# Space-efficient Quantum Computation with Gaussian Systems

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# Raumeffiziente Quantenberechnung mit Gaußschen Systemen

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# Abstract

Quantum computation represents a powerful alternative to, amongst others, handle amounts of data that would pose an intractable problem for our current classical computers. However, scaling quantum computers presents considerable difficulties in experimental implementation, which has so far led to the development of quantum computers whose implemented resources are orders of magnitude too small to be used for practical purposes. Therefore, finding an equivalent simulation of quantum systems with circuits that require fewer resources is of particular interest. One option to achieve such a compaction is to narrow our interest down to Gaussian systems, owing to the fact that the probability distribution describing such systems has characteristics that can be utilized for a more efficient representation. However, notwithstanding the importance of bosonic computing, there is little research in the subfield of bosonic Gaussian circuits that addresses a concrete approach to achieve this aim. This paper therefore reviews previous research on compressed fermionic Gaussian computation in a first instance, and suggests generalizations of these findings to bosonic Gaussian systems in a second instance. Consequently, we will investigate the classical simulatability of both types of circuits, as well as provide evidence that, under certain conditions, the computation of fermionic and bosonic Gaussian systems can be equivalently simulated in polynomial time with logarithmic-sized quantum circuits.

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# CHAPTER I

# Introduction

Owing to the multitude of new counterintuitive insights proposed by quantum mechanics after its emergence in the 20th century, a variety of new approaches to existing research areas such as information processing have surfaced. Quantum computers, first introduced in 1980 [3], are profoundly different from classical common computers, as they utilize precisely these new insights, such as quantum superposition, quantum entanglement, and interference, to process information in a more efficient manner. Among other aspects, one justification for this distinction is that the classical resources to model a generic quantum many-body system, as handled by quantum computers, grow exponentially with the system size. This suggests that, conversely, quantum computation could be relied upon to access solutions to problems that are intractable with classical computation as they require the manipulation of large amounts of information. The factorization problem of integers is an example of such an intractable problem enabled by quantum computations. Although to this day it is considered classically infeasible in polynomial time, i.e. there is no known algorithm to factorize a natural number represented by N bits using poly(N) operations, quantum computers are able to do so using the Shor factorization algorithm [10].

However, in the current state of the art, the implementation of quantum computers is a fairly difficult task as it requires precise manipulation of the system, which has resulted so far in the development of quantum computers with a limited number of qubits [1]. However, the effective count needed for practical applicability is orders of magnitude larger, so there is a particular interest in finding solutions to reduce the resources in terms of qubits and/or the number of quantum gates required to simulate specific quantum systems. A kind of these specific systems that might allow for such a compactified description are Gaussian systems. The idea is that by restricting ourselves to Gaussian states, i.e. thermal states of quadratic Hamiltonians, and to Gaussian transformations, i.e. transformations that leave Gaussian states Gaussian, one could exploit the specific properties of the probability distributions describing such quantum systems to describe their evolution in a more efficient way. Accordingly, this approach could be used to describe the quantum circuits used to realize Gaussian quantum algorithms to achieve new encodings in equivalent circuits that require fewer resources.

This thesis aims to investigate this matter considering two different cases, namely fermionic and bosonic Gaussian systems. Reviewing the work of R. Jozsa et al. [7] on the characterization of matchgate circuits, i.e. equivalent circuits to fermionic Gaussian ones, and proposing an analysis of bosonic Gaussian systems, we attempted to provide evidence for the equivalence of Gaussian and reduced-space circuits, i.e. quantum circuits described by fewer modes than the Gaussian circuit.

This work has been divided into the following parts. The remainder of this chapter will focus on introductory concepts, revolving around our used mathematical toolset; then we will devote Chapter II to reviewing approaches to fermionic compression, i.e. recoding matchgate circuits into circuits with reduced resources. Using these reviewed fermionic methods, we will characterize bosonic Gaussian systems and present evidence for the equivalence between bosonic linear optic and space-reduced circuits in Chapter III. Finally, in Chapter IV we will make concluding remarks and suggest directions for future research.

## I.1 Many-body systems

Consider a system of $N$ indistinguishable particles described by

$$\mathcal{H}_{(N)} = \mathcal{H}_{(1)}^{\otimes N} = \mathcal{H}_{(1)} \otimes \mathcal{H}_{(1)} \otimes \ldots \otimes \mathcal{H}_{(1)} = \text{span}\left\{|\lambda_1\rangle \otimes |\lambda_2\rangle \otimes \ldots \otimes |\lambda_N\rangle\right\} \tag{I.1}$$

then, because of indistinguishability, our subspace of physically meaningful pure states is spanned by

$$|\lambda_1, \lambda_0, \ldots, \lambda_N\rangle = \mathcal{N} \sum_p \zeta_p |\lambda_{p_1}\rangle \otimes |\lambda_{p_2}\rangle \otimes \ldots \otimes |\lambda_{p_N}\rangle \tag{I.2}$$

where $\mathcal{N}$ is the normalization, the sum is going over all permutations of the $N$ indices, and

$$\zeta_p = \begin{cases} 1 & \text{for a symmetric state in the bosonic case} \\ \text{sgn}(p) = \pm 1 & \text{for an asymmetrical state in the fermionic case} \end{cases} \tag{I.3}$$

where sgn(p) is the signature of the permutation p. However, the expression of Eq. I.2 can become somewhat tedious, so we will use the much more practical formalism known as *second quantization* as reviewed in [4]. With this formalism we can represent our state with the occupation number basis (occ. basis) $|n_0, n_1, n_2, \ldots\rangle$ where $n_\lambda$ specifies the number of particles in the state $|lambda\rangle$. Nevertheless, because of the Pauli exclusion principle, we can further specify both symmetry cases separately:

$$n_\lambda \in \begin{cases} \mathbb{N}_0 & \text{for bosons} \\ \{0, 1\} & \text{for fermions.} \end{cases} \tag{I.4}$$

Now, we will define the "ladder operators" connecting these states as

$$\begin{aligned} \hat{a}_\lambda^\dagger |n_1, n_2, n_3, \ldots\rangle &= (n_\lambda + 1)^{\frac{1}{2}} \xi^{S_\lambda} |n_1, \ldots, n_\lambda + 1, \ldots\rangle \\ \hat{a}_\lambda |n_1, n_2, n_3, \ldots\rangle &= (n_\lambda)^{\frac{1}{2}} \xi^{S_\lambda} |n_1, \ldots, n_\lambda - 1, \ldots\rangle \end{aligned} \tag{I.5}$$

where $\xi = +1$ $(-1)$ for bosons (fermions) and $S_\lambda = \sum_i^{\lambda-1} n_i$. With these ladder operators, we now define Hermitian quadratures (Majoranas) as

$$\hat{x}_i \ (\hat{c}_{2i-1}) = \frac{\hat{a}_i^\dagger + \hat{a}_i}{\sqrt{2}}, \quad \hat{p}_i \ (\hat{c}_{2i}) = \mathrm{i}\frac{\hat{a}_i^\dagger - \hat{a}_i}{\sqrt{2}}. \tag{I.6}$$

These operators satisfy the following canonical (anti-)commutation relations (CCR)

$$\begin{cases} [\hat{a}_i, \hat{a}_j]_\xi = [\hat{a}_i^\dagger, \hat{a}_j^\dagger]_\xi = 0 \\ [\hat{a}_i, \hat{a}_j^\dagger]_\xi = \delta_{ij} \end{cases}, \quad \begin{cases} [\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \\ [\hat{x}_i, \hat{p}_j] = \mathrm{i}\delta_{ij} \end{cases}, \quad \Big\{ \{\hat{c}_i, \hat{c}_j\} = \delta_{ij} \tag{I.7}$$

where $[\hat{A}, \hat{B}]_+ = [\hat{A}, \hat{B}] = \hat{A}\hat{B} - (+1) \cdot \hat{B}\hat{A}$ and $[\hat{A}, \hat{B}]_- = \{\hat{A}, \hat{B}\} = \hat{A}\hat{B} - (-1) \cdot \hat{B}\hat{A}$. In Chapter II and Chapter III we will see an advantage in working with vectors with $2N$ operator components

$$\hat{\mathbf{r}} = \begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{a}}^\dagger \end{pmatrix} = \begin{pmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_N \\ \hat{a}_1^\dagger \\ \vdots \\ \hat{a}_N^\dagger \end{pmatrix}, \quad \hat{\mathbf{z}} = \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_N \\ \hat{p}_1 \\ \vdots \\ \hat{p}_N \end{pmatrix}, \quad \hat{\mathbf{c}} = \begin{pmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_{2N} \end{pmatrix} \tag{I.8}$$

because their CCR can be written in a more compact way

$$\begin{cases} [\hat{r}_i, \hat{r}_j]_\xi = \mathbf{\Omega}_{ij} \\ [\hat{r}_i, \hat{r}_j^\dagger]_\xi = \mathbf{\Sigma}^3_{ij} \end{cases}, \quad \Big\{ [\hat{z}_i, \hat{z}_j] = \mathrm{i}\mathbf{\Omega}_{ij} \quad , \quad \Big\{ \{\hat{c}_i, \hat{c}_j\} = (\mathbb{1}^{2N})_{ij} \tag{I.9}$$

with the $2N \times 2N$ matrices

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{0}^N & \mathbb{1}^N \\ -\mathbb{1}^N & \mathbf{0}^N \end{bmatrix}, \quad \mathbf{\Sigma}^3 = \begin{bmatrix} \mathbb{1}^N & \mathbf{0}^N \\ \mathbf{0}^N & -\mathbb{1}^N \end{bmatrix}. \tag{I.10}$$

We can also see that we can express the relation between the ladder operators and the quadratures (Majoranas) as a linear transformation

$$\hat{\mathbf{z}}, \hat{\mathbf{c}} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbb{1}^N & \mathbb{1}^N \\ -\mathrm{i}\mathbb{1}^N & \mathrm{i}\mathbb{1}^N \end{bmatrix} \cdot \hat{\mathbf{r}} = \mathbf{\Lambda} \cdot \hat{\mathbf{r}} \tag{I.11}$$

with $\mathbf{\Lambda}^{-1} = \mathbf{\Lambda}^\dagger \in \mathrm{U(2N)}$.

## I.2 Gaussian systems

We will call a state, represented by its *density matrix* $\hat{\rho}$, a *Gaussian* state if it is the thermal Gibbs state of a quadratic Hamiltonian of the form:

$$\begin{cases} \hat{H}_b = \frac{1}{2}\hat{\mathbf{z}}^T \boldsymbol{h}^z \hat{\mathbf{z}} + (\boldsymbol{d}^z)^T \hat{\mathbf{z}} = \frac{1}{2}\hat{\mathbf{r}}^\dagger \boldsymbol{h}^r \ \hat{\mathbf{r}} + (\boldsymbol{d}^r)^T \hat{\mathbf{r}} & \text{for bosons (bqH)} \\ \hat{H}_f = \mathrm{i}\frac{1}{2}\hat{\mathbf{c}}^T \boldsymbol{g}^c \hat{\mathbf{c}} = \mathrm{i}\frac{1}{2}\hat{\mathbf{r}}^\dagger \boldsymbol{g}^r \ \hat{\mathbf{r}} & \text{for fermions (fqH)} \end{cases} \tag{I.12}$$

where $\boldsymbol{h}^z$ ($\boldsymbol{g}^c$) is a real (skew-)symmetric $2N \times 2N$ matrix, $\boldsymbol{h}^r$ ($\boldsymbol{g}^r$) is a complex (skew-)hermitian $2N \times 2N$ matrix, and $\boldsymbol{d}^{z/r}$ is a real/complex $2N$ dimensional vector, as shown in Appendix A. The absence of linear terms in the fermionic model is due to the parity superselection rule (PSSR) [16], and we omit any overall additive constant, as it would just result in an energy shift and thus not influence the evolution of our system. In fact, it would also not change the ground or thermal state of the Hamiltonian. Thus, any Gaussian state (GS) can be written as

$$\hat{\rho}_G(\beta) = \frac{\mathrm{e}^{-\beta\hat{H}}}{\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}}\right]} =: \frac{\mathrm{e}^{-\beta\hat{H}_{\mathcal{N}}}}{Z} \tag{I.13}$$

where $\hat{H}$ is a quadratic Hamiltonian. However, since the purity condition is $\hat{\rho}^2 = \hat{\rho}$ we can see that, by construction,

$$\begin{cases} \left.\frac{\mathrm{e}^{-\beta\hat{H}_{\mathcal{N}}}}{\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}_{\mathcal{N}}}\right]}\right|_{\beta\in[0,\infty[} & \text{is } \textit{mixed}, \\ \left.\frac{\mathrm{e}^{-\beta\hat{H}_{\mathcal{N}}}}{\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}_{\mathcal{N}}}\right]}\right|_{\beta\to\infty} & \text{is } \textit{pure}. \end{cases} \tag{I.14}$$

*Remark*

In fact, any bosonic state, pure or mixed, represented by $\hat{\rho}$ has an equivalent representation by a function in the quadrature phase space, namely the Wigner characteristic function

$$\chi(\boldsymbol{\xi}) \equiv \mathrm{Tr}\left[\hat{\rho}\exp\left(\mathrm{i}\hat{\boldsymbol{x}}^{\boldsymbol{T}}\boldsymbol{\Omega}\boldsymbol{\xi}\right)\right] \tag{I.15}$$

where $\boldsymbol{\xi} \in \mathbb{R}^{2N}$ and the inverse function given by

$$\hat{\rho} = \frac{1}{(2\pi)^N}\int d^{2N}\boldsymbol{\xi}\chi(\boldsymbol{\xi})\exp\left(-\mathrm{i}\hat{\boldsymbol{x}}^{\boldsymbol{T}}\boldsymbol{\Omega}\boldsymbol{\xi}\right). \tag{I.16}$$

The Wigner characteristic function is the Fourier transform of the normalized nonpositive (quasi-)probability distribution, the Wigner function

$$W(\boldsymbol{x}) = \frac{1}{(2\pi)^{2N}}\int d^{2N}\boldsymbol{\xi}\chi(\boldsymbol{\xi})\exp\left(-\mathrm{i}\boldsymbol{x}^T\boldsymbol{\Omega}\boldsymbol{\xi}\right). \tag{I.17}$$

In fact, for a bosonic Gaussian state, its Wigner function, and therefore its Wigner characteristic function, is as the name of the state indicates, Gaussian [15].

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We will now call a reversible quantum channel *Gaussian* if it preserves the Gaussianity of any Gaussian state. Moreover, because such reversible quantum channels are represented by unitaries, we will represent a Gaussian reversible quantum channel by the *Gaussian unitary* (GU) $\hat{U}_G$. However, because any unitary is generated by a Hamiltonian, we can write according to Eq. I.13

$$\hat{U}_G\hat{\rho}_G\hat{U}_G^\dagger = e^{\mathrm{i}\hat{H}_G}\frac{e^{-\hat{H}}}{Z}e^{-\mathrm{i}\hat{H}_G} = \frac{e^{\mathrm{i}\hat{H}_G}}{Z}e^{-\hat{H}-\mathrm{i}\hat{H}_G+\frac{\mathrm{i}}{2}\left[\hat{H},\hat{H}_G\right]+\frac{\mathrm{i}}{12}\left[\hat{H},\left[\hat{H},\hat{H}_G\right]\right]+\cdots} =: \frac{e^{\mathrm{i}\hat{H}_G}}{Z}e^{\hat{O}_1} \tag{I.18}$$

with $\hat{O}_1 = -\hat{H} - \mathrm{i}\hat{H}_G + \frac{\mathrm{i}}{2}\left[\hat{H}, \hat{H}_G\right] + \frac{\mathrm{i}}{12}\left[\hat{H}, \left[\hat{H}, \hat{H}_G\right]\right] + \cdots$ according to the Baker-Campbell-Hausdorff formula. Consequently, we can rewrite

$$\hat{U}_G \hat{\rho}_G \hat{U}_G^\dagger = \frac{1}{Z} e^{-\hat{O}_1 + \mathrm{i}\hat{H}_G - \frac{\mathrm{i}}{2}\left[\hat{O}_1, \hat{H}_G\right] - \frac{\mathrm{i}}{12}\left[\hat{O}_1, \left[\hat{O}_1, \hat{H}_G\right]\right] + \cdots} =: \frac{e^{-\hat{O}_2}}{Z} \tag{I.19}$$

However, because quantum channels preserve the trace, the sufficient condition for $\frac{e^{-\hat{O}_2}}{Z}$ to be a GS is that all commutators are quadratic or linear. Thus, $\hat{H}_G$ needs to be a *quadratic hamiltonian* in the form of Eq. I.12 as the commutator conserves the polynomial degree due to the CCR in Eq. I.20. Consequently, we can state that bosonic Gaussian unitaries (bGU) as well as fermionic Gaussian unitaries (fGU), are, in fact, unitaries generated by quadratic Hamiltonians.

## I.3 Compressed quantum circuits

Throughout this thesis, we will work with a quantum circuit model (QC) that, depending on whether we are in the fermionic[1] or the bosonic case, can be decomposed into the following components:

- An input state $\hat{\rho}_{in}$ that represents the initial state of our system. For a *qubit circuit*, the pure initial state of the N-entangled particle system is represented in our computational basis, such as, for example, the product state $|0001011\rangle$ for 7 qubits. For a *bosonic circuit* we will directly represent the state of our N-mode system in the occ. basis, such as, for example, $|21073\rangle$ for a pure state of 5 modes. We will assume that we arrange the qubits/modes as a one-dimensional grid and thus we can characterize qubits with a notion of proximity.

- A combination of M two-[2] or one-particle/mode quantum gates, which we will call *elementary gates*, that implement the desired algorithm, represented by the unitaries $\hat{U}_M \cdot \hat{U}_{M-1} \cdot \ldots \cdot \hat{U}_1 =: \hat{U}_{QC}$. We restrict ourselves to two- or one-particle/mode gates to be able to quantify the difficulty to implement the circuit experimentally, moreover we will define the *depth* of our circuit as the longest path in the circuit and thus of the order of the number of such elementary gates.

