Oscillations in Low-Dimensional Cyclic Differential Delay Systems

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Abstract Nonlinear autonomous N-dimensional systems of cyclic differential equations with delays and overall negative feedback are considered. Such systems serve as mathematical models of numerous real world phenomena in physics and laser optics, physiology and mathematical biology, economics and life sciences among others. In the case of lower dimensions $N = 2$ and $N = 3$ sufficient conditions are derived for the oscillation of all solutions about the unique equilibrium. Open problems and conjectures are discussed for the higher dimensional case $N \geq 4$ and for more convoluted sign feedbacks.

1 Introduction

We consider a system of delay differential equations of the form

$$x_i'(t) = -\alpha_i x_i(t) + f_i(x_{i+1}(t - \tau_{i+1})),\ 1 \leq i \leq N, \tag{1}$$

where the functions $f_i(u)$ are real-valued and continuous on $\mathbf{R}$, $f_i \in C(\mathbf{R},\mathbf{R})$, the decay rates $\alpha_i > 0$ are positive, and the delays τ_i are non-negative with the total delay $\tau = \sum_{i=1}^{N} \tau_i > 0$ being positive. The system is a cyclic one with the variables x_{N+1} and τ_{N+1} defined as x_1 and τ_1, respectively, for the index value $i = N$.

Systems of form (1) are used in various applications, including physics and laser optics, physiology and mathematical biology, economics and life sciences among others. In particular, they naturally appear in physiology and mathematical biology [5, 7, 9, 10, 14], where they serve as models of enzyme production [6, 11] or of an

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intracellular circadian rhythm generator [12]. An extensive description of various applications can be found in e.g. [4, 5, 13, 14].

The problem of oscillatory behavior of all solutions in systems of type (1) is a very important one. From the applied point of view, when a system is a mathematical model of a real world phenomenon, it is essential to know whether solutions are monotone (and thus approaching an equilibrium) or they oscillate about the unique equilibrium. The oscillatory behavior is more typical in applied models; it also leads, under proper circumstances, to the existence of periodic motions in the model.

This work is devoted to derivation of sufficient conditions when all solutions of system (1) oscillate. Two partial cases of lower dimension $N = 2$ and $N = 3$ are studied.

2 Preliminaries

In this section we recall some basic notions and facts about system (1), introduce relevant definitions, and derive preliminary results necessary for the exposition and proof of our main results in Section 3.

The phase space of system (1) is the set $\mathbf{X} = C([-\tau_1, 0], \mathbf{R}) \times \ldots \times C([-\tau_N, 0], \mathbf{R})$. For every initial function $\Phi = (\phi_1, \ldots, \phi_N) \in \mathbf{X}, \phi_i \in C([-\tau_i, 0], \mathbf{R}), 1 \le i \le N$, there exists a unique solution $\mathbf{x} = \mathbf{x}(t) = \mathbf{x}(t, \Phi) = (x_1(t), \ldots, x_N(t))$ satisfying system (1) for all $t > 0$. The solution is built by the standard step-by-step integration procedure [1, 3, 8].

We also assume that each nonlinearity f_i satisfies either the positive or negative feedback condition in the sense of the following definition.

Definition 1. We say that function $f(u)$ satisfies the *positive* feedback condition on $\mathbf{R}$ if the following inequality holds

$$u \cdot f(u) > 0 \quad \text{for all} \quad u \in \mathbf{R},\ u \neq 0. \tag{2}$$

Likewise, function $g(u)$ satisfies the *negative* feedback condition if the inequality holds

$$u \cdot g(u) < 0 \quad \text{for all} \quad u \in \mathbf{R},\ u \neq 0. \tag{3}$$

When $i = N$ we set $i + 1 = 1$. If the number of nonlinearities in system (1) satisfying the negative feedback assumption (3) is odd we say that the system possesses the *overall negative feedback*. If it is even (including zero) the system is said to have the *overall positive feedback*.

