stands for the Clarke normal cone ($N := N^C$);

(ii) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}\xi^q, \delta_1[$ and a $\tau \in]0,1[$ such that inequality (2.118) holds for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.116) and (2.119), and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying (2.64) and (2.66), where N stands for the Fréchet normal cone ($N := N^F$).

Corollary 42 Let $\alpha > 0$ and $q \in]0,1]$. The collection $\{\Omega_1,\dots,\Omega_n\}$ is $\alpha-$transversal of order q at $\bar{x}$ with δ_1 and δ_2 if, for some $\gamma > 0$ and any $\omega_i' \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $\xi \in]0,(\alpha\delta_1)^{\frac{1}{q}}[$, one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\alpha^{-1}\xi^q, \delta_1[$ such that inequality (2.120) holds for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\dots,n)$ satisfying (2.40) and (2.41), and all $x_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying (2.64), (2.65) and (2.121), where N stands for the Clarke normal cone ($N := N^C$);

(ii) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}\xi^q, \delta_1[$ and a $\tau \in]0,1[$ such that inequality (2.120) holds for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.40) and (2.119), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.66) and (2.121), where N stands for the Fréchet normal cone ($N := N^F$);

(iii) there exists a $\lambda \in]\alpha^{-1}\xi^q, \delta_1[$ such that inequality (2.122) holds for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.40) and (2.41), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.65) and (2.123), where N stands for the Clarke normal cone ($N := N^C$);

(iv) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}\xi^q, \delta_1[$ and a $\tau \in]0,1[$ such that inequality (2.122) holds for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.40) and (2.119), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.66) and (2.123), where N stands for the Fréchet normal cone ($N := N^F$).

Corollary 43 Let $\alpha > 0$ and $q \in]0,1]$. The collection $\{\Omega_1, \ldots, \Omega_n\}$ is α−transversal of order q at $\bar{x}$ with δ_1 and δ_2 if, for some $\mu > 0$ and any $\omega_i' \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1, \ldots, n)$ and $\xi \in]0, (\alpha\delta_1)^{\frac{1}{q}}[$, one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\alpha^{-1}\xi^q, \delta_1[$ such that conditions (2.120) and (2.124) hold true for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.41) and (2.116), and all $x_i^* \in N^C_{\Omega_i}(\omega_i)$ $(i = 1, \ldots, n)$ satisfying (2.64) and (2.66) with τ defined by (2.125);

(ii) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}\xi^q, \delta_1[$ and a $\hat{\tau} \in]0,1[$ such that conditions (2.120) and (2.126) hold true for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.116) and (2.119), and all $x_i^* \in N^F_{\Omega_i}(\omega_i)$ $(i = 1, \ldots, n)$ satisfying (2.64) and (2.66) with τ defined by (2.127);

(iii) there exists a $\lambda \in]\alpha^{-1}\xi^q, \delta_1[$ such that conditions (2.122) and (2.128) hold true with $\gamma := \mu^{-1}$ and N standing for the Clarke normal cone ($N := N^C$) for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.41) and (2.116), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying $\sum_{i=1}^n x_i^* = 0$, and conditions (2.64) and (2.66) with τ defined by (2.129);

(iv) X is Asplund, and there exist a $\lambda \in]\alpha^{-1}\xi^q, \delta_1[$ and a $\hat{\tau} \in]0,1[$ such that conditions (2.122) and (2.130) hold true with $\gamma := \mu^{-1}$ and N standing for the Fréchet normal cone ($N := N^F$) for all $x, x_i \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying conditions (2.116) and (2.119), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying $\sum_{i=1}^n x_i^* = 0$, and conditions (2.64) and (2.66) with τ defined by (2.131).

2.5.2 Dual Necessary Conditions

Semitransversality

In this and subsequent sections, the sets $\Omega_1, \ldots, \Omega_n$ and function $\varphi \in \mathscr{C}$ are assumed to be convex.

We are going to use the dual norm (2.55) and the function $\widehat{f}$ given by (2.21) with $f\colon X^{n+1} \to \mathbb{R}_+$ defined by (2.20). The next statement provides a dual necessary condition for φ-semitransversality in terms of subdifferentials of $\widehat{f}$.

Proposition 16 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x} \in \cap_{i=1}^n \Omega_i$, and $\varphi \in \mathscr{C}$ be convex with $\varphi'_+(0) > 0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta > 0$, then

$$d_\gamma\big(0, \partial\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})\big) \geq 1 \tag{2.149}$$

for $\gamma := ((\varphi'_+(0))^{-1}+1)^{-1}$, and all $x_i \in X$ $(i=1,\ldots,n)$ satisfying (2.16).

Proof Under the assumptions made, the function $\widehat{f}$ is convex. The assertion follows from Proposition 3(ii) since condition (2.149) is a direct consequence of (2.49). Indeed, for all $x_i \in X$ $(i=1,\ldots,n)$ satisfying (2.16) and all $(x_1^*,\ldots,x_n^*,x^*) \in \partial\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})$, we have:

$$\begin{aligned}
\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma &= \sup_{(u_1,\ldots,u_n,u)\neq 0} \frac{\langle (x_1^*,\ldots,x_n^*,x^*),(u_1,\ldots,u_n,u)\rangle}{\|(u_1,\ldots,u_n,u)\|_\gamma} \\
&= \limsup_{\substack{u_i\to\bar{x}\,(i=1,\ldots,n),\; u\to\bar{x}\\ (u_1,\ldots,u_n,u)\neq(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{-\langle (x_1^*,\ldots,x_n^*,x^*),(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\rangle}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma} \\
&\geq \limsup_{\substack{u_i\to\bar{x}\,(i=1,\ldots,n),\; u\to\bar{x}\\ (u_1,\ldots,u_n,u)\neq(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})-\widehat{f}(u_1,\ldots,u_n,u)}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma} \\
&= \limsup_{\substack{u_i\xrightarrow{\Omega_i}\bar{x}\,(i=1,\ldots,n),\; u\to\bar{x}\\ (u_1,\ldots,u_n,u)\neq(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|x_i\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma} \geq 1.
\end{aligned}$$

□

Remark 34 Condition (2.149) in Proposition 16 is required to hold for all $x_i \in X$ $(i=1,\ldots,n)$ satisfying (2.16). Observe that vectors x_i $(i=1,\ldots,n)$ are not explicitly present in (2.149); they are involved in the definition (2.20) of the function f.

The key condition (2.149) in Proposition 16 involves a subdifferential of the function $\widehat{f}$ given by (2.21) with $f\colon X^{n+1} \to \mathbb{R}_+$ defined by (2.20). Subgradients of $\widehat{f}$ belong to $(X^*)^{n+1}$ and have $n+1$ component vectors $x_1^*,\ldots,x_n^*,x^*$. As it can be seen from the representation (2.55) of the dual norm, the contribution of the vectors $x_1^*,\ldots,x_n^*$ on one hand, and x^* on the other hand to condition (2.149) is different. The next corollary exposes this difference.

Corollary 44 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x} \in \cap_{i=1}^n \Omega_i$, and $\varphi \in \mathscr{C}$ be convex with $\varphi'_+(0) > 0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-semitransversal at $\bar{x}$ with some $\delta > 0$, then, for all $x_i \in X$ $(i=1,\ldots,n)$ satisfying (2.16), and all $(x_1^*,\ldots,x_n^*,x^*) \in \partial\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})$, it holds

$$\|x^*\| \geq 1 - ((\varphi'_+(0))^{-1}+1)\sum_{i=1}^n \|x_i^*\|. \tag{2.150}$$

As a consequence,

$$\liminf_{\substack{x_i^*\to 0\ (i=1,\ldots,n)\\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})}} \|x^*\| \geq 1.$$

Proof The assertion is a direct consequence of Proposition 16 and the representation (2.55) of the dual norm. $\square$

The next 'δ-free' statement is a direct consequence of Corollary 44.

Corollary 45 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-semitransversal at $\bar{x}$, then

$$\liminf_{\substack{x_i\to 0,\ x_i^*\to 0\ (i=1,\ldots,n)\\ \max\limits_{1\le i\le n}\|x_i\|>0,\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})}} \|x^*\| \geq 1.$$

In the convex setting, a partial converse to Theorem 7 is possible.

Theorem 10 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-semitransversal at $\bar{x}$ with some $\delta>0$, then, with $\mu:=(\varphi'_+(0))^{-1}+1$,

$$\varphi'\left(\max_{1\le i\le n}\|x_i\|\right)\left(\left\|\sum_{i=1}^n x_i^*\right\|+\mu\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\bar{x})\right)\right)\geq 1 \tag{2.151}$$

for all $x_i\in X$ $(i=1,\ldots,n)$ satisfying (2.16) and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64) and

$$\sum_{i=1}^n\langle x_i^*,x_i\rangle=\max_{1\le i\le n}\|x_i\|. \tag{2.152}$$

Proof Let $\{\Omega_1,\ldots,\Omega_n\}$ be φ-semitransversal at $\bar{x}$ with some $\delta>0$. Let $\mu:=(\varphi'_+(0))^{-1}+1$ and $\gamma:=\mu^{-1}$. By Proposition 16, condition (2.149) is satisfied for all $x_i\in X$ $(i=1,\ldots,n)$ satisfying (2.16). Observe that $\widehat{f}$ is a sum of the function f given by (2.20) and the indicator function of the set $\Omega_1\times\ldots\times\Omega_n$. Since $\max_{1\le i\le n}\|x_i\|>0$, f is locally Lipschitz continuous near $(\bar{x},\ldots,\bar{x},\bar{x})$. It is a composition of φ and the function

$$\psi(u_1,\ldots,u_n,u):=\max_{1\le i\le n}\|u_i-x_i-u\|,\quad u_1,\ldots,u_n,u\in X. \tag{2.153}$$

By Lemmas 4, 6 and 9, and Remark 5,

$$\begin{aligned}\partial\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})&=\partial f(\bar{x},\ldots,\bar{x},\bar{x})+N_{\Omega_1\times\ldots\times\Omega_n}(\bar{x},\ldots,\bar{x})\times\{0\}\\&=\varphi'\Big(\max_{1\le i\le n}\|x_i\|\Big)\big(\partial\psi(\bar{x},\ldots,\bar{x},\bar{x})+N_{\Omega_1}(\bar{x})\times\ldots\times N_{\Omega_n}(\bar{x})\times\{0\}\big).\end{aligned}$$

By Lemma 8, $(-x_1^*,\ldots,-x_n^*,x^*)\in\partial\psi(\bar{x},\ldots,\bar{x},\bar{x})$ if and only if conditions (2.64) and (2.152) are satisfied and

$$x^*=\sum_{i=1}^n x_i^*. \tag{2.154}$$

Hence, condition (2.151) is a consequence of (2.149). $\square$

Remark 35 (i) The equality $\mu := (\varphi'_+(0))^{-1} + 1$ in Theorem 10 can be replaced by the inequality $\mu \geq (\varphi'_+(0))^{-1} + 1$.

(ii) Conditions (2.151) and (2.152) in Theorem 10 are particular cases of conditions (2.62) and (2.65) in Theorem 7, respectively, corresponding to setting $\omega_1 = \ldots = \omega_n = x := \bar{x}$.

From Theorems 7 and 10, we deduce the following corollary.

Corollary 46 Let $\Omega_1, \ldots, \Omega_n$ be convex subsets of a normed space X, $\bar{x} \in \cap_{i=1}^n \Omega_i$, and $\varphi \in \mathscr{C}^1$ be convex with $\varphi'_+(0) > 0$.

(i) Suppose X is Banach and $\Omega_1, \ldots, \Omega_n$ are closed. $\{\Omega_1, \ldots, \Omega_n\}$ is φ-semitransversal at $\bar{x}$ if inequality (2.62) holds with some $\mu > 0$ for all $x \in X$ and $\omega_i \in \Omega_i$ near $\bar{x}$, and $x_i \in X$ near 0 $(i = 1, \ldots, n)$ satisfying (2.19), and all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64) and (2.65).

(ii) If $\{\Omega_1, \ldots, \Omega_n\}$ is φ-semitransversal at $\bar{x}$, then inequality (2.151) holds with $\mu := (\varphi'_+(0))^{-1} + 1$ for all $x_i \in X$ near 0 $(i = 1, \ldots, n)$ satisfying $\max_{1 \leq i \leq n} \|x_i\| > 0$, and all $x_i^* \in X^*$ $(i = 1, ..., n)$ satisfying (2.64) and (2.152).

A decomposition of the dual necessary transversality condition (2.151) in Theorem 10 can be easily obtained.

Corollary 47 Let $\Omega_1, \ldots, \Omega_n$ be convex subsets of a normed space X, $\bar{x} \in \cap_{i=1}^n \Omega_i$, and $\varphi \in \mathscr{C}^1$ be convex with $\varphi'_+(0) > 0$. If $\{\Omega_1, \ldots, \Omega_n\}$ is φ-semitransversal at $\bar{x}$ with some $\delta > 0$, then, for all $x_i \in X$ $(i = 1, \ldots, n)$ satisfying (2.16), the following conditions hold true:

(i) for all $x_i^* \in N_{\Omega_i}(\bar{x})$ $(i = 1, \ldots, n)$ satisfying (2.64) and (2.152), it holds

$$\varphi'\left(\max_{1 \leq i \leq n} \|x_i\|\right) \left\| \sum_{i=1}^n x_i^* \right\| \geq 1;$$

(ii) for all $x_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (2.64), (2.152) and $\sum_{i=1}^n x_i^* = 0$, it holds

$$\varphi'\left(\max_{1 \leq i \leq n} \|x_i\|\right) \sum_{i=1}^n d\left(x_i^*, N_{\Omega_i}(\bar{x})\right) \geq ((\varphi'_+(0))^{-1} + 1)^{-1}.$$

Subtransversality

In this subsection, we use the function $\widehat{f}$ given by (2.21) with $f\colon X^{n+1} \to \mathbb{R}_+$ defined by (2.33). The next statement provides a dual necessary condition for φ-subtransversality in terms of subdifferentials of $\widehat{f}$.

Proposition 17 Let $\Omega_1, \ldots, \Omega_n$ be convex subsets of a normed space X, $\bar{x} \in \cap_{i=1}^n \Omega_i$, and $\varphi \in \mathscr{C}$ be convex with $\varphi'_+(0) > 0$. If $\{\Omega_1, \ldots, \Omega_n\}$ is φ-subtransversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then

$$d_\gamma\left(0, \partial \widehat{f}(\omega_1, \ldots, \omega_n, x)\right) \geq 1 \tag{2.155}$$

for $\gamma := ((\varphi'_+(0))^{-1} + 1)^{-1}$, and all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ satisfying (2.46).

Proof Under the assumptions made, the function $\widehat{f}$ is convex. The assertion follows from Proposition 4(ii) since condition (2.155) is a direct consequence of (2.50); cf. the proof of Proposition 16. $\square$

Similar to the case of nonlinear semitransversality, the difference in the contribution of components of subgradients of $\widehat{f}$ to the key dual condition (2.155) in Proposition 14 can be exposed.

Corollary 48 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-subtransversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then, for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.46), and all $(x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)$, condition (2.150) holds. As a consequence,

$$\liminf_{\substack{x_i^*\to 0\ (i=1,\ldots,n)\\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)}}\|x^*\|\geq 1.$$

Proof The assertion is a direct consequence of Proposition 14 and the representation (2.55) of the dual norm. $\square$

The next 'δ-free' statement is a direct consequence of Corollary 48.

Corollary 49 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-subtransversal at $\bar{x}$, then

$$\liminf_{\substack{x\to\bar{x},\ \omega_i\xrightarrow{\Omega_i}\bar{x},\ x_i^*\to 0\ (i=1,\ldots,n)\\ \max\limits_{1\leq i\leq n}\|\omega_i-x\|>0,\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)}}\|x^*\|\geq 1.$$

In the convex setting, a partial converse to Theorem 8 is possible.

Theorem 11 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-subtransversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then, with $\mu:=(\varphi'_+(0))^{-1}+1$, inequality (2.95) holds for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.46), and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64) and (2.97).

Proof Let $\{\Omega_1,\ldots,\Omega_n\}$ be φ-subtransversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$. Let $\mu:=(\varphi'_+(0))^{-1}+1$ and $\gamma:=\mu^{-1}$. By Proposition 14, condition (2.155) is satisfied for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.46). Observe that $\widehat{f}$ is a sum of the function f given by (2.33) and the indicator function of the set $\Omega_1\times\ldots\times\Omega_n$. Since $\max_{1\leq i\leq n}\|\omega_i-x\|>0$, f is locally Lipschitz continuous near $(\omega_1,\ldots,\omega_n,x)$. It is a composition of φ and the function

$$\psi(u_1,\ldots,u_n,u):=\max_{1\leq i\leq n}\|u_i-u\|,\quad u_1,\ldots,u_n,u\in X.$$

By Lemmas 4, 6 and 9, and Remark 5,

$$\begin{aligned}\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)&=\partial f(\omega_1,\ldots,\omega_n,x)+N_{\Omega_1\times\ldots\times\Omega_n}(\omega_1,\ldots,\omega_n)\times\{0\}\\ &=\varphi'\Big(\max_{1\leq i\leq n}\|\omega_i-x\|\Big)\left(\partial\psi(\omega_1,\ldots,\omega_n,x)+N_{\Omega_1}(\omega_1)\times\ldots\times N_{\Omega_n}(\omega_n)\times\{0\}\right).\end{aligned}$$

By Lemma 8, $(-x_1^*,...,-x_n^*,x^*)\in\partial\psi(\omega_1,...,\omega_n,x)$ if and only if conditions (2.64), (2.97) and (2.154) are satisfied. Hence, condition (3.132) is a consequence of (2.155). $\square$

Remark 36 (i) The equality $\mu := (\varphi'_+(0))^{-1}+1$ in Theorem 11 can be replaced by the inequality $\mu \geq (\varphi'_+(0))^{-1}+1$.

(ii) The necessary conditions in Theorem 11 correspond to setting $\omega_i = \omega'_i$ $(i=1,\ldots,n)$ and $x = x'$ in the sufficient conditions in Theorem 8.

From Theorems 8 and 11, we deduce the following corollary.

Corollary 50 Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a Banach space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is φ-subtransversal at $\bar{x}$ if and only if inequality (2.95) holds with $\mu := (\varphi'_+(0))^{-1}+1$ for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ near $\bar{x}$ with $\max_{1\leq i\leq n}\|\omega_i - x\|>0$, and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64) and (2.97).

Proof The necessity part is exactly the 'δ-free' version of Theorem 11. To show the sufficiency, observe that the conditions in Theorem 8 are satisfied (for some $\delta_1>0$ and $\delta_2>0$) with $\mu := (\varphi'_+(0))^{-1}+1$, any sufficiently large $\lambda>0$, and $x'=x$ and $\omega'_i=\omega_i$ $(i=1,\ldots,n)$. $\square$

A decomposition of the dual necessary transversality condition in Theorem 11 can be easily obtained.

Corollary 51 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-subtransversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then, for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (2.46), the following conditions hold true:

(i) for all $x_i^*\in N_{\Omega_i}(\omega_i)$ $(i=1,\ldots,n)$ satisfying (2.64) and (2.97), it holds

$$\varphi'\left(\max_{1\leq i\leq n}\|\omega_i - x\|\right)\left\|\sum_{i=1}^n x_i^*\right\|\geq 1;$$

(ii) for all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64), (2.97) and $\sum_{i=1}^n x_i^*=0$, it holds

$$\varphi'\Big(\max_{1\leq i\leq n}\|\omega_i - x\|\Big)\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\omega_i)\right)\geq\left((\varphi'_+(0))^{-1}+1\right)^{-1}.$$

Transversality

In this subsection, we use the function $\widehat{f}$ given by (2.21) with $f\colon X^{n+1}\to\mathbb{R}_+$ defined by (2.20). The next statement provides a dual necessary condition for φ-transversality in terms of subdifferentials of $\widehat{f}$.

Proposition 18 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then

$$d_\gamma\Big(0,\partial\widehat{f}(\omega_1,\ldots,\omega_n,\bar{x})\Big)\geq 1 \tag{2.156}$$

for $\gamma:=((\varphi'_+(0))^{-1}+1)^{-1}$, and all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying (2.48).

Proof Under the assumptions made, the function $\widehat{f}$ is convex. The assertion follows from Theorem 5 since condition (2.156) is a direct consequence of (2.51); cf. the proof of Proposition 16. $\square$

Similar to the case of the other two nonlinear transversality properties, the difference in the contribution of components of subgradients of $\widehat{f}$ to the key dual condition (2.156) in Proposition 15 can be exposed.

Corollary 52 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then, for all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying (2.48), and all $(x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,\bar{x})$, condition (2.150) holds. As a consequence,

$$\liminf_{\substack{x_i^*\to 0\ (i=1,\ldots,n)\\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,\bar{x})}}\|x^*\|\geq 1.$$

Proof The assertion is a direct consequence of Proposition 15 and the representation (2.55) of the dual norm. $\square$

The next 'δ-free' statement is a direct consequence of Corollary 52.

Corollary 53 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-transversal at $\bar{x}$, then

$$\liminf_{\substack{\omega_i\xrightarrow{\Omega_i}\bar{x},\ x_i\to 0,\ x_i^*\to 0\ (i=1,\ldots,n)\\ \max\limits_{1\leq i\leq n}\|\omega_i-x_i-\bar{x}\|>0,\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,\bar{x})}}\|x^*\|\geq 1.$$

In the convex setting, a partial converse to Theorem 9 is possible.

Theorem 12 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then, with $\mu:=(\varphi'_+(0))^{-1}+1$,

$$\varphi'\left(\max_{1\leq i\leq n}\|\omega_i-x_i-\bar{x}\|\right)\left(\left\|\sum_{i=1}^n x_i^*\right\|+\mu\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\omega_i)\right)\right)\geq 1 \tag{2.157}$$

for all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying (2.48), and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64) and

$$\sum_{i=1}^n\langle x_i^*,\bar{x}+x_i-\omega_i\rangle=\max_{1\leq i\leq n}\|\bar{x}+x_i-\omega_i\|. \tag{2.158}$$

Proof Let $\{\Omega_1,\ldots,\Omega_n\}$ be φ-transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$. Let $\mu:=(\varphi'_+(0))^{-1}+1$ and $\gamma:=\mu^{-1}$. By Proposition 15, condition (2.156) is satisfied for all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying (2.48). Observe that $\widehat{f}$ is a sum of the function f given by (2.20) and the indicator function of the set $\Omega_1\times\ldots\times\Omega_n$. Since $\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|>0$, f is locally Lipschitz continuous near $(\omega_1,\ldots,\omega_n,\bar{x})$. It is a composition of φ and the function ψ defined by (2.153). By Lemmas 4, 6, and 9, Remark 5,

$$\partial\widehat{f}(\omega_1,\ldots,\omega_n,\bar{x})=\partial f(\omega_1,\ldots,\omega_n,\bar{x})+N_{\Omega_1\times\ldots\times\Omega_n}(\omega_1,\ldots,\omega_n)\times\{0\}$$
$$=\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)\left(\partial\psi(\omega_1,\ldots,\omega_n,\bar{x})+N_{\Omega_1}(\omega_1)\times\ldots\times N_{\Omega_n}(\omega_n)\times\{0\}\right).$$

By Lemma 8, $(-x_1^*,\ldots,-x_n^*,x^*)\in\partial\psi(\omega_1,\ldots,\omega_n,\bar{x})$ if and only if conditions (2.64), (2.154) and (2.158) are satisfied. Hence, condition (2.156) implies (2.157). □

Remark 37 (i) The equality $\mu:=(\varphi'_+(0))^{-1}+1$ in Theorem 12 can be replaced by the inequality $\mu\ge(\varphi'_+(0))^{-1}+1$.

(ii) Conditions (2.157) and (2.158) in Theorem 12 are particular cases of conditions, respectively, (2.62) and (2.65) in Theorem 9, corresponding to setting $x:=\bar{x}$.

(iii) The necessary conditions in Theorems 10 and 11 correspond to setting, respectively, $\omega_i:=\bar{x}$ $(i=1,\ldots,n)$ and $x_1=\ldots=x_n$ in the necessary conditions in Theorem 12.

From Theorems 9 and 12, we deduce the following corollary.

Corollary 54 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$.

(i) Suppose X is Banach and $\Omega_1,\ldots,\Omega_n$ are closed. $\{\Omega_1,\ldots,\Omega_n\}$ is φ-transversal at $\bar{x}$ if inequality (2.62) holds with some $\mu>0$ for all $x\in X$ and $\omega_i\in\Omega_i$ near $\bar{x}$ and $x_i\in X$ near 0 $(i=1,\ldots,n)$ with $\max_{1\le i\le n}\|\omega_i-x_i-x\|>0$, and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64) and (2.65).

(ii) If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-transversal at $\bar{x}$, then inequality (2.157) holds with $\mu:=(\varphi'_+(0))^{-1}+1$ for all $\omega_i\in\Omega_i$ near $\bar{x}$ and $x_i\in X$ near 0 $(i=1,\ldots,n)$ with $\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|>0$, and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64) and (2.158).

A decomposition of the dual transversality condition (2.151) in Theorem 10 can be easily obtained.

Corollary 55 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is φ-transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then, for all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying (2.48), and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (2.64) and (2.158), the following conditions hold true:

(i) if $x_i^*\in N_{\Omega_i}(\bar{x})$ $(i=1,\ldots,n)$, then $\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)\left\|\sum_{i=1}^n x_i^*\right\|\ge 1$;

(ii) if $\sum_{i=1}^n x_i^*=0$, then

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\bar{x})\right)\ge((\varphi'_+(0))^{-1}+1)^{-1}.$$

Chapter 3

Transversality and Regularity Properties of Set-Valued Mappings

The content of this chapter is based on the publications [65, 67–71].

3.1 Transversality and Regularity

3.1.1 Connections

This section presents connections between transversality properties of collections of sets studied in Chapter 2 and regularity properties of set-valued mappings in the nonlinear setting.

For regularity properties, our model here is a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces. We consider its local regularity properties near a given point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. The nonlinearity in the definitions of the properties is determined by a function $\varphi \in \mathscr{C}$ introduced in Chapter 2.

Regularity of set-valued mappings have been intensively studied for decades due to its numerous important applications. For discussions about the theory, we refer the reader to Chapter 1. Nonlinear regularity properties have also been considered by many authors; cf. [37, 95–97, 113, 132, 133, 144, 154, 191, 219]. The relations between transversality and regularity properties are well known in the linear case [112, 114, 127–129, 134, 138, 139, 142] as well as in the Hölder setting [141]. Below we briefly discuss more general nonlinear models.

Definition 7 The mapping F is

(i) $\varphi-$semiregular at $(\bar{x}, \bar{y})$ if there exists a $\delta > 0$ such that

$$d(\bar{x}, F^{-1}(y)) \leq \varphi(d(y, \bar{y}))$$

for all $y \in Y$ with $\varphi(d(y, \bar{y})) < \delta$;

(ii) $\varphi-$subregular at $(\bar{x},\bar{y})$ if there exist $\delta_1>0$ and $\delta_2>0$ such that

$$d(x,F^{-1}(\bar{y}))\le\varphi(d(\bar{y},F(x)))$$

for all $x\in B_{\delta_2}(\bar{x})$ with $\varphi(d(\bar{y},F(x)))<\delta_1$;

(iii) $\varphi-$regular at $(\bar{x},\bar{y})$ if there exist $\delta_1>0$ and $\delta_2>0$ such that

$$d(x,F^{-1}(y))\le\varphi(d(y,F(x))) \tag{3.1}$$

for all $x\in X$ and $y\in Y$ with $d(x,\bar{x})+d(y,\bar{y})<\delta_2$ and $\varphi(d(y,F(x)))<\delta_1$.

The function $\varphi\in\mathscr{C}$ in the above definition plays the role of a kind of rate or modulus of the respective property. In the Hölder setting, i.e. when $\varphi(t):=\alpha^{-1}t^q$ with $\alpha>0$ and $q>0$, we refer to the respective properties in Definition 7 as $\alpha-$semiregularity, $\alpha-$subregularity and $\alpha-$regularity of order q. These regularity properties have been studied in [96, 97, 132, 141, 144, 154]. It is usually assumed that $q\le 1$. The exact upper bound of all $\alpha>0$ such that a property holds with some $\delta>0$, or $\delta_1>0$ and $\delta_2>0$, is called the *modulus* of this property. We use notations $\mathrm{s_erg}_q[F](\bar{x},\bar{y})$, $\mathrm{srg}_q[F](\bar{x},\bar{y})$ and $\mathrm{rg}_q[F](\bar{x},\bar{y})$ for the moduli of the respective properties. If a property does not hold, then by convention the respective modulus equals 0. With $q=1$ (linear case), the properties are called metric *semiregularity*, *subregularity* and *regularity*, respectively; cf. [58, 81, 116, 129, 172, 202].

The following assertion is a direct consequence of Definition 7.

Proposition 19 If F is $\varphi-$regular at $(\bar{x},\bar{y})$ with some $\delta_1>0$ and $\delta_2>0$, then it is $\varphi-$semiregular at $(\bar{x},\bar{y})$ with $\delta:=\min\{\delta_1,\varphi(\delta_2)\}$ and $\varphi-$subregular at $(\bar{x},\bar{y})$ with δ_1 and δ_2.

Note the combined inequality $d(x,\bar{x})+d(y,\bar{y})<\delta_2$ employed in part (iii) of Definition 7 instead of the more traditional separate conditions $x\in B_{\delta_2}(\bar{x})$ and $y\in B_{\delta_2}(\bar{y})$. This replacement does not affect the property of $\varphi-$regularity itself, but can have an effect on the value of δ_2. Employing this inequality makes the property a direct analogue of the metric characterization of $\varphi-$transversality in Theorem 2 and is convenient for establishing relations between the regularity and transversality properties. The next proposition provides also an important special case when the point x in (3.1) can be fixed: $x=\bar{x}$.

Proposition 20 Let $\delta_1>0$ and $\delta_2>0$. Consider the following conditions:

(a) inequality (3.1) holds for all $x\in B_{\delta_2}(\bar{x})$ and $y\in B_{\delta_2}(\bar{y})$ with $\varphi(d(y,F(x)))<\delta_1$;

(b) inequality (3.1) holds for all $x\in X$ and $y\in Y$ with $d(x,\bar{x})+d(y,\bar{y})<\delta_2$ and $\varphi(d(y,F(x)))<\delta_1$;

(c) $d(\bar{x},F^{-1}(y))\le\varphi(d(y,F(\bar{x})))$ for all $y\in B_{\delta_2}(\bar{y})$ with $\varphi(d(y,F(\bar{x})))<\delta_1$.

Then

(i) (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Moreover, condition (b) implies (a) with $\delta_2' := \delta_2/2$ in place of δ_2.

(ii) If X is a normed space, $Y = X^n$ for some $n \in \mathbb{N}$, $\bar{y} = (\bar{x}_1, \ldots, \bar{x}_n)$ and $F : X \rightrightarrows X^n$ is given by

$$F(x) := (\Omega_1 - x) \times \ldots \times (\Omega_n - x), \quad x \in X, \tag{3.2}$$

where $\Omega_1, \ldots, \Omega_n \subset X$, then (b) $\Leftrightarrow$ (c).

Proof

(i) All the implications are straightforward.

(ii) In view of (i), we only need to prove (c) $\Rightarrow$ (b). Suppose condition (c) is satisfied. Let $x \in X$, $y = (x_1, \ldots, x_n) \in X^n$, $\|x - \bar{x}\| + \|y - \bar{y}\| < \delta_2$ and $\varphi(d(y, F(x))) < \delta_1$. Set $x_i' := x_i + x - \bar{x}$ $(i = 1, \ldots, n)$ and $y' := (x_1', \ldots, x_n')$. Then

$$\begin{aligned} \|y' - \bar{y}\| \le \|y' - y\| + \|y - \bar{y}\| = \|x - \bar{x}\| + \|y - \bar{y}\| < \delta_2, \\ d(x, F^{-1}(y)) = d(x, \cap_{i=1}^n (\Omega_i - x_i)) = d(\bar{x}, \cap_{i=1}^n (\Omega_i - x_i')) = d(\bar{x}, F^{-1}(y')), \\ d(y, F(x)) = \max_{1 \le i \le n} d(x_i, \Omega_i - x) = \max_{1 \le i \le n} d(x_i', \Omega_i - \bar{x}) = d(y', F(\bar{x})). \end{aligned}$$

and, thanks to (c), $d(x, F^{-1}(y)) \le \varphi(d(y, F(x)))$.

□

The set-valued mapping (3.2) plays the key role in establishing relations between the regularity and transversality properties. It was most likely first used by Ioffe in [112]. Observe that $F^{-1}(x_1, \ldots, x_n) = (\Omega_1 - x_1) \cap \ldots \cap (\Omega_n - x_n)$ for all $x_1, \ldots, x_n \in X$ and, if $\bar{x} \in \cap_{i=1}^n \Omega_i$, then $(0, \ldots, 0) \in F(\bar{x})$.

The next statement is a reformulation of the metric characterizations of the transversality properties in Theorem 1. It generalizes and extends the corresponding results in [38, 112, 114, 127–129, 134, 138, 139, 141, 142].

Theorem 13 Let $\Omega_1, \ldots, \Omega_n$ be subsets of a normed space X, F be defined by (3.2), $\bar{x} \in \cap_{i=1}^n \Omega_i$ and $\bar{y} := (0, \ldots, 0) \in X^n$. The collection $\{\Omega_1, \ldots, \Omega_n\}$ is

(i) $\varphi-$semitransversal at $\bar{x}$ with some $\delta > 0$ if and only if F is $\varphi-$semiregular at $(\bar{x}, \bar{y})$ with δ;

(ii) $\varphi-$subtransversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$ if and only if F is $\varphi-$subregular at $(\bar{x}, \bar{y})$ with δ_1 and δ_2;

(iii) $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$ if and only if

$$d(0, F^{-1}(\omega_1 + x_1, \ldots, \omega_n + x_n)) \le \varphi(\|y\|) \tag{3.3}$$

for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1, \ldots, n)$ and $y := (x_1, \ldots, x_n) \in X^n$ with $\varphi(\|y\|) < \delta_1$.

Observe that the condition in part (iii) of Theorem 13 is similar to, but not exactly the one in the definition of $\varphi-$regularity. The next statement shows that the latter corresponds to alternative metric characterizations of $\varphi-$transversality.

Proposition 21 Let $\Omega_1,\ldots,\Omega_n$ be subsets of a normed space X, F be defined by (3.2), $\bar{x}\in\cap_{i=1}^n\Omega_i$, $\bar{y}:=(0,\ldots,0)\in X^n$, $\delta_1>0$ and $\delta_2>0$. The following properties are equivalent:

(i) inequality (3.3) holds for all $\omega_i\in\Omega_i$, $y:=(x_1,\ldots,x_n)\in X^n$ with $\omega_i+x_i\in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\|y\|)<\delta_1$;

(ii) $d(\bar{x},F^{-1}(y))\le\varphi(d(y,F(\bar{x})))$ for all $y\in\delta_2\mathbb{B}_{X^n}$ with $\varphi(d(y,F(\bar{x})))<\delta_1$;

(iii) inequality (3.1) holds for all $x\in X$, $y:=(x_1,\ldots,x_n)\in X^n$ with $x+x_i\in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(d(y,F(x)))<\delta_1$;

(iv) the mapping F is $\varphi-$regular at $(\bar{x},\bar{y})$ with δ_1 and δ_2.

Moreover, if $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then conditions (i)–(iv) hold with any $\delta_1'\in]0,\delta_1]$ and $\delta_2'>0$ satisfying $\varphi^{-1}(\delta_1')+\delta_2'\le\delta_2$ in place of δ_1 and δ_2.

Conversely, if properties (i)–(iv) hold with some $\delta_1>0$ and $\delta_2>0$, then $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with any $\delta_1'\in]0,\delta_1]$ and $\delta_2'>0$ satisfying $\varphi^{-1}(\delta_1')+\delta_2'\le\delta_2$.

Proof With the exception of item (iv), the statement is a reformulation of Theorem 2 in terms of the mapping F. It is easy to see that conditions (iii) and (iv) are equivalent; cf. the hints to the proof of the linear version of this fact in [38]. $\square$

Corollary 56 Let $\Omega_1,\ldots,\Omega_n$ be subsets of a normed space X, F be defined by (3.2), $\bar{x}\in\cap_{i=1}^n\Omega_i$ and $\bar{y}:=(0,\ldots,0)\in X^n$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ if and only if F is $\varphi-$regular at $(\bar{x},\bar{y})$.

In the convex case, conditions (i)–(iv) in Proposition 21 admit simplifications.

Corollary 57 Let $\Omega_1,\ldots,\Omega_n$ be convex subsets of a normed space X, F be defined by (3.2), $\bar{x}\in\cap_{i=1}^n\Omega_i$, $\bar{y}:=(0,\ldots,0)\in X^n$, $\delta_1>0$ and $\delta_2>0$. Conditions (i)–(iv) in Proposition 21 hold if and only if the following equivalent properties hold true:

(i) $F^{-1}(\omega_1+x_1,\ldots,\omega_n+x_n)\cap(\delta_1\mathbb{B})\neq\emptyset$ for all $\omega_i\in\Omega_i$, $y:=(x_1,\ldots,x_n)\in X^n$ with $\omega_i+x_i\in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\|y\|)<\delta_1$;

(ii) $F^{-1}(y)\cap B_{\delta_1}(\bar{x})\neq\emptyset$ for all $y\in\delta_2\mathbb{B}_{X^n}$ with $\varphi(d(y,F(\bar{x})))<\delta_1$;

(iii) $F^{-1}(y)\cap B_{\delta_1}(\bar{x})\neq\emptyset$ for all $x\in X$, $y:=(x_1,\ldots,x_n)\in X^n$ with $x+x_i\in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(d(y,F(x)))<\delta_1$.

The next corollary is a consequence of Theorem 13 and Proposition 21.

Corollary 58 Let $\Omega_1,\ldots,\Omega_n$ be subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, $\varphi\in\mathscr{C}$, and F be defined by (3.2). The collection $\{\Omega_1,\ldots,\Omega_n\}$ is

(i) $\varphi-$semitransversal at $\bar{x}$ if and only if F is $\varphi-$semiregular at $(\bar{x},(0,\ldots,0))$;

(ii) $\varphi-$subtransversal at $\bar{x}$ if and only if F is $\varphi-$subregular at $(\bar{x},(0,\ldots,0))$;

(iii) $\varphi-$transversal at $\bar{x}$ if and only if F is $\varphi-$regular at $(\bar{x},(0,\ldots,0))$.

Remark 38 (i) In the Hölder setting Corollary 58 reduces to [141, Proposition 9].

(ii) Apart from the mapping F defined by (3.2), in the case of two sets other set-valued mappings can be used to ensure similar equivalences between the $\varphi-$transversality and $\varphi-$regularity properties; see [116].

In view of Corollary 58, the nonlinear transversality properties of collections of sets can be viewed as particular cases of the corresponding nonlinear regularity properties of set-valued mappings. We are going to show that the two popular models are in a sense equivalent.

Given an arbitrary set-valued mapping $F:X\rightrightarrows Y$ between metric spaces and a point $(\bar{x},\bar{y})\in\operatorname{gph}F$, we can consider the two sets:

$$\Omega_1:=\operatorname{gph}F,\quad \Omega_2:=X\times\{\bar{y}\} \tag{3.4}$$

in the product space $X\times Y$. Note that $(\bar{x},\bar{y})\in\Omega_1\cap\Omega_2=F^{-1}(\bar{y})\times\{\bar{y}\}$. To establish the relationship between the two sets of properties, we have to assume in the next two theorems that X and Y are normed vector spaces.

Theorem 14 Let X and Y be normed spaces, $F:X\rightrightarrows Y$, $(\bar{x},\bar{y})\in\operatorname{gph}F$, and $\varphi\in\mathscr{C}$. Let Ω_1 and Ω_2 be defined by (3.4), and $\psi(t):=\varphi(2t)+t$ for all $t\geq 0$.

(i) If F is $\varphi-$semiregular at $(\bar{x},\bar{y})$ with some $\delta>0$, then $\{\Omega_1,\Omega_2\}$ is $\psi-$semitransversal at $(\bar{x},\bar{y})$ with $\delta':=\delta+\varphi^{-1}(\delta)/2$.

(ii) If F is $\varphi-$subregular at $(\bar{x},\bar{y})$ with some $\delta_1>0$ and $\delta_2>0$, then $\{\Omega_1,\Omega_2\}$ is $\psi-$subtransversal at $(\bar{x},\bar{y})$ with any $\delta_1'>0$ and $\delta_2'>0$ such that $\varphi(2\psi^{-1}(\delta_1'))\leq\delta_1$ and $\psi^{-1}(\delta_1')+\delta_2'\leq\delta_2$.

(iii) If F is $\varphi-$regular at $(\bar{x},\bar{y})$ with some $\delta_1>0$ and $\delta_2>0$, then $\{\Omega_1,\Omega_2\}$ is $\psi-$transversal at $(\bar{x},\bar{y})$ with any $\delta_1'>0$ and $\delta_2'>0$ such that $\varphi(2\psi^{-1}(\delta_1'))\leq\delta_1$ and $\psi^{-1}(\delta_1')+\delta_2'\leq\delta_2/2$.

Proof Observe that $\psi \in \mathscr{C}$, $\varphi(2\psi^{-1}(t)) + \psi^{-1}(t) = t$ and $\psi(\varphi^{-1}(t)/2) = t + \varphi^{-1}(t)/2$ for all $t \geq 0$.

(i) Let F be $\varphi-$semiregular at $(\bar{x}, \bar{y})$ with some $\delta > 0$. Set $\delta' := \delta + \varphi^{-1}(\delta)/2 = \psi(\varphi^{-1}(\delta)/2)$. Let $\rho \in]0, \delta'[$ and $(u_1, v_1), (u_2, v_2) \in \psi^{-1}(\rho)\mathbb{B}$. Set $y' := \bar{y} + v_1 - v_2$. Observe that

$$\begin{aligned}(\Omega_1 - (u_1, v_1)) \cap (\Omega_2 - (u_2, v_2)) &= (\operatorname{gph} F - (u_1, v_1)) \cap (X \times \{\bar{y} - v_2\}) \\ &= \left(F^{-1}(y') - u_1\right) \times \{\bar{y} - v_2\}.\end{aligned}$$

We haven $\|y' - \bar{y}\| = \|v_1 - v_2\| \leq \|v_1\| + \|v_2\| < 2\psi^{-1}(\rho)$, and consequently, $\varphi(\|y' - \bar{y}\|) < \varphi(2\psi^{-1}(\rho)) < \varphi(2\psi^{-1}(\delta')) = \delta$. By Definition 7(i),

$$\begin{aligned}d(\bar{x}, F^{-1}(y') - u_1) &\leq d(\bar{x}, F^{-1}(y')) + \|u_1\| \\ &\leq \varphi(\|y' - \bar{y}\|) + \|u_1\| < \varphi(2\psi^{-1}(\rho)) + \psi^{-1}(\rho) = \rho,\end{aligned}$$

and consequently,

$$\begin{aligned}d\left((\bar{x}, \bar{y}), (\Omega_1 - (u_1, v_1)) \cap (\Omega_2 - (u_2, v_2))\right) &\leq \max\{d(\bar{x}, F^{-1}(y') - u_1), \|v_2\|\} \\ &< \max\{\rho, \psi^{-1}(\rho)\} = \rho;\end{aligned}$$

hence,

$$(\Omega_1 - (u_1, v_1)) \cap (\Omega_2 - (u_2, v_2)) \cap B_\rho(\bar{x}, \bar{y}) \neq \emptyset. \tag{3.5}$$

By Definition 6(i), $\{\Omega_1, \Omega_2\}$ is $\varphi-$semitransversal at $(\bar{x}, \bar{y})$ with δ'.