- A final measurement on a single particle/mode[3], retrieving the desired information from our probability distribution. For the *qubit case* we represent this measurement, WLOG, by the expectation value of the Pauli Z operator, acting on the k-th qubit line, $\langle \hat{Z}_{k\rangle} \rangle_f$ for the final state $\hat{\rho}_f := \hat{U}_{QC} \hat{\rho}_{in} \hat{U}_{QC}^\dagger$. For the *bosonic case* we measure the expectation value of a *one-body operator* $\langle \hat{O} \rangle_f$ with $\hat{O} = \sum_{\lambda,\lambda'=0}^{N} \hat{a}_\lambda^\dagger \boldsymbol{O}_{\lambda\lambda'} \hat{a}_{\lambda'}$ where $\boldsymbol{O}_{\lambda\lambda'} = \langle \lambda | \hat{O}_{(1)} | \lambda' \rangle$ is the matrix element of $\hat{O}_{(1)}$ that acts on $\mathcal{H}_{(1)}$.

We define the process of *compressing* a QC *efficiently* as recoding it into a new reduced-space QC, i.e. whose number of needed qubits/modes is *reduced*, such that the required

[1] We will show in Section II.2 that fermionic and qubit circuits are equivalent, thus we will refer commonly to fermionic circuit as qubit circuits.
[2] If we talk about a two-particle / mode gate we implicitely talk about a nearest neigbhour (n.n.) gate.
[3] The implication of such a single qubit/mode measurement will be discussed in Chapter IV.2

number of reduced-space elementary gates stays *polynomial* in the initial number of elementary gates.

## Chapter Summary:

- Throughout this thesis, we will mainly use the *operator vectors* $\hat{\mathbf{r}} = \begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{a}}^\dagger \end{pmatrix}$, $\hat{\mathbf{z}} = \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{p}} \end{pmatrix}$ and $\hat{\mathbf{c}} = \begin{pmatrix} \hat{c}_1 & \cdots & \hat{c}_{2N} \end{pmatrix}^T$
- obeying the following *canonical commutation relations* (CCR):

$$\begin{cases} [\hat{r}_i, \hat{r}_j]_\xi = \mathbf{\Omega}_{ij} \\ [\hat{r}_i, \hat{r}_j^\dagger]_\xi = \mathbf{\Sigma}^3_{ij} \end{cases}, \quad \left\{ [\hat{z}_i, \hat{z}_j] = \mathrm{i}\mathbf{\Omega}_{ij} \right. , \quad \left\{ \{\hat{c}_i, \hat{c}_j\} = (\mathbb{1}^{2N})_{ij}. \right. \tag{I.20}$$

- Gaussian *states* are thermal Gibbs states of *quadratic* Hamiltonians.
- Gaussian *unitaries* are generated by *quadratic* Hamiltonians.
- We will simplify a *quantum circuit* to an *initial state* $\hat{\rho}_{in}$, M consecutive *elementary gates*, and a *final measurement* represented by $\langle \hat{Z}_k \rangle_f$ for the fermionic case, and $\langle \hat{O} \rangle_f$, with $\hat{O}$ being a one-body operator, in the bosonic case.
- With *efficient compression* we define the action of re-encoding a QC into a *reduced-space* QC with a depth that stays polynomial in the initial number of elementary gates.

# CHAPTER II

# Compression of fermionic Gaussian circuits

This chapter will be structured as follows. First, in Section II.1 we will introduce the underlying mathematical concept that allows a reduced-space description of fermionic Gaussian circuits (fGC); then in Section II.2 we will discuss the equivalence between fermionic and qubit systems to transfer this mathematical concept to our qubit circuits. This is followed by Section III.2, which reviews the first implication of the reduced description, namely the classical simulatability as proposed by Jozsa et al. [6]. The chapter will then go on to review arguments related to the ones from [7] that fGCs are simulatable by a reduced-space QC in Section III.3 to finally, in Section II.5, outline the ensuing possible encodings, also closely related to [7], to characterize whether fGCs can be compressed efficiently.

## II.1 Compressed description of fGUs

As we saw in Chapter I, to achieve compression, we need to describe the action of our QC with a reduced number of parameters and then implement efficiently this reduced amount of data into a QC. In this section, we will look at the first part, that is, how to describe the action of our circuit with fewer parameters.

Consider a fermionic QC described by the action of the total unitary $\hat{U}_{QC}$ on our initial state $|\psi_i\rangle$ in the Schrödinger picture, then this action can also be interpreted as the evolution of our final measurement observable in the Heisenberg picture as $\hat{U}_{QC}\hat{O}\hat{U}_{QC}^{\dagger}$. However, from the definition of the ladder operators in Eq. I.5 we can follow that $\{\hat{a}_i^{\dagger}, \hat{a}_j\}$, and thus their unitary linear combination $\{\hat{c}_i, \hat{c}_j\}$, are complete sets of *generators* for the operator algebra acting on our fermionic N particle system. Consequently, to describe the action of our fermionic QC, it is sufficient to describe the evolution of our ladder operators or our Majoranas in the Heisenberg picture.

Nevertheless, if we consider a *Gaussian* fermionic circuit and if we recall that fGUs are generated by fqHs, we can see that the evolutions of our Majoranas and our ladder operators have special forms.

**Theorem 1.** *Let $\hat{U}_G$ be a fermionic Gaussian unitary generated by a quadratic Hamiltonian, $\hat{\mathbf{c}}$ the Majorana operator vector, and $\hat{\mathbf{r}}$ the ladder operator vector of a fermionic N-particle system; then:*

$$\hat{U}_G^\dagger \hat{\mathbf{c}} \hat{U}_G = \boldsymbol{R} \cdot \hat{\mathbf{c}} \quad \textit{and} \quad \hat{U}_G^\dagger \hat{\mathbf{r}} \hat{U}_G = \boldsymbol{U} \cdot \hat{\mathbf{r}} \tag{II.1}$$

*where $\boldsymbol{R} \in SO(2N)$ and $\boldsymbol{U} \in U(2N)$.*

*Proof.* Recalling that fGUs are generated by quadratic Hamiltonians of the form

$$\hat{H}_f = \mathrm{i}\frac{1}{2}\hat{\mathbf{c}}^T \boldsymbol{g}^c \hat{\mathbf{c}}, \tag{II.2}$$

we can see $\hat{U}_G^\dagger \hat{c}_k \hat{U}_G$ as the time developed $\hat{c}_k(0) = \hat{c}_k$ in this fqH $\hat{H}_f$. Thus, we can use the Heisenberg equation to get

$$\frac{d\hat{c}_k(t)}{dt} = \mathrm{i}\left[H, \hat{c}_k(t)\right] \overset{\text{Eq. I.12}}{=} -\frac{1}{2}\boldsymbol{g}^c_{ij} \hat{U}_G^\dagger \left(\hat{c}_i \left[\hat{c}_j, \hat{c}_k\right] + \left[\hat{c}_i, \hat{c}_k\right] \hat{c}_j\right) \hat{U}_G. \tag{II.3}$$

Considering that $[A, B] = \{A, B\} - 2BA$ we have

$$\left(\hat{c}_i \left[\hat{c}_j, \hat{c}_k\right] + \left[\hat{c}_i, \hat{c}_k\right] \hat{c}_j\right) = \hat{c}_i\{\hat{c}_j, \hat{c}_k\} - 2\hat{c}_i\hat{c}_k\hat{c}_j + \hat{c}_j\{\hat{c}_i, \hat{c}_k\} - 2\hat{c}_k\hat{c}_i\hat{c}_j \tag{II.4}$$

and because of the CCR of Eq. I.20 we have $\hat{c}_k\hat{c}_i\hat{c}_j = -\hat{c}_i\hat{c}_k\hat{c}_j + \{\hat{c}_k, \hat{c}_i\}\hat{c}_j$ and with $\{\hat{c}_i, \hat{c}_j\} = \delta_{ij}$ we get

$$\frac{d\hat{c}_k(t)}{dt} = -\frac{1}{2}\hat{U}_G^\dagger\, \boldsymbol{g}^c_{ij}(\hat{c}_i\delta_{jk} - \hat{c}_j\delta_{ki})\hat{U}_G = -\frac{1}{2}\hat{U}_G^\dagger \left[\left((\boldsymbol{g}^c)^T - \boldsymbol{g}^c\right) \cdot \hat{c}\right]_k \hat{U}_G. \tag{II.5}$$

Moreover, $\boldsymbol{g}^c$ is skew-symmetric, as shown in Appendix A, thus we get the differential equation

$$\frac{d\hat{\mathbf{c}}(t)}{dt} = \boldsymbol{g}^c \cdot \hat{\mathbf{c}}(t), \tag{II.6}$$

thus by solving it and setting $t = 1$ we retrieve

$$\hat{U}_G^\dagger \hat{\mathbf{c}} \hat{U}_G = e^{\boldsymbol{g}^c} \cdot \hat{\mathbf{c}} \coloneqq \boldsymbol{R} \cdot \hat{\mathbf{c}}. \tag{II.7}$$

However, we know that real skew-symmetric matrices are infinitesimal generators of special orthogonal matrices; hence $e^{\boldsymbol{g}^c} = \boldsymbol{R}$ is an element of $SO(2N)$.

To prove the transformation of $\hat{\mathbf{r}}$ recall that $\hat{\mathbf{c}} = \boldsymbol{\Lambda} \cdot \hat{\mathbf{r}}$, consequently, we have

$$\hat{U}^\dagger \hat{\mathbf{r}} \hat{U} = \hat{U}^\dagger \boldsymbol{\Lambda}^\dagger \hat{\mathbf{c}} \hat{U} = \boldsymbol{\Lambda}^\dagger \cdot \boldsymbol{R} \cdot \boldsymbol{\Lambda} \cdot \hat{\mathbf{r}} \tag{II.8}$$

but because $\boldsymbol{\Lambda} \in U(2N)$, we have $\boldsymbol{\Lambda}^\dagger \cdot \boldsymbol{R} \cdot \boldsymbol{\Lambda} =: \boldsymbol{U} \in U(2N)$. □

The importance of this theorem lies in the fact that even if $\hat{H}$ comprises any possible combination of the basis elements and thus could map onto the entire Hilbert space $\mathcal{H}_{(1)}^{\otimes N}$, one can represent the action of $\hat{U}_G$ on our Majoranas / ladder operators with at most $4N^2$ parameters. This insight is the underlying concept, which allows compression in qubit circuits, and we will now see how to concretely implement it.

## II.2 Fermionic and qubit systems are equivalent

Consider a qubit system of N qubits represented by the computational basis $\{|z_1 z_2 \dots z_N\rangle : z_i = 0, 1\}$ with the usual Pauli operators $\hat{X}_i, \hat{Y}_i, \hat{Z}_i$, then we can interpret this basis as an occ. basis and implement the corresponding ladder operators as [8]

$$\hat{a}_j \equiv -\left(\otimes_{k=1}^{j-1} \hat{Z}_k\right) \otimes \sigma_j \tag{II.9}$$

$\sigma_j =$ indicates $|0\rangle\langle 1|$ that acts on the j-th qubit. This transformation is referred to as the Jordan-Wigner transformation. Similarly, we can implement the Majoranas as [7]:

$$\begin{aligned}
&\sqrt{2}\cdot\hat{c}_1 = \hat{X}\,\hat{I}\cdots\hat{I} \quad \sqrt{2}\cdot\hat{c}_3 = \hat{Z}\,\hat{X}\,\hat{I}\cdots\hat{I} \quad \cdots \quad \sqrt{2}\cdot\hat{c}_{2k-1} = \overbrace{\hat{Z}\ \cdots\hat{Z}}^{j-1 \text{ times}}\,\hat{X}\,\hat{I}\cdots\hat{I} \\
&\sqrt{2}\cdot\hat{c}_2 = \hat{Y}\,\hat{I}\cdots\hat{I} \quad \sqrt{2}\cdot\hat{c}_4 = \hat{Z}\,\hat{Y}\,\hat{I}\cdots\hat{I} \quad \cdots \quad \sqrt{2}\cdot\hat{c}_{2k} = \overbrace{\hat{Z}\ \cdots\hat{Z}}^{j-1 \text{ times}}\,\hat{Y}\,\hat{I}\cdots\hat{I}.
\end{aligned} \tag{II.10}$$

However, because $\{\hat{I}, \hat{X}\,, \hat{Y}\,, \hat{Z}\,\}$ is a basis for any operator, any qubit gate is generated by a Hamiltonian, which can be expressed in terms of Pauli operators and identities. Consequently, we can directly define qubit gates as Gaussian if they are generated by a Hamiltonian that is quadratic in the Majoranas as defined in Eq. II.10. Such a Gaussian qubit gate can be decomposed into Gaussian elementary qubit gates, which, by definition, act only on one or two n.n. qubit lines. Such Gaussian elementary qubit gates are called *matchgates* (MG) and can be written in the form of [7]

$$G(A,B) = \begin{pmatrix} p & 0 & 0 & q \\ 0 & w & x & 0 \\ 0 & y & z & 0 \\ r & 0 & 0 & s \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \tag{II.11}$$

where $A, B \in SU(2)$ or $A, B \in U(2)$. Consequently, we call such Gaussian qubit circuits matchgate circuits (MGC). However, by converting Theorem 1 to the action of matchgates, we can describe the action of a MGC with fewer parameters. We will now discuss the implications of this reduced description.

## II.3 Classical simulatability of MGCs

In the next sections, we review R. Jozsa et al.'s thorough analysis [7] of compressed quantum computation with matchgate circuits, and propose additional aspects originating from a more general fermionic approach.

Consider a circuit of N qubits, M matchgates and some input state $\hat{\rho}_{in}$[1], with $\hat{U}_G := \hat{U}_M \cdot \ldots \cdot \hat{U}_2 \cdot \hat{U}_1$, then

$$\langle \hat{Z}_k \rangle_f = \langle \hat{U}_G^\dagger \hat{Z}_k \hat{U}_G \rangle_{in}. \tag{II.12}$$

Using Eq. II.10 we can see that $\hat{Z}_k = -2\mathrm{i}\hat{c}_{2k-1}\hat{c}_{2k}$, hence

$$\langle \hat{Z}_k \rangle_f = -2\mathrm{i}\langle \hat{U}_G^\dagger \hat{c}_{2k-1} \hat{U}_G \cdot \hat{U}_G^\dagger \hat{c}_{2k} \hat{U}_G \rangle_{in}, \tag{II.13}$$

[1] We will restrict us to circuits where all qubits are acted on by at least one of the M elementary gates, thus $M \geq N/2$

and here we can use Theorem 1 to get

$$\frac{\mathrm{i}}{2}\langle \hat{Z}_k \rangle_f = \boldsymbol{R}_{2k-1,i}\boldsymbol{R}_{2k,j}\langle \hat{c}_i \hat{c}_j \rangle_{in} \tag{II.14}$$

where $\boldsymbol{R}$ is the rotation assimilated with $\hat{U}_G$. Defining $\boldsymbol{\Gamma}_{ij} \coloneqq \langle \hat{c}_i \hat{c}_j \rangle_{in} = \mathrm{Tr}\left[\hat{\rho}_{in}\hat{c}_i\hat{c}_j\right]$ as the *correlation matrix* we can rewrite the expectation value as

$$\frac{\mathrm{i}}{2}\langle \hat{Z}_k \rangle_f = \langle 2k-1 | \, \boldsymbol{R}\boldsymbol{\Gamma}\boldsymbol{R}^T \, | 2k \rangle \, . \tag{II.15}$$

Interestingly, our initial expectation value, where the observable was described by $\mathcal{O}\left(2^N\right)$ parameters, has been reduced to a matrix element of a $2N \times 2N$ matrix. We can even go a step further and discuss the classical computation time of each element in Eq. II.15.

- If we restrict ourselves to an initial *pure product* state $\hat{\rho}_{in} = |\psi_{in}\rangle\langle\psi_{in}|$ with $\psi_{in} = \psi_1 \otimes \cdots \otimes \psi_N$, each entry of the correlation matrix $\boldsymbol{\Gamma}$ can be calculated, for $i \neq j$, as a product of two terms[2] $\langle \psi_i | \hat{c}_i | \psi_i \rangle \cdot \langle \psi_j | \hat{c}_j | \psi_j \rangle = \langle \psi_1 | \mathbb{1} | \psi_1 \rangle \cdot \ldots \cdot \langle \psi_i | \hat{c}_i | \psi_i \rangle \cdot \ldots \cdot \langle \psi_j | \hat{c}_j | \psi_j \rangle \cdot \ldots \cdot \langle \psi_N | \mathbb{1} | \psi_N \rangle$ and for $i = j$ directly as one term $\langle \psi_i | \hat{c}_i^2 | \psi_i \rangle$. Thus, the $4N^2 \leq 16M^2$ entries of $\boldsymbol{\Gamma}$ can be calculated in $\mathcal{O}(N^2)$, hence, $\mathcal{O}(M^2)$ time.

- By decomposing the rotation in Eq. II.15 as $\boldsymbol{R} = \boldsymbol{R}_M \cdot \ldots \cdot \boldsymbol{R}_2 \cdot \boldsymbol{R}_1$ with each $\boldsymbol{R}_i \in SO(4)$ extended to 2n dimensions, corresponding to its matchgate $\hat{U}_i$ acting on the k-th and (k+1)-th qubit, according to [7]:

$$\hat{U}_i^\dagger \hat{c}_j \hat{U}_i = \begin{cases} \sum_{l=2k-1}^{2k+2} \boldsymbol{R}_i[j,l]\hat{c}_l & \text{for } j = 2k-1, 2k, 2k+1, 2k+2 \\ \hat{c}_j & \text{for all other } j \text{ 's} \end{cases} \tag{II.16}$$

  we can compute $\boldsymbol{R}$ as a product of $M$ $2N \times 2N$ matrices in $\mathcal{O}(M^3)$ time.

- Finally, to compute the matrix element in Eq. II.15, we need to sum over at most $\mathcal{O}(N^2) \leq \mathcal{O}(M^2)$ terms.

Consequently, we can classically compute the expectation value $\langle \hat{Z}_k \rangle_f$ in poly$(M)$ time[3], thus, by defining a "polynomial-sized" MGC as a MGC having a number of elementary MGs equal to a polynomial of the number of qubits N, we arrive at the first main result

**Result 1** Any polynomial-sized MGC having an initial state which is a pure product state, can be classically computed in polynomial time.

This is in fact a *remarkable result*, considering that to naively simulate a quantum system by classical means, one would need ressources that are scaling exponentially with the system size. However, we can extend the analysis of Eq. II.15 to find another really interesting proprety.

