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TRABAJO FIN DE GRADO

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Homotopy Theory and homotopy groups

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Contents

Resumen

En este trabajo se desarrolla una introducción a la Teoría de Homotopía, centrada en el estudio de los grupos de homotopía. Primeramente, se introducen ciertas nociones sobre categorías, confiriendo estas nociones un lenguaje natural para establecer algunos conceptos fundamentales de la Topología Algebraica. Después, se presenta una pequeña introducción a ideas básicas de la Teoría de Homotopía, y se muestra su capacidad de desarrollar invariantes topológicos a través de la construcción del grupoide fundamental y el grupo fundamental, junto con el ejemplo del cómputo del grupo fundamental de la circunferencia. Se continúa introduciendo los grupos de homotopía, junto a un breve estudio sobre algunas de sus propiedades. Para finalizar, se introduce una sección dedicada en términos amplios al cómputo de los grupos de homotopía. Se usa esta sección como pretexto para introducir pinceladas sobre temas de estudio en el ámbito de la Teoría de Homotopía, como la clasificación de espacios recubridores, ó la Teoría de Homotopía Estable, buscando dar una pequeña muestra de las direcciones en las que uno puede progresar en el estudio de la teoría general más allá de este trabajo.

Abstract

In this essay, an introduction to Homotopy Theory is developed, centred around the study of the homotopy groups. Firstly, some notions on categories are described, constructing the language to develop several fundamental concepts of Algebraic Topology. Then, we present some basic definitions of Homotopy Theory, and we illustrate its ability to construct topological invariants through the fundamental groupoid and the fundamental group, along with the example of the computation of the fundamental group of the circumference. The homotopy groups are then introduced, together with a brief study of some of their properties. Lastly, a section is included which is dedicated to their computation in broad terms. It is used as pretext to give little insights into several topics of study in the ambit of Homotopy Theory, as the classification of covering spaces or Stable Homotopy Theory, giving a little taste of the directions in which one can further study the general theory beyond this essay.

1 Introduction

Jean Dieudonné attributes ([11]) to Henri Poincaré both the vision of the role Topology was to play in Mathematics, and the introduction, together with mathematicians like Bernhard Riemann and Felix Klein, of the first insights into what was to become Algebraic Topology.

The first challenge of Topology once the notion of homeomorphism was defined, was the classification of topological spaces up to homeomorphism. This was soon realised to be a hopeless goal, as even classes of "simple" spaces, like the subspaces of $\mathbb{R}^2$, resulted too complicated to classify. There was then a shift in attitude, as fields like Algebraic Topology were born with the objective of, not fully classifying all topological spaces, but assigning them invariants that were the same for two homeomorphic spaces.

In 1895, Poincaré published *Analysis situs* ([14]), a treatise through which, together with a series of other papers called the *Complements à l'Analysis Situs*, he established intuitions of what was to be Algebraic Topology, introducing for the first time some concepts of what now is called Homology, and the fundamental group. Poincaré introduced in these papers the Betti numbers, which in *Analysis Situs*, he connected to a notion of "orders of connectedness" established by Enrico Betti. Peter Hilton attributes ([8]) to Pavel Aleksandroff and Heinz Hopf the observation, while they were studying Lefschetz's fixed point theorem, of a connection between the result and these Betti numbers; and he attributes to Emmy Noether the introduction of algebraic concepts to the theory, as she noticed that the Betti numbers corresponded to algebraic invariants of abelian groups associated to the topological space, the homology groups. And so, Algebraic Topology was born.

The notion of homotopy was introduced by L.E.J. Brouwer in 1911 as an auxiliary concept to Homology, in particular regarding the study of the fundamental group, still in a purely homological context. It was through the work of Hopf on maps into spheres in the 1930s ([9]) that Homotopy Theory proper was developed, and Eduard Čech first defined higher homotopy groups in 1935.

From there, Homotopy Theory and its relations to Homology exploded in popularity among the mathematical community. In the 1950s, Samuel Eilenberg and Saunders MacLane introduced Category Theory ([17]) in their work on the foundations of Algebraic Topology, and it quickly spread to many other fields of Mathematics. It did not only became the natural language for Homotopy Theory, but it also generalised notions from Algebraic Topology to other areas of Mathematics, allowing the development of new fields such as Homological Algebra and K Theory.

It was this environment that allowed the work of mathematicians like Daniel Quillen, who developed model categories to generalise the construction of the topological homotopy category ([16]); and Alexander Grothendieck, who, among many other things, pushed the notion that the $\infty-$groupoids, purely algebraic objects that remained to that day without a clear rigorous definition, ought to be defined as to allow them to model topological homotopy types ([7]).

2 Brief notes on Category Theory

Category Theory is a field that engages in abstract study of mathematical structures, and the relationships that arise between them. Its objects of study are categories, which represent collections of mathematical objects, together with morphisms between them.

The mathematical language of this theory has a broad capacity to generalise and translate results and notions between different areas of Mathematics. For example, many Algebraic Topological notions have been abstracted from the study of topological spaces, and successfully implemented in other areas, such as Homology being widely used in the study of groups, and Homotopy being translated to the context of Category Theory itself.

In this essay, we will use notions from Category Theory to formulate how some geometric and topological notions can be given an algebraic structure, and how this process can be understood as describing relations between categories of topological spaces, and categories of algebraic objects, such as groups. These relations can be used to design topological invariants, widely used in the context of the classification of topological spaces.

A brief introduction to Category Theory is developed here. The results described here will be basic enough that any book with a categorical approach will include most of them, like Allen Hatcher's *Algebraic Topology* ([1]).

Definition 2.1 (Category)**.** A category, $\mathcal{C}$, consists of:

- A class $ob(\mathcal{C})$. Each element of $ob(\mathcal{C})$ is called an object of $\mathcal{C}$.
- $\forall A, B \in ob(\mathcal{C})$, a set $hom_{\mathcal{C}}(A, B)$. Each element f of this set is called a morphism or an arrow from A to B, and we can denote it by $f\colon A \to B$. The sets $hom_C(A, B)$ are called the hom-sets of $\mathcal{C}$.

such that they follow these conditions:

- $\forall A, B, C \in ob(\mathcal{C})$, there exists a composition

$$\begin{aligned} \cdot \circ \cdot \colon hom_{\mathcal{C}}(B, C) \times hom_{\mathcal{C}}(A, B) &\longrightarrow hom_{\mathcal{C}}(A, C) \\ (f, g) &\longmapsto f \circ g \end{aligned}$$

- If $A \neq C$ or $B \neq D$, then $hom_{\mathcal{C}}(A, B)$ and $hom_{\mathcal{C}}(C, D)$ are disjoint sets.
- The composition of morphisms is associative.
- $\forall A \in ob(\mathcal{C})$, $\exists 1_A \in hom_{\mathcal{C}}(A, A)$ such that $1_A \circ f = f$, $\forall f \in hom_{\mathcal{C}}(B, A)$ and $g \circ 1_A = g$, $\forall g \in hom_{\mathcal{C}}(A, B)$. These morphisms are called identity morphisms, and in some categories are denoted id_A instead of 1_A.

In general, we can think of the class of objects of a given category as the collection of all mathematical objects of some given form, and their morphisms as select maps between them. We observe some examples of this

Example 2.2. *We define the category* **Set** *as the category such that its objects are the sets and, for two given sets A, B, the hom-set $hom_{\mathbf{Set}}(A, B)$ is the set of maps with domain A and codomain B, and morphism composition is given by usual map composition.*

Observation 2.3. When we state that $ob(\mathcal{C})$ is a class for some category $\mathcal{C}$, it is intentionally allowed to be a proper class instead of a set. Observe that this is required, as if the class of objects was required to be a set, a category of all sets would not exist, as standard Set Theory doesn't allow for universal sets (sets that contain all sets) to exist.

When defining hom-sets, a more general definition of category allows them too to be proper classes. When considering the more general definition, the categories we have defined are usually called locally small categories. As every category in this essay will be locally small, no conflict will arise from this divergence in definitions.

Example 2.4. *We define the category* **Grp** *as the category such that its objects are the groups and, for two given groups,* G, K*, the set of morphisms* $hom_{\mathbf{Grp}}(G, K)$ *is the set of group homomorphisms with domain* G *and codomain* K*, and morphism composition is given by usual map composition.*

Example 2.5. *We define the category* **Top** *as the category such that its objects are the topological spaces and, for two given topological spaces* X *and* Y*,* $hom_{\mathbf{Top}}(X, Y)$ *is the set of continuous maps with domain* X *and codomain* Y*, with morphism composition given by usual map composition.*

Definition 2.6 (Isomorphism)**.** Let $\mathcal{C}$ be a category, and f be a morphism between two objects, $A, B \in ob(\mathcal{C})$. We say f is an isomorphism if $\exists g$ morphism between B and A such that $f \circ g = 1_A$ and $g \circ f = 1_B$.

If two objects, A, B, are such that there exists some $f \in hom_{\mathcal{C}}(A, B)$ with f isomorphism, we say A and B are isomorphic, and we write $A \cong B$.

For some f isomorphism, the associated g is called the inverse of f. For some isomorphism f, its inverse can be denoted f^{-1}.

Proposition 2.7. *Let* $\mathcal{C}$ *be a category, and* $f\colon X \to Y$ *be a morphism in* $\mathcal{C}$*. If* $\exists g\colon Y \to X$ *inverse of* f *in* $\mathcal{C}$*, it is unique.*

We observe that, for example, the concept of isomorphism in the category of sets corresponds with bijections. In the category of topological spaces, isomorphisms correspond with homeomophisms, and in the category of groups, they correspond with group isomorphisms.

Definition 2.8 (Functor / Covariant functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A (covariant) functor $F\colon \mathcal{C} \to \mathcal{D}$ is comprised of:

- A map between classes $F\colon ob(\mathcal{C}) \to ob(\mathcal{D})$.
- $\forall A, B \in ob(\mathcal{C})$, a map $F\colon hom_{\mathcal{C}}(A, B) \to hom_{\mathcal{D}}\big(F(A), F(B)\big)$.

such that these conditions are followed:

- If g and f are morphisms of $\mathcal{C}$ that can be composed, then $F(f \circ g) = F(f) \circ F(g)$.
- $\forall A \in ob(\mathcal{C})$, $F(1_A) = 1_{F(A)}$.

Note 2.9. What has been defined in definition 2.8 is what's called a covariant functor, to distinguish it from contravariant functors.

For a given category, $\mathcal{C}$, we can consider its opposite category, $\mathcal{C}^{op}$, which is the category such that $ob(\mathcal{C}^{op}) = ob(\mathcal{C})$, and $\forall A, B \in ob(\mathcal{C})$, $hom_{\mathcal{C}^{op}}(A, B) = hom_{\mathcal{C}}(B, A)$, where if $f \in hom_{\mathcal{C}^{op}}(A, B)$, $g \in hom_{\mathcal{C}^{op}}(B, C)$, their composition in $\mathcal{C}^{op}$, $g \circ_{\mathcal{C}^{op}} f$, is equal to $f \circ_{\mathcal{C}} g$, their composition in $\mathcal{C}$. In this sense, for each morphism $f\colon A \to B$ in $\mathcal{C}$, we have a correspondence with a morphism $f^{op}\colon B \to A$, such that $f^{op} = f$, with f^{op} being an equal morphism but "with the direction of the arrow reversed".

The relation between covariant and contravariant functors can then be understood as both being essentially the same, but with the latter reversing the direction of the morphisms. Both notions are used in practice, as many important functors are contravariant, but when we refer to functors we will mean covariant functors by default, as all the functors we will describe in this essay are covariant.

Proposition 2.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then, if f is an isomorphism between two objects, $A, B \in \mathcal{C}$, $F(f)$ is an isomorphism between $F(A)$ and $F(B)$. It follows that $A \cong B \implies F(A) \cong F(B)$.*

This observation implies that functors are useful tools for the search of invariants in many different fields.

For example, take the category **Top**. Isomorphisms in this category are homeomorphisms. This means that, if we take a functor $F\colon \mathbf{Top} \to \mathcal{C}$ to some category $\mathcal{C}$, two homeomorphic topological spaces will have isomorphic images through this functor. This can be used to distinguish topological spaces that are not homeomorphic. Consider the following example.

Example 2.11. *Consider $F\colon \mathbf{Top} \to \mathbf{Set}$ such that $F\big((X, \mathcal{T})\big) = X$ and $F(f) = f$ as a map of sets. F is a functor. Then, if two topological spaces are homeomorphic, their base sets are isomorphic in* **Set***, that is, bijective. Obviously, the reciprocal implication is not true.*

This functor is called "forgetful", because it doesn't transform the structure of the objects and morphisms, so much as making them "forget" some of the structure they already had. In this case, the base set is preserved, but the information about the topology on it is lost.