(ii) Let F be $\varphi-$subregular at $(\bar{x}, \bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$. Choose numbers $\delta_1' > 0$ and $\delta_2' > 0$ such that $\varphi(2\psi^{-1}(\delta_1')) \leq \delta_1$ and $\psi^{-1}(\delta_1') + \delta_2' \leq \delta_2$. Let $\rho \in]0, \delta_1'[$ and $(x, y) \in B_{\delta_2'}(\bar{x}, \bar{y})$ with $\psi(\max\{d((x, y), \Omega_1), d((x, y), \Omega_2)\}) < \rho$, i.e. $\|y - \bar{y}\| < \psi^{-1}(\rho)$ and there exists a point $(x_1, y_1) \in \operatorname{gph} F$ such that $\|(x, y) - (x_1, y_1)\| < \psi^{-1}(\rho)$. Then

$$\begin{aligned}\|x_1 - \bar{x}\| &\leq \|x_1 - x\| + \|x - \bar{x}\| < \psi^{-1}(\delta_1') + \delta_2' \leq \delta_2, \\ d(\bar{y}, F(x_1)) &\leq \|y_1 - \bar{y}\| \leq \|y - \bar{y}\| + \|y_1 - y\| < 2\psi^{-1}(\rho),\end{aligned}$$

and consequently, $\varphi(d(\bar{y}, F(x_1))) < \varphi(2\psi^{-1}(\rho)) < \varphi(2\psi^{-1}(\delta_1')) \leq \delta_1$. Choose a positive $\varepsilon < 2\psi^{-1}(\rho) - d(\bar{y}, F(x_1))$. By Definition 7(ii), there exists an $x' \in F^{-1}(\bar{y})$ such that $\|x' - x_1\| < \varphi(d(\bar{y}, F(x_1)) + \varepsilon) < \varphi(2\psi^{-1}(\rho))$. Hence, $(x', \bar{y}) \in \Omega_1 \cap \Omega_2$ and

$$\begin{aligned}\|x - x'\| &\leq \|x_1 - x'\| + \|x - x_1\| < \varphi(2\psi^{-1}(\rho)) + \psi^{-1}(\rho) = \rho, \\ \|y - \bar{y}\| &< \psi^{-1}(\rho) < \rho.\end{aligned}$$

Thus, $\Omega_1 \cap \Omega_2 \cap B_\rho(x, y) \neq \emptyset$. By Definition 6(ii), $\{\Omega_1, \Omega_2\}$ is $\psi-$subtransversal at $(\bar{x}, \bar{y})$ with δ_1' and δ_2'.

(iii) Let F be $\varphi-$regular at $(\bar{x},\bar{y})$ with some $\delta_1>0$ and $\delta_2>0$. Choose numbers $\delta_1'>0$ and $\delta_2'>0$ such that $\varphi(2\psi^{-1}(\delta_1'))\le\delta_1$ and $\psi^{-1}(\delta_1')+\delta_2'\le\delta_2/2$. Let $\rho\in]0,\delta_1'[$, $(x_1,y_1)\in\operatorname{gph}F\cap B_{\delta_2'}(\bar{x},\bar{y})$, $x_2\in B_{\delta_2'}(\bar{x})$ and $(u_1,v_1),(u_2,v_2)\in\psi^{-1}(\rho)\mathbb{B}$. Set $y':=y_1+v_1-v_2$. Then

$$\|x_1-\bar{x}\|+\|y'-\bar{y}\|\le\|v_1\|+\|v_2\|+\|x_1-\bar{x}\|+\|y_1-\bar{y}\|<2\psi^{-1}(\delta_1')+2\delta_2'\le\delta_2,$$
$$\varphi(d(y',F(x_1)))\le\varphi(\|y'-y_1\|)\le\varphi(\|v_1\|+\|v_2\|)<\varphi(2\psi^{-1}(\delta_1'))\le\delta_1.$$

Choose a positive $\varepsilon<2\left(\psi^{-1}(\rho)-\max\{\|v_1\|,\|v_2\|\}\right)$. By Definition 7(iii), there exists an $x'\in F^{-1}(y')$ such that

$$\|x_1-x'\|<\varphi(\|y'-y_1\|+\varepsilon)\le\varphi(2\max\{\|v_1\|,\|v_2\|\}+\varepsilon)<\varphi(2\psi^{-1}(\rho)).$$

Denote $\hat{x}:=x'-x_1-u_1$ and $\hat{y}:=y'-y_1-v_1$. Thus, $(x',y')\in\Omega_1$ and $(\hat{x},\hat{y})\in\Omega_1-(x_1,y_1)-(u_1,v_1)$. At the same time, $\hat{y}=-v_2$ and $(\hat{x},\hat{y})\in\Omega_2-(x_2,\bar{y})-(u_2,v_2)$. Moreover,

$$\|\hat{x}\|\le\|x'-x_1\|+\|u_1\|<\varphi(2\psi^{-1}(\rho))+\psi^{-1}(\rho)=\rho,$$
$$\|\hat{y}\|=\|v_2\|<\psi^{-1}(\rho)<\rho;$$

hence $(x',y')\in\rho\mathbb{B}$. By Definition 6(iii), $\{\Omega_1,\Omega_2\}$ is $\psi-$transversal at $(\bar{x},\bar{y})$ with δ_1' and δ_2'.

$\square$

Theorem 15 Let X and Y be normed spaces, $F:X\rightrightarrows Y$, $(\bar{x},\bar{y})\in\operatorname{gph}F$, and $\varphi\in\mathscr{C}$. Let Ω_1 and Ω_2 be defined by (3.4), and $\psi(t):=\varphi(t/2)$ for all $t\ge0$.

(i) If $\{\Omega_1,\Omega_2\}$ is $\varphi-$semitransversal at $(\bar{x},\bar{y})$ with some $\delta>0$, then F is $\psi-$semiregular at $(\bar{x},\bar{y})$ with δ.

(ii) If $\{\Omega_1,\Omega_2\}$ is $\varphi-$subtransversal at $(\bar{x},\bar{y})$ with some $\delta_1>0$ and $\delta_2>0$, then F is $\psi-$subregular at $(\bar{x},\bar{y})$ with $\delta_1':=\min\{\delta_1,\psi(2\delta_2)\}$ and δ_2.

(iii) If $\{\Omega_1,\Omega_2\}$ is $\varphi-$transversal at $(\bar{x},\bar{y})$ with some $\delta_1>0$ and $\delta_2>0$, then F is $\psi-$regular at $(\bar{x},\bar{y})$ with any $\delta_1'\in]0,\delta_1]$ and $\delta_2'>0$ such that $\psi^{-1}(\delta_1')+\delta_2'\le\delta_2$.

Proof Observe that $\psi\in\mathscr{C}$.

(i) Let $\{\Omega_1,\Omega_2\}$ be $\varphi-$semitransversal at $(\bar{x},\bar{y})$ with some $\delta>0$. By Definition 6(i), condition (3.5) is satisfied for all $\rho\in]0,\delta[$ and $(u_1,v_1),(u_2,v_2)\in\varphi^{-1}(\rho)\mathbb{B}$. Let $y\in Y$ with $\rho_0:=\psi(\|y-\bar{y}\|)<\delta$. Choose a $\rho\in]\rho_0,\delta[$ and observe that

$$\varphi\left(\left\|\left(0,\frac{y-\bar{y}}{2}\right)\right\|\right)=\varphi\left(\frac{\|y-\bar{y}\|}{2}\right)=\psi\left(\|y-\bar{y}\|\right)<\rho.$$

In view of (3.5), we can find $(x_1,y_1) \in \operatorname{gph} F$ and $x_2 \in X$ such that

$$(x_1,y_1) - \left(0, \frac{y-\bar{y}}{2}\right) = (x_2,\bar{y}) - \left(0, \frac{\bar{y}-y}{2}\right) \in B_\rho(\bar{x},\bar{y}).$$

Hence, $y_1 = \bar{y} + 2\frac{y-\bar{y}}{2} = y$, $x_1 \in F^{-1}(y)$, $\|x_1 - \bar{x}\| < \rho$, and consequently, $d(\bar{x}, F^{-1}(y)) < \rho$. Letting $\rho \downarrow \rho_0$, we obtain $d(\bar{x}, F^{-1}(y)) \le \psi(\|y-\bar{y}\|)$. By Definition 7(i), F is $\psi-$semiregular at $(\bar{x},\bar{y})$ with δ.

(ii) Let $\{\Omega_1,\Omega_2\}$ be $\varphi-$subtransversal at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$. By Definition 6(ii), $\operatorname{gph} F \cap (X \times \{\bar{y}\}) \cap B_\rho(x,y) \neq \emptyset$ for all $\rho \in]0,\delta_1[$ and $(x,y) \in B_{\delta_2}(\bar{x},\bar{y})$ with $\varphi(d((x,y),\operatorname{gph} F)) < \rho$ and $\varphi(\|y-\bar{y}\|) < \rho$. Set $\delta_1' := \min\{\delta_1, \psi(2\delta_2)\}$. Let $x \in B_{\delta_2}(\bar{x})$ and $\psi(d(\bar{y},F(x))) < \delta_1'$. Choose a $y \in F(x)$ such that $\rho_0 := \psi(\|\bar{y}-y\|) < \delta_1'$, and a $\rho \in]\rho_0, \delta_1'[$. Set $\hat{y} := \frac{y+\bar{y}}{2}$. Observe that

$$\|\hat{y}-y\| = \|\hat{y}-\bar{y}\| = \frac{\|\bar{y}-y\|}{2} = \frac{\psi^{-1}(\rho_0)}{2} < \frac{\psi^{-1}(\rho)}{2} = \varphi^{-1}(\rho),$$
$$\|\hat{y}-\bar{y}\| < \frac{\psi^{-1}(\rho)}{2} < \frac{\psi^{-1}(\delta_1')}{2} \le \delta_2.$$

Thus, $\rho \in]0,\delta_1[$, $(x,\hat{y}) \in B_{\delta_2}(\bar{x},\bar{y})$, $\varphi(d((x,\hat{y}),\operatorname{gph} F)) \le \varphi(\|\hat{y}-y\|) < \rho$ and $\varphi(\|\hat{y}-\bar{y}\|) < \rho$. Hence, $\operatorname{gph} F \cap (X \times \{\bar{y}\}) \cap B_\rho(x,\hat{y}) \neq \emptyset$, and consequently, $d(x, F^{-1}(\bar{y})) < \rho$. Letting $\rho \downarrow \rho_0$, we obtain $d(x, F^{-1}(\bar{y})) \le \psi(\|\bar{y}-y\|)$. Taking the infimum in the right-hand side of this inequality over $y \in F(x)$, we conclude that F is $\psi-$subregular at $(\bar{x},\bar{y})$ with δ_1' and δ_2 in view of Definition 7(ii).

(iii) Let $\{\Omega_1,\Omega_2\}$ be $\varphi-$transversal at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$, i.e. for all $\rho \in]0,\delta_1[$, $(x',y') \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x},\bar{y})$, $u_1 \in X$ and $v_1, v_2 \in Y$ with $\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\}) < \rho$, it holds

$$\big(\operatorname{gph} F - (x',y') - (u_1,v_1)\big) \cap (X \times \{-v_2\}) \cap (\rho \mathbb{B}) \neq \emptyset,$$

or equivalently, $d\left(x'+u_1, F^{-1}(y'+v_1-v_2)\right) < \rho$. In other words, $d\left(x, F^{-1}(y)\right) < \rho$ for all $\rho \in]0,\delta_1[$, $(x',y') \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x},\bar{y})$, $x \in X$ and $y \in Y$ with $\|x-x'\| < \varphi^{-1}(\rho)$ and $\|y-y'\| < 2\varphi^{-1}(\rho)$. Choose numbers $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ such that $\psi^{-1}(\delta_1') + \delta_2' \le \delta_2$. Let $x \in X$ and $y \in Y$ with $\|x-\bar{x}\| + \|y-\bar{y}\| < \delta_2'$ and $\psi(d(y,F(x))) < \delta_1'$. Choose a $y' \in F(x)$ such that $\rho_0 := \psi(\|y-y'\|) < \delta_1'$ and a $\rho \in]\rho_0, \delta_1'[$. Then $\rho \in]0,\delta_1[$, $(x,y') \in \operatorname{gph} F$, $\|x-\bar{x}\| < \delta_2' < \delta_2$, $\|y'-\bar{y}\| \le \|y'-y\| + \|y-\bar{y}\| < \psi^{-1}(\delta_1') + \delta_2' \le \delta_2$ and $\|y-y'\| < \psi^{-1}(\rho) = 2\varphi^{-1}(\rho)$. Hence, $d\left(x, F^{-1}(y)\right) < \rho$. Letting $\rho \downarrow \rho_0$, we obtain $d(x, F^{-1}(y)) \le \psi(\|y-y'\|)$. Taking the infimum in the right-hand side of this inequality over $y' \in F(x)$, we conclude that F is $\psi-$regular at $(\bar{x},\bar{y})$ with δ_1' and δ_2' in view of Definition 7(iii).

$\square$

The next corollary of Theorems 14 and 15 provides qualitative relations between the regularity and transversality properties.

Corollary 59 Let X and Y be normed spaces, $F : X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, and $\varphi \in \mathscr{C}$. Let Ω_1 and Ω_2 be defined by (3.4), $\psi_1(t) := \varphi(2t) + t$ and $\psi_2(t) := \varphi(t/2)$ for all $t \geq 0$.

(i) If F is $\varphi-$(semi-/sub-)regular at $(\bar{x},\bar{y})$, then $\{\Omega_1,\Omega_2\}$ is ψ_1-(semi-/sub-)transversal at $(\bar{x},\bar{y})$.

(ii) If $\{\Omega_1,\Omega_2\}$ is $\varphi-$(semi-/sub-)transversal at $(\bar{x},\bar{y})$, then F is ψ_2-(semi-/sub-)regular at $(\bar{x},\bar{y})$.

The next statement addresses the Hölder setting. It is a consequence of Theorems 14 and 15 with $\varphi(t) := \alpha^{-1}t^q$ for some $\alpha > 0$, $q > 0$ and all $t \geq 0$.

Corollary 60 Let X and Y be normed spaces, $F : X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\alpha > 0$ and $q > 0$. Let Ω_1 and Ω_2 be defined by (3.4), $\alpha_1 := 2^{-q}\alpha$, $\alpha_2 := 2^q\alpha$, and $\psi(t) := \alpha_1^{-1}t^q + t$ for all $t \geq 0$.

(i) If F is $\alpha-$semiregular of order q at $(\bar{x},\bar{y})$ with some $\delta > 0$, then $\{\Omega_1,\Omega_2\}$ is $\psi-$semitransversal at $(\bar{x},\bar{y})$ with $\delta' := \delta + (\alpha\delta)^{\frac{1}{q}}/2$.

If $\{\Omega_1,\Omega_2\}$ is $\alpha-$semitransversal of order q at $(\bar{x},\bar{y})$ with some $\delta > 0$, then F is α_2-semiregular of order q at $(\bar{x},\bar{y})$ with δ.

(ii) Let $q \leq 1$. If F is $\alpha-$subregular of order q at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then $\{\Omega_1,\Omega_2\}$ is $\psi-$subtransversal at $(\bar{x},\bar{y})$ with any $\delta_1' > 0$ and $\delta_2' > 0$ such that $(2\psi^{-1}(\delta_1'))^q \leq \alpha\delta_1$ and $\psi^{-1}(\delta_1') + \delta_2' \leq \delta_2$.

If $\{\Omega_1,\Omega_2\}$ is $\alpha-$subtransversal of order q at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then F is α_2-subregular of order q at $(\bar{x},\bar{y})$ with $\delta_1' := \min\{\delta_1, \alpha^{-1}\delta_2^q\}$ and δ_2.

(iii) Let $q \leq 1$. If F is $\alpha-$regular of order q at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then $\{\Omega_1,\Omega_2\}$ is $\psi-$transversal at $(\bar{x},\bar{y})$ with any $\delta_1' > 0$ and $\delta_2' > 0$ such that $(2\psi^{-1}(\delta_1'))^q \leq \alpha\delta_1$ and $\psi^{-1}(\delta_1') + \delta_2' \leq \delta_2/2$.

If $\{\Omega_1,\Omega_2\}$ is $\alpha-$transversal of order q at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then F is α_2-regular of order q at $(\bar{x},\bar{y})$ with any $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ such that $2(\alpha\delta_1')^{\frac{1}{q}} + \delta_2' \leq \delta_2$.

In view of Corollary 60, Hölder transversality properties of $\{\Omega_1,\Omega_2\}$ imply the corresponding Hölder regularity properties of F, while Hölder regularity properties of F imply certain 'Hölder-type' transversality properties of $\{\Omega_1,\Omega_2\}$ determined by the function ψ. Utilizing Proposition 1, they can be approximated by proper Hölder (or even linear) transversality properties.

Corollary 61 Let X and Y be normed spaces, $F : X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\alpha > 0$ and $q > 0$. Let Ω_1 and Ω_2 be defined by (3.4) and $\alpha_1 := 2^{-q}\alpha$. If F is $\alpha-$(semi-/sub-) transversal at $(\bar{x},\bar{y})$, then $\{\Omega_1,\Omega_2\}$ is $\alpha'-$(semi-/sub-)transversal of order q' at $\bar{x}$, where:

(i) if $q<1$, then $q'=q$ and α' is any number in $]0,\alpha_1[$;

(ii) if $q=1$, then $q'=1$ and $\alpha' := (1+\alpha_1^{-1})^{-1}$;

(iii) if $q>1$, then $q'=1$ and α' is any number in $]0,1[$.

Thanks to Corollaries 60 and 61, in the case $q\in]0,1]$ we have full equivalence between the two sets of properties. The following corollary recaptures [141, Proposition 10].

Corollary 62 Let X and Y be normed spaces, $F: X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, and $q\in]0,1]$. Let Ω_1 and Ω_2 be defined by (3.4).

(i) $\{\Omega_1,\Omega_2\}$ is $\alpha-$semitransversal of order q at $(\bar{x},\bar{y})$ if and only if F is semiregular of order q at $(\bar{x},\bar{y})$. Moreover,

$$\frac{\mathrm{s_e rg}_q[F](\bar{x},\bar{y})}{\mathrm{s_e rg}_q[F](\bar{x},\bar{y})+2^q} \le \mathrm{s_e tr}_q[\Omega_1,\Omega_2](\bar{x}) \le \frac{\mathrm{s_e rg}_q[F](\bar{x},\bar{y})}{2^q}.$$

(ii) $\{\Omega_1,\Omega_2\}$ is $\alpha-$subtransversal of order q at $(\bar{x},\bar{y})$ if and only if F is subregular of order q at $(\bar{x},\bar{y})$. Moreover,

$$\frac{\mathrm{srg}_q[F](\bar{x},\bar{y})}{\mathrm{srg}_q[F](\bar{x},\bar{y})+2^q} \le \mathrm{str}_q[\Omega_1,\Omega_2](\bar{x}) \le \frac{\mathrm{srg}_q[F](\bar{x},\bar{y})}{2^q}.$$

(iii) $\{\Omega_1,\Omega_2\}$ is $\alpha-$transversal of order q at $(\bar{x},\bar{y})$ if and only if F is regular of order q at $(\bar{x},\bar{y})$. Moreover,

$$\frac{\mathrm{rg}_q[F](\bar{x},\bar{y})}{\mathrm{rg}_q[F](\bar{x},\bar{y})+2^q} \le \mathrm{tr}_q[\Omega_1,\Omega_2](\bar{x}) \le \frac{\mathrm{rg}_q[F](\bar{x},\bar{y})}{2^q}.$$

The next theorem translates nonlinear transversality properties of the collection $\{\Omega_1,\Omega_2\}$ into certain metric properties of the mapping F, which can be used along with those in Definition 7.

Theorem 16 Let X and Y be normed spaces, $F: X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, Ω_1 and Ω_2 be defined by (3.4).

(i) If $\{\Omega_1,\Omega_2\}$ is $\varphi-$semitransversal at $(\bar{x},\bar{y})$ with some $\delta>0$, then

$$d(\bar{x}+u,F^{-1}(\bar{y}+v)) \le \varphi\left(\max\{\|u\|,\|v\|/2\}\right) \tag{3.6}$$

for all $u\in X$ and $v\in Y$ with $\varphi\left(\max\{\|u\|,\|v\|/2\}\right)<\delta$.

(ii) If $\{\Omega_1,\Omega_2\}$ is $\varphi-$subtransversal at $(\bar{x},\bar{y})$ with some $\delta_1>0$ and $\delta_2>0$, then

$$d(x,F^{-1}(\bar{y})) \le \varphi\left(\max\{d((x,y),\operatorname{gph} F),\|y-\bar{y}\|\}\right) \tag{3.7}$$

for all $x\in B_{\delta_2}(\bar{x})$ and $y\in B_{\min\{\varphi^{-1}(\delta_1),\delta_2\}}(\bar{y})$ with $\varphi(d((x,y),\operatorname{gph} F))<\delta_1$.

(iii) If $\{\Omega_1, \Omega_2\}$ is $\varphi-$transversal at $(\bar{x}, \bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then

$$d(x+u, F^{-1}(y+v)) \leq \varphi(\max\{\|u\|, \|v\|/2\}) \tag{3.8}$$

for all $(x,y) \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x}, \bar{y})$, and $u \in X$, $v \in Y$ with $\varphi(\max\{\|u\|, \|v\|/2\}) < \delta_1$.

Moreover, if $\varphi(t) \geq t$ for all $t \in]0, \varphi^{-1}(\delta)[$ in part (i), or $\varphi(t) \geq t$ for all $t \in]0, \varphi^{-1}(\delta_1)[$ in parts (ii) and (iii), then the respective implications hold as equivalences.

Proof

(i) Let $x_1 := (u_1, v_1)$, $x_2 := (u_2, v_2) \in X \times Y$. Then,

$$(\Omega_1 - x_1) \cap (\Omega_2 - x_2) = (F^{-1}(\bar{y} + v_1 - v_2) - u_1) \times \{\bar{y} - v_2\}, \tag{3.9}$$
$$d((\bar{x}, \bar{y}), (\Omega_1 - x_1) \cap (\Omega_2 - x_2)) = \max\{d(\bar{x} + u_1, F^{-1}(\bar{y} + v_1 - v_2)), \|v_2\|\}.$$

Thus, the inequality

$$d((\bar{x}, \bar{y}), (\Omega_1 - x_1) \cap (\Omega_2 - x_2)) \leq \varphi(\max\{\|x_1\|, \|x_2\|\})$$

implies

$$d(\bar{x} + u_1, F^{-1}(\bar{y} + v_1 - v_2)) \leq \varphi(\max\{\|u_1\|, \|u_2\|, \|v_1\|, \|v_2\|\}), \tag{3.10}$$

and if $\varphi(\|v_2\|) < \delta$, the converse implication is true when $\varphi(t) \geq t$ for all $t \in]0, \varphi^{-1}(\delta)[$.

We claim that the following conditions are equivalent:

(a) inequality (3.6) holds for all $u \in X$ and $v \in Y$ with $\varphi(\max\{\|u\|, \|v\|/2\}) < \delta$;

(b) inequality (3.10) holds for all $u_1, u_2 \in X$ and $v_1, v_2 \in Y$ with $\varphi(\max\{\|u_1\|, \|u_2\|, \|v_1\|, \|v_2\|\}) < \delta$.

(a) $\Rightarrow$ (b). Let $u_1, u_2 \in X$ and $v_1, v_2 \in Y$ with $\varphi(\max\{\|u_1\|, \|u_2\|, \|v_1\|, \|v_2\|\}) < \delta$. Then inequality (3.6) holds for u_1 and $v_1 - v_2$ in place of u and v, i.e.

$$d(\bar{x} + u_1, F^{-1}(\bar{y} + v_1 - v_2)) \leq \varphi(\max\{\|u_1\|, \|v_1 - v_2\|\}),$$

and consequently, inequality (3.10) holds.

(b) $\Rightarrow$ (a). Let $u \in X$ and $v \in Y$ with $\varphi(\max\{\|u\|, \|v\|/2\}) < \delta$. Then, inequality (3.10) holds for $u_1 := u$, $u_2 := 0$, $v_1 := v/2$ and $v_2 := -v/2$, which is equivalent to inequality (3.6).

Thus, (a) $\Leftrightarrow$ (b), which, in view of Theorem 1(i), proves the assertion.

(ii) Let $(x,y) \in X \times Y$. Thanks to (3.4), we have

$$d((x,y),\Omega_2) = \|y-\bar{y}\|, \;\; d((x,y),\Omega_1\cap\Omega_2) = \max\{d(x,F^{-1}(\bar{y})),\|y-\bar{y}\|\}.$$

Thus, the inequality

$$d\left((x,y),\Omega_1\cap\Omega_2\right) \le \varphi(\max\left\{d((x,y),\Omega_1),d((x,y),\Omega_2)\right\}$$

implies inequality (3.7), and if $\varphi(\|y-\bar{y}\|) < \delta_1$, the converse implication is true when $\varphi(t) \ge t$ for all $t \in]0,\varphi^{-1}(\delta_1)[$. In view of Theorem 1(ii), this proves the assertion.

(iii) Let $x_1 := (u_1,v_1)$, $x_2 := (u_2,v_2)$, $(x,y) \in X \times Y$ and $z \in X$. Then,

$$\begin{aligned}
&(\Omega_1-(x,y)-x_1)\cap(\Omega_2-(z,\bar{y})-x_2)\\
&\qquad\qquad = (F^{-1}(y+v_1-v_2)-x-u_1)\times\{-v_2\},\\
&d\left((0,0),(\Omega_1-(x,y)-x_1)\cap(\Omega_2-(z,\bar{y})-x_2)\right)\\
&\qquad\qquad = \max\{d(x+u_1,F^{-1}(y+v_1-v_2)),\|v_2\|\}.
\end{aligned}$$

Thus, the inequality

$$d\left((0,0),(\Omega_1-(x,y)-x_1)\cap(\Omega_2-(z,\bar{y})-x_2)\right) \le \varphi\left(\max\{\|x_1\|,\|x_2\|\}\right) \tag{3.11}$$

implies

$$d(x+u_1,F^{-1}(y+v_1-v_2)) \le \varphi(\max\{\|u_1\|,\|u_2\|,\|v_1\|,\|v_2\|\}), \tag{3.12}$$

and, if $\varphi(\|v_2\|) < \delta_1$, the converse implication is true when $\varphi(t) \ge t$ for all $t \in]0,\varphi^{-1}(\delta_1)[$. The same arguments as in the proof of (i) show that inequality (3.12) holds for all $(x,y) \in \Omega_1 \cap B_{\delta_2}(\bar{x},\bar{y})$, $u_1,u_2 \in X$ and $v_1,v_2 \in Y$ with $\varphi(\max\{\|u_1\|,\|u_2\|,\|v_1\|,\|v_2\|\}) < \delta$ if and only if inequality (3.8) holds for all $(x,y) \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x},\bar{y})$, $u \in X$ and $v \in Y$ with $\varphi\left(\max\left\{\|u\|,\|v\|/2\right\}\right) < \delta$. In view of Theorem 1(iii), this proves the assertion.

□

Using the estimates in the proof of Theorem 16, we can also translate the metric characterizations of the nonlinear transversality in Theorem 2 into certain metric conditions involving the set-valued mapping F.

Proposition 22 Let X and Y be normed spaces, $F : X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, Ω_1 and Ω_2 be defined by (3.4), $\delta_1 > 0$ and $\delta_2 > 0$. The following properties are equivalent:

(i) for all $(x,y) \in \operatorname{gph} F$, $u \in X$ and $v_1,v_2 \in Y$ with $\varphi(\max\{\|u\|,\|v_1\|\}) < \delta_1$, $\varphi(\|v_2\|) < \min\{\delta_1,\varphi(\delta_2)\}$, and $(x,y)+(u,v_1) \in B_{\delta_2}(\bar{x},\bar{y})$, it holds

$$d(x+u,F^{-1}(y+v_1-v_2)) \le \varphi(\max\{\|u\|,\|v_1\|,\|v_2\|\}); \tag{3.13}$$

(ii) for all $u \in X$ and $v_1, v_2 \in Y$ with $\max\{\|u\|, \|v_1\|\} < \delta_2$, $\varphi(\|v_2\|) < \min\{\delta_1, \varphi(\delta_2)\}$ and $\varphi(d((\bar{x},\bar{y})+(u,v_1), \mathrm{gph}\, F)) < \delta_1$, it holds

$$d(\bar{x}+u, F^{-1}(\bar{y}+v_1-v_2)) \leq \varphi(\max\{d((\bar{x},\bar{y})+(u,v_1), \mathrm{gph}\, F), \|v_2\|\}); \tag{3.14}$$

(iii) for all (x,y), $(u,v_1) \in X \times Y$ and $v_2 \in Y$ with $(x,y)+(u,v_1) \in B_{\delta_2}(\bar{x},\bar{y})$, $\varphi(d((x,y)+(u,v_1), \mathrm{gph}\, F)) < \delta_1$, and $\varphi(\|y+v_2-\bar{y}\|) < \min\{\delta_1, \varphi(\delta_2)\}$, it holds

$$\begin{aligned} &d(x+u, F^{-1}(y+v_1-v_2)) \\ &\qquad \leq \varphi\left(\max\left\{d\left((x,y)+(u,v_1), \mathrm{gph}\, F\right), \|y+v_2-\bar{y}\|\right\}\right). \end{aligned} \tag{3.15}$$

Moreover, if $\{\Omega_1, \Omega_2\}$ is $\varphi-$transversal at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then conditions (i)–(iii) hold with any $\delta_1' \in]0, \delta_1]$ and $\delta_2' > 0$ satisfying $\varphi^{-1}(\delta_1') + \delta_2' \leq \delta_2$ in place of δ_1 and δ_2.

Conversely, if properties (i)–(iii) hold with some $\delta_1 > 0$ and $\delta_2 > 0$, and $\varphi(t) \geq t$ for all $t \in]0, \varphi^{-1}(\delta_1)[$, then $\{\Omega_1, \Omega_2\}$ is $\varphi-$transversal at $(\bar{x},\bar{y})$ with any $\delta_1' \in]0, \delta_1]$ and $\delta_2' > 0$ satisfying $\varphi^{-1}(\delta_1') + \delta_2' \leq \delta_2$.

Proof

(i) Given $x_1 := (u_1, v_1)$, $x_2 := (u_2, v_2)$, $(x,y) \in X \times Y$ and $z \in X$, inequality (3.11) implies (3.12), and the conditions are equivalent if $\varphi(\|v_2\|) < \delta_1$, and $\varphi(t) \geq t$ for all $t \in]0, \varphi^{-1}(\delta_1)[$. Moreover, given $x, u \in X$ and $y, v_1, v_2 \in Y$, inequality (3.12) holds with $u_1 := u$ for all $u_2 \in X$ with $\varphi(\|u_2\|) < \delta_1$ if and only if inequality (3.13) is satisfied. Hence, condition (i) is equivalent to the one in Proposition 2(i).

(ii) Given $x_1 := (u_1, v_1)$ and $x_2 := (u_2, v_2)$, inequality

$$\begin{aligned} &d\left((\bar{x},\bar{y}), (\Omega_1 - x_1) \cap (\Omega_2 - x_2)\right) \\ &\qquad \leq \varphi\left(\max\{d((\bar{x},\bar{y}), \Omega_1 - x_1), d((\bar{x},\bar{y}), \Omega_2 - x_2)\}\right) \end{aligned}$$

implies

$$d(\bar{x}+u_1, F^{-1}(\bar{y}+v_1-v_2)) \leq \varphi\left(\max\{d((\bar{x},\bar{y})+(u_1,v_1), \mathrm{gph}\, F), \|v_2\|\}\right), \tag{3.16}$$

and the converse implication is true if $\varphi(\|v_2\|) < \delta_1$ and, $\varphi(t) \geq t$ for all $t \in]0, \varphi^{-1}(\delta_1)[$. Observe that inequality (3.16) holds with $u_1 := u$ if and only if inequality (3.14) is satisfied. Hence, condition (ii) is equivalent to the one in Proposition 2(ii).

(iii) Given $x_1 := (u_1, v_1)$, $x_2 := (u_2, v_2)$ and $(x,y) \in X \times Y$, we have representation (3.9), and consequently,

$$\begin{aligned} &d\left((x,y), (\Omega_1 - x_1) \cap (\Omega_2 - x_2)\right) \\ &\qquad = \max\left\{d(x+u_1, F^{-1}(y+v_1-v_2)), \|y+v_2-\bar{y}\|\right\}. \end{aligned}$$

Thus, the inequality

$$d\left((x,y),(\Omega_1-x_1)\cap(\Omega_2-x_2)\right) \\ \leq \varphi\left(\max\left\{d\left((x,y),\Omega_1-x_1\right),d\left((x,y),\Omega_2-x_2\right)\right\}\right)$$

implies inequality (3.15), and the converse implication is true if $\varphi(\|y+v_2-\bar{y}\|)<\delta_1$, and $\varphi(t)\geq t$ for all $t\in]0,\varphi^{-1}(\delta_1)[$. Hence, condition (iii) is equivalent to Proposition 2(iii).

The remaining conclusions follow from Proposition 4 and Theorem 2. □

Thanks to Proposition 6, we can formulate a simplified version of Proposition 22 for the convex case.

Corollary 63 Let X and Y be normed spaces, $F:X\rightrightarrows Y$ have a convex graph, $(\bar{x},\bar{y})\in\operatorname{gph}F$, Ω_1 and Ω_2 be defined by (3.4), $\delta_1>0$, $\delta_2>0$, and $\varphi\in\widehat{\mathscr{C}}_{\delta_1}$. Properties (i)–(iii) in Theorem 2 hold if and only if the following equivalent conditions hold true:

(i) $F^{-1}(y+v_1-v_2)\cap B_{\delta_1}(x+u)\neq\emptyset$ for all $(x,y)\in\operatorname{gph}F$, $u\in X$ and $v_1,v_2\in Y$ with $\varphi(\max\{\|u\|,\|v_1\|\})<\delta_1$, $\varphi(\|v_2\|)<\min\{\delta_1,\varphi(\delta_2)\}$, and $(x,y)+(u,v_1)\in B_{\delta_2}(\bar{x},\bar{y})$;

(ii) $F^{-1}(\bar{y}+v_1-v_2)\cap B_{\delta_1}(\bar{x}+u)\neq\emptyset$ for all $u\in X$ and $v_1,v_2\in Y$ with $\max\{\|u\|,\|v_1\|\}<\delta_2$, $\varphi(\|v_2\|)<\min\{\delta_1,\varphi(\delta_2)\}$, and $\varphi(d((\bar{x},\bar{y})+(u,v_1),\operatorname{gph}F))<\delta_1$;

(iii) $F^{-1}(y+v_1-v_2)\cap B_{\delta_1}(x+u)\neq\emptyset$ for all (x,y), $(u,v_1)\in X\times Y$ and $v_2\in Y$ with $(x,y)+(u,v_1)\in B_{\delta_2}(\bar{x},\bar{y})$, $\varphi(d((x,y)+(u,v_1),\operatorname{gph}F))<\delta_1$, and $\varphi(\|y+v_2-\bar{y}\|)<\min\{\delta_1,\varphi(\delta_2)\}$.

3.1.2 Transversality of a Set-Valued Mapping to a Set

We now discuss nonlinear extensions of the new *transversality properties of a set-valued mapping to a set in the range space* due to Ioffe [114, 116]. In the rest of this section, $F:X\rightrightarrows Y$ is a set-valued mapping between normed spaces, $(\bar{x},\bar{y})\in\operatorname{gph}F$, S is a subset of Y, $\bar{y}\in S$, and $\varphi\in\mathscr{C}$.

Definition 8 The mapping F is

(i) $\varphi-$semitransversal to S at $(\bar{x},\bar{y})$ if $\{\operatorname{gph}F,X\times S\}$ is $\varphi-$semitransversal at $(\bar{x},\bar{y})$, i.e. there exists a $\delta>0$ such that

$$(\operatorname{gph}F-(u_1,v_1))\cap(X\times(S-v_2))\cap B_\rho(\bar{x},\bar{y})\neq\emptyset \tag{3.17}$$

for all $\rho\in]0,\delta[$, $u_1\in X$, $v_1,v_2\in Y$ with $\varphi\left(\max\{\|u_1\|,\|v_1\|,\|v_2\|\}\right)<\rho$;

(ii) $\varphi-$subtransversal to S at $(\bar{x},\bar{y})$ if $\{\operatorname{gph}F,X\times S\}$ is $\varphi-$subtransversal at $(\bar{x},\bar{y})$, i.e. there exist $\delta_1>0$ and $\delta_2>0$ such that

$$\operatorname{gph}F\cap(X\times S)\cap B_\rho(x,y)\neq\emptyset \tag{3.18}$$

for all $\rho\in]0,\delta_1[$ and $(x,y)\in B_{\delta_2}(\bar{x},\bar{y})$ with $\varphi(\max\{d((x,y),\operatorname{gph}F),d(y,S)\})<\rho$;

(iii) $\varphi-$transversal to S at $(\bar{x},\bar{y})$ if $\{\operatorname{gph} F, X\times S\}$ is $\varphi-$transversal at $(\bar{x},\bar{y})$, i.e. there exist $\delta_1>0$ and $\delta_2>0$ such that

$$(\operatorname{gph} F-(x_1,y_1)-(u_1,v_1))\cap(X\times(S-y_2-v_2))\cap(\rho\mathbb{B})\neq\emptyset \tag{3.19}$$

for all $\rho\in]0,\delta_1[$, $(x_1,y_1)\in\operatorname{gph} F\cap B_{\delta_2}(\bar{x},\bar{y})$, $y_2\in S\cap B_{\delta_2}(\bar{y})$, $u_1\in X$, $v_1,v_2\in Y$ with $\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\})<\rho$.

The two-set model $\{\operatorname{gph} F, X\times S\}$ employed in Definition 8 is an extension of the model (3.4), which corresponds to the case when S is a singleton: $S:=\{\bar{y}\}$.

The metric characterizations of the properties in the next two statements are consequences of Theorems 1 and 2, respectively. Each characterization can be used as an equivalent definition for the respective property.

Corollary 64 The mapping F is

(i) $\varphi-$semitransversal to S at $(\bar{x},\bar{y})$ with some $\delta>0$ if and only if

$$d\big((\bar{x},\bar{y}),(\operatorname{gph} F-(x_1,y_1))\cap(X\times(S-y_2))\big)\leq\varphi(\max\{\|x_1\|,\|y_1\|,\|y_2\|\})$$

for all $x_1\in X$, $y_1,y_2\in Y$ with $\varphi(\max\{\|x_1\|,\|y_1\|,\|y_2\|\})<\delta$;

(ii) is $\varphi-$subtransversal to S at $(\bar{x},\bar{y})$ with some $\delta_1>0$ and $\delta_2>0$ if and only if the following equivalent conditions hold:

(a) for all $(x,y)\in B_{\delta_2}(\bar{x},\bar{y})$ with $\varphi\big(\max\{d((x,y),\operatorname{gph} F),d(y,S)\}\big)<\delta_1$, it holds

$$d((x,y),\operatorname{gph} F\cap(X\times S))\leq\varphi(\max\{d((x,y),\operatorname{gph} F),d(y,S)\});$$

(b) for all $(x_1,y_1)\in\operatorname{gph} F\cap B_{\delta_2}(\bar{x},\bar{y})$, $y_2\in S\cap B_{\delta_2}(\bar{y})$ and $u_1\in X$, $v_1,v_2\in Y$ with $\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\})<\delta_1$ and $x_1+u_1\in B_{\delta_2}(\bar{x})$, $y_1+v_1=y_2+v_2\in B_{\delta_2}(\bar{y})$, it holds

$$\begin{aligned}d((0,0),(\operatorname{gph} F-(x_1,y_1)-(u_1,v_1))\cap(X\times(S-y_2-v_2)))\\ \leq\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\});\end{aligned} \tag{3.20}$$

(iii) $\varphi-$transversal to S at $(\bar{x},\bar{y})$ with some $\delta_1>0$ and $\delta_2>0$ if and only if inequality (3.20) holds for all $(x_1,y_1)\in\operatorname{gph} F\cap B_{\delta_2}(\bar{x},\bar{y})$, $y_2\in S\cap B_{\delta_2}(\bar{y})$ and $u_1\in X$, $v_1,v_2\in Y$ with $\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\})<\delta_1$.

Corollary 65 Let $\delta_1>0$ and $\delta_2>0$. The following conditions are equivalent:

(i) for all $(x_1,y_1)\in\operatorname{gph} F\cap B_{\delta_2}(\bar{x},\bar{y})$, $y_2\in S\cap B_{\delta_2}(\bar{y})$ and $u_1\in X$, $v_1,v_2\in Y$ with $x_1+u_1\in B_{\delta_2}(\bar{x})$, $y_1+v_1,y_2+v_2\in B_{\delta_2}(\bar{y})$ and $\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\})<\delta_1$, inequality (3.20) holds true;

(ii) for all $x_1, y_1, y_2 \in \delta_2\mathbb{B}$ with $\varphi(\max\{d((\bar{x},\bar{y}),\operatorname{gph}F-(x_1,y_1)),d(\bar{y},S-y_2)\})<\delta_1$, it holds

$$d\left((\bar{x},\bar{y}),(\operatorname{gph}F-(x_1,y_1))\cap(X\times(S-y_2))\right)$$
$$\leq\varphi\left(\max\{d((\bar{x},\bar{y}),\operatorname{gph}F-(x_1,y_1)),d(\bar{y},S-y_2)\}\right); \quad (3.21)$$

(iii) for all $x, x_1 \in X$, $y, y_1, y_2 \in Y$ such that $x+x_1 \in B_{\delta_2}(\bar{x})$, $y+y_1, y+y_2 \in B_{\delta_2}(\bar{y})$ and $\varphi\left(\max\{d((x,y),\operatorname{gph}F-(x_1,y_1)),d(y,S-y_2)\}\right)<\delta_1$, it holds

$$d\big((x,y),(\operatorname{gph}F-(x_1,y_1))\cap(X\times(S-y_2))\big)$$
$$\leq\varphi\big(\max\{d((x,y),\operatorname{gph}F-(x_1,y_1)),d(y,S-y_2)\}\big).$$

Moreover, if F is $\varphi-$transversal to S at $(\bar{x},\bar{y})$ with some $\delta_1>0$ and $\delta_2>0$, then conditions (i)–(iii) hold with any $\delta_1'\in]0,\delta_1]$ and $\delta_2'>0$ satisfying $\varphi^{-1}(\delta_1')+\delta_2'\leq\delta_2$ in place of δ_1 and δ_2.

Conversely, if conditions (i)–(iii) hold with some $\delta_1>0$ and $\delta_2>0$, then F is $\varphi-$transversal to S at $(\bar{x},\bar{y})$ with any $\delta_1'\in]0,\delta_1]$ and $\delta_2'>0$ satisfying $\varphi^{-1}(\delta_1')+\delta_2'\leq\delta_2$.

Remark 39 In the linear case, i.e. when $\varphi(t):=\alpha t$ for some $\alpha>0$ and all $t\geq 0$, in view of Corollaries 64(ii)(a) and 65(iii), the properties in parts (ii) and (iii) of Definition 8 reduce, respectively, to the ones in [116, Definitions 7.11 and 7.8]. The property in part (i) is new.

The next two statements provide characterizations of the properties in the convex case. They are direct consequences of Propositions 5 and 6, respectively.

