Generators of groups of Hamitonian maps

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Abstract

We prove that analytic Hamiltonian dynamics on tori, annuli, or Euclidean space can be approximated by a composition of nonlinear shear maps where each of the shears depends only on the position or only on the momentum.

1 Statement of the result.

Let $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ denote the n-torus. We endow the $2n$ torus $V := \mathbb{T}^n \times \mathbb{T}^n$ with the canonical coordinates $(q,p) = (q_1, \dots, q_n, p_1, \dots, p_n)$ and symplectic form $\omega = \sum_i dp_i \wedge dq_i$. For a function $H \in C^\infty(V, \mathbb{R})$, the system of differential equations defined by the Hamiltonian H is given by

$$\dot{q} = \partial_p H, \qquad \dot{p} = -\partial_q H. \tag{1.1}$$

The corresponding vector field

$$X_H := (\partial_{p_1} H, \dots, \partial_{p_n} H, -\partial_{q_1} H, \dots, -\partial_{q_n} H),$$

satisfies $\omega(X_H, \cdot) = dH$; it is called the *symplectic gradient* of H. The symplectic gradient defines the *Hamiltonian flow* denoted by ϕ^t_H, the family of time-t maps along the trajectories of system(1.1). Similarly, given a continuous family of functions $H_t \in C^\infty(V, \mathbb{R})$, $t \geq 0$, one defines the *time-dependent Hamiltonian system*

$$(\dot{q}, \dot{p}) = X_{H,t} = (\partial_p H_t,\ -\partial_p H_t). \tag{1.2}$$

The trajectories of this system define the family of maps $\phi^{s,t}_H$, $0 \leq s \leq t$: the solution with the initial condition (q,p) at time s arrives at the point $\phi^{s,t}_H(q,p)$ at time t. Such maps preserve the symplectic form ω. The family of these maps is called the *non-autonomous Hamiltonian flow* of $H := (H_t)_{t\geq 0}$.

A symplectic map is called *a Hamiltonian map* if it is the map $\phi^{0,1}_H$ for a time-dependent Hamiltonian H. We consider the spaces $\mathrm{Ham}^\infty(V)$ of Hamiltonian C^∞-diffeomorphisms and $\mathrm{Ham}^\omega(V)$ of diffeomorphisms defined by real-analytic Hamiltonians H_t which depend on t continuously in $C^\omega(V, \mathbb{R})$.

Recall that the base of the C^ω-topology (the inductive limit topology) on space of real-analytic functions is a collection, taken over all neighborhoods of $V = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ in its complexification $\mathbb{C}^{2n}/\mathbb{Z}^{2n}$, of C^0-open sets of holomorphic functions on such neighborhoods. Thus, a sequence of real-analytic functions ϕ_j converges to a real-analytic function ϕ on V in C^ω iff there exists a

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neighborhood V_ρ of V in the complexification $\mathbb{C}^{2n}/\mathbb{Z}^{2n}$ where ϕ and every ϕ_j for j large enough have their analytic extensions well defined and $\sup_{V_\rho} \|\phi_j - \phi\| \to 0$ as $j \to +\infty$. The space $\mathrm{Ham}^\omega(V)$ is formed by analytic mappings; it is considered with the inductive limit topology as described above.

Both $\mathrm{Ham}^\infty(V)$ and $\mathrm{Ham}^\omega(V)$ are groups (this follows from the identity $\phi_G^{0,t} \circ \phi_H^{0,t} = \phi_K^{0,t}$ where $K_t = G_t + H_t \circ (\phi_{H_2}^{0,t})^{-1}$).

The simplest examples of Hamiltonian maps are given by *vertical* and *horizontal shear maps*:

- a horizontal shear $(q,p) \mapsto (q + \nabla\tau(p), p)$ is the time-1 map for the time-independent Hamiltonian $H(q,p) = \tau(p)$, where $\tau \in C^\omega(\mathbb{T}^n, \mathbb{R})$;
- a vertical shear $(q,p) \mapsto (q, p - \nabla v(q))$ is the time-1 map for the time-independent Hamiltonian $H(q,p) = v(q)$, where $v \in C^\omega(\mathbb{T}^n, \mathbb{R})$.

The system of differential equations defined by Hamiltonian $H(q,p) = \tau(p)$ is

$$\dot{q} = \nabla\tau(p), \quad \dot{p} = 0;$$

its flow map $\phi_H^t : (q,p) \mapsto (q + t\nabla\tau(p), p)$ is a horizontal shear for every $t \in \mathbb{R}$. Similarly, the flow map $\phi_H^t : (q,p) \mapsto (q, p - t\nabla v(q))$ for $H(q,p) = v(q)$ is a verticall shear for every $t \in \mathbb{R}$. We see that the vertical and horizontal shear maps form Abelian subgroups of $\mathrm{Ham}^\omega(V)$, which we denote as $\mathcal{V}$ and, respectively, $\mathcal{T}$.

