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On computation of optimal strategies in oligopolistic markets respecting the cost of change

Jiří V. Outrata · Jan Valdman

Abstract The paper deals with a class of parametrized equilibrium problems, where the objectives of the players do possess nonsmooth terms. The respective Nash equilibria can be characterized via a parameter-dependent variational inequality of the second kind, whose Lipschitzian stability is thoroughly investigated. This theory is then applied to evolution of a oligopolistic market in which the firms adopt their production strategies to changing input costs, while each change of the production is associated with some "costs of change". We examine both the Cournot-Nash equilibria as well as the two-level case, when one firm decides to take over the role of the Leader (Stackelberg equilibrium). The impact of costs of change is illustrated by academic examples.

Keywords Generalized equation · Equilibrium · Cost of Change

1 Introduction

Consider an oligopolistic market, where the data of the production cost functions and constraints of each producer/firm are available to all his rivals. In such a case each producer can compute his optimal non-cooperative Cournot-Nash strategy by solving the corresponding variational inequality, see [13] and [14]. It may happen, however, that in the course of time some external parameters change, e.g., the prices of the inputs or the parameters of the inverse demand function describing the behavior of the customers. In such a case, the strategies should be adjusted but, as thoroughly analyzed in [6], each change is generally associated with some expenses, called *costs of change*. Thus, given a certain *initial strategy profile* (productions of all firms), we face then a different equilibrium model, in which the costs of change enter the objectives of some (or all) producers. Since these costs are typically nonsmooth, the respective variational inequality, describing the new non-cooperative equilibrium, becomes substantially more complicated, both from the theoretical as well as from the numerical point of view. One can imagine that such updates of strategies are performed repetitively. This leads to a discrete-time evolution process, where the firms respond to changing conditions by repetitive solution of the mentioned rather complicated variational inequality (with updated data). As discussed in [14, Chapter 12], it may also happen that one of the producers, having an advantage over the

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others, takes over the role of a Leader and switches to the *Stackelberg strategy*, whereas the remaining firms continue to play non-cooperatively with each other on the lower level as Followers. In this case, our discrete-time evolution process, considered from the point of view of the Leader, amounts to repetitive solution of *mathematical programs with equilibrium constraints* (MPECs) with a nonsmooth objective and the above mentioned variational inequality among the constraints. Further, it is interesting to note that the above described model has, in case of positively homogeneous costs of change, a similar structure as some infinite-dimensional variational systems used in continuum mechanics to model a class of *rate-independent processes* cf., e.g., [12] or [7].

The plan of the paper is as follows. In the preliminary Section 2 we collect the necessary background from variational analysis. Section 3 consists of two parts. In the first one we introduce a general parameter-dependent non-cooperative equilibrium problem which is later used for modeling of the considered oligopolistic market. By employing standard arguments, existence of the respective solutions (equilibria) is shown. In the second part we then consider a parameter-dependent variational system which encompasses the mentioned equilibrium problem and is amenable to advanced tools of variational analysis. In this way one obtains useful stability results concerning the respective *solution map*. They are used in the sequel but they are also important for its own sake.

Thereafter, in Section 4, this equilibrium problem is specialized to a form, corresponding to the oligopolistic market model from [13]. In this case, the solution map is even single-valued and locally Lipschitzian. In Section 5 we then consider a modification of the 5-firm example from [13] with the aim to illustrate the role of costs of change and to describe a possible numerical approach to the computation of the respective equilibria. Whereas Section 5.1 deals with noncooperative Cournot-Nash equilibrium, Section 5.2 concerns the situation, when one of the firms prefers to apply the Stackelberg strategy. In both cases our main numerical tool is the so-called Gauss-Seidel method from [10] tailored to the considered type of problems.

The following notation is employed. Given a vector $x \in \mathbb{R}^n$, $[x]$ denotes the linear subspace generated by x and $[x]^\perp$ stands for its orthogonal complement. For a multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $\mathrm{gph}\, F$ signifies the graph of F, δ_A is the indicatory function of a set A and $\overline{\mathbb{R}} = R \cup \{+\infty\}$ is the extended real line. $\mathbb{B}$ stands for the unit ball and, for a cone K, K° denotes its (negative) polar.

2 Background from variational analysis

Throughout the whole paper, we will make an extensive use of the following basic notions of modern variational analysis.

Definition 1 Let A be a closed set in $\mathbb{R}^n$ and $\bar{x} \in A$. Then

$$T_A(\bar{x}) := \operatorname{Lim\,sup}_{t \searrow 0} \frac{A - \bar{x}}{t}$$

is the *tangent (contingent, Bouligand)* cone to A at $\bar{x}$,

$$\widehat{N}_A(\bar{x}) := (T_A(\bar{x}))^\circ$$

is *regular (Fréchet) normal cone* to A at $\bar{x}$, and

$$N_A(\bar{x}) := \operatorname{Lim\,sup}_{\substack{A \\ x \to \bar{x}}} \widehat{N}_A(x) = \{x^* | \exists\, x_i^* \xrightarrow{A} x^*, x_i^* \in \widehat{N}_A(x_i) \text{ such that } x_i^* \to x^*\}$$

is the *limiting (Mordukhovich) normal cone* to A at $\bar{x}$.

In this definition "Lim sup" stands for the Painlevé-Kuratowski *outer set limit*. If A is convex, then $\widehat{N}_A(\bar{x}) = N_A(\bar{x})$ amounts to the classical normal cone in the sense of convex analysis and we write $N_A(\bar{x})$.

In the sequel, we will also employ the co-called critical cone. In the setting of Definition 1 with an given normal $d^* \in \widehat{N}_A(\bar{x})$, the cone

$$\mathcal{K}_A(\bar{x}, d^*) := T_A(\bar{x}) \cap [d^*]^\perp$$

is called the *critical cone* to A at $\bar{x}$ with respect to d^*.

The above listed cones enable us to describe the local behavior of set-valued maps via various generalized derivatives. Consider a closed-graph multifunction F and the point $(\bar{x}, \bar{y}) \in \text{gph}\, F$.

Definition 2 (i) The multifunction $DF(\bar{x}, \bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, defined by

$$DF(\bar{x}, \bar{y})(u) := \{v \in \mathbb{R}^m | (u, v) \in T_{\text{gph}\, F}(\bar{x}, \bar{y})\}, d \in \mathbb{R}^n,$$

is called the *graphical derivative* of F at $(\bar{x}, \bar{y})$;
(ii) The multifunction $D^* F(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by

$$D^* F(\bar{x}, \bar{y})(v^*) := \{u^* \in \mathbb{R}^n | (u^*, -v^*) \in N_{\text{gph}\, F}(\bar{x}, \bar{y})\}, v^* \in \mathbb{R}^m,$$

is called the *limiting (Mordukhovich) coderivative* of F at $(\bar{x}, \bar{y})$.