Proposition 23 Suppose $\operatorname{gph}F$ and S are convex, $\delta>0$, and $\varphi\in\widehat{\mathscr{C}}_\delta$. The mapping F is

(i) $\varphi-$semitransversal to S at $(\bar{x},\bar{y})$ with δ if and only if

$$(\operatorname{gph}F-(u_1,v_1))\cap(X\times(S-v_2))\cap B_\delta(\bar{x},\bar{y})\neq\emptyset \tag{3.22}$$

for all $u_1\in X$ and $v_1,v_2\in Y$ with $\varphi\left(\max\{\|u_1\|,\|v_1\|,\|v_2\|\}\right)<\delta$;

(ii) $\varphi-$transversal to S at $(\bar{x},\bar{y})$ with $\delta_1:=\delta$ and some $\delta_2>0$ if and only if

$$(\operatorname{gph}F-(x_1,y_1)-(u_1,v_1))\cap(X\times(S-y_2-v_2))\cap(\delta_1\mathbb{B})\neq\emptyset \tag{3.23}$$

for all $(x_1,y_1)\in\operatorname{gph}F\cap B_{\delta_2}(\bar{x},\bar{y})$, $y_2\in S\cap B_{\delta_2}(\bar{y})$, $u_1\in X$ and $v_1,v_2\in Y$ with $\varphi\left(\max\{\|u_1\|,\|v_1\|,\|v_2\|\}\right)<\delta_1$.

Proposition 24 Suppose $\operatorname{gph}F$ and S are convex, $\delta_1>0$, $\delta_2>0$, and $\varphi\in\widehat{\mathscr{C}}_{\delta_1}$. The following properties are equivalent:

(i) condition (3.23) holds for all $(x_1,y_1)\in\operatorname{gph}F$, $y_2\in S$, $u_1\in X$ and $v_1,v_2\in Y$ with $x_1+u_1\in B_{\delta_2}(\bar{x})$, $y_1+v_1, y_2+v_2\in B_{\delta_2}(\bar{y})$ and $\varphi\left(\max\{\|u_1\|,\|v_1\|,\|v_2\|\}\right)<\delta_1$;

(ii) condition (3.22) holds with δ_1 in place of δ for all $u_1 \in \delta_2\mathbb{B}_X$ and $v_1, v_2 \in \delta_2\mathbb{B}_Y$ with $\varphi(\max\{d((\bar{x},\bar{y}), \operatorname{gph} F - (u_1,v_1)), d(\bar{y}, S - v_2)\}) < \delta_1$;

(iii) for all $x, u_1 \in X$ and $y, v_1, v_2 \in Y$ such that $x + u_1 \in B_{\delta_2}(\bar{x})$, $y + v_1, y + v_2 \in B_{\delta_2}(\bar{y})$ and $\varphi(\max\{d((x,y), \operatorname{gph} F - (u_1,v_1)), d(y, S - v_2)\}) < \delta_1$, it holds

$$(\operatorname{gph} F - (u_1,v_1)) \cap (X \times (S - v_2)) \cap B_{\delta_1}(x,y) \neq \emptyset.$$

Moreover, if F is $\varphi-$transversal to S at $(\bar{x},\bar{y})$ with δ_1 and some $\delta_2 > 0$, then properties (i)–(iii) hold with any $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ satisfying $\varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$ in place of δ_1 and δ_2.

Conversely, if properties (i)–(iii) hold with δ_1 and some $\delta_2 > 0$, then F is $\varphi-$transversal to S at $(\bar{x},\bar{y})$ with any $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ satisfying $\varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$.

The set-valued mapping (3.2), crucial for establishing equivalences between transversality properties of collections of sets and the corresponding regularity properties of set-valued mappings, in the setting considered here translates into the mapping $G : X \times Y \rightrightarrows (X \times Y) \times (X \times Y)$ of the following form:

$$G(x,y) := \big(\operatorname{gph} F - (x,y)\big) \times \big(X \times (S - y)\big), \quad (x,y) \in X \times Y. \tag{3.24}$$

Observe that $G^{-1}(x_1,y_1,x_2,y_2) = \big(\operatorname{gph} F - (x_1,y_1)\big) \cap \big(X \times (S - y_2)\big)$ for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and, if $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\bar{y} \in S$, then $\big((0,0),(0,0)\big) \in G(\bar{x},\bar{y})$.

The relationships between the nonlinear transversality and regularity properties in the next statement are direct consequences of Theorem 13 and Proposition 21.

Theorem 17 Let G be defined by (3.24).

(i) F is $\varphi-$semitransversal to S at $(\bar{x},\bar{y})$ with some $\delta > 0$ if and only if G is $\varphi-$semiregular at $\big((\bar{x},\bar{y}),(0,0),(0,0)\big)$ with δ.

(ii) F is $\varphi-$subtransversal to S at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$ if and only if G is $\varphi-$subregular at $\big((\bar{x},\bar{y}),(0,0),(0,0)\big)$ with δ_1 and δ_2.

(iii) If F is $\varphi-$transversal to S at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then G is $\varphi-$regular at $\big((\bar{x},\bar{y}),(0,0),(0,0)\big)$ with any $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ satisfying $\delta_2' + \varphi^{-1}(\delta_1') \le \delta_2$.

Conversely, if G is $\varphi-$regular at $\big((\bar{x},\bar{y}),(0,0),(0,0)\big)$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then F is $\varphi-$transversal to S at $(\bar{x},\bar{y})$ with any $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ satisfying $\delta_2' + \varphi^{-1}(\delta_1') \le \delta_2$.

Remark 40 It is easy to see that the set-valued mapping (3.24) can be replaced in our considerations by the truncated mapping $\mathscr{G} : X \times Y \rightrightarrows X \times Y \times Y$ defined by

$$\mathscr{G}(x,y) := \big(\operatorname{gph} F - (x,y)\big) \times (S - y), \quad (x,y) \in X \times Y.$$

The last mapping admits a simple representation $\mathscr{G}(x,y) = \operatorname{gph}\mathscr{F} - (x,y,y)$, where the set-valued mapping $\mathscr{F} : X \rightrightarrows Y \times Y$ is defined by

$$\mathscr{F}(x) := F(x) \times S, \quad x \in X.$$

It was shown in [116, Theorems 7.12 and 7.9] that in the linear case the subtransversality and transversality of F to S at $(\bar{x},\bar{y})$ are equivalent to the metric subregularity and regularity, respectively, of the mapping $(x,y) \mapsto \mathscr{F}(x) - (y,y)$ at $((\bar{x},\bar{y}),0)$.

In the rest of this section, we assume that X and Y are Banach spaces, $\operatorname{gph} F$ and S are closed, and $\varphi \in \mathscr{C}^1$; δ, δ_1 and δ_2 are given positive numbers.

The dual characterizations of the nonlinear transversality properties in the next three statements are direct consequences of Theorems 7, 8 and 9, respectively.

Theorem 18 The mapping F is $\varphi-$semitransversal to S at $(\bar{x},\bar{y})$ with δ if, for some $\mu > 0$ and any $u_1 \in X$, $v_1, v_2 \in Y$ with $0 < \max\{\|u_1\|, \|v_1\|, \|v_2\|\} < \varphi^{-1}(\delta)$, one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\}), \delta[$ such that

$$\begin{aligned}\varphi'(\max\{\|x_1-u_1-x\|, \|y_1-v_1-y\|, \|y_2-v_2-y\|\})\big(\|x_1^*\| + \|y_1^*+y_2^*\| \\ + \mu\left(d((x_1^*,y_1^*), N_{\operatorname{gph} F}(x_1,y_1)) + d(y_2^*, N_S(y_2))\right)\big) \geq 1 \quad (3.25)\end{aligned}$$

for all $(x,y) \in X \times Y$, $(x_1,y_1) \in \operatorname{gph} F$ and $y_2 \in S$ satisfying

$$\max\{\|x-\bar{x}\|, \|y-\bar{y}\|\} < \lambda, \quad \max\{\|x_1-\bar{x}\|, \|y_1-\bar{y}\|, \|y_2-\bar{y}\|\} < \mu\lambda, \tag{3.26}$$

$$\begin{aligned}0 < \max\{\|x_1-u_1-x\|, \|y_1-v_1-y\|, \|y_2-v_2-y\|\} \\ \leq \max\{\|u_1\|, \|v_1\|, \|v_2\|\}, \quad (3.27)\end{aligned}$$

and all $x_1^* \in X^*$, $y_1^*, y_2^* \in Y^*$ satisfying

$$\|x_1^*\| + \|y_1^*\| + \|y_2^*\| = 1, \tag{3.28}$$

$$\begin{aligned}\langle x_1^*, x+u_1-x_1\rangle + \langle y_1^*, y+v_1-y_1\rangle + \langle y_2^*, y+v_2-y_2\rangle \\ = \max\{\|x_1-u_1-x\|, \|y_1-v_1-y\|, \|y_2-v_2-y\|\}, \quad (3.29)\end{aligned}$$

where N in (3.25) stands for the Clarke normal cone ($N := N^C$);

(ii) X and Y are Asplund spaces, and there exist a $\lambda \in]\varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\})$ and a $\tau \in]0,1[$ such that inequality (3.25) holds with N standing for the Fréchet normal cone ($N := N^F$) for all $(x,y) \in X \times Y$, $(x_1,y_1) \in \operatorname{gph} F$ and $y_2 \in S$ satisfying (3.26) and

$$0 < \max\{\|x_1-u_1-x\|, \|y_1-v_1-y\|, \|y_2-v_2-y\|\} < \varphi^{-1}(\lambda), \tag{3.30}$$

and all $x_1^* \in X^*$, $y_1^*, y_2^* \in Y^*$ satisfying (3.28) and

$$\begin{aligned}\langle x_1^*, x+u_1-x_1\rangle + \langle y_1^*, y+v_1-y_1\rangle + \langle y_2^*, y+v_2-y_2\rangle \\ > \tau\max\{\|x_1-u_1-x\|, \|y_1-v_1-y\|, \|y_2-v_2-y\|\}. \quad (3.31)\end{aligned}$$

Theorem 19 The mapping F is $\varphi-$subtransversal to S at $(\bar{x},\bar{y})$ with δ_1 and δ_2 if, for some $\mu > 0$ and any $(x',y') \in B_{\delta_2}(\bar{x},\bar{y})$ with $0 < \max\{d((x',y'),\mathrm{gph}\,F), d(y',S)\} < \varphi^{-1}(\delta_1)$, one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\varphi(\max\{d((x',y'),\mathrm{gph}\,F), d(y',S)\}), \delta_1[$ such that

$$\varphi'(\max\{\|x_1-x\|,\|y_1-y\|,\|y_2-y\|\})\left(\|x_1^*\|+\|y_1^*+y_2^*\|\right.$$
$$\left.+\mu\left(d\left((x_1^*,y_1^*),N_{\mathrm{gph}\,F}(x_1,y_1)\right)+d(y_2^*,N_S(y_2))\right)\right) \geq 1 \quad (3.32)$$

for all $(x,y) \in X\times Y$ and $(x_1,y_1),(x_1',y_1') \in \mathrm{gph}\,F$, $y_2,y_2' \in S$ satisfying

$$\max\{\|x-x'\|,\|y-y'\|\} < \lambda,\ \max\{\|x_1-x_1'\|,\|y_1-y_1'\|,\|y_2-y_2'\|\} < \mu\lambda, \quad (3.33)$$
$$0 < \max\{\|x_1-x\|,\|y_1-y\|,\|y_2-y\|\}$$
$$\leq \max\{\|x_1'-x'\|,\|y_1'-y'\|,\|y_2'-y'\|\} < \varphi^{-1}(\lambda),$$

and all $x_1^* \in X^*$, $y_1^*,y_2^* \in Y^*$ satisfying (3.28) and

$$\langle x_1^*, x-x_1\rangle + \langle y_1^*, y-y_1\rangle + \langle y_2^*, y-y_2\rangle = \max\{\|x-x_1\|,\|y-y_1\|,\|y-y_2\|\}, \quad (3.34)$$

where N in (3.32) stands for the Clarke normal cone ($N := N^C$);

(ii) X and Y are Asplund spaces, and there exist numbers $\lambda \in]\varphi(\max\{d((x',y'),\mathrm{gph}\,F), d(y',S)\}), \delta_1[$ and $\tau \in]0,1[$ such that inequality (3.32) holds with N standing for the Fréchet normal cone ($N := N^F$) for all $(x,y) \in X\times Y$ and $(x_1,y_1),(x_1',y_1') \in \mathrm{gph}\,F$, $y_2,y_2' \in S$ satisfying (3.33) and $0 < \max\{\|x_1-x\|,\|y_1-y\|,\|y_2-y\|\} < \varphi^{-1}(\lambda)$, and all $x_1^* \in X^*$, $y_1^*,y_2^* \in Y^*$ satisfying (3.28) and

$$\langle x_1^*, x-x_1\rangle + \langle y_1^*, y-y_1\rangle + \langle y_2^*, y-y_2\rangle$$
$$> \tau\max\{\|x-x_1\|,\|y-y_1\|,\|y-y_2\|\}. \quad (3.35)$$

Theorem 20 The mapping F is $\varphi-$transversal to S at $(\bar{x},\bar{y})$ with δ_1 and δ_2 if, for some $\mu > 0$ and any $(x_1',y_1') \in \mathrm{gph}\,F \cap B_{\delta_2}(\bar{x},\bar{y})$, $y_2' \in S\cap B_{\delta_2}(\bar{y})$ and $\xi \in]0,\varphi^{-1}(\delta_1)[$, one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\varphi(\xi),\delta_1[$ such that inequality (3.25) holds with N standing for the Clarke normal cone ($N := N^C$) for all $(x,y) \in X\times Y$, $(x_1,y_1) \in \mathrm{gph}\,F$, $y_2 \in S$ and $u_1 \in X$, $v_1,v_2 \in Y$ satisfying

$$\max\{\|x-\bar{x}\|,\|y-\bar{y}\|\} < \lambda,\ \max\{\|x_1-x_1'\|,\|y_1-y_1'\|,\|y_2-y_2'\|\} < \mu\lambda, \quad (3.36)$$
$$0 < \max\{\|x_1-u_1-x\|,\|y_1-v_1-y\|,\|y_2-v_2-y\|\}$$
$$\leq \max\{\|x_1'-u_1-\bar{x}\|,\|y_1'-v_1-\bar{y}\|,\|y_2'-v_2-\bar{y}\|\} = \xi, \quad (3.37)$$

and all $x_1^* \in X^*$, $y_1^*,y_2^* \in Y^*$ satisfying (3.28) and (3.29);

(ii) X and Y are Asplund spaces, and there exist numbers $\lambda \in]\varphi(\xi), \delta_1[$ and $\tau \in]0,1[$ such that inequality (3.25) holds with N standing for the Fréchet normal cone ($N := N^F$) for all $(x,y) \in X \times Y$, $(x_1,y_1) \in \operatorname{gph} F$, $y_2 \in S$ and $u_1 \in X$, $v_1, v_2 \in Y$ satisfying (3.30) and (3.36), and all $x_1^* \in X^*$, $y_1^*, y_2^* \in Y^*$ satisfying (3.28) and (3.31).

In the Hölder setting, Definition 8 takes the following form.

Definition 9 Let $\alpha > 0$. The mapping F is

(i) $\alpha-$semitransversal of order $q > 0$ to S at $(\bar{x}, \bar{y})$ if there exists a $\delta > 0$ such that condition (3.17) is satisfied for all $\rho \in]0, \delta[$, $u_1 \in X$, $v_1, v_2 \in Y$ with $(\max\{\|u_1\|, \|v_1\|, \|v_2\|\})^q < \alpha\rho$;

(ii) $\alpha-$subtransversal of order $q \in]0,1]$ to S at $(\bar{x}, \bar{y})$ if there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that condition (3.18) is satisfied for all $\rho \in]0, \delta_1[$ and $(x,y) \in B_{\delta_2}(\bar{x}, \bar{y})$ with $(\max\{d((x,y), \operatorname{gph} F), d(y,S)\})^q < \alpha\rho$;

(iii) $\alpha-$transversal of order $q \in]0,1]$ to S at $(\bar{x}, \bar{y})$ if there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that condition (3.19) is satisfied for all $\rho \in]0, \delta_1[$, $(x_1,y_1) \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x}, \bar{y})$, $y_2 \in S \cap B_{\delta_2}(\bar{y})$, $u_1 \in X$, $v_1, v_2 \in Y$ with $(\max\{\|u_1\|, \|v_1\|, \|v_2\|\})^q < \alpha\rho$.

In the linear case, i.e. when $\varphi(t) = \alpha t$ for some $\alpha > 0$ and all $t \geq 0$, the properties in parts (ii) and (iii) of the above definition reduce, respectively, to the ones in [116, Definitions 7.11 and 7.8]. The property in part (i) is new.

Dual characterizations of the Hölder transversality properties follow immediately.

Corollary 66 Let $\alpha > 0$ and $q > 0$. The mapping F is $\alpha-$semitransversal of order q to S at $(\bar{x}, \bar{y})$ with δ if, for some $\mu > 0$ and any $u_1 \in X$, $v_1, v_2 \in Y$ with $0 < \max\{\|u_1\|, \|v_1\|, \|v_2\|\} < (\alpha\delta)^{\frac{1}{q}}$, one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\alpha^{-1}(\max\{\|u_1\|, \|v_1\|, \|v_2\|\})^q, \delta[$ such that

$$q(\max\{\|x_1 - u_1 - x\|, \|y_1 - v_1 - y\|, \|y_2 - v_2 - y\|\})^{q-1}\left(\|x_1^*\| + \|y_1^* + y_2^*\| \right.$$
$$\left. + \mu\left(d((x_1^*, y_1^*), N_{\operatorname{gph} F}(x_1,y_1)) + d(y_2^*, N_S(y_2))\right)\right) \geq \alpha \quad (3.38)$$

for all $(x,y) \in X \times Y$, $(x_1,y_1) \in \operatorname{gph} F$ and $y_2 \in S$ satisfying (3.26) and (3.27), and all $x_1^* \in X^*$, $y_1^*, y_2^* \in Y^*$ satisfying (3.28) and (3.29), where N in (3.38) stands for the Clarke normal cone ($N := N^C$);

(ii) X and Y are Asplund spaces, and there exist a $\lambda \in]\alpha^{-1}(\max\{\|u_1\|, \|v_1\|, \|v_2\|\})^q$ and a $\tau \in]0,1[$ such that inequality (3.38) holds with N standing for the Fréchet normal cone ($N := N^F$) for all $(x,y) \in X \times Y$, $(x_1,y_1) \in \operatorname{gph} F$ and $y_2 \in S$ satisfying (3.26) and

$$0 < \max\{\|x_1 - u_1 - x\|, \|y_1 - v_1 - y\|, \|y_2 - v_2 - y\|\} < (\alpha\lambda)^{\frac{1}{q}}, \quad (3.39)$$

and all $x_1^* \in X^*$, $y_1^*, y_2^* \in Y^*$ satisfying (3.28) and (3.31).

Corollary 67 Let $\alpha > 0$ and $q \in]0,1]$. The mapping F is $\alpha-$subtransversal of order q to S at $(\bar{x},\bar{y})$ with δ_1 and δ_2 if, for some $\mu > 0$ and any $(x',y') \in B_{\delta_2}(\bar{x},\bar{y})$ with $0 < \max\{d((x',y'),\operatorname{gph} F), d(y',S)\} < (\alpha\delta_1)^{\frac{1}{q}}$, one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\alpha^{-1}(\max\{d((x',y'),\operatorname{gph} F), d(y',S)\})^q, \delta_1[$ such that

$$q(\max\{\|x_1-x\|,\|y_1-y\|,\|y_2-y\|\})^{q-1}\left(\|x_1^*\|+\|y_1^*+y_2^*\|\right.$$
$$\left. + \mu\left(d((x_1^*,y_1^*),N_{\operatorname{gph} F}(x_1,y_1)) + d(y_2^*,N_S(y_2))\right)\right) \geq \alpha \quad (3.40)$$

for all $(x,y) \in X \times Y$ and $(x_1,y_1),(x_1',y_1') \in \operatorname{gph} F$, $y_2,y_2' \in S$ satisfying (3.33) and

$$0 < \max\{\|x_1-x\|,\|y_1-y\|,\|y_2-y\|\}$$
$$\leq \max\{\|x_1'-x'\|,\|y_1'-y'\|,\|y_2'-y'\|\} < (\alpha\lambda)^{\frac{1}{q}},$$

and all $x_1^* \in X^*$, $y_1^*,y_2^* \in Y^*$ satisfying (3.28) and (3.29), where N in (3.40) stands for the Clarke normal cone ($N := N^C$);

(ii) X and Y are Asplund spaces, and there exist numbers $\lambda \in]\alpha^{-1}(\max\{d((x',y'),\operatorname{gph} F), d(y',S)\})^q, \delta_1[$ and $\tau \in]0,1[$ such that inequality (3.40) holds with N standing for the Fréchet normal cone ($N := N^F$) for all $(x,y) \in X \times Y$ and $(x_1,y_1),(x_1',y_1') \in \operatorname{gph} F$, $y_2,y_2' \in S$ satisfying (3.33) and $0 < \max\{\|x_1-x\|,\|y_1-y\|,\|y_2-y\|\} < (\alpha\lambda)^{\frac{1}{q}}$, and all $x_1^* \in X^*$, $y_1^*,y_2^* \in Y^*$ satisfying (3.28) and (3.35).

Corollary 68 Let $\alpha > 0$ and $q \in]0,1]$. The mapping F is $\alpha-$transversal of order q to S at $(\bar{x},\bar{y})$ with δ_1 and δ_2 if, for some $\mu > 0$ and any $(x_1',y_1') \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x},\bar{y})$, $y_2' \in S \cap B_{\delta_2}(\bar{y})$ and $\xi \in]0,(\alpha\delta_1)^{\frac{1}{q}}[$, one of the following conditions is satisfied:

(i) there exists a $\lambda \in]\alpha^{-1}(\xi)^q, \delta_1[$ such that inequality (3.38) holds with N standing for the Clarke normal cone ($N := N^C$) for all $(x,y) \in X \times Y$, $(x_1,y_1) \in \operatorname{gph} F$, $y_2 \in S$ and $u_1 \in X$, $v_1,v_2 \in Y$ satisfying (3.36) and (3.37), and all $x_1^* \in X^*$, $y_1^*,y_2^* \in Y^*$ satisfying (3.28) and (3.29);

(ii) X and Y are Asplund, and there exist a $\lambda \in]\alpha^{-1}(\xi)^q, \delta_1[$ and a $\tau \in]0,1[$ such that inequality (3.38) holds with N standing for the Fréchet normal cone ($N := N^F$) for all $(x,y) \in X \times Y$, $(x_1,y_1) \in \operatorname{gph} F$, $y_2 \in S$ and $u_1 \in X$, $v_1,v_2 \in Y$ satisfying (3.36) and (3.39), and all $x_1^* \in X^*$, $y_1^*,y_2^* \in Y^*$ satisfying (3.28) and (3.31).

Remark 41 The combined dual transversality conditions (3.25), (3.32), (3.38) and (3.40) involving normal cones to $\operatorname{gph} F$ and S in the statements above can be 'decomposed' into components; cf. Corollaries 23, 24, 28, 29, 33, 34, 39 and 40.

The next three statements are direct consequences of Theorems 10, 11 and 12, respectively.

Proposition 25 Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces with convex graph, S be a convex subset of Y, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\bar{y} \in S$, and $\varphi \in \mathscr{C}^1$ be convex with $\varphi'_+(0) > 0$. If F is φ-semitransversal to S at $(\bar{x},\bar{y})$ with δ, then, with $\mu := (\varphi'_+(0))^{-1} + 1$,

$$\varphi'(\max\{\|u_1\|,\|v_1\|,\|v_2\|\})\left(\|x_1^*\| + \|y_1^* + y_2^*\| \right.$$
$$\left. + \mu\left(d((x_1^*,y_1^*),N_{\operatorname{gph} F}(\bar{x},\bar{y})) + d(y_2^*,N_S(\bar{y}))\right)\right) \geq 1$$

for all $u_1 \in X$, $v_1, v_2 \in Y$ satisfying $0 < \max\{\|u_1\|,\|v_1\|,\|v_2\|\} < \varphi^{-1}(\delta)$, and all $x_1^* \in X^*$, $y_1^*, y_2^* \in Y^*$ satisfying

$$\|x_1^*\| + \|y_1^*\| + \|y_2^*\| = 1,\ \langle x_1^*,u_1\rangle + \langle y_1^*,v_1\rangle + \langle y_2^*,v_2\rangle = \max\{\|u_1\|,\|v_1\|,\|v_2\|\}. \tag{3.41}$$

Proposition 26 Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces with convex graph, S be a convex subset of Y, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\bar{y} \in S$, and $\varphi \in \mathscr{C}^1$ be convex with $\varphi'_+(0) > 0$. If F is φ-subtransversal to S at $(\bar{x},\bar{y})$ with δ_1 and δ_2, then, with $\mu := (\varphi'_+(0))^{-1} + 1$,

$$\varphi'(\max\{\|x_1 - x\|,\|y_1 - y\|,\|y_2 - y\|\})\left(\|x_1^*\| + \|y_1^* + y_2^*\| \right.$$
$$\left. + \mu\left(d((x_1^*,y_1^*),N_{\operatorname{gph} F}(x_1,y_1)) + d(y_2^*,N_S(y_2))\right)\right) \geq 1$$

for all $(x,y) \in B_{\delta_2}(\bar{x},\bar{y})$ and $(x_1,y_1) \in \operatorname{gph} F$, $y_2 \in S$ satisfying

$$0 < \max\{\|x_1 - x\|,\|y_1 - y\|,\|y_2 - y\|\} < \varphi^{-1}(\lambda),$$

and all $x_1^* \in X^*$, $y_1^*, y_2^* \in Y^*$ satisfying (3.41) and

$$\langle x_1^*, x - x_1\rangle + \langle y_1^*, y - y_1\rangle + \langle y_2^*, y - y_2\rangle = \max\{\|x - x_1\|,\|y - y_1\|,\|y - y_2\|\}.$$

Proposition 27 Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces with convex graph, S be a convex subset of Y, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\bar{y} \in S$, and $\varphi \in \mathscr{C}^1$ be convex with $\varphi'_+(0) > 0$. If F is φ-transversal to S at $(\bar{x},\bar{y})$ with δ_1 and δ_2, then, with $\mu := (\varphi'_+(0))^{-1} + 1$,

$$\varphi'(\max\{\|x_1 - u_1 - \bar{x}\|,\|y_1 - v_1 - \bar{y}\|,\|y_2 - v_2 - \bar{y}\|\})\left(\|x_1^*\| + \|y_1^* + y_2^*\| \right.$$
$$\left. + \mu\left(d((x_1^*,y_1^*),N_{\operatorname{gph} F}(x_1,y_1)) + d(y_2^*,N_S(y_2))\right)\right) \geq 1$$

for all $(x_1,y_1) \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x},\bar{y})$, $y_2 \in S \cap B_{\delta_2}(\bar{y})$ and $u_1 \in X$, $v_1, v_2 \in Y$ with

$$0 < \max\{\|x_1 - u_1 - \bar{x}\|,\|y_1 - v_1 - \bar{y}\|,\|y_2 - v_2 - \bar{y}\|\} < \varphi^{-1}(\delta_1),$$

and all $x_1^* \in X^*$, $y_1^*, y_2^* \in Y^*$ satisfying (3.41) and

$$\langle x_1^*, \bar{x} + u_1 - x_1\rangle + \langle y_1^*, \bar{y} + v_1 - y_1\rangle + \langle y_2^*, \bar{y} + v_2 - y_2\rangle$$
$$= \max\{\|x_1 - u_1 - \bar{x}\|,\|y_1 - v_1 - \bar{y}\|,\|y_2 - v_2 - \bar{y}\|\}.$$

Remark 42 Decompositions of the combined dual necessary transversality conditions in Propositions 25, 26 and 27 can be easily obtained; cf. Corollaries 47, 51 and 55.

3.2 Semitransversality of Collections of Set-Valued Mappings

3.2.1 Definitions and Basic Relationships

The next definition recalls the well-known *extremality* and *stationarity* of collections of sets.

Definition 10 Let $\Omega_1,\ldots,\Omega_n$ be subsets of a normed space X, and $\bar{x}\in\cap_{i=1}^n\Omega_i$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is

(i) locally extremal at $\bar{x}$ if there exists a $\rho\in]0,+\infty]$ such that, for any $\alpha>0$, there exist $x_i\in X$ $(i=1,\ldots,n)$ satisfying $\max_{1\le i\le n}\|x_i\|<\alpha$ and

$$\bigcap_{i=1}^n(\Omega_i-x_i)\cap B_\rho(\bar{x})=\emptyset; \tag{3.42}$$

(ii) stationary at $\bar{x}$ if for any $\alpha>0$, there exist $\rho\in]0,\alpha[$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying (3.42) and

$$\max_{1\le i\le n}\|x_i\|<\alpha\rho. \tag{3.43}$$

Remark 43 (i) Conditions (i) and (ii) in Definition 10 mean that appropriate arbitrarily small shifts of the sets make them nonintersecting (in a neighbourhood of $\bar{x}$). This is a very general model embracing many optimality notions. Note that (i)⇒(ii) holds true, and it can be strict [127, 129]. The two conditions are equivalent when the sets are convex, and in this case we can take $\rho:=+\infty$ in condition (i).

(ii) Unlike condition (i), in condition (ii) the magnitude of the "shifts" of the sets are related to that of the neighbourhood in which the sets become nonintersecting, namely $\max_{1\le i\le n}\|x_i\|/\rho<\alpha$.

The semitransversality of collections of sets in the below definition is the negation of the stationarity.

Definition 11 Let $\Omega_1,\ldots,\Omega_n$ be subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\alpha>0$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\alpha-$semitransversal at $\bar{x}$ if there exists a $\delta>0$ such that

$$\bigcap_{i=1}^n(\Omega_i-x_i)\cap B_\rho(\bar{x})\neq\emptyset \tag{3.44}$$

for all $\rho\in]0,\delta[$ and $x_i\in X$ $(i=1,\ldots,n)$ with satisfying (3.43).

For discussions about the property in Definition 11, we refer the reader to Chapter 1.

From now on, we assume that (X,d) and (X_i,d_i) are metric spaces, $F_i:X_i\rightrightarrows X$ are set-valued mappings, $\bar{x}_i\in X_i$ $(i=1,\ldots,n)$, and $\bar{x}\in\cap_{i=1}^nF_i(\bar{x}_i)$.

Definition 12 The collection $\{F_1,\ldots,F_n\}$ is

(i) local extremal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ if there exists a $\rho \in]0,+\infty]$ such that, for any $\alpha > 0$, there are $x_i \in X_i$ $(i = 1,\ldots,n)$ satisfying $\max_{1\le i\le n}\{d_i(x_i,\bar{x}_i), d(\bar{x},F_i(x_i))\} < \alpha$ and

$$\bigcap_{i=1}^{n} F_i(x_i) \cap B_\rho(\bar{x}) = \emptyset; \tag{3.45}$$

(ii) stationary at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ if for any $\alpha > 0$, there exist $\rho \in]0,\alpha[$ and $x_i \in X_i$ $(i = 1,\ldots,n)$ satisfying (3.45) and

$$\max_{1\le i\le n}\{d_i(x_i,\bar{x}_i), d(\bar{x},F_i(x_i))\} < \alpha\rho. \tag{3.46}$$

The property in part (i) of Definition 12 recaptures [175, Definition 3.3], while the one in part (ii) is new. Both properties can be seen as the generalized versions of the corresponding ones in Definition 10; cf. Proposition 28. It is straightforward in Definition 12 that the implication (i)⇒(ii) holds true.

We are now ready to introduce the concept *semitransversality of collections of set-valued mappings*.

Definition 13 Let $\alpha > 0$. The collection $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ if there exists a $\delta > 0$ such that

$$\bigcap_{i=1}^{n} F_i(x_i) \cap B_\rho(\bar{x}) \neq \emptyset \tag{3.47}$$

for all $\rho \in]0,\delta[$ and $x_i \in X_i$ $(i = 1,\ldots,n)$ satisfying (3.46).

The property in Definition 13 is the negation of the stationarity in Definition 12(ii). Unlike the properties in Definitions 10 and 11 which require the linear structure of the underlying space, the ones in Definitions 12 and 13 are defined on metric spaces, see Example 4.

Proposition 28 The local extreamlity, stationarity, and semitransversality in Definitions 10 and 11 are particular cases of the corresponding ones in Definitions 12 and 13.

Proof We provide a proof for semitransversality properties in Definitions 11 and 13. The proofs for the other cases are analogous.

Let $\Omega_1,\ldots,\Omega_n$ be subsets of a normed space X, $\bar{x} \in \cap_{i=1}^{n}\Omega_i$, and $\alpha > 0$. Define $F_i : X \rightrightarrows X$ $(i = 1,\ldots,n)$ for all $x \in X$ by

$$F_i(x) := \Omega_i - x \;\; (i = 1,\ldots,n). \tag{3.48}$$

Let $\bar{x}_i = 0$, then $F_i(\bar{x}_i) = \Omega_i$ $(i = 1,\ldots,n)$, and consequently, $\bar{x} \in \cap_{i=1}^{n} F_i(\bar{x}_i)$. Suppose $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with some $\delta > 0$. By Definition 13, condition (3.47) holds for all $\rho \in]0,\delta[$ and $x_i \in X_i$ $(i = 1,\dots,n)$ satisfying (3.46). Combining this with (3.48), condition (3.44) holds for all $\rho \in]0,\delta[$ and $x_i \in X$ $(i = 1,\dots,n)$ with satisfying (3.43) since

$$\max_{1\le i\le n} \{\|x_i\|, d(\bar{x}, \Omega_i - x_i)\} = \max_{1\le i\le n} \|x_i\|.$$

By Definition 11, $\{\Omega_1,\dots,\Omega_n\}$ is $\alpha-$semitransversal at $\bar{x}$ with δ. □

Example 4 Let (X,d) be a metric space, (X_i,d_i) be discrete metric spaces, $F_i : X_i \rightrightarrows X$, $\bar{x}_i \in X_i$ $(i = 1,\dots,n)$, $\bar{x} \in \cap_{i=1}^n F_i(\bar{x}_i)$, and $\alpha > 0$. Then, $\{F_1,\dots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\dots,\bar{x}_n)$ with $\delta := \alpha^{-1}$.

Proof Let $\rho \in]0,\delta[$ and $x_i \in X_i$ $(i = 1,\dots,n)$ satisfy (3.46). Let $\delta := \alpha^{-1}$. Then, $d_i(x_i,\bar{x}_i) < \alpha\rho < 1$, and consequently, $x_i = \bar{x}_i$ $(i = 1,\dots,n)$. Hence, condition (3.47) is satisfied. By Definition 13, $\{F_1,\dots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\dots,\bar{x}_n)$ with δ. □

Example 5 Let $\Omega_1 := \mathbb{R}\times\{0\}$, $\Omega_2 := \{0\}\times[0,+\infty[$, $\bar{x} := (0,0) \in \Omega_1\cap\Omega_2$, $\bar{x}_1 = \bar{x}_2 = 0$, $\alpha > 0$ and $\delta > 0$. Define $F_1,F_2 : \mathbb{R} \rightrightarrows \mathbb{R}^2$ for any $x \in \mathbb{R}$ by

$$F_1(x) := \Omega_1, \quad F_2(x) := \begin{cases} \Omega_2 & \text{if } x = 0, \\ \{0\}\times\mathbb{R} & \text{otherwise.} \end{cases}$$

Then $F(\bar{x}_1) = \Omega_1$ and $F(\bar{x}_2) = \Omega_2$. We claim that $\{\Omega_1,\Omega_2\}$ is not $\alpha-$semitransversal at $\bar{x}$ with δ, but $\{F_1,F_2\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\bar{x}_2)$ with δ.

Proof Choose a $\rho \in]0,\delta[$, $x_1 := (0,\alpha\rho)$ and $x_2 := (0,0)$. Then $x_1,x_2 \in (\alpha\rho)\mathbb{B}$, and condition (3.44) is not satisfied. By Definition 11, $\{\Omega_1,\Omega_2\}$ is not $\alpha-$semitransversal at $\bar{x}$ with δ.

Let $x_i \in \mathbb{R}$ $(i = 1,2)$ such that $\max_{i=1,2}\{|x_i - \bar{x}_i|, d(\bar{x}, F_i(x_i))\} < \alpha\rho$, or equivalently, $\max_{i=1,2}|x_i - \bar{x}_i| < \alpha\rho$. By the definition of F_1 and F_2, we have $\bar{x} \in F_1(x_1)\cap F_2(x_2)\cap B_\rho(\bar{x})$, and consequently, condition (3.47) holds true. By Definition 13, $\{F_1,F_2\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\bar{x}_2)$ with δ. □

The following theorem establishes a metric characterization of the property in Definition 13.

Theorem 21 Let $\alpha > 0$. The collection $\{F_1,\dots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\dots,\bar{x}_n)$ with some $\delta > 0$ if and only if

$$\alpha d\left(\bar{x}, \bigcap_{i=1}^n F_i(x_i)\right) \le \max_{1\le i\le n} \{d_i(x_i,\bar{x}_i), d(\bar{x}, F_i(x_i))\} \tag{3.49}$$

for all $x_i \in X_i$ $(i = 1,\dots,n)$ satisfying

$$\max_{1\le i\le n} \{d_i(x_i,\bar{x}_i), d(\bar{x}, F_i(x_i))\} < \alpha\delta. \tag{3.50}$$

Proof Suppose $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with some $\delta>0$. Let $x_i\in X_i$ $(i=1,\ldots,n)$ satisfy (3.50). Let $\rho\in]0,\delta[$ such that $\rho>\alpha^{-1}\max_{1\le i\le n}\{d_i(x_i,\bar{x}_i),d(\bar{x},F_i(x_i))\}$. By Definition 13,

$$d\left(\bar{x},\bigcap_{i=1}^{n}F_i(x_i)\right)<\rho. \tag{3.51}$$

Letting $\rho\downarrow\alpha^{-1}\max_{1\le i\le n}\{d_i(x_i,\bar{x}_i),d(\bar{x},F_i(x_i))\}$, we obtain inequality (3.49).

Conversely, let $\rho\in]0,\delta[$ and $x_i\in X_i$ $(i=1,\ldots,n)$ satisfying (3.46). In view of (3.49), we obtain (3.51), and consequently, condition (3.47) is satisfied. By Definition 13, $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ. $\square$

The next statement is a direct consequence of Theorem 21.

Corollary 69 Let $\Omega_1,\ldots,\Omega_n$ be subsets of a normed space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\alpha>0$. Let F_i $(i=1,\ldots,n)$ be defined by (3.48). The collection $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(0,\ldots,0)$ with some $\delta>0$ if and only if

$$\alpha d\left(\bar{x},\bigcap_{i=1}^{n}(\Omega_i-x_i)\right)\le\max_{1\le i\le n}\|x_i\| \tag{3.52}$$

for all $x_i\in X_i$ $(i=1,\ldots,n)$ satisfying $\max_{1\le i\le n}\|x_i\|<\alpha\delta$.

Remark 44 Corollary 69 recaptures [38, Theorem 2], see also [140, Theorem 3.1(i)].

3.2.2 Slope Necessary and Sufficient Conditions

This section is dedicated to slope necessary and sufficient conditions for the semitransversality in Definition 13.

We are going to employ a metric depending on a parameter $\gamma>0$ given by

$$d_\gamma((u_1,\ldots,u_n,u),(v_1,\ldots,v_n,v)):=\max\left\{d(u,v),\gamma\max_{1\le i\le n}d(u_i,v_i)\right\} \tag{3.53}$$

for any $(u_1,\ldots,u_n,u),(v_1,\ldots,v_n,v)\in X^{n+1}$.

Proposition 29 Let $\delta>0$ and $\alpha>0$.

(i) Suppose X is complete, and F_i $(i=1,\ldots,n)$ are closed-valued. The collection $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ if, for some $\gamma>0$ and all $x_i\in X_i$ $(i=1,\ldots,n)$ satisfying (3.50), $u_i\in X$ $(i=1,\ldots,n)$ satisfying

$$u_i\in F_i(x_i)\ (i=1,\ldots,n),\quad \max_{1\le i\le n}d(u_i,\bar{x})<\alpha\delta, \tag{3.54}$$

it holds

$$\sup_{\substack{v_i'\in F_i(x_i)\ (i=1,\ldots,n),\ x'\in X\\ (v_1',\ldots,v_n',x')\ne(v_1,\ldots,v_n,x)}}\frac{\max\limits_{1\le i\le n}d(v_i,x)-\max\limits_{1\le i\le n}d(v_i',x')}{d_\gamma((v_1,\ldots,v_n,x),(v_1',\ldots,v_n',x'))}\ge\alpha \tag{3.55}$$

for all $x \in X$ and $v_i \in F_i(x_i)$ $(i = 1,\dots,n)$ satisfying

$$d(x,\bar{x}) < \delta, \quad \max_{1\le i\le n} d(v_i,u_i) < \delta/\gamma, \tag{3.56}$$

$$0 < \max_{1\le i\le n} d(v_i,x) \le \max_{1\le i\le n} d(u_i,\bar{x}). \tag{3.57}$$

(ii) If $\{F_1,\dots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\dots,\bar{x}_n)$ with δ, then, with $\gamma := (\alpha+1)^{-1}$,

$$\sup_{\substack{u_i'\in F_i(x_i)\ (i=1,\dots,n),\ x'\in X\\ (v_1',\dots,v_n',x')\ne(v_1,\dots,v_n,\bar{x})}} \frac{\max\limits_{1\le i\le n} d(v_i,\bar{x}) - \max\limits_{1\le i\le n} d(v_i',x')}{d_\gamma((v_1,\dots,v_n,\bar{x}),(v_1',\dots,v_n',x'))} \ge \alpha \tag{3.58}$$

for all $x_i \in X_i$ and $v_i \in F_i(x_i)$ $(i = 1,\dots,n)$ satisfying

$$0 < \max_{1\le i\le n} d(v_i,\bar{x}) \le \max_{1\le i\le n} d_i(x_i,\bar{x}_i) < \alpha\delta. \tag{3.59}$$

The proof below employs two closely related nonnegative functions on X^{n+1} determined by the given points $x_1,\dots,x_n \in X$:

$$f(v_1,\dots,v_n,x) := \max_{1\le i\le n} d(v_i,x), \quad v_1,\dots,v_n,x \in X, \tag{3.60}$$

$$\widehat{f} := f + i_{F_1(x_1)\times\ldots\times F_n(x_n)\times X}. \tag{3.61}$$

Proof

(i) Suppose $\{F_1,\dots,F_n\}$ is not $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\dots,\bar{x}_n)$ with δ, and let $\gamma > 0$ be given. By Definition 13, there exist a $\rho \in]0,\delta[$ and $x_i \in X_i$ $(i = 1,\dots,n)$ satisfying (3.46) such that

$$\bigcap_{i=1}^{n} F_i(x_i) \cap B_\rho(\bar{x}) = \emptyset. \tag{3.62}$$

Then, condition (3.50) is satisfied. By (3.62),

$$\max_{1\le i\le n} d(z_i,z) > 0 \text{ for all } z \in B_\rho(\bar{x}),\ z_i \in F_i(x_i)\ (i = 1,\dots,n). \tag{3.63}$$

In view of (3.46), we can find $u_i \in F_i(x_i)$ $(i = 1,\dots,n)$ such that $\max_{1\le i\le n} d(u_i,\bar{x}) < \alpha\rho < \alpha\delta$, i.e. condition (3.54) is satisfied. Let f and $\widehat{f}$ be defined by (3.60) and (3.61). We have $\widehat{f}(u_1,\dots,u_n,\bar{x}) < \alpha\rho$, and it follows from (3.63) that $\widehat{f}(u_1,\dots,u_n,\bar{x}) > 0$. Choose a number ε such that $f(u_1,\dots,u_n,\bar{x}) < \varepsilon < \alpha\rho$. Applying the Ekeland variational principle (Lemma 1) to the restriction of $\widehat{f}$ to the complete metric space $F_1(x_1)\times\ldots\times F_n(x_n)\times X$ with the $\gamma-$distance (3.53), we can find $x \in X$ and $v_i \in F_i(x_i)$ $(i = 1,\dots,n)$ such that

$$d_\gamma((v_1,\dots,v_n,x),(u_1,\dots,u_n,\bar{x})) < \rho, \quad f(v_1,\dots,v_n,x) \le f(u_1,\dots,u_n,\bar{x}), \tag{3.64}$$

$$f(v_1,\dots,v_n,x) - f(v_1',\dots,v_n',x') \le \frac{\varepsilon}{\rho} d_\gamma((v_1,\dots,v_n,x),(v_1',\dots,v_n',x')) \tag{3.65}$$

for all $(v_1',\ldots,v_n',x')\in F_1(x_1)\times\ldots\times F_n(x_n)\times X$. Conditions (3.64) yield (3.56) and (3.57). Since $\varepsilon/\rho<\alpha$, condition (3.65) contradicts (3.55).