Theorem. *(Main) The group $< \mathcal{V}, \mathcal{T} >$ generated by $\mathcal{V}$ and $\mathcal{T}$ is dense in* $\mathrm{Ham}^\omega(V)$. *In other words, every real-analytic Hamiltonian diffeomorphism of V can be C^ω-approximated by a composition $S_M \circ \ldots \circ S_1$ where $S_j \in \mathcal{T} \cup \mathcal{V}$, $j = 1, \ldots, M$.*

The proof is given in the next Section. Since $\mathrm{Ham}^\omega(V)$ is C^∞-dense in $\mathrm{Ham}^\infty(V)$, we obtain

Corollary 1.1. *The group generated by $\mathcal{V}$ and $\mathcal{T}$ is dense in* $\mathrm{Ham}^\infty(V)$.

Note that any real-analytic function on $\mathbb{R}^n$ can be arbitrarily well approximated, on any given compact, by a periodic function with a sufficiently large period. Therefore, the lifts of Hamiltonian maps of $\mathbb{T}^n \times \mathbb{T}^n$ approximate (on any given compact) Hamiltonian maps of an annulus $\mathbb{T}^n \times \mathbb{R}^n$ or a ball $\mathbb{R}^n \times \mathbb{R}^n$. This implies

Corollary 1.2. *The theorem extends to the cases where $V = \mathbb{T}^n \times \mathbb{R}^n$ and $V = \mathbb{R}^n \times \mathbb{R}^n$: every Hamiltonian map is approximated by a composition of vertical and horizontal shears.*

Along the proof of the main theorem, we will check that the "parametric version" of the results also holds. Namely, we have the following

Corollary 1.3. *Given a compact real-analytic manifold $\mathscr{P}$, every analytic family $f_{\mathscr{P}} = (f_\mu)_{\mu \in \mathscr{P}}$ of Hamiltonian diffeomorphisms of $\mathbb{T}^n \times \mathbb{T}^n$, $\mathbb{T}^n \times \mathbb{R}^n$ or $\mathbb{R}^n \times \mathbb{R}^n$ can be arbitrarily well approximated by analytic families of compositions of vertical and horizontal shears. For any compact set $\mathscr{P}$, every continuous family $f_{\mathscr{P}} = (f_\mu)_{\mu \in \mathscr{P}}$ of analytic Hamiltonian diffeomorphisms of $\mathbb{T}^n \times \mathbb{T}^n$, $\mathbb{T}^n \times \mathbb{R}^n$ or $\mathbb{R}^n \times \mathbb{R}^n$ can be arbitrarily well approximated by continuous families of compositions of vertical and horizontal shears.*

To be precise, we recall that a sequence of continuous families $(f_{j,\mu})_{\mu \in \mathscr{P}}$ of analytic diffeomorphisms on V converges to $(f_\mu)_{\mu \in \mathscr{P}}$ if there exists a complex neighborhood V_ρ of V such that for every $\delta > 0$, for every j large enough, $\sup_{(\mu,x) \in \mathscr{P} \times V_\rho} \|f_\mu(x) - f_{j,\mu}(x)\| < \delta$. Also, we call a family $(f_\mu)_{\mu \in \mathscr{P}}$ analytic if $f_\mu(x)$ is a real-analytic function of μ and x; the convergence in Corollary 1.3 is then in $C^\omega(\mathscr{P} \times V, V)$. These two settings (of continuous and analytic families) seem to be most natural. In order to consider them in a unified way, we adopt from now on a more general setting where $\mathscr{P}$ denotes allways a product:

$$\mathscr{P} = \mathscr{P}_1 \times \mathscr{P}_2$$

of a compact set $\mathscr{P}_1$ and a compact analytic manifold $\mathscr{P}_2$. Also from now on, the considered family $f_{\mathscr{P}} = (f_\mu)_{\mu \in \mathscr{P} := \mathscr{P}_1 \times \mathscr{P}_2}$ will be analytic in μ_2 and continuous with respect to μ_1. Each map $f_\mu(x)$ is the time-1 map of a time-dependent Hamiltonian $H_{\mu,t}(x)$; we assume that H is an analytic function of (μ_2, x) and a continuous (in the topology of $C^\omega(\mathscr{P}_2 \times V, \mathbb{R})$) function of (μ_1, t), and say that the family $f_{\mathscr{P}}$ is *generated* by the family $(H_{\mu,t})_{\mu,t}$.

Remark 1.4. For the ease of presentation, the main theorem is given for $\mathbb{T}^n \times \mathbb{T}^n$ endowed with the standard symplectic form $\omega = \sum_i dp_i \wedge dq_i$, but the proof, with obvious modifications, works also for any symplectic form of the form $\sum_i a_i \; dp_i \wedge dq_i$, with *constant* $a_i > 0$. A natural question is how to extend the results to other symplectic forms on the torus or to other product symplectic manifolds.

The main Theorem implies[1] the work [Tur02] where symplectic maps of $V = \mathbb{R}^{2n}$ were considered in the smooth case. While we found a way to extend the method of [Tur02] to the annulus case, we prefer to present here a more powerful approach, inspired by a technology developed in [BGH22, §2.4-3.2 and app. A] for the non-symplectic case. Similar results for holomorphic automorphisms of $\mathbb{C}^n$, including the volume-preserving case, were obtained in [And90, AL92] and have played an important role in solving several problems of complex analysis, see review in [FK22]. The symplectic result of [Tur02] for $V = \mathbb{R}^{2n}$ was key for the proof of the genericity of the "ultimately rich" (universal) dynamics for certain classes of symplectic and non-symplectic maps [GTS07, GT10, Tur15, GT17] and for the proof of Herman's metric entropy conjecture [BT19]. It has also been used in algorithms for physics-informed machine learning [JZZ+20, BTM20, VWT+22]. The current result and its short constructive proof for the annulus $V = \mathbb{T} \times \mathbb{R}$ enabled to disprove the Birkhoff conjecture in [Ber22].

