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LANDAU AND RAMANUJAN APPROXIMATIONS FOR DIVISOR SUMS AND COEFFICIENTS OF CUSP FORMS

ALEXANDRU CIOLAN, ALESSANDRO LANGUASCO, AND PIETER MOREE

ABSTRACT. In 1961, Rankin determined the asymptotic behavior of the number $S_{k,q}(x)$ of positive integers $n \leq x$ for which a given prime q does not divide $\sigma_k(n)$, the k-th divisor sum function. By computing the associated Euler-Kronecker constant $\gamma_{k,q}$, which depends on the arithmetic of certain subfields of $\mathbb{Q}(\zeta_q)$, we obtain the second order term in the asymptotic expansion of $S_{k,q}(x)$. Using a method developed by Ford, Luca and Moree (2014), we determine the pairs (k,q) with $(k,q-1)=1$ for which Ramanujan's approximation to $S_{k,q}(x)$ is better than Landau's. This entails checking whether $\gamma_{k,q} < 1/2$ or not, and requires a substantial computational number theoretic input and extensive computer usage. We apply our results to study the non-divisibility of Fourier coefficients of six cusp forms by certain exceptional primes, extending the earlier work of Moree (2004), who disproved several claims made by Ramanujan on the non-divisibility of the Ramanujan tau function by five such exceptional primes.

1. Introduction

1.1. **Motivation and historical background.** A set S of positive integers is said to be *multiplicative* if for every pair (m,n) of coprime positive integers we have that $mn \in S$ if and only if $m,n \in S$. (In other words, S is a multiplicative set if and only if the indicator function of S is multiplicative.) An enormous supply of multiplicative sets is provided by taking

$$S = \{n \geq 1 : q \nmid f(n)\}, \tag{1}$$

where f is a multiplicative function and q a prime. (Throughout the paper, the letters p and q will always denote prime numbers.) Several papers (see, e.g., [15, 41, 45, 50, 51, 52, 56, 59]) are concerned with the asymptotic behavior of $S(x)$, the number of positive integers $n \leq x$ that are in S. An important role in understanding this quantity is played by the Dirichlet series

$$L_S(s) := \sum_{n\in S} n^{-s}, \tag{2}$$

which converges for $\Re(s) > 1$.

Here we are interested in the second order behavior of $S(x)$ and, in particular, in the case where $S = \{n \geq 1 : q \nmid \sigma_k(n)\}$, with $\sigma_k(n) = \sum_{d|n} d^k$ being the usual k-th divisor sum function. Our results have applications to the non-divisibility of the Fourier coefficients of six standard cusp forms by so-called *exceptional primes*. The cusp forms that make the object of our study are the normalized generators of the six one-dimensional cusp form spaces for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ (see Table 2). Of these, the *modular discriminant* function

$$\Delta(z) = q_1 \prod_{m=1}^{\infty} (1-q_1^m)^{24} = \sum_{n=1}^{\infty} \tau(n) q_1^n,$$

is perhaps the most well-known (with $z \in \mathbb{H}$, the complex upper half-plane, and $q_1 = e^{2\pi i z}$), its Fourier coefficients $\tau(n)$ being the values of the *Ramanujan tau function*.

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Ramanujan was not the first to consider Δ, but he seems to have been the first to realize that the values of $\tau(n)$ provide an interesting arithmetic sequence. In an "unpublished" manuscript that belongs to the collection of Trinity College, Cambridge, he considered τ modulo various prime powers q^e with $q \in \{2, 3, 5, 7, 23, 691\}$. Except for the case $q = 23$, these congruences involve the divisor sum function (and, often, a power of n), the most famous example in this regard being $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

In 2004 the second order behavior for $f(n) = \tau(n)$ and $q \in \{3, 5, 7, 23, 691\}$ was determined by Moree [34]. One aim of this paper is to put his work in a general framework. First, in that we consider (1) with $f = \sigma_k$ being *any* sum of divisors function and q *any* prime. Second, in that we consider an entire class of cusp forms that share certain properties (e.g., they are normalized simultaneous Hecke eigenforms), out of which Δ is a representative.

The congruences found by Ramanujan for $q \in \{3, 5, 7, 23, 691\}$ are not singular, and certainly not a coincidence. The monumental work of Serre [53, 54] and Swinnerton-Dyer [62, 63] revealed that these primes are only a few out of a much larger, but finite, list of *exceptional primes* modulo which the coefficients of these six cusp satisfy congruences involving divisor sum functions, as shown in Table 2.

1.2. **Euler-Kronecker constants.** In the following we will use $F'/F(s)$ as a shorthand for $F'(s)/F(s)$. If the limit

$$\gamma_S := \lim_{s\to 1^+}\left(\frac{L_S'}{L_S}(s) + \frac{\alpha}{s-1}\right) \tag{3}$$

exists for some $\alpha > 0$, we say that the set S admits an *Euler-Kronecker constant* γ_S. In case $S = \mathbb{N}$, we have $L_S(s) = \zeta(s)$, the *Riemann zeta function*, $\alpha = 1$ and $\gamma_S = \gamma$, the *Euler-Mascheroni constant* (see, for example, Lagarias [25] for a beautiful survey, and Havil [19] for a popular account).

As the following result shows, the Euler-Kronecker constant γ_S determines the second order behavior of $S(x)$,

Classical Theorem 1. *Let S be a multiplicative set. If there are $\rho > 0$ and $0 < \delta < 1$ such that*

$$\sum_{p\le x,\ p\notin S} 1 = \delta \sum_{p\le x} 1 + O_S\left(\frac{x}{\log^{2+\rho} x}\right), \tag{4}$$

then $\gamma_S \in \mathbb{R}$ exists and

$$S(x) = \sum_{n\le x,\ n\in S} 1 = \frac{c_0\, x}{\log^{\delta} x}\left(1 + \frac{(1-\gamma_S)\delta}{\log x}(1 + o_S(1))\right) \tag{5}$$

as $x \to \infty$, where c_0 is a positive constant. If the prime numbers belonging to S are, with finitely many exceptions, precisely those in a finite union of arithmetic progressions, we have, for arbitrary $j \ge 1$,

$$S(x) = \frac{c_0\, x}{\log^{\delta} x}\left(1 + \frac{c_1}{\log x} + \frac{c_2}{\log^2 x} + \cdots + \frac{c_j}{\log^j x} + O_{j,S}\left(\frac{1}{\log^{j+1} x}\right)\right), \tag{6}$$

with $c_0, \ldots, c_j$ constants, $c_0 > 0$ and $c_1 = (1 - \gamma_S)\delta$.

Proof. For the first assertion, see Moree [35, Theorem 4]; for the second, Serre [56, Théorème 2.8]. □

Before stating our main results (Sec. 1.3), we recall some known facts from the literature and we explain what we mean by a "Landau vs. Ramanujan approximation" comparison (Sec. 1.2.2). Our focus is on the special case where S is as in (1), the general case being discussed in greater detail by Moree [36].

1.2.1. **Two claims of Ramanujan.** Put

$$t_j = \begin{cases} 0 & \text{if } q \mid \tau(j), \\ 1 & \text{if } q \nmid \tau(j). \end{cases}$$

For $q \in \{3, 5, 7, 23, 691\}$, in his typical style, Ramanujan makes the following claim in his famous "unpublished" manuscript, perhaps included with his final letter to Hardy (Jan. 12th, 1920), or maybe sent to Hardy in 1923 by Francis Dewsbury, Registrar at the University of Madras.

Claim 1. *It is easy to prove by quite elementary methods that*

$$\sum_{j=1}^{n} t_j = o(n). \tag{7}$$

It can be shown by transcendental methods that

$$\sum_{j=1}^{n} t_j \sim \frac{C_q\, n}{\log^{\delta_q} n} \tag{8}$$

and

$$\sum_{j=1}^{n} t_j = C_q \int_1^n \frac{dx}{\log^{\delta_q} x} + O\left(\frac{n}{\log^{\rho} n}\right), \tag{9}$$

where ρ is any positive number.

Remark 1. We slightly changed the original notation to make it more consistent with ours. In order to stress the dependency on q, we use C_q and δ_q. Ramanujan wrote down the values of δ_q for the above primes q (see Table 1), and he explicitly (and correctly) determined C_3, C_7 and C_{23} (except for a factor $1 - 23^{-s}$ erroneously omitted in case $q = 23$), see Sec. 5 for details.

q	3	5	7	23	691
δ_q	1/2	1/4	1/2	1/2	1/690

TABLE 1. The values δ_q for the primes studied by Ramanujan

Remark 2. Ramanujan [3, p. 8] claims that proving the statement (7) is very similar to showing that $\pi(x) = o(x)$, with $\pi(x)$, the number of primes up to x, and refers to Landau [28]. Thus, one may speculate, what inspired Ramanujan in claiming that the integral in (9) is a better approximation than (8) might have been the fact, of which he was aware, that Gauss's approximation $\mathrm{Li}(x) = \int_2^x dt/\log t$ is a much better estimate for $\pi(x)$ than is $x/\log x$.

Remark 3. For the history of the unpublished manuscript and its wanderings, see Rankin [47]. It was finally made available to the mathematical community in 1999 by Berndt and Ono [3], together with commentaries, proofs and references to the literature. However, the material related to Claim 1 had already been discussed years earlier by Rankin [46, 48].

In his first letter to Hardy (Jan. 16th, 1913), Ramanujan [4, p. 24] had made a claim similar to (9):

Claim 2. *The number of positive integers $A \le n \le x$ that are either squares or can be written as the sum of two squares equals*

$$\mathcal{K} \int_A^x \frac{dt}{\sqrt{\log t}} + \theta(x),$$

where $\mathcal{K} = 0.764\ldots$ and $\theta(x)$ is very small compared with the previous integral. $\mathcal{K}$ and $\theta(x)$ have been exactly found, though complicated...

In his second letter to Hardy (Feb. 27th, 1913), answering his inquiry (see [4, p. 56]), Ramanujan wrote: "the order of $\theta(x)$ which you asked for in your letter is $\sqrt{x}/\sqrt{\log x}$."

See the exposition of Berndt and Rankin [4] for the full text of these two letters.

1.2.2. **Landau vs. Ramanujan.** Let S be the set of natural numbers that can be written as a sum of two squares. The fact that S is a multiplicative set was already known to Fermat. Following Landau, let us denote $S(x)$ by $B(x)$ in this particular case. In 1908, Landau [27] proved (see also [28, pp. 641–669]) that, asymptotically,

$$B(x) \sim \mathcal{K}\frac{x}{\sqrt{\log x}}, \tag{10}$$

a result of which Ramanujan was most likely unaware in 1913.

To honor the contribution of both Landau and Ramanujan, the constant

$$\mathcal{K} = \frac{1}{\sqrt{2}}\prod_{p\equiv 3 \ (\mathrm{mod}\ 4)}(1-p^{-2})^{-1/2} = 0.7642236535892206\ldots \tag{11}$$

is called the *Landau-Ramanujan constant* (cf. Finch [14, Section 2.3]).

For historical reasons delineated in this section, we will call

$$c_0\frac{x}{\log^\delta x} \quad \text{and} \quad c_0\int_2^x \frac{dt}{\log^\delta t}$$

the *Landau* and the *Ramanujan approximation* to $S(x)$, respectively. Further, if the inequality

$$\left|S(x) - c_0\frac{x}{\log^\delta x}\right| < \left|S(x) - c_0\int_2^x \frac{dt}{\log^\delta t}\right|$$

holds for every x sufficiently large, we say that the Landau approximation is better than the Ramanujan approximation (and the other way around if the reverse inequality holds). Partial integration gives us

$$c_0\int_2^x \frac{dt}{\log^\delta t} = \frac{c_0\ x}{\log^\delta x}\left(1+\frac{\delta}{\log x}+O\left(\frac{1}{\log^2 x}\right)\right),$$

and comparison with (5) then yields the following corollary of Classical Theorem 1.

Corollary 1. *If S is a multiplicative set satisfying* (4)*, the associated Euler-Kronecker constant γ_S exists. If $\gamma_S < 1/2$, then Ramanujan's approximation is asymptotically better than Landau's, and the other way around if $\gamma_S > 1/2$.*

Remark 4. If $\gamma = 1/2$, then Landau and Ramanujan give the same approximation up to the second order term. To see which one is closer to the actual value of $S(x)$, one would have to study the higher order terms.

By partial integration we see that Claim 2 implies the asymptotic (10). Nevertheless, one can wonder whether Ramanujan's integral expression provides a better approximation than Landau's asymptotic.

The first to ever consider this question seems to have been Hardy, who in his lectures on Ramanujan's work (see [17, pp. 9, 63]) writes that Ramanujan's "integral has no advantage, as an approximation, over the simpler function $\mathcal{K}x/\sqrt{\log x}$." He also says, see [17, p. 19], "The integral is better replaced by the simpler function...". However, as revealed by Shanks [58], Hardy made his claims based on a flawed paper [61] of his PhD student, Gertrude Stanley.

Going beyond the first order asymptotic behavior, it can be shown (see, e.g., Hardy [17, p. 63]) that, as $x \to \infty$, $B(x)$ has an asymptotic series expansion in the sense of Poincaré of the form

$$B(x) = \mathcal{K}\frac{x}{\sqrt{\log x}}\left(1 + \frac{c_0}{\log x} + \frac{c_1}{\log^2 x} + \cdots + \frac{c_{j-1}}{\log^j x} + O\left(\frac{1}{\log^{j+1} x}\right)\right), \tag{12}$$

where j can be taken arbitrarily large and the c_i are constants. Serre [56] proved a similar result for a larger class of so-called *Frobenian multiplicative functions*. This result implies, in particular, that for the multiplicative set $S_{\tau;q} = \{n \le x : q \nmid \tau(n)\}$ we asymptotically have

$$S_{\tau;q}(x) = C_q\ \frac{x}{\log^{\delta_q} x}\left(1 + \frac{c_0}{\log x} + \frac{c_1}{\log^2 x} + \cdots + \frac{c_{j-1}}{\log^j x} + O\left(\frac{1}{\log^{j+1} x}\right)\right), \tag{13}$$

where q is any of the primes studied by Ramanujan, the constants c_i may depend on the choice of q, and δ_q is given in Table 1. Much earlier, Watson [66] (who had had the unpublished manuscript in his possession for many years) showed that $S_{\tau;691}(x) = O(x(\log x)^{-1/690})$. Both expansions (12) and (13) fit in the framework set up in the opening paragraphs of this article and are special cases of (6).

By partial integration, Claims 1 and 2 imply that expansions of the form (12) and (13) should hold true for $B(x)$, respectively $S_{\tau;q}(x)$. Both claims also imply particular values for the c_i. However, already the values of c_0 from (12) and (13) turn out to be incorrect.

Classical Theorem 2. *For $q \in \{3, 5, 7, 23, 691\}$ the asymptotic* (8) *is correct, cf. Rankin* [46, 48]*, but the refined estimate* (9) *is false, cf. Moree* [34]*, for every $\rho > 1 + \delta_q$, with δ_q as in Table 1. Claim 2 is true for $\theta(x) = O(x \log^{-3/2} x)$, but false for $\theta(x) = o(x \log^{-3/2} x)$, cf. Shanks* [58].

For a more detailed and leisurely account of the historical aspects, see Berndt and Moree [2] or Moree and Cazaran [37]. The latter authors focus on the work done on counting integers represented by quadratic forms other than $X^2 + Y^2$.

The reader might wonder about what happens for the primes not mentioned in Classical Theorem 2. Here it is known, thanks to deep work of Serre [55, 56], that an asymptotic of the form (8) holds. However, the correctness of the refined estimate (9) is an open problem.

1.2.3. **Ramanujan-type congruences and divisor sums.** We now know that 691 and the other primes studied by Ramanujan are only a few out of a larger, but finite set of exceptional primes modulo which certain congruences hold for the six cusp forms given in Table 2. Following the notation used by Ramanujan and, later, by Swinnerton-Dyer, we let Q and R denote the normalized Eisenstein series E_4 and E_6, which, along with Δ, are given by

$$Q = E_4 = 1 + 240\sum_{n\ge1}\sigma_3(n)q_1^n, \quad R = E_6 = 1 - 504\sum_{n\ge1}\sigma_5(n)q_1^n, \quad \Delta = \frac{1}{1728}(E_4^3 - E_6^2).$$

It is an impressive feat that Ramanujan actually found all exceptional primes for Δ.

Weight w	12	16	18	20	22	26
Form	Δ	$Q\Delta$	$R\Delta$	$Q^2\Delta$	$QR\Delta$	$Q^2R\Delta$

TABLE 2. The cusp forms in the "Serre and Swinnerton-Dyer" classification

The weights w appearing here[1] are precisely those for which the associated space of cusp forms of the full modular group is 1-dimensional.

[1]The traditional notation k unfortunately clashes with the subscript used in σ_k.

It is well-known that the coefficients $\tau_w(n)$ of the cusp forms above satisfy the following fundamental properties (for references see the three excellent articles highlighting different aspects of the tau function [39, 48, 63] in the proceedings of the 1987 "Ramanujan Revisited" conference).

Classical Theorem 3. *For $w \in \{12, 16, 18, 20, 22, 26\}$ the following properties hold:*
(1) *τ_w is multiplicative; that is, $\tau_w(mn) = \tau_w(m)\tau_w(n)$ whenever $(m, n) = 1$.*
(2) *if p is prime, then $\tau_w(p^{e+1}) = \tau_w(p)\tau_w(p^e) - p^{k-1}\tau_w(p^{e-1})$ for any $e \geq 2$.*
(3) *$|\tau_w(p)| \leq 2p^{(w-1)/2}$.*

For $\tau(n)$ (which equals $\tau_{12}(n)$, but we will keep our old notation) these properties were conjectured by Ramanujan on basis of very scant numerical material. They were a starting point for amazing and fundamental developments in the 20th and 21st centuries, see, e.g., the book by the Murty brothers [40], or the expository article by Sujatha [60]. In addition, Ramanujan found many other congruences for τ_w involving sums of divisor functions (see [3]). In the years that followed, Deligne [9], Haberland [16], Serre [53, 54] and Swinnerton-Dyer [62, 63] classified all primes q modulo which which congruences hold for τ_w, which are of one of the following types:

(i) $\tau_w(n) \equiv n^v \sigma_{w-1-2v}(n) \pmod q$ for all $(n, q) = 1$, and for some $v \in \{0, 1, 2\}$.
(ii) $\tau_w(n) \equiv 0 \pmod q$ whenever $\left(\frac{n}{q}\right) = -1$.
(iii) $p^{1-w}\tau_w^2(p) \equiv 0, 1, 2$ or $4 \pmod q$ for all primes $p \neq q$.

The complete list of the *exceptional primes* q for each of the forms in Table 2 is given in Section 4. Following convention, we speak about the "Serre and Swinnerton-Dyer" classification.

The congruences of type (i) suggest to investigate the non-divisibility of $n^v \sigma_k(n)$, with v and k arbitrary natural numbers. Note that if $v \geq 1$, we may take without loss of generality $v = 1$. The associated counting functions $S_{k,q}(x) = \sum_{n \leq x,\ q \nmid \sigma_k(n)} 1$ and $S'_{k,q}(x) = \sum_{n \leq x,\ q \nmid n\sigma_k(n)} 1$ are the main functions of interest in this paper.

