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Logarithmically improved regularity criterion for the 3D Hall-MHD equations

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Abstract In this work, we study the blow-up criterion of the smooth solutions of three-dimensional incompressible Hall-Magnetohydrohynamics equations (in short, Hall-MHD). We obtain a logarithmically improved regularity criterion of smooth solutions in terms of the $\dot{B}^0_{\infty,\infty}$ norm. We improve the blow-up criterion for smooth solutions established in [37].

Keywords Hall- Magnetohydrodynamic equations · smooth solutions · Besov space · blow up criterion.

Mathematics Subject Classification 35B65 · 76W05.

1 Introduction

This work focuses on the study of the blow-up criterion of the smooth solutions to the following incompressible Hall-Magnetohydrodynamics (Hall-MHD) system

$$\partial_t u + (u\cdot\nabla)u - \Delta u + \nabla\pi = (B\cdot\nabla)B, \tag{1a}$$

$$\partial_t B + (u\cdot\nabla)B - \Delta B + \nabla\times[(\nabla\mathbb{R}\times B)\times B] = (B\cdot\nabla)u, \tag{1b}$$

$$\nabla\cdot u = \nabla\cdot B = 0, \tag{1c}$$

$$(u,B)(x,0) = (u_0(x), B_0(x)). \tag{1d}$$

where $u = u(x,t) \in \mathbb{R}^3$, $B = B(x,t) \in \mathbb{R}^3, \pi = \pi(x,t) \in \mathbb{R}$ represent the unknown velocity field, the magnetic field and the pressure, respectively. The initial data for the velocity and magnetic fields, given by u_0 and B_0 in system (1), are divergence-free, i.e., $\nabla\cdot u_0 = \nabla\cdot B_0 = 0$.

The Hall-effect term $\nabla\times[(\nabla\times B)\times B]$ represents deviation at a small scale from charge neutrality between the ions and the electrons, and we assume incompressibility $\nabla\cdot u = 0$ of the bulk plasma. The Hall-MHD equations have important applications in fluid mechanics and material sciences, such as, star formation, magnetic reconnection in space plasmas, neutron stars and geo-dynamo (see [20, 26, 29]). Due to these

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various applications, the literature on Hall-MHD equations and in particular on local existence and uniqueness of solutions is already vast and has attracted considerable attention in the community of mathematical fluids dynamics (see, e.g., [1, 5–8]). Due to the presence of Navier-Stokes equations in the system (1), it is a challenging problem to know whether a unique local solution can exist globally. The literature on he Hall-MHD equations was initiated by M. J. Lighthill [26] and subsently developed by various authors with the aim of finding sufficient conditions in terms of the velocity field, the magnetic field, the pressure and their derivatives to ensure the smoothness of the solutions. Parts of them are listed here (see [2, 3, 9–11, 14–19, 22, 23, 31–34, 36] and references therein).

Recently, Z. Ye [37] considered a blow-up criterion in terms of the velocity and the magnetic field in the homogeneous Besov space of negative index. Precisely, if

$$\int_0^T \frac{\|u(\cdot,\tau)\|_{\dot{B}^{-\beta}_{\infty,\infty}}^{\frac{2}{1-\beta}} + \|\nabla B(\cdot,\tau)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}}{\ln(e+\|u(\cdot,\tau)\|_{\dot{B}^{-\beta}_{\infty,\infty}} + \|\nabla B(\cdot,\tau)\|_{\dot{B}^{-\alpha}_{\infty,\infty}})} d\tau < \infty, \tag{2}$$

for $0 < \alpha, \beta < 1$, then the solution (u,B) can be extended smoothly beyond the time T. Here and hereafter, $\dot{B}^{-\alpha}_{\infty,\infty}$ stands for the homogenous Besov space (see, e.g., [30] for the definition). Notice that the limiting case

$$\int_0^T \left(\frac{\|u(\cdot,\tau)\|^2_{\dot{B}^0_{\infty,\infty}}}{\ln(e+\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}})} + \|\nabla B(\cdot,\tau)\|^2_{\dot{B}^0_{\infty,\infty}} \right) d\tau < \infty,$$

was not covered in [37]. Hence, the regularity for (1) with $\alpha = \beta = 0$ is currently open.

The purpose of the present paper is to establish a regularity criterion for the problem (1), which can be r viewed as the end-point case of (2). Our regularity criterion is expressed in terms of the $\dot{B}^0_{\infty,\infty}$ norm. More precisely, we have the following theorem.