Next we turn our attention to a proper convex, lower-semicontinuous (lsc) function $q : \mathbb{R}^n \to \overline{\mathbb{R}}$. Given an $\bar{x} \in \text{dom}\, q$, by $\partial q(\bar{x})$ we denote the classical Moreau-Rockafellar *subdifferential* of q at $\bar{x}$. In this case, for the *subderivative* function $dq(\bar{x}) : \mathbb{R}^n \to \overline{\mathbb{R}}$ ([17, Definition 8.1]) it holds that

$$dq(\bar{x})(w) = q'(\bar{x}; w) := \lim_{\tau \searrow 0} \frac{q(\bar{x} + \tau w) - q(\bar{x})}{\tau} \text{ for all } w \in \mathbb{R}^n.$$

In Section 3 we will employ also second-order subdifferentials and second-order subderivatives of q.

Definition 3 Let $\bar{v} \in \partial q(\bar{x})$. The multifunction $\partial^2 q(\bar{x}, \bar{v}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\partial^2 q(\bar{x}, \bar{v})(v^*) := D^* \partial q(\bar{x}, \bar{v})(v^*), \quad v^* \in \mathbb{R}^n,$$

is called the *second-order subdifferential* of q at $(\bar{x}, \bar{v})$.

If q is separable, i.e., $q(x) = \sum_{i=1}^{n} q_i(x_i)$ with some proper convex, lsc functions $q_i : \mathbb{R} \to \overline{\mathbb{R}}$, then

$$\partial^2 q(\bar{x}, \bar{v})(v^*) = \begin{bmatrix} \partial^2 q_1(\bar{x}_1, \bar{v}_1)(v_1^*) \\ \vdots \\ \partial^2 q_n(\bar{x}_n, \bar{v}_n)(v_n^*) \end{bmatrix},$$

where $\bar{v}_i, v_i^*$ are the ith components of the vectors, $\bar{v}, v^*$, respectively.

Concerning second-order subderivatives ([17, Definition 13.3]), we confine ourselves to the case when q is, in addition, *piecewise linear-quadratic*. This means that $\text{dom}\, q$ can be represented as the union of finitely many polyhedral sets, relative to each of which $q(x)$ is given in the form $\frac{1}{2} \langle x, Ax \rangle + \langle a, x \rangle + \alpha$ for some scalar $\alpha \in \mathbb{R}$, vector $a \in \mathbb{R}^n$ and a symmetric $[n \times n]$ matrix A, cf. [17, Definition 10.20].

In this particular case it has been proved in [17, Proposition 13.9] that, with $\bar{v} \in \partial q(\bar{x})$ and $w \in \mathbb{R}^n$ the second-order subderivative $d^2 q(\bar{x}|\bar{v})$ is proper convex and piecewise linear quadratic and

$$d^2 q(\bar{x}|\bar{v})(w) = q''(\bar{x}; w) + \delta_{K(\bar{x}, \bar{v})}(w), \tag{1}$$

where

$$q''(\bar{x}; w) := \lim_{\tau \searrow 0} \frac{q(\bar{x} + \tau w) - q(\bar{x}) - \tau q'(\bar{x}; w)}{\frac{1}{2} \tau^2}$$

is the *one-sided second directional derivative* of q at $\bar{x}$ in direction w and $K(\bar{x}, \bar{v}) := \{w | q'(\bar{x}; w) = \langle \bar{v}, w \rangle\}$. For a general theory of second-order subderivatives (without our restrictive requirements) the interested reader is referred to [17, Chapter 13 B].

We conclude now this section with the definitions of two important Lipschitzian stability notions for multifunctions which will be extensively employed in the sequel.

Definition 4 Consider a multifunction $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and a point $(\bar{u}, \bar{v}) \in \text{gph}\, S$.

(i) S is said to have the *Aubin property* around $(\bar{u}, \bar{v})$, provided there are neighborhoods $\mathcal{U}$ of $\bar{u}$, $\mathcal{V}$ of $\bar{v}$ along with a constant $\eta \geq 0$ such that

$$S(u_1) \cap \mathcal{V} \subset S(u_2) + \eta \|u_1 - u_2\| \mathbb{B} \text{ for all } u_1, u_2 \in \mathcal{U}.$$

(ii) We say that S has a *single-valued and Lipschitzian localization* around $(\bar{u}, \bar{v})$, provided there are neighborhoods $\mathcal{U}$ of $\bar{u}$, $\mathcal{V}$ of $\bar{v}$ and a Lipschitzian mapping $s : \mathcal{U} \to \mathbb{R}^n$ such that $s(\bar{u}) = \bar{v}$ and

$$S(u) \cap \mathcal{V} = \{s(u)\} \text{ for all } u \in \mathcal{U}.$$

Further important stability notions can be found, e.g., in [4].

3 General equilibrium model: Existence and stability

Consider a non-cooperative game of l players, each of which solves the optimization problem

$$
\begin{aligned}
\text{minimize} \quad & f_i(p, x_i, x_{-i}) + q_i(x_i) \\
\text{subject to} \quad & \\
& x_i \in A_i,
\end{aligned}
\tag{2}
$$

$i = 1, 2, \ldots, l$. In (2), $x_i \in \mathbb{R}^n$ is the *strategy* of the ith player,

$$x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_l) \in (\mathbb{R}^n)^{l-1}$$

is the *strategy profile* of the remaining players and $p \in \mathbb{R}^m$ is a parameter, common for all players. Further, the functions

$$f_i : \mathbb{R}^m \times (\mathbb{R}^n)^l \to \mathbb{R} \quad \text{and} \quad q_i : \mathbb{R}^n \to \mathbb{R}, \quad i = 1, 2, \ldots, l,$$

are continuously differentiable and convex continuous, respectively, and the sets of *admissible strategies* $A_i, i = 1, 2, \ldots, l$, are closed and convex. The objective in (2) is thus the sum of a smooth function depending on the whole strategy profile $x := (x_1, x_2, \ldots, x_l)$ and a convex (not necessarily smooth) function depending only on x_i. Let us recall that, given a parameter vector $\bar{p}$, the strategy profile $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_l)$ is a corresponding *Nash equilibrium* provided

$$\bar{x}_i \in \underset{x_i \in A_i}{\arg\min} \left[f_i(\bar{p}, x_i, \bar{x}_{-i}) + q_i(x_i) \right] \text{ for all } i.$$

Denote by $S : \mathbb{R}^m \rightrightarrows (\mathbb{R}^n)^l$ the solution mapping which assigns each p the corresponding (possibly empty) set of Nash equilibria. The famous Nash Theorem [2, Theorem 12.2] yields the next statement.

Theorem 1 *Given $\bar{p} \in \mathbb{R}^n$, assume that*

(A1) for all admissible values of x_{-i} functions $f_i(\bar{p}, \cdot, x_{-i}), i = 1, 2, \ldots, l$, are convex, and
(A2) sets $A_i, i = 1, 2, \ldots, l$, are bounded.