It is easy to see that the sign assumptions (2) and (3) together with the continuity of f_i imply that $f_i(0) = 0, 1 \le i \le N$. Therefore, system (1) admits the only constant solution $\mathbf{0} = (0, \ldots, 0)$.

We shall make an additional assumption about the smoothness of functions f_i in a neighborhood of zero: each f_i is continuously differentiable for all u such that $|u| \le \delta$ for some $\delta > 0$. Their derivatives satisfy $f_i'(0) = a_i \neq 0, 1 \le i \le N$. The latter inequality describes a generic case for the nonlinearities f_i around the zero equilibrium.

Note that system (1) can be reduced to a standard form where each of the nonlinearities $f_i, 1 \leq i \leq N-1$, satisfies the positive feedback condition (2), while the last nonlinearity f_N satisfies the negative feedback assumption (3) [2]. Indeed, assume that the k-th equation, $k < N$,

$$x_k'(t) = -\alpha_k x_k(t) + f_k(x_{k+1}(t-\tau_{k+1}))$$

is the first one in system (1) where the nonlinearity f_k satisfies the negative feedback condition (3). Introduce then the new component $y_{k+1} =: -x_{k+1}$ and the new nonlinearity $\hat{f}_k(y_{k+1}) = f_k(-y_{k+1})$. One easily sees that $\hat{f}_k$ satisfies the positive feedback condition. The next $(k+1)$-st equation of system (1) should also be rewritten in terms of the new y_{k+1}:

$$\begin{aligned} y_{k+1}'(t) &= -\alpha_{k+1} y_{k+1}(t) - f_{k+1}(x_{k+2}(t-\tau_{k+2})) \\ &= -\alpha_{k+1} y_{k+1}(t) + \hat{f}_{k+1}(x_{k+2}(t-\tau_{k+2})). \end{aligned}$$

If $\hat{f}_{k+1}, k+1 < N$, satisfies the negative feedback condition, then one applies the same procedure of introducing the new variable $y_{k+2} = -x_{k+2}$ to this equation, and renaming the nonlinearity accordingly. If it satisfies the positive feedback condition then one moves to the next equation of the system, and so on until the last equation. The last N-th equation will satisfy the negative feedback assumption since the overall feedback in the system is negative.

Definition 2. Let $\mathbf{x} = (x_1, \ldots, x_N)$ be a solution to system (1). We shall call its k-th component x_k to be oscillatory (about zero) if there exists a sequence $t_n \to \infty, n \in \mathbf{N}$, such that $x_k(t_n) \cdot x_k(t_{n+1}) < 0$. The component x_k will be called non-oscillatory if there exists $T \geq 0$ such that $|x_k(t)| > 0$ for all $t > T$. We exclude from consideration solutions which are identical zero for sufficiently large t: $\mathbf{x} = (0, \ldots, 0)$ $\forall t \geq T$ for some $T \geq 0$.

Lemma 1. *Let $\mathbf{x} = (x_1, \ldots, x_N)$ be an arbitrary solution to system (1).*

(i) If its k-th component x_k is oscillatory then any other component $x_i, i \neq k$, is oscillatory as well;

(ii) If its k-th component x_k is non-oscillatory, so that $x_k(t) \geq 0$ or $x_k(t) \leq 0$ holds for all $t \geq T_1 \geq 0$, then there exists $T_2 \geq T_1$ such that $x_k(t) > 0$ or $x_k(t) < 0$ holds respectively for all $t \geq T_2$;

(iii) If its k-th component x_k is of eventually definite sign, i.e. $x_k(t) > 0$ or $x_k(t) < 0$ for all $t > T$ and some $T \geq 0$, then any other component $x_i, i \neq k$, is also of eventually definite sign;

(iv) Every component x_k of any non-oscillatory solution $\mathbf{x}$ satisfies

$$\lim_{t\to\infty} x_k(t) = \lim_{t\to\infty} x_k'(t) = 0, \; 1 \leq k \leq N. \tag{4}$$

In order to prove Lemma 1 we need several simple facts about solutions of initial value problems for scalar first order ordinary differential equations.