3 Basic definitions and results

Algebraic Topology is the field of study that deals with describing topological invariants with algebraic structure. We will present it through the theory of homotopy, which develops a series of functors between categories of topological spaces and **Grp**, the homotopy groups, which can be used for the study and classification of topological spaces. We will deal in section 5 with the construction and several basic properties of these groups. For the next sections, the main references are Allen Hatcher's *Algebraic Topology* ([1]), which contains most of the results presented in this essay; and Sze-Tsen Hu's *Homotopy Theory* ([10]), which contains alternative approaches to some of the proofs and parts of the theory.

In this essay, we will consider X and Y topological spaces. Any time we describe a topological subspace of $\mathbb{R}^n$ without specifying its topology, we will assume it to be the usual topology.

We will denote by I the interval $[0, 1]$; by D^n the set $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\}$, with $\mathbf{x} := (x_1, x_2, \cdots, x_n)$ and $\|\cdot\|$ the Euclidean norm; by S^n the set $\{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$; and, when context so indicates, we will denote by $*$ a set with a single element, or the topological space on that set together with the trivial topology.

In this essay it is also assumed that $\mathbb{N} = \{1, 2, \cdots\}$. When results related to natural numbers extend to the case $n = 0$, it will be explicitly stated.

Definition 3.1 (Homotopy)**.** Let $f_0, f_1\colon X \to Y$ be continuous maps. A homotopy from f_0 to f_1 is a continuous map $F\colon X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x), \forall x \in X$.

We note that, if we consider for each $t \in I$ the map $f_t \colon X \to Y$ given by $f_t(x) = F(x,t)$ for all $x \in X$, we can think of a homotopy as an indexed family of continuous maps, $\{f_t\}_{t\in I}$. We will indistinctly describe homotopies either way throughout this essay.

If for some $A \subseteq X$ we have that $f_0|_A = f_1|_A$, and $F(a,t) = f_0(a)\ \forall t \in I, \forall a \in A$, we say that F is a homotopy from f_0 to f_1 relative to A.

We will introduce before we move forward an interesting lemma, to which we will turn to continuously, as it allows us simple proofs of the continuity of some maps.

Lemma 3.2 (Gluing lemma)**.** *Let $f \colon X \to Y$ be a map. If $X = U \cup V$, where U, V are open (closed) in X, and $f|_U, f|_V$ are continuous, then f is continuous.*

This result generalises to any open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of X, while it only works in general for finite closed covers.

The importance of such lemma comes from the fact that many of the homotopies we will describe arise from piece-wise maps, and the gluing lemma allows for easy proofs of their continuity.

Definition 3.3 (Homotopic maps)**.** Let $f, g \colon X \to Y$ be continuous maps. We say f and g are homotopic, and denote it $f \simeq g$, if there exists a homotopy from f to g. If $H \colon X \times I \to Y$ is any such homotopy, we can also denote this by $H \colon f \simeq g$ to specify it.

If f and g are homotopic, and the homotopy H is relative to some subspace $A \subseteq X$, we say f and g are homotopic relative to A, and denote this by $f \simeq_A g$, or $H \colon f \simeq_A g$ if we want to specify the homotopy.

We can think of a homotopy between continuous maps as a continuous deformation of one into the other, happening over time as indexed by the parameter t.

We observe that $f \simeq g \iff f \simeq_\emptyset g$, and so, when we are able, we will prove statements using relative homotopy to offer as much generality as possible.

Proposition 3.4. *Let X, Y, Z be topological spaces, $A \subseteq X$, $B \subseteq Y$, $f_0, f_1 \colon X \to Y$, $g_0, g_1 \colon Y \to Z$ continuous maps such that $f_i(A) \subseteq B$, and $f_0 \simeq_A f_1$ and $g_0 \simeq_B g_1$. Then, $g_0 \circ f_0 \simeq_A g_1 \circ f_1$.*

Proof. Let $H \colon f_0 \simeq_A f_1$ and $G \colon g_0 \simeq_B g_1$ be the homotopies between the maps. We will prove the statement by giving the homotopy between the composition maps.

Let $L \colon X \times I \to Z$ be the map given by $L(x,t) = G\left(H(x,t), t\right)$. This map is clearly continuous, $L(x,0) = G(H(x,0),0) = g_0(f_0(x))$, and $L(x,1) = G(H(x,1),1) = g_1(f_1(x))$.

Observe that, if $x \in A$, $H(x,t) = f_0(x)$, $\forall t \in I$, and $f_0(x) \in B$, $\forall x \in A$, and so $L(x,t) = G(f_0(x),t) = g_0(f_0(x))$, $\forall t \in I, \forall x \in A$, implying the homotopy is relative to A. □

Proposition 3.5. *Let $A \subseteq X$ be a subspace. The relation $\simeq_A$ between continuous maps from X to Y is an equivalence relation.*

Proof. Let $f_0, f_1, f_2 \colon X \to Y$ be continuous maps. We must prove $\simeq_A$ is reflexive, symmetric and transitive.

- If we consider $H \colon X \times I \to Y$ given by $H(x,t) = f_0(x)$, we have $H \colon f_0 \simeq_A f_0$, and so $\simeq_A$ is reflexive.

- If we have $F\colon f_0 \simeq_A f_1$, the map $H\colon X \times I \to Y$ given by $H(x,t) = F(x, 1-t)$ is continuous, and so a homotopy between $H(x,0) = f_1(x)$ and $H(x,1) = f_0(x)$, with $H(a,t) = F(a,1-t) = F(a,0)$, $\forall a \in A, \forall t \in I$, and so $H\colon f_1 \simeq_A f_0$, and $\simeq_A$ is symmetric.

- Let $F_1\colon f_0 \simeq_A f_1$ and $F_2\colon f_1 \simeq_A f_2$. Consider the map $H\colon X \times I \to Y$ given by

$$H(x,t) = \begin{cases} F_1(x, 2t) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ F_2(x, 2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

  H is well defined, as both maps are compatible at $t = \frac{1}{2}$, and is continuous by lemma 3.2, as $X \times \left[0, \frac{1}{2}\right], X \times \left[\frac{1}{2}, 1\right]$ are closed, cover $X \times I$ and H is continuous over them. This means H is homotopy, and $H\colon f_0 \simeq_A f_2$, as $H(x,t) = f_0(x)$, $\forall x \in A, \forall t \in I$, as it is equal to either $F_1(x,2t)$ or $F_2(x, 2t-1)$, and both are relative to A. This implies $\simeq_A$ is transitive.

□

Definition 3.6 (Homotopy invariant functor)**.** Let $F\colon \mathbf{Top} \to \mathcal{C}$ be a functor between the category of topological spaces and some category $\mathcal{C}$. We say F is a homotopy invariant functor if, $\forall f, g\colon X \to Y$ morphisms in **Top** such that $f \simeq g$, then $F(f) = F(g)$.

To understand the motivation behind definition 3.6, we first need to introduce the concept of homotopy equivalence of spaces.

Definition 3.7 (Homotopy equivalence. Homotopy type)**.** Let $f\colon X \to Y$ be a continuous map such that there exists a continuous map $g\colon Y \to X$ with $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. Then, we say f is a homotopy equivalence from X to Y.

We write $X \simeq Y$ if there exists a homotopy equivalence from X to Y. This relation is an equivalence relation, and two spaces in the same equivalence class are said to possess the same homotopy type.

Let $X \simeq Y$ be topological spaces. Then, by definition 3.7, we have two continuous maps, f, g, such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. Let $F\colon \mathbf{Top} \to \mathcal{C}$ be a homotopy invariant functor. Then, by definitions 2.8 and 3.6, we have:

$$F(g) \circ F(f) = F(g \circ f) = F(id_X) = id_{F(X)}$$

$$F(f) \circ F(g) = F(f \circ g) = F(id_Y) = id_{F(Y)}$$

This obviously means that $F(f)$ is an isomorphism in $\mathcal{C}$ with inverse $F(g)$, and so we have that $X \simeq Y \implies F(X) \cong F(Y)$ in $\mathcal{C}$.

If we compare homotopy equivalences and homeomorphisms, it can be seen that $X \cong Y \implies X \simeq Y$, and in fact, we can verify that the homotopy equivalence is a strictly weaker equivalence relationship than that of homeomorphism.

Definition 3.8. We say a topological space X is contractible if a single point space $*$ has the same homotopy type as X.

Definition 3.9 (Retract. Deformation retract. Strong deformation retract)**.** Let $A \subseteq X$ be a subspace of X, and $i\colon A \hookrightarrow X$ the canonical inclusion. Let $r\colon X \to A$ be a continuous map.

- We say r is a retract if $r \circ i = id_A$.
- We say r is a deformation retract if it is a retract, and $i \circ r \simeq id_X$.
- We say r is a strong deformation retract if it is a deformation retract, and $i \circ r \simeq_A id_X$.

We also say that A is a retract (deformation retract, strong deformation retract) of X if $\exists r\colon X \to A$ a retract (deformation retract, strong deformation retract).

Observation 3.10. When proving a given subspace $A \subseteq X$ is a (strong) deformation retract, we will normally just give a homotopy between id_X and some continuous map $r\colon X \to X$ (relative to A) with $r(X) \subseteq A$. We will do this by implicitly considering the map $r\colon X \to A$ (restriction of r in codomain to A) as the (strong) deformation retract, and leaving the rest of the conditions in the definition as easily verifiable.

Proposition 3.11. *Deformation retracts are homotopy equivalences.*

Example 3.12. $\mathbb{R} \not\cong *$, *but* $\mathbb{R}$ *is contractible.*

Proof. Evidently, they are not homeomorphic, because their base sets are not bijective. If we consider $F\colon \mathbb{R} \times I \to \mathbb{R}$ given by $F(x,t) = (1-t)x$, it is continuous, and so a homotopy relative to $\{0\}$, which corresponds to a strong deformation retract. This means that $\mathbb{R}$ has the homotopy type of a point, but it is not homeomorphic to a point. □

In fact, this is a particular example of the more general fact that convex subsets of $\mathbb{R}^n$ are contractible. Consider any convex subset $V \subseteq \mathbb{R}^n$, and any point $x_0 \in V$. As V is convex, for any point x, the segment between x_0 and x is contained in V. This means that $tx_0 + (1-t)x \in V$, $\forall t \in I$. If we consider the homotopy $F\colon V \times I \to V$ given by $F(x,t) = tx_0 + (1-t)x$, we have that $\{x_0\}$ is a strong deformation retract of V, which implies V is contractible.

Example 3.13. S^{n-1} *is a strong deformation retract of* $D^n \setminus \{0\}$, *where* 0 *denotes the centre of the disk.*

Proof. Let $r\colon D^n \setminus \{0\} \to D^n \setminus \{0\}$ be given by $r(x) = \frac{x}{\|x\|}$, where we consider the usual euclidean norm, and consider the homotopy $H\colon (D^n \setminus \{0\}) \times I \to D^n \setminus \{0\}$ given by $H(x,t) = (1-t)x + tr(x)$.

Trivially, $H\big((D^n \setminus \{0\}) \times I\big) \subseteq D^n \setminus \{0\}$, so it is well defined, it is continuous, $H(x,t) = (1-t)x + t\frac{x}{\|x\|} = x$, $\forall x \in S^{n-1}$, and $H(x,1) \in S^{n-1}, \forall x \in D^n \setminus \{0\}$. □

Example 3.12 shows that, as we claimed, non-homeomorphic spaces can have the same homotopy type. Then, homotopy invariant functors are those functors that cannot distinguish homotopy equivalences from homeomorphisms. There's an equivalent way of understanding them as functors that only consider the homotopy types of the spaces for determining their images, which is formalised through the concept of the homotopy category.

First some notation. From here on, if $f\colon X \to Y$ is a continuous map, we will use $[f]$ to refer to the equivalence class of continuous maps which are homotopic to f.

If we are to consider homotopies relative to some subspace $A \subseteq X$, then we will denote the class of continuous maps homotopic to f relative to A by $[f]_A$, although we may just denote it $[f]$ if it is clear through context that the homotopy is relative to some the set A.

We will denote by $[X,Y]$ the set $\{[f] \mid f\colon X \to Y \text{ continuous map}\}$, and $[X,Y]_A$ will denote $\{[f]_A \mid f\colon X \to Y \text{ continuous map}\}$.

