SOME REMARKS ON SEMI-CLASSICAL ANALYSIS ON TWO-STEP NILMANIFOLDS

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Abstract. In this paper, we present recent results about the developement of a semiclassical approach in the setting of nilpotent Lie groups and nilmanifolds. We focus on two-step nilmanifolds and exhibit some properties of the weak limits of sequence of densities associated with eigenfunctions of a sub-Laplacian. We emphasize the influence of the geometry on these properties.

1. Introduction

1.1. **Subelliptic operators and subelliptic estimates.** Sub-elliptic operators are an important class of operators containing sub-Laplacians - also known as Hörmander's sums of squares of vector fields [25] that generate the tangent space by iterated commutation. These operators also appear naturally in stochastic analysis as the Kolmogorov equations of stochastic ordinary differential equations are described in terms of second order differential operators which are often sub-Laplacians. In complex geometry, Kohn Laplacian (acting on functions) on Cauchy-Riemann manifolds also gives an example of sub-elliptic operators. More generally, sub-elliptic operators appear in contact geometry, thereby having significant place.

One of their specific properties relies on the sub-elliptic estimates proved independently by Rothschild and Stein [28] on the one hand, and Fefferman and Phong [12], on the other one. While, in the elliptic case, if $\Delta u \in H^s(\mathbb{R}^d)$, then $u \in H^{s+2}(\mathbb{R}^d)$, the gain of regularity is smaller for a sub-elliptic operator $\mathbb{L} = X_1^2 + \cdots + X_p^2$. Indeed, one then has

$$\mathbb{L}u \in H^s(\mathbb{R}^d) \implies u \in H^{s+2/r}(\mathbb{R}^d)$$

where r is the mean length to obtain spanning commutators. The Rothschild and Stein proof in [28] is based on Harmonic analysis on Lie groups, as developed in [20, 28], via a lifting procedure consisting in the construction of a nilpotent stratified Lie group for which the sub-elliptic operator is a sub-Laplacian. It is in that spirit that we work here and we are interested in sublaplacians associated with a special type of manifolds called nilmanifolds, that are naturally attached to a nilpotent Lie group.

1.2. **Analysis on nilmanifolds.** In this paper, as is often the case in harmonic analysis, we restrict our attention to nilpotent Lie groups that are stratified. We will further assume that their step is two later on.

1.2.1. *Stratified Lie groups.* A stratified Lie group G is a connected simply connected Lie group whose (finite dimensional, real) Lie algebra $\mathfrak{g}$ admits an $\mathbb{N}$-stratification into linear subspaces, i.e.

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \quad \text{with} \quad [\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{1+j}, \ \ 1 \leq i \leq j.$$

In this case, the group G and its Lie algebra are nilpotent. Their step of nilpotency is the largest number $s \in \mathbb{N}$ such that $\mathfrak{g}_s$ is not trivial. In this paper, all the nilpotent Lie groups are assumed connected and simply connected.

Once a basis $X_1, \dots, X_n$ for $\mathfrak{g}$ has been chosen, we may identify the points $(x_1, \dots, x_n) \in \mathbb{R}^n$ with the points $x = \exp_G(x_1X_1 + \cdots + x_nX_n)$ in G via the exponential mapping $\exp_G : \mathfrak{g} \to G$. By

choosing a basis adapted to the stratification, we derive the product law from the Baker-Campbell-Hausdorff formula. We can also define the (topological vector) spaces $\mathcal{C}^\infty(G)$ and $\mathcal{S}(G)$ of smooth and Schwartz functions on G identified with $\mathbb{R}^n$. This induces a Haar measure dx on G which is invariant under left and right translations and defines Lebesgue spaces on G, together with a (non-commutative) convolution for functions $f_1, f_2 \in \mathcal{S}(G)$ or in $L^2(G)$,

$$(f_1 * f_2)(x) := \int_G f_1(y) f_2(y^{-1}x) dy, \quad x \in G.$$

The Lie algebra $\mathfrak{g}$ is naturally equipped with the family of dilations $\{\delta_r, r > 0\}$, $\delta_r : \mathfrak{g} \to \mathfrak{g}$, defined by $\delta_r X = r^\ell X$ for every $X \in \mathfrak{g}_\ell$, $\ell \in \mathbb{N}$ [20]. The associated group dilations derive from

$$\delta_r(\exp_G X) = \exp_G(\delta_r X), \quad r > 0, \ X \in \mathfrak{g}.$$

In a canonical way, this leads to a notion of homogeneity for functions (measurable functions as well as distributions) and operators. For instance, the Haar measure is Q-homogeneous where

$$Q := \sum_{\ell \in \mathbb{N}} \ell \dim \mathfrak{g}_\ell$$

is called the *homogeneous dimension* of G. Another example is obtained by identifying the elements of the Lie algebra $\mathfrak{g}$ with the left-invariant vector fields on G: we check readily that the elements of $\mathfrak{g}_j$ are homogeneous differential operators of degree j.

When a scalar product is fixed on the first stratum $\mathfrak{g}_1$ of the Lie algebra $\mathfrak{g}$, the group G is said to be Carnot. The intrinsic sub-Laplacian on G is then the differential operator given by

$$\mathbb{L}_G := V_1^2 + \cdots + V_q^2,$$

for any orthonormal basis $V_1, \ldots, V_q$ of $\mathfrak{g}_1$. We fix such a basis that will be used in different places of the paper.

1.2.2. *Nilmanifolds.* A nilmanifold is the one-sided quotient of a nilpotent Lie group G by a discrete subgroup Γ of G. In this paper, we will choose the left quotient of G and denote it by $M = \Gamma\backslash G$. We will consider compact nilmanifolds, or equivalently cocompact subgroups Γ. We denote by $x \mapsto \dot{x}$ the canonical projection which associates to $x \in G$ its class modulo Γ in M.

Recall that the Haar measure dx on G is unique up to a constant and, once it is fixed, $d\dot{x}$ is the only G-invariant measure on M satisfying for any function $f : G \to \mathbb{C}$, for instance continuous with compact support,

$$\int_G f(x) dx = \int_M \sum_{\gamma \in \Gamma} f(\gamma x) \ d\dot{x}. \tag{1.1}$$

We may allow ourselves to write dx for the measure on M when the variable of integration is $x \in M$ and no confusion with the Haar measure is possible.

The canonical projection $G \to M$ induces a one-to-one correspondence between the set of functions on M with the set of Γ-left periodic functions on G, that is, the set of functions f on G satisfying

$$\forall x \in G, \ \ \forall \gamma \in \Gamma, \ \ f(\gamma x) = f(x).$$

With a function f defined on M, we associate the Γ-left periodic function $f_G : x \mapsto f(\dot{x})$ defined on G. Conversely, a Γ-left periodic function f on G naturally defines a function $f_M : \dot{x} \mapsto f(x)$ on M.

Consider a linear continuous mapping $T : \mathcal{S}(G) \to \mathcal{S}'(G)$ which is invariant under Γ in the sense that

$$\forall F \in \mathcal{S}(G), \ \ \forall \gamma \in \Gamma, \ \ T(F(\gamma\,\cdot)) = (TF)(\gamma\,\cdot).$$

Then it naturally induces [18] an operator T_M on M via

$$T_M f = (T f_G)_M.$$

Furthermore, $T_M : \mathcal{D}(M) \to \mathcal{D}'(M)$ is a linear continuous mapping. Note that if T is invariant under G, then it is invariant under Γ. For instance, any left-invariant differential operator T on G induces a corresponding differential operator T_M on M.

Let us now assume that G is a Carnot group. The intrinsic sub-Laplacian on M is the operator $\mathbb{L}_M$ induced by $\mathbb{L}_G$ on M. It is a differential operator that is essentially self-adjoint on $L^2(M)$; we will keep the same notation for its self-adjoint extension. The spectrum of $-\mathbb{L}_M$ is a discrete and unbounded subset of $[0, +\infty)$. Each eigenspace of $\mathbb{L}_M$ has finite dimension. The constant functions on M form the 0-eigenspace of $\mathbb{L}_M$, see e.g. [18].

1.2.3. *Objectives.* In this paper, we consider nilpotent Lie groups G of step $s = 2$ equipped with a scalar product. They are naturally stratified, (see Section 1.3.1) and so they will also be Carnot. We will focus our attention on sequences of eigenfuctions $(\psi_k)_{k\in\mathbb{N}}$ and eigenvalues $(E_k)_{k\in\mathbb{N}}$ of $\mathbb{L}_M$, ordered in increasing order and repeated according to multiplicity:

$$-\mathbb{L}_M\psi_k = E_k\psi_k, \qquad E_1 \leq E_2 \leq \cdots \leq E_k \leq \cdots, \qquad E_k \underset{k\to\infty}{\longrightarrow} +\infty. \tag{1.2}$$

We are interested in the measures on M that are limit points of the densities $|\psi_k(x)|^2 dx$ as k tends to $+\infty$. Our result extends to operators

$$-\mathbb{L}_M^{\mathbb{U}} = -\mathbb{L}_M + \mathbb{U}(x)$$

where $x \mapsto \mathbb{U}(x)$ is a smooth potential on M. Our analysis will be using a semi-classical approach based on the harmonic analysis on the group G in order to derive invariance properties of these measures.

1.3. **Fourier analysis of step-two groups.** Our semi-classical approach is based on the Fourier theory of the group, as developed in Harmonic analysis (see for example [20, 19]). In the rest of this paper, we will consider only a nilpotent Lie group G of step two and its associated compact nilmanifolds $M = \Gamma\backslash G$.

1.3.1. *Step-two groups.* As G is step two, the derived algebra $\mathfrak{z} := [\mathfrak{g}, \mathfrak{g}]$ lies in the centre of $\mathfrak{g}$. Moreover, denoting by $\mathfrak{v}$ a complement of $\mathfrak{z}$, we have the decomposition:

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}.$$

Note that $\mathfrak{z} = [\mathfrak{v}, \mathfrak{v}]$ and that this decomposition yields a stratification of $\mathfrak{g}$ with $\mathfrak{g}_1 = \mathfrak{v}$, $\mathfrak{g}_2 = \mathfrak{z}$. Hence G is naturally stratified with dilations given by $\delta_\varepsilon(V + Z) = \varepsilon V + \varepsilon^2 Z$ where $\varepsilon > 0$, $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$. Its topological dimension is $n = \dim \mathfrak{v} + \dim \mathfrak{z}$ while the homogeneous dimension is $Q = \dim \mathfrak{v} + 2 \dim \mathfrak{z}$. We also assume that a scalar product has been fixed on $\mathfrak{g}$, and that $\mathfrak{v}$ is an orthogonal complement of $\mathfrak{z}$.

1.3.2. *The dual set.* The dual set $\widehat{G}$ of G is the set of the equivalence classes of the irreducible unitary representations of G. We will often allow ourselves to identify a class of such representations with one of its representatives. Since G is a nilpotent Lie group, its dual is the disjoint union of the (classes of unitary irreducible) representations of dimension one and of infinite dimension:

$$\widehat{G} = \widehat{G}_1 \sqcup \widehat{G}_\infty, \quad \widehat{G}_1 := \{\text{class of } \pi,\ \dim \pi = 1\}, \quad \widehat{G}_\infty := \{\text{class of } \pi,\ \dim \pi = \infty\}.$$

As G is step two, $\widehat{G}_1$ and $\widehat{G}_\infty$ can be described in a relatively simple manner.

(i) The (classes of unitary irreducible) one-dimensional representations are parametrized by the elements $\omega \in \mathfrak{v}^*$ of the dual of $\mathfrak{v}$ and consists of the characters

$$\pi^\omega(x) = \mathrm{e}^{i\omega(V)}, \ \ x = \exp_G(V + Z), \ \ V \in \mathfrak{v}, \ \ Z \in \mathfrak{z}.$$

(ii) The (classes of unitary irreducible) infinite dimensional representations are parametrised by a non-zero element $\lambda \in \mathfrak{z}^* \setminus \{0\}$ of the dual of $\mathfrak{z}$ and another parameter $\nu \in \mathfrak{v}^*$ which we now describe. For any $\lambda \in \mathfrak{z}^\star$, we consider the skew-symmetric bilinear form on $\mathfrak{v}$ defined by

$$\forall\, U, V \in \mathfrak{v}\,, \quad B(\lambda)(U,V) := \lambda([U,V])\,. \tag{1.3}$$

We denote by $\mathfrak{r}_\lambda$ the radical of $B(\lambda)$. The other parameter ν will be in the dual $\mathfrak{r}_\lambda^*$ of this radical.

Using the scalar product on $\mathfrak{g}$, we can construct the representation $\pi^{\lambda,\nu}$ for each $\lambda \in \mathfrak{z}^* \setminus \{0\}$ and $\nu \in \mathfrak{r}_\lambda^*$ as follows. First, we will allow ourselves to keep the same notation for the skew-symmetric form $B(\lambda)$ and the corresponding skew-symmetric linear map on $\mathfrak{v}$. Hence $\mathfrak{r}_\lambda = \ker B(\lambda)$. As $B(\lambda)$ is skew symmetric, we find an orthonormal basis of $\mathfrak{v}$

$$\big(P_1^\lambda, \dots, P_d^\lambda, Q_1^\lambda, \dots, Q_d^\lambda, R_1^\lambda, \dots, R_k^\lambda\big) \qquad \text{with} \quad k = k_\lambda := \dim \mathfrak{r}_\lambda, \quad d = d_\lambda := \frac{\dim \mathfrak{v} - k}{2},$$

where the matrix of $B(\lambda)$ takes the block form

$$\begin{pmatrix} 0_{d,d} & D(\lambda) & 0_{d,k} \\ -D(\lambda) & 0_{d,d} & 0_{d,k} \\ 0_{k,d} & 0_{k,d} & 0_{k,k} \end{pmatrix}. \tag{1.4}$$

Here $D(\lambda)$ is a diagonal matrix with positive diagonal entries depending on λ. Note that $\mathfrak{r}_\lambda = \mathrm{Span}\,\big(R_1^\lambda, \dots, R_k^\lambda\big)$ and we decompose $\mathfrak{v}$ as

$$\mathfrak{v} = \mathfrak{p}_\lambda + \mathfrak{q}_\lambda + \mathfrak{r}_\lambda \quad \text{where} \quad \mathfrak{p}_\lambda := \mathrm{Span}\,\big(P_1^\lambda, \dots, P_d^\lambda\big) \quad , \qquad \mathfrak{q}_\lambda := \mathrm{Span}\,\big(Q_1^\lambda, \dots, Q_d^\lambda\big).$$

One may assume that the above basis for $\mathfrak{v}$ depends continuously on λ.