Theorem 1.1 *Assume that the initial velocity and magnetic field* $(u_0, B_0) \in H^3(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$. *Let* (u,B) *be a local smooth solution to the system (1) for* $0 \le t < T$. *If* (u,B) *satisfies the following condition*

$$\int_0^T \left(\frac{\|u(\cdot,\tau)\|^2_{\dot{B}^0_{\infty,\infty}}}{\ln(e+\|u(\cdot,\tau)\|^2_{\dot{B}^0_{\infty,\infty}})} + \|\nabla B(\cdot,\tau)\|^2_{\dot{B}^0_{\infty,\infty}} \right) d\tau < \infty, \tag{3}$$

then the solution (u,B) *can be extended beyond the time* T.

As a consequence of the fact that $\|u\|_{\dot{B}^0_{\infty,\infty}} \approx \|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}}$, we have the following result:

Corollary 1.1 *Assume that the initial velocity and magnetic field* $(u_0, B_0) \in H^3(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$. *Let* (u,B) *be a local smooth solution to the system (1) for* $0 \le t < T$. *If* (u,B) *satisfies the following condition*

$$\int_0^T \left(\frac{\|\nabla u(\cdot,\tau)\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\ln(e+\|\nabla u(\cdot,\tau)\|^2_{\dot{B}^{-1}_{\infty,\infty}})} + \|\nabla B(\cdot,\tau)\|^2_{\dot{B}^0_{\infty,\infty}} \right) d\tau < \infty,$$

then the solution (u,B) *can be extended beyond the time* T.

2 Preliminaries

To complete the proof of Theorem 1.1, we will need the following logarithmic Sobolev inequality in Besov spaces.

Lemma 2.1 **([25])** *There exists a constant c such that the following inequality*

$$\|f\|_{BMO} \leq c \left[1 + \|f\|_{\dot{B}^0_{\infty,\infty}} \left(\log^+ \|f\|_{H^2} \right)^{\frac{1}{2}} \right]$$

holds for all $f \in H^2(\mathbb{R}^3)$*, where* $H^2(\mathbb{R}^3)$ *denotes the inhomogeneous Sobolev space and*

$$\ln^+ x = \begin{cases} \ln x, & \text{if } \; x > e, \\ 1, & \text{if } \; 0 \leq x \leq e. \end{cases}$$

Here, $BMO(\mathbb{R}^3)$ *is the space of functions of bounded mean oscillations.*

Lemma 2.2 **(Commutator estimate [24])** *Let* $s > 0$, $1 < p < \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ *with* $p_1, q_2 \in [1, +\infty]$ *and* $q_1, p_2 \in (1, +\infty)$*. Then, there exists a constant* $C > 0$ *such that*

$$\|[\Lambda^s, f \cdot \nabla] g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}} \|\Lambda^s g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|\nabla g\|_{L^{q_2}}), \tag{4}$$

for

$$f \in \dot{W}^{1,p_1} \cap \dot{W}^{s,p_2} \quad \text{and} \quad g \in \dot{W}^{s,q_1} \cap L^{q_2}.$$

with $[\Lambda^s, f \cdot \nabla] g = \Lambda^s(f \cdot \nabla g) - f \Lambda^s(\nabla g)$*.*

Next, we introduce the following Kozono-Taniuchi's inequality [12]

Lemma 2.3 **([12])** *Let* $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ *and* $\beta = (\beta_1, \beta_2, \beta_3)$ *are multi-indices with* $|\alpha| = |\beta| = 1$*. Assume that* $f, g \in BMO(\mathbb{R}^3) \cap H(\mathbb{R}^3)^{|\alpha|+|\beta|}$*. Then, there exists a constant* $C > 0$ *such that*

$$\left\| \partial^\alpha f \cdot \partial^\beta g \right\|_{L^2} \leq C \left(\|f\|_{BMO} \left\| (-\Delta)^{\frac{|\alpha|+|\beta|}{2}} g \right\|_{L^2} + \|g\|_{BMO} \left\| (-\Delta)^{\frac{|\alpha|+|\beta|}{2}} f \right\|_{L^2} \right). \tag{5}$$

3 Proof of Theorem 1.1

The proof is based on the establishment of a priori estimates for (u, B) which can then be used to extend a smooth local solution to time T. Under the condition (3), it suffices to show that, there exists a constant $C > 0$ such that

$$\limsup_{t \overset{<}{\to} T} \left(\|u(\cdot, t)\|^2_{H^3} + \|B(\cdot, t)\|^2_{H^3} \right) \leq C < \infty,$$

which is enough to guarantee the extension of smooth solution (u, B) beyond the time T. Troughout the paper, C stands for some real positive constants which may be different in each occurrence.