Then $S(\bar{p}) \neq \emptyset$.

Suppose from now on that (A1) holds true for all p from an open set $\mathcal{B} \subset \mathbb{R}^m$. Then one has that $\mathcal{B} \subset \operatorname{dom} S$ and for $p \in \mathcal{B}$

$$S(p) = \{x \mid 0 \in F(p, x) + Q(x)\}, \tag{3}$$

where

$$F(p, x) = \begin{bmatrix} F_1(p, x) \\ \vdots \\ F_l(p, x) \end{bmatrix} \text{ with } F_i(p, x) = \nabla_{x_i} f_i(p, x_i, x_{-i}), \ i = 1, 2, \ldots, l, \text{ and}$$

$$Q(x) = \partial \tilde{q}(x) \text{ with } \tilde{q}(x) = \sum_{i=1}^{l} \tilde{q}_i(x_i) \text{ and } \tilde{q}_i(x_i) = q_i(x_i) + \delta_A(x_i), \ i = 1, 2, \ldots, l.$$

This follows immediately from the fact that under the posed assumptions the solution set of (2) is characterized by the first-order condition

$$0 \in \nabla_{x_i} f_i(p, x_i, x_{-i}) + \partial \tilde{q}_i(x_i), \ i = 1, 2, \ldots, l.$$

Remark 1 The GE in (3) can be equivalently written down in the form:
For a given $\bar{p}$ find $\bar{x}$ such that

$$\langle F(\bar{p}, \bar{x}), x - \bar{x} \rangle + \tilde{q}(x) - \tilde{q}(\bar{x}) \geq 0 \text{ for all } x.$$

Our equilibrium is thus governed by a variational inequality (VI) of the second kind, cf. [11], [5, page 96].

Next we will concentrate on the (local) analysis of S (given by (3)) under the less restrictive assumptions that, with $s := ln$,

(i) $F : \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}^s$ is continuously differentiable, and
(ii) $Q(\cdot) = \partial \tilde{q}(\cdot)$ for a proper convex, lsc function $\tilde{q} : \mathbb{R}^s \to \overline{\mathbb{R}}$.

In this way, the obtained results will be applicable not only to the equilibrium problem stated above, but to a broader class of parametrized VIs of the second kind. Note that stability of the *generalized equation* (GE)

$$0 \in F(p, x) + \partial \tilde{q}(x) \tag{4}$$

has been investigated, among other works, in [17, Chapter 13] even without any convexity assumptions imposed on $\tilde{q}$. As proved in [17, Theorem 13.48], S has the Aubin property around $(\bar{p}, \bar{x}) \in \text{gph } S$ provided the *adjoint* GE

$$0 \in \nabla_x F(\bar{p}, \bar{x})^T u + \partial^2 \tilde{q}(\bar{x}, -F(\bar{p}, \bar{x}))(u) \tag{5}$$

in variable $u \in \mathbb{R}^s$ has only the trivial solution $u = 0$.

This condition is automatically fulfilled provided $\nabla_x F(\bar{p}, \bar{x})$ is positive definite. Indeed, due to the assumptions imposed on $\tilde{q}$, the mapping $\partial \tilde{q}$ is maximal monotone [17, Theorem 12.17]. We can thus invoke [15, Theorem 2.1], according to which

$$\langle u, v \rangle \geq 0 \ \text{ for all } \ v \in \partial^2 \tilde{q}(\bar{x}, -F(\bar{p}, \bar{x}))(u).$$

The result thus follows from the inequality

$$0 \leq \langle u, \nabla_x F(\bar{p}, \bar{x})^T u \rangle.$$

Let us now derive conditions ensuring the existence of a single-valued and Lipschitzian localization of S around $(\bar{p}, \bar{x})$. To this purpose we employ [4, Theorem 3G.4], according to which this property of S is implied by the existence of a single-valued and Lipschitzian localization of the associated partially linearized mapping $\Sigma : \mathbb{R}^s \rightrightarrows \mathbb{R}^s$ defined by

$$\Sigma(w) := \{x | w \in F(\bar{p}, \bar{x}) + \nabla_x F(\bar{p}, \bar{x})(x - \bar{x}) + \partial \tilde{q}(x)\} \tag{6}$$

around $(0, \bar{x})$. This implication leads immediately to the next statement.

Proposition 1 *Assume that $\nabla_x F(\bar{p}, \bar{x})$ is positive definite. Then S has a single-valued and Lipschitzian localization around $(\bar{p}, \bar{x})$.*

Proof Observe first that, by [17, Theorem 13.48], Σ has the Aubin property around $(0, \bar{x})$ if and only if (5) has only the trivial solution $u = 0$ which, in turn, is ensured by the positive definiteness of $\nabla_x F(\bar{p}, \bar{x})$. So the assertion follows from [4, Theorem 3G.5] provided the mapping

$$\Phi(x) := F(\bar{p}, \bar{x}) + \nabla_x F(\bar{p}, \bar{x})(x - \bar{x}) + \partial \tilde{q}(x)$$

is *locally monotone* at $(\bar{x}, 0)$, i.e., for some neighborhood $\mathcal{U}$ of $(\bar{x}, -F(\bar{x}))$, one has

$$\langle x' - x, \nabla_x F(\bar{p}, \bar{x})(x' - x) \rangle + \langle x' - x, y' - y \rangle \geq 0 \ \forall \ (x, y), (x', y') \in \text{gph } \partial \tilde{q} \cap \mathcal{U}.$$

This holds trivially due to the posed assumptions and we are done. $\qquad \square$

In some situations the assumption of positive definiteness of $\nabla_x F(\bar{p}, \bar{x})$ can be weakened.

Proposition 2 *Assume that $\tilde{q}$ is convex, piecewise linear-quadratic and the mapping*

$$\Xi(w) := \{k \in \mathbb{R}^n | w \in \nabla_x F(\bar{p}, \bar{x})k + \partial\varphi(k)\} \tag{7}$$

with $\varphi(k) := \frac{1}{2}d^2\tilde{q}(\bar{x}| - F(\bar{p}, \bar{x}))(k)$ is single-valued on $\mathbb{R}^s$. Then S has a single-valued and Lipschitzian localization around $(\bar{p}, \bar{x})$.