Example 3.14. *Let X, Y be topological spaces, with Y a convex subspace of $\mathbb{R}^n$. Then, we have $[X, Y] = *$.*

Proof. Let $f, g \colon X \to Y$ be continuous maps, with Y a convex subspace of $\mathbb{R}^n$. It suffices to show that $f \simeq g$, and so they belong to the same homotopy class. Consider the homotopy $H \colon X \times I \to Y$ given by $H(x,t) = tf(x) + (1-t)g(x)$. It is continuous, and so a homotopy, and $H(x,1) = f(x)$ and $H(x,0) = g(x)$, $\forall x \in X$, so we have $H \colon g \simeq f$. $\square$

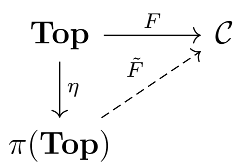
Definition 3.15 (Homotopy category)**.** Let $\pi(\mathbf{Top})$ be the category such that $ob\big(\pi(\mathbf{Top})\big) = ob(\mathbf{Top})$, and for two $X, Y \in ob\big(\pi(\mathbf{Top})\big)$, $hom_{\pi(\mathbf{Top})}(X,Y) = [X,Y]$, and for two composable morphisms in $\pi(\mathbf{Top})$, $[f], [g]$, the composition $[g] \circ [f]$ is defined to be $[g \circ f]$. We call $\pi(\mathbf{Top})$ the homotopy category.

Observe that the composition is well defined due to proposition 3.4. We observe that the identity of an object $X \in ob(\pi(\mathbf{Top}))$, id_X, is equal to $[id_X]$, the class of the identity in **Top**.

This implies that, for two morphisms in $\pi(\mathbf{Top})$, $[f], [g]$, we have that $[g] \circ [f] = id_X \iff [g \circ f] = [id_X] \iff g \circ f \simeq id_X$. This clearly implies that a morphism $[f]$ is an isomorphism in $\pi(\mathbf{Top})$ if and only if f is a homotopy equivalence in **Top**.

This means that two topological spaces, X, Y, are isomorphic in $\pi(\mathbf{Top})$ if and only if they have the same homotopy type.

Proposition 3.16 (Universal property of the homotopy category)**.** *The homotopy category has the universal property that, $\forall F \colon \mathbf{Top} \to \mathcal{C}$ homotopy invariant functor, $\exists! \tilde{F} \colon \pi(\mathbf{Top}) \to \mathcal{C}$ functor such that the diagram*



commutes, where $\eta \colon \mathbf{Top} \to \pi(\mathbf{Top})$ is the projection functor, such that $\eta(X) = X$ and $\eta(f) = [f]$.

Proof. Let $F \colon \mathbf{Top} \to \mathcal{C}$ be a homotopy invariant functor, and construct $\tilde{F} \colon \pi(\mathbf{Top}) \to \mathcal{C}$ by $\tilde{F}(X) = F(X)$, and $\tilde{F}([f]) = F(f)$.

Observe that, if f is homotopy equivalence, $[f]$ is isomorphism, and so $F(f) = \tilde{F}([f])$ is isomorphism. $\square$

Observation 3.17. Although we will not define in this essay a notion of equivalence among categories, if we were to do so, and we let $\mathcal{C}'$ be another category with this property, the functors induced by the result would be enough guarantee $\mathcal{C}'$ to be equivalent to $\pi(\mathbf{Top})$ in the usual way considered in Category Theory.

The objects of study in this essay, the homotopy groups, are constructions that correspond to homotopy invariant functors. These will be constructed in general in 5.1, and the first of them, the fundamental group, will be constructed in the next section.

Before proceeding with that, it is important to note that, even before constructing our first homotopy invariant functor, we have placed strict limits to their ability to classify topological spaces. We have seen that homotopy invariant functors are those for which the image through them of a continuous map $f \colon X \to Y$ which is a homotopy equivalence, is an isomorphism. We have seen that being a homotopy equivalence is strictly weaker than being an isomorphism, so this implies that homotopy invariant functors are not fine enough to discriminate all non-homeomorphic spaces. Even more, we have not introduced any hint to the fact that homotopy types induce a non-trivial partition, most of our examples of homotopy equivalences so far having been for contractible spaces, which could make us wonder if every space is homotopy equivalent to every other space.

We will prove to the contrary in the next section, when we introduce our first homotopy invariant functor, the fundamental group, and show that the space S^1 has non-trivial fundamental group.

4 The fundamental group

4.1 The fundamental groupoid

We will now introduce some definitions that will help us define an interesting functor, the fundamental groupoid, which we will use to illustrate the construction of the fundamental group.

Definition 4.1 (Groupoid)**.** We say the pair $(G, *)$ is a groupoid if G is a set and $*$ is a binary operation on G (not necessarily defined for every pair of elements of G), such that:

- If for elements $a, b, c \in G$, $a * b$ and $b * c$ are defined, then $(a * b) * c$ and $a * (b * c)$ are defined and equal.
- $\forall a \in G$, $\exists a^{-1} \in G$ such that $a * a^{-1}$ and $a^{-1} * a$ are always defined, and such that for all $b \in G$ with $a * b$ defined, $a * b * b^{-1}$ and $a^{-1} * a * b$ are defined, and $a^{-1} * a * b = b$, $a * b * b^{-1} = a$.

We will usually denote a groupoid $(G, *)$ by G for the sake of brevity if it does not induce confusion.

For two groupoids G, G', a map $f\colon G \to G'$ is a groupoid homomorphism if for all $a, b \in G$ with $a * b$ defined, then $f(a) * f(b)$ is defined and equal to $f(a * b)$, and if $f(a^{-1}) = f(a)^{-1}$, $\forall a \in G$.

Equivalently, a groupoid can be defined categorically as a category G such that $ob(G)$ is a set, and $\forall A, B \in ob(G)$, $hom_G(A, B)$ is a set of isomorphisms. The morphisms of G can be thought of as the elements of the groupoid, and the operation corresponds to morphism composition. Here, groupoid homomorphisms correspond to functors between groupoids as categories.

Definition 4.2. **Grpd** is the category such that its objects are the groupoids, and its morphisms are the groupoid homomorphisms.

Definition 4.3 (Path. Product of paths)**.** Let X be a topological space. We say α is a path on X if it is a continuous map from I into X.
For α, β paths such that $\alpha(1) = \beta(0)$, we define the path product $\alpha * \beta\colon I \to X$ as the path given by

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

This map is a path, because $[0, \frac{1}{2}], [\frac{1}{2}, 1]$ are closed sets that cover I with $\alpha(1) = \beta(0)$, and so $\alpha * \beta$ is continuous by lemma 3.2.

Lemma 4.4. *If $p\colon I \to I$ is a homeomorphism, and $\alpha\colon I \to X$ is a path on X, then $\alpha \simeq \alpha \circ p$. In particular, reparametrizations of paths on X are homotopy equivalences. Moreover, if $p(0) = 0$ and $p(1) = 1$, this homotopy is relative to $\partial I = \{0, 1\}$.*

Proof. It is enough to define the homotopy explicitly. If we consider the continuous map $H\colon I \times I \to X$ given by

$$H(x,t) = \alpha\big(tx + (1-t)p(x)\big)$$

we have $H(x, 0) = \alpha\big(p(x)\big) = (\alpha \circ p)(x)$ and $H(x, 1) = \alpha(x)$ for all x, so H is a homotopy from $\alpha \circ p$ to α.

Moreover, if $p(0) = 0$ and $p(1) = 1$, we have $H(0,t) = \alpha(0)$ and $H(1,t) = \alpha(t + 1 - t) = \alpha(1)$, $\forall t \in I$, so this homotopy is relative to ∂I. □

Lemma 4.5. *If $x \in X$ is a point, we denote by $e_x\colon I \to X$ the constant map given by $e_x(t) = x, \forall t \in I$.*

Let $\alpha\colon I \to X$ be a continuous map. Then, we define the inverse path $\overline{\alpha}\colon I \to X$ as the map given by $\overline{\alpha}(t) = \alpha(1-t)$. If $\alpha(0) = x_0$ and $\alpha(1) = x_1$, we have that $\alpha * \overline{\alpha} \simeq_{\partial I} e_{x_0}$, and $\overline{\alpha} * \alpha \simeq_{\partial I} e_{x_1}$.

Proof. Observe that $\alpha * \overline{\alpha}$ is well defined, as $\overline{\alpha}(0) = \alpha(1)$. We prove this case, as the case for $\overline{\alpha} * \alpha$ is equivalent because $\overline{\overline{\alpha}} = \alpha$.

We prove the assertion by giving an explicit homotopy. Let $\phi\colon I \times I \to I$ be given by

$$\phi(x,t) = \begin{cases} 2tx & \text{if } x \in [0, \frac{1}{2}] \\ 2t(1-x) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

ϕ is a continuous map, as $I \times \left[0, \frac{1}{2}\right], I \times \left[\frac{1}{2}, 1\right]$ are closed sets that cover $I \times I$ and both maps are continuous and agree on the intersection, allowing us to apply lemma 3.2, and so $H = \alpha \circ \phi\colon I \times I \to X$ is a homotopy, because it is continuous by composition of continuous maps. Moreover, $\phi(0,t) = 0$ and $\phi(1,t) = 0$, $\forall t \in I$, and so it is a homotopy relative to $\{0, 1\}$. We observe that $H(x, 0) = \alpha(0) = x_0$, $\forall x \in I$, and

$$H(x,1) = \left\{ \begin{array}{rl} \alpha(2x) & \text{if } x \in \left[0, \dfrac{1}{2}\right] \\ \alpha\left(2-2x\right) & \text{if } x \in \left[\dfrac{1}{2}, 1\right] \end{array} \right\} = (\alpha * \overline{\alpha})(x)$$

as $\alpha(2-2x) = \alpha\big(1 - (2x-1)\big) = \overline{\alpha}(2x-1)$. This implies that $H\colon e_{x_0} \simeq_{\partial I} \alpha * \overline{\alpha}$. □

Proposition 4.6. *Let X be a topological space. Let $\Pi X = [I, X]_{\partial I}$. We write its elements as $[\alpha]$ instead of $[\alpha]_{\partial I}$ for the sake of brevity.*

We define the operation on ΠX such that $[\alpha] * [\beta] = [\alpha * \beta]$ when the product $\alpha * \beta$ is defined. Then, $(\Pi X, *)$ is a groupoid, with the inverse given by $[\alpha]^{-1} = [\overline{\alpha}]$.

Proof. We have to prove the items of definition 4.1. We prove them one by one.

- Let $[\alpha], [\beta], [\gamma] \in \Pi X$ be such that $[\alpha] * [\beta]$ and $[\beta] * [\gamma]$ are defined. By definition of $*$, this means the products $\alpha * \beta$ and $\beta * \gamma$ are defined. We have that the end point of $\alpha * \beta$ is the same as that of β, and the starting point of $\beta * \gamma$ is the same as that of β, meaning that $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$ are defined, and so are $[\alpha] * ([\beta] * [\gamma])$ and $([\alpha] * [\beta]) * [\gamma]$.

  The fact they are equal classes comes from the fact that $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ are reparametrizations of each other, and applying lemma 4.4.

- To prove the assertion regarding the inverse, we observe that, by lemma 4.5, $\alpha * \overline{\alpha}$ and $\overline{\alpha} * \alpha$ are always defined, and so are $[\alpha] * [\alpha]^{-1}$ and $[\alpha]^{-1} * [\alpha]$.

  To prove $[\alpha] * [\alpha]^{-1} * [\beta] = [\beta]$ for β a path in X with $\beta(0) = \alpha(0) = x_0$, we observe that, again by lemma 4.5, $\alpha * \overline{\alpha} * \beta \simeq_{\partial I} e_{x_0} * \beta$ with e_{x_0} constant, which is a reparametrization of β, and so the thesis follows by applying lemma 4.4. □

Definition 4.7 (Fundamental groupoid)**.** Let X be a topological space. We call the pair $(\Pi X, *)$ the fundamental groupoid of X.

Lemma 4.8. *Let $\alpha, \beta \colon I \to X$ be paths on X such that $\alpha(1) = \beta(0)$, and $f \colon X \to Y$ be a continuous map. Then, $(f \circ \alpha) * (f \circ \beta) = f \circ (\alpha * \beta)$.*

Proposition 4.9. *A continuous map $f \colon X \to Y$, induces a groupoid homomorphism on the fundamental groupoids, $f_* \colon \Pi X \to \Pi Y$, given by*

$$\begin{aligned} f_* \colon \Pi X &\longrightarrow \Pi Y \\ [\alpha] &\longmapsto [f \circ \alpha] \end{aligned}$$

Proof. We have to prove that f_* is well defined, and that it is a groupoid homomorphism.

