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An Existence Result for Quasi-Equilibrium Problems via Ekeland's Variational Principle

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Abstract This paper deals with the existence of solutions to equilibrium and quasi-equilibrium problems without any convexity assumption. Coverage includes some equivalences to the Ekeland variational principle for bifunctions and basic facts about transfer lower continuity. An application is given to systems of quasi-equilibrium problems.

Keywords Quasi-equilibrium problem, System of quasi-equilibrium problem, Ekeland variational principle, Transfer lower continuity.

Mathematics Subject Classification (2000) 58E30 · 54E50 · 49J40 · 49J27

1 Introduction

To the best of our knowledge, the first appearance of equilibrium problems as we understand them now is due to Muu and Oettli [1] and it was further developed by Blum and Oettli [2]. They are conceptually connected to Ky Fan's minimax inequality [3] which goes back to the equality result of von Neumann [4].

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The term "equilibrium" emphasizes the broad applications that can be analyzed under this framework, such as optimization, variational inequalities, fixed point theory, Nash equilibria, among others, see for instance [5]. Motivated by these applications, many authors have been increasingly interested in studying conditions for the existence of solutions of equilibrium problems. Also, there exists a large literature about existence involving generalized convexity and generalized monotonicity assumptions, see for instance [6–13] and their references therein. The main concepts used in these problems involve the famous minimax inequality due to Ky Fan [3] and the Fan-KKM lemma [14], or the Brézis-Browder-Stampacchia result [15].

Most of the results concerning existence of solutions of equilibrium problems without any convexity assumptions use Ekeland's variational principle, see [16, 17]. Indeed, this principle or its equivalents (see [18]) is a key tool in the theory of variational analysis. Ekeland's result uses completeness of the space as well as the lower semicontinuity of the function under consideration to establish the existence of approximate solutions of minimization problems.

Our main interest concerns quasi-equilibrium problems, that is equilibrium problems with a constraint set depending on the current point. These problems originate from quasi-variational inequalities, that were introduced by Bensoussan and Lions as mentioned by Mosco [19]. Quasi-equilibrium problems have gained more attention lately, perhaps because they model generalized Nash equilibrium problems, which in turn model a large number of real life problems, see e.g. [20] and its references therein. Recent works on the existence of solutions for this kind of problem involving convexity assumptions are given in [21–25]. In [26] an existence result was provided for quasi-equilibrium problems, without any convexity condition, via Ekeland's variational principle.

We aim to move beyond lower semicontinuity by using a weaker notion called *transfer lower continuity* introduced by Tian and Zhou [27] in the field of mathematical analysis for the study of generalizations of the Weierstrass and maximum theorems.

The rest of this article is organized in five main sections. The next section is devoted to the concept of transfer lower continuity, while in Section 3 we establish, for non necessarily lower semicontinuous functions, an Ekeland-type theorem which involves the lower semicontinuous regularization of the given functional. We show that this theorem is equivalent to a theorem by Bianchi et al [16] established for equilibrium problems.

Section 4 is concerned with new existence results of equilibria and quasi-equilibria.

In Section 5 we specialize on systems of quasi-equilibrium problems in complete metric spaces. The origins of the interest for this kind of systems go back to generalized games and the work by Debreu (see [28]). These systems interact with various fields such as economics where they have concrete applications. This is the reason why their study has been developed by several authors, including for instance [29, 30], in connection with systems of quasi-variational inequalities or with systems of vector quasi-equilibrium problems. Our main result in this section is Theorem 5.3 that guarantees the existence of a solution to a system of quasi-equilibrium problems in complete metric spaces.

2 Definitions, Notation and Preliminaries Results

In this section we introduce and remind tools that will be useful throughout the paper and we will use standard notations and terminology from real analysis.

Given a nonempty subset C of a topological space X, a function $h : C \to \mathbb{R}$ is said to be *lower semicontinuous*[1] (lsc for short) if, for each $x \in C$ and each $\lambda \in \mathbb{R}$ such that $h(x) > \lambda$, there exists a neighbourhood V_x of x such that $h(x') > \lambda$, for all $x' \in V_x \cap C$.

Tian and Zhou ([27]) introduced the notion of *transfer lower continuity* (tls, for short). We say that f is tlc if for each $x, y \in C$ such that $h(x) > h(y)$, there exist $y' \in C$ and V_x a neighbourhood of x such that $h(x') > h(y')$, for all $x' \in V_x \cap C$. Trivially, lsc implies tlc.

Given h and $\lambda \in \mathbb{R}$, we denote by $\text{Epi}\, h$ and $S_h(\lambda)$ the *epigraph* and the *lower sub-level* set at level λ of h, respectively, i.e.,

$$\text{Epi}\, h := \{(x,\lambda) \in C \times \mathbb{R} : \ h(x) \le \lambda\} \ \text{ and } \ S_h(\lambda) := \{x \in C : \ h(x) \le \lambda\}.$$

It is well known that a function is lsc if and only if $\text{Epi}\, h$ is closed in $C \times \mathbb{R}$ or equivalently, if and only if $S_h(\lambda)$ is closed in C, for all $\lambda \in \mathbb{R}$, see for instance [32].

We will write $S_h(x)$ instead of $S_h(h(x))$ in order to simplify the notation. Thanks to [27, Lemma 1 and Remark 7] a function h is tlc if and only if

$$\bigcap_{x \in C} S_h(x) = \bigcap_{x \in C} \overline{S_h(x)}, \tag{1}$$

where $\overline{S_h(x)}$ is the closure of $S_h(x)$ in C.

Note that contrary to lower semicontinuity, transfer lower continuity is not closed under addition as the following simple example shows.

Example 2.1 Let $h, g : \mathbb{R} \to \mathbb{R}$ be two functions defined as

$$h(x) := \begin{cases} x+1, \ x < 0 \\ \ \ 2, \quad x = 0 \\ x+3, \ x > 0 \end{cases} \ \text{ and } \ g(x) := -x.$$

It is not hard to observe that both functions are transfer lower continuous. However, $h+g$ fails to be transfer lower continuous. Indeed, the sum function is given by

$$(h+g)(x) = \begin{cases} 1, \ x < 0 \\ 2, \ x = 0 \\ 3, \ x > 0 \end{cases}$$

whose graph is represented in Figure 1.

From Figure 1 the function $h+g$ is not transfer lower continuous at 0.

[1] Introduced by R. Baire, see [31] and the references therein.

