Supplementary Material for 'Highly adiabatic time-optimal quantum driving at low energy cost'

Lluc Garcia,[1,2] Josep Maria Bofill,[1,2] Ibério de P.R. Moreira,[2,3] and Guillermo Albareda[2,4,5]

[1]*Departament de Química Inorgànica i Orgànica, Secció de Química Orgànica, University of Barcelona, Martí i Franquès 1-11, 08028 Barcelona, Spain*

[2]*Institute of Theoretical and Computational Chemistry, University of Barcelona, Martí i Franquès 1-11, 08028 Barcelona, Spain*

[3]*Departament de Ciència de Materials i Química Física, University of Barcelona, Martí i Franquès 1-11, 08028 Barcelona, Spain*

[4]*Ideaded, Carrer de la Tecnologia, 35, 08840 Viladecans, Barcelona, Spain*

[5]*Max Planck Institute for the Structure and Dynamics of Matter and Center for Free-Electron Laser Science, Luruper Chaussee 149, 22761 Hamburg, Germany*

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I. DERIVATION OF THE TIME-OPTIMAL CONTROL SCHEME FOR A TWO-LEVEL SYSTEM

For $|\psi_i\rangle$ and $|\psi'_f\rangle$ being initial and target states in the interaction picture of a two-dimensional Hilbert space, these can be represented on the Bloch sphere by the Bloch vecotrs $\vec{n}_i$ and $\vec{n}'_f(\tau)$ respectively. Then, a state evolving along a great circle (a geodesic) subtending the angle, $\Phi(\tau) = \arccos(\vec{n}_i \cdot \vec{n}'_f(\tau))$, will reach the target state $|\psi'_f(\tau)\rangle$ in the least time τ provided that the angular velocity on the Bloch sphere $\dot{\Phi}(t)$ is the greatest available at any time. Such evolution is dictated by the unitary transformation $\mathcal{U}_c(t) = \exp\left(-i\frac{\Phi(t)}{2}\vec{\sigma}\cdot\vec{n}_\perp(\tau)\right)$, where the rotation axis is defined as $\vec{n}_\perp(\tau) = \frac{\vec{n}_i \times \vec{n}'_f}{|\vec{n}_i \times \vec{n}'_f|}$ and $\vec{\sigma}$ is the vector of Pauli matrices. The Hamiltonian that implements $\mathcal{U}_c(t)$ is:

$$H'_c(t) = \frac{\dot{\Phi}(t)}{2}\vec{\sigma}\cdot\vec{n}_\perp(\tau), \tag{I.1}$$

which is traceless, fulfills the transversality condition $\langle\psi'(t)|\dot{\psi}'(t)\rangle = 0$, and has a Hilbert-Schmidt norm $\|H'_c(t)\| = \sqrt{\mathrm{tr}(H'^2_c(t))} = \sqrt{\dot{\Phi}^2(t)/2}$.

The control Hamiltonian in Eq. (I.1) can now be written in terms of initial and final states $|\psi_i\rangle$ and $|\psi'_f(\tau)\rangle$. For that, we define a new basis through Gram-Schmidt orthogonalization, $\{|\psi_i\rangle, |\bar{\psi}'_f(\tau)\rangle\}$. We then multiply Eq. (I.1) from left and from right by the identity $\mathbb{1} = |\psi_i\rangle\langle\psi_i| + |\bar{\psi}'_f\rangle\langle\bar{\psi}'_f|$, and impose the traceless, transversality and Hilbert-Schmidt norm conditions. We find:

$$H'_c(t) = \frac{i\dot{\Phi}(t)}{2\sqrt{1-s^2}}\left(e^{-i\beta}|\psi'_f(\tau)\rangle\langle\psi_i| - e^{i\beta}|\psi_i\rangle\langle\psi'_f(\tau)|\right), \tag{I.2}$$

where s and β are defined through $\langle\psi_i|\psi'_f(\tau)\rangle = \langle\psi_i|\mathcal{U}_0^\dagger(\tau)|\psi_f\rangle = se^{i\beta}$.