(ii) Suppose $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ. Let $x_i\in X_i$, $v_i\in F_i(x_i)$ $(i=1,\ldots,n)$ satisfy condition (3.59). Let $\gamma:=(\alpha+1)^{-1}$ and $M:=\alpha^{-1}\max_{1\le i\le n}d(v_i,\bar{x})<\delta$. Let $\eta\in]0,1[$, and choose a number $\gamma'\in]\eta\gamma,\gamma[$. Then $(\gamma')^{-1}-\alpha>1$. Choose a $\xi>1$ such that $\xi\le\eta^{-1}$, $\xi\le(\gamma')^{-1}-\alpha$, and $\xi M<\delta$. In view of Definition 13, we have $\cap_{i=1}^n F_i(x_i)\cap B_{\xi M}(\bar{x})\neq\emptyset$, and consequently, there exist $\hat{x}\in X$ and $\hat{v}_i\in F_i(x_i)$ $(i=1,\ldots,n)$ with $\hat{v}_1=\ldots=\hat{v}_n=\hat{x}$ such that $d(\hat{x},\bar{x})<\xi M$. It is obvious that $(\hat{v}_1,\ldots,\hat{v}_n,\hat{x})\neq(v_1,\ldots,v_n,\bar{x})$. Moreover, for all $i=1,\ldots,n$,

$$d(\hat{v}_i,v_i)\le d(\hat{v}_i,\bar{x})+d(v_i,\bar{x})<\xi M+\alpha M\le M(\gamma')^{-1}<M(\eta\gamma)^{-1},$$

and consequently,

$$\begin{aligned} d_\gamma((\hat{v}_1,\ldots,\hat{v}_n,\hat{x}),(v_1,\ldots,v_n,\bar{x})) &= \max_{1\le i\le n}\{d(\hat{x},\bar{x}),\gamma d(\hat{v}_i,v_i)\}\\ &< M\max\left\{\xi,\eta^{-1}\right\}=M\eta^{-1}. \end{aligned}$$

Hence, $\max_{1\le i\le n}d(v_i,\bar{x})>\eta\alpha d_\gamma((\hat{v}_1,\ldots,\hat{v}_n,\hat{x}),(v_1,\ldots,v_n,\bar{x}))$, and consequently,

$$\begin{aligned} &\sup_{\substack{v_i'\in F_i(x_i)\ (i=1,\ldots,n),\ x'\in X\\ (v_1',\ldots,v_n',x')\neq(v_1,\ldots,v_n,\bar{x})}} \frac{\max\limits_{1\le i\le n}d(v_i,\bar{x})-\max\limits_{1\le i\le n}d(v_i',x')}{d_\gamma((v_1,\ldots,v_n,\bar{x}),(v_1',\ldots,v_n',x'))}\\ &\qquad\ge\frac{\max\limits_{1\le i\le n}d(v_i,\bar{x})}{d_\gamma((v_1,\ldots,v_n,\bar{x}),(\hat{u}_1,\ldots,\hat{u}_n,\hat{x}))}>\eta\alpha. \end{aligned}$$

Letting $\eta\uparrow 1$, we arrive at (3.58).

Remark 45 (i) The necessary condition in part (ii) is a particular case of the sufficient condition in part (i) by setting $x:=\bar{x}$, $v_i:=u_i$ $(i=1,\ldots,n)$.

(ii) It is evident from (3.58) and (3.68) that $\gamma:=(\alpha+1)^{-1}$ in part (ii) of Proposition 29 (and in subsequent statement) can be replaced by any positive $\gamma\le(\alpha+1)^{-1}$.

(iii) The closedness assumption in part (i) of Proposition 29 is used to ensure the lower semicontinuity of the indicator function of the set $F_1(x_1)\times\ldots\times F_n(x_n)$.

(iv) Dropping either condition (3.56) or (3.57) makes the sufficient condition in Proposition 29(i) stronger (while weakening the result).

(v) The expressions in the left-hand sides of the inequalities (3.55) and (3.58) are the *nonlocal γ-slopes* $|\nabla\widehat{f}|^{\diamond}_{\gamma}$ (cf. [131, p. 60]) of the function $\widehat{f}$ defined by (3.61) computed at $(v_1,\ldots,v_n,x)$ and $(v_1,\ldots,v_n,\bar{x})$, respectively.

(vi) It suffices to assume in Proposition 29(i) (and subsequent statements) that F_i $(i=1,\ldots,n)$ are closed-valued, respectively, at $x_i\in X_i$ $(i=1,\ldots,n)$ satisfying (3.50).

(vii) Parts (i) and (ii) of Proposition 29 improve [70, Theorem 4.1] and [68, Theorem 3.1(i)], respectively, in the linear setting.

$\square$

The next statement presents a localized version of Proposition 29, which plays a fundamental role in establishing dual characterizations of the property.

Proposition 30 Let $\delta > 0$ and $\alpha > 0$.

(i) Suppose X is complete, and F_i $(i = 1,\ldots,n)$ are closed-valued. The collection $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with some $\delta > 0$ if, for some $\gamma > 0$ and all $x_i \in X_i$, $u_i \in X$ $(i = 1,\ldots,n)$ satisfying (3.50) and (3.54),

$$\limsup_{\substack{v_i' \xrightarrow{F_i(x_i)} v_i\ (i=1,\ldots,n),\ x'\to x\\ (v_1',\ldots,v_n',x')\neq(v_1,\ldots,v_n,x)}} \frac{\max\limits_{1\le i\le n} d(v_i,x) - \max\limits_{1\le i\le n} d(v_i',x')}{d_\gamma((v_1,\ldots,v_n,x),(v_1',\ldots,v_n',x'))} \ge \alpha \tag{3.66}$$

for all $x \in X$ and $v_i \in F_i(x_i)$ $(i = 1,\ldots,n)$ satisfying (3.56) and (3.57).

(ii) Suppose F_i $(i = 1,\ldots,n)$ are convex-valued. If $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with some $\delta > 0$, then

$$\limsup_{\substack{v_i' \xrightarrow{F_i(x_i)} v_i\ (i=1,\ldots,n),\ x'\to \bar{x}\\ (v_1',\ldots,v_n',x')\neq(u_1,\ldots,u_n,\bar{x})}} \frac{\max\limits_{1\le i\le n} d(v_i,\bar{x}) - \max\limits_{1\le i\le n} d(v_i',x')}{d_\gamma((v_1,\ldots,v_n,\bar{x}),(v_1',\ldots,v_n',x'))} \ge \alpha \tag{3.67}$$

with $\gamma := (\alpha+1)^{-1}$ for all $x_i \in X_i$ and $v_i \in F_i(x_i)$ $(i = 1,\ldots,n)$ satisfying (3.59).

Proof

(i) In view of Lemma 2(ii), condition (3.66) implies (3.55).

(ii) The left-hand sides of conditions (3.58) and (3.67) involve the same difference quotient for the function $\widehat{f}$ defined by (3.61). Under the assumptions made, this function is convex. By Lemma 2(iv), the left-hand sides of conditions (3.58) and (3.67) coincide. The assertion is a consequence of Proposition 29(ii).

$\square$

Remark 46 (i) The expressions in the left-hand sides of the inequalities (3.66) and (3.67) are the slopes with respect to the γ-distance (3.68) computed at $(v_1,\ldots,v_n,x)$ and $(v_1,\ldots,v_n,\bar{x})$, respectively, of the function $\widehat{f}$ given by (3.61).

(ii) It suffices to assume in Proposition 30(ii) (and subsequent statements) that F_i $(i = 1,\ldots,n)$ are convex-valued, respectively, at $x_i \in B_{\alpha\delta}(\bar{x}_i)$ $(i = 1,\ldots,n)$.

3.2.3 Dual Necessary and Sufficient Conditions

To establish dual characterizations of the semitransversality, the imaging space X is assumed to have a linear structure. In this case, the $\gamma-$distance (3.53) becomes

$$\|(v_1,\ldots,v_n,v)\|_\gamma := \max\left\{\|v\|,\gamma \max_{1\le i\le n}\|v_i\|\right\}, \quad v,v_1,\ldots,v_n\in X. \tag{3.68}$$

The dual norm on $(X^*)^{n+1}$ corresponding to (3.68) has the following form:

$$\|(v_1^*,\ldots,v_n^*,v^*)\|_\gamma = \|v^*\|+\frac{1}{\gamma}\sum_{i=1}^n\|v_i^*\|, \quad v^*,v_1^*,\ldots,v_n^*\in X^*. \tag{3.69}$$

We will denote by d_γ the distance in $(X^*)^{n+1}$ determined by (3.69).

In this section, we use the function $\widehat{f}$ given by (3.61) with $f: X^{n+1}\to\mathbb{R}\cup\{+\infty\}$ defined by (3.60).

Proposition 31 Let $\delta>0$ and $\alpha>0$.

(i) Suppose X is Banach, and F_i $(i=1,\ldots,n)$ are closed-valued. The collection $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ if, for some $\gamma>0$ and all $x_i\in X_i$, $u_i\in X$ $(i=1,\ldots,n)$ satisfying (3.50) and (3.54), it holds

$$d_\gamma\left(0,\partial\widehat{f}(v_1,\ldots,v_n,x)\right)\ge\alpha \tag{3.70}$$

with $\partial := \partial^C$ for all $x\in X$ and $v_i\in F_i(x_i)$ $(i=1,\ldots,n)$ satisfying (3.56) and (3.57).

If X is Asplund, then the above assertion is valid with $\partial := \partial^F$, and condition (3.57) is replaced by

$$0<\max_{1\le i\le n}\|v_i-x\|<\alpha\delta. \tag{3.71}$$

(ii) Suppose F_i $(i=1,\ldots,n)$ are convex-valued. If $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with some $\delta>0$, then

$$d_\gamma\left(0,\partial\widehat{f}(v_1,\ldots,v_n,\bar{x})\right)\ge\alpha \tag{3.72}$$

with $\gamma := (\alpha+1)^{-1}$ for all $x_i\in X_i$ and $v_i\in F_i(x_i)$ $(i=1,\ldots,n)$ satisfying (3.59).

Proof

(i) Suppose $\{F_1(\bar{x}_1),\ldots,F_n(\bar{x}_n)\}$ is not $\alpha-$semitransversal at $\bar{x}$ with δ. Let $\gamma>0$. By Proposition 30(i), there exist $x_i\in X_i$, $u_i\in X$ $(i=1,\ldots,n)$ satisfying (3.50) and (3.54), $x\in X$ and $v_i\in F_i(x_i)$ $(i=1,\ldots,n)$ satisfying (3.56) and (3.57), and a number $\tau\in]0,\alpha[$ such that

$$\max_{1\le i\le n}\|v_i-x\|-\max_{1\le i\le n}\|v_i'-x'\|\le\tau\|(v_1,\ldots,v_n,x)-(v_1',\ldots,v_n',x')\|_\gamma$$

for all $(v_1',\ldots,v_n',x') \in F_1(x_1)\times\ldots\times F_n(x_n)\times X$ near $(v_1,\ldots,v_n,x)$. In other words, $(v_1,\ldots,v_n,x)$ is a local minimizer of the function

$$(v_1',\ldots,v_n',x') \mapsto \widehat{f}(v_1',\ldots,v_n',x') + \tau\|(v_1,\ldots,v_n,x)-(v_1',\ldots,v_n',x')\|_\gamma. \tag{3.73}$$

Hence, its Fréchet and, as a consequence, Clarke subdifferentials at this point contains the subgradient 0. Observe that (3.73) is the sum of the function $\widehat{f}$ and the Lipschitz continuous convex function $(v_1',\ldots,v_n',x') \mapsto \tau\|(v_1,\ldots,v_n,x)-(v_1',\ldots,v_n',x')\|_\gamma$, and at any point all subgradients $(v_1^*,\ldots,v_n^*,v^*)$ of the latter function satisfy $\|(v_1^*,\ldots,v_n^*,v^*)\|_\gamma \le \tau$. By the Clarke–Rockafellar sum rule (Lemma 4(ii)), there exists a subgradient $(v_1^*,\ldots,v_n^*,v^*) \in \partial^C\widehat{f}(v_1,\ldots,v_n,x)$ such that $\|(v_1^*,\ldots,v_n^*,v^*)\|_\gamma \le \tau < \alpha$. The last inequality contradicts (3.70).

If X is Asplund, then one can employ the fuzzy sum rule (Lemma 4(iii)): for any $\varepsilon > 0$, there exist points $x' \in B_\varepsilon(x), v_i' \in B_\varepsilon(v_i)\cap F_i(x_i)$ $(i=1,\ldots,n)$, and a subgradient $(v_1^*,\ldots,v_n^*,v^*) \in \partial^F\widehat{f}(v_1',\ldots,v_n',x')$ such that $\|(v_1^*,\ldots,v_n^*,v^*)\|_\gamma < \tau+\varepsilon$. The number ε can be chosen small enough so that $\|x'-\bar{x}\| < \delta$, $\max_{1\le i\le n}\|v_i'-v_i\| < \delta/\gamma$, $\max_{1\le i\le n}\|v_i'-x'\| < \alpha\delta$, and $\tau+\varepsilon < \alpha$. The last inequality again yields $\|(v_1^*,\ldots,v_n^*,v^*)\|_\gamma < \alpha$, which contradicts (3.70).

(ii) Under the assumptions made, the function $\widehat{f}$ is convex. Let $\gamma := (\alpha+1)^{-1}$. In view of Proposition 30(ii), for all $x_i \in X_i$ and $v_i \in F_i(x_i)$ $(i=1,\ldots,n)$ satisfying (3.59), and $(v_1^*,\ldots,v_n^*,v^*) \in \partial\widehat{f}(v_1,\ldots,v_n,\bar{x})$, we have

$$\begin{aligned}
&\|(v_1^*,\ldots,v_n^*,v^*)\|_\gamma \\
&\quad= \sup_{(v_1',\ldots,v_n',x)\neq 0} \frac{\langle (v_1^*,\ldots,v_n^*,v^*),(v_1',\ldots,v_n',x')\rangle}{\|(v_1',\ldots,v_n',x')\|_\gamma} \\
&\quad= \limsup_{\substack{v_i'\to v_i\ (i=1,\ldots,n),\ x'\to\bar{x}\\ (v_1',\ldots,v_n',x')\neq(v_1,\ldots,v_n,\bar{x})}} \frac{-\langle (v_1^*,\ldots,v_n^*,v^*),(v_1',\ldots,v_n',x')-(v_1,\ldots,v_n,\bar{x})\rangle}{\|(v_1',\ldots,v_n',x')-(v_1,\ldots,v_n,\bar{x})\|_\gamma} \\
&\quad\ge \limsup_{\substack{v_1'\to v_1\ (i=1,\ldots,n),\ x'\to\bar{x}\\ (v_1',\ldots,v_n',x')\neq(v_1,\ldots,v_n,\bar{x})}} \frac{\widehat{f}(v_1,\ldots,v_n,\bar{x})-\widehat{f}(v_1',\ldots,v_n',x')}{\|(v_1',\ldots,v_n',x')-(v_1,\ldots,v_n,\bar{x})\|_\gamma} \\
&\quad= \limsup_{\substack{v_i'\overset{F_i(x_i)}{\to} v_i\ (i=1,\ldots,n),\ x'\to\bar{x}\\ (v_1',\ldots,v_n',x')\neq(v_1,\ldots,v_n,\bar{x})}} \frac{\max\limits_{1\le i\le n}\|\bar{x}-v_i\| - \max\limits_{1\le i\le n}\|x'-v_i'\|}{\|(v_1,\ldots,v_n,\bar{x})-(v_1',\ldots,v_n',x')\|_\gamma} \ge \alpha.
\end{aligned}$$

The proof is complete. □

The key conditions (3.71) and (3.72) in Proposition 31 involve subdifferentials of the function $\widehat{f}$ given by (3.61). Subgradients of $\widehat{f}$ belong to $(X^*)^{n+1}$ and have $n+1$ component vectors $v_1^*,\ldots,v_n^*,v^*$. As it can be seen from the representation (3.69) of the dual norm, the contribution of the vectors $v_1^*,\ldots,v_n^*$ on one hand and v^* on the other hand to conditions (3.71) and (3.72) is different. The next proposition exposes this difference.

Proposition 32 Let $\delta > 0$ and $\alpha > 0$.

(i) Suppose X is Banach, and F_i $(i = 1,\ldots,n)$ are closed-valued. The collection $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ if, for some $\gamma > 0$ and all $x_i \in X_i$, $u_i \in X$ $(i = 1,\ldots,n)$ satisfying (3.50) and (3.54),

$$\|v^*\| \geq \alpha \tag{3.74}$$

for all $x \in X$, $v_i \in F_i(x_i)$ $(i = 1,\ldots,n)$ satisfying (3.56) and (3.57), and $(v_1^*,\ldots,v_n^*,v^*) \in \partial\widehat{f}(v_1,\ldots,v_n,x)$ with $\sum_{i=1}^n \|v_i^*\| < \alpha\gamma$, where $\partial := \partial^C$.

If X is Asplund, then the above assertion is valid with $\partial := \partial^F$, and condition (3.57) is replaced by (3.71).

(ii) Suppose F_i $(i = 1,\ldots,n)$ are convex-valued. If $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ, then

$$\|v^*\| \geq \alpha - (\alpha+1)\sum_{i=1}^n \|v_i^*\| \tag{3.75}$$

for all $x_i \in X_i$, $v_i \in F_i(x_i)$ $(i = 1,\ldots,n)$ satisfying (3.59), and $(v_1^*,\ldots,v_n^*,v^*) \in \partial\widehat{f}(v_1,\ldots,v_n,\bar{x})$. As a consequence,

$$\liminf_{\substack{v_i^* \to 0\ (i=1,\ldots,n)\\ (v_1^*,\ldots,v_n^*,v^*) \in \partial\widehat{f}(v_1,\ldots,v_n,\bar{x})}} \|v^*\| \geq \alpha.$$

Proof

(i) Suppose $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ, and let $\gamma > 0$ be given. By Proposition 31(i), there exist $x_i \in X_i$, $u_i \in X$ $(i = 1,\ldots,n)$ satisfying (3.50) and (3.54), $x \in X$ and $v_i \in F_i(x_i)$ $(i = 1,\ldots,n)$ satisfying (3.56) and (3.57) ((3.71) if X is Asplund), and a subgradient $(v_1^*,\ldots,v_n^*,v^*) \in \partial^C\widehat{f}(v_1,\ldots,v_n,x)$ ($(v_1^*,\ldots,v_n^*,v^*) \in \partial^F\widehat{f}(v_1,\ldots,v_n,x)$ if X is Asplund) such that $\|(v_1^*,\ldots,v_n^*,v^*)\|_\gamma < \alpha$. By the representation (3.69) of the dual norm, this implies $\sum_{i=1}^n \|v_i^*\| < \alpha\gamma$ and $\|v^*\| < \alpha$. The latter inequality contradicts the assumption.

(ii) The assertion is a direct consequence of Proposition 31(ii) and the representation (3.69) of the dual norm.

□

The next corollary provides the $\delta-$free version of Proposition 32.

Corollary 70 Let $\alpha > 0$.

(i) Suppose X is Banach, and F_i $(i=1,\ldots,n)$ are closed-valued. The collection $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ if

$$\liminf_{\substack{x_i\to\bar{x}_i,\, u_i \overset{F_i(x_i)}{\to} \bar{x},\, x\to\bar{x},\, v_i \overset{F_i(x_i)}{\to} u_i,\, v_i^*\to 0\ (i=1,\ldots,n)\\ 0<\max\limits_{1\le i\le n}\|v_i-x\|,\ (v_1^*,\ldots,v_n^*,v^*)\in\partial\widehat{f}(v_1,\ldots,v_n,x)}} \|v^*\| > \alpha, \tag{3.76}$$

where $\partial := \partial^C$.

If X is Asplund, then the above assertion is valid with $\partial := \partial^F$.

(ii) Suppose F_i $(i=1,\ldots,n)$ are convex-valued. If $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$, then

$$\liminf_{\substack{x_i\to\bar{x}_i,\, v_i\in F_i(x_i),\, v_i^*\to 0\ (i=1,\ldots,n)\\ 0<\max\limits_{1\le i\le n}\|v_i-\bar{x}\|\le\max\limits_{1\le i\le n} d_i(x_i,\bar{x}_i),\ (v_1^*,\ldots,v_n^*,v^*)\in\partial\widehat{f}(v_1,\ldots,v_n,\bar{x})}} \|x^*\| \ge \alpha.$$

The proof of Proposition 31(i) utilizes the sum rules in parts (ii) and (iii) of Lemma 4 to obtain representations of subgradients of the sum function (3.73) in terms of subgradients of the summand functions. Note that one of the summands – the function $\widehat{f}$ – is itself a sum of functions; see (3.61). Next we apply these sum rules again to obtain characterizations of the semitransversality in terms of normals to individual set-valued mappings. This time the difference between the exact sum rule in Lemma 4(ii) and the approximate sum rule in Lemma 4(iii) becomes explicit in the conclusions of the next theorem: compare the exact condition (3.79) and the corresponding approximate condition (3.80).

Theorem 22 Let X be a Banach space, $\delta>0$ and $\alpha>0$. Suppose F_i $(i=1,\ldots,n)$ are closed-valued. The collection $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ if, for some $\mu>0$ and all $x_i\in X_i$, $u_i\in X$ $(i=1,\ldots,n)$ satisfying (3.50) and (3.54), one of the following conditions is satisfied:

(i) with $N := N^C$,

$$\left\|\sum_{i=1}^n v_i^*\right\| + \mu\sum_{i=1}^n d\left(v_i^*, N_{F_i(x_i)}(v_i)\right) \ge \alpha \tag{3.77}$$

for all $x\in X$, $v_i\in F_i(x_i)$ $(i=1,\ldots,n)$ satisfying (3.56) and (3.57), and $v_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying

$$\sum_{i=1}^n \|v_i^*\| = 1, \tag{3.78}$$

$$\sum_{i=1}^n \langle v_i^*, x-v_i\rangle = \max_{1\le i\le n}\|x-v_i\|. \tag{3.79}$$

(ii) X is Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.77) holds with $N := N^F$ for all $x \in X$, $v_i \in F_i(x_i)$ $(i = 1,\ldots,n)$ satisfying (3.56) and (3.70), and $v_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying (3.77) and

$$\sum_{i=1}^{n} \langle v_i^*, x - v_i \rangle > \tau \max_{1 \le i \le n} \|x - v_i\|. \tag{3.80}$$

Proof Suppose $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ. Let $\mu > 0$. By Proposition 31(i), there exist $x_i \in X_i$, $u_i \in X$ $(i = 1,\ldots,n)$ satisfying (3.50) and (3.54), $x \in X$ and $v_i \in F_i(x_i)$ $(i = 1,\ldots,n)$ satisfying (3.56) and (3.57), and a subgradient $(\hat{v}_1^*,\ldots,\hat{v}_n^*,\hat{v}^*) \in \partial \widehat{f}(v_1,\ldots,v_n,x)$ such that

$$\|(\hat{v}_1^*,\ldots,\hat{v}_n^*,\hat{v}^*)\|_\gamma < \alpha, \tag{3.81}$$

where $\gamma := \mu^{-1}$ and ∂ stands for either the Clarke subdifferential (if X is a general Banach space) or the Fréchet subdifferential (if X is Asplund). Recall from (3.61) that $\widehat{f}$ is a sum of two functions: the maximum norm function defined by (3.60) and the indicator function of the set $F_1(x_1) \times \ldots \times F_n(x_n)$. Since $\max_{1 \le i \le n} \|v_i - x\| > 0$, the former is locally Lipschitz continuous near $(v_1,\ldots,v_n,x)$.

(i) X is a Banach space, and $(\hat{v}_1^*,\ldots,\hat{v}_n^*,\hat{v}^*)$ in (3.81) is a Clarke subgradient. By the Clarke–Rockafellar sum rule (Lemma 4(ii)), there exist $(v_1^*,\ldots,v_n^*,v^*) \in \partial^C f(v_1,\ldots,v_n,x)$ and $(\tilde{v}_1^*,\ldots,\tilde{v}_n^*) \in N^C_{F_1(x_1)\times\ldots\times F_n(x_n)}(v_1,\ldots,v_n)$ such that

$$(\hat{v}_1^*,\ldots,\hat{v}_n^*,\hat{v}^*) = (v_1^*,\ldots,v_n^*,v^*) + (\tilde{v}_1^*,\ldots,\tilde{v}_n^*,0). \tag{3.82}$$

By Lemma 6, $\tilde{v}_i^* \in N^C_{F_i(x_i)}(v_i)$ $(i = 1,\ldots,n)$. In view of Lemma 8, conditions (3.78), (3.79) are satisfied, and

$$v^* = \sum_{i=1}^{n} v_i^*. \tag{3.83}$$

Combining (3.81), (3.82) and (3.83), we obtain

$$\left\| \sum_{i=1}^{n} v_i^* \right\| + \mu \sum_{i=1}^{n} d\left(v_i^*, N^C_{F_i(x_i)}(v_i)\right) < \alpha.$$

This contradicts (3.77).

(ii) Let X be Asplund and a number $\tau \in]0,1[$ be given. In view of the fuzzy sum rule (Lemma 4(iii)): for any $\varepsilon > 0$, there exist $x' \in B_\varepsilon(x)$, $v_i' \in B_\varepsilon(v_i) \cap F_i(x_i)$, $v_i'' \in B_\varepsilon(v_i)$ $(i = 1,\ldots,n)$, $(v_1^*,\ldots,v_n^*,v^*) \in \partial^F f(v_1'',\ldots,v_n'',x')$ and $(\tilde{v}_1^*,\ldots,\tilde{v}_n^*) \in N^F_{F_1(x_1)\times\ldots\times F_n(x_n)}(v_1',\ldots,v_n')$, such that

$$\|(\tilde{v}_1^*,\ldots,\tilde{v}_n^*,0) + (v_1^*,\ldots,v_n^*,v^*) - (\hat{v}_1^*,\ldots,\hat{v}_n^*,\hat{v}^*)\|_\gamma < \varepsilon. \tag{3.84}$$

The number ε can be chosen small enough so that

$$\|x' - \bar{x}\| < \delta, \quad \max_{1\le i\le n} \|v_i' - u_i\| < \delta/\gamma, \quad 0 < \max_{1\le i\le n} \|v_i' - x\| < \alpha\delta,$$

and

$$\|(\hat{v}_1^*, \ldots, \hat{v}_n^*, \hat{v}^*)\|_\gamma + \varepsilon < \alpha, \tag{3.85}$$

$$\max_{1\le i\le n} \|v_i'' - v_i'\| < \frac{1-\tau}{2} \max_{1\le i\le n} \|x' - v_i'\|. \tag{3.86}$$

By Lemma 6, $\tilde{v}_i^* \in N^F_{F_i(x_i)}(v_i')$ $(i = 1, \ldots, n)$. In view of Lemma 8, $v_1^*, \ldots, v_n^*, v^*$ satisfy (3.78), (3.83), and

$$\sum_{i=1}^n \langle v_i^*, x' - v_i'' \rangle = \max_{1\le i\le n} \|x' - v_i''\|. \tag{3.87}$$

It follows from (3.86) and (3.87) that

$$\begin{aligned}
\sum_{i=1}^n \langle u_1^*, x' - v_i' \rangle &\ge \sum_{i=1}^n \langle v_i^*, x' - v_i'' \rangle - \max_{1\le i\le n} \|v_i'' - v_i'\| \\
&= \max_{1\le i\le n} \|v_i'' - x'\| - \max_{1\le i\le n} \|v_i'' - v_i'\| \\
&\ge \max_{1\le i\le n} \|x' - v_i'\| - 2 \max_{1\le i\le n} \|v_i'' - v_i'\| \\
&> \tau \max_{1\le i\le n} \|x' - v_i'\|.
\end{aligned}$$

Combining (3.84) and (3.85), we obtain

$$\left\| \sum_{i=1}^n v_i^* \right\| + \mu \sum_{i=1}^n d\left(v_i^*, N^F_{F_i(x_i)}(v_i')\right) < \|(\hat{v}_1^*, \ldots, \hat{v}_n^*, \hat{v}^*)\|_\gamma + \varepsilon < \alpha.$$

This contradicts (3.77).

□

Remark 47 Theorem 22 is a generalized version of Theorem 7 in the linear setting.

The key dual semitransversality condition (3.77) in Theorem 22 combines two conditions on the dual vectors $v_1^*, \ldots, v_n^* \in X^*$: either their sum must be sufficiently far from 0 or the vectors themselves must be sufficiently far from the corresponding normal cones. From the point of view of applications, it can be convenient to have these conditions separated. The next two corollaries show that it can be easily done.

Corollary 71 Let X be a Banach space, $\delta > 0$ and $\alpha > 0$. Suppose F_i $(i = 1, \ldots, n)$ are closed-valued. The collection $\{F_1, \ldots, F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1, \ldots, \bar{x}_n)$ with δ if, for some $\gamma > 0$ and all $x_i \in X_i$, $u_i \in X$ $(i = 1, \ldots, n)$ satisfying (3.50) and (3.54), one of the following conditions is satisfied:

(i) with $N := N^C$,

$$\sum_{i=1}^{n} \|v_i^*\| \geq \alpha \tag{3.88}$$

for all $x \in X$, $v_i \in F_i(x_i)$ $(i = 1,\dots,n)$ satisfying (3.56) and (3.57), and $v_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying (3.78), (3.79) and

$$\sum_{i=1}^{n} d\left(v_i^*, N_{F_i(x_i)}(v_i)\right) < \alpha\gamma; \tag{3.89}$$

(ii) X is Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.88) holds for all $x \in X$, $v_i \in F_i(x_i)$ $(i = 1,\dots,n)$ satisfying (3.56) and (3.70), and $v_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying (3.77), (3.80) and (3.89) with $N := N^F$.

Proof It is sufficient to notice that the assumptions Corollary 71(i) and (ii) imply those in the respective parts of Theorem 22. Indeed, the corollary replaces condition (3.77) by the stronger condition (3.88), which only needs to be satisfied by $v_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying (3.89). Setting $\mu := \gamma^{-1}$, we see that if condition (3.89) is violated, then condition (3.77) is automatically satisfied. □

Corollary 72 Let X be a Banach space, $\delta > 0$ and $\alpha > 0$. Suppose F_i $(i = 1,\dots,n)$ are closed-valued. The collection $\{F_1,\dots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\dots,\bar{x}_n)$ with δ if, for some $\gamma > 0$ and all $x_i \in X_i$, $u_i \in X$ $(i = 1,\dots,n)$ satisfying (3.50) and (3.54), one of the following conditions is satisfied:

(i) with $N := N^C$,

$$\sum_{i=1}^{n} d\left(v_i^*, N_{F_i(x_1)}(v_i)\right) \geq \alpha \tag{3.90}$$

for all $x \in X$, $v_i \in F_i(x_i)$ $(i = 1,\dots,n)$ satisfying (3.56) and (3.57), and $v_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying (3.78), (3.79) and

$$\sum_{i=1}^{n} \|v_i^*\| < \alpha\gamma; \tag{3.91}$$

(ii) X is Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.90) holds with $N := N^F$ for all $x \in X$ and $v_i \in F_i(x_i)$ $(i = 1,\dots,n)$ satisfying (3.56) and (3.70), and all $v_i^* \in X^*$ $(i = 1,\dots,n)$ satisfying (3.77), (3.80) and (3.91).

Proof Cf. the proof of Corollary 71. □

The next statement provides a dual necessary condition for the property, which is partially a converse of Theorem 22.

Theorem 23 Let $\delta > 0$ and $\alpha > 0$. Suppose F_i $(i = 1,\ldots,n)$ are convex-valued. If $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ, then

$$\left\|\sum_{i=1}^{n} v_i^*\right\| + \mu \sum_{i=1}^{n} d\left(v_i^*, N_{F_i(x_i)}(v_i)\right) \geq \alpha \tag{3.92}$$

with $\mu := \alpha + 1$ for all $x_i \in X_i$, $v_i \in F_i(x_i)$ $(i = 1,\ldots,n)$ satisfying (3.59), and $v_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying (3.78) and

$$\sum_{i=1}^{n} \langle v_i^*, \bar{x} - v_i \rangle = \max_{1\leq i\leq n} \|\bar{x} - v_i\|. \tag{3.93}$$

Proof Let $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ. Let $\mu := \alpha + 1$ and $\gamma := \mu^{-1}$. By Proposition 31(ii), condition (3.72) is satisfied for all $x_i \in X_i$, $v_i \in F_i(x_i)$ $(i = 1,\ldots,n)$ satisfying (3.59). Observe that $\widehat{f}$ is a sum of the function f given by (3.60) and the indicator function of the set $F_1(x_1) \times \ldots \times F_n(x_n)$. Since $\max_{1\leq i\leq n} \|v_i - \bar{x}\| > 0$, f is locally Lipschitz continuous near $(v_1,\ldots,v_n,\bar{x})$. In view of Lemmas 4(i) and 6,

$$\begin{aligned}\partial\widehat{f}(v_1,\ldots,v_n,\bar{x}) &= \partial f(v_1,\ldots,v_n,\bar{x}) + N_{F_1(x_1)\times\ldots\times F_n(x_n)}(v_1,\ldots,v_n) \times \{0\}\\ &= \partial f(v_1,\ldots,v_n,\bar{x}) + N_{F_1(x_1)}(v_1) \times \ldots \times N_{F_n(x_n)}(v_n) \times \{0\}.\end{aligned}$$

By Lemma 8, $(v_1^*,\ldots,v_n^*,v^*) \in \partial f(v_1,\ldots,v_n,\bar{x})$ if and only if conditions (3.78), (3.83) and (3.93) are satisfied. Hence, condition (3.92) is a consequence of (3.72). □

Remark 48 (i) It is evident from (3.92) that $\mu := \alpha + 1$ in Theorem 23 can be replaced by any $\mu \geq \alpha + 1$.

(ii) Conditions (3.92) and (3.93) in Theorem 23 are particular cases of conditions (3.77) and (3.79) in Theorem 22, respectively, corresponding to setting $x := \bar{x}$ and $v_i := u_i$ $(i = 1,\ldots,n)$.

(iii) Theorem 23 improves Theorem 10 in the linear setting.

In the convex setting, a similar decomposition of the dual semitransversality condition (3.92) in Theorem 23 can be easily obtained.

Corollary 73 Let $\delta > 0$ and $\alpha > 0$. Suppose F_i $(i = 1,\ldots,n)$ are convex-valued. If $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ, then, for all $x_i \in X_i$, $v_i \in F_i(x_i)$ $(i = 1,\ldots,n)$ satisfying (3.59), the following conditions are satisfied:

(i) for all $v_i^* \in N_{F_i(x_i)}(v_i)$ $(i = 1,\ldots,n)$ satisfying (3.78) and (3.93),

$$\sum_{i=1}^{n} \|v_i^*\| \geq \alpha;$$

(ii) for all $v_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying (3.78), (3.93) and $\sum_{i=1}^{n} v_i^* = 0$,

$$\sum_{i=1}^{n} d\left(v_i^*, N_{F_1(x_i)}(v_i)\right) \geq \alpha.$$

3.2.4 Semitransversality and Semiregularity

In this section, we establish relationships between the semitransversality and semiregularity of set-valued mappings.

Definition 14 Let $F : X \rightrightarrows Y$ be a set-valued mapping between metric spaces, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, and $\alpha > 0$. The mapping F is $\alpha-$semiregular at $(\bar{x},\bar{y})$ if there exists a $\delta > 0$ such that

$$\alpha d(\bar{x},F^{-1}(y)) \leq d(y,\bar{y}) \tag{3.94}$$

for all $y \in B_{\alpha\delta}(\bar{y})$.

For discussions about the property in Definition 14, we refer the reader to Chapter 1. The semiregularity also admits the equivalent geometric reformulation.

Proposition 33 Let $F : X \rightrightarrows Y$ be a set-valued mapping between metric spaces, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\delta > 0$ and $\alpha > 0$. The mapping F is $\alpha-$semiregular at $(\bar{x},\bar{y})$ with δ and α if and only if

$$F^{-1}(y) \cap B_\rho(\bar{x}) \neq \emptyset \tag{3.95}$$

for all $\rho \in]0,\delta[$ and $y \in Y$ with $d(y,\bar{y}) < \alpha\rho$.

Proof Cf. the proof of Theorem 21. □

Let $F_i : X_i \rightrightarrows X$ $(i = 1,\ldots,n)$ be set-valued mappings between metric spaces, $\bar{x}_i \in X_i$ $(i = 1,\ldots,n)$, and $\bar{x} \in \cap_{i=1}^n F_i(\bar{x}_i)$. Define $F : X \rightrightarrows X_1 \times \ldots \times X_n$ by

$$F(x) := F_1^{-1}(x) \times \ldots \times F_n^{-1}(x), \quad x \in X. \tag{3.96}$$

Observe that $(\bar{x},(\bar{x}_1,\ldots,\bar{x}_n)) \in \operatorname{gph} F$ and

$$F^{-1}(x_1,\ldots,x_n) = \bigcap_{i=1}^n F_i(x_i)$$

for all $x_i \in X_i$ $(i = 1,\ldots,n)$.

Theorem 24 Let F be defined by (3.96), $\delta > 0$ and $\alpha > 0$.

(i) If F is $\alpha-$semiregular at $(\bar{x},(\bar{x}_1,\ldots,\bar{x}_n))$ with δ, then $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ.

(ii) Suppose $\max_{1\leq i\leq n} d(\bar{x},F_i(x_i)) \leq \max_{1\leq i\leq n} d_i(x_i,\bar{x}_i)$ for all $x_i \in B_{\alpha\delta}(\bar{x}_i)$ $(i = 1,\ldots,n)$. If $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ, then F is $\alpha-$semiregular at $(\bar{x},(\bar{x}_1,\ldots,\bar{x}_n))$ with δ.

Proof

(i) Suppose F is $\alpha-$semiregular at $(\bar{x},(\bar{x}_1,\ldots,\bar{x}_n))$ with δ. In view of Definition 14,

$$\alpha d\left(\bar{x},\bigcap_{i=1}^{n}F_i(x_i)\right)\leq\max_{1\leq i\leq n}d_i(x_i,\bar{x}_i)\tag{3.97}$$

for all $x_i\in X_i$ $(i=1,\ldots,n)$ satisfying $\max_{1\leq i\leq n}d_i(x_i,\bar{x}_i)<\alpha\delta$. Consequently, inequality (3.49) holds for all $x_i\in X_i$ $(i=1,\ldots,n)$ satisfying (3.50). By Theorem 21, $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ.

(ii) Suppose $\{F_1,\ldots,F_n\}$ is $\alpha-$semitransversal at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$ with δ. By Theorem 21, then inequality (3.49) holds for all $x_i\in X_i$ $(i=1,\ldots,n)$ satisfying (3.50). Under the assumption made, inequality (3.97) is satisfied for all $x_i\in X_i$ $(i=1,\ldots,n)$ with $\max_{1\leq i\leq n}d_i(x_i,\bar{x}_i)<\alpha\delta$. Hence, F is $\alpha-$semiregular at $(\bar{x},(\bar{x}_1,\ldots,\bar{x}_2))$ with δ.

□

In view of Theorem 24, semitransversality of collections of set-valued mappings can be viewed as a particular case of semiregularity of set-valued mappings. We now show that the two models are in a sense equivalent.

Let $F:X\rightrightarrows Y$ be a set-valued mapping between normed spaces, and $(\bar{x},\bar{y})\in\operatorname{gph}F$. Define $F_1,F_2:X\times Y\rightrightarrows X\times Y$ for any $x\in X,y\in Y$ by

$$F_1(x,y):=\operatorname{gph}F-(x,y),\;\;F_2(x,y):=X\times\{\bar{y}\}-(x,y).\tag{3.98}$$

Observe that $(\bar{x},\bar{y})\in F_1(\bar{x}_1)\cap F_2(\bar{x}_2)$, where $\bar{x}_1=\bar{x}_2=(0,0)$.

The following quantitative relations has recently been established in [70].

Theorem 25 Let F_1 and F_2 be defined by (3.98), $\delta>0$ and $\alpha>0$, $\alpha_1:=(2\alpha^{-1}+1)^{-1}$ and $\alpha_2:=2\alpha$.

(i) If F is $\alpha-$semiregular at $(\bar{x},\bar{y})$ with δ, then $\{F_1,F_2\}$ is α_1-semitransversal at $(\bar{x},\bar{y})$ for $(\bar{x}_1,\bar{x}_2)$ with $\min\{\delta(1+\alpha/2),2\alpha^{-1}+1\}$.

(ii) If $\{F_1,F_2\}$ is α_1-semitransversal at $(\bar{x},\bar{y})$ for $(\bar{x}_1,\bar{x}_2)$ with δ, then F is α_2-semiregular at $(\bar{x},\bar{y})$ with δ.

3.2.5 Applications

Our aim in this section is to employ the results in previous sections to formulate characterizations of stationarity of collections of set-valued mappings as well as obtain necessary conditions in multiobjective optimization problems with variable ordering structures.

Stationarity of collections of set-valued mappings

Theorem 26 Let X be a Banach space, and F_i $(i=1,\ldots,n)$ be closed-valued. If $\{F_1,\ldots,F_n\}$ is stationary at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$, then, for every $\delta>0$, $\alpha>0$ and $\mu>0$, there exist $x_i\in B_{\alpha\delta}(\bar{x}_i)$, $u_i\in F(x_i)\cap B_{\alpha\delta}(\bar{x})$ $(i=1,\ldots,n)$ such that the following conditions are satisfied:

(i) with $N:=N^C$,

$$\left\|\sum_{i=1}^{n} v_i^*\right\| + \mu\sum_{i=1}^{n} d\left(v_i^*, N_{F_i(x_i)}(v_i)\right) < \alpha \tag{3.99}$$

for some $x\in B_\delta(\bar{x})$ and $v_i\in F_i(x_i)\cap B_{\delta\mu}(u_i)$ $(i=1,\ldots,n)$ satisfying (3.57), and $v_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (3.78);

(ii) X is Asplund, and there exists a $\tau\in]0,1[$ such that inequality (3.99) holds with $N:=N^F$ for some $x\in B_\delta(\bar{x})$ and $v_i\in F_i(x_i)\cap B_{\delta\mu}(u_i)$ $(i=1,\ldots,n)$ satisfying (3.71), and $v_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (3.80).

Proof The statement follows from Theorem 22 together with the fact that stationarity is the negation of semitransversality. □

The next corollaries are direct consequences of Theorem 26.