Proof Again, by virtue of [4, Theorem 3G.4] it suffices to show that the single-valuedness of Ξ implies the existence of a single-valued and Lipschitzian localization of Σ around $(0, \bar{x})$. Clearly,

$$\text{gph}\,\Sigma = \left\{ (w, x) \left| \begin{bmatrix} x - \bar{x} \\ w - \nabla_x F(\bar{p}, \bar{x})(x - \bar{x}) \end{bmatrix} \in \text{gph}\,\partial\tilde{q} - \begin{bmatrix} \bar{x} \\ - F(\bar{p}, \bar{x}) \end{bmatrix} \right. \right\}$$

so that Σ is a polyhedral multifunction due to our assumptions imposed on $\tilde{q}$, cf. [17, Theorem 12.30]. It follows from [16] (see also [14, Cor.2.5]) that due to the polyhedrality of Σ, it suffices to ensure the single-valuedness of $\Sigma(\cdot) \cap \mathcal{V}$ on $\mathcal{U}$, where $\mathcal{U}$ is a convex neighborhood of $0 \in \mathbb{R}^s$ and $\mathcal{V}$ is a neighborhood of $\bar{x}$. Let us select these neighborhoods in such a way that

$$\text{gph}\,\partial\tilde{q} - \begin{bmatrix} \bar{x} \\ - F(\bar{p}, \bar{x}) \end{bmatrix} = T_{\text{gph}\,\partial\tilde{q}}(\bar{x}, -F(\bar{p}, \bar{x})),$$

which is possible due to the polyhedrality of $\partial\tilde{q}$. Then one has

$$\text{gph}\,\Sigma \cap (\mathcal{U} \times \mathcal{V}) = \{(w, \bar{x} + k) \in \mathcal{U} \times \mathcal{V} | w \in \nabla_x F(\bar{p}, \bar{x})k + D\partial\tilde{q}(\bar{x}, -F(\bar{p}, \bar{x}))(k)\}.$$

Under the posed assumptions for any $k \in \mathbb{R}^n$

$$D\partial\tilde{q}(\bar{x}, -F(\bar{p}, \bar{x}))(k) = \partial\varphi(k),$$

cf. [17, Theorem 13.40], so that $\text{gph}\,\Sigma \cap (\mathcal{U} \times \mathcal{V}) = \{(w, k + \bar{x}) \in \mathcal{U} \times \mathcal{V} | (w, k) \in \text{gph}\,\Xi\}$. Since $D\partial\tilde{q}(\bar{x}, -F(\bar{p}, \bar{x}))(\cdot)$ is positively homogeneous, $\partial\varphi(\cdot)$ is positively homogeneous as well and so the single-valuedness of $\Sigma(\cdot) \cap \mathcal{V}$ on $\mathcal{U}$ amounts exactly to the single-valuedness of Ξ on $\mathbb{R}^s$. $\square$

Note that, by virtue of [8, Theorem 4.4], in the setting of Proposition 2 one has

$$DS(\bar{p}, \bar{x})(h) = \{k | 0 \in \nabla_p F(\bar{p}, \bar{x})h + \nabla_x F(\bar{p}, \bar{x})k + \partial\varphi(k)\} \tag{8}$$

for all $h \in \mathbb{R}^m$.

If $\tilde{q} = \delta_A$ for a convex polyhedral set A, then $\partial\tilde{q}(x) = N_A(x)$ and, by virtue of (1), $\varphi(k) = \delta_{\mathcal{K}_A(\bar{x}, -F(\bar{p}, \bar{x}))}(k)$. It follows that the GE in (7) attains the form

$$w \in \nabla_x F(\bar{p}, \bar{x})k + N_{\mathcal{K}_A(\bar{x}, -F(\bar{p}, \bar{x}))}(k).$$

This is in agreement with [14, Theorem 5.3] and [4, Theorem 4F.1]. Let us illustrate the general case of Proposition 2 via a simple academic

Example 1 Put $m = 2, l = 1, n = 1$ and consider the GE (4), where $F(p, x) = p_1 + p_2 x, \tilde{q}(x) = |x| + \delta_A(x)$ with $A = [0, 1]$ and the reference pair $(\bar{p}, \bar{x}) = ((-1, 1), 0)$. Clearly, $\bar{v} := -F(\bar{p}, \bar{x}) = 1$, and we may again employ formula (1). One has $K(\bar{x}, \bar{v}) = \mathbb{R}_+, f''(\bar{x}, w) = 0$ for any $w \in \mathbb{R}_+$ and so we obtain that

$$\varphi(k) = \frac{1}{2}d^2\tilde{q}(\bar{x}| - F(\bar{p}, \bar{x}))(k) = \delta_{\mathbb{R}_+}(k).$$

It follows that

$$\Xi(w) = \{k | w \in k + \partial\delta_{\mathbb{R}_+}(k)\} = \{k | w \in k + N_{\mathbb{R}_+}(k)\} = \{k \geq 0 | k - w \geq 0, \langle k, k - w \rangle = 0\}. \tag{9}$$

Since Ξ is clearly single-valued on $\mathbb{R}$, we may conclude that the respective S has indeed the single-valued and Lipschitzian localization around $(\bar{p}, \bar{x})$. By virtue of (8)

$$DS(\bar{p}, \bar{x})(h) = \{k | 0 \in h_1 + k + N_{\mathbb{R}_+}(k)\}.$$

Both mappings S and $DS(\bar{p}, \bar{x})$ are depicted in Fig.1.
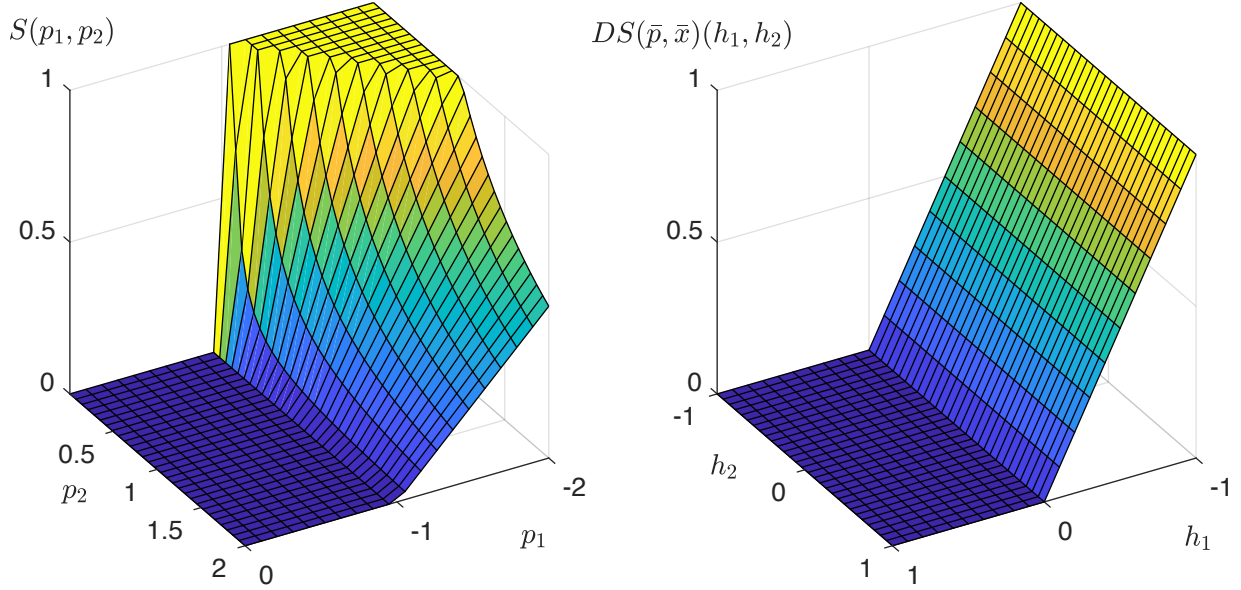
Fig. 1 The set in equilibria $S(p_1, p_2)$ (left) and the graphical derivative $DS(\bar{p}, \bar{x})(h_1, h_2)$ of Example 1.