To prove f_* is well defined, we have to show that if $[\alpha] = [\alpha']$, then $[f \circ \alpha] = [f \circ \alpha']$. By definition, $[\alpha] = [\alpha']$ implies that there exists a homotopy $H \colon \alpha \simeq_{\{0,1\}} \alpha'$. As $f \simeq_X f$, by proposition 3.4, we have that $f \circ \alpha \simeq_{\{0,1\}} f \circ \alpha'$, and so $[f \circ \alpha] = [f \circ \alpha']$.

We have to prove it is a groupoid homomorphism. Taking $[\alpha], [\beta] \in \Pi X$ such that $\alpha(1) = \beta(0)$, we have that $[f \circ (\alpha * \beta)] = [f \circ \alpha] * [f \circ \beta]$ by lemma 4.8, and so $f_*([\alpha] * [\beta]) = f_*([\alpha]) * f_*([\beta])$. Also, $(f \circ \overline{\alpha})(t) = (f \circ \alpha)(1 - t) = \overline{(f \circ \alpha)}(t)$, $\forall t \in I$, and so

$$f_*([\alpha])^{-1} = [\overline{f \circ \alpha}] = [f \circ \overline{\alpha}] = f_*([\overline{\alpha}]) = f_*([\alpha]^{-1}), \ \forall [\alpha] \in \Pi X$$

This implies f_* is a groupoid homomorphism. □

Corollary 4.10 (Functoriality of the fundamental groupoid)**.** $\Pi \colon \mathbf{Top} \to \mathbf{Grpd}$ *given by* $\Pi(X) = \Pi X$ *and* $\Pi(f) = f_*$ *is a functor.*

Proof. We have to prove the conditions in 2.8 and 3.6. First, if $f, g \colon X \to Y$ are continuous maps, applying the induced homomorphisms to each $[\alpha] \in \Pi X$, we have

$$\Pi(g \circ f)([\alpha]) = (g \circ f)_*([\alpha]) = [(g \circ f) \circ \alpha] = g_*([f \circ \alpha]) = (g_* \circ f_*)([\alpha]) = (\Pi(g) \circ \Pi(f))([\alpha])$$

and so $\Pi(g \circ f) = \Pi(g) \circ \Pi(f)$.

The fact that $\Pi(1_X) = 1_{\Pi(X)}$ comes from the fact $id_X \circ \alpha = \alpha$ for all paths α on X, and so

$$\Pi(id_X)([\alpha]) = [id_X \circ \alpha] = [\alpha], \ \forall \alpha \in \Pi X$$ □

Example 4.11. $\Pi * \cong 1$ *(the trivial groupoid with only one element).*

Proof. If we consider a map $\alpha \colon I \to *$, we see that there is only one possible such map. This means that $\Pi *$ trivially has only one element, the class of the constant map. □

We can understand the fundamental groupoid as studying the structure of the paths on a given space. Although we have only shown the fundamental groupoid of a one-point space, and it resulted in the trivial groupoid, fundamental groupoids of spaces are non-trivial in general, and computing them is a very hard endeavour. Observe that, for any given point x in a topological space X, there exists an element $[e_x]_{\partial I} \in \Pi X$, implying that the fundamental groupoid has at least as many elements as points are in the space X. We can use this to see that $\Pi\colon \mathbf{Top} \to \mathbf{Grpd}$ is not a homotopy invariant functor, as $\Pi\mathbb{R}$ has an uncountable number of objects, while $\Pi *$ is trivial, and if Π was a homotopy invariant functor, we would expect the two groupoids to be isomorphic, as $\mathbb{R}$ is contractible by example 3.12.

Instead of giving any more computations of fundamental groupoids, we will use them as a basis for introducing another functor which will be of more use for our purposes: the fundamental group.

The idea of the fundamental group will be to analyse the structure of a subset of all the paths on a given topological space, making it less cumbersome than the fundamental groupoid, and giving it stronger algebraic structure in the process. Another advantage of the fundamental group is that it will yield our first homotopy invariant functor.

4.2 The fundamental group

We will add some more definitions before we introduce the fundamental group.

Definition 4.12 (Based topological space. Pointed map)**.** Let X be a topological space, and $x_0 \in X$ a point. We call (X, x_0) a based topological space, and we say x_0 is the base-point of the space. Pointed maps are continuous maps $f\colon X \to Y$ such that $f(x_0) = y_0$. We denote this by $f\colon (X, x_0) \to (Y, y_0)$. We may, however, still call a map f of that form a continuous map if the additional condition of it being a pointed map is clear enough.

More generally, if $A \subseteq X$ and $B \subseteq Y$, if $f\colon X \to Y$ is a continuous map such that $f(A) \subseteq B$, we will denote it $f\colon (X, A) \to (Y, B)$.

This notation will be further extended, as if we have $C \subseteq A \subseteq X$ and $D \subseteq B \subseteq Y$ topological spaces, then we can denote a continuous map $f\colon (X, A) \to (Y, B)$ such that $f(C) \subseteq D$ by $f\colon (X, A, C) \to (Y, B, D)$.

Definition 4.13. $\mathbf{Top}^*$ is the category such that its objects are the based topological spaces, and its morphisms are the pointed maps with the usual composition.

As the morphisms in the category are base-point preserving, it is natural to require that for two pointed maps to be homotopy equivalent, they be at least homotopic relative to the base-point. This will be a standard assumption when we work with homotopy groups.

Definition 3.6 extends to this category in the following way: a functor $F\colon \mathbf{Top}^* \to \mathcal{C}$ to some category $\mathcal{C}$ will be homotopy invariant if for $f, g\colon (X, x_0) \to (Y, y_0)$ such that $f \simeq_{\{x_0\}} g$, we have $F(f) = F(g)$.

The same way, we will extend definition 3.7 by considering two based spaces to be homotopy equivalent if $\exists f\colon (X, x_0) \to (Y, y_0)$ such that f is a homotopy equivalence with an associated map $g\colon (Y, y_0) \to (X, x_0)$ as before, which also follows that the homotopies in the definition are relative to $\{x_0\}$ and $\{y_0\}$.

We can also extend definition 3.15 in a natural way to consider the homotopy category of based topological spaces, $\pi(\mathbf{Top}^*)$. Homotopy invariant functors on this category are also extended in such way.

Definition 4.14 (Fundamental group)**.** Let (X, x_0) be a based topological space. We define $\pi_1(X, x_0)$ the set of classes of equivalence of paths $\alpha\colon (I, \{0, 1\}) \to (X, x_0)$ up to the relation of homotopy equivalence relative to ∂I. We will denote the elements of $\pi_1(X, x_0)$ by $[\alpha]$ instead of $[\alpha]_{\partial I}$ for the sake of brevity.

We have $\pi_1(X, x_0)$ is a group with operation given by $[\alpha] * [\beta] = [\alpha * \beta]$, $[\alpha]^{-1} = [\overline{\alpha}]$, and trivial element $[e_{x_0}]$, where e_{x_0} is the constant map. We call $\pi_1(X, x_0)$ the fundamental group of (X, x_0).

The observation that it is a group comes from the same arguments to show ΠX was a groupoid, together with the fact that $\alpha * \beta$ is always defined here, because $\alpha(1) = \beta(0) = x_0$. Paths of the form $\alpha\colon (I, \{0, 1\}) \to (X, x_0)$ will be called loops on X based at x_0.

Definition 4.15 (Simple connected space)**.** Let (X, x_0) be a based topological space. We say X is simple connected if X is path-connected and $\pi_1(X, x_0) = 1$, the trivial group with one element.

The reason why we say X is simple connected, and not (X, x_0), is that, as we will prove in proposition 5.13, this is a topological property of X and not dependent on the choice of base-point.

We will not prove here many properties about the fundamental group, because it will be immediately generalised once we introduce higher homotopy groups, of which the fundamental group is just a particular example.

It is important to note now, however, that we will be able to prove that π_1 induces a homotopy invariant functor $\pi_1\colon \mathbf{Top}^* \to \mathbf{Grp}$, and so when we now proceed to compute the fundamental group of S^1, we will be giving our first non-trivial computation of a homotopy invariant functor on $\mathbf{Top}^*$.

4.3 The fundamental group of the circle

We will now introduce our first real computation. Through the process, we will employ the continuous map $p\colon \mathbb{R} \to S^1$ given by $p(t) = (\cos(2\pi t), \sin(2\pi t))$ extensively. The result will employ the following lemma involving it.

Lemma 4.16. *Given some topological space Y, some continuous map $F\colon Y \times I \to S^1$, and some continuous map $\tilde{F}\colon Y \times \{0\} \to \mathbb{R}$ such that $F(y, 0) = (p \circ \tilde{F})(y, 0)$, $\forall y \in Y$, there exists a unique continuous extension $\tilde{F}\colon Y \times I \to \mathbb{R}$ such that $F = p \circ \tilde{F}$.*

This lemma will have to be left without a proof for the sake of simplicity, as it would be highly involved. Nevertheless, it is a particular case of a more general result presented in proposition 6.34, which will be given a proof. Let us now proceed with computing the fundamental group of S^1.

Theorem 4.17 (The fundamental group of the circle)**.** *Consider the space $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. Then,*

$$\pi_1\big(S^1, (1, 0)\big) \cong \mathbb{Z}$$

The isomorphism is given by

$$\begin{aligned} \phi\colon \mathbb{Z} &\longrightarrow \pi_1\big(S^1, (1, 0)\big) \\ n &\longmapsto [\omega_n] \end{aligned}$$

where $\omega_n\colon I \to S^1$ is the loop given by $\omega_n(s) = \big(\cos(2\pi ns), \sin(2\pi ns)\big)$.

Proof. Let $\tilde{\omega}_n \colon I \to \mathbb{R}$ be the map given by $\tilde{\omega}_n(s) = ns$, and observe $\omega_n = p \circ \tilde{\omega}_n$. We have that, if $\tilde{f} \colon I \to \mathbb{R}$ is a path such that $\tilde{f}(0) = 0$ and $\tilde{f}(1) = n$, then $H \colon \tilde{f} \simeq_{\{0,1\}} \tilde{\omega}_n$, with H given by $H(x,t) = t\tilde{f}(x) + (1-t)\tilde{\omega}_n(x)$. Then, $[p \circ \tilde{f}] = [p \circ \tilde{\omega}_n] = [\omega_n]$, and so $\phi(n) = [p \circ \tilde{f}]$ for any such $\tilde{f}$.

The next step will be verifying ϕ is homomorphism. Let $n, m \in \mathbb{Z}$, and $\tau_m \colon \mathbb{R} \to \mathbb{R}$ be given by $\tau_m(x) = m + x$. Then, $\tau_m \circ \tilde{f}$ is a path from m to $m+n$ if $\tilde{f}$ was a path from 0 to n. We observe that $p \circ (\tau_m \circ \tilde{f}) = p \circ \tilde{f}$ for all $m \in \mathbb{Z}$.

This means that, for $\tilde{f}$ path from 0 to n, and $\tilde{g}$ path from 0 to m, as $\tilde{g} * (\tau_m \circ \tilde{f})$ is a path from 0 to $n+m$, we have $\phi(n+m) = [p \circ (\tilde{g} * (\tau_m \circ \tilde{f}))]$. By lemma 4.8, this means $\phi(n+m) = [(p \circ \tilde{g}) * (p \circ \tau_m \circ \tilde{f})]$, and so

$$\phi(n+m) = [(p \circ \tilde{g}) * (p \circ \tau_m \circ \tilde{f})] = [p \circ \tilde{g}] * [p \circ \tau_m \circ \tilde{f}] = [p \circ \tilde{g}] * [p \circ \tilde{f}] = \phi(n) * \phi(m)$$

from which we extract that ϕ is homomorphism.

The fact that it is an isomorphism will follow from two more observations:

- For each $f \colon I \to S^1$ with $f(0) = x_0$, and each choice of $\tilde{x}_0 \in \mathbb{R}$ such that $p(\tilde{x}_0) = x_0$, $\exists! \tilde{f} \colon I \to \mathbb{R}$ with $\tilde{f}(0) = \tilde{x}_0$ and $f = p \circ \tilde{f}$.
- For each homotopy $f_t \colon I \to S^1$ relative to $\{0,1\}$, and each choice of $\tilde{x}_0$ such that $p(\tilde{x}_0) = x_0$, $\exists! \tilde{f}_t \colon I \to \mathbb{R}$ homotopy relative to $\{0,1\}$ such that $f_t = p \circ \tilde{f}_t$, $\forall t \in I$.

These are deduced from lemma 4.16. For the first one, we just consider $Y \times I = \{y_0\} \times I \cong I$ in the statement of the lemma. For the second one, we first apply the first observation to f_0 to find a map $\tilde{f}_0 \colon I \to \mathbb{R}$, and then apply the lemma again for $Y = I$.