Given the form of the time-optimal control Hamiltonian, we can now find the protocol time τ. We recall that the evolution of the state under $H(t) = H_0(t) + H_c(t)$ drives the initial state to the target state up to a phase factor, hence $|\langle\psi_f|\mathcal{U}_0(\tau)\mathcal{U}_c(\tau)|\psi_i\rangle| = 1$ by construction. This relation leads to a closed equation for s and τ:

$$s\cos\left(\frac{\Phi(\tau)}{2}\right) + \sqrt{1-s^2}\sin\left(\frac{\Phi(\tau)}{2}\right) = 1, \tag{I.3}$$

which admits the solution

$$s = \cos\left(\frac{\Phi(\tau)}{2}\right). \tag{I.4}$$

A simple manipulation of this equation allows us to write

$$\Phi(\tau) = 2\arccos(s) = 2\arccos|\langle\psi_i|\psi'_f(\tau)\rangle|, \tag{I.5}$$

and expressing the subtended angle as the integral of the angular velocity, $\Phi(\tau) = \int_0^\tau \dot{\Phi}(t)dt$, and defining $\bar{\dot{\Phi}} = \tau^{-1}\int_0^\tau \dot{\Phi}(t)dt$, finally yields

$$\tau = \frac{2}{\bar{\dot{\Phi}}}\arccos|\langle\psi_i|\psi'_f(\tau)\rangle|. \tag{I.6}$$

II. DERIVATION OF THE FUNCTIONS $\eta(t)$ AND $\zeta(t)$ OF EQ. (3)

In this section, we derive the specific form of the functions $\eta(t)$ and $\zeta(t)$ that define the geodesic path along which initial and final state are connected. The geodesic path in Eq. (3) of the main text can be recast as:

$$|\psi'(t)\rangle = \left(\eta(t) - \frac{se^{i\beta}}{\sqrt{1-s^2}}\zeta(t)\right)|\psi_i\rangle + \frac{\zeta(t)}{\sqrt{1-s^2}}|\psi'_f\rangle, \tag{II.1}$$

in terms of initial and final states.

From the above equation, we can already define the boundary values of the functions of interest. For $t=0$, we have

$$\eta(0) = 1, \tag{II.2}$$
$$\zeta(0) = 0; \tag{II.3}$$

while for $t=\tau$ we have

$$\eta(\tau) - \zeta(\tau)\frac{se^{i\beta}}{\sqrt{1-s^2}} = 0, \tag{II.4}$$
$$\zeta(\tau)\frac{1}{\sqrt{1-s^2}} = e^{i\alpha}, \tag{II.5}$$

where in the second equation we have allowed the final state to acquire a global phase during the evolution. Inserting Eq. (II.5) into (II.4) we get $\eta(\tau) = se^{i(\alpha+\beta)} = s$, where, without loss of generality, we have taken $\alpha = -\beta$ such that $\eta(t)$ is a real function. Thus, we can write:

$$\eta(\tau) = s, \tag{II.6}$$
$$\zeta(\tau) = e^{-i\beta}\sqrt{1-s^2}. \tag{II.7}$$

At his point we can already propose an ansatz that fulfills the above four boundary conditions (Eqs. (II.2), (II.3), (II.6) and (II.7)):

$$\eta(t) = \cos\theta(t), \tag{II.8}$$
$$\zeta(t) = e^{-i\beta}\sin\theta(t), \tag{II.9}$$

where the angle $\theta(t)$ can be fixed by realizing that $|\langle\dot{\psi}'(t)|\dot{\psi}'(t)\rangle| = \dot{\theta}^2(t)$ and then using the form of the control Hamiltonian in Eq. (6) of the main text:

$$\theta(t) = \int_0^t \Delta H'_c(t')dt'. \tag{II.10}$$

A further check on the above ansatz confirms that (i) the initial state $|\psi_i\rangle$ is monotonically brought closer to the final state $|\psi'_f\rangle$ and (ii) that this happens at maximum speed given a particular energy resource of the control $v_z(t)$. The first statement (i) can be proven by noting that the amplitudes accompanying initial and final states in Eq. (II.1) are, respectively, monotonically decreasing and increasing functions of time (from $\theta(0)$ to $\theta(\tau)$). The second statement (ii) can be assessed by realizing that $\langle\psi'(t)|\dot{\psi}'(t)\rangle = 0$, which is the transversality condition of Eq. (5) in the main text and guarantees the minimization of the protocol time τ.