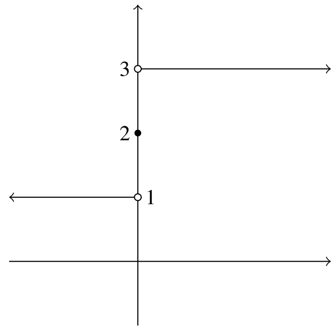

Fig. 1 graph of $h+g$

Given a nonempty subset C of a topological space X and a function $h : C \to \mathbb{R}$, we consider the *minimization problem:*

$$\text{Find } x \in C \text{ such that } h(x) \leq h(y), \text{ for all } y \in C.$$

We denote by $\operatorname{argmin}_C h$ the solution set of the minimization problem associated to h and C, i.e., $\operatorname{argmin}_C h = \{x \in C : \text{ such that } h(x) = \inf_C h\}$. It is important to notice that

$$\operatorname*{argmin}_C h = \bigcap_{x \in C} S_h(x) = \bigcap_{\lambda > \alpha} S_h(\lambda), \tag{2}$$

where $\alpha = \inf_{x \in C} h(x)$. Additionally, If $\operatorname{argmin}_C h = \emptyset$, then

$$\bigcap_{x \in C} \overline{S_h(x)} = \bigcap_{\lambda > \alpha} \overline{S_h(\lambda)}. \tag{3}$$

The following result is an extension of the celebrated Weierstrass theorem.

Theorem 2.1 **([27, Theorem 2])** *Let C be a compact and nonempty subset of a topological space X, and $h : C \to \mathbb{R}$ be a function. Then, the set* $\operatorname{argmin}_C h$ *is nonempty and compact if and only if h is tlc.*

Given a nonempty subset C of a topological space X, it is a basic fact from real analysis that every function $h : C \to \mathbb{R}$ (not necessarily lsc) admits a *lower semicontinuous regularization* $\overline{h} : C \to \mathbb{R} \cup \{-\infty\}$ defined by $\operatorname{Epi} \overline{h} := \overline{\operatorname{Epi} h}$, the closure in $C \times \mathbb{R}$, or equivalently by $\bar{h}(x) = \liminf_{y \to x} h(y) = \sup_U \inf_{y \in U} h(y)$, where U runs over all neighbourhoods of x.

It is well known that for any $x \in C$ and any $\lambda \in \mathbb{R}$

(i) $\overline{h}(x) = \inf\{\lambda \in \mathbb{R} : x \in \overline{S_\lambda(h)}\}$;
(ii) $\overline{h}(x) \leq h(x)$;
(iii) $\overline{S_h(x)} \subset S_{\overline{h}}(x)$;
(iv) $S_{\overline{h}}(\lambda) = \bigcap_{\mu > \lambda} \overline{S_h(\mu)}$;

We will say that a lower semicontinuous regularization is well-defined if it is real valued, that means $\overline{h}(x) > -\infty$, for all $x \in C$, or in other words, if h admits a lsc minorant.

We present now some basic results on transfer lower continuity and on lower semicontinuous regularizations.

Proposition 2.1 *Let C be a nonempty subset of a topological space X and $h : C \to \mathbb{R}$ be a function. If h is tlc, then its lower semicontinuous regularization is well-defined.*

Proof It is enough to consider the case when h is not bounded from below. Then, for each $x \in C$ there exists $y \in C$ such that $h(x) > h(y)$. Since h is tlc there exist V_x a neighbourhood of x and $y' \in C$ such that $h(x') > h(y')$, for all $x' \in V_x$, which in turn implies $\overline{h}(x) \geq h(y')$. Therefore, $\overline{h}$ is well-defined. □

Proposition 2.2 *Let C be a nonempty subset of a topological space X and $h : C \to \mathbb{R}$ be a function. Then, the following holds*

$$\inf_{x \in C} h(x) = \inf_{x \in C} \overline{h}(x).$$

Proof It is clear that $\inf_{x \in C} h(x) \geq \inf_{x \in C} \overline{h}(x)$. If we suppose that

$$\inf_{x \in C} h(x) > \inf_{x \in C} \overline{h}(x),$$

then, there exists $x_0 \in C$ such that $\overline{h}(x_0) < \inf_{x \in C} h(x)$. Thus, there exists $\lambda \in \mathbb{R}$ such that $\lambda < \inf_{x \in C} h(x)$ and $x_0 \in \overline{S_h(\lambda)}$. Now, for each V_{x_0} neighbourhood of x_0 there exists $x' \in V_{x_0} \cap S_h(\lambda)$, so $h(x') \leq \lambda$, which is a contradiction. □

Proposition 2.3 *Let C be a nonempty subset of a topological space X and $h : C \to \mathbb{R}$ be a function. If* $\operatorname{argmin}_C h = \operatorname{argmin}_C \overline{h}$*, then h is tlc. The converse holds, provided that* $\operatorname{argmin}_C h = \emptyset$.

Proof Since $\overline{S_h(x)} \subset S_{\overline{h}}(x)$, for all $x \in C$, by (2) we have

$$\bigcap_{x \in C} \overline{S_h(x)} \subset \bigcap_{x \in C} S_{\overline{h}}(x) = \operatorname*{argmin}_C \overline{h} = \operatorname*{argmin}_C h = \bigcap_{x \in C} S_h(x).$$

Hence, h is transfer lower continuous.

Conversely, by (3) we have

$$\bigcap_{x \in C} S_{\overline{h}}(x) = \bigcap_{\lambda > \alpha} S_{\overline{h}}(\lambda) = \bigcap_{\lambda > \alpha} \left(\bigcap_{\mu > \lambda} \overline{S_h(\mu)} \right) = \bigcap_{\lambda > \alpha} \overline{S_h(\lambda)} = \bigcap_{x \in C} \overline{S_h(x)},$$

where $\alpha = \inf_{x \in C} h(x)$. The result follows from (2) and the transfer lower continuity of h. □

The following example shows that the converse of the previous result is not true in general.

Example 2.2 Let $C = \mathbb{Q}_+$, and $h : C \to \mathbb{R}$ be defined as

$$h(x) := \begin{cases} 0, & x = 0 \\ \dfrac{1}{q}, & x = \dfrac{p}{q} \text{ with } p, q \in \mathbb{N} \text{ coprime} \end{cases}.$$

Since $S_h(0) = \{0\}$ and $0 \in S_h(x)$, for all $x \in C$, we can deduce that h is tlc. Moreover, $\text{argmin}_C h = \{0\}$. On the other hand, the lower semicontinuous regularization of h is the constant function $\overline{h}(x) = 0$, and this implies $\text{argmin}_C \overline{h} = C$.

It is clear that $\text{argmin}_C h = \text{argmin}_C \overline{h}$ when h is lower semicontinuous. However, as the following example shows, the converse is not true in general.

Example 2.3 Let $C = [0,1]$, and $h : C \to \mathbb{R}$ be defined as

$$h(x) := \begin{cases} x, & x \text{ is a rational number,} \\ x+1, & \text{otherwise} \end{cases}.$$

Clearly, h is not lower semicontinuous and its lower semicontinuous regularization is given by $\overline{h}(x) = x$. Moreover, $\text{argmin}_C h = \text{argmin}_C \overline{h} = \{0\}$.

Proposition 2.4 *Let C be a nonempty subset of a topological space X and $h : C \to \mathbb{R}$ be a function such that* $\text{argmin}_C h = \text{argmin}_C \overline{h}$. *If there exists $x \in C$ such that*

$$\overline{h}(x) \leq h(y), \quad \textit{for all } y \in C;$$

then $x \in$ $\text{argmin}_C h$.