Now, we can use these observations to deduce the isomorphism. Let $[f] \in \pi_1\big(S^1, (1,0)\big)$. We have $f \colon I \to S^1$ is a continuous map with $f(1) = f(0) = (1,0)$. By the first observation, $\exists \tilde{f} \colon I \to \mathbb{R}$ such that $f = p \circ \tilde{f}$ and $\tilde{f}(0) = 0$. As $f(1) = (1,0)$, we have that $\tilde{f}(1) = n \in \mathbb{Z} = p^{-1}\big((1,0)\big)$, and so $[f] = [p \circ \tilde{f}] = \phi(n)$ for some $n \in \mathbb{Z}$, and ϕ is surjective.

To prove injectivity, assume $\phi(n) = \phi(m)$. This means there is a homotopy $f_t \colon I \times I \to S^1$ relative to $\{0,1\}$ from ω_n to ω_m. But, by the second observation, there exists a homotopy $\tilde{f}_t \colon I \times I \to \mathbb{R}$ relative to $\{0,1\}$ with $f_t = p \circ \tilde{f}_t$ $\forall t \in I$. As $\tilde{f}_t$ is relative to the endpoints, $\tilde{f}_t(1)$ is independent of t. As we have $n = \tilde{f}_0(1)$ and $m = \tilde{f}_1(1)$, we conclude $n = m$, proving the injectivity of ϕ. □

We must observe, before we carry on, the difficulty involved in proving our first non-trivial example of a fundamental group, even when we have used a lemma left for the moment without a proof, which is technical in itself. This is not out of a deliberate choice of a "bad" case, but out of the fact that calculations of homotopy groups, in general, can be very hard. We will, after we introduce higher homotopy groups, come back to the question of computing these groups.

We can also, before we move on, indulge in some applications of the fundamental group of the circle we have just computed.

Corollary 4.18 (Brouwer Fixed-Point Theorem)**.** *Let $D^2 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$. For any continuous map $f \colon D^2 \to D^2$, $\exists x_0 \in D^2$ such that $f(x_0) = x_0$.*

Proof. We proceed by contradiction, assuming $\exists f \colon D^2 \to D^2$ with $f(x) \neq x, \forall x \in D^2$.

We first define a map $g \colon D^2 \to \partial D^2 = S^1$ by the following process. Consider, for a given $x \in D^2$, the ray that starts at $f(x)$ and crosses x. This will be well defined, as $f(x) \neq x$ for all $x \in D^2$. We define a map $g\colon D^2 \to S^1$, by defining $g(x)$ to be the point at which this ray intersects S^1, $\forall x \in D^2$. We now seek to prove g is continuous.

The points of the ray are given by $f(x) + r\big(x - f(x)\big)$ for some $r \geq 0$. Then, $g(x) = f(x) + r(x)\big(x - f(x)\big)$ for some map $r\colon D^2 \to [0, +\infty)$ such that the resulting $g(x)$ is in S^1, or, equivalently, $\|g(x)\| = 1$. If this r is continuous, g will be continuous, because it is a linear combination of continuous maps.

We have

$$\Big\|f(x) + r(x)\big(x - f(x)\big)\Big\| = 1 \iff$$

$$\Big(f(x) + r(x)\big(x - f(x)\big)\Big) \cdot \Big(f(x) + r(x)\big(x - f(x)\big)\Big) = 1 \iff$$

$$\|f(x)\|^2 + 2r(x)\Big(f(x) \cdot \big(x - f(x)\big)\Big) + \big(r(x)\big)^2\|x - f(x)\|^2 = 1 \iff$$

$$\|x - f(x)\|^2\big(r(x)\big)^2 + 2\Big(f(x) \cdot \big(x - f(x)\big)\Big)r(x) + (\|f(x)\|^2 - 1) = 0$$

Solving for r, we geometrically know there will be two distinct solutions for all x, one positive, corresponding to the point where the line containing $f(x)$ and x intersects S^1 nearer to x, x if $x \in S^1$, and one non-positive, corresponding to the intersection nearer to $f(x)$, or $f(x)$ if $f(x) \in S^1$. Because there will always exist a positive root, and have defined $r(x)$ to be non-negative, we take the positive root at each point x:

$$r(x) = \frac{-f(x) \cdot \big(x - f(x)\big) + \sqrt{\Big(f(x) \cdot \big(x - f(x)\big)\Big)^2 - \|x - f(x)\|^2(\|f(x)\|^2 - 1)}}{\|x - f(x)\|^2}$$

Last expression is well defined and continuous $\forall x \in D^2$, because $\|f(x) - x\| \neq 0$ for all $x \in D^2$ due to the assumption that $f(x) \neq x$.

Now, as g is continuous, we can consider $G\colon D^2 \times I \to D^2$ given by

$$G(x, t) = tg(x) + (1 - t)x$$

G is continuous, $G(x, 0) = x$ and $G(x, 1) = g(x)$. This means $(S^1, (1, 0))$ is a deformation retract of $(D^2, (1, 0))$.

Now, because deformation retracts are homotopy equivalences, and the fact we will prove in next section that $\pi_1\colon \mathbf{Top}^* \to \mathbf{Grp}$ is a homotopy invariant functor, this would imply that

$$\pi_1\big(D^2, (1, 0)\big) \cong \pi_1\big(S^1, (1, 0)\big)$$

but we have computed that $\pi_1\big(S^1, (1, 0)\big) \cong \mathbb{Z}$, and $\pi_1\big(D^2, (1, 0)\big) = 1$ because it is a convex subset of $\mathbb{R}^n$, and so it is contractible, contradiction. □

In the following result, we will use that we can think of $(S^1, (1, 0)) \subseteq (\mathbb{R}^2, (1, 0))$ as a subset of $(\mathbb{C}, 1)$ in the usual way. Then, $S^1 \subseteq \mathbb{C}$ is the set of points $z \in \mathbb{C}$ for which $\|z\| = 1$. In this case, $\omega_n\colon (I, \partial I) \to (S^1, 1)$ is given by $\omega_n(x) = e^{2\pi n i x}$.

Corollary 4.19 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial with coefficients in* $\mathbb{C}$ *has a root in* $\mathbb{C}$.

Proof. We can assume a polynomial $q\colon \mathbb{C} \to \mathbb{C}$ to be of the form

$$q(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0,\ a_i \in \mathbb{C}$$

Assume q has no roots in $\mathbb{C}$, and consider $f_t\colon I \to \mathbb{C}$ given by

$$f_t(x) = \frac{q(te^{2\pi ix})/q(t)}{\|q(te^{2\pi ix})/q(t)\|}$$

We have $\|f_t(x)\| = 1$ for all $t \in \mathbb{R}^+$, $x \in I$. This means that f_t defines a loop in $\big(S^1, 1\big)$ for all t, as $f_t(1) = f_t(0) = 1$. We have f_0 is the trivial constant loop, and so f_{rt} is a homotopy relative to $\{0, 1\}$ from the trivial loop to f_r for any $r \in \mathbb{R}^+$, which implies $[f_r]$ is the trivial element of $\pi_1\big(S^1, (1, 0)\big)$ for all values of r.

Now, we fix a value of r such that $r \geq \sum_{i=0}^{n-1} \|a_i\|$ and $r > 1$. If $\|z\| = r$, we have

$$\|z^n\| = \|z\|^n = \|z\|\|z\|^{n-1} = r\|z\|^{n-1} \geq (\|a_0\| + \|a_1\| + \cdots + \|a_{n-1}\|)\|z^{n-1}\| \geq$$
$$\geq \|a_0\| + \|a_1\|\|z\| + \cdots + \|a_{n-1}\|\|z^{n-1}\| \geq \|a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}\|$$

This means $\|z^n\| \geq \|a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}\|$, and so, $\forall t \in I$,

$$\|z^n + t\big(a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}\big)\| \geq \|z^n\| - t\|a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}\| \geq 0$$

Defining a homotopy between z^n and $q(z)$,

$$q_t(z) = z^n + t\big(a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}\big)$$

this means q_t has no roots z with $\|z\| = r$, $\forall t \in I$.

If we replace q with q_t in the first expression of the proof, we can define $g_t\colon \mathbb{R} \to \mathbb{C}$, given by

$$g_t(x) = \frac{q_t(re^{2\pi ix})/q_t(r)}{\|q_t(re^{2\pi ix})/q_t(r)\|}$$

It is clear g_t defines a homotopy relative to $\{0, 1\}$, with $g_1 = f_r$ and

$$g_0(x) = \frac{r^n e^{2\pi nix} r^{-n}}{\|r^n e^{2\pi nix} r^{-n}\|} = e^{2\pi nix} = \omega_n(x)$$

Now, we know $[f_r]$ is trivial in $\pi_1\big(S^1, 1\big)$, which means that through this homotopy we have $[\omega_n]$ is also trivial. But $[\omega_n] = [\omega_1]^n$, with $[\omega_1]$ a generator of $\pi_1\left(S^1, 1\right)$, which is isomorphic to $\mathbb{Z}$, contradiction. □

5 The homotopy groups

From this chapter on, we will frequently use products of the unit interval, $I^n = I \times \overset{n}{\cdots} \times I$, and some spaces derived from them. We consider their border, ∂I^n, which, if we think of I^n as geometric $n-$cubes, are composed of the union of their faces.

Notations which will be of particular use will be I^{n-1}, which in the context of a subspace of I^n is to be regarded as $I^{n-1} \times \{0\}$, and can be geometrically thought as representing the "bottom face" of the cube; and $J^{n-1} = \overline{\partial I^n \setminus I^{n-1}}$, which can be thought of as the union of the remaining faces.

Other remark on notation is that, when proving the properties of higher homotopy groups, it will be seen that they are abelian for most values of n. This implies that we will from now on use abelian notation when dealing with these cases, which implies that the trivial group, until now denoted 1, may appear denoted by 0.

5.1 The absolute case

The problem that led to the homotopy groups was the problem of classifying the classes of continuous maps from spheres S^n to other topological spaces. For example, of great importance for the development of the theory was Hopf's paper of 1931([9]), in which he showed the set of classes of continuous maps $f\colon S^3 \to S^2$ up to homotopy was infinite. Consider this equivalent way of defining the fundamental group.

We have defined $\pi_1(X, x_0)$ as the set of classes up to homotopy of loops, that is, classes of pointed maps $f\colon (I, \partial I) \to (X, x_0)$ up to homotopy equivalence relative to ∂I. Now, we can obtain S^1 as a quotient space on I by identifying the points 0 and 1. By the universal property of quotient spaces, then, there is a correspondence between continuous maps $f\colon I \to X$ such that $f(0) = f(1)$, and continuous maps $\tilde{f}\colon S^1 \to X$. Then, $\pi_1(X, x_0)$ becomes the set of classes of pointed maps $\tilde{f}\colon (S^1, (1,0)) \to (X, x_0)$ up to homotopy equivalence relative to $(1, 0)$.

We can then think of generalising this to sets $\pi_n(X, x_0)$ given by pointed maps from $(S^n, *)$ to (X, x_0) modulo homotopy relative to the base-point $* = (1, 0, ..., 0)$.

As S^n can be constructed as the quotient given by taking I^n and collapsing ∂I^n to a single point, we will take the equivalent approach of considering classes of continuous maps $f\colon (I^n, \partial I^n) \to (X, x_0)$ up to homotopy equivalence relative to ∂I^n, although viewing the elements of the groups as classes of relative homotopy of pointed maps $f\colon (S^n, *) \to (X, x_0)$ will be useful at some points.

Definition 5.1 ($n-$th homotopy group)**.** Let (X, x_0) be a based topological space, and $n \in \mathbb{N}$. We define $\pi_n(X, x_0)$ as the set of classes of equivalence of continuous maps $\alpha\colon (I^n, \partial I^n) \to (X, x_0)$ up to a relationship of homotopy relative to ∂I^n. When dealing with the class of some map α, we will denote it $[\alpha]$ instead of $[\alpha]_{\partial I^n}$ to simplify the notation.

We will denote $\mathbf{x} := (x_1, x_2, \cdots, x_n)$ for simplicity.

We have that $\pi_n(X, x_0)$ is a group, with operation given by $[\alpha] * [\beta] = [\alpha * \beta]$, $[\alpha]^{-1} = [\overline{\alpha}]$ and the trivial element given by $[e_{x_0}]$, with e_{x_0} the constant map. Here, $\alpha * \beta$ is given by

$$(\alpha * \beta)(\mathbf{x}) = \begin{cases} \alpha\big((2x_1, x_2, \cdots, x_n)\big) & \text{if } x_1 \in \left[0, \frac{1}{2}\right] \\ \beta\big((2x_1 - 1, x_2, \cdots, x_n)\big) & \text{if } x_1 \in \left[\frac{1}{2}, 1\right] \end{cases}$$

and $\overline{\alpha}$ is given for α by $\overline{\alpha}(\mathbf{x}) = \alpha\big((1 - x_1, x_2, \cdots, x_n)\big)$.