In some cases one can apply the following criterion based on Proposition 2 and [17, Theorem 13.9].

Proposition 3 *Assume that $\tilde{q}$ is convex, piecewise linear-quadratic, $\tilde{q}''(\bar{x}; \cdot)$ is convex and, with*

$$K := \{k | q'(\bar{x}; k) = \langle -F(\bar{p}, \bar{x}), k \rangle\},$$

one has

$$\partial \frac{1}{2} d^2 \tilde{q}(\bar{x} | -F(\bar{p}, \bar{x}))(k) = \partial \frac{1}{2} \tilde{q}''(\bar{x}; k) + N_K(k).$$

Further suppose that the matrix $\nabla_x F(\bar{p}, \bar{x})$ is copositive with respect to K. Then S has a single-valued and Lipschitzian localization around $(\bar{p}, \bar{x})$.

Proof We show that the assumptions of Proposition 2 are fulfilled. To this aim consider the quantity

$$V := \langle \nabla_x F(\bar{p}, \bar{x})(k_1 - k_2), k_1 - k_2 \rangle + \langle \xi_1 - \xi_2, k_1 - k_2 \rangle + \langle \eta_1 - \eta_2, k_1 - k_2 \rangle$$

with $\xi_1 \in \partial \frac{1}{2} \tilde{q}''(\bar{x}; k_1), \xi_2 \in \partial \frac{1}{2} \tilde{q}''(\bar{x}; k_2), \eta_1 \in N_K(k_1)$ and $\eta_2 \in N_K(k_2)$. It follows that $k_1, k_2 \in K$ so that, by the imposed assumptions, there is a $\sigma > 0$ such that $V \geq \sigma \|k_1 - k_2\|^2$. Hence the operator on the right-hand side of

$$\xi \in \nabla_x F(\bar{p}, \bar{x})k + \partial \frac{1}{2} [\tilde{q}''(\bar{x}; k)] + N_K(k).$$

is strongly monotone. This implies, by virtue of [17, Proposition 12.54], the single-valuedness of the respective mapping Ξ and we are done. $\square$

4 Optimal strategies of producers

As stated in the Introduction, our motivation for a study of mapping (3) came from an attempt to optimize the production strategies of firms with respect to changing external parameters like input prices, parameters of inverse demand functions etc. These parameters evolve in time and the corresponding adjustments of production strategies have (at least at some producers) to take into account the already mentioned costs of change. The appropriate variant of the GE in (3) (depending on the considered type of market) has thus to be solved at each time step with the updated values of the parameters. In this section we will analyze from this point of view a standard oligopolistic market described thoroughly in [13] and in [14, Chapter

12]. So, in the framework (2) we will assume that n is the number of produced homogeneous commodities, $p = (p_1, p_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}, m_1 + m_2 = m$ and

$$f_i(p, x_i, x_{-i}) = c_i(p_1, x_i) - \langle x_i, \pi(p_2, T) \rangle \tag{10}$$

with $T = \sum\limits_{i=1}^{l} x_i$. Functions $c_i : \mathbb{R}^{m_1} \times \mathbb{R}^n \to \mathbb{R}$ represent the *production costs* of the ith producer and $\pi : \mathbb{R}^{m_2} \times \mathbb{R}^n \to \mathbb{R}$ is the *inverse demand function* which assigns each value of the parameter p_2 and the overall production vector T the price at which the (price-taking) consumers are willing to demand. Additionally, we assume that, with some non-negative reals β_i,

$$q_i(x_i) = \beta_i \|x_i - a_i\|, \quad i = 1, 2, \ldots, l, \tag{11}$$

where $\| \cdot \|$ stands for an arbitrary norm in $\mathbb{R}^n$. Sets $A_i \subset \mathbb{R}^n$ specify the *sets of feasible productions* and functions q_i represent the costs of change associated with the change of production from a given vector a_i to x_i. Thus

$$a_i \in A_i, \quad i = 1, 2, \ldots, l,$$

are "previous" productions which have to be changed taking into account the "new" values of parameters p_1, p_2. Clearly, one could definitely work also with more complicated functions q_i. Let us denote the total costs (negative profits) of the single firms by

$$J_i(p, x_i, x_{-i}) := f_i(p, x_i, x_{-i}) + q_i(x_i), \quad i = 1, 2, \ldots, l.$$

In accordance with [13] and [14] we will now assume for brevity that $n = 1$ (so that $s = l$) and impose the following assumptions:

(S1) $\exists$ an open set $\mathcal{B}_1 \subset \mathbb{R}^{m_1}$ and open sets $\mathcal{D}_i \supset A_i$ such that for for $i = 1, 2, \ldots, l$
  – c_i are twice continuously differentiable on $\mathcal{B}_1 \times \mathcal{D}_i$;
  – $c_i(p_1, \cdot)$ are convex for all $p_1 \in \mathcal{B}_1$.
(S2) $\exists$ an open set $\mathcal{B}_2 \subset \mathbb{R}^{m_2}$ such that
  – π is twice continuously differentiable on $\mathcal{B}_2 \times \operatorname{int} \mathbb{R}_+$;
  – $\vartheta \pi(p_2, \vartheta)$ is a concave function of ϑ for all $p_2 \in \mathcal{B}_2$.
(S3) Sets $A_i \subset \mathbb{R}_+$ are closed bounded intervals and at least one of them belongs to $\operatorname{int} \mathbb{R}_+$.

Note that thanks to (S3) one has that $T > 0$ for any feasible production profile

$$(x_1, x_2, \ldots, x_l) \in \mathbb{R}^l$$

and hence the second term in (10) (representing the revenue) is well-defined.

By virtue of [14, Lemmas 12.1 and 12.2] we conclude that, with F_i and q_i given by (10) and (11), respectively, and $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, the assumptions of Proposition 1 are fulfilled. This means in particular that for all vectors $(a_1, a_2, \ldots, a_i) \in A_1 \times A_2 \times \ldots \times A_l$ the respective mapping $S : (p_1, p_2) \mapsto x$ has a single-valued and Lipschitzian localization around any triple (p_1, p_2, x), where $(p_1, p_2) \in \mathcal{B}_1 \times \mathcal{B}_2$ and $x \in S(p_1, p_2)$. Under the posed assumptions, however, a stronger statement can be established.