III. TIME-OPTIMAL CONTROL OF TWO-QUBITS IN A TIME-INDEPENDENT DRIFT HAMILTONIAN

Here we consider the case of two interacting 1/2-spin qubits in a drift Hamiltonian of the form:

$$H_0 = -\sum_j J_j\sigma_j^{(1)}\sigma_j^{(2)}\ , \tag{III.1}$$

where $j = x, y, z$ and J_j represents the coupling between both spins. It turns out that this Hamiltonian is diagonal in the Bell states basis, and thus:

$$\begin{aligned} H_0 = &-(J_z+J_-)|\Phi^+\rangle\langle\Phi^+| - (J_z-J_-)|\Phi^-\rangle\langle\Phi^-| \\ &+(J_z-J_+)|\Psi^+\rangle\langle\Psi^+| + (J_z+J_+)|\Psi^-\rangle\langle\Psi^-|, \end{aligned} \tag{III.2}$$

with $J_\pm = J_x \pm J_y$.

Our goal is to entangle an initially separable state of the form, $|\psi_i\rangle = |0\rangle \otimes |1\rangle = |01\rangle$, into a Bell state, $|\psi_f\rangle = |\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$, in the minimum possible time. According to Eq. (7) of the main text, the control Hamiltonian in the Schrödinger picture reads:

$$H_c(t) = i\frac{v_z(t)}{\sqrt{1-s^2}}\mathcal{U}_0(t)\Big(e^{-i\beta}\mathcal{U}_0^\dagger(\tau)|\Psi^+\rangle\langle 01| - e^{i\beta}|01\rangle\langle\Psi^+|\mathcal{U}_0(\tau)\Big)\mathcal{U}_0^\dagger(t), \tag{III.3}$$

where $\langle 01|\mathcal{U}_0^\dagger(\tau)|\Psi^+\rangle = se^{i\beta}$. Using the Bell state basis, the above control Hamiltonian can be rewritten as

$$H_c(t) = \frac{1}{\sqrt{2}}\frac{v_z(t)}{\sqrt{1-s^2}}\Big(2\cos\Theta|\Psi^+\rangle\langle\Psi^+| + e^{i(\Theta+2J_+t)}|\Psi^+\rangle\langle\Psi^-| + e^{-i(\Theta+2J_+t)}|\Psi^-\rangle\langle\Psi^+|\Big), \tag{III.4}$$

where $\Theta = \frac{\pi}{2} - \beta + (J_z - J_+)\tau$. The parameters s and β can be obtained from the overlap $\langle 01|\mathcal{U}_0^\dagger(\tau)|\Psi^+\rangle$, which yields:

$$s = \frac{1}{\sqrt{2}} \quad \text{and} \quad \beta = (J_z - J_+)\tau, \tag{III.5}$$

and therefore $\Theta = \pi/2$. The control Hamiltonian in Eq. (III.4) finally reads:

$$H_c(t) = iv_z\Big(e^{i2J_+t}|\Psi^+\rangle\langle\Psi^-| - e^{-i2J_+t}|\Psi^-\rangle\langle\Psi^+|\Big), \tag{III.6}$$

where we have taken the 'full throttle' condition $v_z(t) = v_z$. The least time τ (from Eq. (8) of the main text) then reads:

$$\tau = \frac{1}{v_z}\arccos\left(\frac{1}{\sqrt{2}}\right), \tag{III.7}$$

which gives $\tau = \frac{\pi}{4v_z}$. We may then want to recast the Hamiltonian of Eq. (III.6) in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ i.e.:

$$\begin{aligned} H_c(t) &= 2v_z\sin(2J_+t)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 2v_z\cos(2J_+t)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= v_z\big(\sin(2J_+t)\big(\mathbb{1}\otimes\sigma_z - \sigma_z\otimes\mathbb{1}\big) + \cos(2J_+t)\big(\sigma_y\otimes\sigma_x - \sigma_x\otimes\sigma_y\big)\big)\,. \end{aligned} \tag{III.8}$$