The following elementary result (the proof is immediate from the analysis in Sec. 3.2) greatly simplifies our analysis.

Proposition 1. The prime q divides $\sigma_k(n)$ if and only if it divides $\sigma_{(k,q-1)}(n)$.

Corollary 2. *It is enough to study the non-divisibility problem for $\sigma_k(n)$ with k dividing $q - 1$.*

Definition 1. If the prime q divides $a(n)$ whenever it divides $b(n)$, we write $a(n) \cong b(n) \pmod q$.

Note that $\cong$ is an equivalence relation. In this notation, Proposition 1 can be reformulated as $\sigma_k(n) \cong \sigma_{(k,q-1)}(n) \pmod q$. For example, the congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ implies $\tau(n) \cong \sigma_1(n) \pmod{691}$. For our purposes it is not the actual congruence that is relevant, but the weaker $\cong$ notion. As we shall see, the Serre and Swinnerton-Dyer classification takes, up to $\cong$, a simpler form than with the classical congruence notion.

Ramanujan, in the unpublished manuscript [3, Sec. 19] was likely the first to consider $S_{k,q}(x)$. He made three claims (also reproduced by Rankin [46]). These were later proved by Watson [66]. One of these claims, namely that $S_{k,q}(x) = O(x \log^{-1/(q-1)} x)$ in case k is odd, is discussed by Hardy in his Ramanujan lectures [17, §10.6]. The asymptotic behavior of $S_{k,q}(x)$ for general k was determined by Rankin [45]. Eira Scourfield [50] (in her 1963 PhD thesis, supervised by Rankin) generalized his work by establishing asymptotics in the case where a prescribed prime power is required to exactly divide $\sigma_k(n)$. In a later paper [51] she considered the divisibility of the divisor function by arbitrary fixed integers.

In this paper we will determine the second order behavior of $S_{k,q}(x)$. In particular, one of our main results, Theorem 1, gives a formula for the Euler-Kronecker constant $\gamma_{k,q}$ associated to the non-divisibility of $\sigma_k(n)$ by an odd prime q, which allows one to decide on the "Landau vs. Ramanujan problem" for prescribed k and q. In case $(k, q - 1) = 1$, which holds indeed for most of the exceptional primes of type (i) and the accompanying values k, we can invoke Theorem 4 in order to decide on the "Landau vs. Ramanujan problem."

1.2.4. **Other functions.** Ford, Luca and Moree [15] studied the divisibility of the function $f=\varphi$ by q, with $\varphi(n)$ the Euler totient function, for any odd prime q. They showed that Ramanujan wins for $q\le 67$, and Landau for $q>67$, and were the first to resolve this type of comparison problem for infinitely many cases. Earlier, Spearman and Williams [59] had determined the relevant leading constant C_q by relating it to the arithmetic of the cyclotomic number field $\mathbb{Q}(\zeta_q)$.

More generally, Scourfield [52] considered integer-valued multiplicative functions $f(n)$ with the property that $f(p)=W(p)$ for primes p, with $W(x)$ being a polynomial with integral coefficients. For this class of functions, she obtained an asymptotic expression for $\#\{n\le x : N\nmid f(n)\}$, while Narkiewicz [41] obtained asymptotics for $\#\{n\le x : (f(n),N)=1\}$.

1.3. **Statement of results.** Before stating our results, let us fix some terminology used in the sequel.

Definition 2. Given a divisor m of $q-1$, let K_m be the unique subfield of $\mathbb{Q}(\zeta_q)$ of degree $(q-1)/m$. By $\mathcal{O}_{K_m}, \zeta_{K_m}(s)$ and γ_{K_m}, we denote its associated ring of integers, Dedekind zeta function and Euler-Kronecker constant, respectively.

The uniqueness of K_m is a consequence of Galois theory and $\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})\cong(\mathbb{Z}/q\mathbb{Z})^*$ being cyclic, see Section 2.2. The constant γ_{K_m} is obtained on setting $L_S(s)=\zeta_{K_m}(s)$ and $\alpha=1$ in (3), cf. Section 2.3.

After this rather long introduction, we are now able to state our findings. The first half of Theorem 1 is a special case of Classical Theorem 1, the formulas for $\gamma_{k,q}$ and $\gamma'_{k,q}$ being the novel feature.

Theorem 1. *Let q be an odd prime and $k\ge 1$ an integer. We define*

$$S_{k,q}(x)=\sum_{n\le x,\ q\nmid\sigma_k(n)}1 \quad \text{and} \quad S'_{k,q}(x)=\sum_{n\le x,\ q\nmid n\sigma_k(n)}1.$$

Put $r=(k,q-1)$ and assume that $h=(q-1)/r$ is even. The counting function $S_{k,q}(x)$ satisfies an asymptotic expansion (13) *in the sense of Poincaré with $\delta_q=1/h$. In particular, there is a positive constant $C_{k,q}$ only depending on r and q such that*

$$S_{k,q}(x)=\frac{C_{k,q}\,x}{\log^{1/h}x}\left(1+\frac{1-\gamma_{k,q}}{h\log x}+O_{k,q}\left(\frac{1}{\log^2 x}\right)\right).$$

Here $\gamma_{k,q}$ is the Euler-Kronecker constant of the sum of divisors function $\sigma_k(n)$ and satisfies

$$\gamma_{k,q}=\gamma-\frac{1}{h}(2\gamma_{K_{2r}}-\gamma_{K_r})-\frac{\log q}{h(q-1)}-S(r,q), \tag{14}$$

where γ is the Euler-Mascheroni constant, γ_{K_r} is as in Definition 2,

$$\begin{aligned}S(r,q)=&-\sum_{g_p=1}\frac{(q-1)\log p}{p^{q-1}-1}+\sum_{g_p=1}\frac{q\log p}{p^q-1}-\sum_{g_p\ge 3}\frac{(g_p-1)\log p}{p^{g_p-1}-1}+\sum_{g_p\ge 3}\frac{g_p\log p}{p^{g_p}-1}\\&+\sum_{g_p=2}\frac{\log p}{p^2-1}+\sum_{\substack{g_p\ge 4\\ 2\mid g_p}}\frac{\log p}{p^{g_p/2}-p^{-g_p/2}},\end{aligned} \tag{15}$$

g_p is the multiplicative order of p^r modulo q, and the sums are over primes $p\neq q$.

The counting function $S'_{k,q}(x)$ also satisfies an asymptotic expansion (13) *in the sense of Poincaré with $\delta_q = 1/h$. In particular, we have*

$$S'_{k,q}(x) = \frac{C'_{k,q}\ x}{\log^{1/h} x}\left(1 + \frac{1-\gamma'_{k,q}}{h \log x} + O_{k,q}\left(\frac{1}{\log^2 x}\right)\right),$$

with

$$C'_{k,q} = \left(1 - \frac{1}{q}\right) C_{k,q} \quad \text{and} \quad \gamma'_{k,q} = \gamma_{k,q} + \frac{\log q}{q-1}. \tag{16}$$

Remark 5. The remaining cases where $q = 2$ or h is odd are rather trivial, see Secs. 3.7–3.8.

Remark 6. Consistent with Proposition 1 we have $C_{k,q} = C_{r,q}, C'_{k,q} = C'_{r,q}, \gamma_{k,q} = \gamma_{r,q}$ and $\gamma'_{k,q} = \gamma'_{r,q}$. The constant $C_{k,q}$ was first determined by Rankin [45, p. 38]. For completeness we derive his formula again, in our notation, in Sec. 3.9.

The following is a special case of Corollary 1.

Corollary 3. *A Ramanujan-type claim for $S_{k,q}(x)$ is false if $\gamma_{k,q} \neq 0$. If $\gamma_{k,q} < 1/2$, then the Ramanujan integral approximation for $S_{k,q}(x)$ is asymptotically better than the Landau asymptotic. If $\gamma_{k,q} > 1/2$, then it is the other way around. The same applies for $S'_{k,q}(x)$ and its Euler-Kronecker constant $\gamma'_{k,q}$.*

The proof of Theorem 1 rests on studying the associated Dirichlet series $T(s) := L_S(s)$ (defined in (2)) with $S = \{n \geq 1 : q \nmid \sigma_k(n)\}$, and expressing it in term of Dirichlet L-series and a function which is regular for $\mathrm{Re}(s) > 1/2$. An important aspect in our analysis will be played by the greatest common divisor $r = (k, q-1)$. For small values of r, this is motivated by the congruences involving exceptional primes of type (i), for which we have $r \in \{1, 3, 5\}$. We prove that, for any prescribed r, the Landau approximation is better than the Ramanujan one for all large enough q.

Theorem 2. *There exists an absolute constant c_1 such that for every positive integer r, every prime*

$$q \geq e^{2r(\log r + \log\log(r+2) + c_1)}, \quad \text{with } q \equiv 1 \ (\mathrm{mod}\ 2r),$$

and every positive integer k satisfying $(k, q-1) = r$, the Landau approximation is better than the Ramanujan approximation for both $S_{k,q}(x)$ and $S'_{k,q}(x)$.

The larger the prime q gets, the more the associated Dirichlet series $T(s)$ will resemble the Riemann zeta function, and so the closer the associated Euler-Kronecker constant approximates γ. This is expressed more mathematically in the following theorem.

Theorem 3. *Let q be an odd prime and $k \geq 1$ an integer. Put $r = (k, q-1)$. We have*

$$\gamma_{k,q} = \gamma + O\Big(\frac{\log^3 q}{q^{1/r} \log\log q} + \frac{r \log^2 q}{\sqrt{q}}\Big),$$

where the implied constant is absolute.

Corollary 4. *Let $\epsilon > 0$. There exists a constant $c_1(\epsilon)$ such that*

$$|\gamma_{k,q} - \gamma| < \epsilon,$$

for every positive integer r, every prime

$$q \geq e^{r(3\log r + 2\log\log(r+2) + c_1(\epsilon))}, \quad \text{with } q \equiv 1 \ (\mathrm{mod}\ 2r),$$

and every positive integer k satisfying $(k, q-1) = r$.

Thus for a random choice of k and q, the constant $\gamma_{k,q}$ will be close to γ, and as $\gamma > 1/2$ we deduce that Landau generically wins over Ramanujan. In the special case when $r = 1$, we were able to find the precise value of q beyond which Landau always wins. As this value was not too large, by extensive numerical checks we were also able to determine, for each of the remaining values of q, whether it is the Landau or the Ramanujan approximation that wins.

Theorem 4. *Let $k \geq 1$ be an integer and q an odd prime such that $(k, q-1) = 1$. The Landau approximation for $S_{k,q}(x)$ is better than the Ramanujan one for all primes q other than*

$$q \in \{3, 5, 7, 11, 13, 17, 23, 29, 37, 41, 43, 47, 53, 59, 73\},$$

in which cases the Ramanujan approximation is better. The Landau approximation for $S'_{k,q}(x)$ is better than the Ramanujan one for all primes $q \neq 5$.

It is an exercise in elementary analytic number theory to show that the number of pairs (k, q) with $k, q \leq x$ such that $(k, q-1) = 1$ is asymptotically equal to $Ax^2/\log x$, where $A = 0.37399558\ldots$ is the Artin constant. Thus, in some sense, the probability of the condition $(k, q-1) = 1$ being met, for random integers k and primes q, equals A.

For $r \geq 2$, our upper bound for the values of q beyond which the Landau approximation is certainly better increase rather dramatically (see Sec. 7). Despite the considerable computer resources we had at our disposal, we were not able[2] to run a test on all the remaining primes q, in order to fully answer the question in case $r \in \{3, 5\}$. However, our numerical experiments strongly suggest the following.

Conjecture 1. *If $r = 3$, the Landau approximation for $S_{k,q}(x)$ is better than the Ramanujan one for all primes q other than*

$$q \in \{7, 13, 19, 31, 37, 61, 67, 79, 97, 103, 109, 127, 181\},$$

in which cases the Ramanujan approximation is better. The Landau approximation for $S'_{k,q}(x)$ is better than the Ramanujan one for all primes q other than $q \in \{7, 13, 19, 31, 61, 67, 97, 109\}$.

Conjecture 2. *If $r = 5$, the Landau approximation for $S_{k,q}(x)$ is better than the Ramanujan one for all primes q other than*

$$q \in \{11, 31, 41, 71, 101, 131, 241, 271, 311\},$$

in which cases the Ramanujan approximation is better. The Landau approximation for $S'_{k,q}(x)$ is better than the Ramanujan one for all primes q other than $q \in \{11, 31, 71, 131, 241, 311\}$.

While we were not able to decide on the "Landau vs. Ramanujan" comparison for all primes q in case $r \in \{3, 5\}$, we were nevertheless able, on performing rather involved numerical checks, to answer this question for every exceptional prime q and each of the six cusp forms that we studied.

Theorem 5. *Let $f = \sum_{n \geq 1} \tau_w(n) q_1^n$ be any of the six cusp forms in Table* 2 *and let q be any odd exceptional prime of type* (i) *or* (ii). *If we put*

$$t_n = \begin{cases} 0 & \text{if } q \mid \tau_w(n), \\ 1 & \text{if } q \nmid \tau_w(n), \end{cases}$$

then (8) *holds for some positive numbers C_q and δ_q. However, the Ramanujan-type claim* (9) *is false for any $\rho > 1 + \delta_q$, where $\delta_q = r/(q-1)$ for primes q of type* (i) *(with r given as in Tables 7–8) and $\delta_q = 1/2$ for those of type* (ii).

Ramanujan's approximation is better than Landau's if one of the following is satisfied:

[2]For *every* prescribed $r = (q-1, k)$ it is theoretically possible to decide on the "Landau vs. Ramanujan approximation" for *all* primes q. However, we expect this would require extensive numerical checks, and, very likely, considerable improvements on the algorithms used in this paper.

- $q = 5$;
- $q = 7$ *and* $f \in \{\Delta, Q^2\Delta, Q^2R\Delta\}$;
- $q > 5$ *and* $f = R\Delta$.

In all remaining cases, Landau's approximation is better.

The case where $q = 2$ and exceptional is far more trivial, see Section 4.3 for further details. Thanks to the work of Swinnerton-Dyer and Haberland, see Section 4.5, we know that (iii) only occurs for $w = 16$ and $q = 59$. Here we leave computing the associated Euler-Kronecker constant as a challenge to the interested reader. For some remarks on what happens for non-exceptional primes, see Section 4.6.

1.4. **Outline.** Section 2 contains some prerequisites on the multiplicative order, character theory, factorization of Dedekind zeta functions and splitting of primes in certain number fields (most of these results are well-known facts from algebraic and analytic number theory). In Section 3 we evaluate the generating series and the Euler-Kronecker constant associated to $q \nmid n^v\sigma_k(n)$ and we give the proof of Theorem 1. In Section 4 we discuss the congruences for exceptional primes, and we prove Theorem 5. In Section 5 we look in close detail at the Claim 1 statements in the unpublished manuscript. We present our take on why Ramanujan only wrote down C_3, C_7 and C_{23} explicitly and give a uniform way of deriving his three formulae. Section 6 is dedicated to finding upper and lower bounds for the sum $S(m, q)$. In Sections 7, 8 and 9 we prove Theorems 2, 3 and 4. Section 10 discusses various aspects of the numerical computations that we carried out. Finally, Section 11 discusses possible generalizations of our work and some open questions.

The programs used to obtain the numerical results included in this paper are available under `www.math.unipd.it/~languasc/CLM.html`.

2. Analytic and algebraic preamble

2.1. **Multiplicative orders.** Let us recall that the letters p and q will be used throughout to denote prime numbers. Additionally, we assume q is odd. For a prime $p \neq q$, relevant for our work will be the *multiplicative order* of p modulo q, which is the smallest positive integer f_p such that $p^{f_p} \equiv 1 \pmod{q}$. (The order is more commonly denoted by $\mathrm{ord}_q(p)$; we use f_p for reasons of space and to be consistent with the notation in earlier works, e.g., [15].) Obviously, if satisfies the divisibility property $f_p \mid q - 1$. Since the order is not defined for $p = q$, whenever f_p appears in the sequel, the implicit assumption is that $p \neq q$.

Given a positive integer m, we let g_p be the smallest positive integer such that $p^{g_p m} \equiv 1 \pmod{q}$. In other words, g_p is the order of p^m modulo q. Since this implies that $f_p \mid g_p m$, dividing both sides by (f_p, m), the greatest common divisor of f_p and m, yields $g_p = f_p/(f_p, m)$. We trivially have

$$a^m \equiv -1 \pmod{q} \Leftrightarrow a^{2m} \equiv 1 \pmod{q} \text{ and } a^m \not\equiv 1 \pmod{q}, \tag{17}$$

and we further note that

$$g_p = 1 \Leftrightarrow f_p \mid m \quad \text{and} \quad g_p = 2 \Leftrightarrow f_p \mid 2m \text{ and } f_p \nmid m.$$

Observe that if $q \equiv 1 \pmod{m}$, then g_p is a divisor of h.

We will make several times use of the following elementary result.

Lemma 1. *Let q be an odd prime, let m be a divisor of $q - 1$ and put $h = (q - 1)/m$. Then the equation $x^m \equiv 1 \pmod{q}$ has m solutions modulo q. The equation $x^m \equiv -1 \pmod{q}$ has m solutions if h is even, and no solutions otherwise. If m is even, both congruences have at most $m/2$ prime solutions $p < q - 2$.*

Proof. Left as an exercise. Use the trivial fact that not both p and $q - p$ can be prime (unless $p = 2$). ∎

Lemma 2. *Let $m \geq 1$ be fixed.*
1) *We have $p^{bm} \equiv -1 \pmod{q}$ if and only if g_p is even and $b \equiv g_p/2 \pmod{g_p}$.*
2) *If g_p is even and has an odd divisor $d > 1$, then*

$$\frac{p^{g_p m/2}+1}{p^{g_p m/(2d)}+1} = q m_p, \tag{18}$$

with $m_p \geq 1$ an integer.

Proof. 1) We have to find all b such that $p^{2bm} \equiv 1 \pmod{q}$ and $p^{bm} \not\equiv 1 \pmod{q}$. This is equivalent with $f_p \mid 2bm$ and $f_p \nmid bm$. On dividing both f_p and m by (f_p, m), we see that these two requirements are equivalent with $g_p \mid 2b$ and $g_p \nmid b$. The latter two conditions are fulfilled precisely when g_p is even and b is an odd multiple of $g_p/2$.
2) The left-hand side of (18) is easily seen to be an integer. By part 1), the numerator is divisible by q, while the denominator is not. Assuming otherwise, the order of p^m mod q would divide g_p/d, which contradicts the definition of g_p. □

2.2. **Cyclotomic subfields.** In what follows, we fix an odd prime q and we denote $\mathbb{Q}(\zeta_q)$ by K. By basic algebraic number theory, we have $\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \cong (\mathbb{Z}/q\mathbb{Z})^*$, the latter being a cyclic group. For every divisor m of $q-1$, there is a unique subgroup of order m, which, by the main theorem of Galois theory, corresponds uniquely to a subfield of $\mathbb{Q}(\zeta_q)$.

Definition 3. For any divisor m of $q-1$, we let K_m be the unique subfield of $\mathbb{Q}(\zeta_q)$ of degree $[K_m : \mathbb{Q}] = (q-1)/m$. Certainly, we have $K_1 = K$.