Proof By Proposition 2.3, h is tlc, and by Proposition 2.1 its lower semicontinuous regularization $\overline{h}$ is well-defined. Now, it is clear that $\text{Epi}(h) \subset C \times \left[\overline{h}(x), +\infty\right[$. Thus, we deduce that $x \in \text{argmin}_C \overline{h}$. The result follows. □

Now, we recall some definitions for bifunctions. Given a topological space X and $C \subset X$, a bifunction $f : C \times C \to \mathbb{R}$ is said:

- to have the *triangle inequality property* on C if, for all $x, y, z \in C$ the following holds
$$f(x,y) \leq f(x,z) + f(z,y);$$
- to be *cyclically monotone* on C if, for all $n \in \mathbb{N}$ and all $x_0, x_1, \ldots, x_n \in C$ the following holds ,
$$\sum_{i=0}^{n} f(x_i, x_{i+1}) \leq 0,$$
with $x_{n+1} = x_0$;
- to be *monotone* on C if, for all $x, y \in C$ the following holds
$$f(x,y) + f(y,x) \leq 0;$$
- to be *pseudo-monotone* on C if, for all $x, y \in C$ the following implication holds
$$f(x,y) \geq 0 \quad \Longrightarrow \quad f(y,x) \leq 0;$$

The concept of cyclic monotonicity for bifunctions appeared first in [16]. Many authors studied its properties, see for instance [26, 33, 34]. Recently in [26, 35] the authors used cyclic monotonicity in order to solve equilibrium and quasi-equilibrium problems.

It is clear that cyclic monotonicity implies monotonicity which in turn implies pseudo-monotonicity. Important instances of these kinds of bifunctions are given below.

Example 2.4 Let C be a nonempty subset of a topological space X and $h : C \to \mathbb{R}$ be a function with a well-defined lower semicontinuous regularization. Let us consider two bifunctions $g, f : C \times C \to \mathbb{R}$ defined as

$$g(x,y) := h(y) - h(x) \text{ and } f(x,y) := \overline{h}(y) - h(x). \tag{4}$$

Clearly, f is cyclically monotone and g satisfies the triangular inequality property. Moreover, the following inequality holds: $g \geq f$, that means $g(x,y) \geq f(x,y)$, for all $x, y \in C$.

Given a bifunction $f : C \times C \to \mathbb{R}$, we consider the bifunction $\hat{f} : C \times C \to \mathbb{R}$ given by

$$\hat{f}(x,y) := -f(y,x).$$

Due to [16, Remark 2.2], we note that if f verifies the triangle inequality property, then $\hat{f}$ is cyclically monotone. By [33, Proposition 5.1] cyclic monotonicity of $\hat{f}$ is equivalent to the existence of a function $h : C \to \mathbb{R}$ such that

$$\hat{f}(x,y) \leq h(y) - h(x), \ \forall x, y \in C.$$

For such a function, it is not difficult to check that

$$f(x,y) \geq h(y) - h(x) \geq \hat{f}(x,y).$$

Additionally, if f is monotone, then $f(x,y) = h(y) - h(x)$, for all $x, y \in C$. Hence, f is cyclically monotone.

The following result says that there is a strong relationship between monotonicity and pseudo-monotonicity.

Proposition 2.5 *Let C be a subset of a topological space X and $f : C \times C \to \mathbb{R}$ be a bifunction from $C \times C$ into $\mathbb{R}$. Then, f is pseudo-monotone if and only if there are bifunctions $f_1, f_2 : C \times C \to \mathbb{R}$ satisfying*

$$f(x,y) = f_1(x,y) f_2(x,y), \tag{5}$$

where f_1 is strictly positive and f_2 is monotone.

Proof Assume that f is pseudo-monotone. We denote by D the subset of $C \times C$ where f vanishes. Define bifunctions $f_1, f_2 : C \times C \to \mathbb{R}$ by

$$f_1(x,y) := \begin{cases} 1, & (x,y) \in D \\ |f(x,y)|, & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(x,y) := \operatorname{sign}(f(x,y)).$$

It is clear that f_1 is strictly positive. We affirm that f_2 is monotone. Indeed, for each $x,y \in C$ we have $f_2(x,y) \in \{-1,0,1\}$. So, if $f_2(x,y) = -1$, then it is obvious that $f_2(x,y) + f_2(y,x) \leq 0$. If $f_2(x,y) = 0$ then $f(x,y) = 0$, which in turn implies $f(y,x) \leq 0$ due the pseudo-monotonicity of f. Thus, $f_2(y,x) \in \{-1,0\}$ and this allows us to conclude that $f_2(x,y)+f_2(y,x) \leq 0$. Finally, if $f_2(x,y) = 1$, that means $f(x,y) > 0$, then again by pseudo-monotonicity of f we have $f(y,x) < 0$, in other words $f_2(y,x) = -1$. Hence, $f_2(x,y) + f_2(y,x) = 0$.

The converse is not difficult to prove. □

Remark 2.1 In a similar way to [36, Theorem 2.1], let C be a subset of a topological space X, and let $f : C \times C \to \mathbb{R}$ be a bifunction. If f and $\hat{f}$ are both pseudo-monotone on C, then for all $x,y \in C$ the following equivalence holds

$$f(x,y) = 0 \iff \hat{f}(x,y) = 0.$$

Hence, (x,y) is an element of the set D, where f vanishes, if and only if $(y,x) \in D$. In other words, D is symmetric.

However, contrary to [36, Theorem 2.1] , the converse does not hold, even if we assume the continuity of f. Indeed, consider the bifunction $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$f(x,y) := (y-x)^2.$$

Clearly, f is continuous. Moreover, $f(x,y) = 0$ if and only if $y = x$, which in turn is equivalent to $f(y,x) = 0$. But, the bifunction f is not pseudo-monotone, due to the fact that $f(1,0) = f(0,1) = 1$.

As a direct consequence of the result above, we have the following corollary, which was inspired from [37, Theorem 1.4] for maps in the setting of finite dimensional spaces.

Corollary 2.1 *Let C be a subset of a topological space X and $f : C \times C \to \mathbb{R}$ be a bifunction. Then, f, $\hat{f}$ are pseudo-monotone if and only if there are bifunctions $f_1, f_2 : C \times C \to \mathbb{R}$ such that f_1 is strictly positive and $f_2, \hat{f}_2$ are monotone, satisfying* (5).

3 The Ekeland Variational Principle

We begin this section recalling the celebrated Ekeland variational principle and then its extension to equilibrium problems.

Theorem 3.1 (The Ekeland variational principle [38]) *Let C be a nonempty closed subset of the complete metric space (X,d), and $h : C \to \mathbb{R}$ be a lsc function bounded from below. For every $\varepsilon > 0$, and for any $x_0 \in C$, there exists $\hat{x} \in C$ such that*

$h(\hat{x}) + \varepsilon d(x_0,\hat{x}) \leq h(x_0)$, *and*
$h(x) + \varepsilon d(x,\hat{x}) > h(\hat{x})$, *for all* $x \in C \setminus \{\hat{x}\}$.