2 Proof of the main theorem

We use the Poisson algebra structure on $C^\omega(V, \mathbb{R})$, which is the Hamiltonian counterpart of the Lie algebra structure on the space of vector fields on V. Namely, given two functions $f, g \in C^\omega(V, \mathbb{R})$ the *Poisson bracket* $\{f, g\}$ is the function defined by

$$\{f, g\} = \sum_i \partial_{q_i} f \cdot \partial_{p_i} g - \partial_{q_i} g \cdot \partial_{p_i} f \; .$$

It is easy to check that the Lie bracket of the Hamiltonian vector fields X_f and X_g is the Hamiltonian vector field $X_{\{f,g\}}$.

Cartan's Theorem establishes a correspondence between closed subgroups and Lie sub-algebras for finite-dimensional Lie groups. Certain aspects of this correspondence have been generalized in [BGH22, Prop. B.1] for the group of compactly supported smooth diffeomorphisms. Below is the counterpart for the group of analytic Hamiltonian diffeomorphisms:

Proposition 2.1. *Let a set G be a closed subgroup of* $\mathrm{Ham}^\omega(V)$*. Let $\mathcal{P}(G)$ be the set of all time-independent Hamiltonians $H \in C^\omega(V, \mathbb{R})$ such that their flow maps ϕ_H^t belong to G for all $t \in \mathbb{R}$:*

$$\mathcal{P}(G) := \{H \in C^\omega(V, \mathbb{R}) : \phi_H^t \in G, \; \forall t\} \; .$$

[1] A shear map of $\mathbb{R}^{2n}$ is the composition of two Hénon maps. So the main result implies that compositions of Hénon maps form a dense set in $\mathrm{Ham}^\infty(\mathbb{R}^{2n})$.

Then $\mathcal{P}(G)$ is a closed Lie sub-algebra of $C^\omega(V)$. In other words, it is a closed vector subspace of $C^\omega(V)$ and the Poisson bracket of any two functions from $\mathcal{P}(G)$ also belongs to $\mathcal{P}(G)$.

Proof. First note that $\mathcal{P}(G)$ is closed by continuity of $H \mapsto \phi_H^t$ for every $t \in \mathbb{R}$. Then the proposition follows from the two lemmas below. ☐

Lemma 2.2. *The set $\mathcal{P}(G)$ is a vector space.*

Lemma 2.3. *The vector space $\mathcal{P}(G)$ is a Lie algebra.*

Proof of Lemma 2.2. Let $H \in \mathcal{P}(G)$ and $\lambda \in \mathbb{R}$. For every $t \in \mathbb{R}$, the map $\phi_H^{\lambda t} = \phi_{\lambda H}^t$ belongs to G and so $\lambda H \in \mathcal{P}(G)$. Hence, it suffices to show that $H_1 + H_2 \in \mathcal{P}(G)$, for every $H_1, H_2 \in \mathcal{P}(G)$. As $\phi_{H_1+H_2}^t = \phi_{tH_1+tH_2}^1$, and $tH_1, tH_2 \in \mathcal{P}(G)$ whenever $H_1, H_2 \in \mathcal{P}(G)$ as we just showed, it suffices to check that $\phi_{H_1+H_2}^1 \in G$ for every $H_1, H_2 \in \mathcal{P}(G)$.

On a complex extension V_0 of V, the following holds uniformly as $t \to 0$:

$$\phi_{H_1}^t \circ \phi_{H_2}^t = id + t \cdot \partial_t(\phi_{H_1}^t \circ \phi_{H_2}^t)_{t=0} + O(t^2) = id + t \cdot (X_{H_1} + X_{H_2}) + O(t^2)$$
$$= id + t \cdot \partial_t(\phi_{H_1+H_2}^t)_{t=0} + O(t^2) = \phi_{H_1+H_2}^t \, .$$

In particular, there exists $C > 0$ such that for every $N \geq 1$

$$\sup_{V_0} \|\phi_{H_1+H_2}^{1/N} - \phi_{H_1}^{1/N} \circ \phi_{H_2}^{1/N}\| \leq C \cdot N^{-2} \, . \tag{2.1}$$

Taking sufficiently small complex neighborhoods $V_2 \Subset V_1 \subset V_0$ of V and applying Discretization Lemma 3.1, we infer from (2.1) that

$$\sup_{V_2} \|\phi_{H_1+H_2}^1 - (\phi_{H_1}^{1/N} \circ \phi_{H_2}^{1/N})^N\| \leq C\exp(L)N^{-1} \quad \forall N \geq N_0$$

for some constants L and N_0 (in the Discretization Lemma, put $\varepsilon_N := C/N$, $\phi^{s,t} := \phi_{H_1+H_2}^{t-s}$, and $g_j = \phi_{H_1}^{1/N} \circ \phi_{H_2}^{1/N}$ for all j).