As examples, note that $K_2 = \mathbb{Q}(\zeta_q + \zeta_q^{-1}) = \mathbb{Q}(\cos(2\pi/q))$ and $K_{q-1} = \mathbb{Q}$. The field K_2 is the maximal real subfield of $\mathbb{Q}(\zeta_q)$. Any field K_m with m even is a subfield of K_2, and is therefore real.

By the Kronecker-Weber theorem, every abelian number field is a subfield of some cyclotomic field $\mathbb{Q}(\zeta_n)$. If we restrict n to be a prime, we can realize precisely all extensions of the rationals having a cyclic Galois group that are tamely ramified in one prime and unramified in all other primes (note that in this case $(\mathbb{Z}/n\mathbb{Z})^*$ is always cyclic).

Good introductions to the arithmetic of subfields of $\mathbb{Q}(\zeta_n)$ relevant to this paper can be found in the books by Kato et al. [22, Chp. 1] and Washington [65, Chps. 3-4].

2.2.1. **Splitting of primes.** For certain families of number fields it is not difficult to explicitly work out the Euler product in (21). For this, we need to precisely know how the rational primes split in $\mathcal{O}_{K_m}$.

Lemma 3 (Splitting of primes in K_m)**.** *Let q be an arbitrary odd prime, m an arbitrary divisor of $q-1$ and K_m the number field as in Definition 3. If $p \neq q$ is a prime, the principal ideal $p\mathcal{O}_{K_m}$ factorizes as $p\mathcal{O}_{K_m} = \mathfrak{p}_1 \cdots \mathfrak{p}_g$, where $g = (q-1)/(mf)$ and all prime ideals $\mathfrak{p}_i$ are distinct and of degree f, with f the multiplicative order of p^m modulo q. Furthermore, $q\mathcal{O}_{K_m} = \mathfrak{q}^{(q-1)/m}$ with $\mathfrak{q}$ a prime ideal of norm q.*

Proof. See, e.g., Marcus [31, pp. 76–78] or apply Theorem 5.7 of [22]. □

In case $f = 1$, we say that p *splits completely* in K_m. This happens in K_m if and only if $p^m \equiv 1 \pmod{q}$.

Proposition 2. Let $k \geq 1$ be an integer and q and odd prime. Put $r = (k, q-1)$. We have $q \mid \sigma_k(p)$ if and only if p splits completely in K_{2r}, but does not split completely in K_r.

Proof. Since $q \nmid \sigma_k(q)$ and q is ramified, the assertion is correct for $p = q$, and so we may assume $p \neq q$. By Fermat's little theorem, it suffices to verify the assertion for $k = r$. Notice that $q \mid \sigma_r(p)$ if and only if $p^{2r} \equiv 1 \pmod{q}$ and $p^r \not\equiv 1 \pmod{q}$. By Lemma 3 the proof is then concluded. □

The average behavior of an arithmetic function that is of rather bounded growth is very much influenced (and determined) by its values in the prime numbers. In light of this and Proposition 2, it is not surprising that the fields K_{2r} and K_r play an important role in our results and computations.

2.2.2. **Character theory.** In the following, q is an odd prime and m a divisor of $q-1$.

Definition 4. We let C_m be the subgroup of m-th roots of unity inside $(\mathbb{Z}/q\mathbb{Z})^*$. As a set we have

$$C_m = \{a \in (\mathbb{Z}/q\mathbb{Z})^* : a^m \equiv 1 \pmod{q}\}.$$

We have $K_m = \mathbb{Q}(\sum_{a\in C_m} \zeta_q^a)$. Associated to C_m we define a character group, of Dirichlet characters modulo q, namely

$$X_m = \{\chi : \chi(a) = 1 \text{ for every } a \in C_m\}.$$

Under the Galois correspondence C_m is the group associated to K_m and K_m is the field belonging to X_m. We have $X_m \cong \mathrm{Gal}(K_m/\mathbb{Q})$, cf. Washington [65, p. 22]. Note that X_2 is the set of even characters and that $X_{2m} = \{\chi \in X_m : \chi \text{ is even}\}$ if $m \mid (q-1)/2$. We have $\#C_m = m$ and $\#X_m = (q-1)/m$. The principal character, which we denote by χ_0, is always in X_m. The quadratic character is unique and of order two and so is in X_m if and only if X_m has even order, that is if and only if $(q-1)/m$ is even. For notational convenience we put

$$X_m^* = X_m \setminus \{\chi_0\} = \{\chi \neq \chi_0 : \chi(a) = 1 \text{ for every } a \in C_m\}. \tag{19}$$

A simple observation we will use is the following.

Lemma 4. (i) *If χ is a character modulo m, then*

$$\sum_{a\in C_m} \chi(a) = \begin{cases} m & \text{if } \chi \in X_m, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If $a \in (\mathbb{Z}/q\mathbb{Z})^*$, then*

$$\sum_{\chi\in X_m} \chi(a) = \begin{cases} (q-1)/m & \text{if } a \in C_m, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (i) If $\chi \in X_m$, the claim follows from the definition and the fact that C_m is of order m. If $\chi \notin X_m$, then there exists $b \in C_m$ such that $\chi(b) \neq 1$. Using the group structure of C_m we then infer that

$$\chi(b) \sum_{a\in C_m} \chi(a) = \sum_{a\in C_m} \chi(ba) = \sum_{a\in C_m} \chi(a),$$

and we conclude that $\sum_{a\in C_m} \chi(a) = 0$.

(ii) If $a \in C_m$, the claim follows from the definition and the fact that X_m is of order $(q-1)/m$. If $a \notin C_m$, there exists $\chi_1 \in X_m$ such that $\chi_1(a) \neq 1$. Using the group structure of X_m we then infer that

$$\chi_1(a) \sum_{\chi\in X_m} \chi(a) = \sum_{\chi\in X_m} (\chi_1\chi)(a) = \sum_{\chi\in X_m} \chi(a),$$

and we conclude that $\sum_{\chi\in X_m} \chi(a) = 0$. □

We will often use the trivial observation (17), which implies that if $r \mid (q-1)/2$, then

$$\sum_{i=1}^{r} \chi(a_i) = \sum_{b\in C_{2r}} \chi(b) - \sum_{b\in C_r} \chi(b), \tag{20}$$

where the first sum is over the r solutions $0 < a_i < q$ of $x^r \equiv -1 \pmod{q}$.

2.3. **The Dedekind zeta function.** To any number field K we can associate its *Dedekind zeta function*

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s},$$

defined for $\Re(s) > 1$. Here, $\mathfrak{a}$ runs over non-zero ideals in $\mathcal{O}_K$, the ring of integers of K. It is known that $\zeta_K(s)$ can be analytically continued to $\mathbb{C} \setminus \{1\}$, and has a simple pole at $s = 1$. Over $\mathcal{O}_K$ we have unique factorization into prime ideals, and this leads to the *Euler product identity*

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}}, \tag{21}$$

valid for $\Re(s) > 1$, where $\mathfrak{p}$ runs over the prime ideals in $\mathcal{O}_K$. Around $s = 1$ we have

$$\frac{\zeta_K'}{\zeta_K}(s) = -\frac{1}{s-1} + \gamma_K + O(|s-1|), \tag{22}$$

and thus (3) holds for $\alpha = 0$. On computing the Laurent expansion up to higher order, further constants, known as *Stieltjes constants*, make their appearance (cf. Lagarias [25]).

An alternative formula for γ_K (see, e.g., Hashimoto et al. [18]) is

$$\gamma_K = \lim_{x\to\infty} \Big(\log x - \sum_{N\mathfrak{p}\le x} \frac{\log N\mathfrak{p}}{N\mathfrak{p} - 1} \Big). \tag{23}$$

It shows that the existence of (many) prime ideals in $\mathcal{O}_K$ of small norm has a decreasing effect on γ_K. Taking $K = \mathbb{Q}$ we obtain the well-known formula

$$\gamma_{\mathbb{Q}} = \gamma = \lim_{x\to\infty} \Big(\log x - \sum_{p\le x} \frac{\log p}{p-1} \Big). \tag{24}$$

2.4. L**-series factorizations.** In what follows, we fix an odd prime q and put $K = K_1 = \mathbb{Q}(\zeta_q)$. We want to use more explicit factorizations of Dedekind zeta functions. It is well-known that

$$\zeta_K(s) = \frac{1}{1-q^{-s}} \prod_{p\ne q} \Big(\frac{1}{1-p^{-sf_p}} \Big)^{(q-1)/f_p} = \zeta(s) \prod_{\chi\in X_1^*} L(s,\chi). \tag{25}$$

The first identity in (25) is a consequence of the Euler product identity (21) and the cyclotomic reciprocity law. For any prime $p \ne q$, we put $g_p = f_p/(f_p, m)$ and $g_p' = f_p/(f_p, 2m)$. Note that $g_p' = g_p/2$ if g_p is even and $g_p' = g_p$ otherwise. The following factorization result should be classical, but, to our surprise, we failed to find it in the (many!) algebraic number theory textbooks we consulted.

Proposition 3. If q is an odd prime and m divides $(q-1)/2$, then

$$\zeta_{K_m}(s) = \frac{1}{1-q^{-s}} \prod_{p\ne q} \Big(\frac{1}{1-p^{-sg_p}} \Big)^{(q-1)/(mg_p)} = \zeta(s) \prod_{\chi\in X_m^*} L(s,\chi), \tag{26}$$

and

$$\zeta_{K_{2m}}(s) = \frac{1}{1-q^{-s}} \prod_{p\ne q} \Big(\frac{1}{1-p^{-sg_p'}} \Big)^{(q-1)/(2mg_p')} = \zeta(s) \prod_{\chi\in X_{2m}^*} L(s,\chi) = \zeta(s) \prod_{\substack{\chi\in X_m^* \\ \chi \text{ even}}} L(s,\chi), \tag{27}$$

where X_m^* is defined in (19).

Proof. We recall that K_m is the associated field to X_m. The first identities in both (26) and (27) are a consequence of (21) and Lemma 3; the second ones follow from Theorem 4.3 of Washington [65]. The final identity in (27) follows on noting that $\chi \in X_{2m}$ if and only if $\chi \in X_m$ and χ is even (see Section 2.2.2). $\square$

By comparing local factors in Proposition 3 we immediately obtain the following corollary.

Corollary 5. *If q is an odd prime and m divides $(q-1)/2$, then*

$$\frac{\zeta_{K_{2m}}(s)^2}{\zeta_{K_m}(s)} = \frac{1}{1-q^{-s}} \prod_{2|g_p} \left(\frac{1+p^{-sg_p/2}}{1-p^{-sg_p/2}}\right)^{(q-1)/(mg_p)} = \zeta(s) \prod_{\chi \in X_m^*} L(s,\chi)^{\chi(-1)}. \tag{28}$$

Our next result links γ_{K_m} to the distribution of primes in residue classes modulo q.

Proposition 4. If m is a divisor of $q-1$, then

$$\begin{aligned}\gamma_{K_m} &= -\frac{\log q}{q-1} + \lim_{x\to\infty}\Big(\log x - \frac{q-1}{m} \sum_{\substack{n\le x\\ n^m\equiv 1 \pmod q}} \frac{\Lambda(n)}{n}\Big)\\ &= -\frac{\log q}{q-1} - \frac{q-1}{m}\sum_{g_p\ge 2}\frac{\log p}{p^{g_p}-1} + \lim_{x\to\infty}\Big(\log x - \frac{q-1}{m}\sum_{\substack{p\le x\\ p^m\equiv 1 \pmod q}}\frac{\log p}{p-1}\Big),\end{aligned}$$

where $\Lambda(n)$ is the von Mangoldt function, whose values are $\log p$ if $n = p^j$, with $j \ge 1$, and 0 otherwise.

First proof. Let χ be a non-principal character modulo q. As $x \to \infty$, we have the estimate

$$-\frac{L'}{L}(1,\chi) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n}\chi(n) = \sum_{n\le x}\frac{\Lambda(n)}{n}\chi(n) + o(1).$$

Further, we have the relation (see, e.g., [28, §55] or [64, Corollary 3.9]

$$\gamma = \log x - \sum_{n\le x}\frac{\Lambda(n)}{n} + o(1), \quad x\to\infty,$$

which also can be deduced from (24). Moreover, logarithmic differentiation of the L-function factorization from (26) yields

$$\gamma_{K_m} = \gamma + \sum_{\chi\in X_m^*}\frac{L'}{L}(1,\chi),$$

where we use the fundamental fact due to Dirichlet (1837) that $L(1,\chi)\neq 0$. On applying this identity and remarking that

$$\sum_{\substack{n\le x\\ (n,q)>1}}\frac{\Lambda(n)}{n} = \frac{\log q}{q-1} + o(1)$$

as $x\to\infty$, we now obtain the asymptotic estimates

$$\begin{aligned}\gamma_{K_m} &= \log x - \sum_{n\le x}\frac{\Lambda(n)}{n} - \sum_{\chi\in X_m^*}\sum_{n\le x}\frac{\Lambda(n)}{n}\chi(n) + o(1)\\ &= \log x - \frac{\log q}{q-1} - \sum_{\substack{n\le x\\ (n,q)=1}}\frac{\Lambda(n)}{n} - \sum_{n\le x}\frac{\Lambda(n)}{n}\sum_{\chi\in X_m^*}\chi(n) + o(1)\end{aligned}$$

$$= \log x - \frac{\log q}{q-1} - \sum_{n \le x} \frac{\Lambda(n)}{n} \sum_{\chi \in X_m} \chi(n) + o(1),$$

where in the last step we used that $\chi_0(n) = 1$ if $(n, q) = 1$, and $\chi_0(n) = 0$ otherwise. The first assertion now follows on using part (ii) of Lemma 4.

The second assertion follows from the first on noting that

$$\sum_{\substack{n \le x \\ n^m \equiv 1 \pmod{q}}} \frac{\Lambda(n)}{n} = \sum_{\substack{p^{kg_p} \le x \\ k \ge 1}} \frac{\log p}{p^{kg_p}} = \sum_{\substack{p \le x \\ p^m \equiv 1 \pmod{q}}} \frac{\log p}{p-1} + \sum_{g_p \ge 2} \frac{\log p}{p^{g_p} - 1} + E(x),$$

where

$$|E(x)| \le \sum_{a \ge 2} \sum_{p^a > x} \frac{\log p}{p^a} \le \sum_{a=2}^{\infty} \sum_{n > x^{1/a}} \frac{\log n}{n^a} \ll \sum_{a=2}^{\infty} \frac{\log x}{a^2 x^{1-1/a}} \ll \frac{\log x}{\sqrt{x}}. \qquad \square$$

Alternative proof of the second identity. Apply (23) with $K = K_m$ and the decomposition law in the field K_m given in Lemma 3. $\square$

Note that it is a consequence of Dirichlet's prime number theorem in arithmetic progressions that there exists at least one prime p with $f_p = m$ (in fact, there are infinitely many).

2.5. **The quadratic case.** Relevant for us will also be the particular case where K_m is quadratic. This occurs for $m = (q-1)/2$, when we have $K_{(q-1)/2} = \mathbb{Q}(\sqrt{q^*})$, where $q^* = (-1/q)q$, a field of discriminant q^*. Writing $\chi_{q^*}(\cdot)$ for the Kronecker symbol $\left(\frac{q^*}{\cdot}\right)$, we have

$$\zeta_{K_{(q-1)/2}}(s) = \zeta(s) L(s, \chi_{q^*}),$$

from which we infer that $\gamma_{K_{(q-1)/2}}$, the Euler-Kronecker constant of $K_{(q-1)/2}$, satisfies

$$\gamma_{K_{(q-1)/2}} = \gamma + \frac{L'}{L}(1, \chi_{q^*}). \tag{29}$$

If $q \equiv 3 \pmod{4}$, the field $K_{(q-1)/2}$ is imaginary and we can express $\gamma_{K_{(q-1)/2}}$ in terms of special values of the Dedekind η-function, see Ihara [21, Section 2.2]. Assuming that the Generalized Riemann Hypothesis (GRH) holds, in the same paper, Ihara also proved that $|\gamma_{K_{(q-1)/2}}| \le (2 + o(1)) \log\log q$. Murtada and Murty [38] proved that there are infinitely many q such that $|\gamma_{K_{(q-1)/2}}| \ge \log\log q + O(1)$, and that, under GRH, such a bound can be sharpened to $\log\log q + \log\log\log q + O(1)$. It is conjectured that for all the primes $q \le x$ we have $|\gamma_{K_{(q-1)/2}}| \le \log\log x + \log\log\log x + O(1)$. Further investigations in support of such a conjecture were performed by Lamzouri [26].

3. Preliminary results and proof of Theorem 1

For a prime q, we want to compute the number of positive integers $n \le x$ for which $q \nmid f(n)$, with $f(n) = n^b \sigma_k(n)$, $b \ge 0$ and $k \in \mathbb{N}$. The analysis will split in two cases, depending on whether $b = 0$ or $b \ge 1$. In the latter case, without loss of generalization we may take $b = 1$. As the case where $f(n) = n\sigma_k(n)$ is a trivial variation of the case $f(n) = \sigma_k(n)$, we will only consider it again in the proof of Theorem 1 (see Section 3.6). Let us therefore concentrate for now on studying $f(n) = \sigma_k(n)$.

3.1. **The Dirichlet series $T(s)$.** As already explained in Section 1, we let

$$T(s)=\sum_{n=1}^{\infty}\frac{t_n}{n^s}$$

be the associated Dirichlet series, where

$$t_n=\begin{cases}0 & \text{if } q\mid \sigma_k(n),\\ 1 & \text{if } q\nmid \sigma_k(n).\end{cases}$$

Since $\sigma_k(n)$ is multiplicative, so is t_n, and this further implies that $T(s)$ has an Euler product representation of the form

$$T(s)=\prod_p\sum_{j=0}^{\infty}\frac{t_{p^j}}{p^{js}},$$

where the product runs over all primes p. In light of this, it is enough to study the divisibility of the function $\sigma_k(n)$ by a (fixed) odd prime q only in case n is a prime power.