IV. ON THE NUMERICAL IMPLEMENTATION OF THE CONTROL PROTOCOL

The numerical implementation of the proposed method can be summarized as follows:

i. Given a drift Hamiltonian $\hat{H}_0(t)$, initial and final states $|\psi_i\rangle$ and $|\psi_f\rangle$, and the energy resource of the control $v_z(t)$, we first compute the protocol time τ recursively by means of Eq. (8) (if the full throttle condition $v_z(t) = v_z$ is assumed) or according to $\int_0^\tau v_z(t)dt = \arccos|\langle\psi_i|\psi_f'\rangle|$ (if the energy resource depends on time). Note that this step requires the evaluation of the fidelity $\mathcal{F}_0 \equiv |\langle\psi_i|\psi_f'\rangle|^2$, which in turn entails the computation of $|\psi_f'\rangle = \mathcal{U}_0^\dagger(\tau)|\psi_f\rangle$, where $\mathcal{U}_0(t) = \mathcal{T}\exp\left(-i\int_0^t H_0(t_1)dt_1\right)$ and $\mathcal{T}$ defines the usual time ordering operator.

ii. Given τ, we then construct the control Hamiltonian in the interaction picture $H_c'(t)$ according to Eq. (7).

iii. Given $H_c'(t)$, we can already propagate the initial state $|\psi_i\rangle$ according to the time-dependent Schrödinger equation, $i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$, where $H(t) = H_0(t) + H_c(t)$ and $H_c(t) = \mathcal{U}_0(t)H_c'(t)\mathcal{U}_0^\dagger(t)$.

iv. During the propagation we may compute any observable of interest such as the mean adiabaticity or the cost according to Eqs. (13) and (14) respectively.

In the specific example of the main text, the drift Hamiltonian, $H_0(t)$, is given by Eqs. (11) and (12), and the initial and final states are the instantaneous adiabatic ground states of $H_0(t)$ at $t = 0$ and $t = \tau$ respectively. The first step, (i), is executed using the bisection method with a tolerance tol $= 10^{-5}$. The state propagation in steps (i) and (iii) is carried out using the Runge-Kutta method (4th order) with a time step $dt = 5\times10^{-6}$. The evolution of the control Hamiltonian in the Schrödinger picture was achieved using the same Runge-Kutta method and according to $dH_c/dt = -i[H_0, H_c]$ (after imposing the full throttle condition $v_z(t) = v_z$).

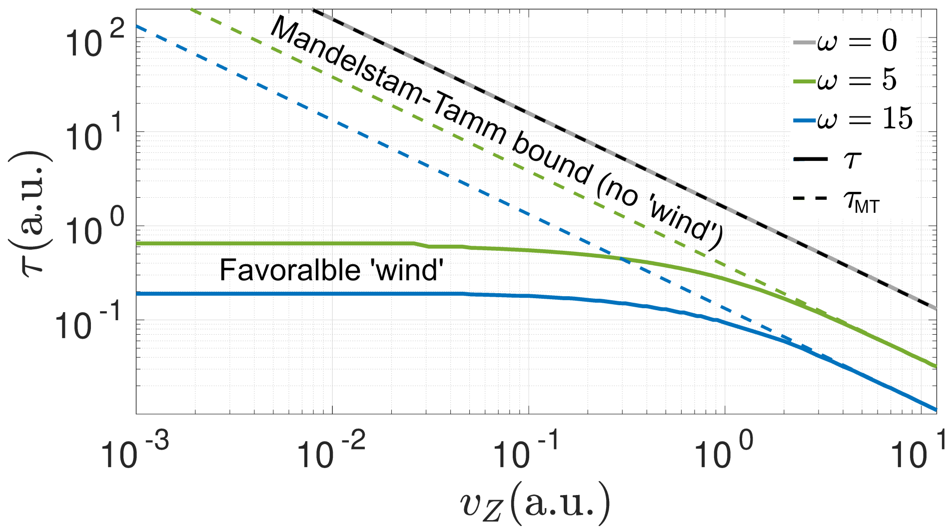

FIG. V.1. Duration of the protocol, τ, in the presence (solid lines) and in the absence (dashed lines) of the drift Hamiltonian $H_{\rm LZ}$ with $\Gamma_{\rm linear}$. These results are presented as a function of the energy disposal of the control, v_z, and for different values of the coupling constant, $\omega = \{0, 5, 15\}$.