We call $(\pi_n(X, x_0), *)$ the $n-$th homotopy group of (X, x_0), and we denote it $\pi_n(X, x_0)$ for simplicity.

We can also extend this to define $\pi_0(X)$ as the set of path-components of X. This case has the inconvenience that it is not given a group structure. However, if we define $\pi_0(X)$ to be trivial when it has only one element (i.e., X is path-connected), then it fits nicely in some results, like 6.10. In this sense, $\pi_0(X, x_0)$ is defined as a based set, that is, a set with one distinguished element, composed of the path-components of X, with the path-component of x_0 as the base element.

5.2 The relative case

We can, in fact, take the definition of $n-$th homotopy groups and generalise it further.

Definition 5.2 (Relative $n-$th homotopy groups)**.** Let (X, x_0) be a based topological space, $x_0 \in A \subseteq X$ subspace of X, and $n \in \mathbb{N}$.

We remember some of the notation introduced at the start of the section.

$$I^{n-1} = \{\mathbf{x} \in I^n \mid x_n = 0\}$$
$$\partial I^n = \{\mathbf{x} \in I^n \mid \exists i \in \{1, 2, \cdots, n\} \text{ such that } x_i \in \{0, 1\}\}$$
$$J^{n-1} = \overline{\partial I^n \setminus I^{n-1}}$$

Using the notation of continuous maps between triples of topological spaces, we can also extend the notion of relative homotopic continuous maps to this setting. Two maps $f, g\colon (X, A, C) \to (Y, B, D)$ are said to be homotopic relative to (A, C) if $f|_C = g|_C$, and there is a homotopy $H\colon f \simeq_C g$ such that $H(x, t) \in B, \forall x \in A, \forall t \in I$. This homotopy relative to (A, C) can be also denoted $H\colon f \simeq_{(A,C)} g$, and it can be proven that $\simeq_{(A,C)}$ is an equivalence relation through an argument similar to that of $\simeq_A$.

We will define $\pi_n(X, A, x_0)$ as the set of equivalence classes of continuous maps of the form $\alpha\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ up to homotopy equivalence relative to $(\partial I^n, J^{n-1})$. When dealing with the class of some map α, we will denote it $[\alpha]$ instead of $[\alpha]_{(\partial I^n, J^{n-1})}$ to simplify the notation.

For all $n \geq 2$, $\pi_n(X, A, x_0)$ is a group, with the operation, neutral element and inverse given by the same expressions as in the absolute case. We call $\pi_n(X, A, x_0)$ the $n-$th homotopy group relative to A, and we say it is a relative homotopy group.

Relative homotopy groups can be thought of as relaxing the condition for its elements that the faces of I^n be mapped to x_0. Instead, we require all of them but one to be mapped to x_0, and only require the face I^{n-1} to be mapped to some subspace A, with $x_0 \in A$.

These groups are of great interest for several reasons, among them because the absolute case is a particular case of the relative one, as $\pi_n(X, \{x_0\}, x_0) = \pi_n(X, x_0)$, and because several basic propositions we are now to prove generalise nicely to them. We will prove the relative versions when we can to show these results with as much generality as possible.

Observe that the condition $n \geq 2$ is necessary in general, as $\pi_1(X, A, x_0)$, while definable as a set, may not have group structure in general with the operation we have defined. However, it trivially does have group structure in the case $A = \{x_0\}$, as it corresponds to the fundamental group.

The case $\pi_0(X, A, x_0)$ is not obvious, and will be given a definition in proposition 5.3 that is different to that of other relative homotopy sets, but fits nicely with future results.

5.3 Basic properties

Proposition 5.3. *Let X, Y be topological spaces, $x_0 \in A \subseteq X, y_0 \in B \subseteq Y$ subspaces and points in them, and $f\colon (X, A, x_0) \to (Y, B, y_0)$ be a continuous map. Let $n \in \mathbb{N} \cup \{0\}$.*

We define the map $f_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ depending on the value of n. If $n \in \mathbb{N}$, then it is given $f_*([\alpha]) = [f \circ \alpha]$.

Before we describe the case $n = 0$, we observe that $\pi_n(X, \{x_0\}, x_0) = \pi_n(X, x_0), \forall n \in \mathbb{N}$, and we define $\pi_0(X, \{x_0\}, x_0)$ to be equal to $\pi_0(X, x_0)$. We define $f_*\colon \pi_0(X, x_0) \to \pi_0(Y, y_0)$ such that, if $[\alpha] \in \pi_0(X, x_0)$ represents the path-component of some point $x \in X$, then $f([\alpha])$ represents the path-component of $f(x)$. If $i\colon (A, x_0) \hookrightarrow (X, x_0)$ is the canonical inclusion, we define the general relative case set for $n = 0$ by $\pi_0(X, A, x_0) = \pi_0(X, x_0) \big/ i_*(\pi_0(A, x_0))$, and $f_*\colon \pi_0(X, A, x_0) \to \pi_0(Y, B, y_0)$ in the same way as the absolute case.

For $n \geq 2$, or $n = 1$ and $A = \{x_0\}, B = \{y_0\}$, f_* will be a group homomorphism. In the rest of cases, the sets don't have group structure, and f_* will be an induced map between sets.

Proof. We prove the case $n \geq 2$. Consider the map f_* as described in the statement. We observe that, for all $[\alpha] \in \pi_n(X, A, x_0)$, $f_*([\alpha]) \in \pi_n(Y, B, y_0)$, as $(f \circ \alpha)(\partial I^n) \subseteq f(A) \subseteq B$, and $(f \circ \alpha)(J^{n-1}) = f(\{x_0\}) = \{y_0\}$.

We have to prove it is well defined and a homomorphism. To prove it is well defined, we consider two α, α' such that $[\alpha] = [\alpha']$ in $\pi_n(X, A, x_0)$. This means we have a homotopy $H\colon (I^n \times I, \partial I^n \times I, J^{n-1} \times I) \to (X, A, x_0)$ with $H\colon \alpha \simeq_{(\partial I^n, J^{n-1})} \alpha'$ homotopy between them. Then, we see that $f \circ H\colon f \circ \alpha \simeq_{(\partial I^n, J^{n-1})} f \circ \alpha'$ as in proposition 3.4.

It is easy to see that $f_*([\alpha] * [\beta]) = f_*([\alpha]) * f_*([\beta])$ by just writing the definition of $\alpha * \beta$ and applying f on it, and we have that

$$f_*([\alpha]^{-1}) = [f \circ \overline{\alpha}] = [\overline{f \circ \alpha}] = f_*([\alpha])^{-1}$$

and so f_* is a group homomorphism for the cases indicated.

It is easy to verify that the map is well defined as a map between sets in the rest of cases. □

Corollary 5.4 (Functoriality of the homotopy groups)**.** *If we consider $\pi_n\colon \mathbf{Top}^* \to \mathbf{Grp}$ given by $\pi_n\big((X, x_0)\big) = \pi_n(X, x_0)$ and $\pi_n(f) = f_*$, we have that π_n is a homotopy invariant functor on $\mathbf{Top}^*$, $\forall n \in \mathbb{N}$.*

If we consider the category $\mathbf{Set}^*$ of based sets, that is, sets together with a distinguished element, (A, a), $\pi_0\colon \mathbf{Top}^* \to \mathbf{Set}^*$ such that the distinguished element of $\pi_0(X, x_0)$ corresponds to the path-component of x_0 is also a homotopy invariant functor.

Proof. We have to prove the items of definitions 2.8 and 3.6 for the case $n \geq 1$.

Firstly, the fact that $\pi_n(f \circ g) = \pi_n(f) \circ \pi_n(g)$ follows applying the corresponding homomorphism to every $[\alpha] \in \pi_n(X, x_0)$, as we have

$$\pi_n(g \circ f)([\alpha]) = [(g \circ f) \circ \alpha] = g_*([f \circ \alpha]) = (g_* \circ f_*)([\alpha]) = \big(\pi_n(g) \circ \pi_n(f)\big)([\alpha])$$

Then, we can prove $\pi_n(1_{(X,x_0)}) = 1_{\pi_n(X,x_0)}$ from the fact that $id_{(X,x_0)} \circ \alpha = \alpha$, implying $[id_{(X,x_0)} \circ \alpha] = [\alpha]$.

The homotopy invariance follows from the fact that, if $H\colon f \simeq_{\{x_0\}} g$ is a relative homotopy equivalence between them, $G\colon X \times I \to Y$ given by $G(x, t) = H\big(\alpha(x), t\big)$ will be a relative homotopy equivalence from $f \circ \alpha$ to $g \circ \alpha$, and so $f_*([\alpha]) = [f \circ \alpha] = [g \circ \alpha] = g_*([\alpha])$ for all $[\alpha] \in \pi_n(X, x_0)$, meaning $f_* = g_*$. □

Observation 5.5. We could also define the category of pairs of topological spaces given by a topological space and a subspace of it (X, A), denoted **PTop**, the category of based pairs of topological spaces (X, A, x_0), denoted $\mathbf{PTop}^*$, define homotopy invariant functors on both of them, and show that the relative case of homotopy groups yields homotopy invariant functors $\pi_n\colon \mathbf{PTop}^* \to \mathbf{Grp}$ for $n \geq 2$, and $\pi_0, \pi_1\colon \mathbf{PTop}^* \to \mathbf{Set}^*$. However, as it would be done in a similar way to what we have already established for the absolute case, and we will not deal with these categories directly, we will not reproduce such argument in this essay.

Proposition 5.6. *Let (X, x_0) be a based topological space. Then, $\pi_n(X, x_0)$ is abelian for all $n \geq 2$.*

Proof. We just have to prove that, for two $[f], [g] \in \pi_n(X, x_0)$, $f * g \simeq_{\partial I^n} g * f$. We will prove this by giving explicit homotopies.

$$H_1\big(\mathbf{x}, t\big) = \begin{cases} f\big(2x_1, (1+t)x_2, \cdots, x_n\big) & \text{if } x_1 \in [0, \frac{1}{2}], x_2 \in [0, \frac{1}{1+t}] \\ g\big(2x_1 - 1, \frac{2x_2 - t}{2-t}, \cdots, x_n\big) & \text{if } x_1 \in [\frac{1}{2}, 1], x_2 \in [\frac{t}{2}, 1] \\ x_0 & \text{otherwise} \end{cases}$$

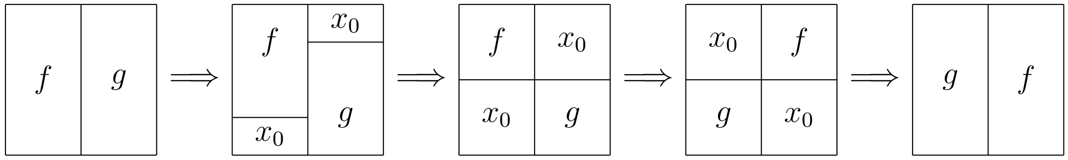


Figure 1: Representation of the homotopies in the proof of proposition 5.6, showing the domains of a piece-wise decomposition of the maps. For example, the first figure represents the domains of f and g in the definition of $f * g$. The first homotopy goes from the first to the third figure, with the second showing a transition step. The second homotopy goes from the third to the fourth, and the last homotopy goes from the fourth to the last. Observe that the lines signalling the borders are all mapped to x_0, which is omitted from the drawing for simplicity.

$$H_2(\mathbf{x}, t) = \begin{cases} f\big(2x_1 - t, 2x_2, \cdots, x_n\big) & \text{if } x_1 \in [\frac{t}{2}, \frac{1+t}{2}], x_2 \in [0, \frac{1}{2}] \\ g\big(2x_1 - 1 + t, 2x_2 - 1, \cdots, x_n\big) & \text{if } x_1 \in [\frac{1-t}{2}, \frac{2-t}{2}], x_2 \in [\frac{t}{2}, 1] \\ x_0 & \text{otherwise} \end{cases}$$

$$H_3(\mathbf{x}, t) = \begin{cases} f\big(2x_1 - 1, (2-t)x_2, \cdots, x_n\big) & \text{if } x_1 \in [\frac{1}{2}, 1] \wedge x_2 \in [0, \frac{1+t}{2}] \\ g\big(2x_1, (2-t)x_2 - 1 + t, \cdots, x_n\big) & \text{if } x_1 \in [0, \frac{1}{2}] \wedge x_2 \in [\frac{1-t}{2-t}, 1] \\ x_0 & \text{otherwise} \end{cases}$$

We observe that, if we take the closure of the domains marked as “otherwise”, the maps still map to x_0, as the intersection of the domains of each map is always mapped to x_0. As counting the closure of those domains, we have closed sets that cover $I^n \times I$, with each H_i agreeing in the intersections, by lemma 3.2, each is continuous and so a homotopy.