Proposition 1. *Consider the initial value problem*

$$u'(t)+\alpha u(t)=b(t), \quad u(t_0)=u_0, \quad t\geq t_0, \tag{5}$$

where $\alpha>0$ is a constant and $b(t)$ is a continuous real-valued function defined for $t\geq t_0$, $b\in C([t_0,\infty),\mathbf{R})$, with $b(t)\not\equiv 0$ for large values of t.

(i) If $u_0\geq 0$ and $b(t)\geq 0$ for all $t\geq t_0$ then there exists $t_1\geq t_0$ such that $u(t)>0$ for all $t\geq t_1$. If $u_0\leq 0$ and $b(t)\leq 0$ for all $t\geq t_0$ then there exists $t_2\geq t_0$ such that $u(t)<0$ for all $t\geq t_2$;

(ii) If $u_0<0$ and $b(t)\geq 0$ for all $t\geq t_0$ then either $u(t)<0$ for all $t\geq t_0$, or there exists $t_1\geq t_0$ such that $u(t_1)=0$. Likewise, if $u_0>0$ and $b(t)\leq 0$ for all $t\geq t_0$ then either $u(t)>0$ for all $t\geq t_0$, or there exists $t_1\geq t_0$ such that $u(t_1)=0$. For either one of these two possibilities the solution $u(t)$ is of definite sign eventually (for all $t\geq T\geq t_0$ and some T).

Proof. The proof of this proposition easily follows from the integral representation of the solution of the initial value problem (5):

$$u(t)=u_0\exp\{-\alpha(t-t_0)\}+\int_{t_0}^{t}\exp\{-\alpha(t-s)\}b(s)\,ds. \tag{6}$$

It is easily seen that when $u_0>0$ and $b(t)\geq 0$ then $u(t)>0\ \forall t\geq t_0$. When $u_0=0$ and $b(t)\geq 0$ (however, $b(t)\not\equiv 0$) then there exists point $t_1\geq t_0$ such that $u(t)>0\ \forall t\geq t_1$ (since the integral value in (6) becomes positive for all large t). The remaining possibilities are considered analogously.

Proposition 2. *Consider the initial value problem*

$$\beta v'(t)+v(t)=c(t), \quad v(t_0)=v_0, \quad t\geq t_0, \tag{7}$$

where $\beta>0$ and $c(t)$ is a continuous function, $c\in C([t_0,\infty),\mathbf{R})$, such that the limit $\lim_{t\to\infty}c(t)=c_0$ is finite. Then the solution $v(t)$ of the initial value problem (7) also has the same limit $\lim_{t\to\infty}v(t)=c_0$ (for any initial value $v_0\in\mathbf{R}$ and any positive parameter value $\beta>0$).

Proof. To prove the limit for any solution we shall show that for arbitrary $\varepsilon>0$ there exists $t_\varepsilon\geq t_0$ such that the solution $v(t)$ satisfies the inclusion $v(t)\in[c_0-\varepsilon,c_0+\varepsilon]$ for all $t\geq t_\varepsilon$.