Theorem 2 *Let $a \in A_1 \times A_2 \times \ldots \times A_l$. Under the posed assumptions (S1)-(S3) the solution mapping S is single-valued and Lipschitzian over $\mathcal{B}_1 \times \mathcal{B}_2$.*

Proof Given the vectors $a_i, i = 1, 2, \ldots, l$, and the parameters p_1, p_2, the GE in (3) attains the form

$$0 \in \begin{bmatrix} \nabla_{x_1} c_1(p_1, x_1) - x_1 \nabla_{x_1} \pi(p_2, T) - \pi(p_2, T) \\ \vdots \\ \nabla_{x_l} c_l(p_1, x_l) - x_l \nabla_{x_l} \pi(p_2, T) - \pi(p_2, T) \end{bmatrix} + \begin{bmatrix} \Lambda_1(x_1 - a_1) \\ \vdots \\ \Lambda_l(x_l - a_l) \end{bmatrix} \tag{12}$$

$$+ \ N_{A_1}(x_1) \times \ldots \times N_{A_l}(x_l),$$

where

$$\Lambda_i(x_i - a_i) = \begin{cases} \beta_i \text{ if } x_i > a_i \\ -\beta_i \text{ if } x_i < a_i \\ [-\beta_i\beta_i] \text{ otherwise.} \end{cases}$$

From [14, Lemma 12.2] and [17, Proposition 12.3] it follows that for any $(p_1, p_2) \in \mathcal{B}_1 \times \mathcal{B}_2$ the first operator on the right-hand side of (12) is strictly monotone in variable x. Moreover, the second one, as the subdifferential of a proper convex function is monotone ([17, Theorem 12.17]). Their sum is strictly monotone by virtue of [17, Exercise 12.4(c)] and so we may recall [17, Example 12.48] according to which $S(p_1, p_2)$ can have no more than one element for any $(p_1, p_2) \in \mathcal{B}_1 \times \mathcal{B}_2$. This, combined with Theorem 1 and the Lipschitzian stability of S mentioned above proves the result. $\square$

In the next section we will be dealing with the mapping $Z : \mathbb{R} \rightrightarrows \mathbb{R}^{l-1}$ which, for given fixed values of a, p_1 and p_2, assigns each vector $x_1 \in A_1$ a solution $(x_2, \ldots, x_l)$ of the GE

$$0 \in \begin{bmatrix} \nabla_{x_2}c_2(p_1, x_2) - \langle x_2 \nabla_{x_2}\pi(p_2, T)\rangle - \pi(p_2, T) \\ \vdots \\ \nabla_{x_l}c_l(p_1, x_l) - \langle x_l \nabla_{x_l}\pi(p_2, T)\rangle - \pi(p_2, T) \end{bmatrix} + \begin{bmatrix} \Lambda_2(x_2 - a_2) \\ \vdots \\ \Lambda_l(x_l - a_l) \end{bmatrix} \tag{13}$$

$$+ \quad N_{A_2}(x_2) \times \ldots \times N_{A_l}(x_l).$$

Variable x_1 enters GE (13) via $T(= \sum_{i=1}^{l} x_i)$. Using the same argumentation as in Theorem 2 we obtain the following result.

Theorem 3 *Let $a_i \in A_i$ for $i = 1, 2, \ldots, l, p_1 \in \mathcal{B}_1$ and $p_2 \in \mathcal{B}_2$. Then, under the assumptions of Theorem 2, mapping Z is single-valued and Lipschitzian over A_1.*

This statement enables us to consider the situation when the first producer decides to replace the non-cooperative by the Stackelberg strategy, cf. [11, page 220]. In this case, to maximize his profit, he has, for given values of a, p_1 and p_2, to solve the MPEC

$$\begin{array}{ll} \underset{x_1}{\text{minimize}} & c_1(p_1, x_1) - \langle x_1, \pi(p_2, T)\rangle + \beta_1|x_1 - a_1| \\ \text{subject to} & \\ & x_{-1} = Z(x_1) \\ & x_1 \in A_1. \end{array} \tag{14}$$

Thanks to Theorem 3 problem (14) can be replaced by the (nonsmooth) minimization problem

$$\begin{array}{ll} \text{minimize} & \Theta(x_1) \\ \text{subject to} & \\ & x_1 \in A_1 \end{array} \tag{15}$$

in variable x_1. In (15), $\Theta : \mathbb{R} \to \mathbb{R}$ is the composition defined by

$$\Theta(x_1) = c_l(p_1, x_l) - \langle x_l, \pi(p_2, x_1 + \mathcal{L}(Z(x_1)))\rangle + \beta_1|x_1 - a_1|, \tag{16}$$

where the mapping $\mathcal{L} : \mathbb{R}^{l-1} \to \mathbb{R}$ is defined by

$$\mathcal{L}(x_2, x_3, \ldots, x_l) = \sum_{i=2}^{l} x_i.$$

Problem (15) is thus a minimization of a locally Lipschitzian function to which various numerical approaches can be applied.
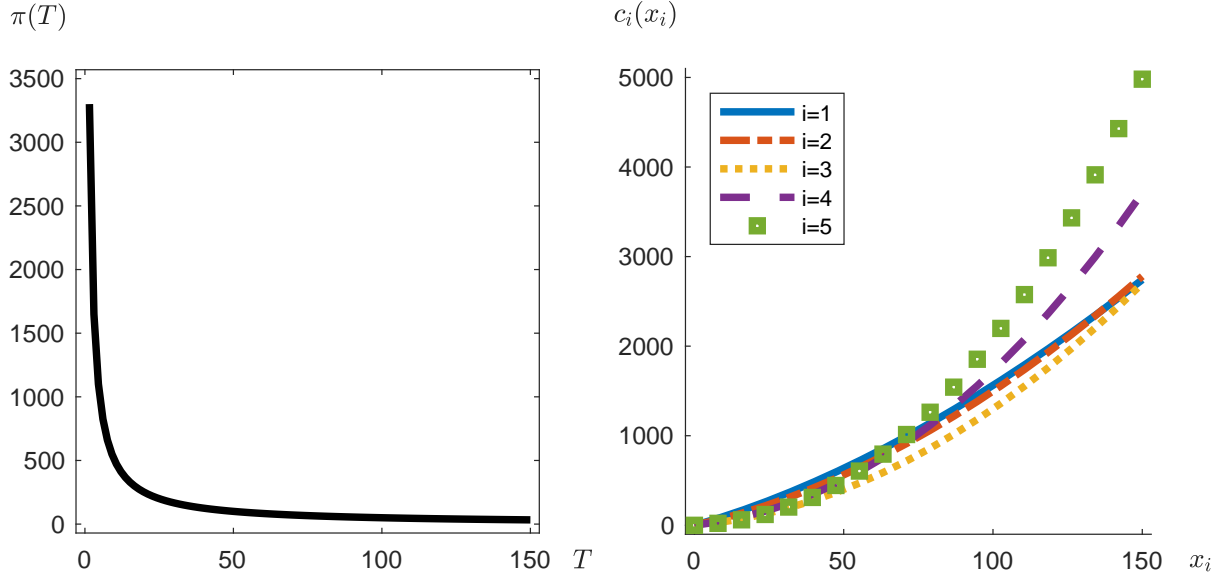
Fig. 2 The inverse demand function $\pi = \pi(\gamma, T)$ with $\gamma = 1$ (left) and production cost functions $c_i = c_i(x_i), i = 1, \ldots, 5$, for $t = 1$ (right).