[2] Because of the PSSR $|\psi_i\rangle = |0\rangle, |1\rangle$, consequently $\langle \psi_i | \hat{c}_i | \psi_i \rangle$ can be directly computed without needing to calculate a superposition sum and thus is only accounted for as one term.

[3] Here we related the computation time of each element to an order of number of gates M because it is defining the depth of the circuit.

## II.4 Reduced-space simulatability of MGCs

The form of Eq. II.15 is suggestive of a possible reduced-space quantum simulability, and since we have already specified that $\boldsymbol{R}$ is an orthogonal rotation, we only need to take a closer look at $\boldsymbol{\Gamma}$ to complete the picture. To do so, we firstly take a look at the ladder operator correlation matrix

$$(\boldsymbol{\Gamma}_{l.o.})_{ij} \coloneqq \langle \hat{r}_i^\dagger \hat{r}_j \rangle = \left[\boldsymbol{\Lambda}^\dagger \boldsymbol{\Gamma} \boldsymbol{\Lambda}\right]_{ij} =: \begin{pmatrix} \boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}} & \boldsymbol{\Gamma}_{\hat{a}\hat{a}} \\ \boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}^\dagger} & \boldsymbol{\Gamma}_{\hat{a}\hat{a}^\dagger} \end{pmatrix}_{ij} \tag{II.17}$$

with

$$\begin{cases} (\boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}})_{ij} = \langle \hat{a}_i^\dagger \hat{a}_j \rangle = \overline{\langle \hat{a}_j^\dagger \hat{a}_i \rangle} = (\boldsymbol{\Gamma}^\dagger_{\hat{a}^\dagger \hat{a}})_{ij} \\ (\boldsymbol{\Gamma}_{\hat{a}\hat{a}^\dagger})_{ij} = \langle \hat{a}_i \hat{a}_j^\dagger \rangle = \overline{\langle \hat{a}_j \hat{a}_i^\dagger \rangle} = (\boldsymbol{\Gamma}^\dagger_{\hat{a}\hat{a}^\dagger})_{ij} \\ (\boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}^\dagger})_{ij} = \langle \hat{a}_i^\dagger \hat{a}_j^\dagger \rangle = \overline{\langle \hat{a}_j \hat{a}_i \rangle} = (\boldsymbol{\Gamma}^\dagger_{\hat{a}\hat{a}})_{ij}. \end{cases} \tag{II.18}$$

Thus, $\boldsymbol{\Gamma}_{l.o.}$ is Hermitian and, therefore, can be diagonalized by a unitary $\boldsymbol{U} \in U(2N)$ according to

$$\boldsymbol{\Gamma}^D_{l.o.} = \boldsymbol{U}^\dagger \boldsymbol{\Gamma}_{l.o.} \boldsymbol{U} = \begin{pmatrix} v_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & v_{2N} \end{pmatrix} \tag{II.19}$$

with $v_i \in \mathbb{R}$. However, as shown in [12], diagonalizing the correlation matrix amounts to finding a canonical transformation of modes, i.e. a set of new modes that also satisfies the CAR, such that the expression in those new modes of the density matrix, for which the correlations of $\boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}}$ are evaluated, represents N decoupled 1-mode systems of the form $\hat{\rho}^D \overset{!}{=} \bigotimes_{k=1}^N \hat{\rho}_k^1$, because then

$$\left(\boldsymbol{\Gamma}^D_{l.o.}\right)_{i \neq j} = \mathrm{Tr}\left[\left(\bigotimes \hat{\rho}_k^1\right) \hat{r}_i \hat{r}_j\right] \overset{i \neq j}{=} \mathrm{Tr}\left[\hat{\rho}_i^1 \hat{r}_i\right] \cdot \mathrm{Tr}\left[\hat{\rho}_j^1 \hat{r}_j\right] \overset{\text{PSSR}}{=} 0. \tag{II.20}$$

However, with this so-called *uncorrelated state*, we can compute

$$\begin{cases} (\boldsymbol{\Gamma}^D_{\hat{a}^\dagger \hat{a}^\dagger})_{ii} = \langle \hat{a}_i^{\dagger 2} \rangle = 0 = \langle \hat{a}_i^2 \rangle = (\boldsymbol{\Gamma}^D_{\hat{a}\hat{a}})_{ii} \\ (\boldsymbol{\Gamma}^D_{\hat{a}^\dagger \hat{a}})_{ii} = \langle \hat{a}_i^\dagger \hat{a}_i \rangle = \langle \hat{N}_i \rangle \\ (\boldsymbol{\Gamma}^D_{\hat{a}\hat{a}^\dagger})_{ii} = \langle \hat{a}_i \hat{a}_i^\dagger \rangle = 1 - \langle \hat{N}_i \rangle \end{cases} \tag{II.21}$$

thus, the diagonal form of the ladder correlation matrix can be written as

$$\boldsymbol{\Gamma}^D_{l.o.} = \begin{pmatrix} \bar{n}_1 & 0 & \cdots & & & \cdots & 0 \\ 0 & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & \bar{n}_N & & & & \\ & & & 1-\bar{n}_1 & & \ddots & \vdots \\ \vdots & & & \ddots & & \ddots & 0 \\ 0 & \dots & & & \dots 0 & & 1-\bar{n}_N \end{pmatrix} \tag{II.22}$$

where $\bar{n}_i = \langle \hat{N}_i \rangle$ is the *mean occupation number* of the i-th uncorrelated mode. However, because of the PSSR the density matrix needs to commute with the parity operator, so

the only 1-mode state which is physically allowed is given, in the computational basis, by

$$\hat{\rho}^1(p) \doteq \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix} \tag{II.23}$$

with $p \in [0,1]$ given by the positive definiteness condition, and because

$$p = \mathrm{Tr}\left[\begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \doteq \mathrm{Tr}\left[\hat{\rho}^1(p)\hat{N}\right] = \bar{n}. \tag{II.24}$$

we can fully describe any 1-mode state by its mean occupation number $\bar{n}$, where,

$$\begin{cases} \hat{\rho}^1_{pur} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \Longrightarrow \quad \bar{n} = p = 0, 1 \quad \text{for } \textit{pure states,} \\ \hat{\rho}^1_{mix}(\bar{n}) = \begin{pmatrix} 1-\bar{n} & 0 \\ 0 & \bar{n} \end{pmatrix}\Big|_{p\in]0,1[} & \Longrightarrow \quad \bar{n} \in ]0,1[ \quad \text{for } \textit{mixed states.} \end{cases} \tag{II.25}$$

Consequently, if we know the uncorrelated form of a state $\hat{\rho}^D$ we can compute its diagonal form from Eq. II.22, and we can also note that for *pure states*, and, what we are going to call, *"equi-occupied mixed states"* where the uncorrelated form can be written as

$$\hat{\rho}^D = \bigotimes_{k=1}^{N} \hat{\rho}_k \quad | \quad \hat{\rho}_k = \hat{\rho}^1_{mix}(p), \hat{\rho}^1_{mix}(1-p), \tag{II.26}$$

$\mathbf{\Gamma}_{l.o.}$ only has *two eigenvalues*, namely 0, 1 for the pure state[4] and $\bar{n}_1$, $1-\bar{n}_1$ for the mixed state.

However, because similarity transformations conserve hermiticity and the eigenspectrum, the same applies also for $\mathbf{\Gamma} = \mathbf{\Lambda}^\dagger \mathbf{\Gamma}_{l.o.} \mathbf{\Lambda}$. Consequently, we can see that $\mathbf{\Gamma}$ satisfies a special and useful condition.

**Theorem 2.** *Let $\mathbf{\Gamma}$ be a correlation matrix of a pure state or a equi-occupied mixed state of a fermionic N-particle system. Then there exists an $\alpha \in \mathbb{C}$ such that:*

$$\mathrm{i}\,(\mathbf{\Gamma} + \alpha \cdot \mathbb{1}) \in O(2N). \tag{II.27}$$

*In fact, with $\bar{n} \in [0,1]$ the expectation value of the one-mode number operator of the uncorrelated state*

$$\alpha = -\frac{1}{2} + \mathrm{i}\sqrt{1 - \left(\bar{n} - \frac{1}{2}\right)^2}. \tag{II.28}$$

[4] This is due to the fact that for the state to be pure, each 1-mode state needs to be pure, hence the eigenvalues can only be 0 and 1.

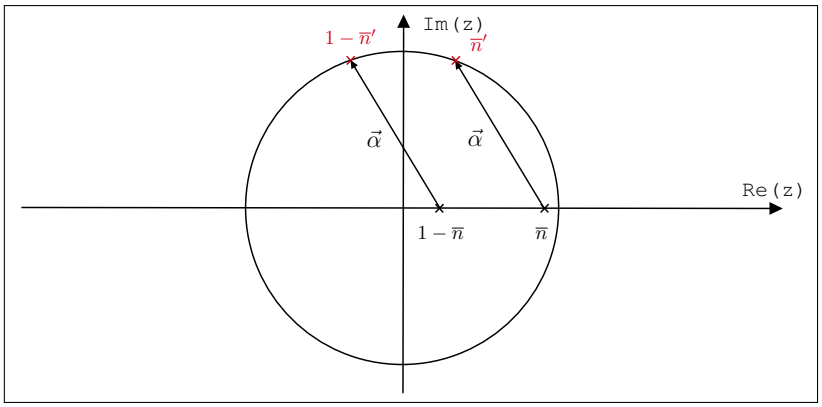


Figure II.1.: Graphical proof of Theorem 2.

*Proof.* Let $\boldsymbol{A}$ be a $2N \times 2N$ Hermitian matrix; thus, there exists a unitary $\boldsymbol{U}$ such that $\boldsymbol{U}^\dagger \boldsymbol{A} \boldsymbol{U} = \boldsymbol{A}^D = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2N})$ with $\lambda_i \in \mathbb{R}$. Hence,

$$\boldsymbol{A} + \alpha \cdot \mathbb{1} = \boldsymbol{U}\boldsymbol{A}^D\boldsymbol{U}^\dagger + \alpha \cdot \boldsymbol{U}\boldsymbol{U}^\dagger \cdot \mathbb{1} = \boldsymbol{U}\left(\boldsymbol{A}^D + \alpha \cdot \mathbb{1}\right)\boldsymbol{U}^\dagger \tag{II.29}$$

and, therefore, $\boldsymbol{A} + \alpha \cdot \mathbb{1}$ is unitary iff $\boldsymbol{A}^D + \alpha \cdot \mathbb{1}$ is unitary. However, this condition can be written as

$$\left(\boldsymbol{A}^D + \alpha \cdot \mathbb{1}\right)^\dagger \left(\boldsymbol{A}^D + \alpha \cdot \mathbb{1}\right) = \mathbb{1}, \tag{II.30}$$

thus,

$$(\lambda_i + \overline{\alpha})(\lambda_i + \alpha) = 1 \quad \forall i = 1, \ldots, 2N \tag{II.31}$$

and solving this equation, we get

$$\lambda_i = \pm\sqrt{1 - \mathrm{Im}\,(\alpha)^2} - \mathrm{Re}\,(\alpha) \quad \forall i. \tag{II.32}$$

Consequently, because for all $\lambda_1, \lambda_2$ such that $|\lambda_1 - \lambda_2| \leq 2$ there exists an $\alpha \in \mathbb{C}$ such that $\lambda_{1,2} = \pm\sqrt{1 - \mathrm{Im}\,(\alpha)} - \mathrm{Re}\,(\alpha)$, and because $\boldsymbol{\Gamma}$ is hermitian, there exists an $\alpha \in \mathbb{C}$ such that $\boldsymbol{\Gamma} + \alpha \cdot \mathbb{1}$ is unitary iff $\boldsymbol{\Gamma}$ only has the two eigenvalues $\bar{n}_1$ and $1 - \bar{n}_1$ with $\bar{n}_1 \in [0, 1]$, which is given for a pure or a equi-occupied mixed state. In fact, if

$$\begin{cases} \bar{n}_i = -\mathrm{Re}\,(\alpha) \pm \sqrt{1 - \mathrm{Im}\,(\alpha)^2} \\ 1 - \bar{n}_i = -\mathrm{Re}\,(\alpha) \mp \sqrt{1 - \mathrm{Im}\,(\alpha)^2}, \end{cases} \tag{II.33}$$

then

$$\begin{cases} \mathrm{Re}\,(\alpha) = -\frac{1}{2} \\ \mathrm{Im}\,(\alpha) = \sqrt{1 - \left(\bar{n}_i - \frac{1}{2}\right)^2}. \end{cases} \tag{II.34}$$

Moreover, for this $\alpha$ we have

$$[\overline{\mathrm{i}(\boldsymbol{\Gamma} + \alpha \cdot \mathbb{1})}]_{ij} = \begin{cases} \overline{\mathrm{i}\langle \hat{c}_i \hat{c}_j \rangle} = -\mathrm{i}\langle \hat{c}_j \hat{c}_i \rangle \overset{\{\hat{c}_j, \hat{c}_i\}=0}{=} \mathrm{i}\langle \hat{c}_i \hat{c}_j \rangle & \text{for } i \neq j \\ \overline{\mathrm{i}\langle \hat{c}_i^2 \rangle} + \overline{\mathrm{i}\alpha} \overset{\hat{c}_i^2 = 1/2}{=} -\sqrt{1 - \left(\bar{n} - \frac{1}{2}\right)^2} & \text{for } i = j \end{cases}, \tag{II.35}$$

hence $\mathrm{i}(\boldsymbol{\Gamma} + \alpha \cdot \mathbb{1})$ is a real unitary for a pure or a equi-occupied mixed state. □

However, because

$$\langle 2k-1|\, \boldsymbol{R}\mathbb{1}\boldsymbol{R}^T\, |2k\rangle = \langle 2k-1|\, \mathbb{1}\, |2k\rangle = 0, \tag{II.36}$$

we have

$$-\frac{1}{2}\langle \hat{Z}_k\rangle_f = \langle 2k-1|\, \boldsymbol{R}\,(\mathrm{i}\boldsymbol{\Gamma} + \mathrm{i}\alpha\cdot\mathbb{1})\,\boldsymbol{R}^T\, |2k\rangle\,, \tag{II.37}$$

and therefore, for pure and equi-occupied mixed states, computing the final expectation value is equivalent to computing the probability distribution of a developed state $|\psi\rangle_f = \hat{U}\,|2k\rangle$ with $\hat{U} \doteq \boldsymbol{R}\,(\mathrm{i}\boldsymbol{\Gamma} + \mathrm{i}\alpha\cdot\mathbb{1})\,\boldsymbol{R}^T \in O(2N)$.

As a result, any MGC with M matchgates and an initial N-qubit state $\hat{\rho}_{in}$ that is an initial pure or equi-occupied mixed state, can be mapped, with the right *encoding*, to a fermionic QC with $\mathcal{O}(\log_2 N)$ qubits, because $\mathcal{O}(\log_2 N)$ qubits are sufficient to represent 2N degrees of freedom[5]. We will now determine the exact encoding of such a compression to characterize the exact number of qubits and elementary operations needed.

## II.5 Encoding of MGCs into reduced-space QCs

Similarly to the argumentation in Section 4.1 of [7], we will now propose evidence, that MGCs can be encoded into reduced-space QCs with a well-defined number of elementary gates.

### II.5.1 Encoding of the majorana rotation

To arrive at such a generalized statement about the equivalence between MGCs and reduced-space QCs, it is important first to be able to express any MGC with a predefined finite set of properties. In fact, following [7], we will assume that the gates of the circuit are drawn from a finite set of elementary MGs which generates the algebra of all possible MGs acting in our circuit, we will call those MGs "generators". Hence, each gate in our MGC can be classified by a triple $(g, i)$, where g gives the position of the generator in the finite set and $i$ indicates on which two qubits (i, i+1) it acts. Hence, any MGC can be classified by M[6] tuples

$$(g_M, i_M)\cdots(g_2, i_2)(g_1, i_1). \tag{II.38}$$

That is, if we can find a finite set of elementary gates generating all our MGs and then find the encoding of each one of them into an elementary reduced-space quantum gate, we have a general translation method between any MGC[7] and their corresponding reduced-space QC, where we are able to quantify its depth by the number of resulting elementary reduced-space gates.

To address the first step, remember from Theorem 1 that every $SO(4)$ rotation has a corresponding MG; however, due to the Euler decomposition [5], every four-dimensional rotation can be decomposed into six rotations, each acting nontrivially in a different two-dimensional subspace of $\mathbb{R}^4$. Moreover, each two-dimensional rotation of a certain

[5] We will assume WLOG that N is a power of 2.
[6] Where M corresponds to the number of already decomposed gates.
[7] As long as we can express them, to an adequate precision, in terms of those generators.

angle can be generated, up to a desired precision, by succession of rotations of an irrational multiple of $\pi$. Consequently, choosing a set of six Matchgates, each corresponding by Theorem 1 to a two-dimensional rotation of an irrational multiple of $\pi$ acting in a different two-dimensional subspace of $\mathbb{R}^4$, we can generate any Matchgate with suitable precision, and thus we have the desired finite set of elementary gates from the previous paragraph.

Now, we deal with the second part, namely the encoding of each of these 6 elementary rotations into an elementary space-reduced quantum gate following [7] once more. To do so, we first need to note that each elementary rotation acting nontrivially in two dimensions only, which we will call $\boldsymbol{R}_t$, is of the form[8]

$$\boldsymbol{R}_t = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & a & & b & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & c & & d & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \tag{II.39}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in O(2)$, hence it is not a tensor product of the form $\boldsymbol{R}_t = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \boldsymbol{R} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$. Consequently, we cannot directly interpret it as a representation of a quantum gate because then it would generally act on all $\log_2 2N$ qubits, and thus it would not be an elementary generator of the reduced-space gate algebra. To solve this problem, we can interpret $\boldsymbol{R}_t$ as a two-dimensional rotation conditioned on the values of the remaining $\log_2 N - 1$ qubits, and then use the algorithm from Section 7.5 of [2] to represent an r-fold controlled single-qubit gate by $\mathcal{O}(r)$ two-qubit gates.