Corollary 74 Let X be a Banach space, and F_i $(i=1,\ldots,n)$ be closed-valued. If $\{F_1,\ldots,F_n\}$ is stationary at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$, then, for every $\delta>0$, $\alpha>0$ and $\gamma>0$, there exist $x_i\in B_{\alpha\delta}(\bar{x}_i)$, $u_i\in F(x_i)\cap B_{\alpha\delta}(\bar{x})$ $(i=1,\ldots,n)$ such that the following conditions are satisfied:

(i)

$$\sum_{i=1}^{n}\|v_i^*\| < \alpha \tag{3.100}$$

for some $x\in B_\delta(\bar{x})$ and $v_i\in F_i(x_i)\cap B_{\delta\mu}(u_i)$ $(i=1,\ldots,n)$ satisfying (3.57), and $v_i^*\in N^C_{F_i(x_i)}(v_i)$ $(i=1,\ldots,n)$ satisfying (3.78);

(ii) X is Asplund, and there exists a $\tau\in]0,1[$ such that inequality (3.100) holds for some $x\in B_\delta(\bar{x})$ and $v_i\in F_i(x_i)\cap B_{\delta\mu}(u_i)$ $(i=1,\ldots,n)$ satisfying (3.71), and $v_i^*\in N^F_{F_i(x_i)}(v_i)$ $(i=1,\ldots,n)$ satisfying (3.80).

Corollary 75 Let X be a Banach space, F_i $(i=1,\ldots,n)$ be closed-valued. If $\{F_1,\ldots,F_n\}$ stationary at $\bar{x}$ for $(\bar{x}_1,\ldots,\bar{x}_n)$, then, for every $\delta>0$, $\alpha>0$ and $\gamma>0$, there exist $x_i\in B_{\alpha\delta}(\bar{x}_i)$, $u_i\in F(x_i)\cap B_{\alpha\delta}(\bar{x})$ $(i=1,\ldots,n)$ such that the following conditions are satisfied:

(i) with $N:=N^C$,

$$\sum_{i=1}^{n} d\left(v_i^*, N_{F_i(x_1)}(v_i)\right) < \alpha \tag{3.101}$$

for some $x \in B_\delta(\bar{x})$ and $v_i \in F_i(x_i) \cap B_{\delta\mu}(u_i)$ $(i = 1, \ldots, n)$ satisfying (3.57), and $v_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (3.78), and

$$\left\| \sum_{i=1}^{n} v_i^* \right\| = 0; \tag{3.102}$$

(ii) X is Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.101) holds with $N := N^F$ for some $x \in B_\delta(\bar{x})$ and $v_i \in F_i(x_i) \cap B_{\delta\mu}(u_i)$ $(i = 1, \ldots, n)$ satisfying (3.71), and $v_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (3.80) and (3.102).

In the convex setting we can establish dual sufficient conditions for stationarity. Note that in this case we just need the image space X being a normed vector space.

Theorem 27 Suppose F_i $(i = 1, \ldots, n)$ are convex-valued. The collection $\{F_1, \ldots, F_n\}$ is stationary at $\bar{x}$ for $(\bar{x}_1, \ldots, \bar{x}_n)$, if for every $\delta > 0$, $\alpha > 0$ and $\mu > 0$, there exist $x_i \in X_i$, $v_i \in F_i(x_i)$ $(i = 1, \ldots, n)$ satisfying (3.59), and $v_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (3.93),

$$\left\| \sum_{i=1}^{n} v_i^* \right\| + \mu \sum_{i=1}^{n} d\left(v_i^*, N_{F_i(x_i)}(v_i)\right) < \alpha.$$

Proof The statement follows from Theorem 23 together with the fact that stationarity is the negation of semitransversality. □

The next statement follows straightforwardly from Theorem 27.

Corollary 76 Suppose F_i $(i = 1, \ldots, n)$ are convex-valued. The collection $\{F_1, \ldots, F_n\}$ is stationary at $\bar{x}$ for $(\bar{x}_1, \ldots, \bar{x}_n)$, if for every $\delta > 0$, $\alpha > 0$ and $\mu > 0$, there exist $x_i \in X_i$, $v_i \in F_i(x_i)$ $(i = 1, \ldots, n)$ satisfying (3.59), the one of the following conditions is satisfied:

(i) there exist $v_i^* \in N_{F_i(x_i)}(v_i)$ $(i = 1, \ldots, n)$ satisfying (3.93) such that

$$\sum_{i=1}^{n} \|v_i^*\| < \alpha;$$

(ii) there exist $v_i^* \in X^*$ $(i = 1, \ldots, n)$ satisfying (3.93) and $\sum_{i=1}^{n} v_i^* = 0$ such that

$$\sum_{i=1}^{n} d\left(v_i^*, N_{F_1(x_i)}(v_i)\right) < \alpha.$$

Necessary conditions in multiobjective optimization

Let $\mathscr{D} : Y \rightrightarrows Y$ be a set-valued mapping on a vector space Y. Define an ordering relation induced from $\mathscr{D}$ denoted by

$$y \leq_{\mathscr{D}} \bar{y} \text{ if and only if } \bar{y} \in y + \mathscr{D}(\bar{y}) \tag{3.103}$$

for any $y, \bar{y} \in Y$. That latter can be rewritten as $y \in \bar{y} - \mathscr{D}(\bar{y})$. The mapping $\mathscr{D}$ is called a *variable ordering structure*. Let us recall that the relation $\mathscr{D}$ is said to be *almost transitivity* if

$$[y_1 \in \operatorname{cl}\mathscr{D}(y_2) \;\&\; y_2 \in \mathscr{D}(y_3)] \Rightarrow y_1 \in \mathscr{D}(y_3)$$

for any $y_1, y_2, y_3 \in Y$. The almost transitivity is widely used in the study of multiobjective optimization problems; cf. [22, 173, 175, 222]. It is worth mentioning that the conventional *Pareto efficiency* [118, 158] is almost transitivity if the cone is convex and pointed [173, Proposition 5.56].

Definition 15 Let Ω be a subset of a vector space Y, and $\bar{y} \in \Omega$. Then $\bar{y}$ is called an *Pareto efficient point* of Ω with respect to $\mathscr{D}$ if there is no $\bar{y} \neq y \in \Omega$ such that $y \leq_{\mathscr{D}} \bar{y}$, i.e. $\Omega \cap (\bar{y} - \mathscr{D}(\bar{y})) = \{\bar{y}\}$.

It is obvious that the concept in Definition 15 reduces to the *Pareto efficiency* when the ordering structure is constant, i.e $\mathscr{D}(y) = K$ for any $y \in Y$ where K is a closed, convex, pointed cone. For intensive discussions about variable ordering structures and related concepts, the reader is referred to [22, 23, 49, 50, 86–88, 90] and references therein.

Let $F : X \rightrightarrows Y$ be a set-valued mapping between Banach spaces, Ω be a nonempty subset of X, and $\mathscr{D}$ is a variable ordering structure on Y. Our interest is the constrained multiobjective optimization problem with geometric constraint of the following form

$$\begin{aligned} \mathscr{P}: \ \text{minimize} \quad & F(x), \\ \text{subject to} \quad & x \in \Omega \ \text{ with respect to } \ \mathscr{D}. \end{aligned}$$

Definition 16 A pair $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is called an *efficient solution* of the problem $\mathscr{P}$ if $\bar{x} \in \Omega$ and $\bar{y}$ is an *efficient point* of $F(\Omega)$ with respect to $\mathscr{D}$, i.e. there is no pair $(x, y) \in \operatorname{gph} F$ with $x \in \Omega$ and $y \neq \bar{y}$ such that $y \leq_{\mathscr{D}} \bar{y}$.

As we already discussed, the primary advantage of the concept extremality/stationarity of collections of set-valued mappings is the ability to describe several optimal notions in vector optimization that the conventional ones cannot do. The next proposition illustrates this observation.

Proposition 34 Let $\mathscr{D}$ be almost transitivity, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ be an efficient solution of $\mathscr{P}$. Let $X_1 := (\bar{y} - \mathscr{D}(\bar{y})) \cup \{\bar{y}\}$, $X_2 := \{0\}$, and $F_i : X_i \rightrightarrows X \times Y$ $(i = 1, 2)$ be defined for any $y \in X_1$ by

$$F_1(y) := \Omega \times (y - \operatorname{cl}\mathscr{D}(y)) \quad \text{and} \quad F_2(0) := \operatorname{gph} F. \tag{3.104}$$

Then the pair $\{F_1, F_2\}$ is stationary at $(\bar{x}, \bar{y})$ for $(\bar{y}, 0)$.

Proof In view of [22, Proposition 2] (see also [175, Example 3.5]), the pair $\{F_1,F_2\}$ is locally extremal at $(\bar{x},\bar{y})$ for $(\bar{y},0)$. Since extremality implies stationarity, we get the conclusion. $\square$

The following statements establish necessary conditions for efficient solutions of the problem $\mathscr{P}$. These results are obtained by employing dual necessary conditions for the stationarity of collections of set-valued mappings. The conditions of this kind are new to the best of our knowledge.

Theorem 28 Let $\operatorname{gph} F$ and Ω be closed, and $\mathscr{D}$ be almost transitivity. If $(\bar{x},\bar{y}) \in \operatorname{gph} F$ be an efficient solution of $\mathscr{P}$, then, for every $\delta > 0$, $\alpha > 0$ and $\mu > 0$, there exist $y \in B_{\alpha\delta}(\bar{y})$, $(x_1,y_1) \in [\Omega \times (y - \operatorname{cl}\mathscr{D}(y))] \cap B_{\alpha\delta}(\bar{x},\bar{y})$, $(x_2,y_2) \in \operatorname{gph} F \cap B_{\alpha\delta}(\bar{x},\bar{y})$ such that the following conditions are satisfied:

(i) with $N := N^C$,

$$\|(x_1^*,y_1^*)+(x_2^*,y_2^*)\|+ \\ \mu\left(d((x_1^*,y_1^*),N_\Omega(x_1,y_1)\times N_{y-\operatorname{cl}\mathscr{D}(y)}(x_1,y_1))+d((x_2^*,y_2^*),N_{\operatorname{gph} F}(x_2,y_2))\right) < \alpha \tag{3.105}$$

for some $(x',y') \in B_\delta(\bar{x},\bar{y})$, $(x_1',y_1') \in [\Omega \times (y - \operatorname{cl}\mathscr{D}(y))] \cap B_{\delta\mu}(x_1,y_1)$ and $(x_2',y_2') \in \operatorname{gph} F \cap B_{\delta\mu}(x_2,y_2)$ satisfying

$$0 < \max_{i=1,2}\|(x_i'-x',y_i'-y')\| \le \max_{i=1,2}\|(x_i-\bar{y},y_i-\bar{y})\| < \alpha\delta, \tag{3.106}$$

and $(x_i^*,y_i^*) \in X^* \times Y^*$ $(i=1,2)$ satisfying

$$\|(x_1^*,y_1^*)\|+\|(x_2^*,y_2^*)\| = 1, \tag{3.107}$$

$$\sum_{i=1}^{2}\langle (x_i^*,y_i^*),(x'-x_i',y'-y_i')\rangle = \max_{i=1,2}\|(x'-x_i',y'-y_i')\|; \tag{3.108}$$

(ii) X and Y are Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.105) holds with $N := N^F$ for some $(x',y') \in B_\delta(\bar{x},\bar{y})$, $(x_1',y_1') \in [\Omega \times (y - \operatorname{cl}\mathscr{D}(y))] \cap B_{\delta\mu}(x_1,y_1)$ and $(x_2',y_2') \in \operatorname{gph} F \cap B_{\delta\mu}(x_2,y_2)$ satisfying

$$0 < \max_{i=1,2}\|(x_i'-x',y_i'-y')\| < \alpha\delta, \tag{3.109}$$

and $(x_i^*,y_i^*) \in X^* \times Y^*$ $(i=1,2)$ satisfying (3.107) and

$$\sum_{i=1}^{2}\langle (x_i^*,y_i^*),(x'-x_i',y'-y_i')\rangle > \tau\max_{i=1,2}\|(x'-x_i',y'-y_i')\|. \tag{3.110}$$

Proof Under the assumptions made the set-valued mappings F_1 and F_2 given by (3.104) are closed-valued. The statements are consequences of Theorem 26 and Proposition 34. $\square$

The following statements follow straightforwardly from Theorem 28.

Corollary 77 Let gph$F$ and Ω be closed, and $\mathscr{D}$ be almost transitivity. If $(\bar{x},\bar{y}) \in$ gph$F$ be an efficient solution of $\mathscr{P}$, then, for every $\delta > 0$, $\alpha > 0$ and $\mu > 0$, there exist $y \in B_{\alpha\delta}(\bar{y})$, $(x_1,y_1) \in [\Omega \times (y - \mathrm{cl}\,\mathscr{D}(y))] \cap B_{\alpha\delta}(\bar{x},\bar{y})$, $(x_2,y_2) \in \mathrm{gph}\,F \cap B_{\alpha\delta}(\bar{x},\bar{y})$ such that the following conditions are satisfied:

(i)

$$\|(x_1^*,y_1^*) + (x_2^*,y_2^*)\| < \alpha \tag{3.111}$$

for some $(x',y') \in B_\delta(\bar{x},\bar{y})$, $(x_1',y_1') \in [\Omega \times (y - \mathrm{cl}\,\mathscr{D}(y))] \cap B_{\delta\mu}(x_1,y_1)$ and $(x_2',y_2') \in \mathrm{gph}\,F \cap B_{\delta\mu}(x_2,y_2)$ satisfying (3.106), and $x_1^* \in N_\Omega^C(x_1)$, $y_1^* \in N^C_{y-\mathrm{cl}\,\mathscr{D}(y)}(y_1)$, $(x_2^*,y_2^*) \in N^C_{\mathrm{gph}\,F}(x_2,y_2)$ satisfying (3.107) and (3.108).

(ii) X and Y are Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.111) holds for some $(x',y') \in B_\delta(\bar{x},\bar{y})$, $(x_1',y_1') \in [\Omega \times (y - \mathrm{cl}\,\mathscr{D}(y))] \cap B_{\delta\mu}(x_1,y_1)$ and $(x_2',y_2') \in \mathrm{gph}\,F \cap B_{\delta\mu}(x_2,y_2)$ satisfying (3.109), and $x_1^* \in N_\Omega^F(x_1)$, $y_1^* \in N^F_{y-\mathrm{cl}\,\mathscr{D}(y)}(y_1)$, $(x_2^*,y_2^*) \in N^F_{\mathrm{gph}\,F}(x_2,y_2)$ satisfying (3.107) and (3.110).

Corollary 78 Let gph$F$ and Ω be closed, and $\mathscr{D}$ be almost transitivity. If $(\bar{x},\bar{y}) \in$ gph$F$ be an efficient solution of $\mathscr{P}$, then, for every $\delta > 0$, $\alpha > 0$ and $\mu > 0$, there exist $y \in B_{\alpha\delta}(\bar{y})$, $(x_1,y_1) \in [\Omega \times (y - \mathrm{cl}\,\mathscr{D}(y))] \cap B_{\alpha\delta}(\bar{x},\bar{y})$, $(x_2,y_2) \in \mathrm{gph}\,F \cap B_{\alpha\delta}(\bar{x},\bar{y})$ such that the following conditions are satisfied:

(i) with $N := N^C$,

$$d((x_1^*,y_1^*), N_\Omega(x_1) \times N_{y-\mathrm{cl}\,\mathscr{D}(y)}(y_1)) + d((x_2^*,y_2^*), N_{\mathrm{gph}\,F}(x_2,y_2)) < \alpha \tag{3.112}$$

for some $(x',y') \in B_\delta(\bar{x},\bar{y})$, $(x_1',y_1') \in [\Omega \times (y - \mathrm{cl}\,\mathscr{D}(y))] \cap B_{\delta\mu}(x_1,y_1)$ and $(x_2',y_2') \in \mathrm{gph}\,F \cap B_{\delta\mu}(x_2,y_2)$ satisfying (3.106), and $(x_i^*,y_i^*) \in X^* \times Y^*$ $(i = 1,2)$ satisfying (3.107), (3.108) and

$$\|(x_1^*,y_1^*) + (x_2^*,y_2^*)\| = 0; \tag{3.113}$$

(ii) X and Y are Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.112) holds with $N := N^F$ for some $(x,y) \in B_\delta(\bar{x},\bar{y})$, $(x_1',y_1') \in [\Omega \times (y - \mathrm{cl}\,\mathscr{D}(y))] \cap B_{\delta\mu}(x_1,y_1)$ and $(x_2',y_2') \in \mathrm{gph}\,F \cap B_{\delta\mu}(x_2,y_2)$ satisfying (3.109), and $(x_i^*,y_i^*) \in X^* \times Y^*$ $(i = 1,2)$ satisfying (3.106), (3.110) and (3.113).

3.3 Uniform Regularity of Set-Valued Mappings and Stability of Implicit Multifunctions

3.3.1 Definitions and Discussions

Many important problems in variational analysis and optimization can be modelled by an inclusion $y \in F(x)$, where F is a set-valued mapping. The behavior of the solution set $F^{-1}(y)$

when y and/or F are perturbed is of special interest. The concepts of *metric regularity* and *subregularity* (cf., e.g., [81, 116, 173]) have been the key tools when studying stability of solutions. In the next definition, we use the names $\alpha-$*regularity* and $\alpha-$*subregularity*, fixing the main quantitative parameter in the conventional definitions of the properties.

Definition 17 Let X and Y be metric spaces, $F: X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, and $\alpha > 0$. The mapping F is

(i) $\alpha-$regular at $(\bar{x},\bar{y})$ if there exist $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that

$$\alpha d(x, F^{-1}(y)) \leq d(y, F(x)) \tag{3.114}$$

for all $x \in B_\delta(\bar{x})$ and $y \in B_\delta(\bar{y})$ with $d(y,F(x)) < \alpha\mu$;

(ii) $\alpha-$subregular at $(\bar{x},\bar{y})$ if there exist $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that

$$\alpha d(x, F^{-1}(\bar{y})) \leq d(\bar{y}, F(x)) \tag{3.115}$$

for all $x \in B_\delta(\bar{x})$ with $d(\bar{y},F(x)) < \alpha\mu$.

Note that δ and μ in the above definition can take infinite values; thus, the definition covers local as well as global properties. This remark applies also to the subsequent definitions. The technical conditions $d(y,F(x)) < \alpha\mu$ and $d(\bar{y},F(x)) < \alpha\mu$ can be dropped (cf. [112, 172]), particularly because the value $\mu = +\infty$ is allowed. This does not affect the properties themselves, but can have an effect on the value of δ.

Inequalities (3.114) and (3.115) provide linear estimates of the distance from x to the solution set of the respective generalized equation via the 'residual' $d(y,F(x))$ or $d(\bar{y},F(x))$. As commented by Dontchev and Rockafellar [81, p.178] 'in applications, the residual is typically easy to compute or estimate, whereas finding a solution might be considerably more difficult'.

Besides their importance in studying stability of solutions to inclusions, regularity type estimates are involved in constraint qualifications for optimization problems, qualification conditions in subdifferential and coderivative calculus, and convergence analysis of computational algorithms [1, 5, 6, 8, 52, 59, 82, 103, 161].

When y is not fixed and can be any point in a neighbourhood of a given point $\bar{y}$, it represents *canonical perturbations* of the inclusion $\bar{y} \in F(x)$. For some applications it can be important to allow also perturbations in the right-hand side. This leads to the need to consider parametric inclusions $\bar{y} \in F(p,x)$ (or even $y \in F(p,x)$, thus, combining the two types of perturbations), where F is a set-valued mapping of two variables, with (nonlinear) perturbations in the right-hand side given by a parameter p from some fixed set P.

Along with the mapping $F: P \times X \rightrightarrows Y$, which is our main object in this chapter, given a point $p \in P$, we consider the mapping $F_p := F(p,\cdot): X \rightrightarrows Y$. Given a $y \in Y$, the mapping

$$p \mapsto G(p) := F_p^{-1}(y) = \{x \in X \mid y \in F(p,x)\} \tag{3.116}$$

can be interpreted as an *implicit multifunction* corresponding to the parametric inclusion $y \in F(p,x)$. When studying implicit multifunctions, it is common to consider 'uniform' versions of the properties in Definition 17 (cf., e.g., [115, Definition 3.1]).

Definition 18 Let X and Y be metric spaces, and P be a set, $F : P \times X \rightrightarrows Y$, $\bar{x} \in X$, $\bar{y} \in Y$, and $\alpha > 0$. The mapping F is

(i) $\alpha-$regular in x uniformly in p over P at $(\bar{x},\bar{y})$ if there exist $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that

$$\alpha d(x, F_p^{-1}(y)) \le d(y, F(p,x)) \tag{3.117}$$

for all $p \in P$, $x \in B_\delta(\bar{x})$ and $y \in B_\delta(\bar{y})$ with $d(y,F(p,x)) < \alpha\mu$;

(ii) $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ if there exist $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that

$$\alpha d(x, F_p^{-1}(\bar{y})) \le d(\bar{y}, F(p,x)) \tag{3.118}$$

for all $p \in P$ and $x \in B_\delta(\bar{x})$ with $d(\bar{y},F(p,x)) < \alpha\mu$.

If P is a singleton, then the properties in Definition 18 reduce to the corresponding conventional regularity properties in Definition 17. Moreover, the subregularity property in Definition 18(ii) coincides in this case with the subregularity property of the mapping (3.116) considered in [56].

Remark 49 (i) If Y is a linear metric space with a shift-invariant metric, in particular, a normed space, then the property in part (i) of Definition 18 reduces to the one in part (ii) with the extended parameter set $\widehat{P} := P \times Y$ and set-valued mapping $\widehat{F}((p,y),x) := F(p,x) - y$, $((p,y),x) \in \widehat{P} \times X$, in place of P and F, respectively. Moreover, in both parts of the definition, it is sufficient to consider the case $\bar{y} := 0$: the general case reduces to it by replacing F with $F - \bar{y}$.

(ii) Unlike Definition 17, in Definition 18 the reference point $(\bar{x},\bar{y})$ is not associated with the graph of F. This is a technical relaxation caused by the fact that $\operatorname{gph} F$ is a subset of a product of three spaces $P \times X \times Y$, and at this stage there is no reference point in P. Definition 19 below is formulated in a more conventional way.

(iii) There exist other concepts of uniform regularity in the literature. For instance, it is not uncommon to talk about uniform regularity when inequality (3.114) holds for all (x,y) in a compact subset of $X \times Y$; cf. [59].

Local (in p) versions of the properties in Definition 18 are of special interest. They correspond to P being a neighbourhood of a point $\bar{p}$ in some metric space; cf., e.g., [115,187].

Definition 19 Let P, X and Y be metric spaces, $F : P \times X \rightrightarrows Y$, $(\bar{p},\bar{x},\bar{y}) \in \operatorname{gph} F$, and $\alpha > 0$. The mapping F is

(i) $\alpha-$regular in x uniformly in p at $(\bar{p},\bar{x},\bar{y})$ if there exist $\eta \in]0,+\infty]$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that inequality (3.117) is satisfied for all $p \in B_\eta(\bar{p})$, $x \in B_\delta(\bar{x})$ and $y \in B_\delta(\bar{y})$ with $d(y,F(p,x)) < \alpha\mu$;

(ii) $\alpha-$subregular in x uniformly in p at $(\bar{p},\bar{x},\bar{y})$ if there exist $\eta \in]0,+\infty]$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that inequality (3.118) is satisfied for all $p \in B_\eta(\bar{p})$ and $x \in B_\delta(\bar{x})$ with $d(\bar{y},F(p,x)) < \alpha\mu$.

We often simply say that F is regular or subregular if the exact value of α in the above definitions is not important. The exact upper bound of all $\alpha > 0$ such that a property in the above definitions is satisfied with some $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ (and $\eta \in]0,+\infty]$), is called the *modulus* (or rate) of the property.

Apart from the main parameter α, providing a quantitative measure of the respective property, the properties in above definitions depend also on the auxiliary parameters δ, η and μ. They control (directly and indirectly) the size of the neighbourhoods of $\bar{x}$ and $\bar{p}$ involved in the definitions. As discussed above, the last parameter can be dropped (together with the corresponding constraints). We keep all the parameters to emphasize their different roles in the definitions and corresponding characterizations. The necessary and sufficient regularity conditions presented in the chapter normally involve the same collection of parameters.

The properties in Definitions 18 and 19 can be interpreted as kinds of Lipschitz-like properties of the implicit multifunction (solution mapping) (3.116). This observation opens a way for numerous applications of the characterizations established in this chapter and many other papers; cf. Section 3.2.3.

Regularity properties of implicit multifunctions were first considered by Robinson [198, 200] when studying stability of solution sets of linear and differentiable nonlinear systems. This initiated a great deal of research by many authors, mostly in normed spaces (and with $\bar{y} := 0$). Dontchev et al. [79, Theorem 2.1] gave a sufficient condition for regularity of implicit multifunctions in terms of graphical derivatives. Ngai et al. [184, 187] employed the theory of error bounds to characterize the property in metric and Banach spaces. In [56, 98, 106, 107, 146, 189, 212] dual sufficient conditions were established in finite and infinite dimensions in terms of Fréchet, limiting, directional limiting and Clarke coderivatives. Chieu et al. [51] established connections between regularity and Lipschitz-like properties of implicit multifunctions.

The regularity properties of the type given in Definitions 18 and 19 are referred to in the literature as *metric regularity* [56, 107, 146], *metric regularity in Robinson's sense* [189, 212], and *Robinson metric regularity* [51, 106] (of implicit multifunctions). We refer the readers to [14, 19, 81, 115, 145, 189] for more discussions and historical comments.

The metric properties in Definitions 18 and 19 admit equivalent geometric characterizations. This is illustrated by the next proposition providing a characterization for the property in Definition 18(ii).

Proposition 35 Let X and Y be metric spaces, and P be a set, $F : P \times X \rightrightarrows Y$, $\bar{x} \in X$, $\bar{y} \in Y$, and $\alpha > 0$. The mapping F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x}, \bar{y})$ with some $\delta \in]0, +\infty]$ and $\mu \in]0, +\infty]$ if and only if

$$F_p^{-1}(\bar{y}) \cap B_\rho(x) \neq \emptyset \tag{3.119}$$

for all $\rho \in]0, \mu[$, $p \in P$ and $x \in B_\delta(\bar{x})$ with $d(\bar{y}, F(p,x)) < \alpha\rho$.

Proof Suppose F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x}, \bar{y})$ with some $\delta \in]0, +\infty]$ and $\mu \in]0, +\infty]$. Let $\rho \in]0, \mu[$, $p \in P$ and $x \in B_\delta(\bar{x})$ with $d(\bar{y}, F(p,x)) < \alpha\rho$. Then $d(\bar{y}, F(p,x)) < \alpha\mu$. By Definition 18(ii), $d(x, F_p^{-1}(\bar{y})) \le \alpha^{-1} d(\bar{y}, F(p,x)) < \rho$. Hence, condition (3.119) is satisfied.

Conversely, suppose $\delta \in]0, +\infty]$ and $\mu \in]0, +\infty]$, and condition (3.119) is satisfied for all $\rho \in]0, \mu[$, $p \in P$ and $x \in B_\delta(\bar{x})$ with $d(\bar{y}, F(p,x)) < \alpha\rho$. Let $p \in P$ and $x \in B_\delta(\bar{x})$ with $d(\bar{y}, F(p,x)) < \alpha\mu$. Choose a ρ satisfying $\alpha^{-1} d(\bar{y}, F(p,x)) < \rho < \mu$. Then, by (3.119), $d(x, F_p^{-1}(\bar{y})) < \rho$. Letting $\rho \downarrow \alpha^{-1} d(\bar{y}, F(p,x))$, we arrive at (3.118), i.e. F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x}, \bar{y})$ with δ and μ. □

The aim of this chapter is not to add some new sufficient or necessary conditions for regularity properties of general set-valued mappings or implicit multifunctions to the large volume of existing ones (although some conditions in the subsequent sections are indeed new), but to propose a unifying general (i.e. not assuming the mapping F to have any particular structure and not using tangential approximations of $\operatorname{gph} F$) view on the theory of regularity, and clarify the relationships between the existing conditions including their hierarchy. We expose the typical sequence of regularity assertions, often hidden in the proofs, and the roles of the assumptions involved in the assertions, in particular, on the underlying space: general metric, normed, Banach or Asplund.

We present a series of necessary and sufficient regularity conditions with the main emphasis (in line with the current trend in the literature) on the latter ones. The (typical) sequence of sufficient regularity conditions is represented by the following chain of assertions, each subsequent assertion being a consequence of the previous one:

(i) nonlocal primal space conditions in complete metric spaces (Theorem 29(ii));

(ii) local primal space conditions in complete metric spaces (Corollary 80(ii));

(iii) subdifferential conditions in Banach and Asplund spaces (Proposition 36);

(iv) normal cone conditions in Banach and Asplund spaces (Theorem 30);

(v) coderivative conditions in Banach and Asplund spaces (Corollaries 83 and 84).

Even if one targets coderivative conditions, they still have to go through the five steps listed above with details often hidden in long proofs. Apart from making the whole process more transparent, which is our main objective, the assertions in (i)–(iv) can be of independent interest, at least theoretically, especially the slope type conditions in (ii) and normal cone conditions in (iv). In combination with tangential approximations of $\operatorname{gph} F$, they are likely to lead to verifiable regularity conditions.

The implications (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (v) in the above list follow immediately from the definitions. The main assertions are the sufficiency of condition (i), and implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). They employ the following fundamental tools of variational analysis:

- *Ekeland variational principle* (sufficiency of condition (i));
- *sum rules* for respective subdifferentials (implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv)).

Thus, all the sufficient conditions on the list are consequences of the Ekeland variational principle, and as such, they are 'outer' conditions, i.e. they need to be checked at points outside the solution set $F_p^{-1}(\bar{y})$.

Most of the sufficient conditions are accompanied by the corresponding necessary ones. The necessary conditions do not require the underlying spaces to be complete and are generally easy consequences of the definitions. With the exception of the general nonlocal condition in Theorem 29(i), such conditions are formulated in normed spaces and assume the graph of F to be convex. In Section 2.5.1, we provide a series of dual necessary regularity conditions for set-valued mappings with closed convex graphs acting between Banach spaces some of which are also sufficient.

In the setting of complete metric spaces, and assuming that $\operatorname{gph} F_p$ is closed for all $p \in P$, the gap between the nonlocal necessary and sufficient subregularity conditions in Theorem 29 is not big: they share the same inequality (3.62); with all the other parameters coinciding, the sufficiency part naturally requires it to hold for all x in a larger set. Unfortunately, unlike the 'full' regularity possessing the well known coderivative criterion (see, e.g., [125, 172]), this is not the case in general with local subregularity conditions unless the graph of F is convex. The sufficient subregularity conditions presented in the chapter are the weakest possible in each group, but can still be far from necessary. As it has been discussed in the literature (see, e.g., a discussion of the equivalent subtransversality property in [138]), the reason for this phenomenon lies in the fact that the subregularity property lacks robustness.

The hot topic of regularity of a set-valued mapping F with a special structure, particularly in the arising in numerous applications such as, e.g., KKT systems and variational inequalities, case when $F = g + G$ with g single valued and G set-valued (typically a normal cone mapping), is outside the scope of the current chapter. Computing 'slopes' and coderivatives of such mappings (or normal cones to their graphs) is usually a difficult job and requires imposing additional assumptions on g and G. This is what people working in this area normally do. We want to emphasize that this type of conditions still fall into the five-point scheme described above.

3.3.2 Slope Necessary and Sufficient Conditions

This section is dedicated to slope necessary and sufficient conditions. For simplicity, we focus on the uniform subregularity property in Definition 18(ii). The corresponding conditions for the property in Definition 18(i) can be formulated in a similar way. Besides, in view of Remark 49(i), in normed spaces (which is our setting in the next section) such conditions can be obtained as consequences of those for the subregularity.

The necessary conditions are deduced directly from the definitions of the respective properties, while the sufficient ones come from the application of the Ekeland variational principle. In the convex case, the conditions are necessary and sufficient.

In this section, P is a nonempty set, X and Y are metric spaces, and $F : P\times X \rightrightarrows Y$. We assume the parameters $\bar{x}\in X$, $\bar{y}\in Y$, $\alpha>0$, $\delta\in]0,+\infty]$ and $\mu\in]0,+\infty]$ to be fixed. In what follows, we employ a collection of functions

$$\psi_p(u,v) := d(v,\bar{y}) + i_{\operatorname{gph} F_p}(u,v), \quad u\in X,\ v\in Y \tag{3.120}$$

depending on a parameter $p\in P$. Along with the standard maximum distance on $X\times Y$, we also use a metric depending on a parameter $\gamma>0$:

$$d_\gamma((u,v),(x,y)) := \max\{d(u,x), \gamma d(v,y)\}, \quad u,x\in X,\ v,y\in Y. \tag{3.121}$$

The next theorem plays a crucial role for the subsequent considerations. The slope and subdifferential/normal cone/coderivative conditions for uniform $\alpha-$subregularity in this chapter are consequences of this theorem.

Theorem 29 (i) If F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with some $\delta\in]0,+\infty]$ and $\mu\in]0,+\infty]$, then

$$\sup_{\substack{(u,v)\in\operatorname{gph} F_p,\,(u,v)\neq(x,y)\\ d(u,\bar{x})<\delta+\mu,\, d(v,\bar{y})<\alpha\mu}} \frac{d(y,\bar{y})-d(v,\bar{y})}{d_\gamma((u,v),(x,y))} \geq \alpha \tag{3.122}$$

for $\gamma:=\alpha^{-1}$, and all $p\in P$, $x\in B_\delta(\bar{x})$ and $y\in Y$ satisfying

$$x\notin F_p^{-1}(\bar{y}), \quad y\in F(p,x)\cap B_{\alpha\mu}(\bar{y}). \tag{3.123}$$

(ii) Suppose X and Y are complete, and $\operatorname{gph} F_p$ is closed for all $p\in P$. If inequality (3.122) holds for some $\gamma>0$, and all $p\in P$, $x\in B_{\delta+\mu}(\bar{x})$ and $y\in Y$ satisfying (3.123), then F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ.

Proof

(i) Suppose F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ. Let $p \in P$, $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfy (3.123), $\gamma := \alpha^{-1}$, and $\eta > 1$. By (3.123) and Definition 18(ii), there exist a $\xi \in]1,\eta[$ such that $\xi d(y,\bar{y}) < \alpha\mu$, and a point $\hat{x} \in F_p^{-1}(\bar{y})$ such that $\alpha d(x,\hat{x}) < \xi d(y,\bar{y})$. Thus, $(\hat{x},\bar{y}) \in \operatorname{gph} F_p$, $(\hat{x},\bar{y}) \neq (x,y)$,

$$d(\hat{x},\bar{x}) \le d(\hat{x},x) + d(x,\bar{x}) < \alpha^{-1}\xi d(y,\bar{y}) + \delta \le \mu + \delta, \quad \text{and}$$
$$d_\gamma((x,y),(\hat{x},\bar{y})) = \max\{d(x,\hat{x}), \gamma d(y,\bar{y})\} \le \alpha^{-1}\max\{\xi,1\} d(y,\bar{y}) = \alpha^{-1}\xi d(y,\bar{y}).$$

Hence,

$$\sup_{\substack{(u,v)\in\operatorname{gph} F_p,\,(u,v)\neq(x,y)\\ d(u,\bar{x})<\delta+\mu,\, d(v,\bar{y})<\alpha\mu}} \frac{d(y,\bar{y}) - d(v,\bar{y})}{d_\gamma((u,v),(x,y))} \ge \frac{d(y,\bar{y})}{d_\gamma((\hat{x},\bar{y}),(x,y))}$$
$$\ge \alpha\xi^{-1} > \alpha\eta^{-1}.$$

Letting $\eta \downarrow 1$, we arrive at (3.122).

(ii) Suppose F is not $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ. By Definition 18(ii), there exist points $p \in P$ and $x \in B_\delta(\bar{x})$ such that

$$d(\bar{y}, F(p,x)) < \alpha \min\{d(x, F_p^{-1}(\bar{y})), \mu\}.$$

Hence, $x \notin F_p^{-1}(\bar{y})$, or equivalently, $\bar{y} \notin F(p,x)$. Set $\mu_0 := \min\{d(x,F_p^{-1}(\bar{y})),\mu\}$. Choose a number ε such that $d(\bar{y},F(p,x)) < \varepsilon < \alpha\mu_0$, and a point $y \in F(p,x)$ such that $d(y,\bar{y}) < \varepsilon$. The function $\psi_p : X \times Y \to \mathbb{R}_+ \cup \{+\infty\}$, defined by (3.120), is lower semicontinuous on $\overline{B}_{\mu+\delta}(\bar{x}) \times \overline{B}_{\alpha\mu}(\bar{y})$. Besides,

$$\psi_p(x,y) = d(y,\bar{y}) < \inf_{\overline{B}_{\mu+\delta}(\bar{x})\times\overline{B}_{\alpha\mu}(\bar{y})} \psi_p + \varepsilon.$$

Let $\gamma > 0$. Applying the Ekeland variational principle (Lemma 1) to the restriction of ψ_p to the complete metric space $\overline{B}_{\mu+\delta}(\bar{x}) \times \overline{B}_{\alpha\mu}(\bar{y})$ with the metric (3.121), we can find a point $(\hat{x},\hat{y}) \in \overline{B}_{\mu+\delta}(\bar{x}) \times \overline{B}_{\alpha\mu}(\bar{y})$ such that

$$d_\gamma((\hat{x},\hat{y}),(x,y)) < \mu_0, \quad \psi_p(\hat{x},\hat{y}) \le \psi_p(x,y), \tag{3.124}$$
$$\psi_p(\hat{x},\hat{y}) \le \psi_p(u,v) + (\varepsilon/\mu_0) d_\gamma((u,v),(\hat{x},\hat{y})) \tag{3.125}$$

for all $(u,v) \in \overline{B}_{\mu+\delta}(\bar{x}) \times \overline{B}_{\alpha\mu}(\bar{y})$. By (3.124), we have $\hat{y} \in F(p,\hat{x})$, and

$$d(\hat{x},\bar{x}) \le d(\hat{x},x) + d(x,\bar{x}) < \mu_0 + \delta \le \mu + \delta,$$
$$d(\hat{y},\bar{y}) \le d(y,\bar{y}) < \varepsilon < \alpha\mu_0 \le \alpha\mu.$$

Besides, $d(\hat{x},x) < \mu_0 \le d(x,F_p^{-1}(\bar{y}))$. This implies $\hat{x} \notin F_p^{-1}(\bar{y})$, and consequently, $\hat{y} \neq \bar{y}$. It follows from (3.125) that

$$\sup_{\substack{(u,v)\in\operatorname{gph} F_p,\,(u,v)\neq(\hat{x},\hat{y})\\ d(u,\bar{x})<\delta+\mu,\, d(v,\bar{y})<\alpha\mu}} \frac{d(\hat{y},\bar{y}) - d(v,\bar{y})}{d_\gamma((u,v),(\hat{x},\hat{y}))} \le \frac{\varepsilon}{\mu_0} < \alpha.$$

The last estimate contradicts (3.122).

$\square$

Remark 50 (i) The expression in the left-hand side of the inequality (3.122) is the nonlocal γ-slope [131, p. 60] at (x,y) of the restriction of the function ψ_p, given by (3.120), to $\operatorname{gph} F_p \cap [B_{\delta+\mu}(\bar{x}) \times B_{\alpha\mu}(\bar{y})]$.

(ii) By the definition of the metric (3.121), if inequality (3.122) is satisfied with a $\gamma > 0$, then it is also satisfied with any $\gamma' \in]0,\gamma[$. This observation is applicable to all slope inequalities in this section.

(iii) The completeness of the space and closedness assumption in part (ii) of Theorem 29 (and the subsequent statements) can be relaxed: it suffices to require that $\operatorname{gph} F_p \cap [\overline{B}_{\delta+\mu}(\bar{x}) \times \overline{B}_{\alpha\mu}(\bar{y})]$ is complete for all $p \in P$.

(iv) The sufficient condition in part (ii) of Theorem 29 is often hidden in the proofs of dual sufficient conditions.

(v) When X and Y are complete, and $\operatorname{gph} F_p$ is closed for all $p \in P$, the gap between the nonlocal necessary and sufficient regularity conditions in parts (i) and (ii) of Theorem 29 is not big: they share the same inequality (3.62); with all the other parameters coinciding, the necessity part (i) guarantees this inequality to hold for all $x \in B_\delta(\bar{x})$, while the sufficiency part (ii) requires it to hold for all x in a larger set $B_{\delta+\mu}(\bar{x})$.

We now illustrate Theorem 29 by applying it to the local (in p) setting in Definition 19(ii). The application is straightforward. We provide a single illustration of this kind, although the other statements in this and the next section are also applicable to this setting.

Corollary 79 Let P be a metric space, and $(\bar{p},\bar{x},\bar{y}) \in \operatorname{gph} F$.

(i) If F is $\alpha-$subregular in x uniformly in p at $(\bar{p},\bar{x},\bar{y})$ with η, δ and μ, then inequality (3.122) holds with $\gamma := \alpha^{-1}$ for all $p \in B_\eta(\bar{p})$, $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfying (3.123).

(ii) Suppose X and Y are complete, and $\operatorname{gph} F_p$ is closed for all $p \in B_\eta(\bar{p})$. If inequality (3.122) holds for some $\gamma > 0$, and all $p \in B_\eta(\bar{p})$, $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfying (3.123), then F is $\alpha-$subregular in x uniformly in p at $(\bar{p},\bar{x},\bar{y})$ with η, δ and μ.

The next statement presents a localized version of Theorem 29.

Corollary 80 (i) Suppose X and Y are normed spaces, and $\operatorname{gph} F_p$ is convex for all $p \in P$. If F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ, then

$$\limsup_{\substack{u\to x,\, v\to y,\, (u,v)\in \operatorname{gph} F_p,\, (u,v)\neq(x,y)\\ d(u,\bar{x})<\delta+\mu,\, d(v,\bar{y})<\alpha\mu}} \frac{d(y,\bar{y})-d(v,\bar{y})}{d_\gamma((u,v),(x,y))} \geq \alpha \tag{3.126}$$

for $\gamma := \alpha^{-1}$, and all $p \in P$, $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfying (3.123).

(ii) Suppose X and Y are complete, and $\operatorname{gph} F_p$ is closed for all $p \in P$. If inequality (3.126) holds for some $\gamma > 0$, and all $p \in P$, $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfying (3.123), then F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ.

Proof In view of Remarks 50(i) and 51(i), assertion (i) follows from Lemma 2(iv) and Theorem 29(i), while assertion (ii) is a consequence of Lemma 2(ii) and Theorem 29(ii). $\square$

Remark 51 (i) The expression in the left-hand side of inequality (3.126) is the γ-slope [131, p. 61] at (x,y) of the restriction of the function ψ_p, given by (3.120), to $\operatorname{gph} F_p \cap [B_{\delta+\mu}(\bar{x}) \times B_{\alpha\mu}(\bar{y})]$.

(ii) The convexity assumption in part (i) of Corollary 80 (and its subsequent statements) can be relaxed: it suffices to require that $\operatorname{gph} F_p \cap [\overline{B}_{\delta+\mu}(\bar{x}) \times \overline{B}_{\alpha\mu}(\bar{y})]$ is convex for all $p \in P$.

(iii) In the particular case when P is a neighborhood of a point $\bar{p}$ in some metric space, part (ii) of Corollary 80 is a quantitative version of [115, Proposition 3.5]. Ngai et al. [187, Theorem 3] established a primal sufficient condition for the property under the assumption that the mapping $F(\cdot,\bar{x})$ is lower semicontinuous at $\bar{p}$.

3.3.3 Dual Necessary and Sufficient Conditions

In this section, we continue studying the mapping $F : P \times X \rightrightarrows Y$ where P is a nonempty set, while X and Y are assumed to be normed spaces. We also assume the parameters $\bar{x} \in X$, $\bar{y} \in Y$, $\alpha > 0$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ to be fixed, and the collection of functions ψ_p be defined by (3.120).