3.2. **Divisibility of σ_k by prime powers.** We want to determine when

$$\sigma_k(p^a)\equiv 0 \pmod{q}.$$

Since clearly $\sigma_k(q^a)\equiv 1 \pmod{q}$, we will assume from now on that $p\neq q$. We have

$$\sigma_k(p^a)=1+p^k+p^{2k}+\cdots+p^{ak}=\frac{p^{k(a+1)}-1}{p^k-1},$$

and we note that the only values of a for which $q\mid\sigma_k(p^a)$ are

$$\begin{cases}a\equiv -1 \pmod{q} & \text{if } f_p\mid k,\\ a\equiv -1 \pmod{h_p} & \text{if } f_p\nmid k,\end{cases}$$

where $h_p=\frac{f_p}{(f_p,k)}$. As $f_p\mid q-1$, we conclude that the only values of a for which $q\mid\sigma_k(p^a)$ are

$$\begin{cases}a\equiv -1 \pmod{q} & \text{if } f_p\mid r,\\ a\equiv -1 \pmod{g_p} & \text{if } f_p\nmid r,\end{cases}\tag{30}$$

where

$$r=(k,q-1)\quad\text{and}\quad g_p=\frac{f_p}{(f_p,r)}.$$

Note that g_p is the multiplicative order of p^r modulo q. This information can be combined into a single congruence by writing

$$a\equiv -1 \pmod{\mu_p},\tag{31}$$

where

$$\mu_p=\begin{cases}q & \text{if } g_p=1,\\ g_p & \text{if } g_p>1.\end{cases}\tag{32}$$

3.3. **An Euler product for $T(s)$.** On using (31) we obtain

$$T(s) = \frac{1}{1-q^{-s}} \prod_{p\neq q} \left(\sum_{j=0}^{\infty} p^{-js} - p^s \sum_{j=1}^{\infty} p^{-j\mu_p s} \right) = \frac{1}{1-q^{-s}} \prod_{p\neq q} \frac{1-p^{-(\mu_p-1)s}}{(1-p^{-s})(1-p^{-\mu_p s})}, \tag{33}$$

and so

$$T(s) = \zeta(s) \prod_{p\neq q} \frac{1-p^{-(\mu_p-1)s}}{1-p^{-\mu_p s}} = \zeta(s)\ D(s) \prod_{g_p=2} (1+p^{-s})^{-1}, \tag{34}$$

where

$$D(s) = \prod_{g_p\neq 2} \frac{1-p^{-(\mu_p-1)s}}{1-p^{-\mu_p s}}. \tag{35}$$

For notational convenience when using f_p, g_p and μ_p, we will always silently assume that $p \neq q$. (Thus, for instance, the product in (35) is taken over the primes $p\neq q$ with $g_p \neq 2$.) The generating series for $T(s)$ was first found by Rankin [45, eq. (11)].

Using logarithmic differentiation we obtain from (34) that

$$\frac{T'}{T}(s) = \frac{\zeta'}{\zeta}(s) + \frac{D'}{D}(s) + \sum_{g_p=2} \frac{\log p}{p^s} - \sum_{g_p=2} \frac{\log p}{p^s(p^s+1)}. \tag{36}$$

For later use, we record that

$$\frac{D'}{D}(1) = \sum_{g_p\neq 2} \frac{(\mu_p-1)\log p}{p^{\mu_p-1}-1} - \sum_{g_p\neq 2} \frac{\mu_p \log p}{p^{\mu_p}-1}. \tag{37}$$

3.4. **Reformulation using L-series.** Our aim is next to relate the first sum on the right-hand side of (36) to logarithmic derivatives of Dirichlet L-series.

Lemma 5. *We have*

$$\sum_{p\equiv a \ (\mathrm{mod}\ q)} \frac{\log p}{p^s} = -\frac{1}{q-1} \sum_{\chi} \overline{\chi}(a) \frac{L'}{L}(s,\chi) - \sum_{b\geq 2} \sum_{p^b\equiv a \ (\mathrm{mod}\ q)} \frac{\log p}{p^{bs}}.$$

Proof. Observe that

$$\sum_{n\equiv a \ (\mathrm{mod}\ q)} \frac{\Lambda(n)}{n^s} = \sum_{p\equiv a \ (\mathrm{mod}\ q)} \frac{\log p}{p^s} + \sum_{b\geq 2} \sum_{p^b\equiv a \ (\mathrm{mod}\ q)} \frac{\log p}{p^{bs}}.$$

We further obtain

$$\sum_{n\equiv a \ (\mathrm{mod}\ q)} \frac{\Lambda(n)}{n^s} = \frac{1}{q-1} \sum_{\chi} \overline{\chi}(a) \sum_{n\geq 1} \frac{\chi(n)\Lambda(n)}{n^s} = -\frac{1}{q-1} \sum_{\chi} \overline{\chi}(a) \frac{L'}{L}(s,\chi),$$

and the proof is completed on combining these two identities. ☐

From now on we assume that $h = (q-1)/r$ is even. This ensures that the equation $x^r \equiv -1 \ (\mathrm{mod}\ q)$ has precisely r solutions $a_1, \ldots, a_r$ with $0 < a_i < q$ (cf. Section 2.1). We observe that

$$\sum_{i} \sum_{b\geq 2} \sum_{p^b\equiv a_i \ (\mathrm{mod}\ q)} \frac{\log p}{p^{bs}} = \sum_{\substack{p^{br}\equiv -1 \ (\mathrm{mod}\ q) \\ b\geq 2}} \frac{\log p}{p^{bs}}.$$

By part 1) of Lemma 2, the contribution of a fixed prime p to the latter sum equals

$$\sum_{n=1}^{\infty} \frac{\log p}{p^{(2n+1)s}} \quad \text{if } g_p = 2, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{\log p}{p^{(2n-1)g_p s/2}} \quad \text{if } g_p \geq 4 \text{ is even,}$$

so that

$$\sum_{\substack{p^{br}\equiv -1 \ (\mathrm{mod}\ q)\\ b\geq 2}} \frac{\log p}{p^{bs}} = \sum_{g_p=2} \frac{\log p}{p^s(p^{2s}-1)} + \sum_{\substack{g_p\geq 4\\ 2|g_p}} \frac{\log p}{p^{g_p s/2}-p^{-g_p s/2}}.$$

Using Lemma 5 and the latter identity, we obtain

$$-\sum_{g_p=2} \frac{\log p}{p^s} = \frac{r\lambda(s)}{q-1} + \sum_{g_p=2} \frac{\log p}{p^s(p^{2s}-1)} + \sum_{\substack{g_p\geq 4\\ 2|g_p}} \frac{\log p}{p^{g_p s/2}-p^{-g_p s/2}},$$

where

$$\lambda(s) = \frac{1}{r}\sum_{\chi} \frac{L'}{L}(s,\chi)\sum_{i=1}^{r} \overline{\chi}(a_i).$$

Combining this with (36) we obtain

$$\frac{T'}{T}(s) = \frac{\zeta'}{\zeta}(s) - \frac{r\lambda(s)}{q-1} - v(s), \tag{38}$$

where

$$v(s) = -\frac{D'}{D}(s) + \sum_{g_p=2} \frac{\log p}{p^s(p^s+1)} + \sum_{g_p=2} \frac{\log p}{p^s(p^{2s}-1)} + \sum_{\substack{g_p\geq 4\\ 2|g_p}} \frac{\log p}{p^{g_p s/2}-p^{-g_p s/2}}.$$

By (20) we have

$$\lambda(s) = \frac{1}{r}\sum_{\chi} \frac{L'}{L}(s,\chi)\Big(\sum_{a\in C_{2r}} \overline{\chi}(a) - \sum_{a\in C_r} \overline{\chi}(a)\Big),$$

and so, by part (i) of Lemma 4,

$$\lambda(s) = 2\sum_{\chi\in X_{2r}} \frac{L'}{L}(s,\chi) - \sum_{\chi\in X_r} \frac{L'}{L}(s,\chi).$$

As $L(s,\chi_0) = \zeta(s)(1-q^{-s})$, we get

$$\frac{L'}{L}(s,\chi_0) = \frac{\zeta'}{\zeta}(s) + \frac{\log q}{q^s-1},$$

therefore

$$\lambda(s) = \frac{\log q}{q^s-1} + 2\Big(\frac{\zeta'}{\zeta}(s) + \sum_{\chi\in X_{2r}^*} \frac{L'}{L}(s,\chi)\Big) - \Big(\frac{\zeta'}{\zeta}(s) + \sum_{\chi\in X_r^*} \frac{L'}{L}(s,\chi)\Big).$$

By Proposition 3, this can be rewritten as

$$\lambda(s) = \frac{\log q}{q^s-1} + 2\frac{\zeta'_{K_{2r}}}{\zeta_{K_{2r}}}(s) - \frac{\zeta'_{K_r}}{\zeta_{K_r}}(s),$$

which, in combination with (38), yields

$$\frac{T'}{T}(s) = \frac{\zeta'}{\zeta}(s) - \frac{r}{q-1}\Big(\frac{\log q}{q^s-1} + 2\frac{\zeta'_{K_{2r}}}{\zeta_{K_{2r}}}(s) - \frac{\zeta'_{K_r}}{\zeta_{K_r}}(s)\Big) - v(s).$$

3.5. **The Euler-Kronecker constant $\gamma_{k,q}$.** For $K=\mathbb{Q}$, K_r and K_{2r} we replace $\zeta_K'/\zeta_K(s)$ by the estimate given in (22). Noting that $v(1)=S(r,q)$, we then obtain, on adding

$$\Big(1-\frac{r}{q-1}\Big)\frac{1}{s-1}$$

to both sides of the resulting identity and on taking the limit as $s\to 1^+$, that

$$\gamma_T=\gamma_{k,q}=\gamma-\frac{r}{q-1}\Big(\frac{\log q}{q-1}+2\gamma_{K_{2r}}-\gamma_{K_r}\Big)-S(r,q).$$

We have thus established the following lemma.

Lemma 6. *Let $r=(k,q-1)$. If $(q-1)/r$ is even, then*

$$\gamma_{k,q}=\gamma-\frac{r}{q-1}\Big(2\gamma_{K_{2r}}-\gamma_{K_r}+\frac{\log q}{q-1}\Big)-S(r,q), \tag{39}$$

where $S(r,q)$ is defined in (15).

Second proof of Lemma 6. Our starting point is the Euler product from (34), which we want to express in terms of Dedekind zeta functions. We do this on using (28), which on splitting off the term with $g_p=2$ rewrites as

$$\frac{\zeta_{K_{2r}}(s)^2}{\zeta_{K_r}(s)}=(1-q^{-s})^{-1}\prod_{g_p=2}(1+p^{-s})^h\prod_{g_p=2}(1-p^{-2s})^{-h/2}E(s)^h,$$

with

$$E(s):=\prod_{\substack{g_p\ge 4\\ 2|g_p}}\Big(\frac{1+p^{-sg_p/2}}{1-p^{-sg_p/2}}\Big)^{1/g_p}.$$

Combining this with (34) yields

$$T(s)^h=(1-q^{-s})^{-1}\zeta(s)^hH(s)^{h/2}\zeta_{K_r}(s)\zeta_{K_{2r}}(s)^{-2}, \tag{40}$$

where

$$H(s):=(D(s)E(s))^2\prod_{g_p=2}(1-p^{-2s})^{-1}.$$

Taking the Laurent series around $s=1$, we obtain

$$\frac{T'}{T}(s)+\Big(1-\frac{1}{h}\Big)\frac{1}{s-1}=\gamma+\frac{1}{2}\frac{H'}{H}(s)-\frac{1}{h}(2\gamma_{K_{2r}}-\gamma_{K_r})-\frac{\log q}{h(q^s-1)}+O(s-1), \tag{41}$$

where we used (22) for each of the three zeta functions involved. We obtain

$$\frac{1}{2}\frac{H'}{H}(1)=\frac{D'}{D}(1)+\frac{E'}{E}(1)-\sum_{g_p=2}\frac{\log p}{p^2-1},$$

with

$$\frac{E'}{E}(1)=-\sum_{\substack{g_p\ge 4\\ 2|g_p}}\frac{\log p}{p^{g_p/2}-p^{-g_p/2}},$$

which, on recalling (37) and (15), shows that

$$\frac{1}{2}\frac{H'}{H}(1)=-S(r,q).$$

We infer that the limit $s\to 1^+$ of the right-hand side in (41) exists and equals the right-hand side of (39). The result then follows on invoking (3) with $\alpha=1-1/h$. ☐

3.6. **The proof of Theorem 1.** With Lemma 6 at our disposal, we are ready to prove Theorem 1.

Proof. We consider $S_{k,q}(x)$ first. The idea is to apply Theorem 1 with $S = \{n : q \nmid \sigma_k(n)\}$, which is a multiplicative set. By Proposition 1 it follows that $S = \{n : q \nmid \sigma_r(n)\}$, with $r = (k, q-1)$. The assumption on h ensures, see Lemma 1, that the equation $x^r \equiv -1 \pmod q$ has r solutions modulo q. A prime p is in S if and only if $p^r \not\equiv -1 \pmod q$. It follows that p is in S if and only if $p = q$ or is in a union of $q-1-r$ arithmetic progressions modulo q. By a strong enough version of the Prime Number Theorem in arithmetic progressions, we then see that (6) is satisfied with $\delta = 1 - r/(q-1)$. Since $L_S(s) = T(s)$, as defined in (33), we infer that $\gamma_S = \gamma_{r,q} = \gamma_{k,q}$. The proof of this case is completed on invoking Lemma 6.

For $S'_{k,q}(x)$ the factor $(1-q^{-s})^{-1}$ in the generating series is not there anymore, and so the associated generating series $T'(s)$ satisfies $T'(s) = (1-q^{-s})T(s)$. Logarithmic differentiation then yields

$$\gamma'_{k,q} = \gamma_{T'} = \frac{\log q}{q-1} + \gamma_T = \frac{\log q}{q-1} + \gamma_{k,q},$$

completing the proof. □

3.7. **The case $q = 2$.** Let $k \geq 1$ be arbitrary. We start by noting that, since $\sigma_k(n) \equiv \sigma_1(n) \pmod 2$, there is no dependency on k. It is not difficult to see that, in the cases $2 \nmid \sigma_k(n)$ and $2 \nmid n\sigma_k(n)$, the generating series equal

$$\frac{1}{1-2^{-s}} \prod_{p>2} \frac{1}{1-p^{-2s}} = (1+2^{-s})\zeta(2s) \quad \text{and} \quad \prod_{p>2} \frac{1}{1-p^{-2s}} = (1-2^{-2s})\zeta(2s),$$

respectively. The functions $S_{k,2}(x)$ and $S'_{k,2}(x)$ count the number of integers of the form $2^e(2m+1)^2 \leq x$ with $e, m \geq 0$, respectively the number of odd squares not exceeding x. It is then an easy exercise to show that

$$S_{k,2}(x) = \Big(1 + \frac{1}{\sqrt{2}}\Big)\sqrt{x} + O(\log x) \quad \text{and} \quad S'_{k,2}(x) = \frac{1}{2}\sqrt{x} + O(1).$$

3.8. **The case $2 \nmid h$.** Let q be an odd prime and $k \geq 1$ an integer. Put $r = (k, q-1)$ and $h = (q-1)/r$. The asymptotic behavior of $S_{k,q}(x)$ in case h is odd was first determined by Rankin [45]. Since $p^{q-1} = p^{rh} \equiv (-1)^h \pmod q$ it follows that $p^r \not\equiv -1 \pmod q$, and so $g_p \neq 2$, and thus (34) simplifies to

$$T(s) = \zeta(s)D(s),$$

where $D(s)$ is defined in (35). It follows that, asymptotically,

$$S_{k,q}(x) \sim D(1)\,x \quad \text{and} \quad S'_{k,q}(x) \sim \Big(1 - \frac{1}{q}\Big)D(1)\,x.$$

Further,

$$\gamma_{k,q} = \gamma + \frac{D'}{D}(1) \quad \text{and} \quad \gamma'_{k,q} = \gamma + \frac{D'}{D}(1) + \frac{\log q}{q-1},$$

with $D'/D(1)$ as in (37).

3.9. **The constants $C_{k,q}$ and $C'_{k,q}$.** Let k, q, r and h be as in Sec. 3.8. Assume that h is even. Recall the definition of X_r^* in (19).

Proposition 5. We have

$$C_{k,q} = \frac{(1-q^{-1})^{-1/h}}{\Gamma(1-1/h)} \prod_{\chi \in X_r^*} L(1,\chi)^{-\chi(-1)/h} \mathfrak{c}_{r,q}, \; C'_{k,q} = \Big(1 - \frac{1}{q}\Big)C_{k,q},$$

with

$$\mathfrak{c}_{r,q} = \prod_{g_p=1} \frac{1-p^{-(q-1)}}{1-p^{-q}} \prod_{g_p=2} \frac{1}{\sqrt{1-p^{-2}}} \prod_{g_p\ge 3} \frac{1-p^{-(g_p-1)}}{1-p^{-g_p}} \prod_{\substack{g_p\ge 4\\ 2|g_p}} \Big(\frac{1+p^{-g_p/2}}{1-p^{-g_p/2}}\Big)^{1/g_p}.$$

Proof. From (40) and (5), we deduce that $T(s)^h = \zeta(s)^{h-1}R(s)$, for some function $R(s)$ that is regular for $\mathrm{Re}(s) > 1/2$ and can be explicitly written down. By a standard application of the (Landau)-Selberg-Delange method, see, e.g., Tenenbaum [64, Chapter II.5], we obtain,

$$C_{k,q} = \frac{R(1)^{1/h}}{\Gamma(1-1/h)},$$

and the proof is easily completed (the details are left to the reader). □

Remark 7. This agrees with Rankin [45, eq. (16)]. However, the constant $C_{1,5}$, which he worked out as an example (and called A), contains a typo; for L_4, in his formula for A, one should read $4L_4$. The constant $C'_{1,5}$ was independently computed by Moree [34].

3.10. **The Euler-Kronecker constant** $\gamma_{(q-1)/2,\,q}$. Let q be an odd prime. As $p^{\frac{q-1}{2}} \equiv \big(\frac{p}{q}\big) \pmod q$ and g_p is the multiplicative order of $p^{\frac{q-1}{2}}$ modulo q, we infer that

$$g_p = \begin{cases} 1 & \text{if } (\frac{p}{q}) = 1, \\ 2 & \text{otherwise.} \end{cases}$$

In this case, formula (33) specializes to

$$T(s) = \frac{1}{1-q^{-s}} \prod_{(\frac{p}{q})=-1} \frac{1}{1-p^{-2s}} \prod_{(\frac{p}{q})=1} \frac{1-p^{-(q-1)s}}{(1-p^{-s})(1-p^{-qs})}. \tag{42}$$

Put $q^* = \big(\frac{-1}{q}\big)q$. Using quadratic reciprocity in the form $\big(\frac{p}{q}\big) = \big(\frac{q^*}{p}\big)$, we infer that

$$T(s)^2 = \frac{\zeta(s)L(s,\chi_{q^*})}{1-q^{-s}} \prod_{(\frac{p}{q})=-1} \frac{1}{1-p^{-2s}} \prod_{(\frac{p}{q})=1} \Big(\frac{1-p^{-(q-1)s}}{1-p^{-qs}}\Big)^2.$$

By the (Landau)-Selberg-Delange method we obtain, noting that $\Gamma(1/2) = \sqrt{\pi}$,

$$C_{\frac{q-1}{2},q} = \sqrt{\frac{qL(1,\chi_{q^*})}{\pi(q-1)}} \prod_{(\frac{p}{q})=-1} \frac{1}{\sqrt{1-p^{-2}}} \prod_{(\frac{p}{q})=1} \frac{1-p^{-(q-1)}}{1-p^{-q}},\ C'_{\frac{q-1}{2},q} = \Big(1-\frac{1}{q}\Big)C_{\frac{q-1}{2},q}.$$

We leave it to the interested reader to check that this coincides with the formulas given in Proposition 5 on setting $k = (q-1)/2$.