Theorem 3.2 (**[16, Theorem 2.1] and [17, Theorem 2.2]**) *Let C be a nonempty closed subset of a complete metric space (X,d) and $f : C \times C \to \mathbb{R}$ be a bifunction. Assume that the following conditions hold*

(i) *f is bounded from below and lsc with respect to its second argument;*
(ii) *$f(x,x) = 0$, for all $x \in C$;*
(iii) *f satisfies the triangle inequality property.*

Then, for all $\varepsilon > 0$ and all $x_0 \in C$, there exists $\hat{x} \in C$ such that

$f(x_0,\hat{x}) + \varepsilon d(x_0,\hat{x}) \leq 0,$
$f(\hat{x},x) + \varepsilon d(x,\hat{x}) > 0$ *for every* $x \in C \setminus \{\hat{x}\}$.

Let us restate Theorem 3.1 in terms of lower semicontinuous regularizations.

Theorem 3.3 *Let C be a nonempty closed subset of the complete metric space (X,d), and $h : C \to \mathbb{R}$ be a function bounded from below. For every $\varepsilon > 0$, and for any $x_0 \in C$, there exists $\hat{x} \in C$ such that*

$\overline{h}(\hat{x}) + \varepsilon d(x_0,\hat{x}) \leq h(x_0)$, *and*
$h(x) + \varepsilon d(x,\hat{x}) > \overline{h}(\hat{x})$, *for all* $x \in C \setminus \{\hat{x}\}$.

For the sake of completeness, we give a self-contained proof of Theorem 3.3 which mimics the proof of Ekeland's Theorem.

Proof Since h is bounded from below, its lsc regularization $\overline{h}$ is well-defined. Without loss of generality we consider $\varepsilon = 1$. Denote by $H(x)$ the set

$$H(x) := \{y \in C : \ \overline{h}(y) + d(y,x) \leq \overline{h}(x)\}.$$

Since the distance is continuous and $\overline{h}$ is lsc, the set $H(x)$ is closed, for every $x \in C$. Moreover, $x \in H(x)$. For each $y \in H(x)$ and any $z \in H(y)$, it is easy to verify that $z \in H(x)$. Hence $y \in H(x)$ implies $H(y) \subset H(x)$. Define $r(x) := \inf_{z \in H(x)} \overline{h}(z)$. For each $z \in H(x)$ we have $\overline{h}(z) + d(z,x) \leq \overline{h}(x)$, which in turn implies $d(x,z) \leq \overline{h}(x) - r(x)$. So, for any $z_1, z_2 \in H(x)$

$$d(z_1,z_2) \leq d(z_1,x) + d(x,z_2) \leq 2(\overline{h}(x) - r(x)).$$

Thus the diameter $\operatorname{diam}(H(x))$ of $H(x)$ satisfies:

$$\operatorname{diam}(H(x)) \leq 2(\overline{h}(x) - r(x)).$$

For $x_0 \in C$, there exists $x_1 \in H(x_0)$ such that

$$\overline{h}(x_1) \leq r(x_0) + \frac{1}{2}.$$

Now, for this x_1 there exists $x_2 \in H(x_1)$ such that

$$\overline{h}(x_2) \leq r(x_1) + \frac{1}{2^2}.$$

Inductively, we define a sequence $\{x_n\}$ of points in C such that $x_{n+1} \in H(x_n)$ and

$$\overline{h}(x_{n+1}) \leq r(x_n) + \frac{1}{2^{n+1}}. \tag{6}$$

On the other hand, we note that

$$r(x_{n+1}) = \inf_{z\in H(x_{n+1})} \overline{h}(z) \geq \inf_{z\in H(x_n)} \overline{h}(z) = r(x_n). \tag{7}$$

Combining (7) and (6) we obtain

$$r(x_n) \leq r(x_{n+1}) \leq \overline{h}(x_{n+1}) \leq r(x_n) + \frac{1}{2^{n+1}}.$$

Therefore, $\operatorname{diam}(H(x_{n+1})) \leq 2(\overline{h}(x_{n+1}) - r(x_{n+1})) \leq \frac{1}{2^n}$, for all $n \in \mathbb{N}$. As a consequence we deduce that

$$\bigcap_{n\in\mathbb{N}} H(x_n) = \{\hat{x}\}.$$

Since $\hat{x} \in H(x_0)$, we have $\overline{h}(\hat{x}) + d(\hat{x}, x_0) \leq \overline{h}(x_0) \leq h(x_0)$. Moreover $\hat{x} \in H(x_n)$ for all $n \in \mathbb{N}$ and since $H(\hat{x}) \subset H(x_n)$ we deduce that $H(\hat{x}) = \{\hat{x}\}$. As a result,

$$x \notin H(\hat{x}) \quad \text{if and only if} \quad x \neq \hat{x}.$$

Therefore, for any $x \in C \setminus \{\hat{x}\}$,

$$\overline{h}(\hat{x}) < \overline{h}(x) + d(x,\hat{x}) \leq h(x) + d(x,\hat{x}).$$

This completes the proof. □

The next theorem shows that the previous results are equivalent.

Theorem 3.4 *Theorems 3.1 through 3.3 are equivalent.*

Proof Theorem 3.1 $\Longleftrightarrow$ Theorem 3.2.

It is clear that Theorem 3.2 implies Theorem 3.1. Reciprocally, for each $\varepsilon > 0$ and $x_0 \in C$, Ekeland's variational principle applied to the function $f(x_0,\cdot)$ gives the existence of $\hat{x} \in C$ such that

$$f(x_0,\hat{x}) \leq f(x_0,x_0) - \varepsilon d(x_0,\hat{x}) \text{ and} \tag{8}$$

$$f(x_0,x) > f(x_0,\hat{x}) - \varepsilon d(\hat{x},x), \ \forall x \in C \setminus \{\hat{x}\}. \tag{9}$$

Since f vanishes on the diagonal of $C \times C$, inequality (8) reduces to

$$f(x_0,\hat{x}) + \varepsilon d(x_0,\hat{x}) \leq 0.$$

On the other hand, according to the triangle inequality property we have $f(x_0,x) \leq f(x_0,\hat{x}) + f(\hat{x},x)$. Thus, inequality (9) reduces to

$$f(\hat{x},x) + \varepsilon d(\hat{x},x) > 0, \ \forall x \in C \setminus \{\hat{x}\}.$$

For the equivalence of Theorem 3.1 and Theorem 3.3, we show that both implications are true.

The first implication follows from $\overline{h} \leq h$ and by applying Theorem 3.1 to $\overline{h}$. The converse follows by applying Theorem 3.3 to the lsc function h, and by remarking that $\overline{h} = h$. □

As a direct consequence of Theorem 3.2 we have the following corollary.