Thus $\phi_{H_1+H_2}^1$ is arbitrarily close to the element $(\phi_{H_1}^{1/N} \circ \phi_{H_2}^{1/N})^N$ of the group G. As G is closed, it follows that $\phi_{H_1+H_2}^1 \in G$, as required. ☐

Proof of Lemma 2.3. It suffices to show that for any $H_1, H_2 \in \mathcal{P}(G)$, the function $H_3 = \{H_1, H_2\}$ belongs to $\mathcal{P}(G)$. Since $\phi_{H_3}^t = \phi_{\{tH_1, H_2\}}^1$ and $tH_1 \in \mathcal{P}(G)$ whenever $H_1 \in \mathcal{P}(G)$, we only need to show that $\phi_{H_3}^1$ belongs to G. On a complex extension V_0 of V, we have, uniformly as $t \to 0$:

$$\phi_{H_j}^t = id + tX_{H_j} + \frac{t^2}{2} DX_{H_j} \cdot X_{H_j} + O(t^3) \, , \; j = 1, 2, 3. \tag{2.2}$$

So,

$$\phi_{H_1}^{t/N} \circ \phi_{H_2}^{t/N} = id + \tfrac{t}{N}(X_{H_1} + X_{H_2}) + \tfrac{t^2}{2N^2}(DX_{H_1} \cdot X_{H_1} + DX_{H_2} \cdot X_{H_2} + 2DX_{H_1} \cdot X_{H_2}) + O(t^3)$$

$$\phi_{H_2}^{-t/N} \circ \phi_{H_1}^{-t/N} = id - \tfrac{t}{N}(X_{H_1} + X_{H_2}) + \tfrac{t^2}{2N^2}(DX_{H_1} \cdot X_{H_1} - DX_{H_2} \cdot X_{H_2} + 2DX_{H_2} \cdot X_{H_1}) + O(t^3).$$

Thus, uniformly on V_0 as $t \to 0$,

$$\phi_{H_1}^{-t} \circ \phi_{H_2}^{-t} \circ \phi_{H_1}^t \circ \phi_{H_2}^t = id + t^2(DX_{H_1} \cdot X_{H_2} - DX_{H_2} \cdot X_{H_1}) + O(t^3). \tag{2.3}$$

One can check that $DX_{H_1}(X_{H_2}) - DX_{H_2}(X_{H_1})$ is the symplectic gradient of $H_3 = \{H_1, H_2\}$. Thus Eq. (2.2) at $j = 3$ and Eq. (2.3) imply the existence of $C > 0$ such that for $N \equiv t^{-2}$ sufficiently large,

$$\sup_{V_0} \left\| \phi_{H_3}^{1/N} - \phi_{H_1}^{-1/\sqrt{N}} \circ \phi_{H_2}^{-1/\sqrt{N}} \circ \phi_{H_1}^{1/\sqrt{N}} \circ \phi_{H_2}^{1/\sqrt{N}} \right\| \leq C \frac{1}{N^{3/2}}. \tag{2.4}$$

From this, taking sufficiently small complex neighborhoods $V_2 \Subset V_1 \subset V_0$, one finds that the assumptions of the Discretization Lemma 3.1 are satisfied with $\varepsilon_N := C/\sqrt{N}$, $\phi^{s,t} := \phi_{H_3}^{t-s}$, and $g_j = \phi_{H_1}^{-1/\sqrt{N}} \circ \phi_{H_2}^{-1/\sqrt{N}} \circ \phi_{H_1}^{1/\sqrt{N}} \circ \phi_{H_2}^{1/\sqrt{N}}$ for all j. This implies that for some L and N_0

$$\sup_{V_2} \|\phi_{H_3}^1 - (\phi_{H_1}^{-1/\sqrt{N}} \circ \phi_{H_2}^{-1/\sqrt{N}} \circ \phi_{H_1}^{1/\sqrt{N}} \circ \phi_{H_2}^{1/\sqrt{N}})^N\| \leq C\exp(L)N^{-1/2}\ , \quad \forall N \geq N_0\ .$$

Thus $\phi_{H_3}^1$ is arbitrarily close to the element $(\phi_{H_1}^{-1/\sqrt{N}} \circ \phi_{H_2}^{-1/\sqrt{N}} \circ \phi_{H_1}^{1/\sqrt{N}} \circ \phi_{H_2}^{1/\sqrt{N}})^N$ of the group G. As G is closed, it follows that $\phi_{H_3}^1 \in G$, as required. □

The space of families of analytic Hamiltonian maps endowed with the composition rule $(f_\mu)_{\mu\in\mathscr{P}} \circ (g_\mu)_{\mu\in\mathscr{P}} = (f_\mu \circ g_\mu)_{\mu\in\mathscr{P}}$ is a group. One can check that the proof of the above proposition does not alter as long as $\mathscr{P}$ is compact, by using the parametric counterpart Corollary 3.2 of Lemma 3.1. Namely, we have