We recall that $K_{(q-1)/2} = \mathbb{Q}(\sqrt{q^*})$. Using Theorem 1 and (29) we obtain

$$\gamma_{\frac{q-1}{2},q} = \frac{1}{2}\gamma_{K_{\frac{q-1}{2}}} - \frac{\log q}{2(q-1)} - S\Big(\frac{q-1}{2},q\Big) = \frac{\gamma}{2} + \frac{1}{2}\frac{L'}{L}(1,\chi_{q^*}) - \frac{\log q}{2(q-1)} - S\Big(\frac{q-1}{2},q\Big).$$

Since

$$S\Big(\frac{q-1}{2},q\Big) = -\sum_{(\frac{p}{q})=1} \log p \,\Big(\frac{q-1}{p^{q-1}-1} - \frac{q}{p^q-1}\Big) + \sum_{(\frac{p}{q})=-1} \frac{\log p}{p^2-1},$$

by formula (15), with $r = (q-1)/2$, we finally obtain

$$\gamma_{\frac{q-1}{2},q} = \frac{\gamma}{2} + \frac{1}{2}\frac{L'}{L}(1,\chi_{q^*}) - \frac{\log q}{2(q-1)} - \sum_{(\frac{p}{q})=-1} \frac{\log p}{p^2-1} + \sum_{(\frac{p}{q})=1} \log p \,\Big(\frac{q-1}{p^{q-1}-1} - \frac{q}{p^q-1}\Big). \tag{43}$$

By Proposition 48 we have $S((q-1)/2,q)>0$. In Section 10 we will describe how to efficiently compute $\gamma_{\frac{q-1}{2},q}$ with high accuracy.

3.10.1. **Cusp form applications.** Let $q\in\{3,7\}$. We consider the non-divisibility of τ by q. Using (43) it can be verified that the formulas for the corresponding Euler-Kronecker constants $-B_t$, as given by Moree [34], satisfy

$$-B_t=\gamma'_{\frac{q-1}{2},q}=\gamma_{\frac{q-1}{2},q}+\frac{\log q}{q-1},$$

as expected. Another relevant case is $q=11$, associated to the form $R\Delta$. Finally, the cases $q=23$ and $q=31$ are relevant for the type (ii) congruences, see Section 4.4.

γ	value
$\gamma_{1,2}$	$-1.370971\ldots$
$\gamma_{1,3}$	$-0.014384\ldots$
$\gamma_{1,5}$	$-0.002812\ldots$
$\gamma_{2,5}$	$0.046145\ldots$
$\gamma_{1,7}$	$0.388115\ldots$
$\gamma_{3,7}$	$-0.092678\ldots$
$\gamma_{1,11}$	$0.282623\ldots$
$\gamma_{5,11}$	$-0.195292\ldots$
$\gamma_{1,13}$	$0.400611\ldots$
$\gamma_{2,13}$	$0.581080\ldots$
$\gamma_{3,13}$	$-0.019200\ldots$
$\gamma_{6,13}$	$0.030107\ldots$

TABLE 3. Euler-Kronecker constants for the smallest primes

4. DIVISIBILITY BY EXCEPTIONAL PRIMES AND PROOF OF THEOREM 5

Recall that Serre and Swinnerton-Dyer proved that the exceptional congruences are of one of the types:

(i) $\tau_w(n)\equiv n^v\sigma_{w-1-2v}(n) \pmod q$ for all $(n,q)=1$, and for some $v\in\{0,1,2\}$.
(ii) $\tau_w(n)\equiv 0 \pmod q$ whenever $\left(\frac{n}{q}\right)=-1$.
(iii) $p^{1-w}\tau_w^2(p)\equiv 0,1,2$ or $4 \pmod q$ for all primes $p\neq q$.

The goal of this section is to prove Theorem 5, our main result on the divisibility of Fourier coefficients of cusp forms. To this end, we invoke Theorem 1 and its corollary for the exceptional primes satisfying condition (i). For primes of type (ii) we have the case $w=12$, $q=23$, already worked out in 2004 by Moree [34], and the case $w=16$, $q=31$, which we work out in Section 4.4. Our techniques do not apply to the primes of type (iii), which satisfy a different sort of congruence criterion (see Section 4.5), and we must therefore skip their analysis.

4.1. **Congruences of type (i).** The exceptional primes $q>w$ all have $v=0$ and are given in Table 4. For $q<w$, Table 5 gives the value of v if q is exceptional, or the word 'No' if not. These tables are taken from Swinnerton-Dyer [62, 63].

w	12	16	18	20	22	26
Form	Δ	$Q\Delta$	$R\Delta$	$Q^2\Delta$	$QR\Delta$	$Q^2R\Delta$
q	691	3617	43867	283, 617	131, 593	657931

TABLE 4. Type (i): Exceptional primes with $q > w$

Form	w	$q=2$	3	5	7	11	13	17	19	23
Δ	12	0	0	1	1	No				
$Q\Delta$	16	0	0	1	1	1	No			
$R\Delta$	18	0	0	2	1	1	1	No		
$Q^2\Delta$	20	0	0	1	2	1	1	No	No	
$QR\Delta$	22	0	0	2	1	No	1	1	No	
$Q^2R\Delta$	26	0	0	2	2	1	No	1	1	No

TABLE 5. Type (i): Value of v for the exceptional primes with $q < w$

4.2. **The behavior of $\tau_w(q)$ for exceptional primes q.** The analysis of Swinnerton-Dyer only pertains to those integers n coprime to the exceptional prime q. We also need to understand the q-divisibility of $\tau_w(q^e)$ for all natural numbers $e \geq 1$. By part (2) of Classical Theorem 3 we have $\tau_w(q^e) \equiv \tau_w(q)^e \pmod{q}$, and so either all $\tau_w(q^e)$ are q-divisible, or none is. Using a program by Martin Raum (Julia/Nemo), but also independently, using Pari/Gp [43], we computed $\tau_w(q)$ modulo q.

Numerical Observation 1. Let q be an exceptional prime for a congruence for τ_w of type (i). If $q < w$, then $q \mid \tau_w(q)$. If $q > w$, then $\tau_w(q) \equiv 1 \pmod{q}$.

Using this numerical fact, the exceptional congruences of type (i) can be easily "lifted" to all integers n.

Proposition 6. Let q be exceptional of type (i) for τ_w. If $q < w$, then $\tau_w(n) \equiv n^{\max\{1,v\}}\sigma_r(n) \pmod{q}$ with $r = (w-1-2v, q-1)$ and v as in Table 5. If $q > w$, then $\tau_w(n) \equiv \sigma_r(n) \pmod{q}$ with $r = (w-1, q-1)$.

Proof. For $v \geq 1$ the first assertion follows since, by assumption, it holds for $(n,q) = 1$ and, in addition, $q \mid \tau_w(q)$. This implies that both sides of the congruence are divisible by q if $(n,q) > 1$. Next, assume $v = 0$. By Table 5 we have $q = 2$ or $q = 3$. Let $r = (w-1, q-1)$. For n odd we have $\tau_w(n) \equiv \sigma_r(n) \equiv n\sigma_r(n) \pmod{2}$. As $\tau_w(2)$ is even, we also have $\tau_w(n) \equiv n\sigma_r(n) \pmod{2}$ for even n. Along the same lines, one checks that $\tau_w(n) \equiv n\sigma_r(n) \pmod{3}$ for $n \not\equiv 2 \pmod{3}$. We claim that $\tau_w(n) \equiv \sigma_r(n) \equiv 0 \equiv n\sigma_r(n) \pmod{3}$ for $n \equiv 2 \pmod{3}$. Such n have a prime power divisor p^e with $p \equiv 2 \pmod{3}$, $p^{e+1} \nmid n$ and $2 \nmid e$. Using the fact that r is odd, we see that $\sigma_r(p^e) \equiv \sum_{j=0}^{e}(-1)^{jr} \equiv 0 \pmod{3}$, and hence $3 \mid \sigma_r(n)$.

In case $q > w$, we have $v = 0$. The assertion follows on noting that the congruence holds for $(n,q) = 1$, and that, in addition, we have $\tau_w(q^e) \equiv \tau_w(q)^e \equiv 1 \equiv \sigma_r(q^e) \pmod{q}$, for every $e \geq 1$, by Numerical Observation 1. □

Recalling Definition 1, we obtain the following corollary.

Corollary 6. *Let q be exceptional of type* (i) *for τ_w. If $q < w$, then $\tau_w(n) \cong n\sigma_r(n) \pmod{q}$ with $r = (w-1-2v, q-1)$ and v as in Table 5. If $q > w$, then $\tau_w(n) \cong \sigma_r(n) \pmod{q}$ with $r = (w-1, q-1)$.*

Remark 8. It is a classical result that $\tau(n) \equiv n\sigma_1(n) \pmod{6}$. Since, coefficient-wise, $Q \equiv R \equiv 1 \pmod{6}$, we infer that $\tau_w(n) \equiv \tau(n) \equiv n\sigma_1(n) \pmod{6}$.

4.2.1. **The case $q < w$.** Corollary 6 makes clear that, disregarding theoretical considerations, working with r (rather than v) is what matters. Doing so leads to Table 6, a variant of Table 5. In Table 7 we give the associated Euler-Kronecker constants with six decimal accuracy.

Form	w	$q=2$	3	5	7	11	13	17	19	23
Δ	12	1	1	1	3	No				
$Q\Delta$	16	1	1	1	1	1	No			
$R\Delta$	18	1	1	1	3	5	3	No		
$Q^2\Delta$	20	1	1	1	3	1	1	No	No	
$QR\Delta$	22	1	1	1	1	No	1	1	No	
$Q^2R\Delta$	26	1	1	1	3	1	No	1	1	No

TABLE 6. Type (i): Value of r for the exceptional primes $q < w$

r	q	$\gamma'_{r,q}$
1	2	$-0.677823\ldots$
1	3	$0.534921\ldots$
1	5	$0.399547\ldots$
1	7	$0.712434\ldots$
3	7	$0.231640\ldots$
1	11	$0.522413\ldots$
5	11	$0.044497\ldots$
1	13	$0.614357\ldots$
3	13	$0.194544\ldots$
1	17	$0.518971\ldots$
1	19	$0.720414\ldots$

TABLE 7. Type (i): Euler-Kronecker constants for $q < w$ related to Table 6

4.2.2. **The case $q > w$.** In this case $v = 0$, $r = (w-1, q-1)$ and the relevant table is Table 8.

form	w	r	q	$\gamma_{r,q}$
Δ	12	1	691	$0.571714\ldots$
$Q\Delta$	16	1	3617	$0.574566\ldots$
$R\Delta$	18	1	43867	$0.57669\ldots.$
$Q^2\Delta$	20	1	283	$0.552571\ldots$
$Q^2\Delta$	20	1	617	$0.567565\ldots$
$QR\Delta$	22	1	131	$0.532695\ldots$
$QR\Delta$	22	1	593	$0.568078\ldots$
$Q^2R\Delta$	26	5	657931	$0.57701\ldots.$

TABLE 8. Type (i): Euler-Kronecker constants for $q > w$ related to Table 4

The computational effort in producing this table was substantial. The computation for $\gamma_{5,657931}$ took the longest, namely about 6 days and 14 hours (Dell OptiPlex-3050 equipped with an Intel i5-7500 processor, 3.40GHz, 16 GB of RAM and running Ubuntu 18.04.5) to determine the value of $S(5, 657931)$; the computation for $\gamma_{K_5}(657931)$ and $\gamma_{K_{10}}(657931)$ took less than 1 second on the same machine. Despite this, we were not able to get more than 5 certified decimal digits.

The computation for $\gamma_{1,43867}$ took less time, namely about 4 days and 15 hours; in this case we were not able to get more than 5 certified decimal digits either.

4.2.3. **The case $f = \Delta$.** In Table 9 we recomputed, with higher precision, the values found in 2004 by Moree [34] for Δ (we give the values of Moree in our notation, which amounts to multiplying his values by minus one). The congruence for $q = 23$ is of type (ii) and is discussed in Section 4.4.

q	type	γ	value	[34]
2	(i)	$\gamma'_{1,2}$	$-0.677823\ldots$	
3	(i)	$\gamma'_{1,3}$	$0.534921\ldots$	$0.5349\ldots$
5	(i)	$\gamma'_{1,5}$	$0.399547\ldots$	$0.3995\ldots$
7	(i)	$\gamma'_{3,7}$	$0.231640\ldots$	$0.2316\ldots$
23	(ii)		$0.216691\ldots$	$0.2166\ldots$
691	(i)	$\gamma_{1,691}$	$0.571714\ldots$	$0.5717\ldots$

Table 9. Euler-Kronecker constants related to Δ

4.3. **The case $q = 2$.** By Proposition 6, cf. Remark 8, we have $\tau_w(n) \equiv n\sigma_1(n) \pmod 2$. Hence $\tau_w(n)$ is odd if and only if n is an odd square, and so

$$\sum_{2\nmid\tau_w(n)} 1 = \frac{1}{2}\sqrt{x} + O(1),$$

see also Sec. 3.7.

4.4. **Congruences of type (ii).** The case $w = 12$ and $q = 23$ is of this type and the analytic number theoretical aspects of the non-divisibility of $\tau(n)$ by 23 are discussed by Ramanujan [3] and Moree [34]. There is only one further case of this type, namely $w = 16$ and $q = 31$. The determination of the Euler-Kronecker constant that we present here works in the same way for $q = 23$ and $q = 31$, and is based on the congruences

$$\tau_w(p) \equiv \begin{cases} 1 \pmod q & \text{if } p = q; \\ 0 \pmod q & \text{if } \left(\frac{p}{q}\right) = -1; \\ -1 \pmod q & \text{if } p = 2X^2 + XY + wY^2/4; \\ 2 \pmod q & \text{if } p = X^2 + XY + wY^2/2, \end{cases} \tag{44}$$

where $w = (q+1)/2$, see Swinnerton-Dyer [62, p. 34], [63, p. 301] or Serre [57] (for $q = 23$). In 1930, a short proof using q-series was given by Wilton [67] for the exceptional prime 23, who also determined the values $\tau(n)$ modulo 23 for every positive integer n. According to Rankin [46], more modern proofs are based on the fact that $\eta(z)\eta(23z)$ is a newform for the group $\Gamma_0(23)$ with multiplier system given by the character $\chi(n) = \left(\frac{n}{23}\right)$. Denote by N_p the number of distinct roots modulo p of the polynomial $x^3 - x - 1$. It is known that $\tau(p) \equiv N_p - 1 \pmod{23}$, cf. Serre [57, p. 437] or [7, pp. 42–43].

Let $\mathcal{S}_1$ denote the set of primes p with $\left(\frac{p}{q}\right) = -1$. Let $\mathcal{S}_2$ and $\mathcal{S}_3$ be the (disjoint) sets of primes represented by the quadratic forms $2X^2 + XY + wY^2/4$, respectively $X^2 + XY + wY^2/2$. Note that the primes p in $\mathcal{S}_2 \cup \mathcal{S}_3$ satisfy $\left(\frac{p}{q}\right) = 1$.

By part (2) of Classical Theorem 3 we have

$$\tau_w(p^{e+1}) = \tau_w(p)\tau_w(p^e) - p^{w-1}\tau_w(p^{e-1}) \equiv \tau_w(p)\tau_w(p^e) - \left(\frac{p}{q}\right)\tau_w(p^{e-1}) \pmod q,$$

e	0	1	2	3	4	5
$p=q$	1	1	1	1	1	1
$p\in\mathcal{S}_1$	1	0	1	0	1	0
$p\in\mathcal{S}_2$	1	-1	0	1	-1	0
$p\in\mathcal{S}_3$	1	2	3	4	5	6

TABLE 10. Value of $\tau_w(p^e)$ modulo q

with $e\ge 1$. Using this recurrence we can easily compute $\tau_w(p^e)$ modulo q, see Table 10. We then deduce that

$$T_{(ii)}(s) := \sum_{q\nmid\tau_w(n)} \frac{1}{n^s} = \frac{1}{1-q^{-s}} \prod_{p\in\mathcal{S}_1} \frac{1}{1-p^{-2s}} \prod_{p\in\mathcal{S}_2} \frac{1+p^{-s}}{1-p^{-3s}} \prod_{p\in\mathcal{S}_3} \frac{1-p^{-(q-1)s}}{(1-p^{-s})(1-p^{-qs})}. \tag{45}$$

By quadratic reciprocity we have

$$\prod_p \frac{1}{1-(\frac{p}{q})p^{-s}} = \prod_p \frac{1}{1-(\frac{-q}{p})p^{-s}} = L(s,\chi_{-q}),$$

where χ_{-q} denotes the quadratic character associated with the Kronecker symbol $\left(\frac{-q}{\cdot}\right)$. Comparison of local factors then shows that the identity

$$T_{(ii)}(s)^2 = \frac{\zeta(s)L(s,\chi_{-q})}{1-q^{-s}} \prod_{p\in\mathcal{S}_1} \frac{1}{1-p^{-2s}} \prod_{p\in\mathcal{S}_2} \left(\frac{1-p^{-2s}}{1-p^{-3s}}\right)^2 \prod_{p\in\mathcal{S}_3} \left(\frac{1-p^{-(q-1)s}}{1-p^{-qs}}\right)^2$$

holds true. Computing the logarithmic derivatives of both sides and taking their limits for $s\to 1^+$, we easily deduce that

$$\gamma_{T_{(ii)}} = \frac{\gamma}{2} + \frac{1}{2}\frac{L'}{L}(1,\chi_{-q}) - \frac{\log q}{2(q-1)} - \sum_{(\frac{p}{q})=-1} \frac{\log p}{p^2-1} + \sum_{p\in\mathcal{S}_2} \log p \left(\frac{2}{p^2-1} - \frac{3}{p^3-1}\right)$$
$$+ \sum_{p\in\mathcal{S}_3} \log p \left(\frac{q-1}{p^{q-1}-1} - \frac{q}{p^q-1}\right). \tag{46}$$

We now want to derive (46) in a different way and, to do so, we start by noticing that (44) can be more compactly written as

$$\tau_w(p) \equiv \begin{cases} \sigma_{(q-1)/2}(p) \pmod q & \text{if } \left(\frac{p}{q}\right)\ne 1 \text{ or } p = X^2+XY+wY^2/2, \\ -1 \pmod q & \text{for all other } p. \end{cases}$$

Let $T(s)$ be the generating series associated to the set $\{n\ge 1 : q\nmid \sigma_{(q-1)/2}(n)\}$. Note that $\gamma_T = \gamma_{(q-1)/2,\,q}$. Comparison of the generating series (47) and (42) shows that

$$T_{(ii)}(s) = T(s) \prod_{p\in\mathcal{S}_2} \frac{(1-p^{-qs})(1-p^{-2s})}{(1-p^{-(q-1)s})(1-p^{-3s})}, \tag{47}$$

which by logarithmic differentiation leads to

$$\gamma_{T_{(ii)}} = \gamma_{(q-1)/2,\,q} + \sum_{p\in\mathcal{S}_2} \log p \left(\frac{2}{p^2-1} - \frac{3}{p^3-1} + \frac{q}{p^q-1} - \frac{q-1}{p^{q-1}-1}\right).$$

On inserting the expression (43) for $\gamma_{(q-1)/2,\,q}$ in the above identity, one obtains (46) upon simplification.

form	w	q	value	Moree [34]
Δ	12	23	$0.216691\ldots$	$0.2166\ldots$
$Q\Delta$	16	31	$0.156105\ldots$	

TABLE 11. Euler-Kronecker constants for type (ii) congruences

Remark 9. The convergence acceleration technique presented in Section 10.2 can be used for the sum over the primes in $\mathcal{S}_1$, but not for the prime sums over the two other sets. Thus, in practice, nothing truly changes for this problem. To get six confirmed decimal digits in Table 11 we truncated the prime sums at $P = 10^9$; each computation required about five minutes using Pari/Gp.