V. TIME-OPTIMAL CONTROL OF A TWO-LEVEL SYSTEM IN A FAVORABLE WIND

Figure V.1 shows the protocol time τ as a function of the energy resource of the control, v_z, for three different values of the coupling constant $\omega = \{0, 5, 15\}$. Solid and dashed lines correspond to Eqs. (8) and (9) of the main text respectively. Figure V.1 is an example of the top inequality in Eq. (10), revealing how $H_{\rm LZ}$ with $\Gamma_{\rm linear}$ acts always as a favorable 'tail wind' in the search for the target state $|g(\tau)\rangle$. In the absence of a drift Hamiltonian, the control is the only driving force during the course of the protocol and hence its duration coincides with the MT bound, which according to Eq. (9) is inversely proportional to the energy disposal of the control v_z. In the presence of $H_{\rm LZ}$, the evolution of the system is expected to be roughly independent of the control for small enough values of v_z (note the asymptotic behavior of the solid lines for $v_z \lesssim 0.1$). An exception occurs for $\omega = 0$. In this limit the instantaneous diabatic and adiabatic levels coincide and a level crossing at $\tau/2$ induces an exchange of roles between ground and excited states, i.e., $\mathcal{F}_0 = 0$. Furthermore, since initial and final states $|g(0)\rangle$ and $|g(\tau)\rangle$ are eigenstates of $H_{\rm LZ}(t)$, the drift Hamiltonian does not exert any force on them and the protocol time reduces again to τ_{MT}, which for orthogonal initial and target states reads $\tau = \tau_{\rm MT} = \pi/2v_z$. In general, for finite values of ω the protocol time is smaller than $\tau_{\rm MT}$ and only at very large values of v_z, i.e., when the role of the drift Hamiltonian is negligible in front of the control Hamiltonian, these differences tend to vanish.

Importantly, no matter how high the energy disposal of the control (v_z) is, the mean energy of the system, $\langle {\rm E}\rangle = \langle\psi(t)|H_{\rm LZ}(t)|\psi(t)\rangle$, remains well bounded and close to the adiabatic evolution path. See for example the cases of $\omega = 0$ (where $\langle {\rm E}\rangle$ is independent of v_z) and $\omega = 1$ (where the shaded green region spans all possible values of v_z) in Fig. V.2.

VI. TIME-OPTIMAL CONTROL OF A DECOUPLED TWO-LEVEL SYSTEM

We here derive an analytical solution to Eqs. (7) and (8) of the main text for the case where the system is immersed in a drift Hamiltonian of the form:

$$H_{\rm LZ}(t) = \Gamma_{\rm linear}(t)\sigma_z, \tag{VI.1}$$

where $\Gamma_{\rm linear}$ has been defined in the main text, and initial and target states are respectively $|\psi_i\rangle = |g(t=0)\rangle = |0(t=0)\rangle$ and $|\psi_f\rangle = |g(t=\tau)\rangle = |1(t=\tau)\rangle$. Let us first write the time-evolution operator associated to $H_{\rm LZ}(t)$:

$$\mathcal{U}_{\rm LZ}(t) = \exp\left(-i\sigma_z \int_0^t \Gamma_{\rm linear}(t')dt'\right) = \exp\left(-i\sigma_z 2t\left(\frac{t}{\tau} - 1\right)\right) = \exp\left(-i\sigma_z \Lambda(t)\right). \tag{VI.2}$$

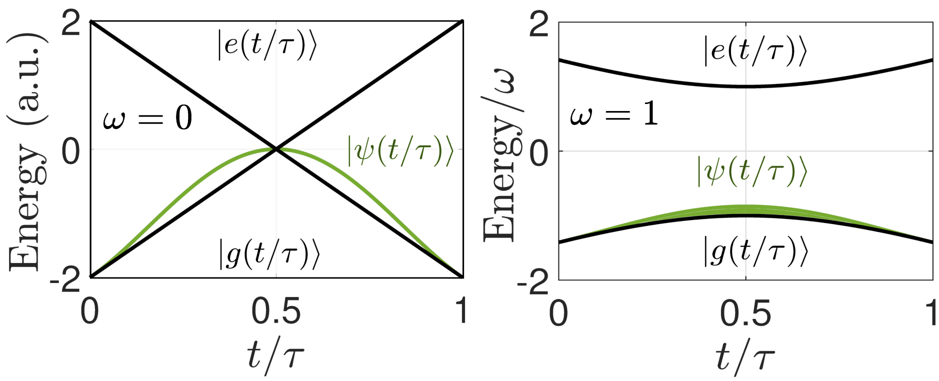

FIG. V.2. Evolution of the energy of the system $\langle \mathrm{E}\rangle = \langle\psi(t)|H_{\mathrm{LZ}}(t)|\psi(t)\rangle$ for all possible values of v_z (shaded green region) and for $\omega = 0$ (left panel) and $\omega = 1$ (right panel). The adiabatic energies of the drift Hamiltonian H_{LZ} are plotted in black as a reference.