The representation $\pi^{\lambda,\nu}$ acts on $L^2(\mathfrak{p}_\lambda)$ via

$$\pi^{\lambda,\nu}(x)\phi(\xi) = e^{i\lambda\left(Z + \left[D(\lambda)^{\frac{1}{2}}\xi + \frac{1}{2}P, Q\right]\right)} e^{i\nu(R)} \phi\left(D(\lambda)^{\frac{1}{2}}\xi + P\right), \quad \phi \in L^2(\mathfrak{p}_\lambda), \quad \xi \in \mathfrak{p}_\lambda, \tag{1.5}$$

where x is written as $x = \exp_G(P + Q + R + Z)$ with $P \in \mathfrak{p}_\lambda$, $Q \in \mathfrak{q}_\lambda$, $R \in \mathfrak{r}_\lambda$, $Z \in \mathfrak{z}$. If $\nu = 0$, we will use the shorthand $\pi^{\lambda,0} = \pi^\lambda$.

With the representations described in (i) and (ii) above, the dual set of G is: $\widehat{G} = \widehat{G}_1 \sqcup \widehat{G}_\infty$ with

$$\widehat{G}_1 = \{\text{class of } \pi^\omega,\, \omega \in \mathfrak{v}^*\} \quad \text{and} \quad \widehat{G}_\infty = \{\text{class of } \pi^{\lambda,\nu},\, \lambda \in \mathfrak{z}^* \setminus \{0\},\, \nu \in \mathfrak{r}_\lambda^*\}.$$

This can be justified in this case with the von Neumann theorem characterising the representations of the Heisenberg groups. Equivalently, we can also use the orbit method which states that there is a one-to-one correspondence between $\pi \in \widehat{G}$ and the co-adjoint orbits $\mathfrak{g}^*/G$. The advantage of the orbit method is that the Kirillov map $\mathfrak{g}^*/G \to \widehat{G}$ is a homeomorphism [7], giving us easy information on the topology of subsets of $\widehat{G}$. Furthermore, one can check that the co-adjoint action of G on $\mathfrak{g}^* = \mathfrak{v}^* \oplus \mathfrak{z}^*$ leaves the $\mathfrak{z}^*$-component invariant. Hence, we can describe the co-adjoint orbit of any $\nu + \lambda \in \mathfrak{g}^* = \mathfrak{v}^* \oplus \mathfrak{z}^*$ by choosing the unique representative as the linear form $\omega = \nu$ if $\lambda = 0$, and $\lambda + \nu$ with $\nu \in \mathfrak{r}_\lambda^*$ if $\lambda \neq 0$. Via the Kirillov map, they correspond respectively to π^ω and $\pi^{\lambda,\nu}$.

1.3.3. *The subsets Ω_k and Λ_0.* As a set, $\mathfrak{z}^* \setminus \{0\}$ decomposes as the disjoint union of

$$\Omega_k := \{\lambda \in \mathfrak{z}^* \setminus \{0\} \;:\; \dim \mathfrak{r}_\lambda = k\}, \quad k \in \mathbb{N}.$$

Observe that $\Omega_k = \emptyset$ when $k > \dim \mathfrak{v}$ and also when $k = \dim \mathfrak{v}$ because if $k_\lambda = \dim \mathfrak{v}$ then $\mathfrak{r}_\lambda = \mathfrak{v}^*$, thus $B_\lambda = 0$ and $\lambda = 0$. We denote by k_0 the smallest $k \in \mathbb{N}$ such that $\Omega_k \neq \emptyset$; roughly speaking, this is the set of $\lambda \in \mathfrak{z}^*$ for which $B(\lambda)$ is of smallest kernel. We have

$$\mathfrak{z}^* \setminus \{0\} = \sqcup_{k_0 \leq k < \dim \mathfrak{v}}\, \Omega_k.$$

We can describe $\cup_{k' \geq k} \Omega_{k'}$ as the set of $\lambda \in \mathfrak{z}^* \setminus \{0\}$ such that all the minors of $B(\lambda)$ (viewed as a matrix in the basis that we have fixed) of order $\leq \dim \mathfrak{v} - k$ cancel, and Ω_k as the subset of $\cup_{k' \geq k} \Omega_{k'}$ formed by the $\lambda' s$ such that at least one minor of order $= \dim \mathfrak{v} - k$ does not vanish. Since

$B(\lambda)$ is linear in λ, $\cup_{k'\geq k}\Omega_{k'}$ is an algebraic variety, and Ω_k is an open subset of it. Moreover, if $\Omega_k \neq \emptyset$ then $\cup_{k'>k}\Omega_{k'}$ is an algebraic subvariety with $\dim \cup_{k'>k}\Omega_{k'} < \dim \cup_{k'\geq k}\Omega_{k'}$. Consequently, Ω_k is an open subset of $\cup_{k'\geq k}\Omega_{k'}$ and it is either empty or dense in $\cup_{k'\geq k}\Omega_{k'}$.

We can decompose each Ω_k into further subsets, according to the multiplicity of the eigenvalues of $B(\lambda)$ viewed as a matrix in a canonical basis. Here, we will be only considering the case $k = k_0$ and denote by Λ_0 the set of $\lambda \in \Omega_{k_0}$ for which $B(\lambda)$ has the maximal number of distinct eigenvalues. Recall that, by the Cauchy residue formula, the multiplicity of a zero z_0 of a polynomial $p(z)$ is equal to $\oint_{|z-z_0|=r} \frac{p'(z)}{p(z)}dz$ for r small enough. Applying this to $\det(B(\lambda)^2 - z)$ in the case of maximal multiplicities implies that the multiplicities of the eigenvalues of $B(\lambda)^2$ for $\lambda \in \Lambda_0$ are locally constant and that the subset Λ_0 is open in Ω_{k_0}. Moreover, by the implicit function theorem, the eigenvalues of $B(\lambda)^2$ can be written locally as smooth functions (even algebraic expressions) of $\lambda \in \Lambda_0$. Similar properties hold for each subset of Ω_{k_0} with fewer constraints on the multiplicities, implying that Λ_0 is dense in Ω_{k_0}.

The Heisenberg groups correspond to the case when $\dim \mathfrak{z} = 1$ while the Heisenberg-type groups are exactly the step-two nilpotent groups G for which $B(\lambda)^2 = -|\lambda|^2 \mathrm{I}_{\mathfrak{v}}$. Heisenberg-type groups and their nilmanifolds have an H-type foliation as in [4], and so do the groups G and their nilmanifolds when, more generally, every $B(\lambda)$, $\lambda \in \mathfrak{z}^* \setminus \{0\}$, has a trivial radical $\mathfrak{r}_\lambda = \{0\}$. Geometrically, these nilmanifolds are contact manifolds when the radicals are all trivial and $\dim \mathfrak{z} = 1$, and they are quasi-contact manifolds when the radicals may not be trivial. The analysis of the properties of weak limits of densities of eigenvalues of the sub-Laplacian for contact manifolds was studied in [10] and for quasi-contact manifold of dimension four with radical generically of dimension one was studied in [29].

As the co-adjoint action is trivial on the $\mathfrak{z}^*$-component, the sets Ω_k may be viewed as the unions of the co-adjoint orbits of $\nu + \lambda \in \mathfrak{g}^* = \mathfrak{v}^* \oplus \mathfrak{z}^*$ with $\lambda \in \Omega_k$, or our chosen representatives for those co-orbits:

$$\Omega_k \sim \{(\lambda, \nu) \in \mathfrak{z}^* \times \mathfrak{v}^*,\ \lambda \in \Omega_k,\ \nu \in \mathfrak{r}_\lambda^*\}, \tag{1.6}$$

and therefore identified via Kirillov's map with the following subset of $\widehat{G}$

$$\Omega_k \sim \{\pi = \pi^{\lambda,\nu} \in \widehat{G}_\infty,\ \lambda \in \Omega_k,\ \nu \in \mathfrak{r}_\lambda^*\}.$$

We also proceed similarly for Λ_0. As subsets of $\widehat{G}_\infty$, they enjoy the same topological properties; for instance, Ω_{k_0} which is an open dense subset of $\widehat{G}_\infty$.

1.3.4. *The Fourier transform.* Let $f \in L^1(G)$, the Fourier transform of f is the field of operators

$$\mathcal{F}(f) := \{\widehat{f}(\pi) : \mathcal{H}_\pi \to \mathcal{H}_\pi, \pi \in \widehat{G}\} \quad \text{given by } \widehat{f}(\pi) = \int_G f(x)\pi(x)^* dx,$$

for any (continuous unitary) representation π of G.

The unitary dual $\widehat{G}$ is a standard Borel space, and there exists a unique positive Borel measure μ on $\widehat{G}$ such that for any continuous function $f : G \to \mathbb{C}$ with compact support we have

$$\int_G |f(x)|^2 dx = \int_{\widehat{G}} \|\widehat{f}(\pi)\|^2_{HS(\mathcal{H}_\pi)} d\mu(\pi).$$

The measure μ is called the Plancherel measure and the formula above the Plancherel formula. For instance, in the case of step-two groups, the Plancherel measure is given by $d\mu(\pi^{\lambda,\nu}) = c_0 \det(D(\lambda))\, d\lambda d\nu$, for a known constant $c_0 > 0$ [11, 27]; note that it is supported on the subsets Ω_{k_0} or even Λ_0 of $\widehat{G}_\infty$ defined in Section 1.3.3.

The Plancherel formula extends the group Fourier transform unitarily to functions $f \in L^2(G)$: their Fourier transforms are then a Hilbert-Schmidt fields of operators satisfying the Plancherel

formula. The group Fourier transform also extends readily to classes of distributions, for instance the distributions with compact support and the distributions whose associated right convolution operators are bounded on $L^2(G)$. If T is the associated operator, we denote by $\widehat{T}$ or $\pi(T)=\widehat{T}(\pi)$ the associated field of operators with

$$\mathcal{F}(Tf)(\pi)=\pi(T)\circ\mathcal{F}f(\pi),\ \ \forall f\in\mathcal{S}(G).$$

In particular, the group Fourier transform extends to left-invariant differential operators.

The considerations above are known for any nilpotent Lie group, and let us consider the case of step-two groups. The group Fourier transform of $f\in L^1(G)$ gives a scalar number at $\pi=\pi^\omega$ and a bounded operator on $\mathcal{H}_{\pi^{\lambda,\nu}}=L^2(\mathfrak{p}^\lambda)$ for $\pi=\pi^{\lambda,\nu}$. It is easy to compute that for the 1-dimensional representation, we have $\pi^\omega(-\mathbb{L}_G)=|\omega|^2$. In the remainder of the paper, we will use the notation $\pi(\mathbb{L})$ and $\widehat{\mathbb{L}}=\{\pi(\mathbb{L}),\pi\in\widehat{G}\}$ and omit the index G in this context. The case of representations of infinite dimension is more involved. The following is known in great generality [19]:

(1) $\mathbb{L}_G$ and $\pi(\mathbb{L})$ for $\pi\in\widehat{G}$ are essentially self-adjoint on $L^2(G)$ and $\mathcal{H}_\pi$; we keep the same notation for their self-adjoint extensions. Hence they both admit spectral decompositions.
(2) For each $\pi\in\widehat{G}\setminus\{1_{\widehat{G}}\}$, the spectrum $\mathrm{sp}(\pi(-\mathbb{L}))$ of $\pi(-\mathbb{L})$ is discrete and lies in $(0,\infty)$ and each eigenspace is finite dimensional, while for $\pi=1_{\widehat{G}}$, $\pi(\mathbb{L})=0$.
(3) Consider the spectral decomposition $\mathbb{P}_\zeta$, $\zeta\geq 0$, of $-\mathbb{L}_G$, i.e. $-\mathbb{L}_G=\int_0^\infty\zeta d\mathbb{P}_\zeta$. For each $\pi\in\widehat{G}\setminus\{1_{\widehat{G}}\}$, the group Fourier transform $\pi(\mathbb{P}_\zeta)$ of the projections $\mathbb{P}_\zeta$ are orthogonal projections of $\mathcal{H}_\pi$. Furthermore, they yield a spectral decomposition of $-\widehat{\mathbb{L}}$: $\pi(-\mathbb{L})=\sum_{\zeta\in\mathrm{sp}(\pi(-\mathbb{L}))}\zeta\pi(\mathbb{P}_\zeta)$.

In the step-two case, some of the properties above are easy to see. Indeed, denoting by

$$\eta_j=\eta_j(\lambda),1\leq j\leq d,\quad\text{with the convention } 0<\eta_1(\lambda)\leq\ldots\leq\eta_d(\lambda),$$

the positive entries of $D(\lambda)=\mathrm{diag}(\eta_1,\ldots,\eta_d)$, we readily compute

$$\pi^{\lambda,\nu}(P_j^\lambda)=\sqrt{\eta_j(\lambda)}\partial_{\xi_j}\ \text{ and }\ \pi^{\lambda,\nu}(Q_j^\lambda)=i\sqrt{\eta_j(\lambda)}\xi_j \tag{1.7}$$

and deduce from the additional observation $\pi^{\lambda,\nu}(R_\ell^\lambda)=i\nu_\ell$, $1\leq l\leq k$.

$$\pi^{\lambda,\nu}(-\mathbb{L})=H(\lambda)+|\nu|^2,$$

where $H(\lambda)$ is the operator on $\mathcal{H}_\lambda$ given by

$$H(\lambda)=\sum_{1\leq j\leq d}\eta_j(\lambda)(-\partial_{\xi_j}^2+\xi_j^2).$$

which is up to multiplicative factors the harmonic oscillator of $L^2(\mathbb{R}^d)$. Recall that Hermite functions give an orthonormal basis of eigenfunctions of $H(\lambda)$ with eigenvalues

$$\zeta(\alpha,\lambda):=\sum_{1\leq j\leq d}(2\alpha_j+1)\eta_j(\lambda),\qquad\alpha\in\mathbb{N}^d, \tag{1.8}$$

see Section 4.2.2. Hence, the spectrum of $\pi^{\lambda,\nu}(-\mathbb{L})$ is $\mathrm{sp}(\pi^{\lambda,\nu}(-\mathbb{L}))=\left\{\zeta(\alpha,\lambda)+|\nu|^2,\ \alpha\in\mathbb{N}^d\right\}$, giving in this special case Property (2) above. Furthermore, the spectral projections $\pi^{\lambda,\nu}(\mathbb{P}_\zeta)$ onto the eigenspaces of $H(\lambda)$ are either zero or orthogonal projections onto subspaces generated by Hermite functions.