The primal and dual parametric product space norms, corresponding to the distance (3.121), have the following form:

$$\|(x,y)\|_\gamma = \max\{\|x\|, \gamma\|y\|\}, \quad x \in X,\ y \in Y. \tag{3.127}$$

$$\|(x^*,y^*)\|_\gamma = \|x^*\| + \frac{1}{\gamma}\|y^*\|, \quad x^* \in X^*,\ y^* \in Y^*. \tag{3.128}$$

We denote by d_γ the distance in $X^* \times Y^*$ determined by (3.128).

Dual Sufficient Conditions

In this subsection, we assume that X and Y are Banach spaces, and $\operatorname{gph} F_p$ is closed for all $p \in P$.

The next subdifferential sufficient condition for uniform $\alpha-$subregularity is a consequence of Corollary 80(ii) thanks to two subdifferential sum rules in Lemma 4.

Proposition 36 Let $\partial := \partial^C$. If

$$d_\gamma(0, \partial\psi_p(x,y)) \geq \alpha \tag{3.129}$$

for some $\gamma > 0$, and all $p \in P$, $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfying (3.123), then F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ.

If X and Y are Asplund, then the above assertion is valid with $\partial := \partial^F$.

Proof Suppose F is not $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ. Let $\gamma > 0$. By Corollary 80(ii), there exist points $p \in P$, $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfying (3.123), and an $\alpha' \in]0,\alpha[$ such that

$$\|y-\bar{y}\| - \|v-\bar{y}\| \le \alpha'\|(u,v)-(x,y)\|_\gamma$$

for all $(u,v) \in \operatorname{gph} F_p \cap [B_{\mu+\delta}(\bar{x}) \times B_{\alpha\mu}(\bar{y})]$ near (x,y). In other words, (x,y) is a local minimizer of the function

$$(u,v) \mapsto \psi_p(u,v) + \alpha'\|(u,v)-(x,y)\|_\gamma. \tag{3.130}$$

By Lemma 5, its Fréchet and, as a consequence, Clarke subdifferential at this point contains 0. Observe that (3.130) is the sum of the function ψ_p and the Lipschitz continuous convex function $(u,v) \mapsto \alpha'\|(u,v)-(x,y)\|_\gamma$, and, by Lemma 7, at any point all subgradients (x^*,y^*) of the latter function satisfy $\|(x^*,y^*)\|_\gamma \le \alpha'$. By Lemma 4(ii), there exists a subgradient $(x^*,y^*) \in \partial^C \psi_p(x,y)$ such that $\|(x^*,y^*)\|_\gamma \le \alpha' < \alpha$, which contradicts (3.129).

Let X and Y be Asplund. Choose an $\varepsilon > 0$ such that

$$\varepsilon < \min\left\{\delta+\mu-\|x-\bar{x}\|, \alpha\mu-\|y-\bar{y}\|, \alpha-\alpha', \|y-\bar{y}\|/2, d(x,F_p^{-1}(\bar{y}))/2\right\}.$$

By Lemma 4(iii), there exist points $x' \in B_\varepsilon(x), y' \in B_\varepsilon(y)$ with $(x',y') \in \operatorname{gph} F_p$, and a subgradient $(x^*,y^*) \in \partial^F \psi_p(x',y')$ such that

$$\|(x^*,y^*)\|_\gamma < \alpha' + \varepsilon < \alpha. \tag{3.131}$$

Besides, $x' \in B_{\delta+\mu}(\bar{x}) \setminus F_p^{-1}(\bar{y})$, $\bar{y} \ne y' \in B_{\alpha\mu}(\bar{y})$ as

$$\|y-\bar{y}\|/2 < \|y'-\bar{y}\|, \quad d(x,F_p^{-1}(\bar{y}))/2 < d(x',F_p^{-1}(\bar{y})),$$
$$\|x'-\bar{x}\| \le \|x'-x\| + \|x-\bar{x}\| < \delta+\mu, \quad \|y'-\bar{y}\| \le \|y'-y\| + \|y-\bar{y}\| < \alpha\mu.$$

It follows from (3.131) that $d_\gamma(0,\partial\psi_p(x',y')) < \alpha$, which contradicts (3.129). $\square$

Remark 52 Condition (3.129) with Fréchet subdifferentials is obviously weaker (hence, more efficient) than its version with Clarke ones. However, it is only applicable in Asplund spaces.

The key condition (3.129) in Proposition 36 involves subdifferentials of the function ψ_p. Subgradients of this function belong to $X^* \times Y^*$ and have two component vectors x^* and y^*. In view of the representation (3.69) of the dual norm on $X^* \times Y^*$, the contributions of the vectors x^* and y^* to the condition (3.129) are different. The next corollary exposes this difference.

Corollary 81 If there exists an $\varepsilon > 0$ such that $\|x^*\| \ge \alpha$ for all $p \in P$, $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfying (3.123), and all $(x^*,y^*) \in \partial^C \psi_p(x,y)$ with $\|y^*\| < \varepsilon$; particularly if

$$\liminf_{\substack{F_p^{-1}(\bar{y}) \not\ni x \to \bar{x},\, F(p,x) \ni y \to \bar{y},\, y^* \to 0 \\ p \in P,\, y \ne \bar{y},\, (x^*,y^*) \in \partial^C \psi_p(x,y)}} \|x^*\| > \alpha,$$

then F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ.

If X and Y are Asplund, then the above assertion is valid with ∂^F in place of ∂^C.

Proof Suppose F is not $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ. Let $\varepsilon > 0$ and $\gamma := \varepsilon/\alpha$. By Proposition 36, there exist $p \in P$, $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfying (3.123), and a subgradient $(x^*,y^*) \in \partial^C \psi_p(x,y)$ ($(x^*,y^*) \in \partial^F \psi_p(x,y)$ if X and Y are Asplund) such that $\|(x^*,y^*)\|_\gamma < \alpha$. In view of the representation of the dual norm (3.69), this implies $\|x^*\| < \alpha$ and $\|y^*\| < \alpha\gamma = \varepsilon$, a contradiction. □

The function ψ_p involved in the subdifferential sufficient conditions for the uniform $\alpha-$subregularity in Proposition 36, is itself a sum of two functions. We are now going to apply the sum rules again to obtain sufficient conditions in terms of Clarke and Fréchet normals to $\operatorname{gph} F_p$.

Theorem 30 The mapping F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ if, for some $\gamma > 0$, and all $p \in P$, $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfying (3.123), one of the following conditions is satisfied:

(i) with $N := N^C$,

$$d_\gamma((0,-y^*), N_{\operatorname{gph} F_p}(x,y)) \geq \alpha \tag{3.132}$$

for all $y^* \in Y^*$ satisfying

$$\|y^*\| = 1, \quad \langle y^*, y-\bar{y}\rangle = \|y-\bar{y}\|; \tag{3.133}$$

(ii) X and Y are Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.132) holds with $N := N^F$ for all $y^* \in Y^*$ satisfying

$$\|y^*\| = 1, \quad \langle y^*, y-\bar{y}\rangle > \tau\|y-\bar{y}\|. \tag{3.134}$$

Proof Suppose F is not $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ. Let $\gamma > 0$. In view of Proposition 36, there exist $p \in P$, $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfying (3.123), and a subgradient $(\hat{x}^*,\hat{y}^*) \in \partial\psi_p(x,y)$ such that $\|(\hat{x}^*,\hat{y}^*)\|_\gamma < \alpha$, where either $\partial := \partial^C$ (if X and Y are general Banach spaces) or $\partial := \partial^F$ (if X and Y are Asplund). Recall from (3.120) that ψ_p is a sum of two functions: the Lipschitz continuous convex function $v \mapsto g(v) := \|v-\bar{y}\|$ and the indicator function of the closed set $\operatorname{gph} F_p$.

(i) By Lemma 4(ii), there exist $y^* \in \partial g(y)$ and $(u^*,v^*) \in N^C_{\operatorname{gph} F_p}(x,y)$ such that $(\hat{x}^*,\hat{y}^*) = (0,y^*) + (u^*,v^*)$. Thus,

$$d_\gamma((0,-y^*), N^C_{\operatorname{gph} F_p}(x,y)) \leq \|(0,y^*) + (u^*,v^*)\|_\gamma = \|(\hat{x}^*,\hat{y}^*)\|_\gamma < \alpha,$$

which contradicts (3.132). Since $y \neq \bar{y}$, by Lemma 7, y^* satisfies conditions (3.133).

(ii) Let X and Y be Asplund, and $\tau \in]0,1[$. By Lemma 4(iii), for any $\varepsilon > 0$, there exist $x_1 \in B_\varepsilon(x)$, $y_1, y_2 \in B_\varepsilon(y)$ with $(x_1,y_1) \in \operatorname{gph} F_p$, and $y^* \in \partial g(y_2)$, $(u^*,v^*) \in N^F_{\operatorname{gph} F_p}(x_1,y_1)$ such that

$$\|(0,y^*) + (u^*,v^*) - (\hat{x}^*,\hat{y}^*)\|_\gamma < \varepsilon. \tag{3.135}$$

The number ε can be chosen small enough to ensure that $x_1 \in B_{\delta+\mu}(\bar{x}) \setminus F_p^{-1}(\bar{y})$, $y_1 \in B_{\alpha\mu}(\bar{y})$, $y_2 \neq \bar{y}$, and

$$\|y_1 - \bar{y}\| \geq \frac{1}{2}\|y - \bar{y}\|, \; \|y_2 - y_1\| < \frac{1-\tau}{4}\|y - \bar{y}\|, \; \|(\hat{x}^*, \hat{y}^*)\|_\gamma + \varepsilon < \alpha.$$

By Lemma 7, we have $\|y^*\| = 1$ and $\langle y^*, y_2 - \bar{y}\rangle = \|y_2 - \bar{y}\|$. Moreover,

$$\|y_2 - y_1\| < \frac{1-\tau}{4}\|y - \bar{y}\| \leq \frac{1-\tau}{2}\|y_1 - \bar{y}\|,$$

and consequently,

$$\begin{aligned}\langle y^*, y_1 - \bar{y}\rangle &\geq \langle y^*, y_2 - \bar{y}\rangle - \|y_2 - y_1\| = \|y_2 - \bar{y}\| - \|y_2 - y_1\| \\ &\geq \|y_1 - \bar{y}\| - 2\|y_2 - y_1\| > \tau\|y_1 - \bar{y}\|.\end{aligned}$$

Making use of (3.135), we obtain

$$d_\gamma((0, -y^*), N^F_{\operatorname{gph} F_p}(x_1, y_1)) \leq \|(0, y^*) + (u^*, v^*)\|_\gamma < \|(\hat{x}^*, \hat{y}^*)\|_\gamma + \varepsilon < \alpha.$$

This contradicts (3.132).

$\square$

Remark 53 (i) Condition (3.132) with Fréchet normal cones is obviously weaker (hence, more efficient) than its version with Clarke ones; cf. Remark 52. However, the Asplund space sufficient condition of uniform $\alpha-$subregularity in part (ii) of Theorem 30 is not necessarily weaker than its general Banach space version in part (i), as it replaces the equality in (3.133) with a less restrictive inequality in (3.134), which involves an additional parameter τ. Of course, τ can be chosen arbitrarily close to 1 making the difference between the constraints (3.133) and (3.134) less significant. The weaker than (3.133) conditions (3.134) employed in part (ii) of Theorem 30 are due to the approximate subdifferential sum rule (Lemma 4(iii)) used in its proof.

(ii) The following alternative sufficient condition is established half way within the proof of part (ii) of Theorem 30:

emph$X$ and Y are Asplund, and, given any $\varepsilon > 0$, inequality (3.132) holds with $N := N^F$ for all $v \in B_\varepsilon(y)$ and all $y^* \in Y^*$ satisfying (3.133) with v in place of y.

It employs the stronger equality conditions (3.133) instead of (3.134), but involves an unknown vector v (arbitrarily close to y). Conditions of this type are used by some authors, but we prefer more explicit ones in Theorem 30(ii) and the statements derived from it.

The qualitative sufficient conditions for uniform regularity follow immediately.

Corollary 82 The mapping F is subregular in x uniformly in p over P at $(\bar{x}, \bar{y})$ if one of the following conditions is satisfied:

(i) $\sup\limits_{\gamma>0} \liminf\limits_{\substack{F_p^{-1}(\bar y)\not\ni x\to\bar x,\, F(p,x)\ni y\to\bar y\\ p\in P,\, y\ne\bar y,\, \|y^*\|=1,\, \langle y^*, y-\bar y\rangle=\|y-\bar y\|}} d_\gamma((0,-y^*), N^C_{\operatorname{gph} F_p}(x,y)) > 0;$

(ii) X and Y are Asplund, and

$$\sup_{\gamma>0,\tau\in]0,1[} \liminf_{\substack{F_p^{-1}(\bar y)\not\ni x\to\bar x,\, F(p,x)\ni y\to\bar y\\ p\in P,\, y\ne\bar y,\, \|y^*\|=1,\, \langle y^*, y-\bar y\rangle>\tau\|y-\bar y\|}} d_\gamma((0,-y^*), N^F_{\operatorname{gph} F_p}(x,y)) > 0.$$

The next example illustrates the sufficient conditions for subregularity in Corollary 82.

Example 6 Let $P = X = Y := \mathbb{R}$, $F(p,x) := \{(p-x)^2\}$ for all $p \in P$ and $x \in X$, and let $\bar y := 0$. By (3.116), $F_p^{-1}(\bar y) = \{p\}$. Thus, $d(x, F_p^{-1}(\bar y)) = |x-p|$ and $d(\bar y, F(p,x)) = (x-p)^2$ for all $p \in P$ and $x \in X$. Hence, for any $\alpha > 0$ and $p \in P$, inequality (3.118) is violated when x sufficiently close to $\bar x := 0$, i.e. the mapping F is not subregular in x uniformly in p over P at $(\bar x, \bar y)$. Observe that the graph $\operatorname{gph} F_p$ is closed for all $p \in P$, and, for any $p \in P$ and $(x,y) \in \operatorname{gph} F_p$,

$$(2(x-p),-1) \in N^C_{\operatorname{gph} F_p}(x,y) = N^F_{\operatorname{gph} F_p}(x,y).$$

Let $p = 0$, $x \neq 0$, $y = x^2$, and $y^* \in \mathbb{R}$ satisfy (3.133) or (3.134), hence, $y^* = 1$, and, for any $\gamma > 0$, $d_\gamma((0,-y^*),(2x,-1)) = 2|x| \to 0$ as $x \downarrow 0$. Both inequalities in Corollary 82 are not satisfied.

Theorem 30 yields sufficient conditions for uniform $\alpha-$subregularity in terms of coderivatives.

Corollary 83 The mapping F is $\alpha-$subregular in x uniformly in p over P at $(\bar x, \bar y)$ with δ and μ if, for some $\eta \in]0,+\infty]$, and all $p \in P$, $x \in B_{\delta+\mu}(\bar x)$ and $y \in Y$ satisfying (3.123), one of the following conditions is satisfied:

(i) with $D^* := D^*_C$, for all $y^* \in Y^*$ satisfying (3.133), it holds

$$d(0, D^*F_p(x,y)(B_\eta(y^*))) \geq \alpha; \tag{3.136}$$

(ii) X and Y are Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.136) holds with $D^* := D^*_F$ for all $y^* \in Y^*$ satisfying (3.134).

Proof Given an $\eta \in]0,+\infty]$, set $\gamma := \alpha^{-1}\eta$. In view of the representations (1.18) of the coderivative and (3.69) of the dual norm, condition (3.132) means that $\|u^*\| + \gamma^{-1}\|v^* - y^*\| \geq \alpha$ for all $v^* \in Y^*$ and $u^* \in D^*F_p(x,y)(v^*)$. The last inequality is obviously satisfied if either $\|u^*\| \geq \alpha$ or $\|v^* - y^*\| \geq \eta$, or equivalently, if $\|u^*\| \geq \alpha$ when $v^* \in B_\eta(y^*)$. $\square$

The coderivative sufficient condition (3.136) can be replaced by its ‘normalized’ (and a little stronger!) version.

Corollary 84 The mapping F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ if, for some $\eta \in]0,1[$, and all $p \in P$, $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfying (3.123), one of the following conditions is satisfied:

(i) with $D^* := D^*_C$,

$$d(0, D^* F_p(x,y)\left(\frac{v^*}{\|v^*\|}\right) \geq \frac{\alpha}{1-\eta} \tag{3.137}$$

for all $y^* \in Y^*$ satisfying (3.133) and $v^* \in B_\eta(y^*)$;

(ii) X and Y are Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.137) holds with $D^* := D^*_F$ for all $y^* \in Y^*$ satisfying (3.134) and $v^* \in B_\eta(y^*)$.

Proof Let $\eta \in]0,1[$, $p \in P$, $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfy (3.123), and $y^* \in Y^*$ satisfy either (3.133) or (3.134). We need to show that, if inequality (3.137) holds for all $v^* \in B_\eta(y^*)$, then inequality (3.136) holds. First note that, in view of (3.133) or (3.134), $\|y^*\| = 1$. Let $v^* \in B_\eta(y^*)$ and $u^* \in D^* F_p(x,y)(v^*)$. Then $\|v^*\| > 1-\eta \in]0,+\infty]$. Thus, condition (3.137) is well defined. Moreover, $u^*/\|v^*\| \in D^* F_p(x,y)(v^*/\|v^*\|)$ and, in view of (3.137), $\|u^*\| \geq \alpha\|v^*\|/(1-\eta) > \alpha$, i.e. inequality (3.136) holds. $\square$

The next qualitative assertion is an immediate consequence of Corollary 83.

Corollary 85 The mapping F is subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ if one of the following conditions is satisfied:

(i) $$\lim_{\delta\downarrow 0} \inf_{\substack{p\in P,\, x\in B_\delta(\bar{x})\setminus F_p^{-1}(\bar{y}),\, \bar{y}\neq y\in F(p,x)\cap B_\delta(\bar{y}) \\ \|y^*\|=1,\, \langle y^*, y-\bar{y}\rangle = \|y-\bar{y}\|}} d(0, D^*_C F_p(x,y)(B_\delta(y^*)) > 0;$$

(ii) X and Y are Asplund, and

$$\lim_{\delta\downarrow 0,\, \tau\uparrow 1} \inf_{\substack{p\in P,\, x\in B_\delta(\bar{x})\setminus F_p^{-1}(\bar{y}),\, \bar{y}\neq y\in F(p,x)\cap B_\delta(\bar{y}) \\ \|y^*\|=1,\, \langle y^*, y-\bar{y}\rangle > \tau\|y-\bar{y}\|}} d(0, D^*_F F_p(x,y)(B_\delta(y^*)) > 0.$$

Remark 54 (i) In the particular case when P is a neighborhood of a given point $\bar{p}$ in a metric space, Corollary 85 (taking into account Remark 53(ii) in some instances) improves [145, Theorem 3.6], [184, Theorem 3.4], [146, Theorem 3.2], [107, Theorem 3.5], [106, Theorem 3.1], [189, Corollary 2.2], [56, Theorem 1] (in the linear setting), and [115, Theorem 4.1(e)].

(ii) Clarke normal cones in this section can be replaced by Ioffe's *G-normal cones* [116].

Dual Necessary Conditions

In this subsection, X and Y are normed spaces, $F : P \times X \rightrightarrows Y$, $\bar{x} \in X$, $\bar{y} \in Y$, $\alpha > 0$, $\delta \in]0,+\infty]$, $\mu \in]0,+\infty]$, and we assume that $\operatorname{gph} F_p$ is convex for all $p \in P$.

The next statement provides a necessary condition for uniform $\alpha-$subregularity in terms of subdifferentials of the function ψ_p defined by (3.120).

Proposition 37 If F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ, then inequality (3.129) is satisfied with $\gamma := \alpha^{-1}$ for all $p \in P$, $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfying (3.123).

Proof Under the assumptions made, the function ψ_p is convex for all $p \in P$. Let $\gamma := \alpha^{-1}$, and $p \in P$, $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfy (3.123). For any $(x^*,y^*) \in \partial\psi_p(x,y)$, we have

$$\begin{aligned}
\|(x^*,y^*)\|_\gamma &= \sup_{(u,v)\neq(0,0)} \frac{\langle(x^*,y^*),(u,v)\rangle}{\|(u,v)\|_\gamma} \\
&= \limsup_{\substack{u\to x,\, v\to y\\ (u,v)\neq(x,y)}} \frac{-\langle(x^*,y^*),(u,v)-(x,y)\rangle}{\|(u,v)-(x,y)\|_\gamma} \\
&\geq \limsup_{\substack{u\to x,\, v\to y\\ (u,v)\neq(x,y)}} \frac{\psi_p(x,y)-\psi_p(u,v)}{\|(u,v)-(x,y)\|_\gamma} \\
&= \limsup_{\substack{u\to x,\, v\to y\\ (u,v)\in \operatorname{gph} F_p,\, (u,v)\neq(x,y)}} \frac{\|y\|-\|v\|}{\|(u,v)-(x,y)\|_\gamma}.
\end{aligned}$$

By Corollary 80(i), we have $\|(x^*,y^*)\|_\gamma \geq \alpha$. Taking the infimum in the left-hand side of the last inequality over $(x^*,y^*) \in \partial\psi_p(x,y)$, we obtain inequality (3.129). $\square$

Combining the above statement with Proposition 36, we obtain a complete subdifferential characterization of uniform $\alpha-$subregularity in the convex setting.

Corollary 86 Let X and Y be Banach, and $\operatorname{gph} F_p$ be closed for all $p \in P$. The mapping F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ if and only if inequality (3.129) holds with $\gamma := \alpha^{-1}$ for all $p \in P$, $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfying (3.123).

As a consequence, F is subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ if and only if

$$\sup_{\gamma>0} \liminf_{\substack{x\to\bar{x},\, y\to\bar{y}\\ p\in P,\, x\notin F_p^{-1}(\bar{y}),\, \bar{y}\neq y\in F(p,x)}} d_\gamma(0,\partial\psi_p(x,y)) > 0.$$

The next corollary follows from Proposition 37 in view of the representation (3.69) of the dual norm.

Corollary 87 If F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ, then $\|x^*\| \geq \alpha(1-\|y^*\|)$ for all $p \in P$, $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfying (3.123), and $(x^*,y^*) \in \partial\psi_p(x,y)$. As a consequence,

$$\liminf_{\substack{x\to\bar{x},\ y\to\bar{y},\, y^*\to 0\\ p\in P,\, x\notin F_p^{-1}(\bar{y}),\, \bar{y}\neq y\in F(p,x),\, (x^*,y^*)\in\partial\psi_p(x,y)}} \|x^*\| \geq \alpha.$$

The next statement gives a partial converse to Theorem 30.

Theorem 31 If F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x},\bar{y})$ with δ and μ, then for all $p \in P$, $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfying (3.123), and $y^* \in Y^*$ satisfying (3.133), inequality (3.132) is satisfied with $\gamma := \alpha^{-1}$.

Proof Observe that ψ_p is the sum of the convex continuous function $v \mapsto g(v) := \|v - \bar{y}\|$ and the indicator function of the convex set $\operatorname{gph} F_p$. By Lemma 4(i), $\partial \psi_p(x,y) = \{0\} \times \partial g(y) + N_{\operatorname{gph} F_p}(x,y)$. The assertion follows from Proposition 37 in view of Lemma 7(ii). $\square$

Combining Theorems 30 and 31, we can formulate a necessary and sufficient characterization of uniform $\alpha-$subregularity in the convex setting.

Corollary 88 Let X and Y be Banach, and $\operatorname{gph} F_p$ be closed for all $p \in P$. The mapping F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x}, \bar{y})$ with δ and μ if and only if for all $p \in P$, $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfying (3.123), and $y^* \in Y^*$ satisfying (3.133), inequality (3.132) is satisfied with $\gamma := \alpha^{-1}$.

As a consequence, F is subregular in x uniformly in p over P at $(\bar{x}, \bar{y})$ if and only if

$$\sup_{\gamma>0} \liminf_{\substack{x\to\bar{x},\, p\in P,\, y\to\bar{y},\, x\notin F_p^{-1}(\bar{y})\\ \bar{y}\neq y\in F(p,x),\, \|y^*\|=1,\, \langle y^*, y-\bar{y}\rangle=\|y-\bar{y}\|}} d_\gamma((0,-y^*), N_{\operatorname{gph} F_p}(x,y)) > 0. \tag{3.138}$$

The next example illustrates the necessary condition in Theorem 31.

Example 7 Let $P = X = Y := \mathbb{R}$, $F(p,x) := \{p - x\}$ for all $p \in P$ and $x \in X$, and let $\bar{x} = \bar{y} := 0$. By (3.116), $F_p^{-1}(\bar{y}) = \{p\}$. Thus, $d(x, F_p^{-1}(\bar{y})) = d(\bar{y}, F(p,x)) = |x - p|$ for all $p \in P$ and $x \in X$. Hence, inequality (3.118) is satisfied for all $p \in P$, $x \in X$, and $\alpha \in]0,1]$, i.e. the mapping F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x}, \bar{y})$ for any $\alpha \in]0,1]$. We have $\operatorname{gph} F_p = \{(x,y) \mid y = p - x\}$ is closed and convex for all $p \in P$, and $N_{\operatorname{gph} F_p}(x,y) = \{(t,t) \mid t \in \mathbb{R}\}$ for any $(x,y) \in \operatorname{gph} F_p$. Let $y^* \in \mathbb{R}$ satisfy (3.133). Then $y^* = 1$ if $y > 0$, and $y^* = -1$ if $y < 0$. It is easy to check that, given a $\gamma > 0$, in both cases the distance $d_\gamma((0,-y^*), N_{\operatorname{gph} F_p}(x,y))$ equals 1 if $\gamma \le 1$, or γ^{-1} if $\gamma > 1$. Hence, condition (3.138) is satisfied, confirming the uniform subregularity of F.

The next statement is a consequence of Theorem 31. It is in a sense a partial converse to Corollary 83.

Corollary 89 If F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x}, \bar{y})$ with δ and μ, then $d(0, D^*F_p(x,y)(B_\eta(y^*))) \ge \alpha(1-\eta)$ for any $\eta \in]0,1[$, all $p \in P$, $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfying (3.123), and $y^* \in Y^*$ satisfying (3.133).

Proof In view of the representations (1.18) of the coderivative and (3.69) of the dual norm, condition (3.132) with $\gamma := \alpha^{-1}$ means that $\|u^*\| + \alpha\|v^* - y^*\| \ge \alpha$ for all $v^* \in Y^*$ and $u^* \in D^*F_p(x,y)(v^*)$. Hence, it yields $\|u^*\| > \alpha(1-\eta)$ if $\|v^* - y^*\| < \eta$. $\square$

Combining the above statement with Corollary 83, we obtain a complete coderivative characterization of uniform $\alpha-$subregularity in the convex setting. It improves [56, Theorem 3] (in the linear case).

Corollary 90 Let X and Y be Banach, and $\operatorname{gph} F_p$ be closed and convex for all $p \in P$. The mapping F is $\alpha-$subregular in x uniformly in p over P at $(\bar{x}, \bar{y})$ if and only if

$$\lim_{\delta\downarrow 0} \inf_{\substack{x\in B_\delta(\bar{x})\setminus F_p^{-1}(\bar{y}),\, \bar{y}\neq y\in F(p,x)\cap B_\delta(\bar{y})\\ p\in P,\, \|y^*\|=1,\, \langle y^*, y-\bar{y}\rangle=\|y-\bar{y}\|}} d(0, D^*F_p(x,y)(B_\delta(y^*))) \ge \alpha.$$

3.3.4 Metric Subregularity, Metric Regularity, and Implicit Multifunctions

In this section, we illustrate the necessary and sufficient conditions for uniform subregularity established in the preceding sections when applied to several conventional properties of set-valued mappings.

Metric Subregularity

The conventional regularity properties in Definition 17 are particular cases of the uniform regularity properties in Definition 18 corresponding to P being a singleton, which practically means that the set-valued mapping F does not involve a parameter.

The next three statements, which are immediate consequences of the corresponding 'parametric' ones in Sections 3.3.2 and 3.3.3, illustrate this observation for the case of sub-regularity. Here X and Y are normed spaces, $F : X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\alpha > 0$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$.

Proposition 38 (i) Suppose $\operatorname{gph} F$ is convex. If F is $\alpha-$subregular at $(\bar{x},\bar{y})$ with δ and μ, then, with $\gamma := \alpha^{-1}$,

$$\limsup_{\substack{u \to x,\, v \to y,\, (u,v) \in \operatorname{gph} F,\, (u,v) \neq (x,y) \\ d(u,\bar{x}) < \delta + \mu,\, d(v,\bar{y}) < \alpha\mu}} \frac{\|y - \bar{y}\| - \|v - \bar{y}\|}{\|(u - x, v - y)\|_\gamma} \geq \alpha \tag{3.139}$$

for all $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfying

$$x \notin F^{-1}(\bar{y}), \quad y \in F(x) \cap B_{\alpha\mu}(\bar{y}). \tag{3.140}$$

(ii) Suppose X and Y are Banach, and $\operatorname{gph} F$ is closed. If inequality (3.139) holds for some $\gamma > 0$, and all $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfying (3.140), then F is $\alpha-$subregular at $(\bar{x},\bar{y})$ with δ and μ.

Proof The statement is a consequence of Corollary 80. □

Proposition 39 (i) Suppose $\operatorname{gph} F$ is convex. If F is $\alpha-$subregular at $(\bar{x},\bar{y})$ with δ and μ, then

$$d_\gamma((0, -y^*), N_{\operatorname{gph} F}(x,y)) \geq \alpha \tag{3.141}$$

for $\gamma := \alpha^{-1}$, and all $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfying (3.140), and $y^* \in Y^*$ satisfying (3.133).

(ii) Suppose X and Y are Banach, and $\operatorname{gph} F$ is closed. The mapping F is $\alpha-$subregular at $(\bar{x},\bar{y})$ with δ and μ if, for some $\gamma > 0$, and all $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfying (3.140), one of the following conditions is satisfied:

(a) inequality (3.141) holds with $N := N^C$ for all $y^* \in Y^*$ satisfying (3.133);

(b) X and Y are Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.141) holds with $N := N^F$ for all $y^* \in Y^*$ satisfying (3.134).

Proof The statement is a consequence of Theorems 30 and 31. □

Proposition 40 (i) Suppose $\operatorname{gph} F$ is convex. If F is $\alpha-$subregular at $(\bar{x},\bar{y})$ with δ and μ, then $d(0, D^*F(x,y)(B_\eta(y^*)) \geq \alpha(1-\eta)$ for any $\eta \in]0,1[$, all $x \in B_\delta(\bar{x})$ and $y \in Y$ satisfying (3.140), and $y^* \in Y^*$ satisfying (3.133).

(ii) Suppose X and Y are Banach, and $\operatorname{gph} F$ is closed. The mapping F is $\alpha-$subregular at $(\bar{x},\bar{y})$ with δ and μ if, for some $\eta \in]0,+\infty]$, and all $x \in B_{\delta+\mu}(\bar{x})$ and $y \in Y$ satisfying (3.140), one of the following conditions is satisfied:

(a) with $D^* := D^*_C$, for all $y^* \in Y^*$ satisfying (3.133), it holds

$$d(0, D^*F(x,y)(B_\eta(y^*)) \geq \alpha; \tag{3.142}$$

(b) X and Y are Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.142) holds with $D^* := D^*_F$ for all $y^* \in Y^*$ satisfying (3.134).

Proof The statement is a consequence of Corollaries 83 and 89. □

Remark 55 (i) In Proposition 38(ii), it is sufficient to assume that X and Y are complete metric spaces (with distances in place of norms in condition (3.139)), or even that $\operatorname{gph} F$ is complete; cf. Remark 50(iii). In this setting, the sufficient condition in Proposition 38(ii) can be viewed as a quantitative version of [131, Corollary 5.8(d)] and [115, Theorem 2.4(a)].

(ii) Proposition 40 improves [154, Theorem 5.3]. In the linear setting, part (ii) of this proposition improves [154, Theorem 3.3], [187, Theorem 6], [56, Theorem 8], [115, Theorem 2.6], and the corresponding parts of [131, Corollary 5.8]. Proposition 40(ii) with condition (a) recaptures [217, Theorem 3.2].

Metric Regularity

The conventional metric regularity is a particular case of the uniform regularity property in Definition 18(i) corresponding to P being a singleton. At the same time, as it follows from the observation in Remark 49(i), in the normed space setting it can be treated as a particular case of the uniform subregularity property in Definition 18(ii) for the set-valued mapping $\widehat{F}(y,x) := F(x) - y$, $(y,x) \in Y \times X$ with y considered as a parameter. Obviously $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if and only if $(\bar{y},\bar{x},0) \in \operatorname{gph} \widehat{F}$.

The next three statements, which are immediate consequences of the corresponding 'parametric' ones in Sections 3.3.2 and 3.3.3, illustrate the above observation. Here X and Y are normed spaces, $F : X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\alpha > 0$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$.

Proposition 41 (i) Suppose $\operatorname{gph} F$ is convex. If F is $\alpha-$regular at $(\bar{x},\bar{y})$ with δ and μ, then

$$\limsup_{\substack{u\to x,\, v\to z,\, (u,v)\in \operatorname{gph} F,\, (u,v)\neq(x,z)\\ d(u,\bar{x})<\delta+\mu,\, d(v,y)<\alpha\mu}} \frac{\|z-y\|-\|v-y\|}{\|(u-x,v-z)\|_\gamma} \geq \alpha \tag{3.143}$$

for $\gamma := \alpha^{-1}$, and all $x\in B_\delta(\bar{x})$, $y\in B_\delta(\bar{y})$ and $z\in Y$ satisfying

$$x\notin F^{-1}(y), \quad z\in F(x)\cap B_{\alpha\mu}(y). \tag{3.144}$$

(ii) Suppose X and Y are Banach spaces, and $\operatorname{gph} F$ is closed. The mapping F is $\alpha-$regular at $(\bar{x},\bar{y})$ with δ and μ if inequality (3.143) holds with some $\gamma>0$ for all $x\in B_{\delta+\mu}(\bar{x})$, $y\in B_\delta(\bar{y})$ and $z\in Y$ satisfying (3.144).

Proof The statement is a consequence of Corollary 80. □

Proposition 42 (i) Suppose $\operatorname{gph} F$ is convex. If F is $\alpha-$regular at $(\bar{x},\bar{y})$ with δ and μ, then

$$d_\gamma((0,-y^*),N_{\operatorname{gph} F}(x,z))\geq \alpha \tag{3.145}$$

for $\gamma := \alpha^{-1}$, and all $x\in B_\delta(\bar{x})$, $y\in B_\delta(\bar{y})$ and $z\in Y$ satisfying (3.144), and $y^*\in Y^*$ satisfying

$$\|y^*\|=1, \quad \langle y^*,z-y\rangle=\|z-y\|. \tag{3.146}$$

(ii) Suppose X and Y are Banach, and $\operatorname{gph} F$ is closed. The mapping F is $\alpha-$regular at $(\bar{x},\bar{y})$ with δ and μ if, for some $\gamma>0$ and all $x\in B_{\delta+\mu}(\bar{x})$, $y\in B_\delta(\bar{y})$ and $z\in Y$ satisfying (3.144), one of the following conditions holds:

(a) inequality (3.145) holds with $N := N^C$ for all $y^*\in Y^*$ satisfying (3.146);

(b) X and Y are Asplund, and there exists a $\tau\in]0,1[$ such that inequality (3.145) holds with $N := N^F$ for all $y^*\in Y^*$ satisfying

$$\|y^*\|=1, \quad \langle y^*,z-y\rangle>\tau\|z-y\|. \tag{3.147}$$

Proof The statement is a consequence of Theorems 30 and 31. □

Proposition 43 (i) Suppose $\operatorname{gph} F$ is convex. If F is $\alpha-$regular at $(\bar{x},\bar{y})$ with δ and μ, then $d(0,D^*F(x,z)(B_\eta(y^*))\geq\alpha(1-\eta)$ for any $\eta\in]0,1[$, all $x\in B_\delta(\bar{x})$, $y\in B_\delta(\bar{y})$ and $z\in Y$ satisfying (3.144), and all $y^*\in Y^*$ satisfying (3.146).

(ii) Suppose X and Y are Banach, and $\operatorname{gph} F$ is closed. The mapping F is $\alpha-$regular at $(\bar{x},\bar{y})$ with δ and μ if, for some $\eta\in]0,+\infty]$ and all $x\in B_{\delta+\mu}(\bar{x})$, $y\in B_\delta(\bar{y})$ and $z\in Y$ satisfying (3.144), one of the following conditions holds:

(a) with $D^* := D^*_C$, for all $y^* \in Y^*$ satisfying (3.146), it holds

$$d(0, D^*F(x,z)(B_\eta(y^*))) \geq \alpha; \tag{3.148}$$

(b) X and Y are Asplund, and there exists a $\tau \in]0,1[$ such that inequality (3.148) holds with $D^* := D^*_F$ for all $y^* \in Y^*$ satisfying (3.147).

Proof The statement is a consequence of Corollaries 83 and 89. □

Remark 56 (i) In the normed space setting, the sufficient condition in Proposition 41(ii) can be viewed as a quantitative version of [112, Theorem 1] and [115, Theorem 2.4(a)].

(ii) Proposition 43(ii) enhances [53, Theorem 3.7]. Part (a) of Proposition 43(ii) improves [106, Corollary 3.1] and [53, Theorem 3.5] (in the linear case), while part (b) improves (in the linear case) [53, Theorem 3.1] and [56, Theorem 7].

Implicit Multifunctions

Now we get back to the implicit multifunction (3.116) and consider its particular case corresponding to the parametric generalized equation $\bar{y} \in F(p,x)$ (with fixed left-hand side), i.e.

$$G(p) := \{x \in X \mid \bar{y} \in F(p,x)\}, \quad p \in P, \tag{3.149}$$

where $F : P \times X \rightrightarrows Y$, and P, X and Y are metric spaces. Stability properties of implicit multifunctions, i.e. solution sets of parametric generalized equations, are of great importance for many applications and have been the subject of numerous publications; cf., e.g., [14, 19, 34, 51, 53, 56, 57, 81, 98, 106, 107, 115, 116, 124, 145, 146, 184, 187, 189, 198, 200, 212]. Here, for illustration, we focus on the most well known perturbation stability property of set-valued mappings called *Aubin property*; cf. [81, 172].

Definition 20 A mapping $G : P \rightrightarrows X$ between metric spaces has the Aubin property at $(\bar{p},\bar{x}) \in \operatorname{gph} G$ with rate $l > 0$ if there exist $\eta \in]0,+\infty]$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that

$$d(x, G(p)) \leq l\, d(p,p')$$

for all $p,p' \in B_\eta(\bar{p})$ with $d(p,p') < \mu$, and $x \in G(p') \cap B_\delta(\bar{x})$.

Similar to Definitions 17, 18 and 19, the inequality $d(p,p') < \mu$ is not essential in the above definition and can be dropped together with the constant μ. We keep them for consistency with the definitions and characterizations in the preceding sections. We also establish connections between the constant μ and the corresponding constants in the other definitions.

Given a point $(\bar{p},\bar{x},\bar{y}) \in \operatorname{gph} F$ and a number $\alpha > 0$, the uniform $\alpha-$subregularity property of F at $(\bar{p},\bar{x},\bar{y})$ in Definition 19(ii) means that there exist $\eta \in]0,+\infty]$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that

$$\alpha d(x,G(p)) \leq d(\bar{y},F(p,x)) \tag{3.150}$$

for all $p \in B_\eta(\bar{p})$ and $x \in B_\delta(\bar{x})$ with $d(\bar{y},F(p,x)) < \alpha\mu$. Several primal and dual sufficient and necessary conditions for this property have been formulated in the preceding sections.

Inequality (3.150) provides an estimate for the distance from x to the value of the implicit multifunction (3.149) at p in terms of the residual of the parametric generalized equation. However, this estimate does not say much about the behaviour of the implicit multifunction. An additional assumption on the mapping F is needed, which would allow one to get rid of F in the right-hand side of the inequality (3.150). This additional assumption is given in the next definition, which is a modification of the second part of [115, Definition 3.1], where we borrow the terminology from. A similar property was considered in [124], where the authors used the name *Lipschitz lower semicontinuity*.

Definition 21 Let $l > 0$. The mapping F is said to $l-$recede in p uniformly in x at $(\bar{p},\bar{x},\bar{y})$ if there exist $\eta \in]0,+\infty]$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that

$$d(\bar{y},F(p,x)) \leq l d(p,p') \tag{3.151}$$

for all $x \in B_\delta(\bar{x})$ and $p,p' \in B_\eta(\bar{p})$ with $d(p,p') < \mu$ and $\bar{y} \in F(p',x)$.

The next statement is a modification of [115, Theorem 3.2].

Proposition 44 Let $\alpha > 0$ and $l > 0$. Suppose that F

- is $\alpha-$subregular in x uniformly in p at $(\bar{p},\bar{x},\bar{y})$ with some $\eta \in]0,+\infty]$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$;
- $l-$recedes in p uniformly in x at $(\bar{p},\bar{x},\bar{y})$ with η, δ and $\alpha\mu/l$.

Then the mapping G given by (3.149) has the Aubin property at $(\bar{p},\bar{x})$ with rate l/α with η, δ and $\alpha\mu/l$.

Proof Let $p,p' \in B_\eta(\bar{p})$ with $d(p,p') < \alpha\mu/l$, and $x \in G(p') \cap B_\delta(\bar{x})$. By (3.149), $\bar{y} \in F(p',x)$. By (3.151), $d(\bar{y},F(p,x)) < \alpha\mu$. Using successively (3.150) and (3.151), we obtain

$$d(x,G(p)) \leq \frac{1}{\alpha} d(\bar{y},F(p,x)) \leq \frac{l}{\alpha} d(p,p').$$

□

Combining Proposition 44 with the sufficient conditions for uniform subregularity formulated in the preceding sections, we can immediately obtain various sufficient conditions for the Aubin property of the implicit multifunction (3.149). The next proposition collects three sufficient conditions arising from Corollary 80(ii), Theorem 30 and Corollary 83, respectively.

Proposition 45 Let P be a metric space, X and Y be complete metric spaces, $F : P \times X \rightrightarrows Y$, $(\bar{p},\bar{x},\bar{y}) \in \operatorname{gph} F$, $G : P \rightrightarrows X$ be given by (3.149). Suppose that $\operatorname{gph} F_p$ is closed for all $p \in B_\eta(\bar{p})$. The mapping G has the Aubin property at $(\bar{p},\bar{x})$ with rate $l > 0$ with η, δ and μ if, for some $l' > 0$, F $l'-$recedes in p uniformly in x at $(\bar{p},\bar{x},\bar{y})$ with η, δ and μ, and one of the following conditions holds true:

(i) there exists a $\gamma > 0$ such that

$$\limsup_{\substack{u\to x,\, v\to y,\, (u,v)\in \operatorname{gph} F_p,\, (u,v)\neq(x,y)\\ d(u,\bar{x})<l\delta+\mu,\, d(v,\bar{y})<l'\mu}} \frac{d(y,\bar{y})-d(v,\bar{y})}{d_\gamma((u,v),(x,y))} \geq \frac{l'}{l}$$

for all p, x and y satisfying

$$p \in B_\eta(\bar{p}),\ x \in B_{\delta+\mu}(\bar{x}) \setminus F_p^{-1}(\bar{y}),\ y \in F(p,x) \cap B_{l'\mu}(\bar{y}); \tag{3.152}$$

(ii) X and Y are Banach, and there exists a $\gamma > 0$ such that, with $N := N^C$,

$$d_\gamma((0,-y^*), N_{\operatorname{gph} F_p}(x,y)) \geq \frac{l'}{l} \tag{3.153}$$

for all p, x and y satisfying (3.152), and all $y^* \in Y^*$ satisfying (3.133);

(iii) X and Y are Asplund, and there exist a $\gamma > 0$ and a $\tau \in]0,1[$ such that condition (3.153) is satisfied with $N := N^F$ for all p, x and y satisfying (3.152), and $y^* \in Y^*$ satisfying (3.134);

(iv) X and Y are Banach, and

$$d(0, D^*F_p(x,y)(B_\eta(y^*))) \geq \frac{l'}{l} \tag{3.154}$$

with $D^* := D^*_C$ for all p, x and y satisfying (3.152), and $y^* \in Y^*$ satisfying (3.133);

(v) X and Y are Asplund, and there exists a $\tau \in]0,1[$ such that condition (3.154) is satisfied with $D^* := D^*_F$ for all p, x and y satisfying (3.152), and $y^* \in Y^*$ satisfying (3.134).