Corollary 2.4. *If $G_{\mathscr{P}}$ is a closed subgroup of the space of families of analytic Hamiltonian maps then the following is a closed sub-algebra of the space of families of functions on V:*

$$\mathcal{P}(G_{\mathscr{P}}) := \{(H_\mu)_{\mu\in\mathscr{P}} : (\phi_{H_\mu}^t)_{\mu\in\mathscr{P}} \in G_{\mathscr{P}},\ \forall t\}\ .$$

We apply Proposition 2.1 to the group G obtained by taking the C^ω-closure of the group $<\mathcal{V}, \mathcal{T}>$ generated by $\mathcal{V}$ and $\mathcal{T}$:

$$G = cl(<\mathcal{V}, \mathcal{T}>)\ .$$

Recall that the groups $\mathcal{V}$ and $\mathcal{T}$ of vertical and horizontal shears consist of the time-1 maps for the time-independent Hamiltonian functions which depend only on p or, respectively, only on q. For such functions, the time-t map belongs to $\mathcal{V}$ or, respectively, $\mathcal{T}$ for all $t \in \mathbb{R}$. Thus, the C^ω Hamiltonians of the form $H(q,p) = \tau(p)$ or $H(q,p) = v(q)$ belong to the Lie algebra $\mathcal{P}(G)$. This implies, by Proposition 2.1, that every Hamiltonian of the form

$$H(q,p) = v_0(q) + \sum_{1\leq r\leq R} \{w_r(q), \{v_r(q), \tau_r(p)\}\} \tag{2.5}$$

lies in $\mathcal{P}(G)$; here $R \geq 0$ and $\tau_r, v_r, w_r \in C^\omega(\mathbb{T}^n, \mathbb{R})$.

We denote the set comprised by C^ω-functions $\mathbb{T}^n \times \mathbb{T}^n \to \mathbb{R}$ which can be represented in the form (2.5) as $\mathring{\mathcal{P}}$. In short one can denote:

$$\mathring{\mathcal{P}} = \mathcal{V} + \{\mathcal{V}, \{\mathcal{V}, \mathcal{T}\}\}\ .$$

As we said, $\mathring{\mathcal{P}} \subset \mathcal{P}(G)$, i.e., for every $H \in \mathring{\mathcal{P}}$ its flow maps ϕ_H^t can be arbitrarily well approximated by compositions of vertical and horizontal shears.

Proposition 2.5. *The set $\mathring{\mathcal{P}}$ is dense in $C^\omega(V, \mathbb{R})$.*

Proof. By Fourier's Theorem, any C^ω function on the torus V can be approximated by a trigonometric polynomial, i.e., a function of the form

$$\sum_{\max_j\{|m_j|,|k_j|\}\le K} \mathrm{Re}(c_{m,k}e^{2\pi i(<m,q>+<k,p>)}),$$

where m and k are integer-valued n-vectors, and $<\cdot,\cdot>$ denotes the inner product. Therefore, it is enough to show that every trigonometric polynomial belongs to $\mathring{\mathcal{P}}$, i.e., it has the form (2.5) for some choice of the functions v, w, and τ. Thus, we choose $v_0(q) = \sum_{\max_j |m_j|\le K} \mathrm{Re}(c_{m,0}e^{2\pi i<m,q>})$, and it remains to show that for every k and m such that $k \neq 0$ the term $\mathrm{Re}(c_{m,k}e^{2\pi i(<m,q>+<k,p>)})$ is a linear combination of terms which can be represented as $\{w_r(q), \{v_r(q), \tau_r(p)\}$.

This is done as follows. Since $k \neq 0$, there exists an index j such that $k_j \neq 0$. Hence there is $\sigma \in \{-1, 1\}$ such that $A = (\sum_{s=1}^n m_s k_s) - \sigma k_j$ is not zero.

We denote $m' = (m'_1, \ldots, m'_n)$, where $m'_j = m_j - \sigma$ and $m'_s = m_s$ if $s \neq j$. Then

$$e^{2\pi i(<m,q>+<k,p>)} = e^{2\pi i<m',q>}e^{2\pi i<k,p>}e^{2\pi i\sigma q_j},$$

so $\mathrm{Re}(c_{m,k}e^{2\pi i(<m,q>+<k,p>)})$ is a linear combination of products of cosines or sines of $2\pi < m', q >$, $2\pi < k, p >$, and $2\pi\sigma q_j$. Hence, it suffices to show that for every (α, β, γ) there exist functions w, v, and τ such that

$$\sin(2\pi < m', q > +\alpha)\ \sin(2\pi < k, p > +\beta)\ \sin(2\pi\sigma q_j + \gamma) = \{w(q), \{v(q), \tau(p)\}. \tag{2.6}$$

For that, we choose

$$v(q) = \frac{\cos(2\pi < m', q > +\alpha)}{2\pi A}, \qquad \tau(p) = -\frac{\sin(2\pi < k, p > +\beta)}{2\pi}, \qquad w(q) = \frac{\cos(2\pi\sigma q_j + \gamma)}{4\pi^2 k_j \sigma}.$$