The properties above hold for any $\lambda\in\mathfrak{z}^*\setminus\{0\}$. Restricting to Λ_0, each $\eta_j(\lambda)$ is a smooth function of $\lambda\in\Lambda_0$ since the η_j^2's are the eigenvalues of $B(\lambda)^2$ which are diagonalisable linear morphisms with eigenvalues of constant multiplicities depending smoothly on λ. Therefore, $\zeta(\alpha,\lambda)$ in (1.8) also depends smoothly on λ in Λ_0.

1.4. **Main result.** Let $x \mapsto \mathbb{U}(x)$ be a smooth potential on M. Let $(\psi_k^{\mathbb{U}})_{k\in\mathbb{N}}$ be a sequence of eigenfunctions of $-\mathbb{L}_M^{\mathbb{U}} = -\mathbb{L}_M + \mathbb{U}$ according to

$$-\mathbb{L}_M^{\mathbb{U}}\psi_k^{\mathbb{U}} = E_k^{\mathbb{U}}\psi_k^{\mathbb{U}}, \quad k \in \mathbb{N}. \tag{1.9}$$

Without loss of generality, we may assume $E_k^{\mathbb{U}} \geq 0$ for all $k \in \mathbb{N}$ (if not, we modify $\mathbb{U}$ by a constant). Let ϱ be a weak limit of the density $|\psi_k^{\mathbb{U}}(x)|^2 dx$, then ϱ decompose according to the structure of $\widehat{G}$ and each of the elements of this decomposition enjoys its own invariances. These invariances are expressed in terms of the elements ω, λ and ν characterizing the points of $\widehat{G}$. We will need the following notation to state the result.

(a) For each $\lambda \in \mathfrak{z}^*$ and $\nu \in \mathfrak{r}_\lambda^*$, we associate

$$\nu \cdot R^\lambda := \nu_1 R_1^\lambda + \cdots + \nu_k R_k^\lambda \ \in \mathfrak{r}_\lambda,$$

where the ν_j's are the coordinates of ν in the dual of the orthonormal basis $(R_1^\lambda, \cdots, R_k^\lambda)$, i.e. $\nu = \nu_1 (R_1^\lambda)^* + \ldots + \nu_k (R_k^\lambda)^*$. This definition is independent of the choice of the orthonormal basis $(R_1^\lambda, \cdots, R_k^\lambda)$ for $\mathfrak{r}_\lambda$.

(b) In the same spirit, for any $\omega \in \mathfrak{v}^*$, we associate

$$\omega \cdot V := \omega_1 V_1 + \cdots \omega_q V_q \ \in \mathfrak{v},$$

where the ω_j's are the coordinates of ω in the dual of an orthonormal basis $(V_1, \cdots, V_q)$: $\omega = \omega_1 V_1^* + \ldots + \omega_q V_q^*$. Here, $q = \dim \mathfrak{v}$. This definition is independent of the choice of the orthonormal basis $(V_1, \cdots, V_q)$ for $\mathfrak{v}$.

(c) If $k_0 = 0$ and $\lambda \in \Lambda_0$, each eigenvalue $\zeta = \zeta(\alpha, \lambda)$ in (1.8) of $\pi^\lambda(\mathbb{L})$ depends smoothly on λ in Λ_0. The vector in $\mathfrak{z}$ corresponding to the gradient at λ is denoted by

$$\nabla_\lambda \zeta(\alpha, \lambda) = \nabla_\lambda \zeta \in \mathfrak{z}.$$

Theorem 1.1 ([15, 17, 16])**.** *Let $(\psi_k^{\mathbb{U}})_{k\in\mathbb{N}}$ be a sequence of eigenfunctions of $-\mathbb{L}_M^{\mathbb{U}} = -\mathbb{L}_M + \mathbb{U}$ according to* (1.9)*. Then a weak limit ϱ of the density $|\psi_k^{\mathbb{U}}(x)|^2 dx$ decomposes as*

$$\varrho = \varrho^{\mathfrak{v}} + \varrho^{\mathfrak{z}} \tag{1.10}$$

with

(1) $\varrho^{\mathfrak{v}}(x) = \displaystyle\int_{\mathfrak{v}^*} \varsigma\,(x, d\omega)$ *where the measure ς is invariant by the flow*

$$(x, \omega) \mapsto (\mathrm{Exp}(s\,\omega \cdot V)x, \omega), \quad s \in \mathbb{R}$$

(2) $\varrho^{\mathfrak{z}}(x) = \displaystyle\sum_{k=0}^{\dim \mathfrak{v} - 1} \int_{(\lambda,\nu)\in\Omega_k} \gamma_k(x, d\lambda, d\nu)$ *with the identification* (1.6) *for Ω_k, with each measure $\gamma_k(x, \lambda, \nu)$ being supported in $M \times \Omega_k$ where it is invariant under the flow given by*

$$(x, (\lambda, \nu)) \longmapsto (\mathrm{Exp}(s\,\nu \cdot R^\lambda)x, (\lambda, \nu)), \quad s \in \mathbb{R}.$$

(3) *Furthermore, in the case when $\Omega_0 \neq \emptyset$, omitting $\nu = 0$,*

$$\gamma_0(x, \lambda) = \sum_{\alpha\in\mathbb{N}} \gamma_0^{(\alpha)}(x, \lambda),$$

with each measure $1_{\lambda\in\Lambda_0}\gamma_0^{(\alpha)}$ being supported on $M \times \Lambda_0$ where it is invariant under the flow given by

$$(x, \lambda) \longmapsto (\mathrm{Exp}(s\,\nabla_\lambda \zeta(\alpha, \lambda))x, \lambda), \quad s \in \mathbb{R}.$$

In the case of the groups of Heisenberg type, $\eta_j(\lambda) = |\lambda|$ for all j, so $\Lambda_0 = \Omega_0 = \mathfrak{z}^* \setminus \{0\}$ and

$$\nabla_\lambda \zeta(\alpha, \lambda) = \mathcal{Z}^\lambda \sum_{j=1}^{\dim \mathfrak{v}/2} (2\alpha_j + 1), \qquad \text{where} \quad \mathcal{Z}^\lambda := |\lambda|^{-1}\lambda^*, \tag{1.11}$$

and $\lambda^* \in \mathfrak{z}$ corresponds to λ by duality via the scalar product. We therefore recover with Theorem 1.1 the results of the first two authors in [15].

Theorem 1.1 is a consequence of Theorem 2.4 below. It is based on a microlocal approach and the measures γ that appear in the statement above are microlocal objects that can be compared with the semi-classical measures introduced in the 90s in the Euclidean context in [24, 21, 22, 23]. The difference here is that the semi-classical calculus we use is based on the Harmonic analysis of the group G and on the Fourier transform introduced via representation theory as presented above. This setting has been introduced in [19] in a microlocal context where no specific semi-classical scale ε is specified. It uses a pseudo-differential calculus with operator-valued symbols that can be composed with the Fourier transform of the functions (that are also operator-valued).

The construction of a pseudodifferential calculus on groups is an old question from the 1980s [30, 5, 6, 9] that have known recent developments with an abstract point of view from the theory of algebra of operators in [31, 32, 33], and with a PDEs approach in [19, 1, 13] with applications in control theory and observability [17]

We conclude this section with some comments about Theorem 1.1. It is noticeable that there is coexistence of two kinds of behaviour, with a splitting of the measure γ corresponding to the different types of elements of $\widehat{G}$. In the context of the Heisenberg group, $\Lambda_0 = \Omega_0 \neq \emptyset$ and $\nabla_\lambda \zeta$ is colinear to $\mathcal{Z}^\lambda$ (see (1.11)) and this is linked to the wave aspect of the sub-Laplacian in this group pointed out in [3, 8, 10, 2]. On other nilpotent Lie groups where $\Omega_0 = \emptyset$, the other vector fields involved, $\nu \cdot R^\lambda$, are more of Schrödinger's type.

Acknowledgements. The authors acknowledges the support of the Leverhulme Trust via Research Project Grant 2020-037. The first author thank Cyril Letrouit for inspiring discussions.

2. Noncommutative semi-classical setting

2.1. **Semi-classical pseudodifferential operators.** We consider the set $\mathcal{A}_0$ of fields of operators $\{\sigma(x,\pi) \in \mathcal{L}(\mathcal{H}_\pi),\ (x,\pi) \in M \times \widehat{G}\}$ such that

$$\sigma(x,\pi) = \mathcal{F}\kappa_x(\pi) = \int_G \kappa_x(z)\pi(z)^* dz,$$

where $x \mapsto \kappa_x(\cdot)$ is in $\mathcal{C}^\infty(M, \mathcal{S}(G))$. We call the function κ_x the *convolution kernel* associated with the symbol σ. In the spirit of the works [2, 19], and when $\varepsilon \ll 1$ is a semi-classical parameter, the ε-quantization of the symbols $\sigma \in \mathcal{A}_0$ is given by

$$\mathrm{Op}_\varepsilon(\sigma) f(x) = \int_{\hat{G}} \mathrm{tr}\left(\pi(x)\sigma(x, \varepsilon\cdot\pi)\widehat{f}(\pi)\right) d\mu(\pi), \quad f \in \mathcal{S}(M), \quad x \in M.$$

Here, $\varepsilon \cdot \pi$ denotes the class in $\widehat{G}$ of the irreducible representation $x \mapsto \pi(\delta_\varepsilon x)$. Setting

$$\kappa_x^\varepsilon(z) = \varepsilon^{-Q}\kappa_x(\delta_{\varepsilon^{-1}} z),$$

the ε-quantization then obeys to

$$\mathrm{Op}_\varepsilon(\sigma) f(x) = \int_G \kappa_x^\varepsilon(y^{-1}x) f(y) dy = \sum_{\gamma\in\Gamma} \int_{y\in M} \kappa_x^\varepsilon(\gamma y^{-1} x) dy, \quad f \in \mathcal{S}(M), \quad x \in M. \tag{2.1}$$

As in the case of groups (see [14]), the family $(\mathrm{Op}_\varepsilon(\sigma))_{\varepsilon>0}$ is a bounded family in $\mathcal{L}(L^2(M))$:

Proposition 2.1. *There exists $C > 0$ such that for all $\sigma \in \mathcal{A}_0$ and $\varepsilon > 0$,*

$$\|\mathrm{Op}_\varepsilon(\sigma)\|_{\mathcal{L}(L^2(M))} \leq \int_G \sup_{x\in M} |\kappa_x(z)|dz.$$

Proof. By Young's convolution inequality

$$\|f * \kappa_x^\varepsilon(\gamma\cdot)\|_{L^2(M)} \leq \| \sup_{x\in M} |\kappa_x^\varepsilon(\gamma\cdot)|\|_{L^1(M)} \|f\|_{L^2(M)},$$

$$\text{with } \| \sup_{x\in M} |\kappa_x^\varepsilon(\gamma\cdot)|\|_{L^1(M)} = \varepsilon^{-Q} \int_M \sup_{x\in M} |\kappa_x(\varepsilon^{-1}\cdot\gamma y)|dy = \int_{\gamma^{-1}M} \sup_{x\in M} |\kappa_x(y)|dy.$$

Therefore, using (2.1), we deduce

$$\begin{aligned}\|\mathrm{Op}_\varepsilon(\sigma)f\|_{L^2(M)} &\leq \sum_{\gamma\in\Gamma} \|f * \kappa_x^\varepsilon(\gamma\cdot)\|_{L^2(M)} \\ &\leq \|f\|_{L^2(M)} \sum_{\gamma\in\Gamma} \int_{\gamma^{-1}M} \sup_{x\in M} |\kappa_x(y)|dy = \|f\|_{L^2(M)} \int_G \sup_{x\in M} |\kappa_x(y)|dy.\end{aligned}$$

□

Besides, this semi-classical pseudodifferential calculus enjoys symbolic calculus (see Proposition 3.6 in [14] in the case of groups and Proposition 2.2 in [17] for the extension to nilmanifolds).

2.2. **Semi-classical measures.** Let us first introduce our notion of operator-valued measures introduced in the earlier papers of the first two authors. We will use the same notation as in those paper, even if it means using the Greek letter Γ for the trace-class operators $\Gamma(x,\pi)$. We think that there is no possible confusion with our current notation for the co-compact discrete subgroup Γ of G, and thus will allow this small conflict of notation

We consider pairs (Γ,γ) consisting in a positive Radon measure γ on $M\times\widehat{G}$ and a measurable field over $(x,\pi)\in M\times\widehat{G}$ of trace-class operators $\Gamma(x,\pi)$ on $\mathcal{H}_\pi$ satisfying

$$\int_{M\times\widehat{G}} \mathrm{Tr}\,|\Gamma(x,\pi)|\,d\gamma(x,\pi) < \infty.$$

We equip the set of such pairs with the equivalence relation $(\Gamma,\gamma)\sim(\Gamma',\gamma')$ given by the existence of a measurable function $f : M\times\widehat{G}\to\mathbb{C}$ such that

$$\gamma' = f\gamma \quad \text{and} \quad \Gamma' = f^{-1}\Gamma, \ \ \gamma - a.e.$$

We denote by $\mathcal{M}_{ov}(M\times\widehat{G})$ the set of equivalence classes for this relation and by $\Gamma d\gamma$ the class of the pair (Γ,γ). If $\Gamma\geq 0$, then we say that the operator valued measure $\Gamma d\gamma$ is positive, and we denote by $\mathcal{M}_{ov}^+(M\times\widehat{G})$ the set of the positive operator-valued measures on $M\times\widehat{G}$. They characterize bounded families in $L^2(M)$ according to the following theorem.