Remark 57 Condition (i) in Proposition 45 can be seen as a quantitative version of [115, Theorem 3.9], while conditions (iv) and (v) improve [115, Theorem 4.1] and [116, Theorem 7.26]. Conditions (ii) and (iii) are new. Note that these two conditions are weaker than (iv) and (v), respectively.

Chapter 4

ERROR BOUNDS REVISITED

The content of the chapter is based on the preprint [66].

4.1 Introduction

Necessary and especially sufficient conditions for error bounds of (extended) real-valued functions have been a subject of intense research for more than half a century due to their numerous applications in optimization and variational analysis, particularly in convergence analysis of iterative algorithms, penalty functions, optimality conditions, weak sharp minima, stability and well-posedness of solutions, (sub)regularity and calmness of set-valued mappings, and subdifferential calculus; see, e.g., the surveys by Jong-Shi Pang [193] and Dominique Azé [12], the recent book by Alexander Ioffe [116], and the most recent papers [18, 131, 133, 137, 155, 156, 211].

A huge number of sufficient and necessary conditions for error bounds have been obtained in the linear [12, 16, 19, 54, 55, 92, 93, 121, 131, 137, 156, 171, 179, 184, 209, 210], as well as more subtle nonlinear (mostly Hölder) [17, 18, 48, 64, 130, 133, 137, 155, 178, 185, 186, 209, 211, 215] settings.

The next definition introduces error bounds in the nonlinear setting, a generalization of the property defined in Definition 4. The nonlinearity is determined by a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(0) = 0$ and $\varphi(t) > 0$ if $t > 0$. The family of all such functions is denoted by $\mathscr{C}$. We denote by $\mathscr{C}^1$ the subfamily of functions from $\mathscr{C}$ which satisfy $\lim_{t\to+\infty} \varphi(t) = +\infty$, and are continuously differentiable on $]0,+\infty[$ with $\varphi'(t) > 0$ for all $t > 0$. Obviously, if $\varphi \in \mathscr{C}^1$, then $\varphi^{-1} \in \mathscr{C}^1$. Observe that, for any $\alpha > 0$ and $q > 0$, the function $t \mapsto \alpha t^q$ on $\mathbb{R}_+$ belongs to $\mathscr{C}^1$.

Definition 22 Suppose X is a metric space, $f : X \to \mathbb{R}_\infty$, and $\varphi \in \mathscr{C}$. The function f admits a $\varphi-$error bound at $\bar{x} \in X$ if there exist $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that

$$d(x,[f \le 0]) \le \varphi(f(x)) \tag{4.1}$$

for all $x \in B_\delta(\bar{x}) \cap [0 < f < \mu]$, and either $\bar{x} \in [f \le 0]$ or $\delta = +\infty$.

Under the conditions of Definition 22, we will often say that f admits a $\varphi-$error bound at $\bar{x}$ with δ and μ. When $\delta < +\infty$, we say that f admits a local $\varphi-$error bound at $\bar{x}$ (with δ and μ). When $\delta = +\infty$ (hence, $B_\delta(\bar{x}) = X$), we say that f admits a global $\varphi-$error bound.

Remark 58 (i) The $\tau-$error bound property in Definition 4 is a particular case of the $\varphi-$error bound property, corresponding to φ being the linear function $t \mapsto \tau^{-1}t$.

(ii) Any function $\varphi \in \mathscr{C}$ can be extended to the whole $\mathbb{R}$ by setting $\varphi(t) = 0$ for all $t < 0$. Then one obviously has $[f \leq 0] = [\varphi \circ f \leq 0]$, and the seemingly more general $\varphi-$error bound property in Definition 22 becomes the conventional $1-$error bound property for the composition function $\varphi \circ f$.

(iii) The requirement $\lim_{t\to+\infty} \varphi(t) = +\infty$ in the definition of the family $\mathscr{C}^1$ is technical. It is only needed to ensure that φ^{-1} is defined on the whole $\mathbb{R}_+$. Both conditions can be weakened.

In all assertions in the current chapter involving Fréchet subdifferentials it is sufficient to assume functions from $\mathscr{C}^1$ to be (not necessarily continuously) differentiable. Continuous differentiability is only needed for assertions involving Clarke subdifferentials.

(iv) It is not unusual to consider nonlinear error bounds with inequality (4.1) in Definition 22 replaced with the following one:

$$\psi(d(x, [f \leq 0])) \leq f(x), \tag{4.2}$$

where $\psi \in \mathscr{C}$. The models (4.1) and (4.2) are obviously equivalent with $\psi = \varphi^{-1}$ as long as one of the functions ψ or φ (hence, also the other one) is strictly increasing, which is the case, in particular, when $\varphi, \psi \in \mathscr{C}^1$.

Similar to the linear case, it makes sense to look for a smaller function $\varphi \in \mathscr{C}$ satisfying inequality (4.1) (for the appropriate set of points $x \in X$). It is not easy to order functions on the whole set $\mathscr{C}$. It is more practical to consider a subset of positive multiples of a given function $\varphi \in \mathscr{C}$. Extending definition (1.13), we define the (local) $\varphi-$error bound modulus of f at $\bar{x}$ as

$$\mathrm{Er}_\varphi f(\bar{x}) := \liminf_{x\to\bar{x},\, f(x)>0} \frac{\varphi(f(x))}{d(x, [f \leq 0])}. \tag{4.3}$$

It is easy to see that this is the reciprocal of the exact lower bound of all $\alpha > 0$ such that f admits an $(\alpha\varphi)-$error bound at $\bar{x}$. When $\varphi(t) = t^q$ for some $q > 0$ and all $t > 0$, definition (4.3) coincides with that of the modulus of $q-$order error bounds [137]. In particular, with $q = 1$ it reduces to (1.13).

The main aim of this chapter is not to add some new sufficient or necessary conditions for error bounds to the large volume of existing ones (although some conditions in the subsequent sections are indeed new even in the linear setting), but to propose a comprehensive unifying general (i.e. not assuming the function f to have any particular structure) view on the theory of error bounds (linear and nonlinear, local and global) and clarify the relationships between the existing conditions including their hierarchy. We expose the roles of the assumptions involved in the error bound assertions, in particular, on the underlying space: general metric, normed, Banach or Asplund. Employing special collections of slope operators defined below, we introduce a succinct form of sufficient error bound conditions, which allows one to combine in a single statement several different assertions: nonlocal and local primal space conditions in complete metric space, and subdifferential conditions in Banach and Asplund spaces.

The hot topics of error bounds for special families of functions and error bounds under uncertainty (see, e.g., [54, 55, 155, 156]) are outside the scope of the current chapter.

The core of the chapter consists of three theorems treating linear (Theorem 32), nonlinear (Theorem 33) and 'alternative' nonlinear (Theorem 34) error bound conditions that seem to cover all existing general error bound results. We show that the 'nonlinear' Theorem 33 is a straightforward consequence of the 'linear' Theorem 32, while the 'alternative nonlinear' Theorem 34 is a straightforward consequence of the conventional Theorem 33. In its turn, the original Theorem 32 is a consequence of a preliminary statement – Proposition 46 – treating the case when x in (1.12) is fixed, while encapsulating all the main arguments used in the general statement. Following Ioffe [112, Basic lemma], separate preliminary 'fixed x' type statements have been formulated by many authors; cf. [116, 137, 184–186, 210]. Proposition 46 seems to be the most comprehensive one.

All the statements have the same structure, each combining several assertions that are mostly well known and are often formulated (and proved) as separate theorems:

(i) sufficient error bound conditions for a lower semicontinuous function on a complete metric space:

   (a) nonlocal primal space conditions;

   (b) infinitesimal primal space conditions in terms of slopes;

   (c) in the setting of a Banach space, dual space conditions in terms of Clarke subdifferentials;

   (d) in the setting of an Asplund space, dual space conditions in terms of Fréchet subdifferentials;

(ii) nonlocal primal space necessary error bound conditions for a (not necessarily lower semicontinuous) function on a metric space (except Theorem 34);

(iii) dual space necessary error bound conditions for a convex function on a normed space in terms of conventional convex subdifferentials.

This chapter seems to be the first attempt to combine the above assertions in a single statement. We believe that it not only makes the presentation shorter, but also clarifies the overall picture: it exposes the relationships between the assertions and the hierarchy of the sufficient conditions in part (i). Most of the assertions in Theorem 32 and to some extent also in Theorems 33 and 34 have been proved multiple times by many authors, often with long multi-page 'from scratch' proofs, and 'new' proofs keep coming. We think that it is time to make a pause and summarize the main ideas behind the assertions.

In the case of the key 'fixed x' type Proposition 46 characterizing linear error bounds, the implication (b) $\Rightarrow$ (a) in part (i) of the above list as well as the necessary conditions in parts (ii) and (iii) follow immediately from the definitions. They are included for the completeness of the picture. The main assertions are the sufficiency of condition (a), and implications (c) $\Rightarrow$ (b) and (d) $\Rightarrow$ (b). They employ the following fundamental tools of variational analysis:

- *Ekeland variational principle* (sufficiency of condition (a));
- *sum rules* for respective subdifferentials (implications (c) $\Rightarrow$ (b) and (d) $\Rightarrow$ (b)).

The seemingly counter-intuitive fact that sufficient conditions for nonlinear error bounds can be deduced from those for the corresponding linear ones was demonstrated by Corvellec et al. using, first, the '*change-of-metric principle*' [17,64], and then the '*change-of-function*' approach [18] (see also [137,185]). Our presentation here largely follows the latter one. We emphasise that throughout the chapter the word 'nonlinear' is used in the conventional sense: 'not necessarily linear'.

In the general nonlinear setting, conventional sufficient local error bound conditions, besides slopes and subdifferentials, naturally involve variable coefficients $\varphi'(f(u))$ computed at appropriate points $u \in [f > 0]$. Several publications have appeared recently proving alternative nonlinear sufficient conditions with coefficients, which involve φ' depending not on values of the function f but on distance $d(u, [f \le 0])$; cf. [48,64,137,211,215]. Such results normally assume certain monotonicity of φ'. We show in Theorem 34 that the alternative sufficient conditions are consequences of the conventional ones. Observe that when φ is linear (the conventional linear case), the coefficients are constant, and there is no difference between 'conventional' and 'alternative' conditions.

4.2 Conventional Linear Error Bound Conditions

The next preliminary statement treats the case when x in the definition of linear error bounds is fixed. It contains all the main ingredients used in the general statement (Theorem 32), the latter being an easy consequence of the first. The nonlinear error bound statements in the subsequent sections are also direct or indirect consequences of the next proposition.

Proposition 46 Suppose X is a metric space, $f : X \to \mathbb{R}_\infty$, $x \in [f > 0]$, and $\tau > 0$.

(i) Let X be complete, f be lower semicontinuous, and $\alpha \in]0,1]$. The error bound inequality (1.12) holds, provided that one of the following conditions is satisfied:

(a) $|\breve{\nabla} f| \in |\mathfrak{D} f|^\dagger$ and $\alpha |\breve{\nabla} f|(u) \ge \tau$ for all $u \in X$ satisfying

$$f(u) \le f(x), \tag{4.4}$$
$$d(u,x) < \alpha d(x,[f \le 0]), \tag{4.5}$$
$$\alpha f(u) < \tau d(u,[f \le 0]), \tag{4.6}$$
$$f(u) < \tau d(x,[f \le 0]); \tag{4.7}$$

(b) X is Asplund and there exists a $\mu > f(x)$ such that $\alpha |\partial^F f|(u) \ge \tau$ for all $u \in X$ satisfying $f(u) < \mu$, and conditions (4.5)–(4.7).

(ii) If the error bound inequality (1.12) holds, then $|\nabla f|^\diamond(x) \ge \tau$.

(iii) Let X be a normed space, and f be convex. If the error bound inequality (1.12) holds, then $|\partial f|(x) \ge \tau$.

All but one of the arguments in the short proof below have been used many times in numerous proofs of this type of assertions.

Proof

(i) Suppose that the error bound inequality (1.12) does not hold, i.e.

$$f(x) < \tau d(x,[f \le 0]). \tag{4.8}$$

Choose a $\tau' \in]0,\tau[$ such that $f(x) < \tau' d(x,[f \le 0])$. By the Ekeland variational principle applied to the lower semicontinuous function $f_+ := \max\{f,0\}$, there exists a point $u \in X$ satisfying (4.4) and (4.5), and such that

$$f_+(u) \le f_+(u') + \alpha^{-1} \tau' d(u',u) \quad \text{for all} \quad u' \in X. \tag{4.9}$$

We show that u satisfies also (4.6) and (4.7), while $\alpha |\nabla f|^\diamond(u) < \tau$. By (4.5), $u \notin [f \le 0]$. Hence, $f_+(u) = f(u)$, and it follows from (4.9) that $\alpha |\nabla f|^\diamond(u) \le \tau' < \tau$, and $\alpha f(u) \le \tau' d(u',u)$ for all $u' \in [f \le 0]$. The last inequality yields (4.6), while (4.8) and (4.4) imply (4.7). This proves the sufficiency of condition (a) with $|\breve{\nabla} f| = |\nabla f|^\diamond$. The sufficiency of this condition with the other components of $|\mathfrak{D} f|^\circ$ and the implication (b) $\Rightarrow$ (a) are consequences of Lemma 2(ii), (v) and (vi).

(ii) is an immediate consequence of the definition of the nonlocal slope.

(iii) follows from (ii) thanks to Lemma 2(iv).

□

Remark 59 (i) In view of the definition of $|\mathfrak{D} f|^\dagger$, condition (a) in Proposition 46(i) combines three separate primal and dual sufficient error bound conditions:

(a1) $\alpha|\nabla f|^{\diamond}(u) \geq \tau$ for all $u \in X$ satisfying conditions (4.4)–(4.7);

(a2) $\alpha|\nabla f|(u) \geq \tau$ for all $u \in X$ satisfying conditions (4.4)–(4.7);

(a3) X is Banach and $\alpha|\partial^C f|(u) \geq \tau$ for all $u \in X$ satisfying conditions (4.4)–(4.7).

Moreover, thanks to parts (ii), (v) and (vi) of Lemma 2, we have (a3) $\Rightarrow$ (a2) $\Rightarrow$ (a1) and (b) $\Rightarrow$ (a2). Thus, condition (a) in Proposition 46(i) can be replaced equivalently with the simpler condition (a1), the weakest of the three sufficient conditions above.

Conditions (a1), (a2), (a3) and (b) represent four types of sufficient error bound conditions frequently appearing in the literature, with each of them having its own area of applicability. Such conditions are often proved independently as separate assertions. Proposition 46(i) seems to be the first attempt to combine them in a single statement.

With obvious minor adjustments, this observation applies to all assertions in this chapter containing multi-component sets of slope operators $|\mathfrak{D}f|^{\circ}$, $|\mathfrak{D}f|^{\dagger}$ or $|\mathfrak{D}f|$.

(ii) The core of Proposition 46(i) is made of the sufficiency of condition (a1), which is a consequence of the Ekeland variational principle, and implications (a3) $\Rightarrow$ (a2) and (b) $\Rightarrow$ (a2), which follow from the sum rules for respective subdifferentials.

(iii) The parameter α in Proposition 46(i) arises naturally from the application of the Ekeland variational principle. In most cases this type of assertions are formulated with $\alpha = 1$. Taking a smaller α, strengthens the slope inequalities in the sufficient conditions at the expense of reducing the set of points satisfying inequality (4.5). This 'trade-off' parameter has been used in several publications [137, 211, 215, 219].

(iv) Restrictions (4.4)–(4.7) on the choice of $u \in X$ and inequality $f(u) < \mu$ in condition (b) in Proposition 46(i) also arise naturally from the application of the Ekeland variational principle. Weakening or dropping any/all of these restrictions produces new (stronger!) sufficient conditions widely used in the literature. This can be particularly relevant in the case of restrictions (4.5)–(4.7) involving distances to the unknown set $[f \leq 0]$. Note that, if restriction (4.5) is dropped, it makes sense checking the resulting sufficient condition with $\alpha = 1$. If a point $\bar{x} \in [f \leq 0]$ is known, restrictions (4.5), (4.6) or (4.7) can be replaced, respectively, with the weaker inequalities:

$$d(u,x) < \alpha d(x,\bar{x}), \quad \alpha f(u) < \tau d(u,\bar{x}) \quad \text{or} \quad f(u) < \tau d(x,\bar{x}).$$

(v) This seems to be the first time that inequality (4.7) appears as a part of sufficient linear error bound conditions. It is a bit surprising since its nonlinear analogues have been exploited in the literature; see, e.g., [215]. We demonstrate in the next theorem that this inequality can be meaningful in the linear setting too, thus, paving the way to the subsequent nonlinear extensions.

(vi) Since $\partial^F f(x) \subset \partial^C f(x)$, the subdifferential slope inequality in condition (a3) obviously implies the one in condition (b). However, the implication (a3) $\Rightarrow$ (b) is not true in general because the restriction $f(u) < \mu$ in (b) is weaker than the corresponding inequality (4.4) in (a). The number $\mu > f(x)$ in condition (b) can be chosen arbitrarily close to $f(x)$, but cannot be replaced with $f(x)$ because of the 'fuzzy' inequality in Lemma 2(vi), which, in turn, is a consequence of the fuzzy sum rule for Fréchet subdifferentials in Lemma 4(iii).

The general error bound statement in the next theorem is a straightforward consequence of Proposition 46.

Theorem 32 Suppose X is a metric space, $f : X \to \mathbb{R}_\infty$, $\bar{x} \in X$, $\tau > 0$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$.

(i) Let X be complete, f be lower semicontinuous, $\alpha \in]0,1]$, and either $\bar{x} \in [f \leq 0]$ or $\delta = +\infty$. Let $|\breve{\nabla} f| \in |\mathfrak{D} f|$. The function f admits a $\tau-$error bound at $\bar{x}$ with $\delta' := \frac{\delta}{1+\alpha}$ and μ, provided that $\alpha|\breve{\nabla} f|(u) \geq \tau$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$ satisfying

$$\max\{\alpha, 1-\alpha\} f(u) < \tau d(u, [f \leq 0]). \tag{4.10}$$

(ii) If f admits a $\tau-$error bound at $\bar{x}$ with δ and μ, then $|\nabla f|^\diamond(u) \geq \tau$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.

(iii) Let X be a normed space, and f be convex. If f admits a $\tau-$error bound at $\bar{x}$ with δ and μ, then $|\partial f|(u) \geq \tau$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.

Proof To prove assertion (i), it suffices to check that, if $x \in B_{\delta'}(\bar{x}) \cap [0 < f < \mu]$ and $u \in X$ satisfies conditions (4.5)–(4.7) and $f(u) < \mu$ (in particular if it satisfies (4.4)), then $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$, and it satisfies conditions (4.10). Indeed, we have $u \in [f < \mu]$. In view of (4.5), $u \in [f > 0]$ and $d(u,x) < \alpha d(x,\bar{x})$. The last inequality together with $x \in B_{\delta'}(\bar{x})$ yield $u \in B_\delta(\bar{x})$. Condition (4.5) obviously implies $d(u, [f \leq 0]) > (1-\alpha) d(x, [f \leq 0])$, and it follows from (4.7) that $(1-\alpha) f(u) < \tau d(u, [f \leq 0])$. Together with (4.6), this gives (4.10). Assertions (ii) and (iii) follow immediately from the corresponding assertions in Proposition 46. □

Remark 60 (i) In view of the definition of $|\mathfrak{D} f|$, Theorem 32(i) combines four separate primal and dual sufficient error bound conditions corresponding to $|\breve{\nabla} f|$ equal to $|\nabla f|^\diamond$, $|\nabla f|$, $|\partial^C f|$ or $|\partial^F f|$ (in appropriate spaces).

(ii) The parameter α in Theorem 32(i) determines a trade-off between the main inequality $\alpha|\breve{\nabla} f|(u) \geq \tau$ (and also inequality (4.10)) and the radius δ' of the neighbourhood of $\bar{x}$ in which the error bound estimate holds; cf. Remark 59(iii). In the conventional case $\alpha = 1$, we have $\delta' = \delta/2$ as it has been observed in numerous publications.

(iii) Weakening or dropping any of the restrictions on u produces new (stronger!) sufficient conditions; cf. Remark 59(iv). This can be particularly relevant in the case of inequality (4.10) involving the distance to the unknown set $[f \le 0]$. If $\bar{x} \in [f \le 0]$, it is common to replace this distance with $d(x,\bar{x})$.

(iv) Sufficient error bound conditions of the type in Theorem 32(i) with the weaker inequality $\alpha f(u) < \tau d(u,[f \le 0])$ in place of (4.10) can be found in the literature (cf., e.g., [137, Theorem 3.4]). The fact that this inequality can be strengthened by replacing α with $\max\{\alpha, 1-\alpha\}$ seems to be observed for the first time here. It is a consequence of condition (4.7) in Proposition 46.

(v) Under the conditions of part (iii) of Theorem 32, one can easily show that, if for all $x \in B_\delta(\bar{x}) \cap [0 < f < \mu]$ condition (1.12) holds as equality, then $|\partial f|(x) = \tau$ for all $x \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.

The local $\tau-$error bound conditions are collected in the next three corollaries.

Corollary 91 Suppose X is a complete metric space, $f : X \to \mathbb{R}_\infty$ is lower semicontinuous, and $\tau > 0$. Let $|\breve{\nabla} f| \in |\mathfrak{D} f|$. The function f admits a local $\tau-$error bound at $\bar{x} \in [f \le 0]$, provided that $|\breve{\nabla} f|(x) \ge \tau$ for all $x \in [f > 0]$ near $\bar{x}$ with $f(x)$ near 0. Moreover, if $|\breve{\nabla} f| = |\nabla f|^\diamond$, then the above condition is also necessary.

Corollary 92 Suppose X is a Banach space, $f : X \to \mathbb{R}_\infty$ is convex lower semicontinuous, and $\tau > 0$. Let $|\breve{\nabla} f| \in |\mathfrak{D} f|$. The function f admits a local $\tau-$error bound at $\bar{x} \in [f \le 0]$ if and only if $|\breve{\nabla} f|(x) \ge \tau$ for all $x \in [f > 0]$ near $\bar{x}$.

Corollary 93 Suppose X is a complete metric space, $f : X \to \mathbb{R}_\infty$ is lower semicontinuous, and $\bar{x} \in [f \le 0]$. Then

$$\operatorname{Er} f(\bar{x}) = \liminf_{x \to \bar{x},\, f(x) \downarrow 0} |\nabla f|^\diamond(x) \ge \liminf_{x \to \bar{x},\, f(x) \downarrow 0} |\nabla f|(x).$$

If X is Banach (Asplund), then $|\nabla f|$ in the above inequality can be replaced with $|\partial^C f|$ ($|\partial^F f|$). If X is Banach and f is convex, then

$$\operatorname{Er} f(\bar{x}) = \liminf_{x \to \bar{x},\, f(x) > 0} |\partial f|(x).$$

Remark 61 The limits

$$\liminf_{x \to \bar{x},\, f(x) \downarrow 0} |\nabla f|^\diamond(x), \quad \liminf_{x \to \bar{x},\, f(x) \downarrow 0} |\nabla f|(x) \quad \text{and} \quad \liminf_{x \to \bar{x},\, f(x) \downarrow 0} |\partial f|(x) \tag{4.11}$$

are referred to in [92, 93, 131] as, respectively, the *strict outer*, *uniform strict outer* and *strict outer subdifferential* slopes of f at $\bar{x}$; cf. *limiting* slopes [112, 116].

The error bound inequalities (1.12), (4.1) and (4.2) correspond to the sublevel set $[f \le 0]$. Definitions 4 and 22 and the corresponding error bound conditions can be easily extended to the case of an arbitrary sublevel set $[f \le c]$ where $c \in \mathbb{R}$. It suffices to replace f in the definitions and statements with $f - c$ (with the corresponding small adjustment in the definition of the nonlocal slope).

The nonlocal slope $|\nabla f|^{\diamond}(x)$ in Proposition 46(ii) cannot in general be replaced with the local one unless f is convex. As observed in [16, proof of Proposition 2.1], this can be done if instead of the fixed error bound inequality (1.12) one considers a family of perturbed ones.

Proposition 47 Suppose X is a metric space, $f : X \to \mathbb{R}_\infty$, $x \in [f > 0]$, and $\tau > 0$. If

$$\tau d(x, [f \le c]) \le f(x) - c \tag{4.12}$$

for all sufficiently large $c < f(x)$, then $|\nabla f|(x) \ge \tau$.

Theorem 32 and Proposition 47 yield the following statement for 'perturbed' error bounds extending [16, Theorem 2.1], [64, Theorem 2.3] and [18, Theorem 3.2].

Proposition 48 Suppose X is a metric space, $f : X \to \mathbb{R}_\infty$, $\bar{x} \in X$, $\tau > 0$, $\delta \in]0, +\infty]$ and $\mu \in]0, +\infty]$.

(i) Let X be complete, f be lower semicontinuous, $\alpha \in]0, 1]$, and either $\bar{x} \in [f \le 0]$ or $\delta = +\infty$. Let $|\check{\nabla} f| \in |\mathfrak{D} f|^{\circ}$. The perturbed error bound inequality (4.12) holds for all $c \in [0, \mu[$ and $x \in B_{\frac{\delta}{1+\alpha}}(\bar{x}) \cap [c < f < \mu]$ provided that $\alpha |\check{\nabla} f|(x) \ge \tau$ for all $x \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.

(ii) If the perturbed error bound inequality (4.12) holds for all $c \in [0, \mu[$ and $x \in B_\delta(\bar{x}) \cap [c < f < \mu]$, then $|\nabla f|(x) \ge \tau$ for all $x \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.

Remark 62 (i) Proposition 48 provides necessary and sufficient conditions for the following *perturbed $\tau-$error bound* property of f at $\bar{x} \in X$ with some $\delta \in]0, +\infty]$ and $\mu \in]0, +\infty]$ such that either $\bar{x} \in [f \le 0]$ or $\delta = +\infty$: inequality (4.12) holds for all $c \in [0, \mu[$ and $x \in B_\delta(\bar{x}) \cap [c < f < \mu]$.

(ii) Thanks to Corollary 91 and Proposition 48, conditions

- $|\nabla f|^{\diamond}(x) \ge \tau$ for all $x \in [f > 0]$ near $\bar{x}$ with $f(x)$ near 0, and
- $|\nabla f|(x) \ge \tau$ for all $x \in [f > 0]$ near $\bar{x}$ with $f(x)$ near 0

provide full characterizations of, respectively, the $\tau-$error bound and the perturbed $\tau-$error bound properties of f at $\bar{x} \in [f \le 0]$. In view of Lemma 2(iv), in the convex case the perturbed $\tau-$error bounds are equivalent to the conventional ones.

(iii) The perturbed $\tau-$error bound property of f at $\bar{x} \in [f \le 0]$ is actually the $\tau-$*metric regularity* of the (truncated) *epigraphical* set-valued mapping $x \mapsto \operatorname{epi} f(x) := \{c \in [0, +\infty[\mid f(x) \le c\}$ at $(\bar{x}, 0)$; cf. [18, Remark 3.2].

4.3 Nonlinear Error Bound Conditions

In view of Remark 58(ii), one can easily deduce from Theorem 32 and Proposition 46 sufficient and necessary conditions for nonlinear error bounds. The sufficient conditions become meaningful when $\varphi \in \mathscr{C}^1$ as in this case one can employ the chain rules in Lemmas 3 and 9.

Theorem 33 Suppose X is a metric space, $f : X \to \mathbb{R}_\infty$, $\bar{x} \in X$, $\varphi \in \mathscr{C}^1$, $\delta \in]0, +\infty]$ and $\mu \in]0, +\infty]$.

(i) Let X be complete, f be lower semicontinuous, $\alpha \in]0, 1]$, and either $\bar{x} \in [f \le 0]$ or $\delta = +\infty$. The function f admits a $\varphi-$error bound at $\bar{x}$ with $\delta' := \frac{\delta}{1+\alpha}$ and μ, provided that one of the following conditions is satisfied:

(a) $\alpha|\nabla(\varphi \circ f)|^\diamond(u) \ge 1$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$ satisfying

$$\max\{\alpha, 1 - \alpha\}\varphi(f(u)) < d(u, [f \le 0]); \tag{4.13}$$

(b) $|\breve{\nabla} f| \in |\mathfrak{D} f|^\circ$ and $\alpha\varphi'(f(u))|\breve{\nabla} f|(u) \ge 1$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$ satisfying condition (4.13).

If φ' is nonincreasing, then $|\mathfrak{D} f|^\circ$ in (b) can be replaced with $|\mathfrak{D} f|$.

(ii) If f admits a $\varphi-$error bound at $\bar{x}$ with δ and μ, then $|\nabla(\varphi \circ f)|^\diamond(u) \ge 1$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.

(iii) Let X be a normed space, and f be convex. If f admits a $\varphi-$error bound at $\bar{x}$ with δ and μ, then $\frac{\varphi(f(u))}{f(u)}|\partial f|(u) \ge 1$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.
If, moreover, φ' is nondecreasing, particularly if φ is convex, then $\varphi'(f(u))|\partial f|(u) \ge 1$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.

Proof Assertions (i) and (ii) are direct consequences of the corresponding assertions in Theorem 32, applied to the composition function $\varphi \circ f$ with $\tau = 1$, and Lemmas 3 and 9. To prove the first part of assertion (iii), it suffices to notice that inequality (1.12) reduces to inequality (4.1) by setting $\tau := \frac{f(x)}{\varphi(f(x))}$ and apply Proposition 46(iii). By the mean value theorem, $\tau^{-1} = \varphi'(\theta)$ for some $\theta \in]0, f(x)[$. If φ' is nondecreasing, then $\tau^{-1} \le \varphi'(f(x))$, which proves the second part. □

Remark 63 (i) In view of the definition of $|\mathfrak{D} f|^\circ$ (or $|\mathfrak{D} f|$), condition (b) in Theorem 33(i) combines three (or four) separate primal and dual sufficient error bound conditions corresponding to $|\breve{\nabla} f|$ equal to $|\nabla f|^\diamond$, $|\nabla f|$, $|\partial^C f|$ or $|\partial^F f|$ (in appropriate spaces).

(ii) With $|\breve{\nabla} f| = |\partial^C f|$ or $|\breve{\nabla} f| = |\partial^F f|$, inequality $\alpha\varphi'(f(u))|\breve{\nabla} f|(u) \ge 1$ in (b) can be interpreted as the Kurdyka–Łojasiewicz property (as defined, e.g., in [9, 32]).

(iii) As in the linear case, parameter α in Theorem 33(i) determines a trade-off between the main inequalities $\alpha|\nabla(\varphi\circ f)|^{\diamond}(u)\geq 1$ in (a) and $\alpha\varphi'(f(u))|\breve{\nabla} f|(u)\geq 1$ in (b) and the radius δ' of the neighbourhood of $\bar{x}$ in which the error bound estimate holds. In the conventional case $\alpha=1$, we have $\delta'=\delta/2$.

(iv) Weakening or dropping any of the restrictions on u produces new (stronger!) sufficient conditions. This can be particularly relevant in the case of inequality (4.13) involving the distance to the unknown set $[f\leq 0]$. If $\bar{x}\in[f\leq 0]$, it is common to replace this distance with $d(u,\bar{x})$.

(v) If the function $t\mapsto\frac{\varphi(t)}{t}$ is nondecreasing on $]0,+\infty[$ (particularly, if φ is convex) then the assumption of differentiability of φ in Theorem 33 can be dropped. As one can observe from the above proof, it suffices to replace $\varphi'(f(u))$ in condition (b) with $\frac{\varphi(f(u))}{f(u)}$.

(vi) Under the conditions of part (iii) of Theorem 33, one can easily show that, if for all $u\in B_\delta(\bar{x})\cap[0<f<\mu]$ condition (4.1) holds as equality, then $\varphi'(f(u))|\partial f|(u)=1$ for all $u\in B_\delta(\bar{x})\cap[0<f<\mu]$; cf. Remark 60(v).

(vii) Employing Proposition 48, one can expand Theorem 33 to cover a *perturbed $\varphi-$error bound* property of f at $\bar{x}\in[f\leq 0]$ as in [18, Theorems 4.1 and 4.2]; cf. Remark 62.

(viii) With $\delta=+\infty$, $\mu<+\infty$, $\alpha=1$ and $\varphi(t):=\tau^{-1}t^q$ for some $\tau>0$ and $q>0$ and all $t>0$ (Hölder case), Theorem 33(i) with $|\breve{\nabla} f|=|\nabla f|$ in condition (b) recaptures [185, Corollary 2.5], while with $|\breve{\nabla} f|=|\partial^F f|$ it recaptures [186, Corollary 2(i)].

(ix) With $\bar{x}\in\operatorname{bd}[f\leq 0]$, $\delta<+\infty$, $\mu=+\infty$, $\alpha=1$, and $\varphi(t):=(\tau)^{-1}t^q$ for some $\tau>0$ and $q>0$ and all $t>0$ (Hölder case), Theorem 33(i) with $|\breve{\nabla} f|=|\partial^F f|$ in condition (b) recaptures [186, Corollary 2(ii)].

(x) With $\bar{x}\in[f\leq 0]$, $\mu=+\infty$, and $\varphi(t):=(\alpha\tau)^{-1}t^q$ for some $\tau>0$ and $q>0$ and all $t>0$ (Hölder case), part (i) of Theorem 33 with $|\breve{\nabla} f|=|\partial^C f|$ and $|\breve{\nabla} f|=|\partial^F f|$ in condition (b) improves [137, Theorem 3.7], while part (iii) partially recaptures and extends [137, Lemma 3.33].

The local $\varphi-$error bound sufficient conditions are collected in the next three corollaries.

Corollary 94 Suppose X is a complete metric space, $f:X\to\mathbb{R}_\infty$ is lower semicontinuous, and $\varphi\in\mathscr{C}^1$. The function f admits a local $\varphi-$error bound at $\bar{x}\in[f\leq 0]$, provided that one of the following conditions is satisfied:

(i) $|\nabla(\varphi\circ f)|^{\diamond}(x)\geq 1$ for all $x\in[f>0]$ near $\bar{x}$ with $f(x)$ near 0;

(ii) $|\breve{\nabla} f|\in|\mathfrak{D}f|^{\circ}$ and $\varphi'(f(x))|\breve{\nabla} f|(x)\geq 1$ for all $x\in[f>0]$ near $\bar{x}$ with $f(x)$ near 0.

Condition (i) is also necessary.
If φ' is nonincreasing, then $|\mathfrak{D}f|^\circ$ in (ii) can be replaced with $|\mathfrak{D}f|$.

Corollary 95 Suppose X is a Banach space, $f : X \to \mathbb{R}_\infty$ is convex lower semicontinuous, $\bar{x} \in [f \le 0]$, and $\varphi \in \mathscr{C}^1$. Consider the following conditions:

(i) f admits a local $\varphi-$error bound at $\bar{x}$;

(ii) $|\nabla(\varphi \circ f)|^\diamond(x) \ge 1$ for all $x \in [f > 0]$ near $\bar{x}$ with $f(x)$ near 0;

(iii) $\varphi'(f(x))|\partial f|(x) \ge 1$ for all $x \in [f > 0]$ near $\bar{x}$ with $f(x)$ near 0;

(iv) $\frac{\varphi(f(x))}{f(x)}|\partial f|(x) \ge 1$ for all $x \in [f > 0]$ near $\bar{x}$ with $f(x)$ near 0.

Then (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv). Moreover, if φ' is nondecreasing, particularly if φ is convex, then all the conditions are equivalent.

Corollary 96 Suppose X is a complete metric space, $f : X \to \mathbb{R}_\infty$ is lower semicontinuous, $\varphi \in \mathscr{C}^1$, $\bar{x} \in [f \le 0]$, and $\operatorname{Er}_\varphi f(\bar{x})$ be defined by (4.3).

(i) The following estimate holds true:

$$\operatorname{Er}_\varphi f(\bar{x}) = \liminf_{x\to\bar{x},\, f(x)\downarrow 0} |\nabla(\varphi \circ f)|^\diamond(x) \ge \liminf_{x\to\bar{x},\, f(x)\downarrow 0} \varphi'(f(x))|\nabla f|(x).$$

If X is Banach (Asplund), then $|\nabla f|$ in the above inequality can be replaced with $|\partial^C f|$ ($|\partial^F f|$).

(ii) If X is Banach and f is convex, then

$$\operatorname{Er}_\varphi f(\bar{x}) \le \liminf_{x\to\bar{x},\, f(x)\downarrow 0} \frac{\varphi(f(x))}{f(x)}|\partial f|(x).$$

Moreover, if φ satisfies

$$\frac{\varphi(t)}{t} \le \gamma\varphi'(t) \quad \text{for some} \quad \gamma \ge 1 \quad \text{and all} \quad t > 0, \tag{4.14}$$

then $\overline{|\partial f|}^{>}_{\varphi}(\bar{x}) \le \operatorname{Er}_\varphi f(\bar{x}) \le \gamma\overline{|\partial f|}^{>}_{\varphi}(\bar{x})$, where

$$\overline{|\partial f|}^{>}_{\varphi}(\bar{x}) := \liminf_{x\to\bar{x},\, f(x)\downarrow 0} \varphi'(f(x))|\partial f|(x); \tag{4.15}$$

as a consequence, f admits a local $(\alpha\varphi)-$error bound at $\bar{x}$ with some α satisfying $\overline{|\partial f|}^{>}_{\varphi}(\bar{x}) \le \alpha^{-1} \le \gamma\overline{|\partial f|}^{>}_{\varphi}(\bar{x})$ if and only if $\overline{|\partial f|}^{>}_{\varphi}(\bar{x}) > 0$.

Remark 64 (i) In Theorem 33(i)(a), Corollary 94(i) and Corollary 95(ii), it suffices to assume that $\varphi \in \mathscr{C}$.

(ii) In the Hölder case, i.e. when $\varphi(t) := \tau^{-1}t^q$ for some $\tau > 0$ and $q > 0$ and all $t > 0$, we have $\varphi'(t) = q\tau^{-1}t^{q-1}$ and $\frac{\varphi(t)}{t} = \tau^{-1}t^{q-1}$. Thus, the implication (iii) $\Rightarrow$ (iv) in Corollary 95 is trivially satisfied when $q \leq 1$, while the opposite implication holds when $q \geq 1$, i.e. φ is convex. Moreover, if $q \leq 1$, then condition (4.14) is satisfied (as equality) with $\gamma = q^{-1}$.

When $\varphi(t) := t^q$ for some $q > 0$ and all $t > 0$, definition (4.15) reduces to [137, (3.13)]. In particular, if $q = 1$, it coincides with the strict outer subdifferential slope of f at $\bar{x}$ given by the last expression in (4.11).

(iii) Condition (4.14) can be replaced by the following weaker condition: $\limsup_{t\downarrow 0} \frac{\varphi(t)}{t\varphi'(t)} < +\infty$.

It can be convenient to reformulate Theorem 33 using the function $\psi := \varphi^{-1} \in \mathscr{C}^1$ instead of φ.

Corollary 97 Suppose X is a metric space, $f : X \to \mathbb{R}_\infty$, $\bar{x} \in X$, $\psi \in \mathscr{C}^1$, $\delta \in]0, +\infty]$ and $\mu \in]0, +\infty]$.

(i) Let X be complete, f be lower semicontinuous, $\alpha \in]0, 1]$, and either $\bar{x} \in [f \leq 0]$ or $\delta = +\infty$. The error bound inequality (4.2) holds for all $x \in B_{\frac{\delta}{1+\alpha}}(\bar{x}) \cap [0 < f < \mu]$, provided that one of the following conditions is satisfied:

(a) $\alpha|\nabla(\psi^{-1} \circ f)|^\diamond(u) \geq 1$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$ satisfying

$$f(u) < \psi((\max\{\alpha, 1-\alpha\})^{-1} d(u, [f \leq 0])); \tag{4.16}$$

(b) $|\breve{\nabla} f| \in |\mathfrak{D} f|^\circ$ and $\alpha|\breve{\nabla} f|(u) \geq \psi'(\psi^{-1}(f(u)))$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$ satisfying condition (4.16).

If φ' is nonincreasing, then $|\mathfrak{D} f|^\circ$ in (b) can be replaced with $|\mathfrak{D} f|$.

(ii) If the error bound inequality (4.2) holds for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$, then $|\nabla(\psi^{-1} \circ f)|^\diamond(u) \geq 1$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.

(iii) Let X be a normed space, and f be convex. If the error bound inequality (4.2) holds for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$, then $|\partial f|(u) \geq \frac{f(u)}{\psi^{-1}(f(u))}$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$. Moreover, if ψ' is nonincreasing, particularly if ψ is concave, then $|\partial f|(u) \geq \psi'(\psi^{-1}(f(u)))$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.

Remark 65 With $\bar{x} \in [f \leq 0]$, $\delta < +\infty$, $\mu = +\infty$ and $\alpha = 1$, Corollary 97(i) with condition (b) and $|\breve{\nabla} f| = |\partial^F f|$ strengthens [211, Theorem 3.2], while with $\delta = \mu = +\infty$ and $\alpha = 1$ it strengthens [211, Theorem 3.3].

4.4 Alternative Nonlinear Error Bound Conditions

In this section, we discuss an alternative set of sufficient and necessary conditions for nonlinear error bounds which instead of values of the given function f employ the distance to the solution set $[f \leq 0]$.

The next simple tool is helpful when comparing the resulting conditions. We have not found a reference for this assertion and provide a short proof for completeness.

Lemma 11 Suppose $\varphi \in \mathscr{C}^1$. If φ' is nonincreasing (nondecreasing) on $]0,+\infty[$, then the function $t \mapsto \frac{\varphi(t)}{t}$ is nonincreasing (nondecreasing) on $]0,+\infty[$, and $\varphi'(t) \leq \frac{\varphi(t)}{t}$ $\left(\varphi'(t) \geq \frac{\varphi(t)}{t}\right)$ for all $t \in]0,+\infty[$.

Proof Let $0 < t_1 < t_2$. If φ' is nonincreasing, then

$$\frac{\varphi(t_1)}{t_1} = \frac{1}{t_1}\int_0^{t_1} \varphi'(t)dt \geq \varphi'(t_1) \geq \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \varphi'(t)dt, \tag{4.17}$$

and consequently,

$$\frac{\varphi(t_2)}{t_2} = \frac{1}{t_2}\left(\varphi(t_1) + \int_{t_1}^{t_2} \varphi'(t)dt\right) \leq \frac{1}{t_2}\left(\varphi(t_1) + \frac{t_2 - t_1}{t_1}\varphi(t_1)\right) = \frac{\varphi(t_1)}{t_1}. \tag{4.18}$$

Similarly, if φ' is nondecreasing, we have the opposite inequalities in (4.17) and (4.18). □

Theorem 34 Suppose X is a metric space, $f : X \to \mathbb{R}_\infty$, $\bar{x} \in X$, $\varphi \in \mathscr{C}^1$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$.