Proof If (3) holds, one can deduce that for any small constant $\varepsilon > 0$, there exists $T_0 = T_0(\varepsilon) < T$ such that

$$\int_{T_0}^{T} \left(\frac{\|u(\cdot, \tau)\|^2_{\dot{B}^0_{\infty,\infty}}}{\ln(e + \|u(\cdot, \tau)\|^2_{\dot{B}^0_{\infty,\infty}})} + \|\nabla B(\cdot, \tau)\|^2_{\dot{B}^0_{\infty,\infty}} \right) d\tau \leq \varepsilon. \tag{6}$$

Firstly, testing (1a) by u and using (1c), we infer that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 \, dx + \int_{\mathbb{R}^3} |\nabla u|^2 \, dx = \int_{\mathbb{R}^3} (B \cdot \nabla) B \cdot u dx. \tag{7}$$

Testing (1b) by B and using (1c), we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|B|^2\,dx+\int_{\mathbb{R}^3}|\nabla B|^2\,dx=\int_{\mathbb{R}^3}(B\cdot\nabla)u\cdot Bdx, \tag{8}$$

which follows from the simple fact

$$\int_{\mathbb{R}^3}\nabla\times((\nabla\times B)\times B)\cdot Bdx=\int_{\mathbb{R}^3}((\nabla\times B)\times B)\cdot(\nabla\times B)dx=0.$$

Summing up (7) and (8), it follows that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(|u|^2+|B|^2)dx+\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla B|^2)dx=0, \tag{9}$$

where the following identities have been applied, due to $\nabla\cdot u=\nabla\cdot B=0$,

$$\int_{\mathbb{R}^3}(B\cdot\nabla)B\cdot udx+\int_{\mathbb{R}^3}(B\cdot\nabla)u\cdot Bdx=0,$$

Integrating (9) in time, we get

$$\sup_{0<t<T}(\|u(t)\|_{L^2}^2+\|B(t)\|_{L^2}^2)+\int_0^T(\|\nabla u(t)\|_{L^2}^2+\|\nabla B(t)\|_{L^2}^2)dt\le\|u_0\|_{L^2}^2+\|B_0\|_{L^2}^2.$$

Next, we are going to derive estimates of ∇u and ∇B. Multiplying the first equation of (1a) by $-\Delta u$, after integration by parts and taking the divergence free property into account, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla u|^2\,dx+\int_{\mathbb{R}^3}|\Delta u|^2\,dx\\
&=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\cdot\nabla udx+\int_{\mathbb{R}^3}\nabla(B\cdot\nabla B)\cdot\nabla udx\\
&=-\sum_{i=1}^{3}\int_{\mathbb{R}^3}[(\partial_i u\cdot\nabla)u]\cdot\partial_i udx\\
&+\sum_{i=1}^{3}\int_{\mathbb{R}^3}[(\partial_i B\cdot\nabla)B]\cdot\partial_i udx+\sum_{i=1}^{3}\int_{\mathbb{R}^3}[(B\cdot\nabla)\partial_i B]\cdot\partial_i udx.
\end{aligned} \tag{10}$$

Similarly, multiplying the second one by $-\Delta B$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla B|^2\,dx+\int_{\mathbb{R}^3}|\Delta B|^2\,dx\\
&=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla B)\cdot\nabla Bdx+\int_{\mathbb{R}^3}\nabla(B\cdot\nabla u)\cdot\nabla Bdx-\int_{\mathbb{R}^3}\nabla((\nabla\times B)\times B)\cdot\nabla(\nabla\times B)dx.\\
&=-\sum_{i=1}^{3}\int_{\mathbb{R}^3}[(\partial_i u\cdot\nabla)B]\cdot\partial_i Bdx+\sum_{i=1}^{3}\int_{\mathbb{R}^3}[(\partial_i B\cdot\nabla)u]\cdot\partial_i Bdx+\sum_{i=1}^{3}\int_{\mathbb{R}^3}[(B\cdot\nabla)\partial_i u]\cdot\partial_i Bdx\\
&-\int_{\mathbb{R}^3}[\nabla(\nabla\times((\nabla\times B)\times B)))]\cdot\nabla Bdx.
\end{aligned} \tag{11}$$