Corollary 3.1 *Let C be a nonempty closed subset of a complete metric space (X,d) and $f : C\times C \to \mathbb{R}$ be a bifunction. Assume that there exists a bifunction $g : C\times C \to \mathbb{R}$ such that:*

(i) $f \geq g$;
(ii) *g is bounded from below and lsc with respect to its second argument;*
(iii) *g vanishes on the diagonal of $C\times C$;*
(iv) *g satisfies the triangle inequality property.*

Then, for all $\varepsilon > 0$, and all $x_0 \in C$, there exists $\hat{x} \in C$ such that

$g(x_0,\hat{x}) + \varepsilon d(x_0,\hat{x}) \leq 0$, *and*
$f(\hat{x},x) + \varepsilon d(x,\hat{x}) > 0$, *for every $x \in C\setminus\{\hat{x}\}$.*

The conclusion of Corollary 3.1 is similar to the one in [26, Theorem 2.4], where instead of supposing that g satisfies the triangle inequality property, the authors considered g defined as in (4).

4 New Existence Results of Equilibria and Quasi-Equilibria

We begin this section by recalling the definitions of equilibrium and Minty equilibrium problems, respectively.

4.1 Equilibrium Problems

Let C be a nonempty subset of a topological space X and $f : C\times C \to \mathbb{R}$, be a given bifunction. We denote by $\mathrm{EP}(f,C)$ *the solution set* of the *equilibrium problem*, introduced by Blum and Oettli in [2],

$$\text{Find} \quad x \in C \quad \text{such that} \quad f(x,y) \geq 0, \text{ for all } y \in C. \tag{10}$$

In a similar way, $\mathrm{MEP}(f,C)$ denotes the *solution set* of the so-called *Minty equilibrium problem*

$$\text{Find} \quad x \in C \quad \text{such that} \quad f(y,x) \leq 0, \text{ for all } y \in C. \tag{11}$$

Clearly, they satisfy

$$\mathrm{EP}(f,C) = \mathrm{MEP}(\hat{f},C) \text{ and } \mathrm{EP}(\hat{f},C) = \mathrm{MEP}(f,C).$$

Provided that $f(x,y) \geq h(y) - h(x)$ for some function $h : C \to \mathbb{R}$, which implies that $\hat{f}$ is cyclically monotone, we may observe that

$$\mathrm{MEP}(f,C) \subset \operatorname*{argmin}_C h \subset \mathrm{EP}(f,C). \tag{12}$$

Moreover, if f is pseudo-monotone, then the above inclusions are actually equalities.

Remark 4.1 If the bifunction f vanishes on the diagonal of $C\times C$, then

$$x\in \mathrm{EP}(f,C)\Leftrightarrow x\in \operatorname*{argmin}_{C} f(x,\cdot) \ \text{ and } \ x\in \mathrm{MEP}(f,C)\Leftrightarrow x\in \operatorname*{argmin}_{C} \hat{f}(x,\cdot).$$

Moreover,

$$\mathrm{EP}(f,C)\subset \bigcup_{x\in C}\operatorname*{argmin}_{C} f(x,\cdot) \ \text{ and } \ \mathrm{MEP}(f,C)\subset \bigcup_{y\in C}\operatorname*{argmin}_{C} \hat{f}(y,\cdot).$$

Theorem 4.1 *Let C be a compact and nonempty subset of a topological space X, and $f: C\times C\to\mathbb{R}$ be a bifunction. If there exists a tlc function $h: C\to\mathbb{R}$ with*

$$f(x,y)\geq h(y)-h(x),\ \textit{for all } x,y\in C;$$

then, the set $\mathrm{EP}(f,C)$ *is nonempty.*

Proof From Theorem 2.1, the set $\operatorname{argmin}_C h$ is nonempty. The result follows from (12). □

The previous result was given in [26, Theorem 3.4], but instead of considering the transfer lower continuity of h, the authors assumed lower semicontinuity.

Example 4.1 Let $h:[0,2]\to\mathbb{R}$ be defined as

$$h(x):=\begin{cases} x, & 0\leq x<1\\ 2, & x=1\\ x+2, & 1<x\leq 2\end{cases}$$

Its graph is shown in Figure 2.

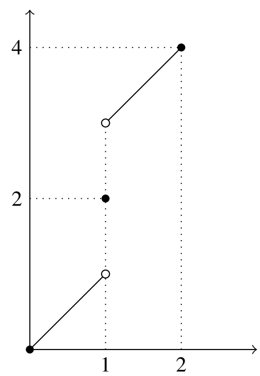

Fig. 2 graph of h

Clearly, h is not lsc. However, it is tlc. Indeed, since $S(x)=[0,x]$, for any $x\in[0,2]$, relation (1) implies that h is tlc. As the interval $[0,2]$ is a compact set, for any bifunction $f:[0,2]\times[0,2]\to\mathbb{R}$, which satisfies

$$f(x,y)\geq h(y)-h(x),$$

the set $\mathrm{EP}(f,[0,2])$ is nonempty, due to Theorem 4.1. It is important to notice that we cannot apply [26, Theorem 3.4]. Moreover, it is important to note that Theorem 4.1 is not a consequence of [39, Theorem 4.1], due to the lack of any continuity assumption for f.

As a direct consequence we have the following corollary, which is a generalization of [35, Theorem 3.1].

Corollary 4.1 *Let C be a compact and nonempty subset of a topological space X, and $f : C \times C \to \mathbb{R}$ be a bifunction. If there exists a tlc function $h : C \to \mathbb{R}$ with*

$$f(x,y) \leq h(y) - h(x), \text{ for all } x,y \in C;$$

then, the set $\mathrm{MEP}(f,C)$ is nonempty.

4.2 **Quasi-Equilibrium Problems**

Given a subset C of a complete metric space (X,d), a bifunction $f : C \times C \to \mathbb{R}$ and a set-valued mapping $K : C \rightrightarrows C$, we denote by $\mathrm{QEP}(f,K)$ the solution set of the so-called *quasi-equilibrium problem*:

$$\text{Find} \quad x \in C \quad \text{such that} \quad x \in K(x) \ \wedge \ f(x,y) \geq 0, \text{ for all } y \in K(x). \tag{13}$$

Lemma 4.1 *Let C be a nonempty closed subset of a complete metric space (X,d), $K : C \rightrightarrows C$ be a set-valued mapping and $h : C \to \mathbb{R}$ be a function bounded from below. We assume that for every $\varepsilon > 0$, and for any $x_0 \in C$ the following implication holds: for all $x \in C$*

$$\overline{h}(x) + \varepsilon d(x,x_0) \leq h(x_0) \implies \exists y \in K(x),\ h(y) + \varepsilon d(x,y) \leq \overline{h}(x).$$

Then, there exists $\hat{x} \in \mathrm{Fix}(K)$[2] satisfying

$\overline{h}(\hat{x}) + \varepsilon d(x_0,\hat{x}) \leq h(x_0)$, and
$h(x) + \varepsilon d(x,\hat{x}) > \overline{h}(\hat{x})$, for all $x \in C \setminus \{\hat{x}\}$.