However, to be able to use this algorithm, the two-dimensional rotation should correspond to a single qubit gate, but, because $\boldsymbol{R}_t$ is acting arbitrarily on two out of four dimensions, they do not need to be adjacent; hence, depending on the representation of the dimension $1, \ldots, 2n$ by $\log_2 2N$ qubits, $\boldsymbol{R}_t$ will generally act on the subspace of two qubits that are arbitrarily distant[9] from each other, controlled by the other $\log_2 N - 1$ qubits. Consequently, we need to conjugate[10] $\boldsymbol{R}_t$ by an adequate sequence of $X_i$ and $CX_{ij}$[11] gates to make it act in the two-dimensional subspace of a single qubit. However, because the two qubits will be arbitrarily separated for a given representation of the dimensions, the length of such a sequence will vary depending on which dimension $\boldsymbol{R}_t$ acts on and could be of order of $\log_2 2N$.

To overcome this issue, instead of representing each dimension $1, \ldots, 2N$ to a string of lenght $z \coloneqq \log_2 2N$ with the usual binary representation, we associate each dimension

[8] In the case where the rotation is acting on the first and fourth dimension.
[9] In terms of nearest neigbhours.
[10] In the sence of a similarity transformation.
[11] With $X_i$ the Pauli X gate acting on the i-th qubit and CX the CNOT gate acting on the i-th qubit and controlling the j-th qubit.

to a string of length z according to the Gray code [9]

$$\text{Binary:} \begin{cases} |1\rangle & \Longrightarrow & |0,0,\ldots,0,0,0\rangle \\ |2\rangle & \Longrightarrow & |0,0,\ldots,0,0,1\rangle \\ |3\rangle & \Longrightarrow & |0,0,\ldots,0,1,0\rangle \\ |4\rangle & \Longrightarrow & |0,0,\ldots,0,1,1\rangle \\ |5\rangle & \Longrightarrow & |0,0,\ldots,1,0,0\rangle \\ & \vdots & \end{cases}, \quad \text{Gray:} \begin{cases} |1\rangle & \Longrightarrow & |0,0,\ldots,0,0,0\rangle \\ |2\rangle & \Longrightarrow & |0,0,\ldots,0,0,1\rangle \\ |3\rangle & \Longrightarrow & |0,0,\ldots,0,1,1\rangle \\ |4\rangle & \Longrightarrow & |0,0,\ldots,0,1,0\rangle \\ |5\rangle & \Longrightarrow & |0,0,\ldots,1,1,0\rangle \\ & \vdots & \end{cases} \tag{II.40}$$

This ensures that two adjacent dimensions correspond to two strings having Hamming distance 1, consequently, $\boldsymbol{R}_t$ acts on the subspace spanned by two Gray strings having Hamming distance $\leq 3$. Thus, they are denoted by two strings that have at least $z-3$ common values $s_i = 0, 1$ and, at most, three different values $d_i^1, d_i^2 = 0, 1 \ : \ d_i^1 \neq d_i^2$.

As an example, let $\boldsymbol{R}_t$ act on the subspace spanned by

$$\begin{cases} |s_1, 1, s_3, 0, 1, s_6, s_7, \ldots, s_z\rangle \\ |s_1, 0, s_3, 1, 0, s_6, s_7, \ldots, s_z\rangle \end{cases} \tag{II.41}$$

then, applying $CX_{52} \cdot CX_{42} \cdot X_4$, with each gate controlled on the values $s_1, s_3, \ldots, s_z$, we get

$$\begin{cases} |s_1, 1, s_3, 0, 0, s_6, s_7, \ldots, s_z\rangle \\ |s_1, 0, s_3, 0, 0, s_6, s_7, \ldots, s_z\rangle \, . \end{cases} \tag{II.42}$$

Consequently,

$$(CX_{52} \cdot CX_{42} \cdot X_4)\boldsymbol{R}_t(CX_{52} \cdot CX_{42} \cdot X_4)^{-1} \tag{II.43}$$

is a sequence of r-fold controlled gates that act in the subspace of the third qubit, with $r \leq z$.

In fact, repeating this for two generic strings of Hamming distance $\leq 3$ can be done by finding the sequence of X gates to bring $\left|\boldsymbol{d}^1\right\rangle \to |\boldsymbol{0}\rangle$[12], hence bringing $\left|\boldsymbol{d}^2\right\rangle \to |\boldsymbol{1}\rangle$[13], and then applying a sequence of CX gates to bring $\left|\boldsymbol{d}^2\right\rangle \to |\boldsymbol{e}_1\rangle$ with $\boldsymbol{e}_i$ the i-th standard basis vector of the computational basis. However, because $\boldsymbol{d}^i$ is at most a three-dimensional vector, we need at most a sequence of three X and two CX gates. As a result, by adding an ancillary qubit and using the algorithm of Section 7.5 of [2], we can encode $\boldsymbol{R}_t$ in a circuit of $\mathcal{O}(\log_2 N)$ two qubit gates; hence, assuming that we expressed our MGC in terms of M generators, we can encode $\boldsymbol{R}$ from

$$\frac{\mathrm{i}}{2}\langle \hat{Z}_k \rangle_f = \langle 2k-1| \, \boldsymbol{R}\boldsymbol{\Gamma}\boldsymbol{R}^T \, |2k\rangle \tag{II.44}$$

into $\mathcal{O}(M \cdot \log_2 N)$ elementary gates that act on $z+1$ qubits.

[12] Here $\boldsymbol{d}^j$ denotes the vector having the differing values $d_i^j$ of the gray string i as components, thus its dimension is equal to the hamming distance of string 1 and 2.

[13] $|\boldsymbol{1}\rangle \coloneqq |1, \ldots, 1\rangle$

Furthermore, because in the classification $(g_k, i_k)$ of each generator the length of $g_k$, indicating the position of the generator in the finite set, is constant and the length of $i_k$, indicating which of the N qubit the generator acts on, is of $\mathcal{O}(\log_2 N)$, the *translation* of each generator to its reduced-space elementary gate, can be done classically in space $\mathcal{O}(\log_2 N)$.

### II.5.2 Encoding of the correlation matrix

Now that we have managed to find an encoding for our rotation $\boldsymbol{R}$, all that remains is to be able to encode our correlation matrix $\boldsymbol{\Gamma}$ or better the rotation given by $\mathrm{i}\boldsymbol{\Gamma} + \mathrm{i}\alpha\mathbb{1}$. For a general pure or equi-occupied mixed state, $\mathrm{i}\boldsymbol{\Gamma} + \mathrm{i}\alpha\mathbb{1}$ is a 2N-dimensional rotation; therefore, according to the generalization of Euler angles [5], it can be decomposed into $2N^2 - N$ two-dimensional rotations.

Recalling the previous process, we can see that $\mathrm{i}\boldsymbol{\Gamma}+\mathrm{i}\alpha\mathbb{1}$ can be encoded into $\mathcal{O}(N^2 \log_2 N)$ elementary reduced-space gates by adding an additional ancillary qubit. Consequently, we arrive at the desired destination, that is,

**Result 2** The final measurement of any N-qubit MGC that can be decomposed into M generators that act on an initial state that is pure or equi-occupied mixed can be computed by a reduced-space QC that has $\mathcal{O}(M^2 \log_2 N)$ elementary gates that act on $(\log_2 2N + 3)$ qubits.

*Remark*

However, as we can see, the number of elementary quantum gates needed to implement the correlation matrix for a generic pure or equi-occupied mixed state is *growing quadratically in M*; but in fact, for the vacuum state $|\mathbf{0}\rangle$, the expectation value of $\hat{c}_i\hat{c}_j$ is only non-zero for i's and j's such that $\hat{c}_i\hat{c}_j = \hat{c}_k^2$ or $\hat{c}_i\hat{c}_j = \mathrm{i}2 \cdot \hat{Z}_k$, hence the correlation matrix takes the form

$$\boldsymbol{\Gamma} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\mathrm{i}} & & & & & & \\ \frac{\mathrm{i}}{2} & \frac{1}{2} & & & & & & \\ & & \frac{1}{2} & \frac{1}{2\mathrm{i}} & & & & \\ & & \frac{\mathrm{i}}{2} & \frac{1}{2} & & & & \\ & & & & \ddots & & & \\ & & & & & & \frac{1}{2} & \frac{1}{2\mathrm{i}} \\ & & & & & & \frac{\mathrm{i}}{2} & \frac{1}{2} \end{pmatrix}, \tag{II.45}$$

consequently, in the usual binary representation, it represents the tensor product $\begin{pmatrix} \frac{1}{2} & \frac{1}{2\mathrm{i}} \\ \frac{\mathrm{i}}{2} & \frac{1}{2} \end{pmatrix} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$, therefore $\boldsymbol{\Gamma} + \mathrm{i}\alpha\mathbb{1}$ is also a tensor product, and the whole discussion about an r-fold controlled operation is redundant. However, in the used Gray representation, Eq. II.45 does not represent a tensor; therefore, we will need to use the translational

circuit presented in [9]. This $\mathcal{O}(\log_2 N)$ circuit maps the Gray code to the binary code, so conjugating $\mathrm{i}\boldsymbol{\Gamma} + \mathrm{i}\alpha\mathbb{1}$ by it will make it a tensor product form in the Gray representation. As a result, we can compute our $\boldsymbol{\Gamma}$ with a QC of depth $\mathcal{O}(\log_2 N)$ instead of $\mathcal{O}(M^2 \log_2 N)$, thus the whole MGC can be simulated with $\mathcal{O}(M \cdot \log_2 N)$ elementary gates. The reason why this result is interesting is that qubit circuits with an arbitrary initial occ. basis state can be maped to a qubit circuit with an initial vacuum state with $\mathcal{O}(N^2)$ elementary operations [7], and hence we can generalize the computation with a vacuum state correlation matrix to any initial basis state.

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Furthermore, to encode the correlation matrix into a circuit of elementary quantum gates, we first not only need to classically calculate it but also compute its decomposition; however, for a general pure or equi-occupied mixed state, this calculation can become arbitrarily complicated. It is, in fact, dependent on the number of superpositions, which we will look at a little closer in Section III.2. Therefore, the statement of result 2 is not totally true; in practice, our initial state is only allowed to have a certain number of superpositions defined by the available classical resources; consequently, the use of the vacuum state is even more attractive.

## Chapter Summary:

- We can fully characterize the action of a fGU circuit in the Heisenberg picture by its action on the Majorana vector:

$$\hat{U}_G^\dagger \hat{\mathbf{c}} \hat{U}_G = \boldsymbol{R} \cdot \hat{\mathbf{c}} \quad \text{with} \quad \boldsymbol{R} \in SO(2N). \tag{II.46}$$

- The fermionic occ. basis and the qubit computational basis can be connected by defining the Majoranas as

$$\begin{aligned} \sqrt{2} \cdot \hat{c}_{2k-1} &= \overbrace{\hat{Z} \cdots \hat{Z}}^{j-1 \text{ times}} \hat{X}\, \mathbb{1} \cdots \mathbb{1} \\ \sqrt{2} \cdot \hat{c}_{2k} &= \overbrace{\hat{Z} \cdots \hat{Z}}^{j-1 \text{ times}} \hat{Y}\, \mathbb{1} \cdots \mathbb{1}. \end{aligned} \tag{II.47}$$

- *Matchgates* are the qubit equivalents of elementary fGUs and a QC with M matchgate operators are called *matchgate circuits* (MGC).

- The final expectation value, computed by a matchgate circuit, can be written as

$$\frac{\mathrm{i}}{2} \langle \hat{Z}_k \rangle_f = \langle 2k-1 |\, \boldsymbol{R} \boldsymbol{\Gamma} \boldsymbol{R}^T \,| 2k \rangle \,, \tag{II.48}$$

where $\boldsymbol{R}$ is the *rotation* associated with the MGC and $\boldsymbol{\Gamma}_{ij} = \langle \hat{c}_i \hat{c}_j \rangle_{in}$ is the *correlation matrix*.

- **Result 1** If the initial state of a polynomial sized matchgate circuit is a pure product state, it is *classically simulatable* in *polynomial time*.
- We can rewrite the final expectation value as

$$-\frac{1}{2}\langle\hat{Z}_k\rangle_f = \langle 2k-1|\,\boldsymbol{R}\,(\mathrm{i}\boldsymbol{\Gamma}+\mathrm{i}\alpha\cdot\mathbb{1})\,\boldsymbol{R}^T\,|2k\rangle\,, \tag{II.49}$$

  thus, if $\hat{\rho}_{in}$ is pure or equi-occupied mixed, $\boldsymbol{R}\,(\mathrm{i}\boldsymbol{\Gamma}+\mathrm{i}\alpha\cdot\mathbb{1})\,\boldsymbol{R}^T$ is a real unitary and we can *map* the MGC to a QC on $\mathcal{O}(\log_2 N)$ qubits.
- The rotation $\boldsymbol{R}$ can be encoded into $\mathcal{O}(M\cdot\log_2 N)$ elementary gates acting on $\log_2 N+1$ qubits. Furthermore, the translation can be be done classically in space $\mathcal{O}(\log_2 N)$.
- **Result 2** The final measurement of any N-qubit MGC with a pure or equi-occupied mixed initial state can be computed using a $\mathcal{O}(\log_2 N)$-qubit QC with depth $\mathcal{O}(M^2\log_2 N)$.

# CHAPTER III

## Compression of bosonic Gaussian circuits

Despite the relevance of bosonic linear optics, which, as we will see in Section III.3, are a special case of bosonic Gaussian systems, to date and to our knowledge only a few references in the literature [14] have dealt with a concrete approach to the compression of bosonic Gaussian circuits (bGC). Consequently, this chapter discusses whether and under what conditions a bGC can be compressed efficiently. The sections will be structured analogously to Chapter II: Section III.1 presents the underlying mathematical concept that allows a reduced-space description of bGUs; in Section III.2, we provide evidence for the classical simulatability, to then, in Section III.3 discuss conditions for reduced-space quantum simulatability. Finally, in Section III.4 we will explore the encoding possibilities to charaterize whether bGCs can be simulated in polynomial time.

### III.1 Compressed description of bGUs

Similarly to the fermionic case, $\{\hat{a}_i, \hat{a}_j^\dagger\}$ and their unitary transformation $\{\hat{z}_i, \hat{z}_j\}$ are complete sets of *generators*[1] for the operator algebra acting on our bosonic system. Therefore, it is sufficient to describe the action of the unitary $\hat{U}_{QC}$, which represents the QC, on our ladder operator vector $\hat{\mathbf{r}}$ or our quadrature vector $\hat{\mathbf{z}}$ to classify the action of the quantum gates on the inital state $\hat{\rho}_{in}$ in the Schrödinger picture.

Recalling that bGUs are generated by quadratic Hamiltonians, as described in Eq. I.12, we can see that once again the evolution of our ladder operators and our quadratures have specific forms.

**Theorem 3.** *Consider an N-mode bosonic system with $\hat{U}_G$ a bosonic Gaussian unitary generated by a bqH of the form of*

$$\hat{H}_b = \frac{1}{2}\hat{\mathbf{z}}^T \boldsymbol{h}^z \hat{\mathbf{z}} + (\boldsymbol{d}^z)^T \hat{\mathbf{z}} = \frac{1}{2}\hat{\mathbf{r}}^\dagger \boldsymbol{h}^r \hat{\mathbf{r}} + (\boldsymbol{d}^r)^T \hat{\mathbf{r}} \tag{III.1}$$

*with $\hat{\mathbf{r}}$ the ladder operator vector and $\hat{\mathbf{z}}$ the quadrature operator vector; then:*

$$\hat{U}_G^\dagger \hat{\mathbf{r}} \hat{U}_G = \boldsymbol{S}^r \cdot \hat{\mathbf{r}} + \boldsymbol{\beta}^r \quad \textit{and} \quad \hat{U}_G^\dagger \hat{\mathbf{z}} \hat{U}_G = \boldsymbol{S}^z \cdot \hat{\mathbf{z}} + \boldsymbol{\beta}^z \tag{III.2}$$

[1] In fact this statement can become critical if we consider infinite number of particles, therefore we will disregard this case throughout the coming chapters.