We shall show first that if a solution enters a sufficiently small neighborhood of value c then it must stay there for all forward times. That is if the above claim about the solution $v(t)$ is not valid for a particular choice of $\beta>0, v_0\in\mathbf{R}$, and a sufficiently small $\varepsilon_0>0$ then the solution $v(t)$ must satisfy $v(t)\notin[c_0-\varepsilon_0,c_0+\varepsilon_0]$ for all $t\geq T_1\geq t_0$ for some T_1. Indeed, given $\varepsilon_0>0$ one can choose T_1 large enough such that the inclusion $c(t)\in(c_0-\varepsilon_0,c_0+\varepsilon_0)$ holds for all $t\geq T_1$. If there exists a point $t_1\geq T_1$ such that $v(t_1)\in[c_0-\varepsilon_0,c_0+\varepsilon_0]$ then $v(t)\in[c_0-\varepsilon_0,c_0+\varepsilon_0]$ must hold for all $t\geq t_1$. Indeed, assume that $t_2\geq t_1$ is the first point of exit of the solution $v(t)$ from the interval $[c_0-\varepsilon_0,c_0+\varepsilon_0]$. To be definite, assume that $v(t_2)=c_0+\varepsilon_0$, and $v(t) > c_0 + \varepsilon_0$ for all $t \in (t_2, t_2 + \delta)$ for some $\delta > 0$. Then the interval $(t_2, t_2 + \delta)$ also contains a point t_3 such that $v(t_3) > c_0 + \varepsilon_0$ and $v'(t_3) > 0$. On the other hand, according to the equation, $v'(t_3) = \frac{1}{\beta}[c(t_3) - v(t_3)] < 0$, a contradiction. The other possibility $v(t_2) = c_0 - \varepsilon_0$ leads to a contradiction in a similar way.

Therefore, we can assume next that there exists $T_2 \geq t_0$ such that $c(t) \in [c_0 - \varepsilon_0, c_0 + \varepsilon_0]$ and $v(t) \notin [c_0 - \varepsilon_0, c_0 + \varepsilon_0]$ for all $t \geq T_2$. To be definite, assume that $v(t) > c_0 + \varepsilon_0 \; \forall t \geq T_2$. Equation (7) then implies that $\beta v'(t) = c(t) - v(t) < 0$ for $t \geq T_2$, therefore the solution $v(t)$ is monotone decreasing. Set $v_0 = \lim_{t\to\infty} v(t) \geq c_0 + \varepsilon_0$. By using the limit values for functions $c(t)$ and $v(t)$ the last inequality yields

$$\beta v'(t) = c(t) - v(t) < c_0 + \sigma - (v_0 - \sigma) = c_0 - v_0 + 2\sigma < 0$$

for any sufficiently small $\sigma > 0$ and all $t \geq t_\sigma$ for some large t_σ. The latter implies that $\lim_{t\to\infty} v(t) = -\infty$, a contradiction with $v(t) \to v_0 \geq c_0 + \varepsilon_0$. The other possibility $v(t) < c_0 - \varepsilon_0 \;\; \forall t \geq T_2$ is treated analogously leading to a contradiction in a similar way. This completes the proof of the proposition.

Note that Proposition 2 can also be proved by using the variation of constant formula for the solution of the initial value problem (7).

Now we are in position to prove Lemma 1.

Proof. We shall prove first that when a solution $\mathbf{x} = (x_1, \ldots, x_N)$ to system (1) is non-oscillatory, so either $x_k(t) \geq 0$ or $x_k(t) \leq 0$ holds for all $t \geq T_1$ and some $k \in \{1, 2, \ldots, N\}$, then there exists $T_2 \geq T_1$ such that in fact the strict inequalities hold: either $x_k(t) > 0$ or $x_k(t) < 0$ for all $t \geq T_2$. Besides, for every other component $x_i, i \neq k$, there exists time moment s_i such that either $x_i(t) > 0$ or $x_i(t) < 0$ holds for all $t \geq s_i$.