5 Numerical experiments

We consider an example from [14, Section 12.1] enhanced by a nonsmooth term reflecting the cost of change. We have five firms (i.e., $l = 5$) supplying production quantities (productions)

$$x_1, x_2, \ldots, x_5$$

of one (i.e., $n = 1$) homogeneous commodity to a common market with the inverse demand function

$$\pi(\gamma, T) = 5000^{1/\gamma} T^{-1/\gamma},$$

where γ is a positive parameter termed *demand elasticity*. In our tests, however, this parameter will be fixed ($\gamma = 1$). The resulting inverse demand function is depicted in Figure 2 (left). The production cost functions have the form

$$c_i(b_i, x_i) = b_i x_i + \frac{\delta_i}{\delta_i + 1} K_i^{-1/\delta_i} x_i^{(1+\delta_i)/\delta_i},$$

where $b_i, \delta_i, K_i, i = 1, \ldots, 5$, are positive parameters. For brevity we assume that only the parameters b_i, reflecting the impact of the input prices on the production costs, evolve in time, whereas parameters δ_i, K_i attain the same constant values as in [14, Table 12.1]. The cost of change will arise only at the Firms 1, 2 and 3 with different multiplicative constants

$$\beta_1 = 0.5, \quad \beta_2 = 1, \quad \beta_3 = 2.$$

At the remaining firms any change of production does not incure additional costs ($\beta_4 = \beta_5 = 0$). We will study the behaviour of the market over three time intervals, $t \in \{1, 2, 3\}$ with the initial productions (at $t = 0$)

$$a_1 = 47.81, \quad a_2 = 51.14, \quad a_3 = 51.32, \quad a_4 = 48.55, \quad a_5 = 43.48,$$

corresponding to the standard Cournot-Nash equilibrium with the parameters taken over from [13].

The evolution of parameters b_i is displayed in Table 1 and the production cost functions for $t = 1$ are depicted in Figure 2 (right).

	i	1	2	3	4	5
t=1	b_i	9	7	3	4	2
t=2	b_i	10	8	5	4	2
t=3	b_i	11	9	8	4	2

Table 1 Time dependent input parameters for the production costs.

	i	1	2	3	4	5
t=0	a_i	47.81	51.14	51.32	48.55	43.48
t=1	x_i	49.41	51.14	54.24	48.05	43.09
	$-J_i$	377.23 (-0.80)	459.95	639.95 (-5.83)	503.44	507.09
t=2	x_i	49.41	51.14	54.24	48.05	43.09
	$-J_i$	328.62	408.81	537.30	503.44	507.09
t=3	x_i	45.71	51.14	51.58	48.76	43.64
	$-J_i$	286.75 (-1.85)	379.76	386.92 (-5.31)	527.22	527.81

Table 2 Cournot-Nash equilibria.

	i	1	2	3	4	5
t=0	a_i	47.81	51.14	51.32	48.55	43.48
t=1	x_i	54.95	51.14	53.59	47.52	42.68
	$-J_i$	380.49 (-3.57)	443.52	619.80 (-4.54)	486.00	491.88
t=2	x_i	53.09	51.14	53.59	47.72	42.84
	$-J_i$	329.49 (-0.93)	398.58	523.65	492.55	497.60
t=3	x_i	53.05	50.46	50.77	48.11	43.14
	$-J_i$	289.65 (-0.02)	356.57 (-0.68)	364.33 (-5.64)	505.29	508.71

Table 3 Stackelberg-Cournot-Nash equilibria. Firm 1 is a leader.

5.1 Cournot-Nash equilibria

For the computation of respective Cournot-Nash equilibria we make use of a "nonsmooth" variant of the Gauss-Seidel method described in [10, Algorithm 2]. Using the notation of Section 4, in the main step S.2 one computes the iteration $x^k = (x_1^k, \ldots, x_l^k)$ via l nonsmooth (but convex) consecutive minimizations

$$x_i^k \in \operatorname*{arg\,min}_{x_i \in A_i} f_i(p, x_1^k, \ldots, x_{i-l}^k, x_i, x_{i+1}^{k-1}, \ldots, x_l^{k-1}) + q_i(x_i), i = 1, 2, \ldots, l, \tag{17}$$

where p is the given parameter and $x^{k-1} = (x_1^{k-1}, \ldots, x_l^{k-1})$ is the previous iteration.
A respective modification of the convergence result [10, Theorem 6.1] takes the following form.

Theorem 4 *In addition to the assumptions posed in Section 4 suppose that the sequence x_k converges for $k \to \infty$ to some $x^* = (x_1^*, \ldots, x_l^*) \in \mathbb{R}^l$. Then x^* is a Cournot-Nash equilibrium.*

Proof Observe that for $i = 1, 2, \ldots, l$ the points x_i^k fulfill the optimality condition

$$0 \in \nabla_{x_i} f_i(p, x_1^k, \ldots, x_i^k, x_{i+l}^{k-1}, \ldots, x_l^{k-1}) + \partial q_i(x_i^k) + N_{A_i}(x_i^k).$$

Thus, there are elements $\xi_i^k \in \partial q_i(x_i^k)$ and $\eta_i^k \in N_{A_i}(x_i^k)$ such that, by taking subsequences (without relabeling) if necessary, $\xi_i^k \to \xi_i^*, \eta_i^k \to \eta_i^*$, satisfying the conditions

$$0 \in \nabla_{x_i} f_i(p, x_1^*, \ldots, x_l^*) + \xi_i^* + \eta_i^*, \xi_i^* \in \partial q_i(x_i^*), \eta_i^* \in N_{A_i}(x_i^*), \tag{18}$$

$i = 1, 2, \ldots, l$. This follows from the outer semicontinuity of the subdifferential mapping and from the uniform boundedness of the subdifferentials ∂q_i. Conditions (18) say that x^* is a Cournot-Nash equilibrium and we are done. □

As the stopping rule we employ an approximative version of the optimality conditions (18).