(i) Let X be complete, f be lower semicontinuous, φ' be nonincreasing, $\alpha \in]0,1]$, and either $\bar{x} \in [f \leq 0]$ or $\delta = +\infty$. Let $|\breve{\nabla} f| \in |\mathfrak{D} f|$. The function f admits a $\varphi-$error bound at $\bar{x}$ with $\delta' := \frac{\delta}{1+\alpha}$ and μ, provided that

$$\alpha\varphi'(\varphi^{-1}((\max\{\alpha, 1-\alpha\})^{-1} d(u,[f \leq 0])))|\breve{\nabla} f|(u) \geq 1 \tag{4.19}$$

for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$ satisfying condition (4.13).

(ii) Let X be a normed space, and f be convex. If f admits a $\varphi-$error bound at $\bar{x}$ with δ and μ, then $\frac{d(u,[f\leq 0])}{\varphi^{-1}(d(u,[f\leq 0]))}|\partial f|(u) \geq 1$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.
If, moreover, φ' is nondecreasing, particularly if φ is convex, then
$\varphi'(\varphi^{-1}(d(u,[f \leq 0])))|\partial f|(u) \geq 1$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.

Proof

(i) The assertion is a consequence of Theorem 33(i). It suffices to notice that, by the monotonicity of φ', we have $\varphi'(\varphi^{-1}((\max\{\alpha, 1-\alpha\})^{-1} d(u,[f \leq 0]))) \leq \varphi'(f(u))$ for all $u \in X$ satisfying condition (4.13).

(ii) The first part follows from Proposition 46(iii) applied with $\tau := \frac{\varphi^{-1}(d(x,[f\le 0]))}{d(x,[f\le 0])}$. If φ' is nondecreasing, then $(\varphi^{-1})'$ is nonincreasing and, by Lemma 11, $\tau \ge (\varphi^{-1})'(d(x,[f\le 0])) = 1/\varphi'(\varphi^{-1}(d(x,[f\le 0])))$. This proves the second part.

□

Remark 66 (i) In view of the definition of $|\mathfrak{D}f|$, Theorem 34(i) combines four separate primal and dual sufficient error bound conditions corresponding to $|\breve{\nabla}f|$ equal to $|\nabla f|^{\diamond}$, $|\nabla f|$, $|\partial^C f|$ or $|\partial^F f|$ (in appropriate spaces).

(ii) Compared to Theorem 33(i), the statement of Theorem 34(i) contains an additional assumption that φ' is nonincreasing This assumption is satisfied, e.g., in the Hölder setting, i.e. when $\varphi(t) := \tau^{-1}t^q$ for some $\tau > 0$ and $q \in]0,1]$, and all $t \ge 0$. In the linear case, i.e. when $q = 1$, the sufficient conditions in Theorems 33 and 34 are equivalent, and reduce to the corresponding ones in Theorem 32.

(iii) Under the assumption that φ' is nonincreasing and with $\alpha = 1$, the local sufficient conditions in Theorem 34(i) are in a sense weaker than the corresponding conventional ones in Theorem 33(i). Indeed, if $\alpha = 1$ and, given some $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$, condition (b) in Theorem 33(i) is satisfied with some $|\breve{\nabla}f| \in |\mathfrak{D}f|^{\circ}$, then, by Theorem 33(i), $\varphi^{-1}(d(u,[f\le 0])) \le f(u)$ for all $u \in B_{\delta'}(\bar{x})$, where $\delta' := \delta/2$. (As a consequence, inequality (4.13) is violated for all $u \in B_{\delta'}(\bar{x})$.) Then, thanks to the monotonicity of φ', for all $u \in B_{\delta'}(\bar{x})$, we have $\varphi'(\varphi^{-1}(d(u,[f\le 0]))) \ge \varphi'(f(u))$, and consequently, inequality (4.19) is satisfied with the same $|\breve{\nabla}f|$.

(iv) In view of the monotonicity of φ^{-1} and φ', one can replace $\max\{\alpha, 1-\alpha\}$ in (4.13) and (4.19) in Theorem 34(i) with any positive $\beta \le \max\{\alpha, 1-\alpha\}$. In particular, one can take $\beta := \alpha$ or $\beta := 1-\alpha$ (if $\alpha < 1$). The resulting sufficient conditions are obviously stronger (hence, less efficient) than those in the current statement.

(v) With $\bar{x} \in [f\le 0]$, $\mu = +\infty$, $\alpha < 1$, and $\varphi(t) := (\alpha^q(1-\alpha)^{1-q}\tau)^{-1}t^q$ for some $\tau > 0$ and $q > 0$ and all $t > 0$ (Hölder case), Theorem 34(i) with $|\breve{\nabla}f| = |\partial^F f|$ and $|\breve{\nabla}f| = |\partial^C f|$ improves and strengthens [137, Theorem 3.11].

The local $\varphi-$error bound sufficient conditions arising from Theorem 34(i) are collected in the next three corollaries.

Corollary 98 Suppose X is a complete metric space, $f : X \to \mathbb{R}_\infty$ is lower semicontinuous, $\varphi \in \mathscr{C}^1$, and φ' is nonincreasing. Let $|\breve{\nabla}f| \in |\mathfrak{D}f|$. The function f admits a local $\varphi-$error bound at $\bar{x} \in [f\le 0]$, provided that

$$\varphi'(\varphi^{-1}(d(x,[f\le 0])))|\breve{\nabla}f|(x) \ge 1$$

for all $x \in [f > 0]$ near $\bar{x}$ with $f(x)$ near 0.

Corollary 99 Suppose X is a Banach space, $f : X \to \mathbb{R}_\infty$ is convex lower semicontinuous, $\bar{x} \in [f \le 0]$, $\varphi \in \mathscr{C}^1$ and φ' is nonincreasing. Consider the following conditions:

(i) f admits a local $\varphi-$error bound at $\bar{x}$;

(ii) $\varphi'(\varphi^{-1}(d(x,[f \le 0])))|\partial f|(x) \ge 1$ for all $x \in [f > 0]$ near $\bar{x}$ with $f(x)$ near 0;

(iii) $\frac{d(x,[f\le 0])}{\varphi^{-1}(d(x,[f\le 0]))}|\partial f|(x) \ge 1$ for all $x \in [f > 0]$ near $\bar{x}$ with $f(x)$ near 0.

Then (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). If φ is linear, then all the conditions are equivalent.

Corollary 100 Suppose X is a complete metric space, $f : X \to \mathbb{R}_\infty$ is lower semicontinuous, $\varphi \in \mathscr{C}^1$, φ' is nonincreasing, and $\bar{x} \in [f \le 0]$.

(i) The following estimate holds true:

$$\mathrm{Er}_\varphi f(\bar{x}) \ge \liminf_{x\to\bar{x},\, f(x)\downarrow 0} \varphi'(\varphi^{-1}(d(x,[f \le 0])))|\nabla f|(x).$$

If X is Banach (Asplund), then $|\nabla f|$ in the first inequality can be replaced with $|\partial^C f|$ ($|\partial^F f|$).

(ii) If X is Banach and f is convex, then

$$\mathrm{Er}_\varphi f(\bar{x}) \le \liminf_{x\to\bar{x},\, f(x)\downarrow 0} \frac{d(x,[f \le 0])}{\varphi^{-1}(d(x,[f \le 0]))}|\partial f|(x).$$

Moreover, if φ satisfies condition (4.14), then $\widehat{|\partial f|}{}^{>}_{\varphi}(\bar{x}) \le \mathrm{Er}_\varphi f(\bar{x}) \le \gamma \widehat{|\partial f|}{}^{>}_{\varphi}(\bar{x})$, where

$$\widehat{|\partial f|}{}^{>}_{\varphi}(\bar{x}) := \liminf_{x\to\bar{x},\, f(x)\downarrow 0} \varphi'(\varphi^{-1}(d(x,[f \le 0])))|\partial f|(x);$$

as a consequence, f admits a local $(\alpha\varphi)-$error bound at $\bar{x}$ with some α satisfying $\widehat{|\partial f|}{}^{>}_{\varphi}(\bar{x}) \le \alpha^{-1} \le \gamma \widehat{|\partial f|}{}^{>}_{\varphi}(\bar{x})$ if and only if $\widehat{|\partial f|}{}^{>}_{\varphi}(\bar{x}) > 0$.

Remark 67 In view of Remark 66(iii), when φ' is nonincreasing, each of the sufficient conditions in Corollaries 98 and 99 is implied by the corresponding condition in Corollaries 94 and 95, respectively. Similarly, condition $\overline{|\partial f|}{}^{>}_{\varphi}(\bar{x}) > 0$ implies $\widehat{|\partial f|}{}^{>}_{\varphi}(\bar{x}) > 0$.

In view of Remark 64(ii), the next statement is a consequence of Corollaries 96 and 100. It recaptures [137, Corollary 3.35].

Corollary 101 Suppose X is a Banach space, $f : X \to \mathbb{R}_\infty$ is convex lower semicontinuous, $\bar{x} \in [f \le 0]$, and $q \in]0,1]$. The following assertions are equivalent:

(i) the function f admits a local error bound of order q at $\bar{x}$, i.e. there exist $\tau > 0$ and $\delta > 0$ such that

$$\tau d(x,[f \le 0]) \le (f(x))^q \quad \text{for all} \quad x \in B_\delta(\bar{x}) \cap [f > 0],$$

(ii) $\liminf\limits_{x\to\bar{x},\, f(x)>0} (f(x))^{q-1}|\partial f|(x) > 0.$

(iii) $\liminf\limits_{x\to\bar{x},\, f(x)>0} (d(x,[f\le 0]))^{1-\frac{1}{q}}|\partial f|(x) > 0.$

It can be convenient to reformulate Theorem 34 using the function $\psi := \varphi^{-1} \in \mathscr{C}^1$ instead of φ. Obviously φ' is nonincreasing if and only if ψ' is nondecreasing.

Corollary 102 Suppose X is a metric space, $f : X \to \mathbb{R}_\infty$, $\bar{x} \in X$, $\psi \in \mathscr{C}^1$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$.

(i) Let X be complete, f be lower semicontinuous, ψ' be nondecreasing, $\alpha \in]0,1]$, $\beta := \max\{\alpha, 1-\alpha\}$, and either $\bar{x} \in [f \le 0]$ or $\delta = +\infty$. Let $|\breve{\nabla} f| \in |\mathfrak{D} f|$. The error bound inequality (4.2) holds for all $x \in B_{\frac{\delta}{1+\alpha}}(\bar{x}) \cap [0 < f < \mu]$, provided that

$$\alpha|\breve{\nabla} f|(u) \ge \psi'(\beta^{-1} d(u,[f\le 0])) \tag{4.20}$$

for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$ satisfying $f(u) < \psi(\beta^{-1} d(u,[f \le 0]))$.

(ii) Let X be a normed space, and f be convex. If the error bound inequality (4.2) holds for all $x \in B_\delta(\bar{x}) \cap [0 < f < \mu]$, then $|\partial f|(u) \ge \frac{\psi(d(u,[f\le 0]))}{d(u,[f\le 0])}$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$. Moreover, if ψ' is nonincreasing, particularly if ψ is concave, then $|\partial f|(u) \ge \psi'(d(u,[f \le 0]))$ for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$.

Remark 68 (i) Condition (4.20) can itself be interpreted as an error bound estimate. Unlike the conventional upper estimates (4.1) and (4.2) for the distance to the set $[f \le 0]$ in terms of the values of the function f, the inequality provides an estimate for this distance in terms of the appropriate slopes of the function f.

(ii) Remark 66(iv) is applicable to Corollary 102(i).

(iii) With $\bar{x} \in [f \le 0]$, $\delta < +\infty$, $\mu = +\infty$, and $\alpha < 1$, Corollary 102(i) with appropriate slopes strengthens [215, Theorem 3.1 and Proposition 3.3] and [211, Theorems 3.1 and 3.4]. Note that in [215] the function ψ is assumed convex that does not seem to be necessary. It is also worth observing that the rather complicated statements in [215, Theorem 3.1 and Proposition 3.3] become simpler if the two parameters $\beta > 0$ and $\tau > 0$ involved in them are replaced with a single parameter $\alpha := \beta/(\tau+\beta) \in]0,1[$ and one employs the function $t \mapsto \psi(\tau\alpha t)$ instead of ψ.

A special case of Corollary 102, which can be of interest, is when the function ψ is defined via another function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ as follows: $\psi(t) := \int_0^t \nu(s)ds$ $(t \ge 0)$.

Corollary 103 Suppose X is a complete metric space, $f : X \to \mathbb{R}_\infty$ is lower semicontinuous, $\bar{x} \in X$, $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, $\nu(t) > 0$ for all $t > 0$, $\int_0^{+\infty} \nu(t)dt = +\infty$,

$\delta \in]0,+\infty]$, $\mu \in]0,+\infty]$, $\alpha \in]0,1]$, $\beta := \max\{\alpha, 1-\alpha\}$, and either $\bar{x} \in [f \le 0]$ or $\delta = +\infty$. Let $|\breve{\nabla} f| \in |\mathfrak{D} f|$. The error bound inequality

$$\int_0^{d(x,[f\le 0])} \nu(t)dt \le f(x)$$

holds for all $x \in B_{\frac{\delta}{1+\alpha}}(\bar{x}) \cap [0 < f < \mu]$, provided that

$$\alpha|\breve{\nabla} f|(u) \ge \nu(\beta^{-1} d(u,[f \le 0]))$$

for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$ satisfying $f(u) < \int_0^{\beta^{-1}d(u,[f\le 0])} \nu(s)ds$.

Remark 69 With $\delta = +\infty$ and $\alpha = 1$, Corollary 103 with $|\breve{\nabla} f| = |\nabla f|$ recaptures [64, Theorem 4.3] and [18, Theorem 7.1]. With $\delta < +\infty$, $\mu = +\infty$ and $\alpha = 1$, it recaptures [17, Corollary 4.1].

4.5 Subregularity of Set-Valued Mappings

In this section, we illustrate the sufficient and necessary conditions for nonlinear error bounds by applying them to characterizing the nonlinear version of the ubiquitous property of *subregularity* (cf. [81, 116]) of set-valued mappings. Nonlinear subregularity has many important applications and has been a subject of intense research in recent years; cf. [130, 133, 154, 174, 219].

Below, $F : X \rightrightarrows Y$ is a set-valued mapping between metric spaces, and $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Recall that $X \times Y$ is assumed to be equipped with the maximum distance.

Definition 23 Let $\varphi \in \mathscr{C}$. The mapping F is

(i) $\varphi-$subregular at $(\bar{x},\bar{y})$ if there exist $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that

$$d(x,F^{-1}(\bar{y})) \le \varphi(d(\bar{y},F(x)))$$

for all $x \in B_\delta(\bar{x})$ with $d(\bar{y},F(x)) < \mu$;

(ii) graph $\varphi-$subregular at $(\bar{x},\bar{y})$ if there exist $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that

$$d(x,F^{-1}(\bar{y})) \le \varphi(d((x,\bar{y}),\operatorname{gph} F)) \tag{4.21}$$

for all $x \in B_\delta(\bar{x})$ with $d((x,\bar{y}),\operatorname{gph} F) < \mu$.

Remark 70 Instead of the standard maximum distance employed in (4.21), it is common when studying regularity properties of mappings to consider parametric distances of the type $d_\rho((x,y,(u,v)) := \max\{d(x,u), \rho d(y,v)\}$, where $\rho > 0$; cf. [13, 16, 116, 130, 131, 133]. This usually gives an additional degree of freedom and leads to sharper conditions. We avoid doing it here just for simplicity as our main purpose in this section is to provide some illustrations. All the conditions below can be easily extended to parametric distances.

The nonlinear property in part (i) of Definition 23 is a quite common extension of the conventional subregularity; cf., e.g., [133], while the property in part (ii) extends the (linear) graph subregularity studied by Jourani and Thibault [122]. In the linear case, the two properties are equivalent. The next proposition establishes quantitative relations between the properties in the nonlinear setting.

Proposition 49 Let $\varphi \in \mathscr{C}$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$.

(i) If F is graph $\varphi-$subregular at $(\bar{x},\bar{y})$ with δ and μ, then it is $\varphi-$subregular at $(\bar{x},\bar{y})$ with δ and μ.

(ii) Suppose $\varphi(t) \le t$ for all $t \in]0,\delta[$. If F is $\varphi-$subregular at $(\bar{x},\bar{y})$ with δ and μ, then it is graph $\psi-$subregular at $(\bar{x},\bar{y})$ with $\psi(t) := 2t$ $(t \ge 0)$, $\delta' = \min\{\delta,\mu\}/2$ and any $\mu' \in]0,+\infty]$.

Proof

(i) Suppose F is graph $\varphi-$subregular at $(\bar{x},\bar{y})$ with δ and μ. Let $x \in B_\delta(\bar{x})$ with $d(\bar{y},F(x)) < \mu$. Then $d((x,\bar{y}),\operatorname{gph} F) \le d(\bar{y},F(x))$. Thus, $d((x,\bar{y}),\operatorname{gph} F) < \mu$ and, in view of Definition 23(ii), $d(x,F^{-1}(\bar{y})) \le \varphi(d((x,\bar{y}),\operatorname{gph} F)) \le \varphi(d(\bar{y},F(x)))$. Hence, F is $\varphi-$subregular at $(\bar{x},\bar{y})$ with δ and μ.

(ii) Suppose F is $\varphi-$subregular at $(\bar{x},\bar{y})$ with δ and μ. Let $x \in B_{\delta'}(\bar{x})$, where $\delta' = \min\{\delta,\mu\}/2$. Observe that $d((x,\bar{y}),\operatorname{gph} F) \le d(x,\bar{x})$ and, if $d((u,v),(x,\bar{y})) \le d(x,\bar{x})$, then $d((u,v),(\bar{x},\bar{y})) \le 2d(x,\bar{x}) < 2\delta'$. Hence,

$$d((x,\bar{y}),\operatorname{gph} F) = \inf_{(u,v)\in \operatorname{gph} F\cap B_{2\delta'}(\bar{x},\bar{y})} \max\{d(x,u),d(\bar{y},v)\}.$$

For any $(u,v) \in \operatorname{gph} F \cap B_{2\delta'}(\bar{x},\bar{y})$, we have $u \in B_\delta(\bar{x})$, and $d(\bar{y},F(u)) \le d(\bar{y},v) < 2\delta' = \min\{\delta,\mu\}$. Thus, by Definition 23(i) and the assumption on φ,

$$d(u,F^{-1}(\bar{y})) \le \varphi(d(\bar{y},F(u))) \le d(\bar{y},F(u)) \le d(\bar{y},v),$$

and consequently,

$$d(x,F^{-1}(\bar{y})) \le d(x,u) + d(\bar{y},v) \le 2d((x,\bar{y}),(u,v)) = \psi(d((x,\bar{y}),(u,v))).$$

Taking infimum over $(u,v) \in \operatorname{gph} F \cap B_{2\delta'}(\bar{x},\bar{y})$, we arrive at $d(x,F^{-1}(\bar{y})) \le \psi(d((x,\bar{y}),\operatorname{gph} F))$. Hence, F is graph $\psi-$subregular at $(\bar{x},\bar{y})$ with δ' and any $\mu' \in]0,+\infty]$.

$\square$

Observe that graph subregularity of F in Definition 23(ii) is precisely the error bound property of the Lipschitz continuous function $x \mapsto d((x,\bar{y}),\operatorname{gph} F)$. When applied to this function, formulas (1.19) and (1.20) lead to the following definitions of the primal and dual slopes of the set-valued mapping F at $\bar{x}$:

$$|\nabla F|^{\diamond}(x) := \sup_{u\neq x} \frac{[d((x,\bar{y}),\operatorname{gph} F) - d((u,\bar{y}),\operatorname{gph} F)]_+}{d(u,x)},$$

$$|\nabla F|(x) := \limsup_{u\to x,\, u\neq x} \frac{[d((x,\bar{y}),\operatorname{gph} F) - d((u,\bar{y}),\operatorname{gph} F)]_+}{d(u,x)},$$

$$|\partial^C F|(x) := d(0,\partial^C d((\cdot,\bar{y}),\operatorname{gph} F)(x)), \quad |\partial^F F|(x) := d(0,\partial^F d((\cdot,\bar{y}),\operatorname{gph} F)(x)).$$

Note that they differ from the corresponding slopes used in [130, 131, 133]. We use below the collections of slope operators of F (being realizations of the corresponding collections of f) defined recursively as follows:

(i) $|\mathfrak{D}F|^{\circ} := \{|\nabla F|\}$, $|\mathfrak{D}F| = |\mathfrak{D}F|^{\dagger} := \{|\nabla F|^{\diamond}, |\nabla F|\}$;

(ii) if X is Banach, then $|\mathfrak{D}F|^{\circ} := |\mathfrak{D}F|^{\circ} \cup \{|\partial^C F|\}$, $|\mathfrak{D}F| = |\mathfrak{D}F|^{\dagger} := |\mathfrak{D}F| \cup \{|\partial^C F|\}$;

(iii) if X is Asplund, then $|\mathfrak{D}F|^{\circ} := |\mathfrak{D}F|^{\circ} \cup \{|\partial^F F|\}$, $|\mathfrak{D}F| := |\mathfrak{D}F| \cup \{|\partial^F F|\}$.

The next two statements are consequences of Theorems 33 and 34, respectively. Their first parts extend [121, Theorem 2.4] and [116, Theorem 2.53].

Proposition 50 Suppose X and Y are metric spaces, $\varphi \in \mathscr{C}^1$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$.

(i) Let X be complete and $\alpha \in]0,1]$. The mapping F is graph $\varphi-$subregular at $(\bar{x},\bar{y})$ with $\delta' := \frac{\delta}{1+\alpha}$ and μ, provided that one of the following conditions is satisfied:

(a) $\alpha|\nabla(\varphi \circ d((\cdot,\bar{y}),\operatorname{gph} F))|^{\diamond}(u) \geq 1$ for all $u \in B_{\delta}(\bar{x}) \setminus F^{-1}(\bar{y})$ with $d((u,\bar{y}),\operatorname{gph} F) < \mu$ satisfying

$$\max\{\alpha, 1-\alpha\}\varphi(d((u,\bar{y}),\operatorname{gph} F)) < d(u,F^{-1}(y)); \tag{4.22}$$

(b) $|\breve{\nabla} F| \in |\mathfrak{D}F|^{\circ}$ and $\alpha\varphi'(d((u,\bar{y}),\operatorname{gph} F))|\breve{\nabla} F|(u) \geq 1$ for all $u \in B_{\delta}(\bar{x}) \setminus F^{-1}(\bar{y})$ with $d((u,\bar{y}),\operatorname{gph} F) < \mu$ satisfying condition (4.22).

If φ' is nonincreasing, then $|\mathfrak{D}F|^{\circ}$ in (b) can be replaced with $|\mathfrak{D}F|$.

(ii) If F is $\varphi-$graph regular at $(\bar{x},\bar{y})$ with δ and μ, then $|\nabla(\varphi \circ d((\cdot,\bar{y}),\operatorname{gph} F))|^{\diamond}(u) \geq 1$ for all $u \in B_{\delta}(\bar{x}) \setminus F^{-1}(\bar{y})$ with $d((u,\bar{y}),\operatorname{gph} F) < \mu$.

(iii) Let X and Y be normed spaces, and $\operatorname{gph} F$ be convex. If F is graph $\varphi-$subregular at $(\bar{x},\bar{y})$ with δ and μ, then $\frac{\varphi(d((u,\bar{y}),\operatorname{gph} F))}{d((u,\bar{y}),\operatorname{gph} F)}|\partial F|(u) \geq 1$ for all $u \in B_{\delta}(\bar{x}) \setminus F^{-1}(\bar{y})$ with $d((u,\bar{y}),\operatorname{gph} F) < \mu$.
If, moreover, φ' is nondecreasing, particularly if φ is convex, then $\varphi'(d((u,\bar{y}),\operatorname{gph} F))|\partial F|(u) \geq 1$ for all $u \in B_{\delta}(\bar{x}) \setminus F^{-1}(\bar{y})$ with $d((u,\bar{y}),\operatorname{gph} F) < \mu$.

Proposition 51 Suppose X and Y are metric spaces, $\varphi \in \mathscr{C}^1$, $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$.

(i) Let X be complete, φ' be nonincreasing, and $\alpha \in]0,1]$. Let $|\check{\nabla}F| \in |\mathfrak{D}F|$. The mapping F is graph $\varphi-$subregular at $(\bar{x},\bar{y})$ with $\delta' := \frac{\delta}{1+\alpha}$ and μ, provided that

$$\alpha\varphi'(\varphi^{-1}((\max\{\alpha,1-\alpha\})^{-1}d(u,F^{-1}(\bar{y}))))|\check{\nabla}F|(u) \geq 1$$

for all $u \in B_\delta(\bar{x}) \setminus F^{-1}(\bar{y})$ with $d((u,\bar{y}),\mathrm{gph}\,F) < \mu$ satisfying condition (4.22).

(ii) Let X and Y be normed spaces, and $\mathrm{gph}\,F$ be convex. If F is graph $\varphi-$subregular at $(\bar{x},\bar{y})$ with δ and μ, then $\frac{d(u,F^{-1}(\bar{y}))}{\varphi^{-1}(d(u,F^{-1}(\bar{y})))}|\partial F|(u) \geq 1$ for all $u \in B_\delta(\bar{x}) \setminus F^{-1}(\bar{y})$ with $d((u,\bar{y}),\mathrm{gph}\,F) < \mu$.
If, moreover, φ' is nondecreasing, particularly if φ is convex, then $\varphi'(\varphi^{-1}(d(u,F^{-1}(\bar{y}))))|\partial F|(u) \geq 1$ for all $u \in B_\delta(\bar{x}) \setminus F^{-1}(\bar{y})$ with $d((u,\bar{y}),\mathrm{gph}\,F) < \mu$.

4.6 Convex Semi-Infinite Optimization

In this section, we consider a canonically perturbed convex semi-infinite optimization problem:

$$\begin{aligned} P(c,b):\quad & \text{minimize} \quad \psi(x) + \langle c,x\rangle \\ & \text{subject to} \quad g_t(x) \leq b_t,\ t \in T, \end{aligned}$$

where $x \in \mathbb{R}^n$ is the vector of variables, $c \in \mathbb{R}^n$, $\langle\cdot,\cdot\rangle$ represents the usual inner product in $\mathbb{R}^n$, T is a proper compact subset of a metric space Z, $\psi : \mathbb{R}^n \to \mathbb{R}$ and $g_t : \mathbb{R}^n \to \mathbb{R}$ $(t \in T)$ are convex functions, and the function $(t,x) \mapsto g_t(x)$ is assumed to be lower semicontinuous on $T \times \mathbb{R}^n$, and $b \in C(T,\mathbb{R})$ (the space of continuous functions from T to $\mathbb{R}$). In this setting, the pair $(c,b) \in \mathbb{R}^n \times C(T,\mathbb{R})$ is regarded as the parameter to be perturbed. The norm in this parameter space is given by $\|(c,b)\| := \max\{\|c\|,\|b\|_\infty\}$, where $\|\cdot\|$ is any given norm in $\mathbb{R}^n$ and $\|b\|_\infty := \max_{t\in T}|b_t|$.

The *solution*, *feasible set* and *level set* mappings corresponding to the above problem are the set-valued mappings defined, respectively, by

$$\mathscr{S}(c,b) := \{x \in \mathbb{R}^n \mid x \text{ solves } P(c,b)\}, \quad (c,b) \in \mathbb{R}^n \times C(T,\mathbb{R}), \tag{4.23}$$

$$\mathscr{F}(b) := \{x \in \mathbb{R}^n \mid g_t(x) \leq b_t,\ t \in T\}, \quad b \in C(T,\mathbb{R}),$$

$$\mathscr{L}(\alpha,b) := \{x \in \mathscr{F}(b) \mid \psi(x) + \langle\bar{c},x\rangle \leq \alpha\}, \quad (\alpha,b) \in \mathbb{R} \times C(T,\mathbb{R}). \tag{4.24}$$

If $c := \bar{c}$ in (4.23) is fixed, then $\mathscr{S}$ reduces to the partial solution mapping $\mathscr{S}_{\bar{c}} : C(T,\mathbb{R}) \rightrightarrows \mathbb{R}^n$ given by $\mathscr{S}_{\bar{c}}(b) = \mathscr{S}(\bar{c},b)$.

Our goal in this section is to use nonlinear error bound conditions for analyzing nonlinear *calmness* of the mappings $\mathscr{S}$, $\mathscr{S}_{\bar{c}}$ and $\mathscr{L}$.

Definition 24 Let $F : Y \rightrightarrows X$ be a set-valued mapping between metric spaces, $(\bar{y},\bar{x}) \in \mathrm{gph}\,F$, and $\varphi \in \mathscr{C}$. The mapping F is $\varphi-$calm at $(\bar{y},\bar{x})$ if there exist $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ such that

$$d(x,F(\bar{y})) \leq \varphi(d(y,\bar{y})) \tag{4.25}$$

for all $y \in Y$ with $d(y,\bar{y}) < \mu$ and $x \in F(y) \cap B_\delta(\bar{x})$.

Calmness of set-valued mappings plays an important role in optimization theory, cf. [81, 116]. It is easy to check that F is $\varphi-$calm at $(\bar{y},\bar{x}) \in \operatorname{gph} F$ with some δ and μ if and only if F^{-1} is $\varphi-$subregular at $(\bar{x},\bar{y})$ with the same δ and μ.

Cánovas et al. [47] examined the problem $P(c,b)$ in the particular case when g_t $(t \in T)$ are linear functions and $\psi \equiv 0$, and obtained estimates for the calmness modulus. In [46], Cánovas et al. studied the modulus of metric regularity of the solution mapping. Kruger et al. [137] established characterizations of Hölder calmness of the solution mapping by employing the error bound theory. Motivated by the latter paper, we establish characterizations of the calmness in the nonlinear setting.

From now on, we assume a point $((\bar{c},\bar{b}),\bar{x}) \in \operatorname{gph}\mathscr{S}$ to be given. Obviously, if $\mathscr{S}$ is $\varphi-$calm at $((\bar{c},\bar{b}),\bar{x})$ for some $\varphi \in \mathscr{C}$, then $\mathscr{S}_{\bar{c}}$ is $\varphi-$calm at $(\bar{b},\bar{x})$. We are going to employ the following convex and continuous function:

$$f(x) := \max\{\psi(x) - \psi(\bar{x}) + \langle \bar{c}, x - \bar{x}\rangle,\ \sup_{t\in T}(g_t(x) - \bar{b}_t)\}, \quad x \in \mathbb{R}^n. \tag{4.26}$$

Observe that

$$\begin{aligned} \mathscr{S}(\bar{c},\bar{b}) &= [f = 0] = [f \le 0] = \mathscr{L}(\psi(\bar{x}) + \langle \bar{c},\bar{x}\rangle, \bar{b}), \\ f_+(x) &= d((\psi(\bar{x}) + \langle \bar{c},\bar{x}\rangle, \bar{b}), \mathscr{L}^{-1}(x)) \quad \text{for all} \quad x \in \mathbb{R}^n. \end{aligned} \tag{4.27}$$

As a consequence, we have the following statement.

Proposition 52 Let $\varphi \in \mathscr{C}$. The mapping $\mathscr{L}$ is $\varphi-$calm at $(\bar{x}, (\psi(\bar{x}) + \langle \bar{c},\bar{x}\rangle, \bar{b}))$ with some $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$ if and only if f admits a $\varphi-$error bound at $\bar{x}$ with the same δ and μ.

The assertions in the next proposition are extracted from [137, Proposition 4.5 & Theorem 4.7] and their proofs. The set of *active indices* at $x \in \mathscr{F}(b)$ is defined by $T_b(x) := \{t \in T \mid g_t(x) = b_t\}$. The problem $P(c,b)$ satisfies *Slater condition* if there exists an $\hat{x} \in \mathbb{R}^n$ such that $g_t(\hat{x}) < b_t$ for all $t \in T$.

Proposition 53 Let $P(\bar{c},\bar{b})$ satisfy the Slater condition.

(i) There exist $\delta > 0$, $\mu > 0$ and $M > 0$ such that

$$\psi(x) - \psi(\bar{x}) + \langle \bar{c}, x - \bar{x}\rangle \le M\|(c,b) - (\bar{c},\bar{b})\| \tag{4.28}$$

for all $(c,b) \in B_\mu(\bar{c},\bar{b})$ and $x \in \mathscr{S}(c,b) \cap B_\delta(\bar{x})$.

(ii) If $x^n \to \bar{x}$ with $f(x^n) \downarrow 0$, then

(a) there exists a sequence $\{b^n\}_{n\in\mathbb{N}} \subset C(T,\mathbb{R})$ such that $x^n \in \mathscr{F}(b^n)$ and $\|b^n - \bar{b}\|_\infty \le Nf(x^n)$ for some $N > 0$ and all $n \in \mathbb{N}$;

(b) there exist a finite subset $T_0 \subset \cap_{n\in\mathbb{N}} T_{b^n}(x^n)$, and $\gamma_t > 0$, $u_t \in \partial g_t(\bar{x})$ $(t \in T_0)$ and $u \in \partial \psi(\bar{x})$ such that $-(\bar{c}+u) \in \sum_{t\in T_0} \gamma_t u_t$.

The next proposition extending [137, Propositions 4.4] gives sufficient conditions for nonlinear calmness of the level set mapping $\mathscr{L}$.

Proposition 54 Let $P(\bar{c},\bar{b})$ satisfy the Slater condition and $\varphi \in \mathscr{C}^1$ satisfy the following condition:

$$\forall N > 0 \quad \exists \gamma > 0 \quad \text{such that} \quad \frac{\varphi(Nt)}{t} \le \gamma \varphi'(t) \quad \text{for all} \quad t > 0. \tag{4.29}$$

If $\mathscr{L}$ is not $(\alpha\varphi)-$calm at $((\psi(\bar{x}) + \langle \bar{c},\bar{x}\rangle,\bar{b}),\bar{x})$ for all $\alpha > 0$, then there exist sequences $x^n \to \bar{x}$ and $\{b^n\}_{n\in\mathbb{N}} \subset C(T,\mathbb{R})$ such that $x^n \in \mathscr{F}(b^n)$ $(n \in \mathbb{N})$, $f(x^n) \downarrow 0$ and

$$\lim_{n\to+\infty} \frac{\varphi(\|b^n - \bar{b}\|_\infty)}{d(x^n, \mathscr{S}_{\bar{c}}(\bar{b}))} = 0. \tag{4.30}$$

Proof Suppose $\mathscr{L}$ is not $(\alpha\varphi)-$calm at $((\psi(\bar{x}) + \langle \bar{c},\bar{x}\rangle,\bar{b}),\bar{x})$ for all $\alpha > 0$. By Corollary 96(ii), there exists a sequence $x^n \to \bar{x}$ with $f(x^n) \downarrow 0$ such that

$$\lim_{n\to+\infty} \varphi'(f(x^n))|\partial f|(x^n) = 0.$$

By Lemma 2(iv),

$$|\partial f|(x^n) = |\nabla f|^\diamond(x^n) \ge \sup_{u\in\mathscr{S}(\bar{c},\bar{b})} \frac{f(x^n)}{\|x^n-u\|} = \frac{f(x^n)}{d(x^n,\mathscr{S}(\bar{c},\bar{b}))}, \quad n \in \mathbb{N}.$$

By Proposition 53(ii), there exist a sequence $\{b^n\}_{n\in\mathbb{N}} \subset C(T,\mathbb{R})$ and a number $N \in \mathbb{N}$ such that $x^n \in \mathscr{F}(b^n)$ and $\|b^n - \bar{b}\|_\infty \le N f(x^n)$ for all $n \in \mathbb{N}$. Then, with $\gamma > 0$ corresponding to N in view of condition (4.29), we have

$$\begin{aligned} 0 &\le \lim_{n\to+\infty} \frac{\varphi(\|b^n-\bar{b}\|_\infty)}{d(x^n,\mathscr{S}(\bar{c},\bar{b}))} \le \lim_{n\to+\infty} |\partial f|(x^n) \frac{\varphi(\|b^n-\bar{b}\|_\infty)}{f(x^n)} \\ &\le \lim_{n\to+\infty} |\partial f|(x^n) \frac{\varphi(Nf(x^n))}{f(x^n)} \le \gamma \lim_{n\to+\infty} |\partial f|(x^n)\varphi'(f(x^n)) = 0. \end{aligned}$$

This completes the proof. □

Remark 71 (i) Condition (4.29) implies condition (4.14). It is satisfied, e.g., in the Hölder case, i.e. when $\varphi(t) = \tau^{-1}t^q$ for some $\tau > 0$ and $q \in]0,1]$, or more generally, when $\varphi(t) = \tau^{-1}(t^q + \beta t)$ for some $\tau > 0$, $\beta > 0$ and $q \in]0,1]$.

(ii) Condition (4.29) can be replaced by the following weaker condition: for any $N > 0$, $\limsup_{t\downarrow 0} \frac{\varphi(Nt)}{t\varphi'(t)} < +\infty$.

Proposition 55 Let $P(\bar{c},\bar{b})$ satisfy the Slater condition, and $\varphi \in \mathscr{C}$. If f admits a $\varphi-$error bound at $\bar{x}$ with some $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$, then there exists an $\alpha > 0$ such that $\mathscr{S}$ is $\phi_\alpha-$calm at $((\bar{c},\bar{b}),\bar{x})$ with some $\delta' \in]0,\delta[$ and $\mu' \in]0,\mu[$, where $\phi_\alpha(t) := \varphi(\alpha t)$ $(t \ge 0)$.

Proof Let f have a $\varphi-$error bound at $\bar{x}$ with some $\delta \in]0,+\infty]$ and $\mu \in]0,+\infty]$. By Proposition 53(i), there exist $\delta' \in]0,\delta[$, $\mu' \in]0,\mu[$ and $M > 0$ such that inequality (4.28) holds for all $(c,b) \in B_{\mu'}(\bar{c},\bar{b})$ and $x \in \mathscr{S}(c,b) \cap B_{\delta'}(\bar{x})$. It follows from (4.1), (4.27) and (4.28) that, for all $(c,b) \in B_{\mu'}(\bar{c},\bar{b})$ and $x \in \mathscr{S}(c,b) \cap B_{\delta'}(\bar{x}) \cap [f > 0]$ (hence, $x \in \mathscr{F}(b)$),

$$\begin{aligned} d(x,\mathscr{S}(\bar{c},\bar{b})) =& d(x,[f \leq 0]) \leq \varphi(f(x)) \\ \leq& \varphi(\max\{[\psi(x) - \psi(\bar{x}) + \langle \bar{c}, x - \bar{x}\rangle]_+, \sup_{t\in T}[b_t - \bar{b}_t]_+\}) \\ \leq& \varphi(\max\{M,\mu'\}\|(c-\bar{c},b-\bar{b})\|) = \varphi(\alpha\|(c-\bar{c},b-\bar{b})\|), \end{aligned}$$

where $\alpha := \max\{M,\mu'\}$. Hence, $\mathscr{S}$ is $\phi_\alpha-$calm at $((\bar{c},\bar{b}),\bar{x})$ with δ' and μ'. □

Proposition 56 Let ψ and g_t $(t \in T)$ be linear, $P(\bar{c},\bar{b})$ satisfy the Slater condition, and $\varphi \in \mathscr{C}^1$ satisfy condition (4.29). If $\mathscr{S}_{\bar{c}}$ is $\varphi-$calm at $(\bar{b},\bar{x})$, then there exists an $\alpha > 0$ such that $\mathscr{L}$ is $(\alpha\varphi)-$calm at $((\psi(\bar{x}) + \langle \bar{c},\bar{x}\rangle,\bar{b}),\bar{x})$.

Proof Suppose $\mathscr{L}$ is not $(\alpha\varphi)-$calm at $((\psi(\bar{x}) + \langle \bar{c},\bar{x}\rangle,\bar{b}),\bar{x})$ for all $\alpha > 0$. By Proposition 53(ii) and Proposition 54, there exist sequences $x^n \to \bar{x}$ and $b^n \to \bar{b}$ such that $x^n \in \mathscr{F}(b^n)$ $(n \in \mathbb{N})$, and conditions (4.30) and (b) in Proposition 53(ii) are satisfied. By continuity, we can assume that $P(\bar{c},b^n)$ satisfies the Slater condition for all sufficiently large n. It readily follows from condition (b) in Proposition 53(ii) in the linear setting that $x^n \in \mathscr{S}_{\bar{c}}(b^n)$ for all sufficiently large n. Thus, $\mathscr{S}_{\bar{c}}$ is not $\varphi-$calm at $(\bar{b},\bar{x})$. □

Future Research

In this section, we briefly describe some potential research directions on the topic of transversality, regularity and error bounds.

1) Borwein et al. [35] recently examine convergence rates of the cyclic projection algorithm for solving semialgebraic convex feasibility problems under the Hölder subtransversality qualification condition. This approach is further investigated by Drusvyatskiy et al. [84] in the convex ill-posed semidefinite setting. It would be good to analyze the aforementioned models in the case when the convexity assumption is dropped. We think that the the nonlinear (in particular, Hölder) transversality property (a specific property) can be employed for the convergence analysis of these algorithms.

2) The affine hull regularity property is originally defined by Phan [195] and subsequently used by Dao and Phan [72] as a qualification condition for the linear convergence of generalized Douglas-Rachford (DR) algorithms. It would be interesting to study the Hölder version of the property by formulating necessary and sufficient conditions for the property in terms of primal and dual objects based on the generalized differentiation theory. The property then will be applied to convergence analysis of the DR algorithm and its variants for solving nonconvex feasibility problems.

3) The growing interest in computational algorithms and optimization leads to the development of transversality/regularity theory. While subtransversality and transversality have been studied for decades, their siblings such as affine hull regularity [195], intrinsic transversality [83, 134, 205], and tangential transversality [30] have come to life very recently. It is well known that transversality implies the three properties which imply subtransversality, and intrinsic transversality is equivalent to subtransversality in the convex setting. To make the picture of the transversality theory clear (hopefully complete), we believe that the following questions should be taken into account for future clarifications.

    (a) What are the connections of affine hull regularity/intrinsic transversality/tangential transversality?

    (b) What can be said about the above relations when the sets under consideration are convex and/or the underlying space is finite-dimensional?

(c) Can primal (slope, geometric, metric) and dual (subdifferential, coderivative, normal cone) characterizations of the tangential transversality be formulated?

4) Metric regularity properties of set-valued mappings and their variants have been topics of intensive study in the literature. However, most of the existing results are dedicated to the very general setting when $F : X \rightrightarrows Y$ is an arbitrarily set-valued mapping between metric spaces. It would be useful to analyze particular cases when the set-valued mapping under consideration posses one of the following structures:

   (a) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ where $\operatorname{gph} F$ is a semialgebraic (in particular, polyhedral) set in $\mathbb{R}^{n+m}$;

   (b) $F := f + N_\Omega$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a single-valued mapping, Ω is a given convex set in $\mathbb{R}^n$, and N_Ω stands the normal cone in the sense of convex analysis.

   The established results are expected to profit from these specific structures. Part (a) is motivated by recent achievements by Lee and Pham [147], while part (b) is a continuation of our recent work [71].

5) Recently, Karkhaneei and Mahdavi-Amiri [123] study necessary and/or sufficient conditions for weak sharp minima of extended-real-valued functions on manifolds, which improve the previous work of Li et al. [151]. It is standard that error bounds and weak sharp minima are strongly connected. Hence, we think that it is possible to establish (primal and dual) characterizations of error bounds of extended-real-valued [66] and vector-valued functions [28] on manifolds in both linear and nonlinear settings by employing variational analysis techniques on manifolds [94]. The established results can be used to analyze convergence rates of computational methods on manifolds.

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