*where*

$$\boldsymbol{S}^r = e^{-\mathrm{i}\boldsymbol{\Sigma}^3\boldsymbol{h}^r} \in Sp(2N,\mathbb{C}), \quad \boldsymbol{S}^z = e^{\boldsymbol{\Omega}\boldsymbol{h}^z} \in Sp(2N,\mathbb{R}) \quad \textit{and} \quad \boldsymbol{\beta}^{r/z} \in \mathbb{C}^{2N}/\mathbb{R}^{2N}. \tag{III.3}$$

*Proof.* Analogously to Theorem 1, for the *ladder operator vector*, we can interpret $\hat{U}_G^\dagger \hat{\mathbf{r}} \hat{U}_G = e^{-\mathrm{i}\hat{H}\cdot t}\hat{\mathbf{r}}e^{\mathrm{i}\hat{H}\cdot t} = \hat{\mathbf{r}}(t)$ with $\hat{H}$ the corresponding bqH to $\hat{U}_G$ and $t = 1$. Thus, using the Heisenberg equation

$$\frac{d\hat{r}_k(t)}{dt} = \mathrm{i}[\hat{H}, \hat{r}_k(t)] = \mathrm{i}\hat{U}_G^\dagger(\frac{1}{2}\overbrace{\boldsymbol{h}^r_{ij}[\hat{r}_i^\dagger \hat{r}_j, k]}^{\mathbf{II}} + \overbrace{\boldsymbol{d}^r_i[\hat{r}_i, \hat{r}_k]}^{\mathbf{I}})\hat{U}_G, \tag{III.4}$$

however, because of the CCR in Eq. I.20, we have

$$\mathbf{I} = \boldsymbol{d}^r_i \boldsymbol{\Omega}_{ik} = \left[\boldsymbol{\Omega}^T \cdot \boldsymbol{d}^r\right]_k, \tag{III.5}$$

and

$$\mathbf{II} = \boldsymbol{h}^r_{ij}\boldsymbol{\Omega}_{jk}\hat{r}_i^\dagger - \boldsymbol{h}^r_{ij}\boldsymbol{\Sigma}^3_{ik}\hat{r}_j = \left[\boldsymbol{\Omega}^T(\boldsymbol{h}^r)^T \cdot \hat{\mathbf{r}}^\dagger - (\boldsymbol{\Sigma}^3)^T\boldsymbol{h}^r \cdot \hat{\mathbf{r}}\right]_k. \tag{III.6}$$

Moreover, as shown in Appendix A, for the ladder vector expression of bqHs, $\boldsymbol{h}^r$ is a Hermitian matrix of the block form

$$\boldsymbol{h}^r = \left.\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \overline{\boldsymbol{B}} & \overline{\boldsymbol{A}} \end{pmatrix}\right|_{\substack{\boldsymbol{A}^\dagger = \boldsymbol{A} \\ \boldsymbol{B}^T = \boldsymbol{B}}}, \tag{III.7}$$

thus, **II** simplifies to

$$\mathbf{II} = -2\left[\boldsymbol{\Sigma}^3\boldsymbol{h}^r \cdot \hat{\mathbf{r}}\right]_k. \tag{III.8}$$

Solving the differential equation and setting $t = 1$, we retrieve

$$\hat{U}_G^\dagger \hat{\mathbf{r}} \hat{U}_G = e^{-\mathrm{i}\boldsymbol{\Sigma}^3\boldsymbol{h}^r} \cdot \hat{\mathbf{r}} - \mathrm{i}\boldsymbol{\Omega} \cdot \boldsymbol{d}^r. \tag{III.9}$$

Similarly, for the *quadrature operator vector*, we have

$$\frac{d\hat{z}_k(t)}{dt} = \mathrm{i}[\hat{H}, \hat{z}_k(t)] \overset{\text{Eq. I.12}}{=} \mathrm{i}\hat{U}_G^\dagger(\frac{1}{2}\overbrace{\boldsymbol{h}^z_{ij}[\hat{z}_i\hat{z}_j, k]}^{\mathbf{II}} + \overbrace{\boldsymbol{d}^z_i[\hat{z}_i, \hat{z}_k]}^{\mathbf{I}})\hat{U}_G, \tag{III.10}$$

with

$$\mathbf{I} = \mathrm{i}\boldsymbol{d}^z_i\boldsymbol{\Omega}_{ik} = \mathrm{i}\left[\boldsymbol{\Omega}^T \cdot \boldsymbol{d}^z\right]_k \tag{III.11}$$

and

$$\mathbf{II} = \mathrm{i}\boldsymbol{h}^z_{ij}\boldsymbol{\Omega}_{jk}\hat{z}_i + \mathrm{i}\boldsymbol{h}^z_{ij}\boldsymbol{\Omega}_{ik}\hat{z}_j = \mathrm{i}\left[\left(\boldsymbol{\Omega}^T(\boldsymbol{h}^z)^T + \boldsymbol{\Omega}^T\boldsymbol{h}^z\right) \cdot \hat{\mathbf{r}}\right]_k. \tag{III.12}$$

Using that for the quadrature expression of a bqH, $\boldsymbol{h}$ is real and symmetric, by solving the differential equation and setting $t = 1$, we get

$$\hat{U}_G^\dagger \hat{\mathbf{z}} \hat{U}_G = e^{\boldsymbol{\Omega}\boldsymbol{h}^z} \cdot \hat{\mathbf{z}} + \boldsymbol{\Omega} \cdot \boldsymbol{d}^z. \tag{III.13}$$

However, the generators of the symplectic group $Sp(2N, \boldsymbol{F})$ over a field $\boldsymbol{F} = \mathbb{R}, \mathbb{C}$ are of the block matrix form

$$\boldsymbol{G} = \left.\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & -\boldsymbol{A}^T \end{pmatrix}\right|_{\substack{\boldsymbol{B}^T = \boldsymbol{B} \\ \boldsymbol{C}^T = \boldsymbol{C}}}, \tag{III.14}$$

and thus,

$$e^{-\mathrm{i}\boldsymbol{\Sigma}^3\boldsymbol{h}^r} = \exp\left[-\mathrm{i}\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ -\overline{\boldsymbol{B}} & -\overline{\boldsymbol{A}}\end{pmatrix}\right]\Bigg|_{\substack{\boldsymbol{A}^\dagger=\boldsymbol{A}\\ \boldsymbol{B}^T=\boldsymbol{B}}} \tag{III.15}$$

is a *complex symplectic matrix*; and because $\boldsymbol{h}^z = \begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{B}^T & \boldsymbol{D}\end{pmatrix}\Big|_{\substack{\boldsymbol{A}^T=\boldsymbol{A}\\ \boldsymbol{D}^T=\boldsymbol{D}}} \in \mathbb{R}^{2N\times 2N}$,

$$e^{\boldsymbol{\Omega}\boldsymbol{h}^z} = \exp\left[\begin{pmatrix}\boldsymbol{B}^T & \boldsymbol{D}\\ -\boldsymbol{A} & -\boldsymbol{B}\end{pmatrix}\right]\Bigg|_{\substack{\boldsymbol{A}^T=\boldsymbol{A}\\ \boldsymbol{D}^T=\boldsymbol{D}}} \tag{III.16}$$

is a *real symplectic matrix*. □

Consequently, equivalently to the fGCs, we can fully describe the action of our bGC on the initial state by the action of a $2N \times 2N$ symplectic matrix on the operator vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$. Interestingly, in the bosonic case, we have an additional *displacement vector* $\boldsymbol{\beta}^{r/z}$ that originates from the displacement term $\boldsymbol{d}^r\hat{\mathbf{r}}/\boldsymbol{d}^z\hat{\mathbf{z}}$ in the bqH representation in Eq. I.12, which is not present in the fermionic case.

Furthermore, Theorem 3 is also consistent with the fact that such Gaussian transformations, generated by a quadratic Hamiltonian, produce canonical transformations that conserve the CCR [11]. The reason is that, first, anti-commutator relations, for the fermionic case, would not allow for a transformation with a displacement constant, wherein commutator relations, for the bosonic case, would allow for it, since $[\hat{\mathbf{r}}, \boldsymbol{\beta}^{r/z}] = 0$. And second, for the transformation $\boldsymbol{S}^{r/z}$ to conserve the CCR it needs to conserve $\boldsymbol{\Omega}$ as

$$\boldsymbol{S}^{r/zT}\boldsymbol{\Omega}\boldsymbol{S}^{r/z} = \boldsymbol{\Omega} \tag{III.17}$$

which is, by definition, the symplectic group.

From Theorem 3 we can follow that a description in $\mathcal{O}(N^2)$ of our $\mathcal{O}\big(2^N\big)$-parameter system is possible and can be utilized, similarly to the fermionic case, to classically simulate our bGC, as we will see in Section III.2, and even to simulate it, for some conditions, with a new logarithmic-sized quantum circuit, as shown in Section III.3.

## III.2 Classical simulatability of bGCs

Consider a quantum circuit with N bosonic modes, M elementary bosonic Gaussian gates, and the initial state $\hat{\rho}_{in}$. Again, we will only consider bGCs where there is no mode that is not acted upon by an elementary gate, thus $M \geq \frac{N}{2}$. The purpose of this circuit is to provide information on the probability distribution of our final state by measuring the expectation value of a one-body operator

$$\hat{O} = \sum_{\lambda,\lambda'=0}^{N} \hat{a}_\lambda^\dagger \boldsymbol{O}_{\lambda\lambda'}\hat{a}_{\lambda'} = \sum_{\lambda,\lambda'=0}^{N} \hat{r}_\lambda^\dagger \boldsymbol{O}_{\lambda\lambda'}\hat{r}_{\lambda'} \tag{III.18}$$

where $\boldsymbol{O}_{\lambda\lambda'}$ is the matrix element of an observable acting on the Hilbert space of one particle $\mathcal{H}_{(1)}$. By denoting the whole unitary unitary acting on the initial state as

$\hat{U}_G = \hat{U}_M \cdot \ldots \cdot \hat{U}_1$ we can write this expectation value as

$$\langle \hat{O} \rangle_f = \sum_{\lambda,\lambda'=1}^{N} \boldsymbol{O}_{\lambda\lambda'} \langle \hat{U}_G^\dagger \hat{r}_\lambda^\dagger \hat{U}_G \cdot \hat{U}_G^\dagger \hat{r}_{\lambda'} \hat{U}_G \rangle_{in}, \tag{III.19}$$

then, using Theorem 3 we get

$$\langle \hat{O} \rangle_f = \sum_{\lambda,\lambda'=1}^{N} \boldsymbol{O}_{\lambda\lambda'} \sum_{i,j=1}^{2N} \left[ \overline{\boldsymbol{S}^r_{\lambda i}} \boldsymbol{S}^r_{\lambda' j} \langle \hat{r}_i^\dagger \hat{r}_j \rangle_{in} + \boldsymbol{S}^r_{\lambda' j} \overline{\boldsymbol{\beta}^r_i} \langle \hat{r}_j \rangle_{in} + \overline{\boldsymbol{S}^r_{\lambda i}} \boldsymbol{\beta}^r_j \langle \hat{r}_i^\dagger \rangle_{in} + \overline{\boldsymbol{\beta}^r_i} \boldsymbol{\beta}^r_j \right] \tag{III.20}$$

where $\boldsymbol{S}^z$ is the symplectic map and $\boldsymbol{\beta}^r$ is the displacement assimilated with $\hat{U}_G$. Consequently, by defining the Hermitian matrix $\boldsymbol{\Gamma} \coloneqq \langle \hat{r}_i^\dagger \hat{r}_j \rangle_{in}$ as in Section II.3, we can rewrite the expectation value as

$$\langle \hat{O} \rangle_f = \sum_{\lambda,\lambda'=1}^{N} \boldsymbol{O}_{\lambda\lambda'} \left( \langle \lambda | \, \boldsymbol{S}^r \boldsymbol{\Gamma} \boldsymbol{S}^{r\dagger} \, | \lambda' \rangle + \sum_{i,j=1}^{N} \left[ \overline{\boldsymbol{\beta}^r_i} [\boldsymbol{S}^r \langle \hat{\mathbf{r}} \rangle]_{\lambda'} + \boldsymbol{\beta}^r_j [\overline{\boldsymbol{S}^r} \langle \hat{\mathbf{r}}^\dagger \rangle]_\lambda + \overline{\boldsymbol{\beta}^r_i} \boldsymbol{\beta}^r_j \right] \right). \tag{III.21}$$

This shows that, similar to the fermionic case, the expectation value is equal to an expression of 2N-dimensional matrix elements, and thus we can approach this bGC with a time-efficient classical computation. In fact,

- if we restrict ourselves again to an initial occ. basis state $|n_1, \ldots, n_N\rangle$, computing the correlation matrix $\boldsymbol{\Gamma}$ is equivalent to computing 2N terms $\langle n_i | \hat{r}_i^\dagger \hat{r}_i | n_i \rangle$ because for $i \neq j$ the matrix element is zero. Therefore $\boldsymbol{\Gamma}$ can be classically computed in $\mathcal{O}(N)$ thus, at most, $\mathcal{O}(M)$ time.

- Similarly, the 2N terms of $\langle \hat{\mathbf{r}} \rangle$ and $\langle \hat{\mathbf{r}}^\dagger \rangle$ are zero and do not need to be accounted for.

- The symplectic transformation and the displacement vector that act on $\boldsymbol{\Gamma}$ or $\langle \hat{\mathbf{r}} \rangle_{in} / \langle \hat{\mathbf{r}}^\dagger \rangle_{in}$ can be decomposed into $\boldsymbol{S}^r = \boldsymbol{S}^r_M \cdot \ldots \cdot \boldsymbol{S}^r_1$ with each $\boldsymbol{S}^r_i \in Sp(4, \mathbb{C})$ extended with identities to 2N dimensions and $\boldsymbol{\beta}^r = \boldsymbol{\beta}^r_M + \ldots + \boldsymbol{\beta}^r_1$ with each $\boldsymbol{\beta}^r_i \in \mathbb{C}^4$ zero-extended to 2N dimensions. Each $\boldsymbol{S}^r_i$ and $\boldsymbol{\beta}^r_i$ corresponds to the elementary bGU $\hat{U}_i$ acting on the k-th and (k+1)-th mode, according to (Theorem 3):

  $$\hat{U}_i^\dagger \hat{r}_j \hat{U}_i = \begin{cases} \sum_{l \in \{k\}} \boldsymbol{S}^r_i[j,l] \hat{r}_l + \boldsymbol{\beta}^r_i[j] & \text{for } j \in \{k\} \coloneqq \{k, k+1, k+N, k+N+1\} \\ \hat{r}_j & \text{for all other } j \text{ 's.} \end{cases} \tag{III.22}$$

  As a result, $\boldsymbol{S}^r$ is a product of M $2N \times 2N$ matrices and $\boldsymbol{\beta}^r$ is a sum of M 2N-dimensional vectors, so they can be calculated, respectively, in $\mathcal{O}(M^3)$ and $\mathcal{O}(M^2)$ time.

- Finally, we can calculate the expectation value $\langle \hat{O} \rangle_f$ as the sum of $4N^4 = \mathcal{O}(M^4)$ terms.

In summary, we can compute the final expectation value $\langle \hat{O} \rangle$ of any bGC in poly$(M)$, thus, as in the fermionic case, we have

**Result 3** Any polynomial-sized bGC having an initial state that is a pure occ. basis state, can be classically computed in *polynomial time*.

This is once again a *remarkable result* as the naive simulation of a quantum circuit would expect exponentially scaling resources. However, as indicated by the form of Eq. III.21, a re-simulation with a new space-restricted QC is also possible as we will see in the next section.

## III.3 Reduced-space simulatability of bGCs

Looking at Eq. III.21

$$\langle \hat{O} \rangle_f = \sum_{\lambda,\lambda'=1}^{N} \boldsymbol{O}_{\lambda\lambda'} \left( \langle \lambda | \, \boldsymbol{S}^r \boldsymbol{\Gamma} \boldsymbol{S}^{r\dagger} \, | \lambda' \rangle + \sum_{i,j=1}^{N} \left[ \overline{\boldsymbol{\beta}_i^r} [\boldsymbol{S}^r \langle \hat{\mathbf{r}} \rangle]_{\lambda'} + \boldsymbol{\beta}_j^r [\overline{\boldsymbol{S}^r} \langle \hat{\mathbf{r}}^\dagger \rangle]_\lambda + \overline{\boldsymbol{\beta}_i^r} \boldsymbol{\beta}_j^r \right] \right) \tag{III.23}$$

we can see that for $\boldsymbol{S}^r, \boldsymbol{\Gamma} \in U(2N)$ and $\boldsymbol{\beta}^r = \mathbf{0}$ we find a form of $\langle \hat{O} \rangle$ corresponding to the fermionic quantum compressed case, wherefore we will take a closer look at these elements hereafter.

### III.3.1 Unitarity of the symplectic transformation

In fact, the first condition, namely $\boldsymbol{S}^r \in U(2N) \wedge \boldsymbol{\beta}^r = 0$, is satisfied by bGUs that are generated by special bqHs.

**Theorem 4.** *Consider an N-mode bosonic system with $\hat{U}_G = e^{\mathrm{i}\hat{H}}$ a unitary generated by a bosonic quadratic Hamiltonian $\hat{H}$ and $\hat{\mathbf{r}}$ the ladder operator vector, then*

$$\hat{U}_G^\dagger \hat{\mathbf{r}} \hat{U}_G = \boldsymbol{U} \cdot \hat{\mathbf{r}}, \tag{III.24}$$

*with $\boldsymbol{U} \in U(2N)$ iff $\hat{H}$ is a particle number-conserving Hamiltonian. In fact, $\boldsymbol{U}$ is a block matrix of the form*

$$\boldsymbol{U} = \begin{pmatrix} \boldsymbol{U}_N & \\ & \overline{\boldsymbol{U}_N} \end{pmatrix} \tag{III.25}$$

*with $\boldsymbol{U}_N \in U(N)$.*

*Proof.* We recall from Theorem 3 that the canonical transformation for $\hat{\mathbf{r}}$ is of the form

$$\hat{U}_G^\dagger \hat{\mathbf{r}} \hat{U}_G = \boldsymbol{S}^r \cdot r + \boldsymbol{\beta}^r \tag{III.26}$$

with

$$\boldsymbol{S}^r = e^{-\mathrm{i}\boldsymbol{\Sigma}^3 \boldsymbol{h}^r} = \exp\left[ -\mathrm{i} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ -\overline{\boldsymbol{B}} & -\overline{\boldsymbol{A}} \end{pmatrix} \right] \Bigg|_{\substack{\boldsymbol{A}^\dagger = \boldsymbol{A} \\ \boldsymbol{B}^T = \boldsymbol{B}}}, \tag{III.27}$$

because

$$\boldsymbol{h}^r = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \overline{\boldsymbol{B}} & \overline{\boldsymbol{A}} \end{pmatrix}_{\substack{\boldsymbol{A}^\dagger = \boldsymbol{A} \\ \boldsymbol{B}^T = \boldsymbol{B}}} . \tag{III.28}$$

However, $\boldsymbol{S}^r$ is unitary iff $-\mathrm{i}\boldsymbol{\Sigma}^3\boldsymbol{h}^r$ is skew-hermitian, meaning that

$$(-\mathrm{i}\boldsymbol{\Sigma}^3\boldsymbol{h}^r)^\dagger \overset{(\boldsymbol{h}^r)^\dagger = \boldsymbol{h}^r}{=} \mathrm{i}\boldsymbol{h}^r\boldsymbol{\Sigma}^3 \overset{!}{=} +\mathrm{i}\boldsymbol{\Sigma}^3\boldsymbol{h}^r \quad \Longleftrightarrow \quad [\boldsymbol{h}^r, \boldsymbol{\Sigma}^3] = 0, \tag{III.29}$$

but because $\boldsymbol{\Sigma}^3$ is a block diagonal matrix $\not\propto \mathbb{1}$, it follows that $\boldsymbol{h}^r$ also needs to be a block diagonal matrix, hence

$$\boldsymbol{h}^c = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \overline{\boldsymbol{A}} \end{pmatrix}_{\boldsymbol{A}^\dagger = \boldsymbol{A}} . \tag{III.30}$$