This unitary operator allows us to find $s = |\langle\psi_i|\mathcal{U}^\dagger_{\mathrm{LZ}}(\tau)|\psi_f\rangle| = 0$, and bearing in mind the functional form of the control Hamiltonian of Eq. (7), this yields:

$$H'_c(t) = v_z\sigma_y. \tag{VI.3}$$

In the Schrödinger picture, the above equation reads

$$H_c(t) = \mathcal{U}_{\mathrm{LZ}}(t)H'_c(t)\mathcal{U}^\dagger_{\mathrm{LZ}}(t) = v_z \exp\left(-i\sigma_z\Lambda(t)\right)\sigma_y \exp\left(i\sigma_z\Lambda(t)\right) = v_z\left(\cos\left(2\Lambda(t)\right)\sigma_y - \sin\left(2\Lambda(t)\right)\sigma_x\right), \tag{VI.4}$$

where in the last equality we have taken into account the commutation relations of the Pauli matrices.

Given the above control Hamiltonian, we can now determine the dynamics of the system:

$$|\psi(t)\rangle = \mathcal{U}_{\mathrm{LZ}}(t)\mathcal{U}_c(t)|\psi_i\rangle\rangle = \exp\left(-i\sigma_z\Lambda(t)\right)\exp\left(-i\sigma_y\int_0^t v_z(t')dt'\right)|\psi_i\rangle = \exp\left(-i\sigma_z\Lambda(t)\right)\exp\left(-i\sigma_y v_z t\right)|\psi_i\rangle. \tag{VI.5}$$

Notice that in the last equality we have taken the 'full throttle' condition $v_z(t) = v_z$.

Given the above result we can now look for an analytical expression of the adiabaticity $\mathcal{A}(t) = |\langle g(t)|\psi(t)\rangle|^2$. For that, we first notice that the instantaneous adiabatic ground state evolves as:

$$|g(t)\rangle = \begin{cases}\mathcal{U}_{\mathrm{LZ}}(t)|0\rangle & 0 \le t \le \tau/2,\\ \mathcal{U}_{\mathrm{LZ}}(t)\mathcal{U}_{\mathrm{LZ}}(\tau/2)|1\rangle & \tau/2 \le t \le \tau,\end{cases} \tag{VI.6}$$

where $|0\rangle$ and $|1\rangle$ are the ground and excited states at $t = 0$, respectively. Using Eq. (VI.5) and Eq. (VI.6), the adiabaticity can be cast as the following piecewise expression:

$$\mathcal{A}(t) = \begin{cases}|\langle 0|\mathcal{U}^\dagger_{\mathrm{LZ}}(t)|\psi(t)\rangle|^2 = \cos^2\left(\frac{\pi}{2}\frac{t}{\tau}\right) & 0 \le t \le \tau/2,\\ |\langle 1|\mathcal{U}^\dagger_{\mathrm{LZ}}(t)|\psi(t)\rangle|^2 = \sin^2\left(\frac{\pi}{2}\frac{t}{\tau}\right) & \tau/2 \le t \le \tau,\end{cases} \tag{VI.7}$$

where we have used that $\tau = \pi/2v_z$. Finally, the mean adiabaticity can be written as

$$\bar{\mathcal{A}} = \frac{1}{\tau}\int_0^\tau \mathcal{A}(t)dt = \frac{1}{\tau}\int_0^{\tau/2}\cos^2\left(\frac{\pi}{2}\frac{t}{\tau}\right)dt + \frac{1}{\tau}\int_{\tau/2}^{\tau}\sin^2\left(\frac{\pi}{2}\frac{t}{\tau}\right)dt = \frac{2+\pi}{2\pi} \approx 0.82. \tag{VI.8}$$

VII. FORM OF THE CONTROL HAMILTONIAN FOR A DRIVEN TWO-LEVEL SYSTEM

The most general form of the control Hamiltonian that drives an initial state into a target state under the influence of H_{LZ} can be written as:

$$H_c(t) = \mu(t)\sigma_x - \gamma(t)\sigma_y + \epsilon(t)\sigma_z. \tag{VII.1}$$