The obtained results are summarized in Table 2 and show the productions and profits (negative total costs) of all firms at single time instances. In parentheses we display the costs of change (with negative signs) which decrease the profits of the Firms 1, 2, 3 in case of any change of their production strategy.

Note that firms 1 and 3 increased significantly their productions in time 1 and decreased them in time 3, whereas Firm 2, due to the cost of change, kept its production unchanged during the whole time.

Figure 3 shows the total cost functions J_i at time 1. Note that the equilibrium production of Firm 2 lies in a kink point because its cost of change is zero. Expectantly, functions J_i are smooth for $i = 4, 5$.
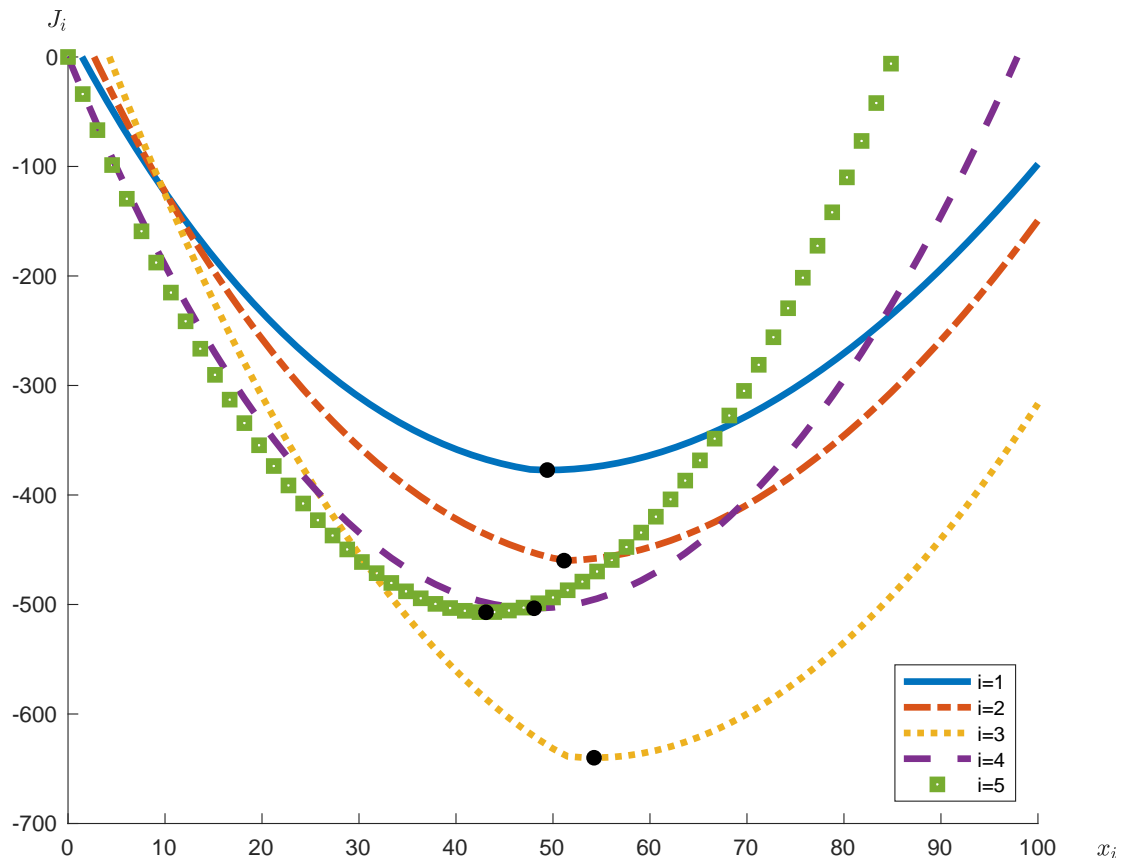

Fig. 3 The total cost functions and Cournot-Nash equilibria for t=1.

5.2 Stackelberg-Cournot-Nash equilibria

Next we will consider the same market as in the previous section where, however, the first producer decides now to replace the non-cooperative by the Stackeberg strategy. This leads to the minimization of Θ over A_1, cf. [14, page 220]. MPEC (14) is, however, more complicated than the MPECs solved with this approach in [14] because the presence of costs of change makes the lower-level equilibrium much more difficult and the objective is not continuously differentiable.

On the other hand, since $n = 1$, we may use for the minimization of Θ any suitable routine for nonsmooth constrained univariate minimization and the same routice may be used inside the Gass-Seidel method for the solution of the "lower-level" problem, needed in the computations of the values of Θ. To this aim we used an inbuilt Matlab function `fminbnd` ([3]).

Remark 2 Numerical results of both subsections 5.1, 5.2 were generated by own Matlab code available freely for downloading and testing at:

$$https:\,//www.\,mathworks.\,com/matlabcentral/fileexchange/72771\,.$$

The code is flexible and allows for easy modifications to different models.

The obtained results are summarized in Table 3. They are quite different from their counterparts in Table 2 and show that, switching to the Stackelberg strategy, Firm 1 substantially improves its profit. In contrast to the noncooperative strategy, it has now to change its production at each time step and also Firm 2, who preserved in the Cournot-Nash case the same production over the whole time, is now forced to change it at $t = 2$. Of course, our data are purely academic and can hardly be used for some economic interpretations. On the other hand, the results are sound and show the potential of the suggested techniques to be applied also in some more realistic situations.

Conclusion

In the first half of the paper we have studied a parametrized variational inequality of the second kind. In this form, one can write down, for example, a condition which characterizes solutions of some parameter-dependent Nash equilibrium problems. By using standard tools of variational analysis sufficient conditions have been derived ensuring the existence of a single-valued and Lipschitzian localization of the respective solution mapping. Apart from post-optimal analysis, the obtained results can be used in computation of respective equilibria for given values of the parameter via continuation ([1]) or Newton-type methods ([9]).

The second part of the paper has been inspired, on one hand, by the successful theory of rate-independence processes ([12],[7]) and, on the other hand, by the important economic paper [6]. It turns out that in some market models the cost of change of the production strategy can be viewed as the economic counterpart of the dissipation energy, arising in rate independent dissipative models of nonlinear mechanics of solids. Cost of change (dissipation energy) occurs further, e.g., in modeling the behavior of some national banks who try to regulate the inflation rate, among other instruments, via buying or selling suitable amounts of the domestic currency on international financial markets [18].

The considered market model, obtained by augmenting the cost of change to a standard model from [13] possesses very strong stability properties and is amenable to various numerical approaches. Both in the case of the Cournot-Nash equilibrium as well as in the two-level (Stackelberg) case we have used rather simple techniques based on the "nonsmooth" line search from [3]. In case of multiple commodities and/or more complicated costs of change one could employ more sophisticated approaches based either on the second-order subdifferentials or on the second subderivatives discussed in Section 3. This will be the subject of our future work.

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