To be definite, assume that $x_1(t) \geq 0 \; \forall t \geq T_1$ and $x_1(t) \not\equiv 0$ (other possibilities are considered similarly). Then the inequality $f_N(x_1(t - \tau_1)) \leq 0$ (and $\not\equiv 0$) holds for all large t. The last equation of system (1) can be represented in the integral form as follows

$$x_N(t) = x_N(t_0)\exp\{-\alpha_N(t - t_0)\} + \int_{t_0}^{t} \exp\{-\alpha_N(t - s)\} f_N(x_1(s - \tau_1))\, ds. \quad (8)$$

One applies now Proposition 1 to conclude that either $x_N(t) > 0$ or $x_N(t) < 0$ holds eventually, since the kernel of the integral in the representation (8) is non-positive and is not identical zero eventually. Note that similarly to 8 any other equation of system (1) has its integral representation as follows

$$x_k(t) = x_k(t_0)\exp\{-\alpha_k(t - t_0)\} + \int_{t_0}^{t} \exp\{-\alpha_k(t - s)\} f_k(x_{k+1}(s - \tau_{k+1}))\, ds. \quad (9)$$

Using next the $(N-1)$-st equation of the system, and its analogous representation in a form of integral equation (9) one finds that either $x_{N-1}(t) > 0$ or $x_{N-1}(t) < 0$ holds eventually. Going up along equations of system (1) one completes the proof of the claim for all the components $x_k, 1 \leq k \leq N$.

We shall show next that all the components $x_i, 1 \le i \le N$, of the non-oscillatory solution $\mathbf{x} = (x_1, \ldots, x_N)$ converge to zero together with their derivatives. To be definite assume that $x_1(t) > 0\, \forall t \ge t_0$. Consider the last equation of system (1). Suppose first that $x_N(t) > 0$ holds for all $t \ge t_N$. Then $x_N'(t) = -\alpha_N x_N(t) + f_N(x_1(t-\tau_1)) < 0$ is satisfied for all large t. Therefore, the finite limit $\lim_{t\to\infty} x_N(t) = x_N^0 \ge 0$ exists. By using the second from the last equation of system (1), $x_{N-1}'(t) = -\alpha_{N-1}x_{N-1}(t) + f_{N-1}(x_N(t-\tau_N))$, its integral representation in the form of (9), and Proposition 2, one sees that the limit of the component $x_{N-1}(t)$ exists with $\lim_{t\to\infty} x_{N-1}(t) = f_N(x_N^0) =: x_{N-1}^0$. Likewise, $\lim_{t\to\infty} x_{N-2}(t) = f_{N-2}(x_{N-1}^0) =: x_{N-2}^0$, and finally the limit of the first component is $\lim_{t\to\infty} x_1(t) = f_2(x_2^0) =: x_1^0$. Using again the last equation of system (1) and Proposition 2 one finds that $\lim_{t\to\infty} x_N(t) = f_N(x_1^0) =: x_N^0$. Therefore, the constant x_N^0 satisfies the recursive equation

$$x_N^0 = f_N(x_1^0) = f_N \circ f_1(x_2^0) = \ldots = f_N \circ f_1 \circ \ldots \circ f_{N-1}(x_N^0).$$

Since function $F(u) = f_N \circ f_1 \circ \ldots \circ f_{N-1}(u)$ satisfies the negative feedback condition (3) the only solution of the equation $F(u) = u$ is $u = 0$. Therefore, $x_1^0 = x_2^0 = \ldots = x_N^0 = 0$. Also, one easily finds that $\lim_{t\to\infty} x_k'(t) = \lim_{t\to\infty}[-\alpha_k x_k(t) + f_k(x_{k+1}(t-\tau_{k+1}))] = 0$. This completes the proof of the lemma.

3 Main Results

In this section we consider two particular cases of system (1) when $N = 2$ and $N = 3$. We establish sufficient conditions for the oscillatory behavior of all solutions in the system. The complete proof is provided for the case $N = 2$. The very same ideas for the proof are applicable for the three-dimensional systems, however, an outline is only given for the more involved case $N = 3$, due to the length of considerations.