Summing up (10) and (11) and using the fact that

$$\sum_{i=1}^{3}\int_{\mathbb{R}^3}[(B\cdot\nabla)\partial_i B]\cdot\partial_i udx+\sum_{i=1}^{3}\int_{\mathbb{R}^3}[(B\cdot\nabla)\partial_i u]\cdot\partial_i Bdx=0,$$

yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)+\|\Delta u\|_{L^2}^2+\|\Delta B\|_{L^2}^2\\
&=-\sum_{i=1}^{3}\int_{\mathbb{R}^3}[(\partial_i u\cdot\nabla)u]\cdot\partial_i u dx+\sum_{i=1}^{3}\int_{\mathbb{R}^3}[(\partial_i B\cdot\nabla)B]\cdot\partial_i u dx-\sum_{i=1}^{3}\int_{\mathbb{R}^3}[(\partial_i u\cdot\nabla)B]\cdot\partial_i B dx\\
&+\sum_{i=1}^{3}\int_{\mathbb{R}^3}[(\partial_i B\cdot\nabla)u]\cdot\partial_i B dx-\int_{\mathbb{R}^3}[\nabla(\nabla\times((\nabla\times B)\times B)))]\cdot\nabla B dx\\
&=I_1+I_2+I_3+I_4+I_5.
\end{aligned}
\tag{12}
$$

In the following calculations, we use the following interpolation inequality [27] :

$$
\|\nabla u\|_{L^4}^2\leq C\|u\|_{\dot{B}^0_{\infty,\infty}}\|\Delta u\|_{L^2}. \tag{13}
$$

Using (13), the Hölder and the Young inequalities, permits to obtain an upper bound for I_1 :

$$
\begin{aligned}
I_1&\leq\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2\\
&\leq C\|u\|_{\dot{B}^0_{\infty,\infty}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\\
&\leq\frac{1}{4}\|\Delta u\|_{L^2}^2+C\|u\|_{\dot{B}^0_{\infty,\infty}}^2\|\nabla u\|_{L^2}^2.
\end{aligned}
\tag{14}
$$

Using the following interpolation inequality [12]:

$$
\|\nabla B\|_{L^4}^2\leq C\|\nabla B\|_{L^2}\|\nabla B\|_{BMO}\cdot, \tag{15}
$$

we derive

$$
\begin{aligned}
I_2+I_3+I_4&\leq C\|\nabla u\|_{L^2}\|\nabla B\|_{L^4}^2\\
&\leq C\|\nabla u\|_{L^2}\|\nabla B\|_{L^2}\|\nabla B\|_{BMO}\\
&\leq C\left(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)\|\nabla B\|_{BMO}\\
&\leq C\left(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)(1+\|\nabla B\|_{BMO}^2).
\end{aligned}
\tag{16}
$$

Using the following cancellation property

$$
\int_{\mathbb{R}^3}((\nabla(\nabla\times B))\times B))\cdot\nabla(\nabla\times B)dx=0,
$$

we deduce from (15) that

$$
\begin{aligned}
I_5&=\int_{\mathbb{R}^3}\nabla[(\nabla\times B)\times B]\cdot\nabla(\nabla\times B)-(\nabla(\nabla\times B)\times B))\cdot\nabla(\nabla\times B)dx\\
&=-\sum_{i=1}^{3}\int_{\mathbb{R}^3}[\partial_i((\nabla\times B)\times B)-(\partial_i(\nabla\times B))\times B]\cdot\partial_i(\nabla\times B)dx\\
&=-\sum_{i=1}^{3}\int_{\mathbb{R}^3}(\nabla\times B)\times\partial_i B\cdot\partial_i(\nabla\times B)dx\\
&\leq C\|\nabla B\|_{L^4}^2\|\Delta B\|_{L^2}\\
&\leq C\|\nabla B\|_{BMO}\|\nabla B\|_{L^2}\|\Delta B\|_{L^2}\\
&\leq\frac{1}{2}\|\Delta B\|_{L^2}^2+C\|\nabla B\|_{BMO}^2\|\nabla B\|_{L^2}^2.
\end{aligned}
\tag{17}
$$

Inserting (14), (16) and (17) into (12) yields

$$\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)+\|\Delta u\|_{L^2}^2+\|\Delta B\|_{L^2}^2$$
$$\leq C(1+\|u\|_{\dot{B}^0_{\infty,\infty}}^2+\|\nabla B\|_{BMO}^2)(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2). \tag{18}$$