Proposition 2.2 ([14, 15])**.** *Let $(\psi^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^2(M)$. There exist a subsequence $\varepsilon_k\to 0$ as $k\to\infty$, and an operator-valued measure $\Gamma d\gamma\in\mathcal{M}_{ov}^+(M\times\widehat{G})$ satisfying*

$$\forall\sigma\in\mathcal{A}_0, \qquad \left(\mathrm{Op}_{\varepsilon_k}(\sigma)\psi^{\varepsilon_k},\psi^{\varepsilon_k}\right) \underset{k\to\infty}{\longrightarrow} \int_{M\times\widehat{G}} \mathrm{Tr}\left(\sigma(x,\pi)\Gamma(x,\pi)\right) d\gamma(x,\pi).$$

Continuing with the setting of the statement above, we say then that the operator-valued measure $\Gamma d\gamma$ is a *semi-classical measure* of $(\psi^\varepsilon)_{\varepsilon>0}$ at the scale ε. A given family $(\psi^\varepsilon)_{\varepsilon>0}$ may have several semi-classical measures, depending on different subsequences $(\varepsilon_k)_{k\in\mathbb{N}}$. The knowledge of all these families indicates the obstruction to strong convergence in $L^2(M)$ of the family $(\psi^\varepsilon)_{\varepsilon>0}$.

The scale ε is particularly interesting for analyzing the oscillations of a family $(\psi^\varepsilon)_{\varepsilon>0}$ that satisfies weighted Sobolev estimates such as

$$\exists s, C>0,\ \ \forall \varepsilon>0,\ \ \|(-\varepsilon^2\mathbb{L}_M)^{\frac{s}{2}}\psi^\varepsilon\|_{L^2(M)}\leq C. \tag{2.2}$$

Indeed, one can then link the weak limits of the energy densities with the semi-classical measures:

Proposition 2.3 ([15])**.** *Assume $(\psi^\varepsilon)_{\varepsilon>0}$ satisfies* (2.2) *and that $\Gamma d\gamma$ is a semi-classical measure of $(\psi^\varepsilon)_{\varepsilon>0}$ for the subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$. Then for all $\phi\in\mathcal{C}^\infty(M)$,*

$$\limsup_{k\to+\infty}\int_M \phi(x)|\psi^{\varepsilon_k}(t,x)|^2dx=\int_{M\times\widehat{G}}\phi(x)\mathrm{Tr}\left(\Gamma(x,\pi)\right)d\gamma(x,\pi). \tag{2.3}$$

2.3. **Application to quantum limits.** Let us now come back to the sequence (1.2) of eigenfunctions $(\psi_k^{\mathbb{U}})_{k\in\mathbb{N}}$ of the sub-Laplacian operator $-\mathbb{L}_M^{\mathbb{U}}=-\mathbb{L}_M+\mathbb{U}(x)$ for a compact nilmanifod $M=\Gamma\backslash G$ whose underlying group G is step two. Denoting by $E_k^{\mathbb{U}}$ the associated sequence of eignevalue; we set

$$\varepsilon_k=(E_k^{\mathbb{U}})^{-1/2}$$

we obtain a semi-classical scale such that the sequence $(\psi_k^{\mathbb{U}})_{k\in\mathbb{N}}$ is ε_k-oscillating. Thus any weak limit ϱ of the energy density $|\psi_k^{\mathbb{U}}(x)|^2dx$ is the marginal of a semi-classical measure $\Gamma d\gamma$ of the family $(\psi_k^{\mathbb{U}})_{k\in\mathbb{N}}$ according to (2.3). Therefore, the properties of the semi-classical measures of the sequence $(\psi_k^{\mathbb{U}})_{k\in\mathbb{N}}$ will reflect on any weak limit of the energy density.

We now omit the index $k\in\mathbb{N}$ and focus on the semi-classical measures of a family of normalized functions $(\psi^\varepsilon)_{\varepsilon>0}$ that satisfy

$$-\varepsilon^2\mathbb{L}_M^{\mathbb{U}}\psi^\varepsilon=\psi^\varepsilon, \tag{2.4}$$

where $\mathbb{U}\in\mathcal{C}^\infty(M)$ is a potential on M.

As G is a nilpotent Lie group, the elements of $\mathcal{M}_{ov}^+(M\times\widehat{G})$ split into two parts

$$\Gamma d\gamma\ =\ \mathbf{1}_{M\times\widehat{G}_1}\Gamma d\gamma\ +\ \mathbf{1}_{M\times\widehat{G}_\infty}\Gamma d\gamma$$

In particular, on $M\times\widehat{G}_1$, we may assume $\Gamma=1$, while on $M\times\widehat{G}_\infty$, the trace-class operator $\Gamma(x,\pi^{\lambda,\nu})$ acts on $\mathcal{H}_{\pi^{\lambda,\nu}}=L^2(\mathfrak{p}_\lambda)$ in the case of a step-two group G.

We can already observe that the decomposition (1.10) in Theorem 1.1 is due to the split above: the measure $\varrho^{\mathfrak{v}}$ is the restriction of $\Gamma d\gamma$ to $M\times\widehat{G}_1$, while the restriction to $M\times\widehat{G}_\infty$ yields a more involved measure $\varrho^{\mathfrak{z}}$. The invariance then comes from the theorem below. In this statement, we will allow ourselves to use the identifications (see Sections 1.3.2 and 1.3.3):

$$\widehat{G}_1\sim\mathfrak{v}^*\qquad\text{and}\qquad\widehat{G}_\infty\sim\sqcup_{k=k_0}^{\dim\mathfrak{v}-1}\Omega_k.$$

Theorem 2.4. *Let $(\psi^\varepsilon)_{\varepsilon>0}$ be a family of normalized functions satisfying* (2.4) *and $\Gamma d\gamma$ one of its semi-classical measures. Then we have the following properties:*

(i) *Localization:*

$$\pi(\mathbb{L})\Gamma(x,\pi)=\Gamma(x,\pi)\pi(\mathbb{L})=-\Gamma(x,\pi),\qquad\gamma(x,\pi)\,a.e.$$

which implies

(1) *The scalar measure $\mathbf{1}_{M\times\widehat{G}_1}\gamma$ on $M\times\widehat{G}_1$ is supported in $\{(x,\pi^\omega)\in M\times\widehat{G}_1,|\omega|=1\}$.*

(2) *Setting $\Gamma_\zeta:=\mathbf{1}_{M\times\widehat{G}_\infty}\widehat{\mathbb{P}}_\zeta\Gamma$ for each $\zeta>0$, we have $\Gamma(x,\pi)=\sum_{\zeta\in\mathrm{sp}(\pi(-\mathbb{L}))}\Gamma_\zeta(x,\pi)$ for γ-almost every $(x,\pi)\in M\times\widehat{G}_\infty$. Moreover, it satisfies $\zeta\Gamma_\zeta d\gamma=\Gamma_\zeta d\gamma$ in $\mathcal{M}_{ov}^+(M\times\widehat{G})$. In other words, $\zeta=1$ on the support of the measure $\mathrm{Tr}(\Gamma_\zeta(x,\pi))\gamma(x,\pi)$.*

(ii) *Invariance:*

(1) *The scalar measure* $\mathbf{1}_{M\times\widehat{G}_1}\gamma$ *is invariant under the flow*

$$(x,\pi^\omega)\longmapsto(\mathrm{Exp}(s\omega\cdot V)x,\pi^\omega),\quad s\in\mathbb{R}.$$

(2) (a) *For each* $\zeta>0$*, the operator valued measure* $\Gamma_\zeta d\gamma=\mathbf{1}_{M\times\widehat{G}_\infty}\widehat{\mathbb{P}}_\zeta\Gamma d\gamma$ *is supported in* $M\times\widehat{G}_\infty$ *where it is invariant under the flow*

$$(x,\pi^{\lambda,\nu})\longmapsto(\mathrm{Exp}(s\nu\cdot R^\lambda)x,\pi^{\lambda,\nu}),\quad s\in\mathbb{R}.$$

(b) *Assume* $\Omega_0\neq\emptyset$*. For each* $\zeta>0$ *parametrized smoothly by* λ*, the operator valued measure* $\mathbf{1}_{M\times\Lambda_0}\Gamma_\zeta d\gamma$ *is supported on* $M\times\Lambda_0$ *where it is invariant under the flow*

$$(x,\pi^\lambda)\longmapsto(\mathrm{Exp}(s\nabla_\lambda\zeta)x,\pi^\lambda),\quad s\in\mathbb{R}.$$

Note that the flow invariances may be different for various ζ in Part (2) (b). This was already observed on the groups of Heisenberg type where $\Omega_0=\Lambda_0=\mathfrak{z}^*\setminus\{0\}\sim\widehat{G}_\infty$ (see [15, 17]). The invariance of Point (2)(a) is empty in that case since the flow map of (2)(a) reduces to identity on Ω_0.

Theorem 2.4 implies Theorem 1.1 through the identification that has been mentioned above:

$$\varrho^{\mathfrak{v}}(x)=\int_{\omega\in\mathfrak{v}^*}d\gamma(x,\pi^\omega)\qquad\text{and}\qquad\varrho^{\mathfrak{z}}(x)=\int_{\pi\in\widehat{G}_\infty}\mathrm{Tr}(\Gamma(x,\pi))d\gamma(x,\pi).$$

2.4. **Main ideas of the proof.** Theorem 2.4 is inspired by the results [14, 17] where the group G was assumed to be of Heisenberg type. We follow here the ideas developed in these papers and extend them to general two-step groups. We explain below the main elements of the proof that rely on technical lemmata that are discussed in Section 4.

One can notice that, formally,

$$-\varepsilon^2\mathbb{L}_M^{\mathbb{U}}=-\mathrm{Op}_\varepsilon(\pi(\mathbb{L}))+\varepsilon^2\mathrm{Op}_\varepsilon(\mathbb{U}),\tag{2.5}$$

which implies that the term involving the potential $\mathbb{U}$ is of lower order than the operator $\varepsilon^2\mathbb{L}_M$ itself. For both the proof of the localisation results and the invariance ones, we start from some relations coming from the $(\psi^\varepsilon)_{\varepsilon>0}$ being eigenfunctions of the subLaplacian. We then use symbolic calculus as developed in [14, 17] to analyse these algebraic relations and compute precisely the symbols involved in the calculus. Finally, passing to the limit $\varepsilon\to0$, we investigate what the resulting equations mean for the semi-classical measure. We restrict ourselves to the zone $\widehat{G}_1$ or $\widehat{G}_\infty$ by using symbol belonging to the von Neumann algebra generated by $\mathcal{A}_0$. Another important ingredient of the proof consists in analyzing the different behavior of symbols that commute with $\widehat{\mathbb{L}}$ and those who don't. These technical points are developed in Section 4.

(i) *Localization.* Let $\sigma\in\mathcal{A}_0$. By the definition of the family $(\psi^\varepsilon)_{\varepsilon>0}$, we have (by equation (2.4))

$$\left(\mathrm{Op}_\varepsilon(\sigma)(-\varepsilon^2\mathbb{L}_M^{\mathbb{U}}\psi^\varepsilon),\psi^\varepsilon\right)_{L^2(M)}=\left(\mathrm{Op}_\varepsilon(\sigma)\psi^\varepsilon,-\varepsilon^2\mathbb{L}_M^{\mathbb{U}}\psi^\varepsilon\right)_{L^2(M)}=\left(\mathrm{Op}_\varepsilon(\sigma)\psi^\varepsilon,\psi^\varepsilon\right)_{L^2(M)}.$$

By passing to the limit and using (2.5), the definition of the semi-classical measures as in Proposition 2.2 and the properties of the calculus [14, 17], give that any semi-classical measure $\Gamma d\gamma$ of $(\psi^\varepsilon)_{\varepsilon>0}$ satisfies

$$\begin{aligned}\int_{M\times\widehat{G}}\mathrm{Tr}\left(\sigma(x,\pi)\pi(\mathbb{L})\Gamma(x,\pi)\right)d\gamma(x,\pi)&=\int_{M\times\widehat{G}}\mathrm{Tr}\left(\pi(\mathbb{L})\sigma(x,\pi)\Gamma(x,\pi)\right)d\gamma(x,\pi)\\&=-\int_{M\times\widehat{G}}\mathrm{Tr}\left(\sigma(x,\pi)\Gamma(x,\pi)\right)d\gamma(x,\pi).\end{aligned}\tag{2.6}$$

This readily implies the first localization property in (i). The rest of (i) follows as we can now apply (2.6) not only to symbols σ in $\mathcal{A}_0$, but also in the von Neumann algebra generated by $\mathcal{A}_0$, in

particular to $\mathbf{1}_{M\times\widehat{G}_1}\sigma$ and to $\mathbf{1}_{M\times\widehat{G}_\infty}\sigma$, see Lemma 4.5. Furthermore, (2.6) shows the commutation of Γ with $\widehat{\mathbb{L}}$ so also with the spectral projectors $\widehat{\mathbb{P}}_\zeta$ for $\zeta>0$. Therefore, with the notation of Section 4.1, our classical measure $\Gamma d\gamma$ is in $\mathcal{M}_{ov}(M\times\widehat{G})^{(\widehat{\mathbb{L}})}$, the subspace of semi-classical measures that commute with $\widehat{\mathbb{L}}$. Hence, by the analysis in Section 4.1, we only need to consider symbols σ in $\mathcal{B}_0$ which is the space of the symbols in $\mathcal{A}_0$ that commutes with $\widehat{\mathbb{L}}$.