Proof By Theorem 3.3, for each $\varepsilon > 0$ and x_0, there exists $\hat{x} \in C$ such that

$\overline{h}(\hat{x}) + \varepsilon d(x_0,\hat{x}) \leq h(x_0)$, and
$h(x) + \varepsilon d(x,\hat{x}) > \overline{h}(\hat{x})$, for all $x \in C \setminus \{\hat{x}\}$.

It is enough to show that $\hat{x}$ is a fixed point of K. From the first inequality and the assumed implication, there exists $y \in K(\hat{x})$ such that

$$h(y) + \varepsilon d(\hat{x},y) \leq \overline{h}(\hat{x}).$$

Supposing $y \neq \hat{x}$ leads to a contradiction with the second inequality, and therefore, we derive that $y = \hat{x} \in K(\hat{x})$. □

[2] $\mathrm{Fix}(K)$ denotes the set of fixed points of K.

Given f and K, we notice that if there exists a function $h : C \to \mathbb{R}$ such that $f(x,y) \geq h(y) - h(x)$ (in other words, $\hat{f}$ is cyclically monotone), then

$$\operatorname*{argmin}_C h \cap \operatorname{Fix}(K) \subset \operatorname{QEP}(f,K). \tag{14}$$

The following result is an extension of [26, Theorem 3.11].

Theorem 4.2 *Let C be a nonempty closed subset of a complete metric space (X,d), let $K : C \rightrightarrows C$ be a set-valued mapping, and let $f : C \times C \to \mathbb{R}$ be a bifunction. Let us assume that the following conditions hold.*

(i) $\operatorname{Fix}(K)$ *is compact and nonempty;*
(ii) *there exists a bounded from below function $h : C \to \mathbb{R}$ such that*

$$\operatorname*{argmin}_C h = \operatorname*{argmin}_C \overline{h} \ \ \text{and} \ \ f(x,y) \geq h(y) - h(x), \ \ \text{for all } x,y \in C.$$

Suppose that for each $\varepsilon > 0$ and each $x_0 \in X$ the following implication holds: for all $x \in C$

$$\overline{h}(x) + \varepsilon d(x,x_0) \leq h(x_0) \implies \exists y \in K(x), \ h(y) + \varepsilon d(x,y) \leq \overline{h}(x).$$

Then, the set $\operatorname{QEP}(f,K)$ *is nonempty.*

Proof Fix $x_0 \in C$. By Lemma 4.1, for each $n \in \mathbb{N}$, there exists $x_n \in \operatorname{Fix}(K)$ such that

$$h(x) + \frac{1}{n} d(x,x_n) \geq \overline{h}(x_n), \ \text{for all } x \in C.$$

Since $\operatorname{Fix}(K)$ is compact, without loss of generality, we can assume that $(x_n)_{n\in\mathbb{N}}$ converges to $\hat{x} \in \operatorname{Fix}(K)$. We claim that $\hat{x} \in \operatorname{argmin}_C h$. Indeed, as the distance function is continuous and $\overline{h}$ is lsc, we have

$$h(x) \geq \overline{h}(\hat{x}), \ \text{for all } x \in C.$$

By Proposition 2.4, $\hat{x} \in \operatorname{argmin}_C h$. The result follows from (14). □

It is important to note that the previous result is not a consequence of [39, Theorem 4.3], because neither h is lower semicontinouous, nor K is upper semicontinuous.

As a direct consequence of Theorem 4.2 we derive the following.

Corollary 4.2 *Let C be a nonempty closed subset of a complete metric space (X,d), $K : C \rightrightarrows C$ be a set-valued mapping, and let $h : C \to \mathbb{R}$ be a function. Let us assume that the following conditions hold.*

(i) $\operatorname{Fix}(K)$ *is compact and nonempty;*
(ii) *h is a function bounded from below such that* $\operatorname{argmin}_C h = \operatorname{argmin}_C \overline{h}$.

Suppose that for each $\varepsilon > 0$, and each $x_0 \in X$ the following implication holds: for all $x \in C$

$$\overline{h}(x) + \varepsilon d(x,x_0) \leq h(x_0) \implies \exists y \in K(x), \ h(y) + \varepsilon d(x,y) \leq \overline{h}(x).$$

Then, there exists $\hat{x} \in \operatorname{Fix}(K)$ such that

$$h(\hat{x}) \leq h(x), \ \textit{for all } x \in K(\hat{x}).$$

The previous result is known as the existence of solutions to a *quasi-optimization* problem. Important results about the existence of solution of this kind of problem were presented in [40, Propositions 4.2 and 4.5] and [25, Corollary 3.2] under continuity and quasi-convexity assumptions.

5 System of Quasi-Equilibrium Problems

Let I be an index set. For each $i \in I$, we consider a complete metric space (X_i, d_i), a nonempty closed subset C_i of X_i and a set-valued mapping $K_i : C_i \rightrightarrows C_i$. We define the set-valued mapping $K : C \rightrightarrows C$ by

$$K(x) := \prod_{i \in I} K_i(x^i),$$

where $C = \prod C_i$ and $x = (x^i)_{i \in I}$. By a *system of quasi-equilibrium problems* we understand the problem of finding

$$\hat{x} \in \text{Fix}(K) \text{ such that } f_i(\hat{x}, y^i) \geq 0 \text{ for all } y \in K(\hat{x}), \tag{15}$$

where the $f_i : C \times C_i \to \mathbb{R}$ are given. It is important to see that

$$\text{Fix}(K) = \prod_{i \in I} \text{Fix}(K_i).$$

In the particular case when for each $i \in I$, $K_i(x^i) = C_i$, for all $x^i \in C_i$, we obtain the known *system of equilibrium problems*.

The following result generalizes [26, Theorem 4.2], [17, Proposition 4.2] and [16, Proposition 2].

Theorem 5.1 *For each $i \in I$, let C_i be a nonempty compact subset of a topological space X_i, and let each $f_i : C \times C_i \to \mathbb{R}$ be a bifunction such that*

$$f_i(x, y^i) \geq h_i(y^i) - h_i(x^i), \ \forall x, y \in C \tag{16}$$

holds for some transfer lower continuous function $h_i : C_i \to \mathbb{R}$ that is also bounded from below. Then, the system of equilibrium problems admits at least one solution.

Proof For each $i \in I$, we apply Theorem 2.1 and obtain $\hat{x}^i \in \text{argmin}_{C_i} h_i$. Thus, from (16), $\hat{x} = (\hat{x}^i)$ is a solution of the system of equilibrium problems. □

Remark 5.1 Condition (16) is equivalent to the following: for any i from I and any positive integer m and any $x_1, x_2, \ldots, x_m \in C$ it holds

$$\sum_{j=1}^{m} f_i(x_j, x_{j+1}^i) \geq 0 \tag{17}$$

where $x_{m+1} = x_1$. It follows from the same steps of the proof of [33, Proposition 5.1].