Applying the standard Gronwall's inequality to (18), one gets for any $t\in[T_0,T)$

$$\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla B(\cdot,t)\|_{L^2}^2+\int_{T_0}^t(\|\Delta u(\cdot,\tau)\|_{L^2}^2+\|\Delta B(\cdot,\tau)\|_{L^2}^2)d\tau$$
$$\leq\left(\|\nabla u(\cdot,T_0)\|_{L^2}^2+\|\nabla B(\cdot,T_0)\|_{L^2}^2\right)\exp\left(C\int_{T_0}^t(1+\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2+\|\nabla B(\cdot,\tau)\|_{BMO}^2)d\tau\right).$$

For any $t\in[T_0,T)$, we note

$$F(t)=\sup_{T_0\leq\tau\leq t}\left(\|u(\cdot,\tau)\|_{H^3}^2+\|B(\cdot,\tau)\|_{H^3}^2\right).$$

It should be noted that $F(t)$ is a nondecreasing function. Using Lemma 2.1, and the fact that $H^3(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)\hookrightarrow\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$, one obtains

$$\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla B(\cdot,t)\|_{L^2}^2+\int_{T_0}^t(\|\Delta u(\cdot,\tau)\|_{L^2}^2+\|\Delta B(\cdot,\tau)\|_{L^2}^2)d\tau$$
$$\leq C(T_0)\exp\left(\widetilde{C}\int_{T_0}^t\left[\frac{\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2}{\ln(e+\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2)}\ln(e+\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2)+\|\nabla B(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2\ln(e+\|B(\cdot,\tau)\|_{H^3}^2)\right]d\tau\right)$$
$$\leq C(T_0)\exp\left(\widetilde{C}\int_{T_0}^t\left[\frac{\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2}{\ln(e+\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2)}\ln(e+\|u(\cdot,\tau)\|_{L^\infty}^2)+\|\nabla B(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2\ln(e+\|B(\cdot,\tau)\|_{H^3}^2)\right]d\tau\right)$$
$$\leq C(T_0)\exp\left(\widetilde{C}\int_{T_0}^t\left[\frac{\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2}{\ln(e+\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2)}+\|\nabla B(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2\right]\ln(e+\|u(\cdot,\tau)\|_{H^3}^2+\|B(\cdot,\tau)\|_{H^3}^2)d\tau\right)$$
$$\leq C(T_0)\exp\left(\widetilde{C}\int_{T_0}^t\left[\frac{\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2}{\ln(e+\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2)}+\|\nabla B(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2\right]\ln(e+F(\tau))d\tau\right)$$
$$\leq C(T_0)\exp\left(\widetilde{C}\int_{T_0}^t\left[\frac{\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2}{\ln(e+\|u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2)}+\|\nabla B(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2\right]d\tau\ln(e+F(t))\right)$$
$$\leq C(T_0)\exp\left(\widetilde{C}\varepsilon\ln(e+F(t))\right)\leq C(T_0)\,(e+F(t))^{\widetilde{C}\varepsilon}, \tag{19}$$

where $\widetilde{C}$ is an absolute constant and $C(T_0)$ depends on $\|\nabla u(\cdot,T_0)\|_{L^2},\|\nabla B(\cdot,T_0)\|_{L^2},T_0,T$. Finally under the H^1 estimates of ∇u and ∇B, we will show that

$$F(t)\leq C<\infty,\ \text{ for any } t\in[T_0,T).$$

Applying the operator ∇^3 to equations (1a) and (1b), multiplying the resulting equations by $\nabla^3 u$ and $\nabla^3 B$ respectively, adding them up and using the incompressible conditions $\nabla\cdot u=\nabla\cdot B=0$, yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\left\|\nabla^3 u\right\|_{L^2}^2+\left\|\nabla^3 B\right\|_{L^2}^2)+\left\|\nabla^4 u\right\|_{L^2}^2+\left\|\nabla^4 B\right\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}\nabla^3(u\cdot\nabla u)\cdot\nabla^3 u dx-\int_{\mathbb{R}^3}\nabla^3(u\cdot\nabla B)\cdot\nabla^3 B dx\\
&+\int_{\mathbb{R}^3}\nabla^3(B\cdot\nabla u)\cdot\nabla^3 B dx+\int_{\mathbb{R}^3}\nabla^3(B\cdot\nabla B)\cdot\nabla^3 u dx-\int_{\mathbb{R}^3}\nabla^3((\nabla\times B)\times B)\cdot\nabla^3(\nabla\times B)dx\\
&-\int_{\mathbb{R}^3}\left[\nabla^3,u\cdot\nabla\right]u\cdot\nabla^3 u dx-\int_{\mathbb{R}^3}\left[\nabla^3,u\cdot\nabla\right]B\cdot\nabla^3 B dx+\int_{\mathbb{R}^3}\left[\nabla^3,B\cdot\nabla\right]u\cdot\nabla^3 B dx\\
&+\int_{\mathbb{R}^3}\left[\nabla^3,B\cdot\nabla\right]B\cdot\nabla^3 B dx-\int_{\mathbb{R}^3}\nabla^3[\nabla\times((\nabla\times B)\times B)]\cdot\nabla^3 B dx\\
&=J_1+J_2+J_3+J_4+J_5.
\end{aligned}
$$