4.5. **Congruences of type (iii).** Haberland [16], using Galois cohomological methods, in part III of a series of papers, proved that the case $w = 16$ and $q = 59$ is of this type. He thus established a conjecture of Swinnerton-Dyer who had earlier proved that there cannot be further cases of this type. Later Boylan [6], and Kiming and Verrill [23] gave different proofs. The relevant algebraic field is non-abelian with a non solvable Galois group, and so a factorization of $T(s)$ as given in this paper, solely in terms of Dirichlet L-series and a regular factor, is not expected to exist. We have to leave computing the associated Euler-Kronecker constant as an open problem.

4.6. **Non-divisibility for non-exceptional primes.** The Fourier coefficient $\tau_w(p)$ can be computed by evaluating it modulo q for enough small prime q and using the bound $|\tau_w(p)| \le 2p^{(w-1)/2}$. The main result of the book [12] is that this can be done in polynomial time in $\log p$. This requires also studying congruences for non-exceptional primes, which turns out to be way more difficult than for the exceptional primes and is worked out in a relatively explicit way by Bosman [12, Ch. 7] for some small primes. Put

$$g(x) = x^{12} - 4x^{11} + 55x^9 - 165x^8 + 264x^7 - 341x^6 + 330x^5 - 165x^4 - 55x^3 + 99x^2 - 41x - 111.$$

He proves, for example that for $q \neq 11$ we have $11 \mid \tau(q)$ if and only if the prime q decomposes in the number field $\mathbb{Q}[x]/(g(x))$ as a product of primes of degree 1 and 2, with degree 2 occurring at least once. He uses these results to show that if $\tau(n) = 0$, then $n > 2 \cdot 10^{19}$, making some progress towards Lehmer's conjecture that $\tau(n) \neq 0$.

4.7. **Proof of Theorem 5.** For the exceptional congruences of type (i) and (ii) we determine the associated Euler-Kronecker constants with enough precision to ensure that they are non-zero. It follows that the corresponding variant of Ramanujan's Claim 1 is false for any $r > 1 + \delta_q$. In each case we also compute them with more than enough precision to decide whether they are greater than 1/2 (in which case Landau wins) or not; see Section 10 for the algorithms employed in our numerical computations, and Tables 7–9 for the values.

5. A detailed look at the non-divisibility claims in the unpublished manuscript

q	δ_q	E.P.	C_q	pp.	Sec.
3	+	+	+	22–23	11
5	+	+		06–08	2
7	+	+	+	11–12	6
23	+	−	−	36–37	17
691	+			24–25	12

TABLE 12. Correctness of non-divisibility claims from the unpublished manuscript

Table 12 lists all the non-divisibility claims similar to Claim 1 made by Ramanujan in the unpublished manuscript. They all involve the τ function (not listed are those cases where he only claimed bounds of the form $O(n/\log^\delta n)$). The " $+$ " entry indicates a correct claim, the " $-$ " a false one, whereas no entry indicates that no claim was made. The first column concerns the value of δ_q (see Table 1), the second the Euler product of the generating series, the third the value of the constant C_q, and the two remaining ones give the pages numbers and section numbers in [3] where the specific claims can be found. Rankin, using resuls from his paper [45], confirmed the correctness of C_3, C_7 and the δ_q column [46, p. 10]. However, C_{23} needs minor correction (as first pointed out by Moree [34]). The square of Ramanujan's Euler product (17.6) for the generating series equals the right-hand side of (47), but with the factor $(1-23^{-s})^{-1}$ replaced by $(1-23^{-s})$ (it is clear from his writing that when he writes "all primes of the form $23a^2+b^2$," he excludes the prime 23). The asymptotic constant associated to his Euler product he calculated correctly, but it has to be multiplied by 23/22 in order to obtain the true C_{23}.

The Dirichlet series $T_q(s)$ with $q \in \{3, 7, 23\}$ are the easiest in the sense that they satisfy $T_q(s)^2 = \zeta_M(s)A(s)$, with M quadratic (in fact, with $M = \mathbb{Q}(\sqrt{-q})$) and $A(s)$ a regular function for $\mathrm{Re}(s) > 1/2$. In this case, we have $h = 2$ and $\delta_q = 1/2$. As we have $\delta_q = q/(q^2-1)$ for non-exceptional q (see, e.g. Serre [56, p. 229]), it follows that for the tau function there are no further primes with this property. For these three primes, $T_q(s)$ can be related to the generating series associated to $\{n \ge 1 : q \nmid \tau_{\frac{q-1}{2}}(n)\}$, and we find

$$C_3 = C'_{1,3},\ C_7 = C'_{3,7},\ C_{23} = C_{11,23} \prod_{p\in\mathcal{S}_2} \frac{(1-p^{-23})(1-p^{-2})}{(1-p^{-22})(1-p^{-3})},$$

where the latter equality is immediate from (47). Using $L(1,\chi_{-3}) = \pi/\sqrt{27}$, $L(1,\chi_{-7}) = \pi/\sqrt{7}$, and $L(1,\chi_{-23}) = 3\pi/\sqrt{23}$, where χ_{-q} is the quadratic character modulo q, in combination with (45), we get precisely the expressions found by Ramanujan (with the caveat pointed out above for $q = 23$).

The five Euler products alluded to in Table 12 are the tip of an iceberg, Ramanujan's work being abundant with them; for an overview, see [1] or [44]. Therefore it comes as no surprise that his unpublished manuscript also contains more Euler products than those considered here.

6. Bounding $S(m,q)$

Before we begin, let us first recall that m is a divisor of $q-1$ such that $h = (q-1)/m$ is even and

$$S(m,q) = -\sum_{g_p \ne 2}\left(\frac{(\mu_p-1)\log p}{p^{\mu_p-1}-1} - \frac{\mu_p \log p}{p^{\mu_p}-1}\right) + \sum_{\substack{g_p\ge 4\\ 2\mid g_p}} \frac{\log p}{p^{g_p/2}-p^{-g_p/2}} + \sum_{g_p=2}\frac{\log p}{p^2-1}, \tag{48}$$

with g_p being the multiplicative order of p^m modulo q, and μ_p as in (32). Our bounds are given in Lemmas 8 and 9. They have terms with $q^{-1/m}$ in the denominator, and thus require $m = o(\log q)$ for them to tend to zero. Thus, one has to think of m as at most slowly growing with q. Note that $g_p \mid h$, where h, for the reason just given, will be close in size to $q-1$. To avoid technical complications that would bring no gain, we mostly use $g_p \le q-1$ in the sequel.

6.1. An upper bound for $S(m,q)$.

In order to prove Theorem 2 we need an upper bound for $S(m,q)$, which, for any fixed m, tends to zero as $q \to \infty$. This is provided by Lemma 8.

6.1.1. A trivial estimate.

Noticing that for $j \ge 3$ we have

$$\frac{p^j-1}{p^{j-1}-1} > p > \frac{3}{2} \ge \frac{j}{j-1},$$

the argument of the first sum in (48) is seen to be positive. It thus suffices to find upper bounds for the second and third sum in (48). We further observe that

$$\sum_{\substack{g_p\ge 4\\ 2|g_p}} \frac{\log p}{p^{g_p/2}-p^{-g_p/2}} + \sum_{g_p=2} \frac{\log p}{p^2-1} \le \sum_{\substack{p<q-2,\ g_p\ge 4\\ 2|g_p}} \frac{\log p}{p^{g_p/2}-p^{-g_p/2}} + \sum_{\substack{p<q-2\\ g_p=2}} \frac{\log p}{p^2-1} + \sum_{p\ge q-2} \frac{\log p}{p^2-1},$$

and denote the latter three sums by $S_1(q)$, $S_2(q)$, respectively $S_3(q)$.

6.1.2. **The sums $S_1(q)$ and $S_2(q)$.** We first give a rough estimate of the sum $S_0(q;\alpha)$ of the terms in $S_1(q)$ for which $g_p \ge \alpha$, where we will choose α later (think of α as being of size $O(\log q)$). The remainder we denote by $S_1(q;\alpha)$.

In the sequel we will make use of the fact that $\log y/(y-1)$ is decreasing for $y>1$, and hence so is $\log x/(x^j-1) = \log x^j/(j(x^j-1))$, with $j \ge 1$ any fixed real number and $x>1$. We have

$$S_0(q;\alpha) := \sum_{\substack{p<q-2,\ g_p\ge\alpha\\ 2|g_p}} \frac{\log p}{p^{g_p/2}-p^{-g_p/2}} \le \sum_{\substack{p<q-2\\ \alpha\le g_p\le h}} \frac{\log p}{p^{g_p/2}-1} \le \sum_{j=\lceil\alpha\rceil}^{h} \frac{jm\log 2}{2^{j/2}-1} \le \frac{q^2\log 2}{2(2^{\alpha/2}-1)},$$

where we use that there are at most jm primes $p<q-2$ with $g_p=j$. We split $S_1(q;\alpha)$ as

$$S_1(q;\alpha) = \sum_{\substack{p<q-2\\ g_p=2^e,\ e\ge 2\\ g_p<\alpha}} \frac{\log p}{p^{g_p/2}-p^{-g_p/2}} + \sum_{\substack{p<q-2,\ 2|g_p\\ P(g_p)>1,\ g_p<\alpha}} \frac{\log p}{p^{g_p/2}-p^{-g_p/2}} = S_{1,1}(q;\alpha)+S_{1,2}(q;\alpha),$$

where $P(g_p)$ denotes the largest odd divisor of g_p. Let $j\ge 1$ be an integer. We have

$$\sum_{\substack{p<q-2\\ g_p=4j}} \frac{\log p}{p^{g_p/2}-p^{-g_p/2}} < \sum_{\substack{p<q-2\\ g_p=4j}} \frac{\log p}{p^{g_p/2}-1} < \frac{jm\log((q-1)^{1/(2jm)})}{(q-1)^{1/m}-1} < \frac{\log q}{2((q-1)^{1/m}-1)}. \tag{49}$$

To see this, we note that for any prime p satisfying $g_p=4j$ we have $p^{2jm}\equiv -1 \pmod q$, hence $p^{2jm}\ge q-1$, and so $p^{g_p/2}=p^{2j}\ge (q-1)^{1/m}$. The second inequality now follows on noting that, by Lemma 1, there are at most jm primes $p<q-2$ satisfying the congruence. First assume that $\nu_2(h)\ge 2$, where ν_2 is the 2-adic valuation. If p contributes to $S_{1,1}(q;\alpha)$, then $g_p=2^e$ with $2\le e\le \min\{\nu_2(h),\log\alpha/\log 2\}$, and we thus infer, on invoking the estimate (49), that

$$S_{1,1}(q;\alpha) < \frac{\alpha_1\log q}{(q-1)^{1/m}-1)}, \quad \text{with } \alpha_1 = \min\Big\{\frac{\nu_2(h)-1}{2}, \frac{\log\alpha}{\log 4}\Big\}.$$

If $\nu_2(h)=1$, the sum $S_{1,1}(q;\alpha)$ is zero and hence the latter estimate also (trivially) holds.

We now turn our attention to $S_{1,2}(q;\alpha)$, and the plan is to compare $(p^{g_p/2}-p^{-g_p/2})^m$ with $p^{g_pm/2}+1$, which we know to be divisible by q by part 1) of Lemma 2. If $0<\beta<1$ is fixed, it is easy to see that

$$\min_{0<x\le\beta} \frac{(1-x^2)^m}{1+x^m} = \frac{(1-\beta^2)^m}{1+\beta^m} \ge \frac{(1-\beta^2)^m}{(1+\beta)^m} = (1-\beta)^m, \tag{50}$$

which holds for any integer $m\ge 1$. Since for every prime p contributing to $S_{1,2}(q)$ we have $p^{-g_p/2}\le 1/8$, on applying (50) with $x=p^{-g_p/2}$ and $\beta=1/8$ we obtain

$$(p^{g_p/2}-p^{-g_p/2})^m \ge c^m(p^{g_pm/2}+1), \tag{51}$$

with $c=7/8$. Using (51) and (18) with $d=P(g_p)>1$, we conclude that

$$(p^{g_p/2}-p^{-g_p/2})^m \ge c^m(p^{g_pm/2}+1) \ge c^m q(p^{g_pm/(2d)}+1) \ge c^m q(p^m+1) \ge q(cp)^m.$$

Taking m-th roots and noting that there are at most $m\alpha^2/2$ primes p with $g_p < \alpha$ and $(\log x)/x$ is decreasing for $x \ge e$, we now infer that

$$S_{1,2}(q) \le \frac{8}{7q^{1/m}} \sum_{p<m\alpha^2/2} \frac{\log p}{p} < \frac{8\log(m\alpha^2/2)}{7q^{1/m}},$$

where we used the estimate $\sum_{p<x}(\log p)/p < \log x$, valid for $x > 1$, due to Rosser and Schoenfeld [49, (3.24)].

A minor variation of the argument leading to the chain of inequalities in (49) gives

$$S_2(q) = \sum_{\substack{p<q-2\\ g_p=2}} \frac{\log p}{p^2-1} < \frac{m\log((q-1)^{1/m})}{(q-1)^{2/m}-1} < \frac{\log q}{(q-1)^{2/m}-1}.$$

Remark 10. We used several times the fact that there are at most jm primes $p < q$ for which $g_p = j$. In fact, there are at most $\phi(j)m$ primes with $g_p = j$. This would lead, at the cost of mathematical complication, to only a tiny improvement, and so we abstained from implementing it.

6.1.3. **The sum $S_3(q)$.** The next lemma implies that, for $q \ge 7$,

$$S_3(q) < \frac{1.053}{q-2.1},$$

which is rather sharp, as by the Prime Number Theorem we asymptotically have $S_3(q) \sim q^{-1}$.

Lemma 7. *For $x \ge 3$, we have*

$$\sum_{p>x} \frac{\log p}{p^2-1} < \frac{1.053}{x}.$$

Proof. Put $\vartheta(x) = \sum_{p\le x} \log p$ and $x_0 = 7481$. For $x \ge x_0$ one has $0.98 \cdot x \le \vartheta(x) \le 1.01624 \cdot x$, as was shown by Rosser and Schoenfeld [49, Theorems 9 and 10]. From this, one easily infers that, for $x \ge x_0$,

$$\sum_{p>x} \frac{\log p}{p^2} = \int_x^\infty \frac{d\vartheta(t)}{t^2} = -\frac{\vartheta(x)}{x^2} + 2\int_x^\infty \frac{\vartheta(t)}{t^3}dt \le \frac{-0.98x}{x^2} + 2\cdot 1.01624 \int_x^\infty \frac{dt}{t^2} \le \frac{1.0525}{x}.$$

Since $p^2 - 1 = p^2(1-p^{-2}) \ge p^2(1-x^{-2})$ for $p > x$, for $x \ge x_0$ we obtain that

$$\sum_{p>x} \frac{\log p}{p^2-1} < \frac{1}{1-x^{-2}} \sum_{p>x} \frac{\log p}{p^2} \le \frac{1.0525}{x(1-x_0^{-2})} \le \frac{1.053}{x}.$$

For $x < 7481$, we explicitly calculate the sum using

$$\sum_{p>x} \frac{\log p}{p^2-1} = -\frac{\zeta'(2)}{\zeta(2)} - \sum_{p\le x} \frac{\log p}{p^2-1} < 0.569961 - \sum_{p\le x} \frac{\log p}{p^2-1}. \qquad \square$$

6.1.4. **Upper estimates for $S(m,q)$.** Since there is no prime $p \equiv -1 \pmod q$ with $p < q-2$, we note that $S_2(q) = 0$ in case $m = 1$. Notice that if α is at most twice the smallest odd prime factor of h, then $S_{1,2}(q;\alpha) = 0$. On recalling that $S(m,q) = S_0(q;\alpha) + S_{1,1}(q;\alpha) + S_{1,2}(q;\alpha) + S_2(q;\alpha) + S_3(q)$ and inserting the estimates for these sums derived above, we arrive at the following result for $q \ge 7$ and prime; for $q = 3$ and $q = 5$ we verified the upper bound numerically.

Lemma 8. *Let q be an odd prime, m a divisor of $(q-1)/2$ and $h = (q-1)/m$. Then, for any $3 \le \alpha \le q-1$,*

$$S(m,q) < \frac{\alpha_1 \log q}{(q-1)^{1/m}-1} + \frac{8\log(m\alpha^2)}{7q^{1/m}} + \frac{1.053}{q-2.1} + \frac{q^2\log 2}{2(2^{\alpha/2}-1)} + \frac{\log q}{(q-1)^{2/m}-1},$$

with $\alpha_1 = \min\{(\nu_2(h)-1)/2, \log\alpha/\log 4\}$. *If* α *is at most twice the smallest odd prime factor of* h, *then the second term can be dropped. The final term can be dropped if* $m = 1$.

Corollary 7. *We have* $S(m,q) \ll (\log q)(\log\log q)q^{-1/m}$, *where the implicit constant is absolute.*

Proof. This follows on setting $\alpha = 10\log q$ (for example) and using the trivial bound $m \le q$ in the numerator of the second term. ☐

We point out that in case h satisfies $h \equiv 2 \pmod 4$ and has only odd prime factors exceeding $10\log q$ (for example), we have the sharper bound $S(m,q) \ll 1/q + (\log q)q^{-2/m}$.

6.2. **Lower bound for** $S(m,q)$**.** In order to prove Theorem 3 we need not only the upper bound for $S(m,q)$ given in Lemma 8, but also a lower bound. This is provided by Lemma 9. A tedious analysis gives that for $j \ge 4$ and $p \ge 2$ always

$$-\left(\frac{(j-1)}{p^{j-1}-1} - \frac{j}{p^j-1}\right) + \frac{1}{p^{j/2}-p^{-j/2}} \ge 0. \tag{52}$$

We are thus left with finding an upper bound for

$$T(q) := \sum_{2\nmid g_p} \frac{(\mu_p-1)\log p}{p^{\mu_p-1}-1}.$$

As a digression, we make the following observation.

Proposition 7. If h is a power of two, then $S(m,q) > 0$.

Proof. If h is a power of two, then so is g_p (which divides h). It follows that $T(q) = 0$. By Dirichlet's theorem on primes in arithmetic progression, the final sum in the formula (48) for $S(m,q)$ is strictly positive. ☐

Observe that

$$T(q) \le \sum_{\substack{g_p\ge 3,\, 2\nmid g_p\\ p<q}} \frac{(g_p-1)\log p}{p^{g_p-1}-1} + \sum_{\substack{g_p\ge 3,\, 2\nmid g_p\\ p>q}} \frac{(g_p-1)\log p}{p^{g_p-1}-1} + \sum_{g_p=1} \frac{(q-1)\log p}{p^{q-1}-1},$$

which we denote by $T_1(q), T_2(q)$ and $T_3(q)$, respectively. We have

$$T_2(q) \le \sum_{p>q} \frac{2\log p}{p^2-1} + \sum_{p>q} \frac{q\log p}{p^4-1} \ll \frac{1}{q} + \frac{1}{q^2}.$$

Reasoning as before, cf. the derivation of (49), we deduce

$$T_3(q) \le \sum_{\substack{p<q\\ p^m\equiv 1 \pmod q}} \frac{q\log p}{p^{q-1}-1} + \sum_{p>q} \frac{q\log p}{p^{q-1}-1} \ll \frac{\log q}{q^{h-1}} + \frac{1}{q^{q-3}} \ll \frac{\log q}{q^{h-1}}. \tag{53}$$

We write $T_1(q) = T_{1,1}(q;\alpha) + T_{1,2}(q;\alpha)$, where the first sum runs over the terms of $T_1(q)$ with $g_p < \alpha$, where α will be chosen later. We have

$$T_{1,2}(q;\alpha) \le \sum_{\substack{p<q\\ \alpha\le g_p\le h}} \frac{g_p\log p}{p^{g_p-1}-1} \le \sum_{j=\lceil\alpha\rceil}^{h} \frac{j^2 m\log 2}{2^{j-1}-1} \ll \frac{mh^3}{2^{\alpha-1}} \ll \frac{qh^2}{2^\alpha}. \tag{54}$$

The sum $T_{1,1}(q;\alpha)$, we write as $V_1(q;\alpha) + V_2(q;\alpha)$, where in the first sum we impose the additional condition that g_p is a prime itself.