We denote by $\mathrm{SEP}(f_i, C_i, I)$ the solution set of (15), when $K_i(x^i) = C_i$, for all $x^i \in C_i$. If I is a finite index set, as a particular case, we define the bifunction $f : C \times C \to \mathbb{R}$ by

$$f(x,y) := \sum_{i \in I} f_i(x, y^i). \tag{18}$$

The next result says that a system of equilibrium problems is equivalent to a particular equilibrium problem under suitable assumptions.

Proposition 5.1 *Assume that I is a finite index set and f is defined as (18). Then* $\mathrm{SEP}(f_i, C_i, I) \subset \mathrm{EP}(f, C)$*. The equality holds provided that $f_i(x, x^i) = 0$, for all $i \in I$.*

Proof Let $x \in \mathrm{SEP}(f_i, C_i, I)$ and $y \in C$. For each $i \in I$, we have

$$f_i(x, y^i) \geq 0.$$

Thus $f(x,y) \geq 0$. Hence $x \in \mathrm{EP}(f, C)$.

Conversely, let $x \in \mathrm{EP}(f, C)$, $i \in I$ and $y^i \in C_i$. We take $z \in C$ such that $z^i = y^i$ and $z^j = x^j$, for all $j \in I \setminus \{i\}$. So,

$$0 \leq f(x,y) = \sum_{j \in I} f_j(x, z^j) = f_i(x, y^i).$$

Therefore, $x \in \mathrm{SEP}(f_i, C_i, I)$. □

Given a finite index set I and for each $i \in I$, we consider a compact subset C_i of a topological space and a function $f_i : C \times C_i \to \mathbb{R}$. We say that the family of functions $\{f_i\}_{i \in I}$ has the *transfer lower continuity property* if there exists a tlc function $h : C \to \mathbb{R}$ such that the bifunction f defined in (18) satisfies

$$f(x,y) \geq h(y) - h(x). \tag{19}$$

Remark 5.2 Two remarks are needed.

(i) Let f be defined by (18), where the family $\{f_i\}_{i \in I}$ has the transfer lower continuity property. Then, the bifunction $\hat{f}$ is cyclically monotone.
(ii) If for each $i \in I$, the function f_i is usc in its second argument, and the relation (19) holds, then the family of functions $\{f_i\}_{i \in I}$ has the transfer lower continuity property. This is due to [26, Theorem 2.16].

Below we present a result similar to Theorem 5.1.

Theorem 5.2 *Assume that I is a finite index set and the family of functions $\{f_i\}_{i \in I}$ has the transfer lower continuity property. If $f_i(x, x^i) = 0$, for all $x \in C$ and all $i \in I$, then the set* $\mathrm{SEP}(f_i, C_i, I)$ *admits at least one element.*

Proof It follows from Theorem 4.1 and Proposition 5.1. □

After proving the existence of solutions to systems of equilibrium problems, we can conclude this section by turning our attention to systems of quasi-equilibrium problems. The proof of the next result follows the one in [26, Theorem 3.11].

Theorem 5.3 *For each $i \in I$, let C_i be a nonempty closed subset of a complete metric space (X_i, d_i), $K_i : C_i \rightrightarrows C_i$ be a set-valued mapping, and let $f_i : C \times C_i \to \mathbb{R}$ be a function such that (16) holds for some $h_i : C_i \to \mathbb{R}$, bounded from below such that* $\operatorname{argmin}_{C_i} h_i = \operatorname{argmin}_{C_i} \overline{h}_i$. *If* $\operatorname{Fix}(K)$ *is compact and, for any $\varepsilon > 0$, any $x_0 \in C$, and any $i \in I$ the following implication holds*

$$\overline{h}_i(x^i) + \varepsilon d_i(x^i, x_0^i) \le h_i(x_0^i) \implies \exists y^i \in K_i(x^i),\ h_i(y^i) + \varepsilon d_i(x^i, y^i) \le \overline{h}_i(x^i),$$

then (15) has a solution.

Proof For each $i \in I$, $x_0 \in C$, and $n \in \mathbb{N}$, we apply Lemma 4.1 and we obtain the existence of a fixed point of K_i, say $\hat{x}_n^i$, such that

$$h_i(x^i) + \frac{1}{n} d_i(x^i, \hat{x}_n^i) \ge \overline{h}_i(\hat{x}_n^i), \text{ for all } x^i \in C_i.$$

Since $\operatorname{Fix}(K_i)$ is compact, without loss of generality, assume that $(\hat{x}_n^i)_{n\in\mathbb{N}}$ converges to $\hat{x}^i \in \operatorname{Fix}(K_i)$. By continuity of d_i and lower semicontinuity of $\overline{h}_i$, we have

$$h_i(x^i) \ge \overline{h}_i(\hat{x}^i), \text{ for all } x_i \in C_i.$$

Due to Proposition 2.4, we deduce that $\hat{x}^i \in \operatorname{argmin}_{C_i} h_i$,. The result follows from considering $\hat{x} = (\hat{x}^i) \in \operatorname{Fix}(K)$ and (16). □

6 Conclusions

Our aim in the present paper was to study the existence of equilibria and quasi-equilibria, in the setting of metric spaces. We achieved this goal by using the Ekeland variational principle and by dropping usual convexity assumptions. Our results extend many results that can be found in the literature (e.g. [26]). We also proved the existence of solutions for systems of quasi-equilibrium problems in the setting of metric spaces.

Further research could be done regarding other types of transfer continuities such as transfer weakly lower continuity and quasi transfer lower continuity (see [27]). Natural extension of this work to generalized Nash equilibrium problems and quasi-variational inequalities could also be considered in the future.