Now, we have:

$$\{v(q), \tau(p)\} =< \nabla_q v, \nabla_p \tau >= \sum_{s=1}^n \frac{m'_s k_s}{A} \sin(2\pi < m', q > +\alpha)\cos(2\pi < k, p > +\beta)$$

and as $A = \sum_{s=1}^n m'_s k_s$,

$$\{v(q), \tau(p)\} = \sin(2\pi < m', q > +\alpha)\cos(2\pi < k, p > +\beta).$$

As w is a function of only one variable, q_j, we have:

$$\{w(q), \{v(q), \tau(p)\}\} = w'(q_j)\partial_{p_j}\{v(q), \tau(p)\},$$

which gives Eq. (2.6) since $\partial_{p_j}\{v(q), \tau(p)\} = -2\pi \cdot k_j \cdot \sin(2\pi < m', q > +\alpha)\sin(2\pi < k, p > +\beta)$ and $w'(q_j) = -\frac{\sin(2\pi\sigma q_j+\gamma)}{2\pi k_j}$. □

Proposition 2.5 implies that the flow maps ϕ_H^t of any time-independent Hamiltonian $H \in C^\omega(V, \mathbb{R})$ can be arbitrarily well approximated by the flow maps of some Hamiltonians from $\mathring{\mathcal{P}}$. Since $\mathring{\mathcal{P}} \subset \mathcal{P}(G)$ and $\mathcal{P}(G)$ is closed subset of $C^\omega(V, \mathbb{R})$, this gives $\mathcal{P}(G) = C^\omega(V, \mathbb{R})$ and so:

Proposition 2.6. *For every time-independent Hamiltonian H its time-t maps can, for every t, be arbitrarily well approximated by compositions of vertical and horizontal shears.*

Using Corollary 2.4 and the fact that the approximation given by the proof of Proposition 2.5 is based on the Fourier's decomposition which depends analytically on the function considered, we obtain

Corollary 2.7. *Every family of time-independent Hamiltonian functions $(H_\mu)_{\mu\in\mathscr{P}}$, the family of its time-t maps can, for every t, be arbitrarily well approximated by families of compositions of vertical and horizontal shears.*

To finish the proof of the theorem, we now show that the same is true for time-dependent Hamiltonians. First, we prove

Lemma 2.8. *Every diffeomorphism $f \in \mathrm{Ham}^\omega(V)$ can be approximated by a composition of flow maps defined by time-independent Hamiltonians. More precisely, if f is the flow map $\phi_H^{0,1}$ of a time-dependent Hamiltonian $H = (H_t)_{t\geq 0}$, then there is a complex neighborhood W of V such that for every $\eta > 0$, the holomorphic extension $f|_W$ is η-close, for all sufficiently large N, to the composition of time-$1/N$ maps $\phi^{1/N}_{H_{j/N}}$ for the time-independent Hamiltonians $H_{j/N}$:*

$$\sup_W \left\| f - \phi^{1/N}_{H_{(N-1)/N}} \circ \ldots \circ \phi^{1/N}_{H_{1/N}} \circ \phi^{1/N}_{H_0} \right\| < \eta \; .$$

Proof. We have:

$$f = \phi_H^{0,1} = \phi_H^{(N-1)/N,1} \circ \phi_H^{(N-2)/N,(N-1)/N} \circ \ldots \circ \phi_H^{1/N,2/N} \circ \phi_H^{0,1/N} .$$

Using the compactness of the time interval $[0,1]$, there exists a complex neighborhood V_0 of V and a sequence $\varepsilon_N \to 0$ such that the flow maps for time-independent Hamiltonians satisfy

$$\sup_{V_0} \left\| \phi^{1/N}_{H_t} - id - \frac{1}{N} X_{H_t} \right\| \leq \frac{\varepsilon_N}{2N}, \quad \forall 0 \leq t \leq 1, \forall N \geq 1,$$

and the non-autonomous flow maps satisfy

$$\sup_{V_0} \left\| \phi_H^{t,t+1/N} - id - \frac{1}{N} X_{H,t} \right\| = \sup_{V_0} \left\| \int_t^{t+1/N} X_{H,s} \circ \phi_H^{t,s} ds - \frac{1}{N} X_{H,t} \right\| \leq \frac{\varepsilon_N}{2N}, \quad \forall 0 \leq t \leq 1 - 1/N.$$

By Eq. (1.2), the vector fields $X_{H,t}$ and X_{H_t} are equal. Thus, summing these two estimates taken at $t = j/N$, we obtain:

$$\sup_{V_0} \left\| \phi_H^{j/N,(j+1)/N} - \phi^{1/N}_{H_{j/N}} \right\| \leq \frac{\varepsilon_N}{N}, \quad \forall 0 \leq j \leq 1 - 1/N \; .$$

From this, it easy to find complex neighborhoods $W = V_2 \Subset V_1 \subset V_0$ which satisfy the assumptions of Lemma 3.1 with $\phi^{s,t} := \phi_H^{s,t}$ and $g_j := \phi^{1/N}_{H_{j/N}}$. This implies the sought bound: there exists $L > 0$ such that for all N large enough

$$\sup_{V_2} \left\| \phi_H^{0,1} - g_{N-1} \circ \ldots \circ g_0 \right\| \leq \exp(L) \cdot \varepsilon_N \; .$$