(ii) *Invariance.* We now take advantage of the fact that for all $\sigma\in\mathcal{A}_0$,

$$\Big([\mathrm{Op}_\varepsilon(\sigma),-\varepsilon^2\mathbb{L}_M^{\mathbb{U}}]\psi^\varepsilon,\psi^\varepsilon\Big)_{L^2(M)}=0. \tag{2.7}$$

Setting $\pi(V)\cdot V:=\sum_{j=1}^q\pi(V_j)V_j$ for any orthonormal basis of $V_1,\dots,V_q$ of $\mathfrak{v}$, a computation gives for $\sigma\in\mathcal{A}_0$,

$$\frac{1}{\varepsilon}[\mathrm{Op}_\varepsilon(\sigma),-\varepsilon^2\mathbb{L}_M^{\mathbb{U}}]=-\frac{1}{\varepsilon}\mathrm{Op}_\varepsilon([\sigma,\pi(\mathbb{L})])+2\,\mathrm{Op}_\varepsilon(\pi(V)\cdot V\sigma)+\varepsilon\,\mathrm{Op}_\varepsilon(\mathbb{L}\sigma)+\varepsilon[\mathrm{Op}_\varepsilon(\sigma),\mathbb{U}(x)]. \tag{2.8}$$

For symbols $\sigma\in\mathcal{B}_0$ (which then commute with $\widehat{\mathbb{L}}$), the term in $\frac{1}{\varepsilon}$ in the right-hand side vanishes and we deduce by passing to the limit that any semi-classical measure $\Gamma d\gamma$ of $(\psi^\varepsilon)_{\varepsilon>0}$ satisfies

$$\forall\sigma\in\mathcal{B}_0,\qquad\int_{M\times\widehat{G}}\mathrm{Tr}\left(\pi(V)\cdot V\sigma(x,\pi)\Gamma(x,\pi)\right)d\gamma(x,\pi)=0. \tag{2.9}$$

Let us prove Part (2)(a). As for Part (1), we can apply this to the elements $\mathbf{1}_{M\times\widehat{G}_1}\sigma$ and to $\mathbf{1}_{M\times\widehat{G}_\infty}\sigma$ of the von Neumann algebra generated by $\mathcal{B}_0$, see Lemma 4.5. We obtain first that (2.9) holds with integration over $M\times\widehat{G}_1$; Part (ii)(1) then follows from this and Corollary 4.4. Then, we obtain that (2.9) holds with integration on $M\times\widehat{G}_\infty$. This yields

$$\begin{aligned}0&=\int_{M\times\widehat{G}_\infty}\mathrm{Tr}\left(\pi(V)\cdot V\sigma(x,\pi)\Gamma(x,\pi)\right)d\gamma(x,\pi)\\&=\int_{M\times\widehat{G}_\infty}\sum_{\zeta\in\mathrm{sp}(\pi(\mathbb{L})}\mathrm{Tr}\left(\pi(\mathbb{P}_\zeta)(\pi(V)\cdot V)\pi(\mathbb{P}_\zeta)\;\sigma(x,\pi)\Gamma(x,\pi)\right)d\gamma(x,\pi),\end{aligned} \tag{2.10}$$

since $\sum_{\zeta\in\mathrm{sp}(\pi(\mathbb{L})}\pi(\mathbb{P}_\zeta)$ is the identity operator on $\mathcal{H}_\pi$ and $\pi(\mathbb{P}_\zeta)=\pi(\mathbb{P}_\zeta)^2$ commutes with $\sigma(x,\pi)$ and $\Gamma(x,\pi)$. Furthermore, for $\pi=\pi^{\lambda,\nu}$, it follows from Section 4 (see (4.4))

$$\forall\zeta>0,\qquad\pi(\mathbb{P}_\zeta)(\pi(P^\lambda)\cdot P^\lambda)\pi(\mathbb{P}_\zeta)=0,\qquad\pi(\mathbb{P}_\zeta)(\pi(Q^\lambda)\cdot Q^\lambda)\pi(\mathbb{P}_\zeta)=0.$$

Hence (2.10) becomes

$$\forall\sigma\in\mathcal{B}_0,\qquad\int_{M\times\widehat{G}_\infty}\mathrm{Tr}\left(\nu\cdot R^\lambda\sigma(x,\pi^{\lambda,\nu})\Gamma(x,\pi^{\lambda,\nu})\right)d\gamma(x,\pi^{\lambda,\nu})=0.$$

This implies Part (ii)(2)(a) by Proposition 4.2 as $\Gamma d\gamma$ is in the set $\mathcal{M}_{ov}(M\times\widehat{G})^{(\widehat{\mathbb{L}})}$ of operator-valued measures that commute with $\widehat{\mathbb{L}}$.

Let us prove Part (2)(b). We now assume $\Omega_0\neq\emptyset$. Indeed, on Ω_0, the analysis above does not yield anything since $\nu\cdot R^\lambda=0$ on Ω_0. We will need the following observation:

Lemma 2.5. *If $\sigma\in\mathcal{A}_0$ and $\eta\in\mathcal{S}(\mathfrak{z}^*)$, then the symbol $\sigma\eta$ given by $(\sigma\eta)(x,\pi^{\lambda,\nu})=\sigma(x,\pi^{\lambda,\nu})\eta(\lambda)$ is in $\mathcal{A}_0$. If $\sigma\in\mathcal{B}_0$ then $\sigma\eta\in\mathcal{B}_0$.*

Proof. If $\kappa_x(y)$ is the kernel of σ, then we check readily that $(y_\mathfrak{v},y_\mathfrak{z})\mapsto(\kappa_x(y_\mathfrak{v},\cdot)*_\mathfrak{z}\mathcal{F}_\mathfrak{z}^{-1}\eta)(y_\mathfrak{z})$ is the kernel of $\sigma\eta$. The rest follows. □

By Lemma 2.5, if $\sigma_1 \in \mathcal{B}_0$ and if $\eta \in \mathcal{S}(\mathfrak{z}^*)$ is supported in the dense open subset Ω_0 of $\mathfrak{z}^* \setminus \{0\}$, then $\sigma := \sigma_1\eta$ is supported in $M \times \Omega_0$. Moreover, by Lemma 4.6, there exists a symbol $T\sigma \in \mathcal{A}_0$ such that

$$\pi(V) \cdot V\sigma = [T\sigma, \pi(-\mathbb{L})].$$

Therefore, using the additional fact

$$[\mathrm{Op}_\varepsilon(T\sigma), \mathbb{U}(x)] = O(\varepsilon) \text{ in } \mathcal{L}(L^2(M)),$$

the equation (2.8) gives

$$\begin{aligned}\frac{1}{\varepsilon}\left(\mathrm{Op}_\varepsilon(\pi(V)\cdot V\sigma)\psi^\varepsilon, \psi^\varepsilon\right)_{L^2(M)} &= \frac{1}{\varepsilon}\left(\mathrm{Op}_\varepsilon([T\sigma, \pi(-\mathbb{L})])\psi^\varepsilon, \psi^\varepsilon\right)_{L^2(M)} \\ &= \frac{1}{\varepsilon}\left([\mathrm{Op}_\varepsilon(T\sigma), -\varepsilon^2\mathbb{L}_M^{\mathbb{U}}]\psi^\varepsilon, \psi^\varepsilon\right)_{L^2(M)} - 2\left(\mathrm{Op}_\varepsilon((\pi(V)\cdot V)\circ T\sigma)\psi^\varepsilon, \psi^\varepsilon\right)_{L^2(M)} + O(\varepsilon).\end{aligned}$$

By (2.7), the first term of the right-hand side is 0 and we have

$$\frac{1}{\varepsilon}\left(\mathrm{Op}_\varepsilon(\pi(V)\cdot V\sigma)\psi^\varepsilon, \psi^\varepsilon\right)_{L^2(M)} = -2\left(\mathrm{Op}_\varepsilon((\pi(V)\cdot V)\circ T\sigma)\psi^\varepsilon, \psi^\varepsilon\right)_{L^2(M)} + O(\varepsilon).$$

Plugging this expression of $\left(\mathrm{Op}_\varepsilon(\pi(V)\cdot V\sigma)\psi^\varepsilon, \psi^\varepsilon\right)_{L^2(M)}$ in (2.8) and using one more time (2.7), we finally get

$$\begin{aligned}O(\varepsilon) &= \frac{2}{\varepsilon}\left(\mathrm{Op}_\varepsilon(\pi(V)\cdot V\sigma)\psi^\varepsilon, \psi^\varepsilon\right)_{L^2(M)} + \left(\mathrm{Op}_\varepsilon(\mathbb{L}\sigma)\psi^\varepsilon, \psi^\varepsilon\right)_{L^2(M)} \\ &= \left(\mathrm{Op}_\varepsilon(-4\pi(V)\cdot V\circ T\sigma + \mathbb{L}\sigma)\psi^\varepsilon, \psi^\varepsilon\right)_{L^2(M)}.\end{aligned}$$

We now pass to the limit $\varepsilon \to 0$ and transform the latter equation according to the equality

$$-4\pi(V)\cdot V\circ T\sigma + \mathbb{L}\sigma = i\sum_{\zeta\in\mathrm{Sp}\widehat{\mathbb{L}}} \nabla_\lambda\zeta\,\sigma\,\widehat{\mathbb{P}}_\zeta.$$

induced by Corollary 4.8 and the fact that $\sigma \in \mathcal{B}_0$. We are left with

$$\sum_{\zeta\in\mathrm{Sp}\widehat{\mathbb{L}}}\int_{M\times\Omega_0} \mathrm{Tr}\left(\nabla_\lambda\zeta\,\sigma(x,\pi^\lambda)\Gamma_\zeta(x,\pi^\lambda)\right)d\gamma(x,\pi^\lambda) = 0,$$

and the relation holds for all $\sigma = \sigma_1\eta$ with $\sigma_1 \in \mathcal{B}_0$ and $\eta \in \mathcal{S}(\mathfrak{z}^*)$ supported in the dense open set Ω_0. This concludes the proof.

3. Geometric invariance

In this section, we address the geometric invariance of the objects that we have introduced above.

3.1. **Nilmanifolds as filtered manifolds.** A stratified Lie group G carries a natural filtration on its Lie algebra given by

$$\mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \cdots \subset \mathfrak{h}_k = \mathfrak{g} = T_eG, \qquad \text{with} \qquad \mathfrak{h}_j = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_j.$$

One can view the nilmanifold M as a *filtered manifold* with associated filtration of subbundles

$$H^1_{\dot{x}} \subset H^2_{\dot{x}} \subset \cdots \subset H^r_{\dot{x}} = T_{\dot{x}}M, \quad x \in G, \quad [H^i, H^j] \subset H^{i+j},\ 1 \le i+j \le r, \tag{3.1}$$

given by $H^i_{\dot{x}} = d\pi_\Gamma \circ dL_x(\mathfrak{h}_i)$. Here $\pi_\Gamma : G \to M = \Gamma\backslash G$ is the quotient map and $L_x : G \to G$ is the left-translation. In fact, G induces a left-invariant stratification by the subbundles $d\pi_\Gamma \circ d_{L_x}(\mathfrak{g}_i)$ of TM in the obvious way, but such a stratification will not respect the Lie bracket of vector fields on M unless one restricts to left-invariant vector fields. What's more, we will see that the semi-classical calculus only depends on the filtration, and not on the stratification or the metric.

When G is step 2, we have $\mathfrak{h}_1 = \mathfrak{v}$ and $\mathfrak{h}_2 = \mathfrak{g}$. In this case, the data of the filtration on G is almost the same as a stratification except that one forgets the second stratum $\mathfrak{g}_2 = \mathfrak{z}$. On M, the

filtration is given by a single step 2 bracket generating subbundle $H^1 \subset TM$ without a preferred complement.

3.2. **Filtration preserving maps.** Let U be an open subset of M and $\Phi : U \to M$ a smooth map on M. We introduce two definitions.

Definition 3.1. (1) The smooth map Φ is said to preserve the filtration at $\dot{x} \in U$ when

$$d_{\dot{x}}\Phi\left(H^i_{\dot{x}}\right) \subseteq H^i_{\Phi(\dot{x})}, \quad i = 1, \dots, r.$$

(2) The map Φ is Pansu differentiable at the point $\dot{x}$ when for any $z \in G$,

$$\lim_{\varepsilon \to 0} \delta_{\varepsilon^{-1}}\left(\Phi(\dot{x})^{-1}\Phi(\dot{x}\delta_\varepsilon z)\right) = \lim_{\varepsilon \to 0} \delta_{\varepsilon^{-1}}\left(\Phi_{\dot{x}}(\delta_\varepsilon z)\right) =: \mathrm{PD}_{\dot{x}}\Phi(z). \tag{3.2}$$

(3) The map Φ is uniformly Pansu differentiable on U if it is Pansu differentiable at every point in U, and the limit (3.2) holds locally uniformly on $U \times G$.

Remark 3.2. Taking $U \subset M$ to be a sufficiently small neighborhood of $\dot{x} \in M$, we may consider U as a neighborhood of $x \in G$ and lift Φ to a smooth map $\Phi_G : U \subset G \to G$. Then the above definition is equivalent to saying Φ_G is Pansu differentiable (resp. uniformly Pansu differentiable) at $x \in G$ (resp. on $U \times G$).

On a neighborhood $U \subset M$ sufficiently small to identify with a neighborhood in G, the notions of Pansu differentiability and filtration preservation are related via the following result [16]:

Theorem 3.3 ([16])**.** *The map Φ is uniformly Pansu differentiable on U if and only if Φ preserves the filtration at every point $x \in U$.*

This result relates a morally algebraic property, Pansu differentiability, to a geometric property of being filtration-preserving. Consequently, the diffeomorphisms Φ we consider in the sequel are uniformly Pansu differentiable, and the transformation of pseudodifferential operators by the pullback associated with Φ will involve the Pansu derivative of Φ. This leads us to employ the osculating Lie group and Lie algebra bundles in the next section.