3.1 Two Dimensional Systems

Consider the two-dimensional case $N = 2$ of system (1)

$$\begin{aligned} x_1'(t) &= -\alpha_1 x_1(t) + f_1(x_2(t-\tau_2)) \\ x_2'(t) &= -\alpha_2 x_2(t) + f_2(x_1(t-\tau_1)). \end{aligned} \tag{10}$$

Since it is in the standard form f_1 satisfies the positive feedback assumption (2) while f_2 satisfies the negative feedback assumption (3). Introduce the following quantities: $a = a_1 \cdot a_2 > 0, \tau_1 + \tau_2 = \tau > 0$, where $f_1'(0) = a_1 > 0, f_2'(0) = -a_2 < 0$.

Theorem 1. *Suppose that the inequality $a\tau > \max\{\alpha_1, \alpha_2\}$ is satisfied. Then all nontrivial solutions of system (10) oscillate.*

Proof. Consider consecutively all the possibilities for non-oscillatory solutions of system (10).

(i) Assume first that inequalities $x_1(t) > 0$ and $x_2(t) > 0$ hold eventually. Then by Lemma 1 (iv) one has that

$$\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = \lim_{t\to\infty} x_1'(t) = \lim_{t\to\infty} x_2'(t) = 0. \tag{11}$$

The second equation of system (10) shows that $x_2'(t) < 0$ eventually, so $x_2(t)$ is monotone decreasing to zero for large t. The first equation of (10) can be written in the form $(1/\alpha_1)x_1'(t) = -x_1(t) + (1/\alpha_1)f_1(x_2(t-\tau_2))$. Since $(1/\alpha_1)f_1(x_2(t-\tau_2)) > 0$ and is decreasing to zero as $x_2 \to 0^+$ one sees that the inequality $x_1(t) \leq (1/\alpha_1)f_1(x_2(t-\tau_2))$ holds for all sufficiently large t.

Assume now that for arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ the values of t are chosen to be large enough, $t \geq T$, so that the following inequalities hold:

$$f_2(x_1(t-\tau_1)) \leq \left[f_2'(0)+\varepsilon_1\right] x_1(t-\tau_1) \text{ and } f_1(x_2(t-\tau_2)) \geq \left[f_1'(0)-\varepsilon_2\right] x_2(t-\tau_2).$$

Integrate now the second equation of system (10) over the interval $[t-\tau, t]$:

$$\begin{aligned} x_2(t) - x_2(t-\tau) &= -\alpha_2 \int_{t-\tau}^{t} x_2(s)\,ds + \int_{t-\tau}^{t} f_2(x_1(s-\tau_1)\,ds \leq \\ &- \alpha_2 x_2(t)\tau + \left[f_2'(0)+\varepsilon_1\right] \int_{t-\tau}^{t} x_1(s-\tau_1)\,ds \leq \\ &- \alpha_2 x_2(t)\tau + \left[f_2'(0)+\varepsilon_1\right] \frac{1}{\alpha_1} \int_{t-\tau}^{t} f_1(x_2(s-\tau))\,ds \leq \\ &- \alpha_2 x_2(t)\tau + \frac{\tau}{\alpha_1}\left[f_2'(0)+\varepsilon_1\right]\left[f_1'(0)-\varepsilon_2\right] x_2(t-\tau)). \end{aligned}$$

Therefore, we obtain the inequality

$$x_2(t)\left[1+\alpha_2\tau\right] \leq x_2(t-\tau)\left\{1+\frac{\tau}{\alpha_1}\left[f_2'(0)+\varepsilon_1\right]\left[f_1'(0)-\varepsilon_2\right]\right\}.$$

In the case when $1+\frac{\tau}{\alpha_1}[f_2'(0)+\varepsilon_1][f_1'(0)-\varepsilon_2] < 0$ is satisfied we arrive at a contradiction with $x_2(t) > 0$. This will clearly be the case when the inequality $\tau a > \alpha_1$ is satisfied and $\varepsilon_1, \varepsilon_2$ are sufficiently small.