Nevertheless, if we recall the expression for the Hamiltonian generating the bGU, we can see that for this kind of block diagonal matrix $\boldsymbol{h}^r$ the first term of Eq. I.12 only has terms in $\hat{a}_i^\dagger \hat{a}_j$:

$$\boldsymbol{h}^c_{ij}\hat{r}_i^\dagger \hat{r}_j = \boldsymbol{A}_{ij}\hat{a}_i^\dagger \hat{a}_j + \overline{\boldsymbol{A}}_{ij}\hat{a}_i\hat{a}_j^\dagger = 2\boldsymbol{A}_{ij}\hat{a}_i^\dagger \hat{a}_j + c \cdot \hat{I} \tag{III.31}$$

where the constant $c = \mathrm{Tr}\,[\boldsymbol{A}]$ results in a non-essential energy shift and thus can be neglected. Hence, $\boldsymbol{S}^r$ is unitary iff the first term of the bqH $\hat{H}$ that generates the bGU is number-preserving. Moreover, with $\boldsymbol{h}^r$ in block diagonal form, we can write $\boldsymbol{S}^r$ as

$$\boldsymbol{S}^r = e^{-\mathrm{i}\boldsymbol{\Sigma}^3\boldsymbol{h}^r} = \begin{pmatrix} \boldsymbol{U}_N & \\ & \overline{\boldsymbol{U}_N} \end{pmatrix}_{\boldsymbol{U}_N = e^{-\mathrm{i}\boldsymbol{A}}} \tag{III.32}$$

with $\boldsymbol{U}_N \in U(N)$ because $\boldsymbol{A}$ is hermitian. However, $\boldsymbol{\beta}^r = -\mathrm{i}\boldsymbol{\Omega} \cdot \boldsymbol{d}^r$, where $\boldsymbol{d}^r$ is the generator of linear terms in Eq. I.12 thus $\boldsymbol{\beta}^r = \boldsymbol{0}$ iff $\hat{H}$ does not have linear terms. Consequently,

$$\hat{U}_G^\dagger \hat{\mathbf{r}} \hat{U}_G = \boldsymbol{U} \cdot \hat{\mathbf{r}} \tag{III.33}$$

iff both terms of $\hat{H}$ in Eq. I.12 are *number-preserving*. □

In fact, such a bGU that produces a unitary transformation of the ladder operators corresponds to the action of a linear interferometer on our N-mode bosonic system. Thus, from Eq. III.21 we can see that for any bosonic QC that uses *linear optical elements* to implement the gates experimentally, we can express the final measurement as

$$\langle \hat{O} \rangle_f = \sum_{\lambda,\lambda'=1}^{N} \boldsymbol{O}_{\lambda\lambda'} \langle \lambda | \boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{U}^\dagger | \lambda' \rangle . \tag{III.34}$$

However, the sum only goes over the first $N \times N$ block matrix elements of $\boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{U}^\dagger$, which because $\boldsymbol{U}$ is a block diagonal matrix, can be written as

$$\left(\boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{U}^\dagger\right)_{ij} = \left(\boldsymbol{U}_N \boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}} \boldsymbol{U}_N^\dagger\right)_{ij} \quad \text{for} \quad i,j = 1,\ldots,N \tag{III.35}$$

where $(\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}})_{ij} = \langle \hat{a}_i^\dagger \hat{a}_j \rangle$ which we will call the "reduced correlation matrix" (rCM). Thus,

$$\langle \hat{O} \rangle_f = \sum_{\lambda,\lambda'=1}^{N} \boldsymbol{O}_{\lambda\lambda'} \langle \lambda | \boldsymbol{U}_N \boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}} \boldsymbol{U}_N^\dagger | \lambda' \rangle . \tag{III.36}$$

Consequently, we can conclude that if we find conditions such that $\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}}$ is unitary, we can apply the same method as in Chapter II and compute this expectation value with a reduced-space quantum circuit by computing the probabilities of a time-developed state $|\psi_f\rangle = \hat{U}\,|\lambda'\rangle$ with $\hat{U} \doteq \boldsymbol{U}_N \mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}} \boldsymbol{U}_N^\dagger$.

However, we can further note that, using $\boldsymbol{O}_{\lambda\lambda'} = \langle\lambda|\,\hat{O}_{(1)}\,|\lambda'\rangle$ we can rewrite Eq. III.36 value as

$$\langle\hat{O}\rangle_f = \mathrm{Tr}\left[\hat{O}_{(1)} \cdot \left(\boldsymbol{U}_N \mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}} \boldsymbol{U}_N^\dagger\right)^T\right], \tag{III.37}$$

consequently, it is very similar to another quantum case, namely the expectation value of $\hat{O}_{(1)}$ for a state represented by the density matrix that takes the form $\mathbf{\Gamma}^T_{\hat{a}^\dagger\hat{a}}$ and develops over time by the unitary $\overline{\boldsymbol{U}_N}$. Therefore, we have *two different approaches* to consider, which we will discuss in the next two sections.

### III.3.2 Unitarity of the rCM

In this subsection, we will deal with the first part, namely finding conditions such that $\boldsymbol{U}_N \mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}} \boldsymbol{U}_N^\dagger$ represents a unitary to apply the method of Section II.4.

The very first intuition would be to use Theorem 2, however, even if it is very useful for the fermionic correlation matrix, because an initial state will only have occ. numbers 0 and 1, it is not the case for a general bosonic state. To see that, we firstly take a look at the eigenvalues of the rCM. In fact, because $(\mathbf{\Gamma}^\dagger_{\hat{a}^\dagger\hat{a}})_{ij} = \overline{\langle\hat{a}_j^\dagger\hat{a}_i\rangle} = (\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}})_{ij}$, the rCM is Hermitian, thus there exists a unitary transformation such that

$$\mathbf{\Gamma}^D_{\hat{a}^\dagger\hat{a}} = \boldsymbol{U}\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}}\boldsymbol{U}^\dagger = \begin{pmatrix} v_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & v_N \end{pmatrix}. \tag{III.38}$$

However, for each unitary $\boldsymbol{U}$ there exists a Hermitian matrix $\boldsymbol{A}$ such that

$$\boldsymbol{U} = e^{-\mathrm{i}\boldsymbol{A}}, \tag{III.39}$$

consequently, according to Theorem 4, $\boldsymbol{U}$ corresponds to the canonical transformation of the ladder operator vector $\hat{\mathbf{r}}$ caused by the conjugation with the bGU $\hat{U}_G$ generated by $\hat{H}(\boldsymbol{A}) = \overline{\boldsymbol{A}}_{ij}\hat{a}_i^\dagger\hat{a}_j + \boldsymbol{A}_{ij}\hat{a}_i\hat{a}_j^\dagger$ according to

$$\boldsymbol{U}\cdot\hat{\boldsymbol{a}}^\dagger + \overline{\boldsymbol{U}}\cdot\hat{\boldsymbol{a}} = \hat{U}_G^\dagger\hat{\mathbf{r}}\hat{U}_G. \tag{III.40}$$

Hence,

$$[\boldsymbol{U}\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}}\boldsymbol{U}^\dagger]_{ij} = \langle\boldsymbol{U}_{il}\hat{a}_l^\dagger\overline{\boldsymbol{U}}_{jk}\hat{a}_k\rangle = \langle\hat{U}_G^\dagger\hat{a}_i^\dagger\hat{U}_G\cdot\hat{U}_G^\dagger\hat{a}_j\hat{U}_G\rangle = \mathrm{Tr}\left[\hat{a}_i^\dagger\hat{a}_j\hat{U}_G\hat{\rho}\hat{U}_G^\dagger\right]. \tag{III.41}$$

*Remark*

For $\mathbf{\Gamma}_{ij} = \langle \hat{r}_i \hat{r}_j \rangle$, it is not given that there exists a corresponding bGU for any diagonalizing unitary $\boldsymbol{U}\mathbf{\Gamma}\boldsymbol{U}^\dagger = \mathbf{\Gamma}^D$, because bGUs only generate unitaries of the block diagonal form

$$\boldsymbol{U}_{2N} = \begin{pmatrix} \boldsymbol{U}_N & \\ & \overline{\boldsymbol{U}_N} \end{pmatrix}. \tag{III.42}$$

---

Thus, diagonalizing $\mathbf{\Gamma}_{\hat{a}^\dagger \hat{a}}$ once again amounts to finding a basis in which the state is not correlated and therefore $\hat{U}_G \hat{\rho} \hat{U}_G^\dagger = \bigotimes_{k=1}^N \hat{\rho}_k^1$. Consequently,

$$\mathbf{\Gamma}^D_{\hat{a}^\dagger \hat{a}} = \begin{pmatrix} \langle \hat{N}_1 \rangle & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \langle \hat{N}_N \rangle \end{pmatrix} \tag{III.43}$$

with $\langle \hat{N}_i \rangle \geq 0$ the *mean occupation number* of the i-th decoupled mode. Consequently, because Theorem 2 is only applicable to a Hermitian matrix that has only two eigenvalues $\lambda_1, \lambda_2$ such that $|\lambda_1 - \lambda_2| \leq 2$, it would only be applicable to the bosonic case if the initial state only has two decoupled mode occ. numbers separated by less than two. Hence, for example, any initial state of the form $|0, 0, 0, n\rangle : n > 2$ cannot be considered; thus, the theorem is far too restrictive to generalize our results sufficiently. Thence, we will introduce another property of the correlation matrix.

**Theorem 5.** *Let $\mathbf{\Gamma}_{\hat{a}^\dagger \hat{a}}$ be a rCM of an initial state that has at most two different mean occ. numbers $\bar{n}_1$ and $\bar{n}_2$ in its uncorrelated form, then there exist an $\alpha \in \mathbb{R}$ and a $\gamma \in \mathbb{R}^+$ such that:*

$$\gamma \cdot \mathbf{\Gamma}_{\hat{a}^\dagger \hat{a}} + \alpha \cdot \mathbb{1} \in U(N). \tag{III.44}$$

*In fact,*

$$\alpha = -\frac{\bar{n}_1 + \bar{n}_2}{\Delta \bar{n}} \quad \textit{and} \quad \gamma = \frac{2}{\Delta \bar{n}} \tag{III.45}$$

*where $\Delta \bar{n} := |\bar{n}_2 - \bar{n}_2|$.*

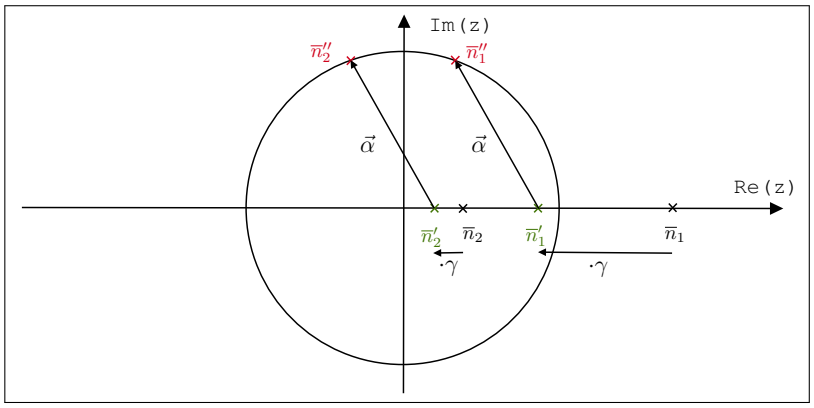


Figure III.1.: Graphical proof of Theorem 5.

*Proof.* Let $\boldsymbol{A}$ be a $N \times N$ Hermitian matrix; thus, there exists a unitary $\boldsymbol{U}$ such that $\boldsymbol{U}\boldsymbol{A}\boldsymbol{U}^\dagger = \boldsymbol{A}^D = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ with $\lambda_i \in \mathbb{R}$. Hence, similarly to the proof of Theorem 2, $\gamma\boldsymbol{A} + \alpha\mathbb{1}$ is unitary iff

$$(\gamma\lambda_i + \overline{\alpha})(\gamma\lambda_i + \alpha) = 1. \tag{III.46}$$

Therefore, solving for $\lambda_i$ we get

$$\gamma\lambda_i = -\mathrm{Re}\,(\alpha) \pm \sqrt{1 - \mathrm{Im}\,(\alpha)^2}, \tag{III.47}$$

consequently, because for all $\lambda_1, \lambda_2 \in \mathbb{R}$ there exist an $\alpha \in \mathbb{C}$ and a $\gamma \in \mathbb{R}^+$ such that

$$\lambda_{1,2} = \frac{1}{\gamma}\left(-\mathrm{Re}\,(\alpha) \pm \sqrt{1 - \mathrm{Im}\,(\alpha)^2}\right), \tag{III.48}$$

for all $\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}}$ of an initial state that has at most two different mean occ. numbers $\bar{n}_1$ and $\bar{n}_2$, there exist an $\alpha \in \mathbb{C}$ and a $\gamma \in \mathbb{R}^+$ such that

$$\gamma \cdot \boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}} + \alpha \cdot \mathbb{1} \in U(N). \tag{III.49}$$

Furthermore, for

$$\begin{cases} \bar{n}_1 = \frac{1}{\gamma}\left(-\mathrm{Re}\,(\alpha) - \sqrt{1 - \mathrm{Im}\,(\alpha)^2}\right) \\ \bar{n}_2 = \frac{1}{\gamma}\left(-\mathrm{Re}\,(\alpha) + \sqrt{1 - \mathrm{Im}\,(\alpha)^2}\right), \end{cases} \tag{III.50}$$

we have

$$\begin{cases} \Delta\bar{n} = \frac{2}{\gamma}\sqrt{1 - \mathrm{Im}\,(\alpha)^2}\Big|_{\Delta\bar{n} := |n_1 - n_2|} \\ \bar{n}_1 + \bar{n}_2 = -\frac{2\mathrm{Re}(\alpha)}{\gamma} \end{cases}, \tag{III.51}$$

hence,

$$\alpha(\gamma) = -\frac{\gamma(\bar{n}_1 + \bar{n}_2)}{2} + \mathrm{i}\sqrt{1 - \left(\frac{\gamma\Delta\bar{n}}{2}\right)^2}. \tag{III.52}$$

Consequently, if $\alpha$ is real, $\gamma = \frac{2}{\Delta\bar{n}}$, thus $\alpha = -\frac{\bar{n}_1 + \bar{n}_2}{\Delta\bar{n}} \in \mathbb{R}$. □

This result is indeed quite useful because we have

$$\langle\lambda|\,\boldsymbol{U}_N(\gamma\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}} + \alpha\mathbb{1})\boldsymbol{U}_N^\dagger\,|\lambda'\rangle = \gamma \cdot \langle\lambda|\,\boldsymbol{U}_N\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}}\boldsymbol{U}_N^\dagger\,|\lambda'\rangle + \alpha \cdot \delta_{\lambda\lambda'}, \tag{III.53}$$

hence, we can follow from Eq. III.36 that

$$\gamma \cdot \langle\hat{O}\rangle_f + \alpha \cdot \mathrm{Tr}\left[\hat{O}_{(1)}\right] = \sum_{\lambda,\lambda'=1}^{N} \boldsymbol{O}_{\lambda\lambda'}\,\langle\lambda|\,\boldsymbol{U}_N(\gamma\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}} + \alpha\mathbb{1})\boldsymbol{U}_N^\dagger\,|\lambda'\rangle\,. \tag{III.54}$$

Consequently, we arrive at our desired result, computing the final expectation value of a bGC is equivalent to computing a sum of the probabilistic results of a developed state $|\psi_f\rangle = \hat{U}\,|\lambda'\rangle$ with $\hat{U} \doteq \boldsymbol{U}_N(\gamma\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}} + \alpha\mathbb{1})\boldsymbol{U}_N^\dagger$.

As a result, any bGC with N modes, M *number conserving* elementary bGUs and an initial state that has at most two different mean occ. numbers in its uncorrelated form can be mapped, with the right *encoding*, to a QC with $\mathcal{O}(\log_2 N)$ qubit or modes. However, we can achieve this result with another approach based on another very useful property of the rCM, which we will discuss now.

### III.3.3 Positive semi-definiteness of the correlation matrix

We saw that, by imposing certain conditions, we have a unitary expression for the rCM; however, correlation matrices, in general, and in particular the rCM $\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}}$, have another useful property that can be utilized.

**Theorem 6.** *Consider an N-mode bosonic system with the raising and lowering operators acting on the i-th mode $\hat{a}_i^\dagger$ and $\hat{a}_i$ respectively, and some state $\hat{\rho}$. Then the correlation matrix $\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}}$, defined by the expectation value of the state $\hat{\rho}$*

$$(\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}})_{ij} = \langle \hat{a}_i^\dagger \hat{a}_j \rangle, \tag{III.55}$$

*is positive semi-definite.*

*Proof.* Consider some operator vector $\hat{\mathbf{s}} \coloneqq (\hat{s}_1, \ldots, \hat{s}_N)$ and some complex vector $\boldsymbol{c} \coloneqq (c_o, \ldots, c_n) \in \mathbb{C}^N$. Now, let us define the correlation matrix $\mathbf{\Gamma}$ of $\hat{\mathbf{s}}$ as the expectation value matrix of the state $\hat{\rho}$ as

$$(\mathbf{\Gamma})_{ij} \coloneqq \langle \hat{s}_i^\dagger \hat{s}_j \rangle. \tag{III.56}$$

Then we have

$$\boldsymbol{c}^\dagger \cdot \mathbf{\Gamma} \cdot \boldsymbol{c} = \sum_{ij} \overline{c_i} c_j \mathrm{Tr}\left[\hat{s}_i^\dagger \hat{s}_j \hat{\rho}\right] = \mathrm{Tr}\left[\sum_{ij} \overline{c_i} \hat{s}_i^\dagger c_j \hat{s}_j \hat{\rho}\right], \tag{III.57}$$

and by defining $\hat{\boldsymbol{\xi}} \coloneqq \sum_i c_i \hat{s}_i$, we can write

$$\boldsymbol{c}^\dagger \cdot \mathbf{\Gamma} \cdot \boldsymbol{c} = \mathrm{Tr}\left[\hat{\boldsymbol{\xi}}^\dagger \hat{\rho} \hat{\boldsymbol{\xi}}\right]. \tag{III.58}$$

However, $\hat{\rho}$ is a positive semi-definite density matrix and can thus be decomposed as $\hat{\rho} = \hat{\chi}^\dagger \hat{\chi}$, hence

$$\boldsymbol{c}^\dagger \cdot \mathbf{\Gamma} \cdot \boldsymbol{c} = \mathrm{Tr}\left[(\hat{\chi}\hat{\boldsymbol{\xi}})^\dagger (\hat{\chi}\hat{\boldsymbol{\xi}})\right], \tag{III.59}$$

but because $(\hat{\chi}\hat{\boldsymbol{\xi}})^\dagger(\hat{\chi}\hat{\boldsymbol{\xi}})$ is also positive semi-definite, and because the trace is base independent, thus it is the sum of the eigenvalues, we can follow that

$$\boldsymbol{c}^\dagger \cdot \mathbf{\Gamma} \cdot \boldsymbol{c} \geq 0 \quad \forall \boldsymbol{c} \in \mathbb{C}^N. \tag{III.60}$$

Consequently, $\mathbf{\Gamma} \geq 0$, and therefore $\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}}$ is also *positive semi-definite.* □

Interestingly, this shows that not only $\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}}$ is positive semi-definite, but also $\mathbf{\Gamma}$, the bosonic expectation matrix of $\hat{\mathbf{r}}$, and it can also be proved similarly that the fermionic case is equivalent, thus $\mathbf{\Gamma}$ and $\mathbf{\Gamma}_{l.o.}$ of Chapter II are also positive semi-definite.