We have also that $H_i(\partial I^n, t) = \{x_0\}$, $\forall t \in I$, $i = 1, 2, 3$; and $H_1(x, 0) = (f * g)(x)$, $H_1(x, 1) = H_2(x, 0)$, $H_2(x, 1) = H_3(x, 0)$ and $H_3(x, 1) = (g * f)(x)$ for all $x \in I^n$, so we have proven the required equivalence. □

We must remark two things about this result. The first, is that the fundamental group of a based topological space is not abelian in general. In fact, in 6.3, we will show that any given group G is the fundamental group of some path-connected topological space.

The second is that, while this result doesn't directly translate to relative homotopy groups, it does for $n \geq 3$. In this proof, we are reproducing what's called an Eckmann-Hilton argument, by “sliding f and g to change their positions”. The homotopies in the proof above can be better visualised through figure 1.

However, for relative homotopy groups, we have that the base of the cube is mapped to A in a way we “cannot control”, as it complicates creating the blocks that are mapped to x_0 in figure 1. In the relative case, in the figure, the bottom edge would not necessarily be mapped to x_0 but to A in a more general way, and so creating the bottom-left block x_0 may not be doable in general. This implies that one more coordinate is needed to reproduce the argument. However, this comment is not enough to prove $\pi_2(X, A, x_0)$ is not abelian in general. For this, we will give a counterexample in proposition 6.61.

Definition 5.7 (Homotopy extension property)**.** Let (X, A) be a pair of topological spaces, with A subspace of X. We say that the pair (X, A) has the homotopy extension property if for each Y, each homotopy $f_t \colon A \to Y$ and each continuous map $\widetilde{f}_0 \colon X \to Y$ such that

$f_0 = \tilde{f}_0|_A$, $\exists \tilde{f}_t \colon X \to Y$ homotopy such that $f_t = \tilde{f}_t|_A$ for all $t \in I$, and $\tilde{f}_0$ agrees as expected by notation.

We say $\tilde{f}_t$ is an extension of f_t and $\tilde{f}_0$.

Observation 5.8. Let A be a closed subset of X. If we consider the equivalent notation for homotopies, either by indexed families of continuous maps, $f_t \colon X \to Y$, or continuous maps of the form $F \colon X \times I \to Y$, the homotopy extension property can be equivalently described as every continuous map $F \colon (X \times \{0\}) \cup (A \times I) \to Y$ having a continuous extension of the form $\tilde{F} \colon X \times I \to Y$.

This is because we can consider the map $F \colon (X \times \{0\}) \cup (A \times I) \to Y$ given by

$$F(x,t) = \begin{cases} \tilde{f}_0(x) & \text{if } t = 0 \\ f_t(x) & \text{if } x \in A \end{cases}$$

which is continuous because both maps agree on their intersection, and applying lemma 3.2, as both domains are closed when A is closed. A continuous extension of this map, $\tilde{F} \colon X \times I \to Y$, is equivalent to the homotopy extension we wanted.

Lemma 5.9 (Characterisation of the homotopy extension property)**.** *A pair (X, A) of topological spaces, with A a closed subspace of X, has the homotopy extension property if and only if $(A \times I) \cup (X \times \{0\})$ is a retract of $X \times I$.*

Proof. Assume (X, A) has the homotopy extension property. If we take $Y = (X \times \{0\}) \cup (A \times I)$, the identity on Y, $id_Y \colon (X \times \{0\}) \cup (A \times I) \to (X \times \{0\}) \cup (A \times I)$, has a continuous extension, we will denote $r \colon X \times I \to (X \times \{0\}) \cup (A \times I)$. It can be seen that r suffices as a retract.

Assume now that there is some continuous map $r \colon X \times I \to (A \times I) \cup (X \times \{0\})$ which defines a retract. If we have a continuous map of the form $F \colon (X \times \{0\}) \cup (A \times I) \to Y$, $\tilde{F} = F \circ r \colon X \times I \to Y$ is a suitable homotopy extension. □

Lemma 5.10. *The pair $(I^n, \partial I^n)$ has the homotopy extension property for all $n \in \mathbb{N}$.*

Proof. As ∂I^n is the union of a finite number of closed faces, it is a closed set of I^n, and we just have to prove that $(I^n \times \{0\}) \cup (\partial I^n \times I)$ is a retract of $I^n \times I$.

Consider $r \colon I^n \times I \to (I^n \times \{0\}) \cup (\partial I^n \times I)$ given by the following process. Take the point $P = \left(\left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right), 2\right) \in I^n \times \mathbb{R}$. The image of a point $(\mathbf{x}, t) \in I^n \times I$, $r(\mathbf{x}, t)$, is given by the intersection with $(I^n \times \{0\}) \cup (\partial I^n \times I)$ of the ray that starts in P and crosses $(\mathbf{x}, t)$. This can be visualised for $n = 2$ in figure 2.

It can be verified that the map r is continuous, and that it is a retract. □

Lemma 5.11. *Let (X, A) be a pair of topological spaces, where $A \subseteq X$ is a closed subset, which has the homotopy extension property. Then, for any topological space W, $(X \times W, A \times W)$ also has the homotopy extension property.*

Proof. By lemma 5.9, $(X \times \{0\}) \cup (A \times I)$ is a retract of $X \times I$. If $r \colon X \times I \to (X \times \{0\}) \cup (A \times I)$ defines this retract, $r' \colon (X \times I) \times W \to (X \times \{0\} \times W) \cup (A \times I \times W)$ given by $r'(x, y) = (r(x), y)$ suffices. □

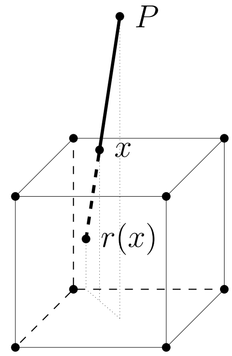


Figure 2: Image of a point $\mathbf{x}$ through the retract from $I^2 \times I$ to $(\partial I^2 \times I) \cup (I^2 \times \{0\})$

Proposition 5.12. *Let (X, x_0) be a based topological space. If $\alpha\colon I \to X$ is a path with $\alpha(0) = x_0$ and $\alpha(1) = y_0$, and $[f] \in \pi_n(X, x_0)$ any element, we define the operation $\alpha \cdot [f]$ by considering the homotopy $f_t\colon \partial I^n \to X$ given by $f_t(\mathbf{x}) = \alpha(t), \forall \mathbf{x} \in \partial I^n$, the map $\tilde{f}_0\colon I^n \to X$ given by $\tilde{f}_0 = f$, and setting $\alpha \cdot [f] = [\tilde{f}_1]$, where $\tilde{f}_1$ is the final map of any homotopy extension of f_t and $\tilde{f}_0$, which is guaranteed to exist by lemma 5.10.*

If we consider the map $\beta^n_\alpha\colon \pi_n(X, x_0) \to \pi_n(X, x_0)$ given by $\beta^n_\alpha([f]) = \alpha \cdot [f]$, we have that it is well defined, and it follows these properties for all $n \in \mathbb{N}$, for all $\alpha, \alpha'\colon I \to X$ paths on X:

1. $\alpha \simeq_{\partial I^n} \alpha' \implies \beta^n_\alpha = \beta^n_{\alpha'}$.
2. $\beta^n_\alpha([f] * [g]) = \beta^n_\alpha([f]) * \beta^n_\alpha([g]), \forall [f], [g] \in \pi_n(X, x_0)$.
3. $\beta^n_{e_{x_0}}([f]) = [f], \forall [f] \in \pi_n(X, x_0)$.
4. $\beta^n_\alpha \circ \beta^n_{\alpha'} = \beta^n_{\alpha' * \alpha}$.

Proof. We first start by showing that, if $\alpha \simeq_{\partial I} \alpha'$, then $\alpha \cdot [f] = \alpha' \cdot [f]$ for all $[f] \in \pi_n(X, x_0)$. If $\tilde{f}_t, \tilde{f}'_t$ are the homotopy extensions involved in the constructions of $\alpha \cdot [f]$ and $\alpha' \cdot [f]$, we can define $H\colon I^n \times I \to X$ as given by

$$H(\mathbf{x}, t) = \begin{cases} \tilde{f}_{1-2t}(\mathbf{x}) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \tilde{f}'_{2t-1}(\mathbf{x}) & \text{otherwise} \end{cases}$$

We observe that $H(\mathbf{x}, t) = (\overline{\alpha} * \alpha')(t)$ if $\mathbf{x} \in \partial I^n$, and by assumption that $\alpha \simeq_{\partial I} \alpha'$, we have that there exists a homotopy $W\colon \overline{\alpha} * \alpha' \simeq_{\partial I} e_{x_0}$.

As by lemma 5.11 we have that the pair $(I^n \times I, \partial I^n \times I)$ has the homotopy extension property, if we take $\tilde{g}_0\colon I^n \times I \to X$ given by $\tilde{g}_0(\mathbf{x}, s) = H(\mathbf{x}, s)$, and $g_t\colon \partial I^n \times I \to X$, given by $g_t(\mathbf{x}, s) = W(s, t)$, we have that $\exists \tilde{g}_t\colon I^n \times I \to X$ homotopy extension. Consider the following map $R\colon I^n \times I \to X$ given by

$$R(\mathbf{x}, t) = \begin{cases} \tilde{g}_0(\mathbf{x}, 3t) & \text{if } t \in \left[0, \frac{1}{3}\right] \\ \tilde{g}_{3t-1}(\mathbf{x}, 1) & \text{if } t \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ \tilde{g}_1(\mathbf{x}, 3 - 3t) & \text{if } t \in \left[\frac{2}{3}, 1\right] \end{cases}$$

The domains are closed sets that cover $I^n \times I$, and the maps coincide and are continuous in their domain, so G is a homotopy by lemma 3.2. We have $R(\mathbf{x}, 0) = \tilde{f}_1(\mathbf{x})$, $R(\mathbf{x}, 1) = \tilde{f}'_1(\mathbf{x})$ and $R(\partial I^n \times I) = \{x_0\}$, so $R\colon \tilde{f}_1 \simeq_{\partial I^n} \tilde{f}'_1$.

We observe that, as $\alpha \simeq_{\partial I} \alpha$, this implies that any homotopy extension we might find in the construction will yield the same class $\alpha \cdot [f]$ for any fixed map f, so we have proven that the operation is well defined with respect to any choice of homotopy extension in the construction of $\alpha \cdot [f]$.

To prove that $L\colon f \simeq_{\partial I^n} f' \implies \alpha \cdot [f] = \alpha \cdot [f']$, we observe that, if $\tilde{f}'_t$ is the homotopy extension in the construction of $\alpha \cdot [f']$, the homotopy $\tilde{L}\colon I^n \times I \to X$ given by

$$\tilde{L}(\mathbf{x}, t) = \begin{cases} L(\mathbf{x}, 1-2t) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \tilde{f}'_{2t-1}(\mathbf{x}) & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

is indeed a homotopy, as the domains are closed, the maps are continuous and agree and they cover $I^n \times I$, and we have $L = \tilde{L}|_{\partial I^n \times I}$ is given by $L(\mathbf{x}, t) = e_{x_0} * \alpha$, $\tilde{L}(\mathbf{x}, 0) = L(\mathbf{x}, 0) = f(\mathbf{x})$, and so $\tilde{L}$ coincides with the extension in the construction of $(e_{x_0} * \alpha) \cdot [f]$. Considering that $\alpha \simeq_{\partial I} e_{x_0} * \alpha$, this implies $\alpha \cdot [f] = \alpha \cdot [f']$, as both constructions coincide at $t = 1$. With the arguments given until now, we have proven that the operation is well defined and follows the first property.

To prove $\alpha \cdot [f * g] = (\alpha \cdot [f]) * (\alpha \cdot [g])$, we consider that if $\tilde{f}_t, \tilde{g}_t\colon I^n \times I \to X$ are homotopy extensions for the construction of $\alpha \cdot [f]$ and $\alpha \cdot [g]$, we observe that the homotopy $h_t\colon I^n \times I \to X$ given by $h_t = \tilde{f}_t * \tilde{g}_t$ for each $t \in I$ is compatible with the construction of $\alpha \cdot [f * g]$.