For the particular case of $\omega = 0.1$ and Γ_{linear}, the time-dependence of the functions $\mu(t)$, $\gamma(t)$ and $\epsilon(t)$ is shown in Fig. VII.1 for different values of the least time τ (this is to say for different values of the energy resource of the control v_z). For the sake of comparison, we also plot the corresponding counterdiabatic term $H_{\text{CD}}(t)$, that for the LZ model with $\Gamma_{\text{linear}}(t)$, takes the specific form [1]:

$$H_{\text{CD}}(t) = -\frac{\dot{\Gamma}_{\text{linear}}(t)\omega}{2\big(\omega^2 + \Gamma^2_{\text{linear}}(t)\big)}\sigma_y, \tag{VII.2}$$

where $\dot{\Gamma}_{\text{linear}}(t) = \partial_t \Gamma_{\text{linear}}(t) = 4/\tau$, and the coefficients of σ_x and σ_z are zero. In the three panels of Fig. VII.1, we observe that the coefficients are either antisymmetric (the case of $\epsilon(t)$) or symmetric (the case of $\mu(t)$ and $\gamma(t)$) with respect to the midpoint of the time axis (at $t = \tau/2$) where an avoided crossing of the adiabatic levels occurs with an energy gap 2ω. The coefficients in Eq. (VII.1) tend to zero progressively as τ increases (i.e. the energy resource of the control decreases). This behavior indicates that the control is less relevant at low speeds, as expected. Moreover, notice that the three quantities, μ, γ and ϵ, are connected through the Zermelo velocity, $v_z = \sqrt{\text{tr}\big(H_c^2(t)\big)/2}$, which can be written as:

$$v_z(t) = \sqrt{\mu^2(t) + \gamma^2(t) + \epsilon^2(t)}. \tag{VII.3}$$

Above, we have used that $H_c^2(t) = \big(\mu^2(t) + \gamma^2(t) + \epsilon^2(t)\big)\mathbb{1}$ and thus $\text{tr}\big(H_c^2(t)\big) = 2\big(\mu^2(t) + \gamma^2(t) + \epsilon^2(t)\big)$. Equation (VII.3) establishes a connection between the three panels of Fig. VII.1 for the control Hamiltonian. Regarding the counterdiabatic term (VII.2), it is relevant to highlight the fact that it takes considerably higher values than the corresponding control term of our protocol, as can be seen in the first panel.

The practical implementation of the above control Hamiltonian depends on the system under consideration. As a first example, consider the situation where a spin 1/2 is immersed in a time-varying magnetic field $\vec{B}(t)$. The spin-field interaction Hamiltonian can be written as:

$$H_S(t) = -\xi\vec{B}(t)\cdot\vec{S} = -\frac{\xi}{2}\big(B_x(t)\sigma_x + B_y(t)\sigma_y + B_z(t)\sigma_z\big), \tag{VII.4}$$

where $\xi = gq/2m$, with g the g-factor and m and q the mass and the charge of the particle, respectively. Comparing Eqs. (VII.1) and (VII.4), it is simple to establish a one-to-one correspondence between the components of the magnetic field and the control Hamiltonian:

$$B_x(t) = -\frac{2\mu(t)}{\xi}, \quad B_y(t) = \frac{2\gamma(t)}{\xi}, \quad B_z(t) = -\frac{2\epsilon(t)}{\xi}. \tag{VII.5}$$

As a second example, we may consider a Bose–Einstein condensate (BEC) in an optical lattice. Under appropriate conditions the wavefunction of the BEC in the periodic potential of the optical lattice can be approximated by considering only the two lowest energy bands [2, 3], which exhibit an avoided crossing at the edge of the first Brillouin zone (see Fig. V.2) and thus realize a Landau-Zener Hamiltonian of the form in Eq. (11) of the main text, where terms accompanying σ_z and σ_x can be controlled through the quasimomentum p and the depth V_0 of the optical lattice, respectively. To make a given evolution of a two-level system introducing the control Hamiltonian of Eq. (VII.1), one needs to add an interaction term corresponding to a σ_y Pauli matrix. In practice, this term can be implemented by introducing an additional interaction into the system, for example, through an extra laser or microwave field. In the case of atoms in an optical lattice, σ_y components can be realized by adding a second optical lattice shifted with respect to the first by $d_L/4$ (d_L is the lattice spacing) [4]. It can be shown, however, that the effect of this extra field can also be achieved through an appropriate transformation $\Gamma \to \Gamma'$ and $\omega \to \omega'$, such that no extra field is necessary [4]. This transformation, which should be independent of the physical system under consideration (and could be considered as well for the spin in a magnetic field case), means that the resulting protocol is intrinsically more stable, as there will be no problems associated, for example, with phase fluctuations between the fields [4].