□

If, in the above proof, we consider parametric families and employ Corollary 3.3 instead of Lemma 3.1, we obtain

Corollary 2.9. *Every family of analytic Hamiltonian diffeomorphisms $(f_\mu)_{\mu\in\mathscr{P}}$ can be approximated by compositions of families of flow maps defined by time-independent Hamiltonians taken from the family of Hamiltonians that generates $(f_\mu)_{\mu\in\mathscr{P}}$.*

Now, let f be the time-1 map for a time-dependent Hamiltonian $(H_t)_{0\le t\le 1}$. For every frozen value of $s\in[0,1]$, consider the function $H_s : V\to\mathbb{R}$ as a time-independent Hamiltonian and take its time-t flow map. This defines a family of maps

$$(\phi^t_{H_s})_{0\le s,t\le 1}. \tag{2.7}$$

It is a continuous 2-parameter family of time-independent Hamiltonian flow maps; hence by Corollary 2.7, there exists a complex neighborhood W_0 of V such that for every $\eta>0$ there is a continuous family $(g^t_s)_{0\le s,t\le 1}$ of compositions of vertical and horizontal shears, such that:

$$\sup_{W_0}\|\phi^t_{H_s}-g^t_s\|\le\eta\ ,\quad \forall 0\le s,t\le 1\ .$$

Thus, for any complex neighborhood $W_1\Subset W_0$, if η is small enough, then the differences

$$(\phi^{1/N}_{H_{(N-1)/N}}\circ\ldots\circ\phi^{1/N}_{H_{j/N}}-\phi^{1/N}_{H_{(N-1)/N}}\circ\ldots\circ\phi^{1/N}_{H_{(j+1)/N}}\circ g^{1/N}_{j/N})\circ g^{1/N}_{(j-1)/N}\circ\ldots g^{1/N}_0$$

are uniformly small (for all N and all $j=1,\ldots,N-1$) on W_1.

Summing this over $1\le j\le N-1$, we obtain that $\phi^{1/N}_{H_{(N-1)/N}}\circ\ldots\circ\phi^{1/N}_{H_{1/N}}\circ\phi^{1/N}_{H_0}$ and $g^{1/N}_{(N-1)/N}\circ\ldots g^{1/N}_0$ are uniformly close on W_1. Now, we invoke Lemma 2.8, which gives the existence of a complex neighborhood $W_2\subset W_1$ such that for every $\delta>0$, if N is sufficiently large, then f is $\delta/2$ -close to $\phi^{1/N}_{H_{(N-1)/N}}\circ\ldots\circ\phi^{1/N}_{H_{1/N}}\circ\phi^{1/N}_{H_0}$ on W_2. Thus, for N large enough, we have

$$\sup_{W_2}\left\|f-g^{1/N}_{(N-1)/N}\circ\ldots g^{1/N}_0\right\|\le\delta.$$

Since W_2 does not depend on δ, it follows that $\phi^{0,1}_H$ can be approximated arbitrarily well by a composition of vertical and horizontal shears. This proves the theorem. Corollary 1.3 is proved exactly in the same way, just the family (2.7) now also depends on the additional parameters $\mu\in\mathscr{P}$ and, instead of Lemma 2.8 we employ its prametric version Corollary 2.7.

3 Bounds on compositions

The following lemma was used several times in the proof above. It works actually on any analytic manifold and the dynamics does not need to be Hamiltonian nor real. Let V_0 be a complex manifold and V_1 a neighborhood of a compact subset V_2 of V_0: $V_2\Subset V_1\subset V_0$.

Lemma 3.1. (Discretization lemma) *For any $L>0$ and any positive sequence $\varepsilon_N\to 0$, there is N_0 such that the following property holds true for every $N\ge N_0$.*

(i) Let $X:=(X_t)_{t\in[0,1]}$ be any time-dependent vector field, holomorphic on V_0 and continuously dependent on time, such that its flow $(\phi^{s,t})_{0\le s\le t\le 1}$ is defined on V_1 and its derivative is bounded by L:

$$V_1^t:=\phi^{0,t}(V_1)\subset V_0\quad\textit{and}\quad\sup_{x\in V_1^t}\|\partial_x X_t\|\le L\quad\forall t\in[0,1].$$

(ii) *Let any sequence of analytic maps g_j, $0 \le j \le N-1$, be defined on $V_1^{j/N}$ and satisfying*

$$\sup_{V_1^{j/N}} \left\| \phi^{j/N,(j+1)/N} - g_j \right\| < \frac{\varepsilon_N}{N}\ , \quad \forall 0 \le j < N\ . \tag{3.1}$$

Then the composition $g_{N-1} \circ \ldots \circ g_0$ is well-defined on V_2 and satisfies

$$\sup_{V_2} \left\| \phi^{0,1} - g_{N-1} \circ \ldots \circ g_0 \right\| < \exp(L) \cdot \varepsilon_N. \tag{3.2}$$

Proof. We show, by induction in k, that for every $k = 1, \ldots, N$ the composition $g_{k-1} \circ \ldots \circ g_0$ is well-defined on V_2 and satisfies

$$\sup_{V_2} \left\| \phi^{0,k/N} - g_{k-1} \circ \ldots \circ g_0 \right\| < \frac{1 + \ldots + \exp(kL/N)}{N} \cdot \varepsilon_N \tag{3.3}$$

for all $N \ge N_0$. Obviously, this gives (3.2) at $k = N$.