Combining Hölder's inequality, the commutator estimate (4) and the Gagliardo-Nirenberg inequality

$$\left\|\nabla^3 u\right\|_{L^4}\le C\left\|\nabla^2 u\right\|_{L^4}^{\frac{1}{8}}\left\|\nabla^4 u\right\|_{L^4}^{\frac{7}{8}},$$

gives an upper bound estimate of J_1:

$$
\begin{aligned}
J_1&=-\int_{\mathbb{R}^3}\left[\nabla^3,u\cdot\nabla\right]u\cdot\nabla^3 u dx\le\left\|\left[\nabla^3,u\cdot\nabla\right]u\right\|_{L^{\frac{4}{3}}}\left\|\nabla^3 u\right\|_{L^4}\\
&\le C(\|\nabla u\|_{L^2}\left\|\nabla^3 u\right\|_{L^4}+\left\|\nabla^3 u\right\|_{L^4}\|\nabla u\|_{L^2})\left\|\nabla^3 u\right\|_{L^4}\\
&\le C\|\nabla u\|_{L^2}\left\|\nabla^3 u\right\|_{L^4}^2\\
&\le C\|\nabla u\|_{L^2}\left\|\nabla^2 u\right\|_{L^2}^{\frac{1}{4}}\left\|\nabla^4 u\right\|_{L^2}^{\frac{7}{4}}\\
&\le\frac{1}{6}\left\|\nabla^4 u\right\|_{L^2}^2+C\|\nabla u\|_{L^2}^8\left\|\nabla^2 u\right\|_{L^2}^2.
\end{aligned}
$$

Similarly, the terms J_2, J_3 and J_4 can be bounded from above as follows:

$$
\begin{aligned}
J_2&=-\int_{\mathbb{R}^3}\left[\nabla^3,u\cdot\nabla\right]B\cdot\nabla^3 B dx\\
&\le C\left(\|\nabla u\|_{L^2}\left\|\nabla^3 B\right\|_{L^4}+\left\|\nabla^3 u\right\|_{L^4}\|\nabla B\|_{L^2}\right)\left\|\nabla^3 B\right\|_{L^4}\\
&\le C\|\nabla u\|_{L^2}\left\|\nabla^2 B\right\|_{L^2}^{\frac{1}{4}}\left\|\nabla^4 B\right\|_{L^2}^{\frac{7}{4}}+C\|\nabla B\|_{L^2}\left\|\nabla^2 u\right\|_{L^4}^{\frac{1}{8}}\left\|\nabla^4 u\right\|_{L^4}^{\frac{7}{8}}\left\|\nabla^2 B\right\|_{L^4}^{\frac{1}{8}}\left\|\nabla^4 B\right\|_{L^4}^{\frac{7}{8}}\\
&\le\frac{1}{6}\left\|\nabla^4 u\right\|_{L^2}^2+\frac{1}{6}\left\|\nabla^4 B\right\|_{L^2}^2+C(\|\nabla u\|_{L^2}^8+\|\nabla B\|_{L^2}^8)(\left\|\nabla^2 u\right\|_{L^2}^2+\left\|\nabla^2 B\right\|_{L^2}^2);
\end{aligned}
$$

$$
\begin{aligned}
J_3&=\int_{\mathbb{R}^3}\left[\nabla^3,B\cdot\nabla\right]u\cdot\nabla^3 B dx\\
&\le C\left(\|\nabla B\|_{L^2}\left\|\nabla^3 u\right\|_{L^4}+\left\|\nabla^3 B\right\|_{L^4}\|\nabla u\|_{L^2}\right)\left\|\nabla^3 B\right\|_{L^4}\\
&\le C\|\nabla B\|_{L^2}\left\|\nabla^3 u\right\|_{L^4}\left\|\nabla^3 B\right\|_{L^4}+C\|\nabla u\|_{L^2}\left\|\nabla^3 B\right\|_{L^4}^2\\
&\le\frac{1}{6}\left\|\nabla^4 u\right\|_{L^2}^2+\frac{1}{6}\left\|\nabla^4 B\right\|_{L^2}^2+C(\|\nabla u\|_{L^2}^8+\|\nabla B\|_{L^2}^8)(\left\|\nabla^2 u\right\|_{L^2}^2+\left\|\nabla^2 B\right\|_{L^2}^2);
\end{aligned}
$$