(ii) Assume next that inequalities $x_1(t) > 0$ and $x_2(t) < 0$ are satisfied eventually. As in part (i) one has the limits (11). The first equation of system (10) shows that $x_1'(t) < 0$ so $x_1(t)$ is decreasing to zero. The second equation of the system implies that $x_2'(t) > 0$ eventually, so $x_2(t)$ is increasing with $x_2(t) \leq (1/\alpha_2)f_2(x_1(t-\tau_1))$ satisfied for all large t. Now integrate the first equation of the system over the interval $[t-\tau, t]$, assuming similar smallness of $\varepsilon_1, \varepsilon_2$ as in part (i) above:

$$x_1(t) - x_1(t-\tau) = -\alpha_1 \int_{t-\tau}^{t} x_1(s)\,ds + \int_{t-\tau}^{t} f_1(x_2(s-\tau_2)\,ds \leq$$

$$
\begin{aligned}
&-\ \alpha_1 \tau x_1(t)\,ds + \int_{t-\tau}^{t} [f_1'(0)-\varepsilon_1] x_2(s-\tau_2)\,ds \le \\
&-\ \alpha_1 \tau x_1(t)\,ds + \left[f_1'(0)-\varepsilon_1\right] \int_{t-\tau}^{t} (1/\alpha_2) f_2(x_1(s-\tau))\,ds \le \\
&-\ \alpha_1 \tau x_1(t)\,ds + \frac{1}{\alpha_2}\left[f_1'(0)-\varepsilon_1\right]\left[f_2'(0)+\varepsilon_2\right] \int_{t-\tau}^{t} x_1(s-\tau)\,ds \le \\
&-\ \alpha_1 \tau x_1(t)\,ds + \frac{\tau}{\alpha_2}\left[f_1'(0)-\varepsilon_1\right]\left[f_2'(0)+\varepsilon_2\right] x_1(t-\tau).
\end{aligned}
$$

The last inequality implies that the following estimate holds

$$
x_1(t)\,[1+\alpha_1\tau] \le x_1(t-\tau)\left\{1+\frac{\tau}{\alpha_2}\left[f_1'(0)-\varepsilon_1\right]\left[f_2'(0)+\varepsilon_2\right]\right\}.
$$

Therefore when the condition $a\tau > \alpha_2$ is satisfied the latest inequality leads to a contradiction with $x_1(t) > 0$.

(iii) Two remaining subcases, $\{x_1(t) < 0, x_2(t) < 0\}$ and $\{x_1(t) < 0, x_2(t) > 0\}$ are symmetric to those treated above in cases (i) and (ii), respectively. The details of the proof are derived along the same lines, with a contradiction obtained to the assumption that $x_1(t) < 0$. They are left to the reader.

3.2 Three Dimensional Systems

Consider the three-dimensional case $N = 3$ of system (1)

$$
\begin{aligned}
x_1'(t) &= -\alpha_1 x_1(t) + f_1(x_2(t-\tau_2)) \\
x_2'(t) &= -\alpha_2 x_2(t) + f_2(x_3(t-\tau_3)) \\
x_3'(t) &= -\alpha_3 x_1(t) + f_3(x_1(t-\tau_1)).
\end{aligned}
\tag{12}
$$

Since it is in the standard form f_1 and f_2 satisfy the positive feedback assumption (2) while f_3 satisfies the negative feedback assumption (3). Introduce the following quantities: $a = a_1 a_2 a_3 > 0, \tau_1+\tau_2+\tau_3 = \tau > 0$ where $f_1'(0) = a_1 > 0, f_2'(0) = a_2 > 0, f_3'(0) = -a_3 < 0$.