3.3. **Schwartz Vertical Densities.** For a filtered manifold M, the *osculating Lie algebra bundle* $\mathfrak{G}M$ (and the *osculatig Lie group bundle* $\mathbb{G}M := \exp(\mathbb{G}M)$), defined in [16], play the role of the tangent bundle. When $M = \Gamma\backslash G$, with the filtration 3.1, the fibers of $\mathfrak{G}M$ and $\mathbb{G}M$, are all isomorphic to $\mathfrak{g}$ and G respectively. In particular, we have canonical identifications

$$\mathfrak{G}M \cong M \times \mathfrak{g} \text{ and } \mathbb{G}M \cong M \times G.$$

The Haar measure on each fibers $\mathbb{G}_{\dot{x}}M$ is given by $d_{\dot{x}}z = dz$ and the dual sets by $\widehat{\mathbb{G}}_{\dot{x}}M = \widehat{G}$.

To any semi-classical pseudodifferential operator on a compact nilmanifold $M = \Gamma\backslash G$, its convolution kernel κ may be viewed as an element of $C^\infty(M, \mathcal{S}(G))$. However, this is not the right space for the general case of filtered manifolds. Indeed, Theorem 3.4 together with Equation (3.5) below will imply that as a pseudodifferential operator transforms under diffeomorphisms on M preserving the filtration, its associated convolution kernel transforms like a density on the osculating group bundle. We show that it is natural to view the convolution kernels κ as elements of the bundle of *Schwartz vertical densities* on $\mathbb{G}M$, rather than functions in $C^\infty(M, \mathcal{S}(G))$. We briefly elaborate below.

Let $\mathcal{V}(\mathbb{G}M)$ be the vertical bundle of $\mathbb{G}M$, that is, the kernel of the map $\mathbb{G}M \to M$. Let $|\Lambda|\mathcal{V}(\mathbb{G}M)$ be the bundle of vertical densities, that is, the bundle over $\mathbb{G}M$ whose fibers are densities in the vertical spaces. Let $\mathcal{S}(\mathbb{G}M) = \coprod_{\dot{x}\in M} \mathcal{S}(\mathbb{G}_{\dot{x}}M)$ be the Fréchet vector bundle over M whose fibers are Schwartz class functions. Furthermore, let $\mathcal{S}(\mathbb{G}M, |\Lambda|\mathcal{V})$ be the Fréchet bundle over M whose fibers are Schwartz class densities on the vertical space. As in [16], denote by $\Gamma_c\left(\mathcal{S}\left(\mathbb{G}M, |\Lambda|\mathcal{V}\right)\right)$, the space of its smooth compactly supported sections, which we call the

Schwartz vertical densities. After making a choice of Haar measure on G, this space is identified with $C^\infty(M, \mathcal{S}(G))$.

Indeed, by left-invariance, we identify the fibers of $\mathcal{V}(\mathbb{G}M)$ with the Lie algebra $\mathfrak{g}$ and fibers of $|\Lambda|\mathcal{V}(\mathbb{G}M)$ with $|\Lambda|\mathfrak{g}$, the set of densities on the Lie algebra:

$$\mathcal{V}(\mathbb{G}M) \cong (M \times G) \times \mathfrak{g}, \quad \text{whence} \quad |\Lambda|\mathcal{V}(\mathbb{G}M) \cong (M \times G) \times |\Lambda|\mathfrak{g}. \tag{3.3}$$

The above trivializations give the identification

$$\Gamma_c\left(\mathcal{S}\left(\mathbb{G}M, |\Lambda|\mathcal{V}\right)\right) \cong C^\infty(M, \mathcal{S}(G, |\Lambda|\mathfrak{g}))$$

And a choice of Haar measure on G gives $\mathcal{S}(G, |\Lambda|\mathfrak{g}) \cong \mathcal{S}(G)$.

For a choice of Haar measure dz on G, which in turn gives a Haar system $\{d_{\dot{x}}z\}$ on $\mathbb{G}M$ through (3.3), the identifications of vertical Schwartz densities with functions is given explicitly by

$$\kappa \in \Gamma_c\left(\mathcal{S}\left(\mathbb{G}M, |\Lambda|\mathcal{V}\right)\right) : \dot{x} \mapsto \kappa_{\dot{x}} = \widetilde{\kappa}_{\dot{x}} d_{\dot{x}}z, \quad \widetilde{\kappa}_{\dot{x}} \in \mathcal{S}(\mathbb{G}_{\dot{x}}M).$$

The symbols $\sigma \in \mathcal{A}_0$ are defined as the images of the elements $\kappa \in \Gamma_c\left(\mathcal{S}\left(\mathbb{G}M, |\Lambda|\mathcal{V}\right)\right)$ by the fiberwise Fourier transform:

$$\dot{x} \mapsto \sigma(\dot{x}, \pi) = \int_{z \in \mathbb{G}_{\dot{x}}M} \widetilde{\kappa}_{\dot{x}}(z)\pi(z)^* d_{\dot{x}}z, \ \ \pi \in \widehat{\mathbb{G}}_{\dot{x}}M. \tag{3.4}$$

Since our convolution kernels are densities, the integral (3.4) is independent of the choice of Haar measure.

3.4. **Semi-classical pseudodifferential calculus and filtration diffeomorphisms.** We keep the notations of the preceding section except we suppose $\Phi : U \subset M \to M$ is a diffeomorphism onto its image. Let J_Φ be the Jacobian of Φ. We associate with Φ

(i) a unitary transformation $\mathcal{U}_\Phi$ of $L^2(U)$ induced by Φ

$$\mathcal{U}_\Phi(f) := J_\Phi^{1/2} f \circ \Phi, \ \ f \in L^2(U),$$

(ii) a map $\mathcal{I}_\Phi$ on the space of Schwartz vertical densities that extends to an isometry of $L^1(|\Lambda|\mathcal{V}(\mathbb{G}M))$

$$(\mathcal{I}_\Phi \kappa)_x(z) := J_\Phi(x)\, \kappa_{\Phi(x)}(\mathrm{PD}_x\Phi(z)), \ \ \forall (x, z) \in U \times G.$$

We are interested in the properties of the operator $\mathcal{U}_\Phi \circ \mathrm{Op}_\varepsilon(\sigma) \circ \mathcal{U}_\Phi^{-1}$, in particular in the asymptotics in ε of its semi-classical pseudodifferential symbol. The structure of the latter and the way it can be deduced from σ will give information of the geometric nature of the objects. Indeed, Φ induces several geometric transformations:

(i) Φ induces a map on representations

$$\widehat{\mathbb{G}}\Phi : \left\{ \begin{array}{ccc} U \times \widehat{G} & \longrightarrow & \Phi(U) \times \widehat{G} \\ (x, \pi) & \longmapsto & \left(\Phi(x), \pi \circ (\mathrm{PD}_x\Phi)^{-1}\right) \end{array} \right.$$

(ii) The *generalized canonical transformation* $\widehat{\mathbb{G}}\Phi$ induces a pull-back on symbols

$$(\widehat{\mathbb{G}}\Phi)^* \sigma(x, \pi) := \sigma(\widehat{\mathbb{G}}\Phi(x, \pi)).$$

The maps $\widehat{\mathbb{G}}\Phi$ and $\mathcal{I}_\Phi$ are intertwined by the group Fourier Transform: If $\sigma(x, \pi) = \widehat{\kappa_x}(\pi)$ for all $x \in U \subset M$ and $\pi \in \widehat{\mathbb{G}}_x M$, then for any filtration preserving differmorphism $\Phi : U \to M$

$$(\widehat{\mathbb{G}}\Phi)^* \sigma(x, \pi) = \widehat{\mathcal{I}_\Phi \kappa_x}(\pi), \quad x \in U \subset M, \ \pi \in \widehat{G} = \widehat{\mathbb{G}}_x M. \tag{3.5}$$

These two maps are involved in the description of the first term of the expansion of the semi-classical symbol of the operator $\mathcal{U}_\Phi \circ \mathrm{Op}_\varepsilon(\sigma) \circ \mathcal{U}_\Phi^{-1}$:

Theorem 3.4 ([16]). *Assume that Φ is filtration preserving on U. Then in $\mathcal{L}(L^2(U))$,*

$$\mathcal{U}_\Phi \circ \mathrm{Op}_\varepsilon(\sigma) \circ \mathcal{U}_\Phi^{-1} = \mathrm{Op}_\varepsilon\left((\widehat{\mathbb{G}}\Phi)^*\sigma\right) + O(\varepsilon).$$

Remark 3.5. Theorem 3.4 establishes the geometric invariance of the semi-classical calculus by filtration preserving differmorphisms Φ. In particular, Φ does not need to preserve the action of G on M, or even preserve the gradation.

The results of this section suggest that the semi-classical symbols we defined in Section 2.1 ought to be the natural generalization of symbols for arbitrary filtered manifolds. So defined, the semi-classical symbols are invariant under generalized canonical transformations of $\widehat{\mathbb{G}}M$ associated to differmorphisms preserving the filtration on M.

4. Technical tools

This section is devoted to several technical results used in the proof of Theorem 2.4.

4.1. Some C^*-algebras and their properties.

4.1.1. *The von Neumann algebra $L^\infty(M\times\widehat{G})$.* A measurable symbol $\sigma = \{\sigma(x,\pi) : (x,\pi) \in M\times\widehat{G}\}$ is said to be bounded when there exists a constant $C > 0$ such that for $dxd\mu(\pi)$-almost all $(x,\pi) \in M \times \widehat{G}$, we have $\|\sigma(x,\pi)\|_{\mathcal{H}_\pi} \leq C$. We denote by $\|\sigma\|_{L^\infty(M\times\widehat{G})}$ the smallest of such constant $C > 0$ and by $L^\infty(M \times \widehat{G})$ the space of bounded measurable symbols. We check readily that $\|\cdot\|_{L^\infty(M\times\widehat{G})}$ is a norm on $L^\infty(M \times \widehat{G})$ which is a C^*-algebra. We will later use the fact that it is a von Neumann algebra.

4.1.2. *The C^*-algebra $\mathcal{A}$ and its topological dual.* Clearly, $\mathcal{A}_0$ is a subspace of $L^\infty(M \times \widehat{G})$. Its closure denoted by $\mathcal{A}$ for the norm $\|\cdot\|_{L^\infty(M\times\widehat{G})}$ is a sub-C^*-algebra of $L^\infty(M \times \widehat{G})$. Its topological dual $\mathcal{A}^*$ is isomorphic to the Banach space of operator-valued measures $\mathcal{M}_{ov}(M \times \widehat{G})$ via

$$\mathcal{M}_{ov}(M \times \widehat{G}) \ni \Gamma d\gamma \mapsto \ell_{\Gamma d\gamma} \in \mathcal{A}^*, \qquad \ell_{\Gamma d\gamma}(\sigma) := \int_{M\times\widehat{G}} \mathrm{Tr}\left(\sigma(x,\pi)\Gamma(x,\pi)\right) d\gamma(x,\pi).$$

Moreover, the isomorphism is isometric:

$$\|\ell_{\Gamma d\gamma}\|_{\mathcal{A}^*} = \|\Gamma d\gamma\|_{\mathcal{M}_{ov}(M\times\widehat{G})}, \quad \text{where} \quad \|\Gamma d\gamma\|_{\mathcal{M}_{ov}} := \int_{M\times\widehat{G}} \mathrm{Tr}\,|\Gamma(x,\pi)|\, d\gamma(x,\pi),$$

and the positive linear functionals on $\mathcal{A}$ are the $\ell = \ell_{\Gamma d\gamma}$'s with $\Gamma d\gamma \geq 0$.

4.1.3. *The C^*-algebra $\mathcal{B}$ and its topological dual.* Let $\mathcal{B}_0$ be the subspace of $\mathcal{A}_0$ of symbols commuting with $\widehat{\mathbb{L}}$. Clearly $\mathcal{B}_0$ contains all the symbols of the form $a(x)\psi(\widehat{\mathbb{L}})$, $a \in \mathcal{C}^\infty(M)$, $\psi \in \mathcal{S}(\mathbb{R})$, by Hulanicki's theorem (see [26]):

Theorem 4.1 (Hulanicki). *The convolution kernel of a spectral multiplier $\psi(\mathbb{L}_G)$ of $\mathbb{L}_G$ for a Schwartz function $\psi \in \mathcal{S}(\mathbb{R})$ is Schwartz on G.*

We denote by $\mathcal{B}$ the closure of $\mathcal{B}_0$ for the norm $\|\cdot\|_{L^\infty(M\times\widehat{G})}$. Property (2) of $\widehat{\mathbb{L}}$ recalled in Section 1.3.4 implies that $\mathcal{B}$ is the subspace of $\mathcal{A}$ of symbols commuting with every $\widehat{\mathbb{P}}_\zeta$, $\zeta > 0$. We check readily that $\mathcal{B}$ is a sub-C^*-algebra of $\mathcal{A}$ and that $\mathcal{B}_0 = \mathcal{A}_0 \cap \mathcal{B}$. The next statement identifies the topological dual of $\mathcal{B}$:

Proposition 4.2. *Via $\Gamma d\gamma \mapsto \ell_{\Gamma d\gamma}|_{\mathcal{B}}$, the topological dual $\mathcal{B}^*$ of $\mathcal{B}$ is isomorphic with the closed subspace $\mathcal{M}_{ov}(M\times\widehat{G})^{(\widehat{\mathbb{L}})}$ of operator valued measures $\Gamma d\gamma \in \mathcal{M}_{ov}(M\times\widehat{G})$ such that the operator Γ commutes with $\widehat{\mathbb{P}}_\zeta$ for all $\zeta>0$, in the sense that*

$$\forall \zeta>0 \qquad \pi(\mathbb{P}_\zeta)\Gamma(x,\pi)=\Gamma(x,\pi)\pi(\mathbb{P}_\zeta) \quad \textit{for } \gamma-\textit{almost all } (x,\pi)\in M\times\widehat{G}.$$

Proof. **Step 0.** We observe that if two pairs (Γ,γ) and (Γ_1,γ_1) are equivalent and one of them satisfies the commutative condition with $\widehat{\mathbb{P}}_\zeta$ for all $\zeta>0$, then so does the other. Hence, $\mathcal{M}_{ov}(M\times\widehat{G})^{(\widehat{\mathbb{L}})}$ is a well defined subset of $\mathcal{M}_{ov}(M\times\widehat{G})$. One checks that it is a closed subspace of $\mathcal{M}_{ov}(M\times\widehat{G})$.