$$
\begin{aligned}
J_4&=-\int_{\mathbb{R}^3}\left[\nabla^3,B\cdot\nabla\right]B\cdot\nabla^3 B dx\le\left\|\left[\nabla^3,B\cdot\nabla\right]B\right\|_{L^{\frac{4}{3}}}\left\|\nabla^3 B\right\|_{L^4}\\
&\le C\|\nabla B\|_{L^2}\left\|\nabla^3 B\right\|_{L^4}^2\\
&\le\frac{1}{6}\left\|\nabla^4 B\right\|_{L^2}^2+C\|\nabla B\|_{L^2}^8\left\|\nabla^2 B\right\|_{L^2}^2.
\end{aligned}
$$

From the following cancellation property

$$\int_{\mathbb{R}^3} \left(\nabla^3(\nabla\times B)\times B\right)\cdot\nabla^3(\nabla\times B)dx = 0,$$

and the Leibnitz formula, one has that

$$\begin{aligned}
J_5 &= -\int_{\mathbb{R}^3} \nabla^3((\nabla\times B)\times B)\cdot\nabla^3(\nabla\times B)dx\\
&\le \left|\int_{\mathbb{R}^3}\left[\nabla^3((\nabla\times B)\times B)\cdot\nabla^3(\nabla\times B) - \left(\nabla^3(\nabla\times B)\times B\right)\cdot\nabla^3(\nabla\times B)\right]dx\right|\\
&\le C\left|\int_{\mathbb{R}^3}(\nabla^2(\nabla\times B)\times\nabla B)\cdot\nabla^3(\nabla\times B)dx\right|\\
&\quad +C\left|\int_{\mathbb{R}^3}(\nabla(\nabla\times B)\times\nabla^2 B)\cdot\nabla^3(\nabla\times B)dx\right|\\
&\quad +C\left|\int_{\mathbb{R}^3}((\nabla\times B)\times\nabla^3 B)\cdot\nabla^3(\nabla\times B)dx\right|\\
&= J_5^1 + J_5^2 + J_5^3.
\end{aligned}$$

In order to handle J_5^1 and J_5^3, we recall the following property of Hardy space $\mathscr{H}^1$ and BMO [13] (see also [21]) :

$$\int_{\mathbb{R}^3} fghdx \le C\|f\|_{BMO}\|gh\|_{\mathscr{H}^1} \le C\|f\|_{BMO}\|g\|_{L^2}\|h\|_{L^2}. \tag{20}$$

The two terms J_5^1 and J_5^3 can be estimated by using (20)

$$\begin{aligned}
J_5^1 &\le C\|\nabla B\|_{BMO}\left\|\nabla^2(\nabla\times B)\cdot\nabla^3(\nabla\times B)\right\|_{\mathscr{H}^1}\\
&\le C\|\nabla B\|_{BMO}\left\|\nabla^3 B\right\|_{L^2}\left\|\nabla^4 B\right\|_{L^2}\\
&\le \frac{1}{6}\left\|\nabla^4 B\right\|_{L^2}^2 + C\|\nabla B\|_{BMO}^2\left\|\nabla^3 B\right\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
J_5^3 &\le C\|\nabla\times B\|_{BMO}\left\|\nabla^3 B\cdot\nabla^3(\nabla\times B)\right\|_{\mathscr{H}^1}\\
&\le C\|\nabla B\|_{BMO}\left\|\nabla^3 B\right\|_{L^2}\left\|\nabla^4 B\right\|_{L^2}\\
&\le \frac{1}{6}\left\|\nabla^4 B\right\|_{L^2}^2 + C\|\nabla B\|_{BMO}^2\left\|\nabla^3 B\right\|_{L^2}^2.
\end{aligned}$$