We always have that q divides $(p^{g_p m}-1)/(p^m-1)$, a quotient which is bounded above by $2p^{(g_p-1)m}$. Letting $\sigma'(n;\alpha)$ denote the sum of the prime divisors $p_i<\alpha$ of n, we then obtain in the usual way

$$V_1(q;\alpha) \ll \frac{\sigma'(q-1;\alpha)\log q}{q^{1/m}}.$$

We finally turn our attention to $V_2(q;\alpha)$. The plan is to compare $p^m(p^{g_p-1}-1)^m$ with $p^{g_p m}-1$, which is of course divisible by q. For $p\geq 2$ and $g_p\geq 3$, we find, with $c=3/4$,

$$p^m(p^{g_p-1}-1)^m \geq c^m(p^{g_p m}-1) \geq c^m q(p^{g_p m/d}-1) \geq c^m q(p^{3m}-1) \geq 7qc^m p^{3m}/8,$$

where we use that g_p is a composite odd integer, and so it must have a divisor $1<d<g_p$. Taking m-th roots we infer that

$$V_2(q;\alpha) \ll \frac{1}{q^{1/m}}\sum_{g_p\leq\alpha}\frac{g_p\log p}{p^2} \ll \frac{\alpha}{q^{1/m}}.$$

We trivially have

$$\sigma'(q-1;\alpha) \leq \sum_{p\leq\alpha} p \ll \frac{\alpha^2}{\log\alpha}. \tag{55}$$

Gathering all the bounds together and setting $\alpha=10\log q$, we obtain that there is an absolute constant $c_1>0$ such that

$$-S(m,q) \leq c_1\frac{\sigma'(q-1;10\log q)\log q}{q^{1/m}},$$

which on invoking (55) leads to the following conclusion.

Lemma 9. *There is an absolute constant $c_2>0$ such that*

$$-S(m,q) \leq \frac{c_2\log^3 q}{q^{1/m}\log\log q}.$$

Remark 11. In case h satisfies $h\equiv 2 \pmod 4$ and has only odd prime factors exceeding $C\log q$ we can do much better and using the estimates (53) and (54) obtain

$$-S(m,q) \leq T_3(q)+T_{1,2}(q;C\log q) \ll \frac{\log q}{q^{h-1}}+\frac{qh^2}{q^{C\log 2}} \ll \frac{h^2}{q^{C\log 2-1}}.$$

The final estimate follows on noting that $C\log 2 < C\log q \leq h/2$.

Remark 12. Suppose there are infinitely many primes $q\equiv 3 \pmod 4$ with $q-1$ squarefree and having all its odd prime divisors in the interval $[\log q, 10\log q]$. Note that for these primes $\sigma'(q-1;10\log q) \gg \log^2 q/(\log\log q)$ and so the upper bound (55) with $\alpha=10\log q$ is sharp.

Remark 13. It is also possible to do the estimation without making use of inequality (52). For that, we have to bound from above the sum

$$\sum_{g_p\geq 3}\frac{(\mu_p-1)\log p}{p^{\mu_p-1}-1},$$

where now the terms with $2\mid g_p$ are included as well. Reasoning as in the derivation of (49), we find that

$$\sum_{\substack{g_p\geq 4\\ 2\mid g_p}}\frac{(g_p-1)\log p}{p^{g_p-1}-1} \ll \frac{\log^3 q}{q^{3/2}},$$

which is swamped by the major contribution to the error term for $-S(m,q)$.

7. Proof of Theorem 2

The arguments are inspired by the proof of [15, Theorem 1] and are related to the number of zeros of Dirichlet L-series in certain regions near the line $\Re(s) = 1$. McCurley [32, Theorem 1.1] showed that, for every q, the region

$$\Re(s) \geq 1 - \frac{1}{R \log \max\{q, q|\Im(s)|, 10\}},$$

where $R = 9.645908801$, contains at most one zero β_0 of $\prod_{\chi \,(\mathrm{mod}\, q)} L(s,\chi)$. If β_0 is exceptional, it must be real, simple, and satisfy $L(\beta_0, \chi_q) = 0$, where χ_q is the real, nonprincipal quadratic character modulo q. We will need an explicit version of Page's theorem [42] giving a lower bound for β_0. For this we use the one established by Ford et al. [15].

Lemma 10 ([15, Lemma 3])**.** *If $q \geq 10\,000$ is prime and β_0 an exceptional zero, then*

$$\beta_0 \geq 1 - \frac{3.125 \min\{2\pi, (\log q)/2\}}{\sqrt{q}\log^2 q} \geq 0.9983.$$

Let q be a prime, a an integer coprime with q, and let

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \,(\mathrm{mod}\, q)}} \Lambda(n)$$

be the Chebyshev ψ-function. The following modification of [15, Lemma 9] is an essential ingredient in our arguments (the notation used is as introduced in the beginning of this section).

Lemma 11. *Let $q \geq 10\,000$ be a prime and a a fixed integer coprime with q. For $x \geq \exp(R\log^2 q)$ we have*

$$\left|\psi(x; q, a) - \frac{x}{q-1}\right| \leq \frac{1.012 x^{\beta_0}}{q} + \frac{8}{9} x \sqrt{\frac{\log x}{R}} \exp\Big(-\sqrt{\frac{\log x}{R}}\Big),$$

where the first term is there only if there is an exceptional zero β_0.

Proof. For $a = 1$ this is [15, Lemma 9]. The proof depends heavily on earlier work of McCurley [32], whose arguments work for arbitrary a coprime to q. This allows for an easy adaptation of the proof in [15] to any a coprime with q as well. ☐

Remark 14. For primes $q \equiv 1 \,(\mathrm{mod}\, 2r)$, our interest is more precisely in

$$\sum_{\substack{n \leq x \\ n^r \equiv -1 \,(\mathrm{mod}\, q)}} \Lambda(n).$$

One could hope that this can be expressed as a linear combination of $\psi(x, \chi) = \sum_{n \leq x} \Lambda(n)\chi(n)$ not involving the quadratic character modulo q, thus avoiding the contribution of the possible exceptional zero β_0. However, this is not the case by the remark after Definition 4.

With these ingredients in place, we can finally prove Theorem 2.

Proof of Theorem 2. Recall that $r = (k, q-1)$. The equation $x^r \equiv -1 \,(\mathrm{mod}\, q)$ has precisely r solutions $a_1, \ldots, a_r$, with $0 < a_i < q$ (cf. Section 2.1). On combining Proposition 4 and Lemma 6, we have

$$\gamma_{k,q} = \gamma - \sum_{i=1}^{r} \lim_{x \to \infty} \Big(\frac{\log x}{q-1} - \sum_{\substack{n \leq x \\ n \equiv a_i \,(\mathrm{mod}\, q)}} \frac{\Lambda(n)}{n}\Big) - S(r, q). \tag{56}$$

Writing $E(t;q,a):=\psi(t;q,a)-t/(q-1)$, where $(a,q)=1$, and using a partial summation argument, we obtain

$$\lim_{x\to\infty}\Big(\sum_{\substack{y<n\le x\\ n\equiv a\ (\mathrm{mod}\ q)}}\frac{\Lambda(n)}{n}-\frac{\log(x/y)}{q-1}\Big)=\int_y^\infty\frac{E(t;q,a)}{t^2}dt-\frac{E(y;q,a)}{y}.$$

Invoking Lemma 11, we obtain, for any $y\ge\exp(R\log^2 q)$ the estimate

$$\Big|\int_y^\infty\frac{E(t;q,a)}{t^2}dt-\frac{E(y;q,a)}{y}\Big|\le\frac{1.012(2-\beta_0)y^{\beta_0-1}}{(1-\beta_0)q}+\frac{8}{9}\frac{2RW^2+(4R+1)W+4R}{e^W},\tag{57}$$

with $W=\sqrt{\log y/R}$ and where the first term can be left out if there is no exceptional zero β_0. On ignoring the summands from (56) with $n\le y$, we can now use (56)–(57) and $\beta_0\ge 0.9983$, to obtain

$$\gamma_{k,q}\ge\gamma-r\Big(\frac{\log y}{q-1}+\frac{1.015}{D\sqrt{q}}y^{-\frac{D}{\sqrt{q}\log^2 q}}\log^2 q+\frac{8}{9}\frac{2RW^2+(4R+1)W+4R}{e^W}\Big)-S(r,q),\tag{58}$$

for any $q\ge 10\,000$, where $D=3.125\min\{2\pi,\log q/2\}$ and $y=\exp(1.44R\log^2 q)$. The largest of the terms in between the brackets in (58) is coming from the exceptional zero and is $O(q^{-1/2}\log^2 q)$. Using Corollary 7 we thus conclude that there exist absolute constants c_2 and c_3 such that

$$\gamma_{k,q}\ge\gamma-c_2\frac{r\log^2 q}{\sqrt{q}}-c_3\frac{\log^2 q}{q^{1/r}}=\gamma-F(q),$$

say. It is easy to see that there exists an absolute constant c_1 such that $F(q)<0.077$ and hence $\gamma_{k,q}>1/2$ for any

$$q\ge e^{2r(\log r+\log\log(r+2)+c_1)},\quad \text{with } q\equiv 1\ (\mathrm{mod}\ 2r).$$

By Corollary 3 it then follows that the Landau approximation is better for any such value q. Using (16) we have $\gamma'_{k,q}=\gamma_{k,q}+\log q/(q-1)>\gamma_{k,q}$, and so we obtain the same conclusion for $\gamma'_{k,q}$. □

Remark 15. Let $q_0(r)$ be the minimal prime such that $\gamma_{r,q}>1/2$ for $q\ge q_0(r)$ using (58) and Lemma 8. Choosing $C=10$, a numerical evaluation of such formulae gave $q_0(1)=28\,537$; $q_0(2)=1\,160\,893$; $q_0(3)=2\,089\,575\,931$; $q_0(r)>10^{10}$ for $r\ge 4$. These bounds are too large in order for $S(r,q)$ to be evaluated over the whole range $3\le q\le q_0(r)$, q prime, $q\equiv 1\ (\mathrm{mod}\ 2r)$, as described in Section 10; in fact there we will explain that we are able to compute $S(r,q)$ only for $1\le r\le 6$ and $3\le q\le 3000$. However, for $r=1$ we can also use some already computed data on γ_{K_1} and γ_{K_2} to prove that $\gamma_{1,q}>1/2$ for every odd prime $q\in[q_1(1),q_0(1)]$, where $q_1(1)<3000$; see the proof of Theorem 4. Unfortunately, the cases with $r\ge 2$ are well beyond our computational capabilities and hence we presently cannot settle the truth of Conjectures 1–2.

8. Proof of Theorem 3

Our proof will make use of the following result.

Proposition 8. If $y\ge 10q$ and $q\ge 11$, then

$$\sum_{\substack{2q<p\le y\\ p\equiv a\ (\mathrm{mod}\ q)}}\frac{\log p}{p-1}\le\frac{2\log y+2(\log q)\log\log(y/q)}{q-1}.$$

Proof. In [15, Prop. 6] this is proved for $a=1$. As it hinges on the Montgomery-Vaughan sharpening of the Brun-Titchmarsh theorem, which holds for arbitrary progressions, it trivially generalizes. □

Proof of Theorem 3. The argument leading to (58) is easily adapted to obtain, for any $y \geq \exp(R\log^2 q)$, the upper bound

$$\gamma_{k,q} \leq \gamma + r\Big(\frac{\log y}{q-1} + \frac{1.015}{D\sqrt{q}} y^{-\frac{D}{\sqrt{q}\log^2 q}} \log^2 q + \frac{8}{9}\frac{2RW^2+(4R+1)W+4R}{e^W}\Big) - S(r,q) + T_q(y), \quad (59)$$

with

$$T_q(y) = \sum_{\substack{n\leq y \\ n^r\equiv -1 \ (\mathrm{mod}\ q)}} \frac{\Lambda(n)}{n}.$$

Note that for the lower bound we had dropped the sum $T_q(y)$. Put $y_1 = \exp(1.44R\log^2 q)$. Using Proposition 8, we deduce that

$$T_q(y_1) \leq T_q(2q) + \sum_{\substack{2q<p\leq y_1 \\ p^r\equiv -1 \ (\mathrm{mod}\ q)}} \frac{\log p}{p-1} \ll \frac{\log q}{q^{1/r}} + \frac{\log^2 q}{q}.$$

This estimate, together with (58) and (59), then yields

$$\gamma_{k,q} = \gamma - S(r,q) + O\Big(\frac{r\log^2 q}{\sqrt{q}} + \frac{\log q}{q^{1/r}}\Big).$$

Taking into account the upper and lower bound for $S(r,q)$ provided by Lemmas 8 and 9, the proof is completed. □

9. Proof of Theorem 4

We work here under the assumption that $r = 1$; that is, we study the divisibility of $n^v\sigma_k(n)$ by primes q such that $(k, q-1) = 1$. We will follow the same argument used in Theorem 2 to prove that Landau wins for large enough primes q, but, in addition, we will be able to treat all the remaining primes q and to conclude, in each case, whether the Landau or the Ramanujan approximation is better. For this, we will need the upper estimate established in Lemma 8 and the following sandwich bounds for γ_{K_1} and γ_{K_2}.

Lemma 12 ([30, Section 6])**.** *For* $3 \leq q < 30\,000$, *we have*

$$0.3145 \cdot \log q \leq \gamma_{K_1} \leq 1.6270 \cdot \log q,$$
$$0.5254 \cdot \log q \leq \gamma_{K_2} \leq 1.4263 \cdot \log q.$$

Proof. The values γ_{K_1} and γ_{K_2} (denoted by $\mathfrak{G}_q$ and $\mathfrak{G}_q^+$ in [30]) are the Euler-Kronecker constants of the fields $K_1 = \mathbb{Q}(\zeta_q)$ and $K_2 = \mathbb{Q}(\zeta_q + \zeta_q^{-1})$ respectively, see Section 2.2. The lower and upper estimates given here are taken from [30, Section 6]. □

We are now ready to prove Theorem 4.

Proof of Theorem 4. Setting $r = 1$ in (58) gives

$$\gamma_{1,q} \geq \gamma - \frac{\log y}{q-1} - \frac{1.015}{D\sqrt{q}} y^{-\frac{D}{\sqrt{q}\log^2 q}} \log^2 q - \frac{8}{9}\frac{2RW^2+(4R+1)W+4R}{e^W} - S(1,q), \qquad (60)$$

where $q \geq 10\,000$ and we recall that $D = 3.125\min\{2\pi, \log q/2\}$, $y = \exp(1.44R\log^2 q)$, $W = \sqrt{\log y/R}$ and $R = 9.645908801$. A quick numerical check using Lemma 8 and (60) reveals that $\gamma_{1,q} > 1/2$ for $q \geq 29\,100$. In the remaining q-range we use the alternative expression

$$\gamma_{1,q} = \gamma - \frac{\log q}{(q-1)^2} - \frac{2\gamma_{K_2} - \gamma_{K_1}}{q-1} - S(1,q), \qquad (61)$$

which comes from taking $r = 1$ in (14). Inserting the upper bound for $S(1, q)$ given in Lemma 8 in (61), we obtain a lower bound for $\gamma_{1,q}$, which, using Lemma 12 and a numerical verification, is seen to exceed $1/2$ for $600 < q \le 29\,000$. Thus, to prove the first part of the statement, it remains to check it for $3 \le q \le 600$, which we do by a direct numerical evaluation of the quantities appearing in (61). Using (16) we have $\gamma'_{1,q} = \gamma_{1,q} + \log q/(q-1) > \gamma_{1,q}$ and hence, for $q > 600$, the second part of Theorem 4 follows immediately. In the remaining q-range a numerical verification completes the proof. □

10. On the numerical computations

All the numerical results presented in this paper were obtained using the following considerations. The computation of $\gamma_{k,q}$ naturally splits in two parts: the evaluation of the pair γ_{K_r}, $\gamma_{K_{2r}}$, and that of $S(r, q)$, where $r = (k, q-1)$ and $h = (q-1)/r$ is even (and so $r \mid (q-1)/2$). In fact, both problems can be handled in a more general setting, i.e., for each $m \mid (q-1)/2$.

We first remark that a logarithmic differentiation of the L-function factorization from (26)–(27) yields

$$\gamma_{K_m} = \gamma + \sum_{\chi \in X_m^*} \frac{L'}{L}(1, \chi), \quad \gamma_{K_{2m}} = \gamma + \sum_{\chi \in X_{2m}^*} \frac{L'}{L}(1, \chi).$$

These formulae suggest that γ_{K_m} and $\gamma_{K_{2m}}$ can be computed by adapting the approach presented in [29, 30]. Indeed, using techniques from [29, 30] we can get the values of $L'/L(1, \chi)$ for every non-principal Dirichlet character mod q. So, after having obtained the list of the divisors m of $(q-1)/2$, in order to get γ_{K_m} and $\gamma_{K_{2m}}$, it is enough to sum $L'/L(1, \chi)$ on every non-principal character of X_m and, respectively, X_{2m}. Such sets of characters can be described in the following way: recalling that q is prime, it is enough to get g, a primitive root of q, and χ_1, the Dirichlet character mod q given by $\chi_1(g) = \exp(2\pi i/(q-1))$, to see that the set of the non-principal characters mod q is $\{\chi_1^j : j = 1, \dots, q-2\}$. In order for χ_1^j to be in X_m, we need that $\chi_1^j(a) = 1$ for every $a \in C_m$. But $a \in C_m$ if and only if it can be written as $a \equiv g^b \pmod q$, with $b = \ell(q-1)/m$ for some $\ell = 0, \dots, m-1$. Hence $\chi_1^j \in X_m$ implies that $\chi_1^j(a) = \exp(2\pi i j b/(q-1)) = \exp(2\pi i j \ell/m) = 1$ for every $\ell = 0, \dots, m-1$, and this is equivalent to $m \mid j$. Summarizing, we can say that $X_m = \{\chi_0\} \cup \{\chi_1^j : j = 1, \dots, q-2;\ m \mid j\}$. This characterization, albeit elementary, is particularly useful in practice since the condition $m \mid j$ can be easily checked by a computer program.

Recalling that $\chi \in X_{2m}$ if and only if $\chi \in X_m$ and χ is even, we observe that $\gamma_{K_{2m}}$ can be obtained without further efforts by storing the sum over even characters used for γ_{K_m}.