*Remark*

Another noteworthy matter is the relation of this positive semi-definiteness and the *uncertainty principle.* In fact, if we define the bosonic covariance matrix $\boldsymbol{V}$ as

$$\mathbf{\Gamma}_{ij} = \langle \hat{r}_i \hat{r}_j \rangle = \frac{1}{2}\langle \{\hat{r}_i, \hat{r}_j\} \rangle + \frac{1}{2}\mathbf{\Sigma}^3_{ij} =: \boldsymbol{V}_{ij} + \frac{1}{2}\mathbf{\Sigma}^3_{ij}, \tag{III.61}$$

then the sufficient condition that the state satisfies the uncertainty principle is given by [11]

$$\boldsymbol{c}^\dagger \cdot \left( \boldsymbol{V} + \frac{1}{2}\boldsymbol{\Sigma}^3 \right) \cdot \boldsymbol{c} \geq 0 \quad \forall \boldsymbol{c} \in \mathbb{C}^{2N}. \tag{III.62}$$

Thus, the positive semi-definiteness of the correlation matrix, which arises from the positive semi-definiteness of the density matrix, can be interpreted as *equivalent* to the uncertainty principle.

---

Knowing that $\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}} \geq 0$, and setting the condition that $\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}} > 0$, we can now rewrite Eq. III.37 as

$$\langle \hat{O} \rangle_f = \mathrm{Tr}\left[\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}}\right] \cdot \mathrm{Tr}\left[ \hat{O}_{(1)} \cdot \left( \boldsymbol{U}_N \frac{\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}}}{\mathrm{Tr}\left[\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}}\right]} \boldsymbol{U}_N^\dagger \right)^T \right] \tag{III.63}$$

with $\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}}/\mathrm{Tr}\left[\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}}\right]$ positive (semi-)definite, Hermitian, and trace one. Therefore, we can interpret this expression for $\langle \hat{O} \rangle_f$ as the expectation value of $\hat{O}_{(1)}$ of some state developed in time by $\overline{\boldsymbol{U}_N}$ represented by the density matrix $\hat{\rho}_{(1)} = (\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}}/\mathrm{Tr}\left[\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}}\right])^T$.

Moreover, the trace of the rCM can be directly computed from the initial state as

$$\mathrm{Tr}\left[\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}}\right] = \sum_{i=1}^{N} \langle \hat{a}_i^\dagger \hat{a}_i \rangle = \sum_{i=1}^{N} \langle \hat{N}_i \rangle \coloneqq \bar{n} \tag{III.64}$$

with $\langle \hat{N}_i \rangle =$ the *mean occ. number* of the i-th mode and thus $\bar{n}$ the *total (mean) particle number* of the initial state, hence the condition for the trace to be non-zero simplifies to the condition that the initial state *cannot* be the *vacuum state*. Consequently, we can write the final measurement of our bGC as

$$\langle \hat{O} \rangle_f = \bar{n} \cdot \mathrm{Tr}\left[ \hat{O}_{(1)} \hat{U}_N \hat{\rho}_{(1)} \hat{U}_N^\dagger \right] \tag{III.65}$$

where $\hat{U}_N \doteq \overline{\boldsymbol{U}_N}$ and $\hat{\rho}_{(1)} \doteq \boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}}^T/\bar{n}$. Therefore, we arrive at a similar result to the one in the previous subsection, namely that, with the right *encoding*, any bGC acting with M *number preserving* elementary bGUs on N modes can be maped to a QC with $\mathcal{O}(\log_2 N)$ qubits or modes, but this time we do not need to impose conditions on the initial state except the condition that it should not be the vacuum state, which, for a number preserving (np.) circuit, is not that deplorable.

*Remark*

The physical idea behind III.65 is quite interesting; in fact, if we interpret $\boldsymbol{\Gamma}_{\hat{a}^\dagger\hat{a}}/\bar{n}$ as a density matrix, it will represent the statistical mixture of the state of *one particle* being in each of the N modes given the state of $\bar{n}$ particles, similarly, $\hat{O}_{(1)}$ will represent the observable $\hat{O}$ for *one particle* in the N modes. Hence, calculating the expectation value of our $\bar{n}$-particle observable $\hat{O}$ for the statistics of $\bar{n}$ particles is equivalent to calculating the expectation value of the 1-particle-observable $\hat{O}_{(1)}$ for the mean statistics of one particle and then multiply it by the total number of particles $\bar{n}$.

Furthermore, the beauty of a bGC is that the action of its unitary can be mapped bijectively to its action on the rCM. Therefore, instead of computing the expectation value by calculating the correlation of a time-developed state, and thus needing to calculate the time development of the $\bar{n}$ particles, we can compute the whole process in the 1-particle picture by implementing the action of the bGU as the time development of the 1-particle state $\hat{U}_N \hat{\rho}_{(1)} \hat{U}_N^\dagger$.

---

We will now determine the exact encoding of such a mapping in order to characterize the exact number of qubits/modes and elementary operations required and hence discuss whether it is really an efficient compression.

## III.4 Encoding of bGCs into reduced-space QCs

We will now propose evidence that, similarly to MGCs, bGCs can be encoded into reduced-space QCs with a well-defined number of elementary gates as well. However, even if the quantum simulation could be done equivalently with a bosonic circuit as with a qubit circuit, we will propose an encoding in the qubit circuit *only*, because the form of the reduced-space QC to be used in this chapter is very similar to the one in Chapter II and thus we can use the already established encoding from Section II.5.

### III.4.1 Encoding of the unitary transformation

In fact, similarly to Eq. III.22, each elementary np. bGU $\hat{U}_i$ acting on the k-th and (k+1)-th qubit corresponds to a *two-dimensional* unitary $\boldsymbol{U}_i^t$ according to:

$$\hat{U}_i^\dagger \hat{a}_j \hat{U}_i = \begin{cases} \sum_{l=k}^{k+1} \boldsymbol{U}_i^t[j,l] \hat{a}_l & \text{for } j = k, k+1 \\ \hat{a}_j & \text{for all other } j \text{ 's,} \end{cases} \tag{III.66}$$

and hence can be written as

$$\boldsymbol{U}_i^t = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \tag{III.67}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U(2)$. Consequently, we can directly interpret each $\boldsymbol{U}_i^t$ as a $\log_2 N - 1$-fold controlled 1-qubit gate, and similarly to Section II.5, using the method from [2], we can encode $\boldsymbol{U}_N = \boldsymbol{U}_N^M \cdot \ldots \cdot \boldsymbol{U}_N^1$ from Eq. III.36

$$\langle \hat{O} \rangle_f = \sum_{\lambda,\lambda'=1}^{N} \boldsymbol{O}_{\lambda\lambda'} \langle \lambda | \boldsymbol{U}_N \boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}} \boldsymbol{U}_N^\dagger | \lambda' \rangle . \tag{III.68}$$

into a circuit of $\mathcal{O}(M \cdot \log_2 N)$ elementary gates acting on $\log_2 N + 1$ qubits.

### III.4.2 Encoding of the unitary rCM

In this subsection, we will devote the argumentation to the *first method* of reduced-space simulatability, namely, where we compute the final expectation value with Eq. III.54

$$\gamma \cdot \langle \hat{O} \rangle_f + \alpha \cdot \mathrm{Tr}\left[\hat{O}_{(1)}\right] = \sum_{\lambda,\lambda'=1}^{N} \boldsymbol{O}_{\lambda\lambda'} \langle \lambda | \boldsymbol{U}_N (\gamma \boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}} + \alpha \mathbb{1}) \boldsymbol{U}_N^\dagger | \lambda' \rangle \,. \tag{III.69}$$

However, because we have already found the encoding for $\boldsymbol{U}_N$, all that remains is to find an encoding for the N-dimensional unitary given by $\gamma \boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}} + \alpha \mathbb{1}$. In fact, interpreting it as a quantum gate acting on $\log_2 N$ qubits, we can decompose it according to Section 8 of [2] into $\mathcal{O}\big((\log_2 N)^3 \cdot 4^{\log_2 N}\big) = \mathcal{O}(N^2 (\log_2 N)^3)$ two-qubit gates using no ancillary qubit. Consequently, we arrive at our fourth main result

**Result 4** The final measurement of any N-qubit np. bGC that can be decomposed into M generators that act on an initial state that has at most two different occ. numbers in its uncorrelated form can be computed by a reduced-space QC that has $\mathcal{O}(M^2 (\log_2 N)^3)$ elementary gates that act on $(\log_2 N + 2)$ qubits.

*Remark*

We can even go a step further and take a look at a specific initial state, namely $|n, 0, \ldots, 0\rangle$. In fact, because, compared to a qubit circuit, the vacuum state $|\mathbf{0}\rangle$ is rather useless as an initial state for a number-conserving bosonic circuit, we need another more generalizable state. However, by applying a series of n.n. "hopping gates" on an initial state, which we define as $\hat{h}_i = \hat{a}_i \hat{a}_{i+1}^\dagger$, any occ. basis state can be achieved from an initial state $|n, 0, \ldots, 0\rangle$, therefore, it could be capable of fulfilling this role of a generalized initial state. However, computing the rCM with this state $|n, 0, \ldots, 0\rangle$ we get the real diagonal form

$$\boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}} = \mathrm{diag}(n, 0, \ldots, 0), \tag{III.70}$$

hence, with $\alpha = -n/n = -1$ and $\gamma = 2/n$ $\gamma$, $\boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}} + \alpha \mathbb{1}$ is a real unitary of the form

$$\gamma \boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}} + \alpha \mathbb{1} = \mathrm{diag}(-1, 1, \ldots, 1) \in O(N). \tag{III.71}$$

Consequently, by representing the dimensions with the Gray code, we can directly interpret it as a $-\hat{Z}_1$ gate controlled on all the other $\log_2 N - 1$ qubits. Therefore, using Lemma 7.1 of [2], we can encode it in a circuit of $2^{\log_2 N} - 2 = N - 2$ elementary gates. Consequently, if our initial state is $|n, 0, \ldots, 0\rangle$, then the final expectation value can be simulated by a QC using $\mathcal{O}(M \cdot \log_2 N)$ elementary gates.

### III.4.3 Encoding of the density matrix rCM

For the *second method*, we interpret $\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}}$ or rather $\mathbf{\Gamma}^T_{\hat{a}^\dagger\hat{a}}$ as a density matrix; hence the difficulty in this process will mainly be to first calculate the reduced correlations classically and then prepare a state that corresponds to this rCM, interpreted as a density matrix, up to a desired precision. However, because the experimental implementation is, at first sight, very case-specific, we will devote this subsection mainly to the discussion about the classical simulation complexity. Nevertheless, recalling Eq. III.65

$$\langle \hat{O} \rangle_f = \bar{n} \cdot \mathrm{Tr}\left[\hat{O}_{(1)} \hat{U}_N \hat{\rho}_{(1)} \hat{U}_N^\dagger\right], \tag{III.72}$$

we see that we do not only need to implement the rCM as a density matrix but also the one-body operator $\hat{O}_{(1)}$ as a final observable, however, assuming that the N-mode final observable is defined beforehand, we can directly compute the $N^2$ elements of $\hat{O}_{(1)}$ classically. Thus, it only remains to find the right experimental implementation; however, similarly to the rCM, it seems to be quite problem-specific, and therefore we will not discuss any general approach.

Each element of the rCM can be calculated from $\langle \hat{a}_i^\dagger \hat{a}_j \rangle$, consequently, as we saw in Section III.2 for an *occ. basis state* $|n_1, \ldots, n_n\rangle$ each element is non-zero iff $i = j$, hence, the rCM takes the form

$$\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}} = \mathrm{diag}(n_1, \ldots, n_n). \tag{III.73}$$

In fact, for any pure or mixed *uncorrelated state* with occupation numbers $\bar{n}_1, \ldots, \bar{n}_N$, the rCM can be defined directly as the matrix

$$\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}} = \mathrm{diag}(\bar{n}_1, \ldots, \bar{n}_N). \tag{III.74}$$

However, if we consider a general pure state expressed in the occ. basis as[2]

$$|\psi\rangle = \sum_{i=1}^{No} |\boldsymbol{n}_i\rangle \tag{III.75}$$

with $No$ the number of superpositions and $\boldsymbol{n}_i = (n_1^i, \ldots, n_n^i)$ we can see that the complexity of the computation of the rCM directly depends on the number of superpositions; hence for a general state it becomes impossible to classically compute it, as the number of possible superpositions tend to infinity.

*Remark*

In fact, even if we restrict ourselves to a pure *eigenstate* of the total occ. number operator $\hat{N} = \sum_i \hat{a}_i^\dagger \hat{a}_i$, with eigenvalue the total particle number (tp. number) $n$[3], then each superposition have a fixed total particle number $n = \|\boldsymbol{n}_i\|_1 \ \forall i$, thus we can compute the maximal number of possible superpositions $No^{max}$ by doing a simple combinatorial process. In fact, if we consider a state having n particles that we decompose as

$$n = \overbrace{1+1}^{2} + 1 + \overbrace{1+1+1+1}^{4} + 1 + 1 + \cdots + 1 = 2 + 1 + 4 + 1 + 1 + \cdots + 1 \tag{III.76}$$

[2] Up to a normalization.

[3] We will once again omit the bar because $n$ does not have a variance for an eigenstate of $\hat{N}$.

then we can decompose n as a sum of any number of "+"s. However, if this state represents N modes, n needs to be decomposed by $N-1$ "+"s, thus the question of how many occ. basis state have the same tp. number is equivalent to the number of ways to arrange $N-1$ "+"s in a sequence of n "1"s and $N-1$ "+"s. Consequently,

$$No^{max} = \binom{n+N-1}{N-1} = \frac{(n+N-1)!}{(N-1)!n!}. \tag{III.77}$$

However, we can see that this expression does not scale polynomially with the size of the system, which increases with the number of modes N and the tp. number $n$. In fact, if we make the approximation that the state has the same total number of particles as it has modes, hence $n = N$, we can rewrite $No^{max}$ as

$$No^{max} = \binom{2N-1}{N-1} = \frac{N}{2N}\binom{2N}{N}, \tag{III.78}$$

and using the Stirling approximation for large Ns, $N! \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$, we find that

$$No^{max} \sim \frac{2^{2N-1}}{\sqrt{N\pi}}, \tag{III.79}$$

hence, the possible number of superpositions grows *exponentially* with the number of modes N. Consequently, for a *generic* pure eigenstate of $\hat{N}$, we will need, for each term of $\mathbf{\Gamma}$ or $\langle\hat{\mathbf{r}}\rangle$, to compute a sum of a $\mathcal{O}(No^{max}N^2)$ number of terms that is growing exponentially in N. As a result, even for a generic state with a fixed particle number, we *cannot* compute $\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}}$ classically in polynomial time.

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However, we can solve this problem by restricting ourselves to states[4] that have a set number of superpositions $No = \text{poly}(N)$ in the occ. basis, because then the classical computation of $\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}}$ is feasible in $\mathcal{O}(\text{poly}(N)\cdot N^2) = \text{poly}(N)$ time. However, this restriction is also relevant for the *previous method* in which we encode the rCM as a unitary because to do so we first need to classically compute the rCM and its decomposition $\gamma\mathbf{\Gamma}_{\hat{a}^\dagger\hat{a}} + \alpha\mathbb{1}$, before being able to encode it into a reduced-space QC. Consequently, we *will not* account for this restriction for the second method more than for the first; therefore, we can write the fifth and final result of this thesis as

**Result 5** The final measurement of any N-qubit np. bGC that can be decomposed into M generators that act on any initial state different from the vacuum state can be computed by a reduced-space QC that has $\mathcal{O}(M\cdot\log_2 N)$ elementary gates that act on $\log_2 N + 2$ qubits.

[4] Which do not need to be eigenstates of $\hat{N}$.

## Chapter Summary:

- Similarly to the fermionic case, we can fully characterize the action of a bGU circuit in the Heisenberg picture by its action on the ladder operator vector
$$\hat{U}_G^\dagger \hat{\mathbf{r}} \hat{U}_G = \boldsymbol{S}^r \cdot \hat{\mathbf{r}} + \boldsymbol{\beta}^r \tag{III.80}$$
where $\boldsymbol{S}^r$ is a *complex symplectic matrix* and the *displacement vector* $\boldsymbol{\beta}^r$ is a complex vector of 2N dimensions.

- **Result 3** A *polynomial-sized* bGC with a final measurement represented by the expectation value of the observable
$$\hat{O} = \sum_{\lambda,\lambda'=0}^{N} \hat{a}_\lambda^\dagger \boldsymbol{O}_{\lambda\lambda'} \hat{a}_{\lambda'} = \sum_{\lambda,\lambda'=0}^{N} \hat{r}_\lambda^\dagger \boldsymbol{O}_{\lambda\lambda'} \hat{r}_{\lambda'} \tag{III.81}$$
and an initial state that is a basis state of the occ. basis $|\psi_{in}\rangle = |n_1, \ldots, n_N\rangle$ is *classically simulatable* in *polynomial time*.

- For a *particle number preserving* bGC we can write the final expectation value as
$$\langle \hat{O} \rangle_f = \sum_{\lambda,\lambda'=1}^{N} \boldsymbol{O}_{\lambda\lambda'} \langle \lambda | \boldsymbol{U}_N \boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}} \boldsymbol{U}_N^\dagger | \lambda' \rangle = \mathrm{Tr}\left[ \hat{O}_{(1)} \hat{U}_N \boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}} \hat{U}_N^\dagger \right] \tag{III.82}$$
with $\boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a} ij} = \langle \hat{a}_i^\dagger \hat{a}_j \rangle$ the rCM, $\hat{U}_N$ an N-dimensional unitary corresponding to the transformation of $\hat{\mathbf{r}}$ that is induced by the bGC, and $\hat{O}_{(1)}$ is defined by $\boldsymbol{O}_{\lambda\lambda'} = \langle \lambda | \hat{O}_{(1)} | \lambda' \rangle$.