The term J_5^2 can be estimated as follows. By using the Hölder inequality, (5) with $|\alpha| = |\beta| = 1$ and the Young inequality, we obtain

$$\begin{aligned}
J_5^2 &\le C\left\|\nabla^3(\nabla\times B)\right\|_{L^2}\left\|\nabla^2 B\cdot\nabla^2 B\right\|_{L^2}\\
&\le C\left\|\nabla^4 B\right\|_{L^2}\left(\|\nabla B\|_{BMO}\left\|\nabla^3 B\right\|_{L^2} + \|\nabla B\|_{BMO}\left\|\nabla^3 B\right\|_{L^2}\right)\\
&\le \frac{1}{6}\left\|\nabla^4 B\right\|_{L^2}^2 + C\|\nabla B\|_{BMO}^2\left\|\nabla^3 B\right\|_{L^2}^2\\
&\le \frac{1}{6}\left\|\nabla^4 B\right\|_{L^2}^2 + C\left((1+\|\nabla B\|_{\dot{B}^0_{\infty,\infty}}^2\ln(e+\|B\|_{H^3})\right)\left\|\nabla^3 B\right\|_{L^2}^2.
\end{aligned}$$

Combining the previous estimates, one obtains

$$\begin{aligned}
&\frac{d}{dt}(\left\|\nabla^3 u\right\|_{L^2}^2 + \left\|\nabla^3 B\right\|_{L^2}^2)\\
&\le C(\|\nabla u\|_{L^2}^8 + \|\nabla B\|_{L^2}^8)(\left\|\nabla^2 u\right\|_{L^2}^2 + \left\|\nabla^2 B\right\|_{L^2}^2)\\
&+C\left((1+\|\nabla B\|_{\dot{B}^0_{\infty,\infty}}^2\ln(e+\|B\|_{H^3})\right)\left\|\nabla^3 B\right\|_{L^2}^2,
\end{aligned}$$

which together with the basic energy estimate (9) yields

$$\begin{aligned}
&\frac{d}{dt}(\|u(\cdot,t)\|_{H^3}^2+\|B(\cdot,t)\|_{H^3}^2)\\
&\leq C(\|\nabla u\|_{L^2}^8+\|\nabla B\|_{L^2}^8)(\left\|\nabla^2 u\right\|_{L^2}^2+\left\|\nabla^2 B\right\|_{L^2}^2)\\
&+C\left((1+\|\nabla B\|_{\dot{B}^0_{\infty,\infty}}^2\ln(e+\|B\|_{H^3})\right)\|B\|_{H^3}^2.
\end{aligned}$$

Integrating this inequality over (T_0,t), we get by (19) that

$$\begin{aligned}
&\|u(\cdot,t)\|_{H^3}^2+\|B(\cdot,t)\|_{H^3}^2\\
&\leq \|u(\cdot,T_0)\|_{H^3}^2+\|B(\cdot,T_0)\|_{H^3}^2+C\int_{T_0}^t\|\nabla B(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2\ln(e+\|B(\cdot,\tau)\|_{H^3})\,\|B(\cdot,\tau)\|_{H^3}^2\,d\tau\\
&+C(T_0)\int_{T_0}^t[e+F(\tau)]^{4C\varepsilon}\,(\left\|\nabla^2 u(\cdot,\tau)\right\|_{L^2}^2+\left\|\nabla^2 B(\cdot,\tau)\right\|_{L^2}^2)d\tau\\
&\leq \|u(\cdot,T_0)\|_{H^3}^2+\|B(\cdot,T_0)\|_{H^3}^2+C\int_{T_0}^t\|\nabla B(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2\ln(e+F(\tau))F(\tau)d\tau\\
&+C(T_0)\sup_{\tau\in(T_0,t)}[e+F(\tau)]^{4C\varepsilon}\int_{T_0}^t(\left\|\nabla^2 u(\cdot,\tau)\right\|_{L^2}^2+\left\|\nabla^2 B(\cdot,\tau)\right\|_{L^2}^2)d\tau\\
&\leq \|u(\cdot,T_0)\|_{H^3}^2+\|B(\cdot,T_0)\|_{H^3}^2+C\int_{T_0}^t\|\nabla B(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2\ln(e+F(\tau))F(\tau)d\tau\\
&+C(T_0)\,[e+F(t)]^{5C\varepsilon}
\end{aligned}\tag{21}$$

Choosing ε small enough such that $\varepsilon<\frac{1}{5C}$, from the above inequality we derive

$$e+F(t)\leq e+F(T_0)+C\int_{T_0}^t\|\nabla B(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}}^2(e+F(\tau))\ln(e+F(\tau))d\tau+C(T_0)(e+F(t)).$$

Applying the standard Log-Gronwall argument (for instance, Chapter 3 of [4]), one can conclude that

$$F(t)\leq C-e<\infty,\ \text{ for any } t\in[T_0,T).$$

This implies that $(u,B)\in L^\infty(0,T;H^3(\mathbb{R}^3))$. Thus, (u,B) can be extended smoothly beyond $t=T$. This completes the proof of Theorem 1.1.

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