In this way it is then possible to evaluate every γ_{K_m} and $\gamma_{K_{2m}}$ with essentially the same computational cost needed to get γ_{K_1} and γ_{K_2}. Using Pari/Gp [43] we implemented this, with a precision of 30 decimal digits, for each odd prime $q \le 3000$ and $1 \le m \le 6$; this required about 33 minutes of computing time. For $q > 3000$, the use of the Fast Fourier Transform algorithm is mandatory, as explained in [29, 30]; the accuracy of the latter procedure is commented on in [30].

We did not perform such FFT computations for $m \ge 2$, since in these cases we would not be able to prove an analogue of Theorem 4. To reach this goal, in fact, we should obtain the values of $S(m, q)$ for q up to very large bounds, see Remark 15, which is currently infeasible because it is much harder to compute $S(m, q)$, which is defined as in (32) but using $m \mid (q-1)/2$ and $g_p = f_p/(f_p, m)$. The prime sums involved were computed up to a certain bound P and then we estimated the remaining tails exploiting the summation functions of Pari/Gp [43]. The slow decay ratio of some of the summands in $S(m, q)$ prevents us from obtaining a very good accuracy. However, by choosing $P = 10^8$ first, and then using $P = 10^9$ or $P = 10^{10}$ if necessary, we were able to handle all the cases $3 \le q \le 3000$ with $1 \le m \le 6$ and $m \mid (q-1)/2$, with sufficient accuracy to determine the winner in the "Landau vs. Ramanujan" problem; this required about a week of computation time. For the cases in Tables 3, 7, 8, 9 and 11, with the choice $P = 10^8$ resulting in a final accuracy less than 6 decimal digits, we repeated the computation using $P = 10^9$ or $P = 10^{10}$. Some practical tricks were used to improve the actual running time of this part. First, for a fixed odd prime q we scanned the set of primes $p \le P$ just once storing the partial results of each sum in the definition of $S(m, q)$ in a matrix having a row for each requested m (the largest possible set in our implementation of this part is $1 \le m \le 6$). Second, to have the sharpest possible estimate for the "tails", for every q we stored the set of g_p-values used in the previous procedure so that the evaluated upper bound for such tails were based just on the effectively used g_p and not over every divisor of $q - 1$. The computations with $P = 10^8$ for every odd prime up to 3000 and $1 \le m \le 6$ were performed on the Dell Optiplex machine already mentioned and required about 40 hours of computing time. The ones with $P \in \{10^9, 10^{10}\}$ were performed on six machines of the cluster of the Dipartimento di Matematica of the University of Padova; in this case the total computing time amounted to 45 days.

10.1. **Accelerated convergence formulae for $\gamma_{k,r}$.** For any $J \ge 2$ we rewrite (36) as

$$\frac{T'}{T}(s) = \frac{\zeta'}{\zeta}(s) + \frac{D'}{D}(s) + \sum_{g_p=2} \frac{\log p}{p^s} - \sum_{j=2}^{J} (-1)^j \sum_{g_p=2} \frac{\log p}{p^{js}} + (-1)^J \sum_{g_p=2} \frac{\log p}{p^{Js}(p^s+1)}.$$

By Lemma 5, sums of the form $\sum_{p \equiv a \ (\mathrm{mod}\ q)} (\log p) p^{-js}$ can be expressed in terms $L'/L(js, \chi)'s$ and sums of the same type, but with j replaced by $2j$. The upshot is that we can write the right-hand side in terms of $L'/L(js, \chi)'s$ with $j \le J$ and with an error term of the form $O(\sum_p (\log p) p^{-(J+1)s})$. The same applies to the logarithmic derivative $D'/D(s)$. This reasoning suggests that we can express $T(s)$ itself in terms of $L'/L(js, \chi)'s$ with $j \le J$ and a regular function for $\mathrm{Re}(s) > 1/(J+1)$. In the next section we confirm this supposition.

10.1.1. **Higher level L-factorability of $T(s)$.**

Definition 5. Let q be a fixed odd prime. We say a Dirichlet-series $F(s)$ is *L-factorable of level ℓ* if there are integers j, e, e_{χ,j_1} such that

$$F(s)^j = \zeta(s)^e R(s) \prod_{j_1=1}^{\ell} \prod_{\chi} L(j_1 s, \chi)^{e_{\chi,j_1}}, \tag{62}$$

with $R(s)$ a regular function for $\mathrm{Re}(s) > 1/(\ell + 1)$ and where χ runs over the non-principal characters modulo q. We say that a set of primes $\mathcal{P}$ is L-factorable of level ℓ if $\prod_{p \in \mathcal{P}} (1 - p^{-s})^{-1}$ is L-factorable of level ℓ.

Notice that the product of two L-factorable functions of level ℓ is L-factorable of level ℓ again. It is a classical fact that the set of primes splitting completely in any prescribed subfield of $\mathbb{Q}(\zeta_q)$ is L-factorable of level 1. Thus the set of primes with $f_p = 1$ is L-factorable of level 1. The regular part $R(s)$ consists of Euler products of the form $\prod_{f_p=j} (1 - p^{-e_j s})^{-1}$, with $e_j \ge 2$. Each of these is L-factorable of level 2, with the new regular part $R(s)$ consisting of Euler products of the form $\prod_{f_p=j} (1 - p^{-e_j s})^{-1}$, with $e_j \ge 3$. We conclude that for arbitrary $\ell \ge 1$ the set of primes that split completely in any subfield of $\mathbb{Q}(\zeta_q)$ is L-factorable of level ℓ. Given m dividing $q - 1$, the set of primes with f_p dividing m is also L-factorable of level ℓ, as this is the set of primes that split completely in K_m. By inclusion-exclusion we then infer that the set of primes p with $f_p = m$ is L-factorable of level ℓ.

Proposition 9. Let q be an odd prime and $k \ge 1$ an arbitrary integer. Then $T(s) := \sum_{q \nmid \sigma_k(n)} n^{-s}$ is L-factorable of level ℓ, with $\ell \ge 1$ arbitrary.

Proof. The Euler product (33) for $T(s)$ consists of Euler products of the form $\prod_{g_p=m}(1-p^{-e_m s})^{-1}$, with m running over the divisors of $q-1$ and with $e_m \geq 1$. Recalling that $g_p = f_p/(f_p, r)$ with $r = (k, q-1)$, we see that the set of primes with $g_p = m$, is a union of sets of primes of the form $f_p = m_i$. Each of these prime sets is L-factorable of level l and hence so is $T(s)$. □

As usual, let $h = (q-1)/r$. From (40) and (5) it follows that $T(s)$ is L-factorable of level 1, with $F(s) = T(s)$, $j = h$ and $e = h - 1$ in (62).

Remark 16. Proposition 9 can also be proved using the theory developed in Ettahri et al. [13] (communication by Olivier Ramaré).

10.2. **Special cases.** In certain special cases the convergence can be improved. We start by noting that for $k \geq 1$ we have

$$\prod_{(\frac{D}{p})=-1} \frac{1}{(1-p^{-ks})^2} = \frac{\zeta(ks)}{L(ks,\chi_D)} \prod_{p|D}(1-p^{-ks}) \prod_{(\frac{D}{p})=-1} \frac{1}{(1-p^{-2ks})}$$

and

$$\prod_{(\frac{D}{p})=1} \frac{1}{(1-p^{-ks})^2} = L(ks,\chi_D)\frac{\zeta(ks)}{\zeta(2ks)} \prod_{p|D}(1+p^{-ks})^{-1} \prod_{(\frac{D}{p})=1} \frac{1}{(1-p^{-2ks})}.$$

To see this we partition the primes p according to the Legendre symbol $\left(\frac{D}{p}\right)$ and verify that, in each case, the Euler product factor at p on the left-hand side equals that on the right-hand side.

By logarithmic differentiation we obtain that, for $k \geq 2$,

$$\sum_{(\frac{D}{p})=-1} \frac{\log p}{p^k-1} = \sum_{(\frac{D}{p})=-1} \frac{\log p}{p^{2k}-1} + \frac{1}{2}\left(\frac{L'}{L}(k,\chi_D) - \frac{\zeta'}{\zeta}(k) - \sum_{p|D}\frac{\log p}{p^k-1}\right) \tag{63}$$

and

$$\sum_{(\frac{D}{p})=1} \frac{\log p}{p^k-1} = \sum_{(\frac{D}{p})=1} \frac{\log p}{p^{2k}-1} - \frac{1}{2}\left(\frac{L'}{L}(k,\chi_D) + \frac{\zeta'}{\zeta}(k) - 2\frac{\zeta'}{\zeta}(2k) + \sum_{p|D}\frac{\log p}{p^k+1}\right). \tag{64}$$

Assume that $r = (q-1)/2$ and $q \equiv 3 \pmod 4$. In this case the condition $g_p = 2$ is equivalent with $p^{(q-1)/2} \equiv \left(\frac{p}{q}\right) = -1$. By quadratic reciprocity we have $\left(\frac{p}{q}\right) = \left(\frac{-q}{p}\right)$. Using (63) we conclude that

$$\sum_{g_p=2} \frac{\log p}{p^2-1} = \sum_{g_p=2} \frac{\log p}{p^4-1} + \frac{1}{2}\left(\frac{L'}{L}(2,\chi_{-q}) - \frac{\zeta'}{\zeta}(2) - \frac{\log q}{q^2-1}\right).$$

Moreover, in this case, the condition $g_p = 1$ is equivalent to $\left(\frac{p}{q}\right) = 1$; so, inserting formula (64) for $k = q-1$ and $k = q$ into (15), we can improve the convergence ratio of this sum too. In fact, both formulae (63) and (64) can be iterated several times. Implementing this strategy we were able to compute $\gamma_{(q-1)/2,q}$, as described in Section 3.10, for each odd prime $q \leq 3000$ with an accuracy of 50 decimal digits in less than 123 seconds of computing time.

The previous argument requires to compute $L'/L(j,\chi)$, for $j \geq 2$. To obtain such values we can use (q odd prime, $\Re(s) > 1$) that

$$L(s,\chi) = q^{-s}\sum_{a=1}^{q-1}\chi(a)\zeta(s,a/q) \quad \text{and} \quad L'(s,\chi) = -(\log q)L(s,\chi) + q^{-s}\sum_{a=1}^{q-1}\chi(a)\zeta'(s,a/q),$$

where $\zeta(s,x)$ is the Hurwitz zeta function and $\zeta'(s,x) := \frac{\partial\zeta}{\partial s}(s,x)$, $\Re(s) > 1$, $x > 0$. Hence

$$\frac{L'}{L}(j,\chi) = -\log q + \frac{\sum_{a=1}^{q-1}\chi(a)\zeta'(j,a/q)}{\sum_{a=1}^{q-1}\chi(a)\zeta(j,a/q)}. \tag{65}$$

10.3. **An application of the convergence acceleration technique: the Shanks constant.** In 1964, Shanks [58] was the first to use (63) to study the behavior of $B(x)$, the number of integers less or equal to x that are the sum of two squares. We show now how this works and how to improve some of the known results on this problem. Shanks obtained that

$$B(x) = \frac{\mathcal{K}x}{\sqrt{\log x}}\Big(1 + \frac{c}{\log x} + O\Big(\frac{1}{\log^2 x}\Big)\Big),$$

as $x \to \infty$, where $\mathcal{K}$ is the Landau-Ramanujan constant (see (11)) and

$$c = \frac{1}{2} + \frac{\log 2}{4} - \frac{\gamma}{4} - \frac{1}{4}\frac{L'}{L}(1,\chi_{-4}) + \frac{1}{2}\sum_{p\equiv 3 \ (\mathrm{mod}\ 4)} \frac{\log p}{p^2-1}, \tag{66}$$

with $\chi_{-4}(\cdot) = \big(\frac{-4}{\cdot}\big)$ being the quadratic Dirichlet character modulo 4. The associated Euler-Kronecker constant γ_{SB} satisfies $\gamma_{SB} = 1 - 2c$ by Theorem 1. Iteratively using (63) $J_c \geq 1$ times, we obtain that

$$\sum_{p\equiv 3 \ (\mathrm{mod}\ 4)} \frac{\log p}{p^2-1} = \frac{1}{2}\sum_{j=1}^{J_c}\Big(\frac{L'}{L}(2^j,\chi_{-4}) - \frac{\zeta'}{\zeta}(2^j) - \frac{\log 2}{2^{2^j}-1}\Big) + \sum_{p\equiv 3 \ (\mathrm{mod}\ 4)} \frac{\log p}{p^{2^{J_c+1}}-1}, \tag{67}$$

which, for $J_c = 2$, gives eq. (18) of [58]. Shanks wrote b instead of $\mathcal{K}$, and obtained a very similar formula whose truncated form can be written as follows:

$$\sum_{p\equiv 3 \ (\mathrm{mod}\ 4)} \log\Big(1 - \frac{1}{p^2}\Big) = -\sum_{j=1}^{J_b}\frac{1}{2^j}\log\Big(\frac{\zeta(2^j)(1-2^{-2^j})}{L(2^j,\chi_{-4})}\Big) + \frac{1}{2^{J_b}}\sum_{p\equiv 3 \ (\mathrm{mod}\ 4)} \log\Big(1 - \frac{1}{p^{2^{J_b+1}}}\Big),$$

where $J_b \geq 1$ is an integer. A straightforward argument proves that the last sum in (67) does not exceed $(\log 3)\cdot 4^{1-2^{J_c}}/3$, so that in order to show that this term is less that $10^{-\alpha}$, it is enough to choose

$$J_c > \frac{1}{\log 2}\log\left(1 + \frac{\alpha\log 10 + \log\log 3 - \log 3}{2\log 2}\right).$$

Similar remarks applies to J_b too. For example, for $\alpha = 100$, it is enough to choose $J_b = J_c = 8$. Since χ_{-4} is an odd primitive character, we can write $L'/L(1,\chi_{-4})$ in terms of the $\log\Gamma$-function and of the first χ-Bernoulli number, see, e.g., [29, §3]. Straightforward computations give

$$\frac{L'}{L}(1,\chi_{-4}) = \gamma + 2\log 2 + 3\log\pi - 4\log\Gamma\Big(\frac{1}{4}\Big),$$

and the needed Gamma-value can be obtained using the Arithmetic-Geometric Mean (AGM) inequality, see, e.g., Borwein-Zucker [5]. The contribution of $L'/L(2^j,\chi_{-4})$ can be evaluated using (65), which in this case becomes

$$\frac{L'}{L}(2^j,\chi_{-4}) = -2\log 2 + \frac{\zeta'(2^j,1/4) - \zeta'(2^j,3/4)}{\zeta(2^j,1/4) - \zeta(2^j,3/4)}. \tag{68}$$

We remark that for $j = 1$ the denominator in the previous equation is an integer multiple of the *Catalan constant* G, since it is well-known that $16G = \zeta(2,1/4) - \zeta(2,3/4)$.

Inserting (67) and (68) into (66), we obtain an explicit formula that can be directly used in any mathematical software in which the Hurwitz zeta function is implemented. Using Pari/Gp, for instance, and choosing $J_c = 8$, we can obtain at least 100 correct decimal digits of c (and, in fact, also for $\mathcal{K}$) in about 38 milliseconds; choosing $J_c = 11$ we get at least 1000 correct decimal digits in less than 4 seconds of computation time; in about 383 minutes, with $J_c = 16$, we can get at least 31000 correct decimal digits (such computations were performed on the Dell Optiplex machine previously mentioned, using up to 12GB of RAM). In OEIS, the Landau-Ramanujan constant $\mathcal{K}$

appears as A064533, with about 125000 digits available, while the Shanks constant c is mentioned as A227158, with about 5000 digits available.

We finally remark that Ettahri, Ramaré and Surel [13] further boosted the idea of Shanks and gave it a more systematic setting using group theory.

11. Outlook

11.1. **Generalizations.** The following result of Datskovsky and Guerzhoy [8] shows that Ramanujan type congruences abound.

Classical Theorem 4. *For any even integer $w \geq 12$ there exists a nonzero cusp form $f = \sum a(n)q_1^n$ of weight w with rational Fourier coefficients $a(n)$, so that for every $n \geq 1$ we have $v_q(a(n) - \sigma_{w-1}(n)) \geq 1$, where q can be any prime divisor of the numerator of the reduced fraction $\frac{B_w}{2w}$, v_q is the q-adic valuation and B_w denotes the w-th Bernoulli number.*

In case $\dim S_w = 1$, it is easy to deduce from this that for f we can take the unique cusp form of weight w normalized so that $a(1) = 1$. This allows one to obtain the type (i) congruences satisfying $w > q$ without a coprimality condition and gives an alternative proof of the second statement in Proposition 6.

For some congruence subgroups $\Gamma_0(N)$, Ramanujan-type congruences are known where the relevant Fourier coefficients satisfy $a(n) \equiv \sigma_k(n) \pmod{q}$ for all n coprime to N, see, e.g., Kulle [24]. The associated generating series will be as $T(s)$ above, except for some possible modified Euler product factors at primes p dividing N. These factors can be easily logarithmically differentiated and we can express the Euler-Kronecker constant as $\gamma_{k,q}$ plus possibly a sum of terms involving the primes p dividing N. Dummigan and Fretwell [11] gave a result similar to Classical Theorem 4 for $\Gamma_0(p)$, with p prime.

The divisor sums arise as Fourier coefficients of Eisenstein series. Over the years, many generalized Eisenstein series have been considered, for example $E_w^{\psi,\xi}$, which involves two Dirichlet characters ψ and ξ (see Diamond and Shurman [10, Thm. 4.5.1]). Its Fourier coefficients are of the form $\sum_{d|n} \psi(\frac{n}{d})\chi(d)d^{w-1}$ and can likely also be dealt with using our methods. The non-divisibility asymptotics, in the special case where ψ is the principal character, were determined by Scourfield [50, 52]. Here, if χ is a Dirichlet character modulo N, then the divisor sum $\sum_{d|n} \chi(d)d^{w-1}$ is the n-th Fourier coefficient of the Eisenstein series of weight w and character χ on $\Gamma_0(N)$, see, e.g., the book [7, p. 17].

11.2. **Regarding our conjectures.** One might hope that Conjectures 1 and 2 can be proved under GRH. Indeed, the analysis of the "Landau vs. Ramanujan problem" using GRH (pioneered by Ihara [20]) is technically far less demanding; for this, compare Moree [35] (on GRH) with Ford et al. [15] (unconditional). However, in our case, the bottleneck is represented by the behavior and slow decay rate of $S(3, q)$ and $S(5, q)$.

11.3. **Some open questions.**

- Solve the "Landau vs. Ramanujan problem" for non-exceptional primes.
- What are the optimal upper bounds for the prime sums $S(m, q)$? How do they behave on average (with m fixed)?
- Consider the number $N(x)$ of pairs (k, q) with $1 \leq k, q \leq x$ with q prime for which Landau wins, that is, for which $\gamma_{k,q} > 1/2$. Is it true that Landau wins almost always in the sense that asymptotically $N(x) \sim x^2/\log x$?
- Given (any) $\epsilon > 0$, are there k and q such that $|\gamma_{k,q} - 1/2| < \epsilon$?
- What is the average behavior of $\gamma_{k,q}$ for q fixed?
- How is $\gamma_{(q-1)/2,q}$ distributed as q runs over the primes?

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