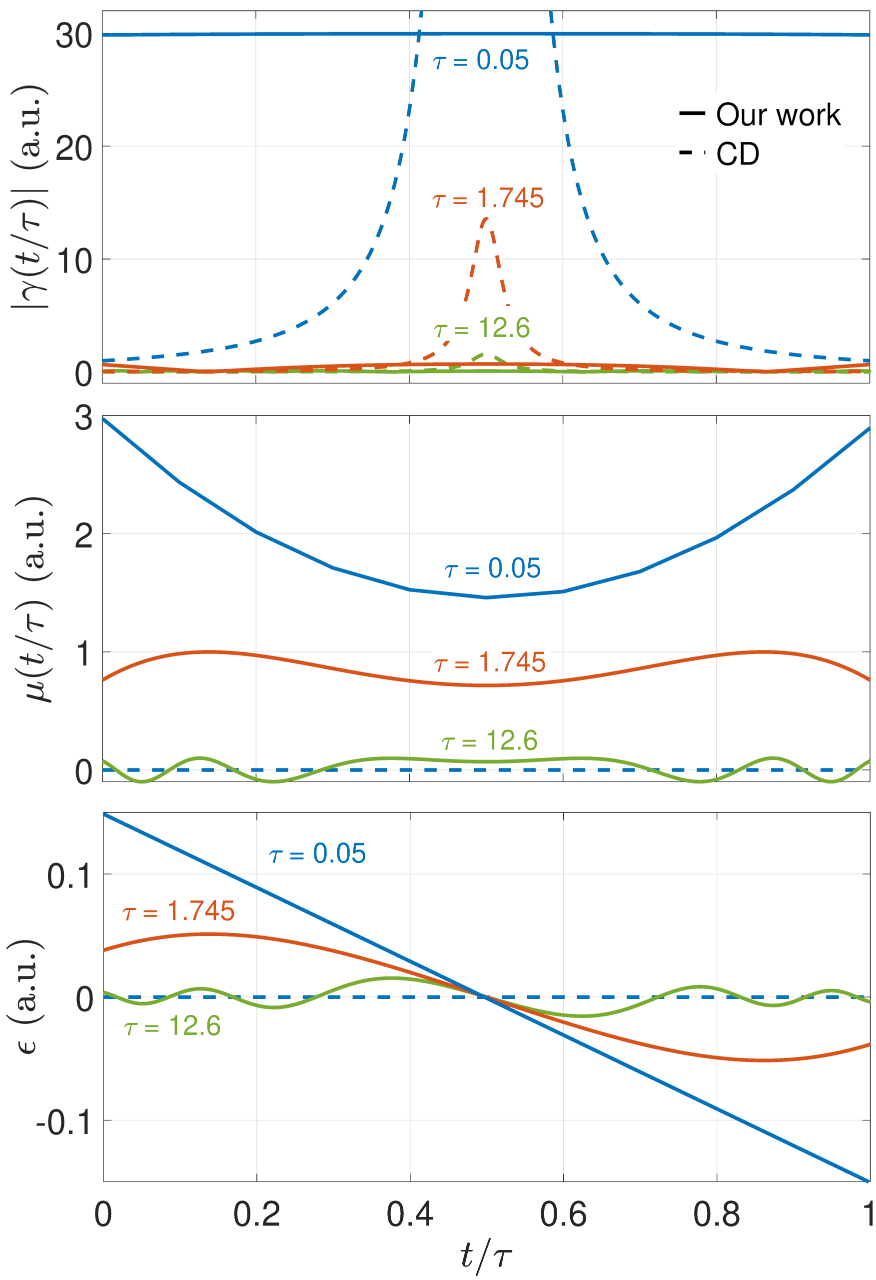

FIG. VII.1. Time dependence of each component of the control Hamiltonian in Eq. (VII.1) as a function of the normalized time, t/τ, and for different values of the protocol time τ (solid lines). Dashed lines correspond to the counterdiabatic field of Eq. (VII.2).

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