Theorem 2. *Suppose that the inequality $a\,\tau > \max\{\alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_2\alpha_3\}$ is satisfied. Then all nontrivial solutions of system (10) oscillate.*

Proof. The proof of this theorem in very similar to that of Theorem 1. One has to consider the following three principal subcases for the eventual signs of the components x_1, x_2, x_3 of a non-oscillating solution $\mathbf{x}$: $\{x_1 > 0, x_2 > 0, x_3 > 0\}$, $\{x_1 > 0, x_2 > 0, x_3 < 0\}$, and $\{x_1 > 0, x_2 < 0, x_3 < 0\}$. The remaining five subcases are symmetric opposite or similar to those three, and are considered along the same lines. For example, the case $\{x_1 > 0, x_2 > 0, x_3 > 0\}$ leads to the following integral equation for the component x_3, when the last equation of the system is integrated over the interval $[t-\tau, t]$,

$$x_3(t) - x_3(t-\tau) = -\alpha_3 \int_{t-\tau}^{t} x_3(s)\,ds + \int_{t-\tau}^{t} f_3(x_1(s-\tau_1)\,ds,$$

and to the following two inequalities for the components x_1 and x_2

$$x_1(t) \geq \frac{1}{\alpha_1} f_1(x_2(t-\tau_2)), \quad x_2(t) \geq \frac{1}{\alpha_2} f_2(x_3(t-\tau_3)).$$

Substituting the latter into the former, one derives a contradiction with the assumption $x_3(t) > 0$, when the inequality $\tau a > \alpha_1 \alpha_2$ is satisfied. The other two principal subcases lead to a similar contradiction when the other two assumptions are in place, $\tau a > \alpha_1 \alpha_3$ and $\tau a > \alpha_2 \alpha_3$. We leave details to the reader.

4 Discussion

Theorems 1 and 2 provide simple and verifiable sufficient conditions for the oscillation of all solutions of system (1) in cases $N = 2$ and $N = 3$. In the case when the feedback functions f_1, f_2, f_3 are fixed, and the rates of decay of all the components are bounded above, $\max_i \alpha_i \leq \alpha_0$ for some fixed constant $\alpha_0 > 0$, a sufficiently large overall delay $\tau = \sum_{i=0}^{N}$ in the system forces all its solutions to oscillate. We believe that an analogue of these two theorems is valid in the case of general dimension N. However, we are not in a position to provide a complete proof at this time. The ideas used in the proof of Theorems 1 and 2 cannot be extended to the case $N \geq 4$, due to the variety and complexity of all the subcases. Therefore, we are only in a position to state the following conjecture.

Set $a = -a_1 a_2 \ldots a_{N-1} a_N > 0$ and $\tau = \tau_1 + \ldots + \tau_N > 0$, where $f_i'(0) = a_i > 0, 1 \leq i \leq N-1$ and $f_N'(0) = a_N < 0$. Given positive $\alpha_1, \ldots, \alpha_N$ introduce the following quantities: $\Lambda_i = \prod_{k \neq i} \alpha_k, 1 \leq i \leq N$.

Conjecture 1. Suppose that the inequality $a\tau > \max\{\Lambda_1, \ldots, \Lambda_N\}$ is satisfied. Then all nontrivial solutions of system (1) oscillate.

Another interesting and challenging problem is to derive sufficient conditions for the oscillation of all solutions in cyclic type systems when either a positive or a negative type feedback is in place between any two consecutive components x_k and x_{k+1}, however, all other components are also involved on every step. In the simplest case of dimension $N = 2$ such system would have the form

$$x_1'(t) = -\alpha_1 x_1(t) + f_1(x_1(t-\tau_1), x_2(t-\tau_2))$$
$$x_2'(t) = -\alpha_2 x_2(t) + f_2(x_1(t-\tau_1), x_2(t-\tau_2)),$$

where the nonlinearities f_1 and f_2 satisfy the positive and negative feedback assumptions, respectively, in the following sense:

$$v \cdot f_1(u,v) > 0 \;\; \forall (u,v) \in \mathbf{R}^2, v \neq 0 \qquad u \cdot f_2(u,v) < 0 \;\; \forall (u,v) \in \mathbf{R}^2, u \neq 0.$$

This problem can be generalized to the case of arbitrary dimension N. This oscillation problem and the above conjectured Conjecture 1 represent a program for future research.

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