Step 1. Let $\ell\in\mathcal{B}^*$. By the Hahn-Banach theorem, this functional extends to $\tilde{\ell}\in\mathcal{A}^*$, i.e. $\tilde{\ell}_{|\mathcal{B}}=\ell$. Denote by $\Gamma d\gamma\in\mathcal{M}_{ov}(M\times\widehat{G})$ the corresponding operator-valued measure: $\tilde{\ell}=\ell_{\Gamma d\gamma}$. Now set

$$\Gamma_1(x,\pi):=\sum_{\zeta\in\mathrm{sp}(\pi(\mathbb{L}))}\pi(\mathbb{P}_\zeta)\Gamma(x,\pi)\pi(\mathbb{P}_\zeta).$$

The operator-valued measure $\Gamma_1 d\gamma$ is a well defined element of $\mathcal{M}_{ov}(M\times\widehat{G})$ satisfying the condition of commutativity with $\widehat{\mathbb{L}}$ so $\Gamma_1 d\gamma\in\mathcal{M}_{ov}(M\times\widehat{G})^{(\widehat{\mathbb{L}})}$. Let us show that it coincides with $\ell_{\Gamma d\gamma}$ on $\mathcal{B}$. Let $\sigma\in\mathcal{B}$. Since $\sum_{\zeta\in\mathrm{sp}(\pi(\mathbb{L}))}\pi(\mathbb{P}_\zeta)$ is the identity operator on $\mathcal{H}_\pi$, we have

$$\begin{aligned}\ell_{\Gamma d\gamma}(\sigma)&=\int_{M\times\widehat{G}}\sum_{\zeta\in\mathrm{sp}(\pi(\mathbb{L}))}\mathrm{Tr}\left(\sigma(x,\pi)\Gamma(x,\pi)\pi(\mathbb{P}_\zeta)\right)d\gamma(x,\pi)\\&=\int_{M\times\widehat{G}}\sum_{\zeta\in\mathrm{sp}(\pi(\mathbb{L})-}\mathrm{Tr}\left(\sigma(x,\pi)\pi(\mathbb{P}_\zeta)\Gamma(x,\pi)\pi(\mathbb{P}_\zeta)\right)d\gamma(x,\pi),)\end{aligned}$$

since $\pi(\mathbb{P}_\zeta)=\pi(\mathbb{P}_\zeta)^2$ commutes with $\sigma(x,\pi)$. We recognise $\ell_{\Gamma_1 d\gamma}(\sigma)$ on the right-hand side. We have obtained that any $\ell\in\mathcal{B}^*$ may be written as the restriction to $\mathcal{B}$ of $\ell_{\Gamma_1 d\gamma}$, for some $\Gamma_1 d\gamma\in\mathcal{M}_{ov}(M\times\widehat{G})^{(\widehat{\mathbb{L}})}$.

In other words, we have proved that $\Gamma d\gamma\mapsto\ell_{\Gamma d\gamma}|_{\mathcal{B}}$ maps $\mathcal{M}_{ov}(M\times\widehat{G})^{(\widehat{\mathbb{L}})}$ onto $\mathcal{B}^*$. This map is continuous and linear. It remains to show that it is injective.

Side step. Let us open a parenthesis. The von Neumann algebra $L^\infty(M\times\widehat{G})$ is a C^* algebra containing $\mathcal{A}$ and we denote by $vN\mathcal{A}$ the von Neumann algebra generated by $\mathcal{A}$. This means that $vN\mathcal{A}$ is the closure of $\mathcal{A}$ for the strong operator topology in $L^\infty(M\times\widehat{G})$. We are going to use this von Neumann algebra by considering the natural unique extension of $\ell=\ell_{\Gamma d\gamma}\in\mathcal{A}^*$ to a continuous linear functional on the von Neumann algebra $vN\mathcal{A}$ of $\mathcal{A}$.

Since $\mathcal{B}\subset\mathcal{A}$, we also have $vN\mathcal{B}\subset vN\mathcal{A}$ where $vN\mathcal{B}$ denotes the von Neumann algebra generated by $\mathcal{B}$. Moreover, $vN\mathcal{B}$ is the subspace of the symbols $\sigma\in vN\mathcal{A}$ commuting with $\widehat{\mathbb{P}}_\zeta$ $dxd\mu(\pi)$-almost everywhere for every , $\zeta>0$.

Finally, we observe that for $\zeta>0$ and $\sigma\in\mathcal{A}$, the symbol $\pi(\widehat{\mathbb{P}}_\zeta)\sigma\pi(\widehat{\mathbb{P}}_\zeta)$ is in $vN\mathcal{A}$. Indeed, using Hulanicki's theorem (Theorem 4.1) together with $\mathcal{S}(G)*\mathcal{S}(G)\subset\mathcal{S}(G)$, we obtain that if $\sigma\in\mathcal{A}_0$ then for any $\psi_1,\psi_2\in\mathcal{S}(\mathbb{R})$, the symbol $\psi_1(\widehat{\mathbb{L}})\sigma\psi_2(\widehat{\mathbb{L}})$ is in $\mathcal{A}_0$. Taking limits for suitable sequences of σ,ψ_1,ψ_2 implies that the symbol $\pi(\widehat{\mathbb{P}}_\zeta)\sigma\pi(\widehat{\mathbb{P}}_\zeta)$ is in $vN\mathcal{A}$ for any $\sigma\in\mathcal{A}$.

Step 2. Let us now consider $\Gamma d\gamma\in\mathcal{M}_{ov}(M\times\widehat{G})^{(\widehat{\mathbb{L}})}$ such that $\ell:=\ell_{\Gamma d\gamma}$ vanishes on $\mathcal{B}$. We want to show $\ell=0$. We extend ℓ to a functional L on $vN\mathcal{A}$. This functional vanishes on $vN\mathcal{B}$. We set

$$L_\zeta(\sigma):=\int_{M\times\widehat{G}}\mathrm{Tr}\left(\sigma(x,\pi)\pi(\mathbb{P}_\zeta)\Gamma(x,\pi)\right)d\gamma(x,\pi),\qquad\zeta>0.$$

We check readily that $\zeta \mapsto L_\zeta(\sigma)$ defines a complex measure on $[0,\infty)$ with total mass that is smaller or equal to $\|\sigma\|_{L^\infty(M\times\widehat{G})}\|\Gamma d\gamma\|_{\mathcal{M}_{ov}}$. Moreover, $\ell(\sigma) = \int_0^{+\infty} L_\zeta(\sigma)$ since $\sum_{\zeta\in \mathrm{sp}(\pi(\mathbb{L})} \pi(\mathbb{P}_\zeta)$ is the identity operator on $\mathcal{H}_\pi$.

Using $\mathbb{P}_\zeta^2 = \mathbb{P}_\zeta$ and the commutation of Γ with $\pi(\mathbb{P}_\zeta)$ $d\gamma$-a.e., together with trace property, we obtain

$$L_\zeta(\sigma) = \int_{M\times\widehat{G}} \mathrm{Tr}\left(\pi(\mathbb{P}_\zeta)\sigma(x,\pi)\pi(\mathbb{P}_\zeta)\Gamma(x,\pi)\right) d\gamma(x,\pi) = L_\zeta(\pi(\mathbb{P}_\zeta)\sigma(x,\pi)\pi(\mathbb{P}_\zeta))$$

with $\pi(\mathbb{P}_\zeta)\sigma(x,\pi)\pi(\mathbb{P}_\zeta) \in vN\mathcal{B}$. Arguing as above (in the side step), we deduce $L_\zeta = 0$, whence $L = 0$ and $\ell = 0$. This implies the injectivity of $\Gamma d\gamma \mapsto \ell_{\Gamma d\gamma}|_{\mathcal{B}}$ on $\mathcal{M}_{ov}(M\times\widehat{G})^{(\widehat{\mathbb{L}})}$. □

The proof above has an important consequence regarding the restriction of symbols to $M\times\widehat{G}_1$, a notion we now explain.

4.1.4. *Restriction of symbols to $M\times\widehat{G}_1$.* The restriction $\sigma|_{M\times\widehat{G}_1}$ of $\sigma\in\mathcal{A}$ with kernel $\kappa_x(y)$, to $M\times\widehat{G}_1$ is given by

$$\sigma|_{M\times\widehat{G}_1}(x,\omega) = \sigma(x,\pi^\omega) = \mathcal{F}_{\mathfrak{v}}\int_{\mathfrak{z}} \kappa_x(\cdot,z)dz(\omega), \quad (x,\omega)\in M\times\mathfrak{v}^*,$$

having identified $\widehat{G}_1$ with $\mathfrak{v}^*$. Moreover, we can therefore identify

$$\mathcal{A}|_{M\times\widehat{G}_1} := \{\sigma|_{M\times\widehat{G}_1}, \sigma\in\mathcal{A}\}$$

with a sub-space of $\mathcal{C}_0(M\times\mathfrak{v}^*)$. In fact, we can show

Lemma 4.3. *We have $\mathcal{C}_0(M\times\mathfrak{v}^*) = \mathcal{A}|_{M\times\widehat{G}_1}$.*

Proof. Any element of $\mathcal{C}_0(M\times\mathfrak{v}^*)$ may be viewed as a limit for the supremum norm on $M\times\mathfrak{v}^*$ of $\mathcal{F}_{\mathfrak{v}}\kappa_x^{(j)}(\omega)$ for a sequence of kernels $\kappa^{(j)}\in\mathcal{C}^\infty(M,\mathcal{S}(\mathfrak{v}))$. We then consider the sequence of symbols $\sigma_j(x,\pi) = \pi(\kappa_x^{(j)}\eta)$ with $\eta\in\mathcal{S}(\mathfrak{z})$ satisfying $\mathcal{F}_{\mathfrak{z}}\eta(0) = \int_{\mathfrak{z}}\eta(Z)dz = 1$. We check readily that $\sigma_j|_{M\times\widehat{G}_1}(x,\omega) = \mathcal{F}_{\mathfrak{v}}\kappa_x^{(j)}(\omega)$. □

For a subspace S of $\mathcal{A}$, we denote by

$$S|_{M\times\widehat{G}_1} := \{\sigma|_{M\times\widehat{G}_1}, \sigma\in S\}$$

the resulting subspace in $\mathcal{A}|_{M\times\widehat{G}_1}$. The proof of Lemma 4.3 shows that if $\bar{S}$ denotes the closure of S in the C^*-algebra $\mathcal{A}$, then $\bar{S}|_{M\times\widehat{G}_1}$ is the closure of $S|_{M\times\widehat{G}_1}$ in $\mathcal{C}_0(M\times\mathfrak{v}^*)$, that is, given by the supremum norm on $M\times\mathfrak{v}^*$. Hence $\bar{S}|_{M\times\widehat{G}_1}\subset\mathcal{A}|_{M\times\widehat{G}_1}$.

We will need the following property regarding the restriction of the symbols in $\mathcal{A}_0$ and $\mathcal{B}_0$ to $M\times\widehat{G}_1$; its proof relies on the proof of Proposition 4.2:

Corollary 4.4. *The following commutative C^* algebras coincide:*

$$\bar{\mathcal{B}}_0|_{M\times\widehat{G}_1} = \mathcal{B}|_{M\times\widehat{G}_1} = \bar{\mathcal{A}}_0|_{M\times\widehat{G}_1} = \mathcal{A}|_{M\times\widehat{G}_1} = \mathcal{C}_0(M\times\mathfrak{v}^*).$$

Proof. Clearly, $\bar{\mathcal{B}}_0|_{M\times\widehat{G}_1} = \mathcal{B}|_{M\times\widehat{G}_1} \subset \bar{\mathcal{A}}_0|_{M\times\widehat{G}_1} = \mathcal{A}|_{M\times\widehat{G}_1} = \mathcal{C}_0(M\times\mathfrak{v}^*)$. It remains to show the converse inequality.

Let ℓ be a continuous linear functional on $\mathcal{C}_0(M\times\mathfrak{v}^*)$. This is given by integration against a complex Radon measure γ_1. Consider the operator-valued measure $\Gamma d\gamma\in\mathcal{M}_{ov}(M\times\widehat{G})$ defined by $1_{M\times\widehat{G}_\infty}\Gamma d\gamma = 0$ and $1_{M\times\widehat{G}_1}\Gamma d\gamma = \gamma_1$, that is,

$$\ell_{\Gamma d\gamma}(\sigma) = \int_{M\times\mathfrak{v}^*} \sigma|_{M\times\widehat{G}_1}(x,\pi^\omega)\ d\gamma_1(\omega), \quad \sigma\in\mathcal{A}.$$

We observe that Γ commutes with $\widehat{\mathbb{P}}_\zeta$, $\zeta > 0$. Hence, if $\ell = 0$ on $\mathcal{B}|_{M\times\widehat{G}_1}$ then $\ell_{\Gamma d\gamma} \equiv 0$ on $\mathcal{B}$ and therefore also on $\mathcal{A}$ by Proposition 4.2, or rather Step 2 of its proof; this implies $\Gamma d\gamma = 0$ thus $\gamma_1 = 0$ and $\ell = 0$. By the Hahn-Banach theorem, this shows that $\mathcal{B}|_{M\times\widehat{G}_1} = \mathcal{C}_0(M \times \mathfrak{v}^*)$. □

4.1.5. *Some elements of $vN\mathcal{A}$ and $vN\mathcal{B}$.* We will need the following properties:

Lemma 4.5. *If $\sigma \in \mathcal{A}_0$ then $\mathbf{1}_{M\times\widehat{G}_1}\sigma$ and $\mathbf{1}_{M\times\widehat{G}_\infty}\sigma$ are in $vN\mathcal{A}$. Similarly, if $\sigma \in \mathcal{B}_0$ then $\mathbf{1}_{M\times\widehat{G}_1}\sigma$ and $\mathbf{1}_{M\times\widehat{G}_\infty}\sigma$ are in $vN\mathcal{B}$.*

Proof. We consider $\sigma\eta$ as in Lemma 2.5 with a sequence of functions $\eta \in \mathcal{S}(\mathfrak{z}^*)$ satisfying $\eta(0) = 1$ and with support shrinking to $\{0\}$. We check readily that if $\sigma \in \mathcal{A}_0$ then the limit of these $\sigma\eta$ for the strong operator topology will be $\mathbf{1}_{M\times\widehat{G}_1}\sigma$ which is therefore in $vN\mathcal{A}$. It will also be the case for $\mathbf{1}_{M\times\widehat{G}_\infty}\sigma = \sigma - \mathbf{1}_{M\times\widehat{G}_1}\sigma$. The case of $\mathcal{B}_0$ follows. □

4.2. **The lowering and raising operators associated with $H(\lambda)$.**

4.2.1. *Preliminaries.* Before proving several useful identities, we introduce some notations. If π_1, π_2 are two representations of $\mathfrak{g}$, and $A : \mathfrak{v} \to \mathfrak{v}$ is a linear morphism, then we set

$$(A\pi_1(V)) \cdot \pi_2(V) = \sum_{j,k} A_{j,k}\pi_1(V_k) \otimes \pi_2(V_j) \in \mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2}$$

where $(A_{j,k})$ is the matrix representing A in the orthonormal basis (V_j). We can check that this is independent of the orthonormal basis (V_j). If the context is clear, we may allow ourselves to omit the notation for the tensor product $\otimes$ and may swap the order in the tensor product.