Note that by Grönwall's inequality

$$\sup_{x \in V_1^s} \|\partial_x \phi^{s,t}\| \le \exp(L(t-s)), \qquad 0 \le s \le t \le 1. \tag{3.4}$$

In particular, the maps $\phi^{s,t}$ are uniformly continuous, hence there is $\eta > 0$ (depending only on L) such that V_1^t contains the η-neighborhood of $V_2^t := \phi^{0,t}(V_2)$ for every $0 \le t \le 1$.

Now, assume (3.3) is true for some $k \le N$ (it is true at $k = 1$ by assumption). This implies

$$\sup_{V_2} \left\| \phi^{0,k/N} - g_{k-1} \circ \ldots \circ g_0 \right\| < \exp(L)\varepsilon_N,$$

hence for all sufficiently large N:

$$\sup_{V_2} \left\| \phi^{0,k/N} - g_{k-1} \circ \ldots \circ g_0 \right\| < \eta$$

Therefore, the image $g_{k-1} \circ \ldots \circ g_0(V_2)$ lies in the η-neighborhood of $V_2^{k/N}$. This is a subset of $V_1^{k/N}$ where g_k is defined by assumption, so the composition $g_k \circ \ldots \circ g_0$ is well-defined on V_2, as required. Since

$$\phi^{0,(k+1)/N} = \phi^{k/N,(k+1)/N} \circ \phi^{0,k/N},$$

it follows from (3.4) and (3.3) that

$$\sup_{V_2} \left\| \phi^{0,(k+1)/N} - \phi^{k/N,(k+1)/N} \circ g_{k-1} \circ \ldots \circ g_0 \right\| < \exp(L/N) \frac{1 + \ldots + \exp(kL/N)}{N} \cdot \varepsilon_N.$$

By (3.1) at $j = k$, we have

$$\sup_{V_2} \left\| (\phi^{k/N,(k+1)/N} - g_k) \circ g_{k-1} \circ \ldots \circ g_0 \right\| < \frac{1}{N} \cdot \varepsilon_N.$$

Summing up these two inequalities, we obtain inequality (Eq. (3.3)) at $k+1$, i.e., we complete the induction step. □

Let us emphasis that the bound on the above lemma depends only on L and $V_2 \Subset V_1 \subset V_0$. So it implies immediately the following for family parametrized by a set E (not necessarily topological).

Corollary 3.2. *For any $L > 0$ and any positive sequence $\varepsilon_N \to 0$, there is N_0 such that the following property holds true for any $N \ge N_0$.*

Let $(X_\mu)_{\mu\in E}$ be any families of time dependent vector fields X_μ and let $(g_{j,\mu})_{\mu\in E}$, $0 \le j \le N-1$, be any families of maps such that X_μ and $(g_{j,\mu})_{0\le i\le N-1}$ satisfy assumptions (i) and (ii) of Lemma 3.1 for every $\mu \in E$. Then each of the composition $g_{N-1,\mu} \circ \ldots \circ g_{0,\mu}$ is well defined on V_2 and satisfies the following estimate with the flow $\phi_\mu^{0,1}$ of X_μ:

$$\sup_{V_2} \left\|\phi_\mu^{0,1} - g_{N-1,\mu} \circ \ldots \circ g_{0,\mu}\right\| < \exp(L) \cdot \varepsilon_N.$$

Now assume that $V_2 \subset V_1 \subset V_0$ are complex extension of a compact real analytic manifold V and that E is of the form $E = \mathscr{P}_1 \times \tilde{\mathscr{P}}_2$ where $\mathscr{P}_1$ is a compact set and $\tilde{\mathscr{P}}_2$ a complex extension of an analytic compact manifold $\mathscr{P}_2$. We obtain immediately

Corollary 3.3. *For any $L > 0$ and any positive sequence $\varepsilon_N \to 0$, there is N_0 such that the following property holds true for any $N \ge N_0$. Let $(X_\mu)_{\mu\in\mathscr{P}}$ be any families of time dependent C^ω-vector fields on V and let $(g_{j,\mu})_{\mu\in\mathscr{P}}$, $0 \le j \le N-1$, be any continuous families of C^ω maps of V which all extend to $\mathscr{P}_1 \times \tilde{\mathscr{P}}_2$ and such that X_μ and $(g_{j,\mu})_{0\le i\le N-1}$ satisfy assumptions (i) and (ii) of Lemma 3.1 for every $\mu \in \mathscr{P}_1 \times \tilde{\mathscr{P}}_2$. Then each of the composition $g_{N-1,\mu} \circ \ldots \circ g_{0,\mu}$ is well defined on V_2 and satisfies the following estimate with the flow $\phi_\mu^{0,1}$ of X_μ:*

$$\sup_{V_2} \left\|\phi_\mu^{0,1} - g_{N-1,\mu} \circ \ldots \circ g_{0,\mu}\right\| < \exp(L) \cdot \varepsilon_N.$$

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