- If the initial state has at most two different occ. numbers in its uncorrelated form, we can rewrite the final expectation value as
$$\gamma \cdot \langle \hat{O} \rangle_f + \alpha \cdot \mathrm{Tr}\left[ \hat{O}_{(1)} \right] = \sum_{\lambda,\lambda'=1}^{N} \boldsymbol{O}_{\lambda\lambda'} \langle \lambda | \boldsymbol{U}_N (\gamma \boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}} + \alpha \mathbb{1}) \boldsymbol{U}_N^\dagger | \lambda' \rangle \tag{III.83}$$
with $\boldsymbol{U}_N (\gamma \boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}} + \alpha \mathbb{1}) \boldsymbol{U}_N^\dagger \in U(N)$, thus, we can *map* the bGC to a QC on $\mathcal{O}(\log_2 N)$ qubits.

- If the initial state is not the vacuum state, we can rewrite the final expectation value as
$$\langle \hat{O} \rangle_f = \bar{n} \cdot \mathrm{Tr}\left[ \hat{O}_{(1)} \hat{U}_N \hat{\rho}_{(1)} \hat{U}_N^\dagger \right] \tag{III.84}$$
where $\hat{U}_N \doteq \overline{\boldsymbol{U}_N}$ and $\hat{\rho}_{(1)} \doteq \boldsymbol{\Gamma}_{\hat{a}^\dagger \hat{a}}^T / \bar{n}$ with $\bar{n}$ the total number of particles, thus we can *map* the bGC to a QC on $\mathcal{O}(\log_2 N)$ qubits with a *second* method.

- **Result 4** The final measurement of any N-mode bGC with an initial state having *at most two different* occ. numbers in its uncorrelated form can be computed using a $\mathcal{O}(\log_2 N)$-qubit QC with depth $\mathcal{O}(M^2(\log_2 N)^3)$.
- **Result 5** The final measurement of any N-mode bGC with an initial state that is *not the vacuum* state can be computed using a $\mathcal{O}(\log_2 N)$-qubit QC with depth $\mathcal{O}(M \cdot \log_2 N)$.

# CHAPTER IV

# Conclusion

In this chapter, we will present a brief summary of the thesis in Section IV.1, and then we will discuss limitations of the reasoning we used as well as possible future research directions in Section IV.2.

## IV.1 Summary

The goal of this thesis was to provide evidence that Gaussian quantum circuits can be efficiently simulated with reduced-space quantum circuits. To this end, we considered the fermionic and bosonic cases separately.

For the *fermionic case*, due to the Jordan-Wigner transformation (II.2), a qubit circuit can represent a fermionic quantum circuit; therefore, we derived the following results for a model of a N qubit circuit consisting of a sequence of M elementary matchtgates, which is the qubit equivalent of fermionic Gaussian unitaries, acting on an initial state, and a final measurement represented by the expectation value of the Pauli operator $\hat{Z}_k$ acting on the k-th qubit.

We showed that, firstly, if the initial state is a *pure product* state, any of these matchgate circuits can be simulated by a classical circuit in poly($M$) time. Therefore, any polynomial-sized matchgate circuit having an initial pure product state can be classically simulated in polynomial time (II.3).

Moreover, any of these matchgate circuits that have an initial state that is either a *pure* or a *"equi-occupied mixed"* state can be simulated with a quantum circuit with $\mathcal{O}(\log_2 N)$ qubits and $\mathcal{O}(M^2 \log_2 N)$ elementary gates (II.5).

For the *bosonic case*, we used a circuit model that consists of an initial state acted on by a sequence of M elementary bosonic Gaussian and then evaluated with an expectation value of a one-body operator $\hat{O} = \sum_{\lambda\lambda'} \hat{a}^\dagger_\lambda \boldsymbol{O}_{\lambda,\lambda'} \hat{a}_{\lambda'}$ where $\boldsymbol{O}$ is the matrix representation of the one-particle operator $\hat{O}_{(1)}$.

We showed that any of these bosonic model circuits that have an initial state that is a *pure occupation basis state* can be simulated by a classical circuit in poly($M$) time. Therefore, similarly to the fermionic case, any polynomial-sized bosonic Gaussian circuit having an initial pure occupation basis state can be classically simulated in polynomial time (III.2).

Furthermore, we found *two methods*, which have slightly different conditions, to simulate any *particle number preserving* bosonic Gaussian circuits with a reduced-space qubit circuit comprising $\mathcal{O}(\log_2 N)$ qubits and having two different orders of the number of elementary gates. *Firstly*, if the initial state has at most two different occupation numbers in its uncorrelated form, we can implement the correlation matrix as a unitary acting on $\log_2 N$ modes and thus simulate the bosonic Gaussian circuit with a quantum circuit comprising $\mathcal{O}(M^2(\log_2 N)^3)$ elementary gates. And *secondly*, if the initial state is not the vacuum state, we can implement the correlation matrix as a density matrix of an initial state of $\log_2 N$ modes and thus simulate the bosonic Gaussian circuit with a quantum circuit comprising $\mathcal{O}(M \cdot \log_2 N)$ elementary gates (III.4).

However, comparing both the fermionic and bosonic cases, we can recognize resemblances allowing for generalizations. For example, as noted in Section III.4, one could also have considered the encoding of bosonic, but also fermionic Gaussian circuits into a reduced-space *bosonic* quantum circuit.

Furthermore, similarly, we can see that the compression methods used in this thesis can be, by adapting the conditions, generalized from one to the other case directly. For example, because $\hat{Z}_k = \hat{a}_k\hat{a}_k^\dagger - \hat{a}_k^\dagger\hat{a}_k$, we can see that the second bosonic encoding method can be utilized for the fermionic circuit under the condition that the initial state is not the vacuum state. This would, firstly, allow one to drop the condition on the initial state to be pure or equi-occupied mixed that we have for the method of Section II.5, and secondly, allow for an efficient simulation with $\mathcal{O}(M \cdot \log_2 N)$ elementary gates. As a result, this would mean that both fermionic and bosonic Gaussian systems can be, under certain conditions, equivalently simulated in polynomial time with logarithmic sized quantum circuits.

## IV.2 Limitations and Outlook

The main limitation of this work comes from the form of the measurements used; in fact, in both the bosonic and fermionic cases, we limited ourselves to one-particle measurements. However, one can simply see that with these types of measured observables, we cannot achieve the measurement result of a projector on a given state $|n_1, \ldots, n_i\rangle$ such that $i \geq 2$.

This is particularly unfortunate in the bosonic case, because any circuit composed of linear optical elements is in fact a bosonic Gaussian circuit; thus, one could hope to compress *boson sampling circuits*, which can be used to evaluate the permanents of unitaries [13]. However, a general boson sampling circuit has a final measurement represented by a projection on some state written in the occ. basis as $|\boldsymbol{x}\rangle \coloneqq |x_1, \ldots, x_N\rangle$.

However, such a projector can be decomposed as

$$|\boldsymbol{x}\rangle\langle\boldsymbol{x}| = |x_1\rangle\langle x_1| \otimes \cdots \otimes |x_N\rangle\langle x_N| \tag{IV.1}$$

and thus expressed in terms of ladder operators and the vacuum projector

$$|\boldsymbol{x}\rangle\langle\boldsymbol{x}| = (\hat{a}_1^\dagger)^{x_1} \cdots (\hat{a}_N^\dagger)^{x_N} |\mathbf{0}\rangle\langle\mathbf{0}| (\hat{a}_N)^{x_N} \cdots (\hat{a}_1)^{x_1}. \tag{IV.2}$$

Using that for a finite number of particles n, the vacuum projector $|0\rangle\langle 0|$ can be expressed as

$$|0\rangle\langle 0| = \prod_{k\neq n} \left( \frac{\hat{N} - k}{n - k} \right) \Bigg|_{n=0}, \tag{IV.3}$$

we see that $|\boldsymbol{x}\rangle\langle\boldsymbol{x}|$ can be expressed as a polynomial of ladder operators of order $\geq 2$. Consequently, this projector does not fall into our set of single-particle observable, and thus our methods *cannot* compress general boson sampling circuits. In fact, such an observable would replace the correlation matrix with a tensor having the same order as the polynomial of the projector.

However, Gaussian states can be completely characterized by their correlation matrix $\boldsymbol{\Gamma}$ and their means $\langle\hat{\mathbf{r}}\rangle$[1] [12][15], consequently, if the initial state of our Gaussian circuit is Gaussian, there should exist a way to decompose such a tensor of polynomial order into a polynomial of correlation matrices and thus by applying a procedure similar to the one used in Chapter III we could compress bosonic sampling circuits having an initial state that is Gaussian.

Another direction which could be quite interesting to pursue is the idea of expressing a generic quantum circuit with an exponentially-spaced Gaussian quantum circuit. The reason is that, if in the future Gaussian gates were more easily realizable experimentally, then one could translate any circuit into a Gaussian circuit, which could be useful even if the Gaussian circuit uses a number of qubits that is exponentially larger. In fact, this has already been done for the fermionic case by R. Jozsa et al.'s [7], but for the bosonic case has not been explored in depth. Actually, in the bosonic case, not only would it be helpful if Gaussian unitaries were more easily realizable but also, in general, one could translate any qubit/bosonic circuit to a *linear optical quantum circuit*.

[1] For a fermionic state their mean is zero because of the PSSR, hence we will not account for it.

# APPENDIX A

# Appendix

In this appendix, we show some properties of the matrices of Eq. I.12.

## I.1 The bqH

For the bosonic case, recall that the Hamiltonian expressed in terms of quadratures takes the form

$$\hat{H}_b = \frac{1}{2}\hat{\mathbf{z}}^T \boldsymbol{h}^z \hat{\mathbf{z}} + (\boldsymbol{d}^z)^T \hat{\mathbf{z}}, \tag{A.1}$$

because $\hat{\mathbf{z}}$ has a block vector form, it turns out to be useful to consider $\boldsymbol{h}^z$ as the block matrix

$$\boldsymbol{h}^z = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}. \tag{A.2}$$

Now, let us define the block matrix

$$\widetilde{\boldsymbol{h}^z} = \begin{pmatrix} \boldsymbol{A} + \boldsymbol{Y}^1 & \boldsymbol{B} + \boldsymbol{X} \\ \boldsymbol{C} - \boldsymbol{X}^T & \boldsymbol{D} + \boldsymbol{Y}^2 \end{pmatrix} \tag{A.3}$$

where $\boldsymbol{Y}^i$ is skew-symmetric, then

$$2\hat{H}_b(\widetilde{\boldsymbol{h}^z}) - 2(\boldsymbol{d}^z)^T \hat{\mathbf{r}} = \boldsymbol{h}^z_{ij}\hat{z}_i\hat{z}_j + \boldsymbol{X}_{ij}\hat{x}_i\hat{p}_j - \boldsymbol{X}_{ji}\hat{p}_i\hat{x}_j + \boldsymbol{Y}^1_{ij}\hat{x}_i\hat{x}_j + \boldsymbol{Y}^2_{ij}\hat{p}_i\hat{p}_j. \tag{A.4}$$

However, because of the CCR we have $\boldsymbol{X}_{ji}\hat{p}_i\hat{x}_j = \boldsymbol{X}_{ij}\hat{x}_i\hat{p}_j + \mathrm{i}\boldsymbol{X}_{ij}\delta_{ij}$ and $\boldsymbol{Y}^1_{ij}\hat{x}_i\hat{x}_j = -\boldsymbol{Y}^1_{ij}\hat{x}_i\hat{x}_j = 0 = \boldsymbol{Y}^2_{ij}\hat{p}_i\hat{p}_j$, thus

$$2\hat{H}_b(\widetilde{\boldsymbol{h}^z}) - 2(\boldsymbol{d}^z)^T \hat{\mathbf{r}} = \boldsymbol{h}^z_{ij}\hat{z}_i\hat{z}_j - \underbrace{\boldsymbol{X}_{ij}\delta_{ij}}_{=\text{ const.}}. \tag{A.5}$$

Consequently, $\hat{H}_b(\widetilde{\boldsymbol{h}^z})$ and $\hat{H}_b(\boldsymbol{h}^z)$ describe the same *physical* Hamiltonian, so $\boldsymbol{X}, \boldsymbol{Y}^1$ and $\boldsymbol{Y}^2$ can be freely chosen. Therefore, by choosing

$$\begin{cases} 2\boldsymbol{X} = \boldsymbol{C}^T - \boldsymbol{B} \\ 2\boldsymbol{Y}^1 = \boldsymbol{A}^T - \boldsymbol{A} \\ 2\boldsymbol{Y}^2 = \boldsymbol{D}^T - \boldsymbol{D} \end{cases}, \tag{A.6}$$

we can rewrite $\boldsymbol{h}^z$ WLOG as the symmetric matrix

$$\boldsymbol{h}^z = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{D} \end{pmatrix}\Bigg|_{\substack{\boldsymbol{A}^T=\boldsymbol{A} \\ \boldsymbol{D}^T=\boldsymbol{D}}} \tag{A.7}$$

where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}$ have been redefined. Moreover, $\hat{H}_b$ has to be Hermitian, therefore $(\boldsymbol{h}^z)^\dagger = \boldsymbol{h}^z$ and $\overline{\boldsymbol{d}^z} = \boldsymbol{d}^z$, thus $\boldsymbol{h}^z$ is *real* and *symmetric* and $\boldsymbol{d}^z$ is also *real*.

For the expression of the Hamiltonian in terms of bosonic ladder operators we have

$$\hat{H}_b = \frac{1}{2}\hat{\mathbf{r}}^\dagger \boldsymbol{h}^r \hat{\mathbf{r}} + (\boldsymbol{d}^r)^T \hat{\mathbf{r}}. \tag{A.8}$$

Hence, we can proceed similarly and get the free parameters $\boldsymbol{X}, \boldsymbol{Y}^1$ and $\boldsymbol{Y}^2$ for the block matrix form

$$\widetilde{\boldsymbol{h}^r} = \begin{pmatrix} \boldsymbol{A}+\boldsymbol{X} & \boldsymbol{B}+\boldsymbol{Y}^1 \\ \boldsymbol{C}+\boldsymbol{Y}^2 & \boldsymbol{D}-\boldsymbol{X}^T \end{pmatrix} \tag{A.9}$$

thus, by choosing

$$\begin{cases} 2\boldsymbol{X} = \boldsymbol{D}^T - \boldsymbol{A} \\ 2\boldsymbol{Y}^1 = \boldsymbol{B}^T - \boldsymbol{B} \\ 2\boldsymbol{Y}^2 = \boldsymbol{C}^T - \boldsymbol{C} \end{cases}, \tag{A.10}$$

we can rewrite $\boldsymbol{h}^c$ WLOG as the block matrix

$$\boldsymbol{h}^r = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{A}^T \end{pmatrix}\Bigg|_{\substack{\boldsymbol{B}^T=\boldsymbol{B} \\ \boldsymbol{C}^T=\boldsymbol{C}}}. \tag{A.11}$$

Furthermore, for $\hat{H}_b$ to be Hermitian, $\boldsymbol{h}^r$ needs to be Hermitian, thus

$$\boldsymbol{h}^r = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \overline{\boldsymbol{B}} & \overline{\boldsymbol{A}} \end{pmatrix}\Bigg|_{\substack{\boldsymbol{B}^T=\boldsymbol{B} \\ \boldsymbol{A}^\dagger=\boldsymbol{A}}}. \tag{A.12}$$

## I.2 The fqH

For the fermionic case the Hamiltonian expressed in terms of Majoranas takes the form

$$\hat{H}_f = \mathrm{i}\frac{1}{2}\hat{\mathbf{c}}^T \boldsymbol{g}^c \hat{\mathbf{c}}. \tag{A.13}$$

In this case, however, we will not describe the matrix using a block-form approach, instead, because $\{\hat{a}_i^\dagger, \hat{a}_i\} = 1$

$$\begin{cases} \hat{c}_{2k}^2 = \frac{1}{2}(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger) = \frac{1}{2} \\ \hat{c}_{2k-1}^2 = \frac{1}{2}(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger) = \frac{1}{2} \end{cases}, \tag{A.14}$$

thus, our degree of freedom in the fermionic case translates to $\boldsymbol{g}_{ii}^c = 0$. Moreover, because $\{\hat{c}_i, \hat{c}_j\} = 0$ for $i$, we have

$$\boldsymbol{g}_{ij}^c \hat{c}_i \hat{c}_j \overset{i\neq j}{=} -\boldsymbol{g}_{ij}^c \hat{c}_j \hat{c}_i \quad \Longrightarrow \quad (\boldsymbol{g}^c)^T = -\boldsymbol{g}^c, \tag{A.15}$$

and thus, because $\hat{H}_f^\dagger = \hat{H}_f$ we get $(\boldsymbol{g}^c)^\dagger = -\boldsymbol{g}^c$, thus $\boldsymbol{g}^c$ is *real* and *skew-symmetric*.

However, if we recall that $\hat{\mathbf{c}} = \boldsymbol{\Lambda} \cdot \hat{\mathbf{r}}$, we have

$$\hat{H}_f = \mathrm{i}\frac{1}{2}\hat{\mathbf{r}}\boldsymbol{\Lambda}^\dagger \boldsymbol{g}^c \boldsymbol{\Lambda}\hat{\mathbf{r}} =: \mathrm{i}\frac{1}{2}\hat{\mathbf{r}}\boldsymbol{g}^r\hat{\mathbf{r}}, \tag{A.16}$$

thus $\boldsymbol{g}^r = \boldsymbol{\Lambda}^\dagger \boldsymbol{g}^c \boldsymbol{\Lambda}$. Moreover, because $(\boldsymbol{\Lambda}^\dagger \boldsymbol{g}^c \boldsymbol{\Lambda})^\dagger = -\boldsymbol{\Lambda}^\dagger \boldsymbol{g}^c \boldsymbol{\Lambda}$, $\boldsymbol{g}^r$ is a *complex skew-hermitian* matrix.

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