With $A = \mathrm{id}_{\mathfrak{v}}$, π_1 being the regular representation of $\mathfrak{g}$ on $L^2(M)$ and $\pi_2 = \pi \in \widehat{G}$, this yields the super-operator $V \cdot \pi(V)$ acting on $\mathcal{A}_0$. If we restrict this to $M \times \widehat{G}_1$, i.e. $\pi_2 = \pi^\omega \in \widehat{G}_1$, this defines $\omega \cdot V$ acting on $\mathcal{C}^\infty(M, \mathcal{S}(\mathfrak{v}^*)) \sim \mathcal{A}_0|_{M\times\widehat{G}_1}$.

4.2.2. *Technical computations.* Here, we assume that $k = 0$ and consider $\lambda \in \Omega_0$. Following Appendix B in [15], instead of the basis P_j^λ, Q_j^λ, $1 \le j \le d$, we will use the fields

$$W_j^\lambda := \frac{1}{2}(P_j^\lambda - iQ_j^\lambda) \qquad \text{and} \qquad \overline{W}_j^\lambda := \frac{1}{2}(P_j^\lambda + iQ_j^\lambda). \tag{4.1}$$

Direct computations show using equation (1.7), $\pi(P_j^\lambda) = \sqrt{\eta_j(\lambda)}\partial_{\xi_j}$ and $\pi(Q_j^\lambda) = i\sqrt{\eta_j(\lambda)}\xi_j$, so we obtain

$$\pi^\lambda(W_j^\lambda) = \frac{\sqrt{\eta_j(\lambda)}}{2}(\partial_{\xi_j} + \xi_j) \qquad \text{and} \qquad \pi^\lambda(\overline{W}_j^\lambda) = \frac{\sqrt{\eta_j(\lambda)}}{2}(\partial_{\xi_j} - \xi_j).$$

In particular, these new fields coincide up to normalisation with the lowering and raising operators of the harmonic oscillators $(-\partial_{\xi_j}^2 + \xi_j^2)$. Consequently, the family of Hermite functions $(h_\alpha)_{\alpha\in\mathbb{N}^d}$ given by

$$h_\alpha(\xi_1, \dots, \xi_d) = h_{\alpha_1}(\xi_1) \dots h_{\alpha_d}(\xi_d), \quad \text{where} \quad h_n(\xi) = \frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}}} e^{\frac{\xi^2}{2}} \frac{d}{d\xi}(e^{-\xi^2}), \quad n \in \mathbb{N},$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$ that satisfies:

$$\pi^\lambda(W_j^\lambda)h_\alpha = \sqrt{\frac{\eta_j(\lambda)}{2}}\sqrt{\alpha_j}h_{\alpha-\mathbf{1}_j} \qquad \pi^\lambda(\overline{W}_j^\lambda)h_\alpha = -\sqrt{\frac{\eta_j(\lambda)}{2}}\sqrt{\alpha_j+1}h_{\alpha+\mathbf{1}_j}. \tag{4.2}$$

Here, $\mathbf{1}_j$ denotes the multi-index with j-th coordinate 1 and 0 elsewhere. We also have extended the notation h_α to $\alpha \in \mathbb{Z}^d$ with $h_\alpha = 0$ if $\alpha \notin \mathbb{N}^d$. We then deduce easily

$$\left[\pi^\lambda(W_j^\lambda), \pi^\lambda(-\mathbb{L})\right] = 2\eta_j(\lambda)\pi^\lambda(W_j^\lambda) \quad \text{and} \quad \left[\pi^\lambda(\overline{W}_j^\lambda), \pi^\lambda(-\mathbb{L})\right] = -2\eta_j(\lambda)\pi^\lambda(\overline{W}_j^\lambda). \tag{4.3}$$

and that both the operators $\widehat{\mathbb{P}}_\zeta \pi(W_j^\lambda)\widehat{\mathbb{P}}_\zeta$ and $\widehat{\mathbb{P}}_\zeta \pi(\overline{W}_j^\lambda)\widehat{\mathbb{P}}_\zeta$ are zero. Consequently, we also have

$$\widehat{\mathbb{P}}_\zeta \pi(P_j^\lambda)\widehat{\mathbb{P}}_\zeta = 0 \text{ and } \widehat{\mathbb{P}}_\zeta \pi(Q_j^\lambda)\widehat{\mathbb{P}}_\zeta = 0, \ \forall \lambda \in \mathfrak{z}^* \setminus \{0\}, \ \forall j \in \{1, \cdots, d\}, \ \forall \zeta \in \mathbb{R}, \tag{4.4}$$

Following the ideas and notation from Section 4.2.1, we define the operator

$$T = \frac{i}{2}(B(\lambda)^{-1} V \cdot \pi^\lambda(V)), \quad \lambda \in \Omega_0,$$

acting on the space of symbols in $\mathcal{A}_0$ restricted to $M \times \Omega_0$. This may also be viewed as acting on the space of symbols in $\mathcal{A}_0$ which are supported in $M \times \Omega_0$. The properties above imply:

Lemma 4.6. (1) *For any $\sigma \in \mathcal{A}_0$, we have on $M \times \Omega_0$:*

$$[T\sigma, \pi(-\mathbb{L})] = \pi(V) \cdot V\sigma$$

(2) *For any $\lambda \in \Omega_0$ and $\zeta > 0$, using the shorthand $\pi(\mathbb{P}_\zeta)$ for $\mathrm{id}_{L^2(G)} \otimes \pi(\mathbb{P}_\zeta)$, we have*

$$\pi^\lambda(\mathbb{P}_\zeta)\Big(V \cdot \pi^\lambda(V)) \circ T\Big)\pi^\lambda(\mathbb{P}_\zeta) = \frac{1}{4}\mathbb{L} - \frac{i}{4}\sum_{j=1}^d (2\alpha_j + 1)[P_j^\lambda, Q_j^\lambda]\pi^\lambda(\mathbb{P}_\zeta).$$

Proof. Since $P_j^\lambda = \overline{W}_j^\lambda + W_j^\lambda$, and $Q_j^\lambda = \frac{1}{i}(\overline{W}_j^\lambda - W_j^\lambda)$, we deduce for $\pi = \pi^\lambda$, $\lambda \in \Omega_0$,

$$V \cdot \pi(V) = 2\sum_{j=1}^d \left(W_j^\lambda \pi(\overline{W}_j^\lambda) + \overline{W}_j^\lambda \pi(W_j^\lambda)\right).$$

As $B(\lambda)Q_j^\lambda = \eta_j(\lambda)P_j^\lambda$ and $B(\lambda)P_j^\lambda = -\eta_j(\lambda)Q_j^\lambda$, we obtain

$$(B(\lambda)^{-1}V) \cdot \pi(V) = \sum_j^d \frac{1}{\eta_j}\left(-P_j^\lambda \pi(Q_j^\lambda) + Q_j^\lambda \pi(P_j^\lambda)\right) = \frac{2}{i}\sum_{j=1}^d \frac{1}{\eta_j}\left(\overline{W}_j^\lambda \pi(W_j^\lambda) - W_j^\lambda \pi(\overline{W}_j^\lambda)\right).$$

By (4.3), we check readily Part (1).

For Part (2), we may assume that $\pi^\lambda(\mathbb{P}_\zeta) \neq 0$, that is, ζ is in the spectrum of the harmonic oscillator $\pi^\lambda(\mathbb{L})$, or in other words $\zeta = \sum_j \eta_j(\lambda)(2\alpha_j + 1)$ for some $\alpha \in \mathbb{N}^d$. For any such index α and for an arbitrary vector $w_1 \in \mathcal{S}(G)$, by the computations above and (4.2), we see with $\pi = \pi^\lambda$:

$$\begin{aligned}
&\pi(\mathbb{P}_\zeta)\big(V \cdot \pi(V)\big) \circ \big(B(\lambda)^{-1}V \cdot \pi(V)\big)w_1 \otimes h_\alpha \\
&\quad = \frac{4}{i}\sum_{j_1, j_2} \eta_{j_2}^{-1}\pi(\mathbb{P}_\zeta)\big(W_{j_1}^\lambda \pi(\overline{W}_{j_1}^\lambda) + \overline{W}_{j_1}^\lambda \pi(W_{j_1}^\lambda)\big)\big(\overline{W}_{j_2}^\lambda \pi(W_{j_2}^\lambda) - W_{j_2}^\lambda \pi(\overline{W}_{j_2}^\lambda)\big)w_1 \otimes h_\alpha \\
&\quad = \frac{2}{i}\sum_j \left(\overline{W}_j^\lambda W_j^\lambda(\alpha_j + 1) - W_j^\lambda \overline{W}_j^\lambda \alpha_j\right) w_1 \otimes h_\alpha.
\end{aligned}$$

We can simplify each term in the sum above with:

$$\overline{W}_j^\lambda W_j^\lambda(\alpha_j + 1) - W_j^\lambda \overline{W}_j^\lambda \alpha_j = \frac{1}{4}((P_j^\lambda)^2 + (Q_j^\lambda)^2) - \frac{i}{4}(2\alpha_j + 1)[P_j^\lambda, Q_j^\lambda].$$

Part (2) follows □

We recall that the maps $\lambda \mapsto \eta_j(\lambda)$, $j = 1, \ldots, d$, are smooth in Λ_0. Moreover, if $\lambda_0 \in \Lambda_0$, one can choose the vectors P_j^λ, Q_j^λ, $j = 1, \ldots, d$, so that they depend smoothly on λ in a neighborhood of λ_0. We then have the following result.

Lemma 4.7. *Let P_j^λ, Q_j^λ, $j = 1, \ldots, d$, be smooth eigenvectors in an open subset U of U. Then we have*

$$[P_j^\lambda, Q_j^\lambda] = \nabla_\lambda \eta_j(\lambda) \in \mathfrak{z}, \qquad j = 1, \ldots, d, \quad \lambda \in \Lambda_0.$$

Proof of Lemma 4.7. The differentiation of the equality $B(\lambda)Q_j^\lambda = \eta_j(\lambda)P_j^\lambda$ with respect to λ gives

$$\forall \lambda' \in \mathfrak{z}^* \qquad B(\lambda')Q_j^\lambda + B(\lambda)\,\lambda'\cdot\nabla_\lambda Q_j^\lambda = \lambda'\cdot\nabla_\lambda\eta_j(\lambda)P_j^\lambda + \eta_j(\lambda)\,\lambda'\cdot\nabla_\lambda P_j^\lambda.$$

Taking the scalar product with P_j^λ and using $B(\lambda)^t = -B(\lambda)$ with $-B(\lambda)P_j^\lambda = \eta_j(\lambda)Q_j^\lambda$, we obtain:

$$(B(\lambda')Q_j^\lambda, P_j^\lambda) + \eta_j(\lambda)(\lambda'\cdot\nabla_\lambda Q_j^\lambda, Q_j^\lambda) = \lambda'\cdot\nabla_\lambda\eta_j(\lambda)(P_j^\lambda, P_j^\lambda) + \eta_j(\lambda)(\lambda'\cdot\nabla_\lambda P_j^\lambda, P_j^\lambda).$$

Now, $(Q_j^\lambda, Q_j^\lambda) = 1 = (P_j^\lambda, P_j^\lambda)$. Differentiating this with respect to λ yields $(\lambda'\cdot\nabla_\lambda Q_j^\lambda, Q_j^\lambda) = 0 = (\lambda'\cdot\nabla_\lambda P_j^\lambda, P_j^\lambda)$, and we have for all $\lambda' \in \mathfrak{z}^*$

$$(B(\lambda')Q_j^\lambda, P_j^\lambda) = \lambda'\cdot\nabla_\lambda\eta_j(\lambda).$$

Since the left-hand side is equal to $\lambda'([Q_j^\lambda, P_j^\lambda])$ by definition of $B(\lambda')$, the conclusion follows. □

The two lemmata above imply readily:

Corollary 4.8. *Using $\zeta = \zeta(\alpha,\lambda) = \sum_{j=1}^d \eta_j(\lambda)(2\alpha_j+1)$, we deduce that for the choice of orthonormal basis of Lemma 4.7, we have*

$$\pi^\lambda(\mathbb{P}_\zeta)\Big(V\cdot\pi^\lambda(V))\circ T\Big)\pi^\lambda(\mathbb{P}_\zeta) = \frac{1}{4}\mathbb{L} - \frac{i}{4}\nabla_\lambda\zeta.$$

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