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References

1. Muu, L., Oettli, W.: Convergence of an adaptive penalty scheme for finding constrained equilibria. Nonlinear Analysis: Theory, Methods & Applications **18**(12), 1159 – 1166 (1992). DOI https://doi.org/10.1016/0362-546X(92)90159-C. URL http://www.sciencedirect.com/science/article/pii/0362546X9290159C
2. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Student **63**(1-4), 123–145 (1994)
3. Fan, K.: A minimax inequality and applications. In: Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin), pp. 103–113 (1972)
4. von Neumann, J.: Zur theorie der gesellschaftsspiele. Mathematische Annalen **100**(1), 295–320 (1928). DOI 10.1007/BF01448847. URL https://doi.org/10.1007/BF01448847
5. Ansari, Q., Köbis, E., Yao, J.C.: Vector Variational Inequalities and Vector Optimization: Theory and Applications. Springer International (2017)
6. Bianchi, M., Pini, R.: A note on equilibrium problems with properly quasimonotone bifunctions. J. Global Optim. **20**, 67–76 (2001)
7. Iusem, A., Sosa, W.: New existence results for equilibrium problems. Nonlinear Anal. **52**, 621–635 (2003)
8. Iusem, A., Kassay, G., Sosa, W.: On certain conditions for the existence of solutions of equilibrium problems. Math. Program. **116**, 259–273 (2009)
9. Iusem, A., Kassay, G., Sosa, W.: An existence result for equilibrium with some surjectivity consequences. J. Convex Anal. **16**, 807–826 (2009)
10. Nasri, M., Sosa, W.: Equilibrium problems and generalized Nash games. Optimization **60**, 1161–1170 (2011)
11. Castellani, M., Giuli, M.: On equivalent equilibrium problems. J. Optim. Theory Appl. **147**, 157–168 (2010)
12. Castellani, M., Giuli, M.: Refinements of existence results for relaxed quasimonotone equilibrium problems. J. Global Optim. **57**, 1213–1227 (2013)
13. Cotrina, J., García, Y.: Equilibrium problems: existence results and applications. Set-Valued Var. Anal. **26**, 159–177 (2018). DOI 10.1007/s11228-017-0451-6
14. K. Fan: A generalization of Tychonoff's fixed point theorem. Math. Ann. **142**, 305–310 (1961)
15. Brézis, H., Nirenberg, L., Stampacchia, G.: A remark on Ky Fan's minimax principle. Boll. Un. Mat. Ital. (4) **6**, 293–300 (1972)
16. Bianchi, M., Kassay, G., Pini, R.: Existence of equilibria via Ekeland's principle. J. Math. Anal. **305**, 502–512 (2005)
17. Aleche, B., Radulescu, V.: The Ekeland variational principle for equilibrium problems revisited and applications. Nonlinear Anal. RWA **23**, 17–25 (2015)
18. Oettli, W., Théra, M.: Equivalents of Ekeland's principle. Bull. Austral. Math. Soc. **48**(3), 385–392 (1993). DOI 10.1017/S0004972700015847. URL https://doi.org/10.1017/S0004972700015847
19. Mosco, U.: Implicit variational problems and quasi variational inequalities. In: J.P. Gossez, E.J. Lami Dozo, J. Mawhin, L. Waelbroeck (eds.) Nonlinear Operators and the Calculus of Variations, pp. 83–156. Springer Berlin Heidelberg, Berlin, Heidelberg (1976)
20. Facchinei, F., Kanzow, C.: Generalized Nash equilibrium problems. 4OR **5**(3), 173–210 (2007). DOI 10.1007/s10288-007-0054-4. URL https://doi.org/10.1007/s10288-007-0054-4
21. Cotrina, J., Zúñiga, J.: Quasi-equilibrium problems with non-self constraint map. J. Global Optim. **75**, 177–197 (2019). URL https://doi.org/10.1007/s10898-019-00762-5
22. Cotrina, J., Hantoute, A., Svensson, A.: Existence of quasi-equilibria on unbounded constraint sets. Optimization **0**(0), 1–18 (2020). DOI 10.1080/02331934.2020.1778690. URL https://doi.org/10.1080/02331934.2020.1778690
23. Aussel, D., Cotrina, J., Iusem, A.: An existence result for quasi-equilibrium problems. J. Convex Anal. **24**, 55–66 (2017)
24. Castellani, M., Giuli, M., Pappalardo, M.: A Ky Fan minimax inequality for quasiequilibria on finite-dimensional spaces. J Optim Theory Appl **179**, 53–64 (2018)
25. Cotrina, J., Zúñiga, J.: A note on quasi-equilibrium problems. Oper Res Lett. **46**, 138–140 (2018)

26. Castellani, M., Giuli, M.: Ekeland's principle for cyclically monotone equilibrium problems. Nonlinear Anal. RWA **32**, 213–228 (2016)
27. Tian, G.Q., Zhou, J.: Transfer continuities, generalizations of the Weierstrass and maximum theorems: a full characterization. J. Math. Econ. **24**, 281–303 (1995)
28. Debreu, G.: A social equilibrium existence theorem. Proceedings of the National Academy of Sciences **38**(10), 886–893 (1952). DOI 10.1073/pnas.38.10.886. URL https://www.pnas.org/content/38/10/886
29. Ansari, Q.H., Idzik, A., Yao, J.C.: Coincidence and fixed point theorems with applications. Topol. Methods Nonlinear Anal. **15**(1), 191–202 (2000). URL https://projecteuclid.org:443/euclid.tmna/1471873918
30. Ansari, Q.H., Chan, W.K., Yang, X.Q.: The system of vector quasi-equilibrium problems with applications. Journal of Global Optimization **29**(1), 45–57 (2004). DOI 10.1023/B:JOGO.0000035018.46514.ca. URL https://doi.org/10.1023/B:JOGO.0000035018.46514.ca
31. Penot, J.P., Théra, M.: Semicontinuous mappings in general topology. Arch. Math. (Basel) **38**(2), 158–166 (1982). DOI 10.1007/BF01304771. URL https://doi.org/10.1007/BF01304771
32. Choquet, G.: Cours d'analyse. Tome II: Topologie. Espaces topologiques et espaces métriques. Fonctions numériques. Espaces vectoriels topologiques. Deuxième édition, revue et corrigée. Masson et Cie, Éditeurs, Paris (1969)
33. Hadjisavvas, N., Khatibzadeh, H.: Maximal monotonicity of bifunctions. Optimization **59**(2), 147–160 (2010)
34. Alizadeh, M.H., Bianchi, M., Hadjisavvas, N., Pini, R.: On cyclic and n-cyclic monotonicity of bifunctions. J. Global Optim. **60**(4), 599–616 (2014). DOI 10.1007/s10898-013-0113-7. URL https://doi.org/10.1007/s10898-013-0113-7
35. Giuli, M.: Cyclically monotone equilibrium problems and Ekeland's principle. Decis. Econ. Finance **40**(1-2), 231–242 (2017). DOI 10.1007/s10203-017-0188-6. URL https://doi.org/10.1007/s10203-017-0188-6
36. Bianchi, M., Schaible, S.: An extension of pseudolinear functions and variational inequality Problems. J. Optim. Theory Appl. **104**, 59–71 (2000)
37. Bianchi, M., Hadjisavvas, N., Schaible, S.: On pseudomonotone maps T for which $-T$ is also pseudomonotone. J Convex Anal. **10**(1), 149–168 (2003)
38. Ekeland, I.: On the variational principle. J. Math. Anal. Appl. **47**, 324–353 (1974). DOI 10.1016/0022-247X(74)90025-0. URL https://doi.org/10.1016/0022-247X(74)90025-0
39. Al-Homidan, S., Ansari, Q.H., Kassay, G.: Vectorial form of ekeland variational principle with applications to vector equilibrium problems. Optimization **69**(3), 415–436 (2020). DOI 10.1080/02331934.2019.1589469. URL https://doi.org/10.1080/02331934.2019.1589469
40. Aussel, D., Cotrina, J.: Quasimonotone quasivariational inequalities: existence results and applications. J. Optim. Theory Appl. **158**, 637–652 (2013)