Lastly, to prove $\beta^n_\alpha \circ \beta^n_{\alpha'} = \beta^n_{\alpha * \alpha'}$, we observe that, if $\tilde{f}_t$ is the homotopy from the construction of $\alpha \cdot [f]$, and $\tilde{f}'_t$ is the homotopy from $\alpha \cdot [\tilde{f}_1]$, the homotopy h_t given by

$$h_t(\mathbf{x}) = \begin{cases} \tilde{f}'_{2t}(\mathbf{x}) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \tilde{f}_{2t-1}(\mathbf{x}) & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

has $h_1 = \tilde{f}'_1$, and h_t is compatible with the extension in the construction of $(\alpha * \alpha') \cdot [f]$. □

Proposition 5.13. *Let X be a topological space, $x_0, y_0 \in X$, with x_0 and y_0 being in the same path-component. Then, $\pi_n(X, x_0) \cong \pi_n(X, y_0)$, $\forall n \in \mathbb{N}$.*

Proof. Let $\alpha\colon I \to X$ be a path from x_0 to y_0. It exists because they are in the same path-component. Then, consider $\beta^n_\alpha\colon \pi_n(X, x_0) \to \pi_n(X, y_0)$ as in proposition 5.12. By the second property in that proposition, we have that β^n_α is a homomorphism, and by the first, third and fourth properties, it has inverse $\beta^n_{\overline{\alpha}}$, and so it is an isomorphism. □

Observation 5.14. The definition of the maps β^n_α and their properties will be used later. In particular, the maps $\beta^n\colon \pi_1(X, x) \to \mathit{Aut}\,(\pi_n(X, x))$, given by $\beta^n([\alpha]) = \beta^n_\alpha$, will define interesting group actions, which we will explore in proposition 6.15. These actions can be extended to the action of $\pi_1(X, x)$ on itself by conjugation.

Proposition 5.15. *Let X be a topological space, $x_0 \in A \subseteq X$. If A_0, X_0 represent the path-components of x_0 in A and X, then $\pi_n(X, A, x_0) \cong \pi_n(X_0, A_0, x_0)$, for all $n \in \mathbb{N} \cup \{0\}$.*

Proof. Trivial from the fact I^n, I^{n+1} are path-connected, and so the image of every continuous map from I^n, and every homotopy of all such maps, are contained in the path-component of the base-point. □

The two last propositions imply that homotopy groups are limited to the scope of the path-component of the base-point, and that inside of a given path-component, they are independent of the choice of base-point. This means that homotopy groups only depend on a choice of path-component for a given space X. This allows us to, in abuse of notation, write $\pi_n(X)$ instead of $\pi_n(X, x_0)$ when X is a path-connected space. This abuse may be used even in contexts when we don't know if X is path-connected, but specifying the base-point is not required.

There exists an analogous result to that of proposition 5.13 regarding relative homotopy groups, as for x, y in the same path-component of $A \subseteq X$, $\pi_n(X, A, x) \cong \pi_n(X, A, y)$ for all $n \in \mathbb{N}$, and so when A is path-connected, we just write $\pi_n(X, A)$.

These results also show that, in the case of restricting to path components, the homotopy groups of X coincide with those of its subspaces. This, however, does not mean that homotopy groups behave nicely with subspaces. We have even shown that, while $S^1 \subseteq D^2$, we have $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(D^2) = 1$. We will come back to the homotopy groups of subspaces, and their relation with relative homotopy groups, after introducing proposition 6.10.

Homotopy groups can be used to generalise the notion of simple connection, in a way in which we now can see is a topological property of the spaces, as now we know that for path-connected spaces, this property is independent of the choice of base-point.

Definition 5.16 ($n-$connected pairs. $n-$connected spaces)**.** Let (X, A) be a pair of X together with a subspace $A \subseteq X$. If X is path-connected, we will say that (X, A) is a $0-$connected pair. We will say that the pair (X, A) is $n-$connected if it is $0-$connected and $\pi_i(X, A) = 0$, $\forall i \leq n$.

Taking $A = \{x_0\}$, for x_0 any point in X, we can extend this notion to the $0-$connectedness and $n-$connectedness of topological spaces. Trivially, $0-$connectedness is equivalent to path-connectedness, and $1-$connectedness is equivalent to simple connection.

Proposition 5.17. *Let $\{(X_\alpha, x_\alpha)\}_{\alpha \in \mathcal{A}}$ be a collection of path-connected based topological spaces, indexed by some set $\mathcal{A}$. Then, if we consider the based topological space corresponding to the product space, $\big(\Pi_{\alpha \in \mathcal{A}} X_\alpha, (x_\alpha)_{\alpha \in \mathcal{A}}\big)$, we have that, $\forall n \in \mathbb{N}$,*

$$\pi_n\Big(\prod_{\alpha \in \mathcal{A}} X_\alpha\Big) \cong \prod_{\alpha \in \mathcal{A}} \pi_n(X_\alpha)$$

Proof. By the product's universal property, a continuous map $f\colon I^n \to \prod_{\alpha \in \mathcal{A}} X_\alpha$ corresponds to a family of continuous maps $\{f_\alpha\colon I^n \to X_\alpha\}$, so every element of $\pi_n\big(\prod_{\alpha \in \mathcal{A}} X_\alpha\big)$ has an associated element in every $\pi_n(X_\alpha)$.

Also, this follows for maps $f\colon I^{n+1} \to \prod_{\alpha \in \mathcal{A}} X_\alpha$, and so if we have two maps $f, g\colon I^n \to \prod_{\alpha \in \mathcal{A}} X_\alpha$ with $H\colon f \simeq_{\partial I^n} g$, it induces a homotopy $H_\alpha\colon f_\alpha \simeq_{\partial I^n} g_\alpha$ for each $\alpha \in \mathcal{A}$. This means that the classes $[f] \in \pi_n\big(\Pi_{\alpha \in \mathcal{A}} X_\alpha\big)$ correspond exactly with elements $([f_\alpha])_{\alpha \in \mathcal{A}} \in \Pi_{\alpha \in \mathcal{A}} \pi_n(X_\alpha)$ in a way that clearly preserves the group structure. □

Example 5.18. *If we consider the torus $\mathbf{T} = S^1 \times S^1$, $\pi_1(\mathbf{T}) = \mathbb{Z} \times \mathbb{Z}$.*

Summarising the behaviour of homotopy groups with the basic constructions (subspaces, product spaces and quotient spaces), the second is the only one that behaves as expected in general, as quotient spaces can also lead to unexpected results.

As an example of quotients not behaving as one might expect, with what we have already seen, both the interval I and a two point space $\{0, 1\}$ (choosing either of the points as the base-point) have trivial fundamental groups, but the quotient space of I by collapsing $\{0,1\}$ to a point, $\frac{I}{\{0,1\}}$, is homeomorphic to S^1, which has non-trivial fundamental group (in fact, we know it is isomorphic to $\mathbb{Z}$).

6 The computation of homotopy groups

The computation of higher homotopy groups is very hard in general, and it has sparked over the years many subfields of study within Homotopy Theory. Many notions and results, devised to compute homotopy groups in particular cases, ended up leading to interesting theory worth studying on its own.

We will use the excuse of showing some ways higher homotopy groups are computed, to both display the difficulty of calculations in the field, and give insight into how some subfields of Homotopy Theory are studied, which can be sought after by an interested reader who want to deepen their understanding of the theory lying beyond this essay.

In this section we will show some interesting relationships between homotopy groups, which allow to translate information from known groups into unknown ones; we will show Seifert-van Kampen's theorem and how it's used to compute fundamental groups; we will generalise the properties of the map $p\colon \mathbb{R} \to S^1$ employed in the computation of the fundamental group of the circle to the notion of covering spaces, whose classification is an interesting subject of study; we will give some notions into the importance of the homotopy of CW-complexes for the field; and we will show some glimpses of the problem of the homotopy groups of spheres, the Freudenthal suspension theorem, and the field of Stable Homotopy Theory.

Some parts of this section will deal heavily with CW-complexes, and so we will give a refresher into their definition, and some interesting properties of their topology in the context of Homotopy Theory.

6.1 The topology and homotopy of CW-complexes

Definition 6.1 ($n-$cells. CW-complexes)**.** A $0-$cell is a one-point space. An $n-$cell in a topological space X is the image of an $n-$ball B^n through a continuous map $\Phi\colon D^n \to X$, where B^n corresponds to the interior of the disk D^n.

A CW-complex is a topological space X, together with a sequence of subspaces, $X_0 \subseteq X_1 \subseteq \cdots \subseteq \cup_n X^n = X$, constructed from cells through the following process:

1. X^0 is a discrete space, with points regarded as $0-$cells.

2. We construct the space X^n from X^{n-1} by attaching $n-$cells, $\{e^n_\alpha\}_{\alpha\in\mathcal{A}_n}$, to X^{n-1}, through maps $\varphi_\alpha\colon S^{n-1} \to X^{n-1}$. For this, we regard S^{n-1} as the border of a disk D^n, and φ_α as describing how the disk is attached. We take the disjoint sum $X^{n-1} \sqcup D^n$ and then the quotient that identifies $x \sim \varphi_\alpha(x)$ for each $x \in \partial D^n = S^{n-1}$. X^n is defined by reproducing this process simultaneously for each e^n_α (α is understood to uniquely determine the cell, but the superscript n is included for clearly conveying the size of the cell).

3. If $X \neq X^n$ for any finite $n \in \mathbb{N} \cup \{0\}$, then it is taken to be the limit case $X = \cup_n X^n$. In this case, the topology used on X is usually called the weak topology in the literature, which corresponds to the strong topology with respect to the family of

canonical inclusions $\{X^n \hookrightarrow X\}_{n=0}^{\infty}$. Equivalently, a set $A \subseteq X$ is open (closed) if and only if $A \cap X^n$ is open (closed) in X^n for each $n \in \mathbb{N} \cup \{0\}$.

For a given e_α^n of a CW-complex, the map Φ_α given by the composition $D^n \hookrightarrow X^{n-1} \sqcup D^n \to X^n \hookrightarrow X$ is called the characteristic map of e_α^n.

The spaces X^n involved in the construction of a CW-complex are called its $n-$skeleton.

Definition 6.2 (CW-subcomplex. CW-pair)**.** If $A \subseteq X$ is a closed union of cells of X, we say A is a subcomplex of X. In particular, X^n is the biggest subcomplex with cells of size at most n. A pair (X, A) of a CW-complex and a subcomplex $A \subseteq X$ is called a CW-pair.

Proposition 6.3. *Let $A \subseteq X$ be a subset of a CW-complex X. Then, A is open (closed) in X if and only if $\Phi_\alpha^{-1}(A)$ is open (closed) in D^n, $\forall \alpha \in \mathcal{A}$, $\forall n \in \mathbb{N}$.*

Proof. We prove the result for open sets, as it is equivalent for closed sets. One implication is obvious due to the continuity of characteristic maps.

Assume that $\Phi_\alpha^{-1}(A)$ is open for each $\alpha \in \mathcal{A}$, and that $A \cap X^i$ is open $\forall i < n$. Then, as $\Phi_\alpha^{-1}(A)$ is open in D_α^n, $A \cap X^n$ is open in X^n by definition of the quotient topology, as $A \cap X^n = (A \cap X^{n-1}) \cup \Phi_\alpha\left(\Phi_\alpha^{-1}(A)\right)$. This implies A is open in X^n for all $n \in \mathbb{N}$ by induction, and so we are done. □

CW-complexes are topological spaces with some nice properties. We will now proceed to prove some of them.

Proposition 6.4. *Let X be a CW-complex. Then, X is a T_4 space.*

Proof. Points are closed in X, as their pre-images through the characteristic maps are always closed, by applying proposition 6.3, implying it is a T_1 space.

Let $A \subseteq X$ be a subset of X. We define an open neighbourhood $N_\epsilon(A)$ of A for each map $\epsilon\colon \mathcal{A} \to \mathbb{R}^+$ by the following process. We define $N_\epsilon^0(A) = A \cap X^0$. Assume N_ϵ^i defined for $i < n$. Then, we will define $N_\epsilon^n(A)$ by giving two open sets $A_\alpha, B_\alpha \; \forall \alpha \in \mathcal{A}_n$, and defining

$$N_\epsilon^n(A) = N_\epsilon^{n-1}(A) \cup \bigcup_{\alpha \in \mathcal{A}_n} \left(A_\alpha \cup B_\alpha\right)$$

Here, A_α will be an open set in the interior of the disk, $D^n \setminus \partial D^n$, and B_α will be another open set "along the border" that is contained in $D^n \setminus \{0\}$, where 0 denotes the centre of the disk. A_α will be a neighbourhood of the points of A inside the disk, and B_α will be a neighbourhood of the points of the border that are mapped to $N_\epsilon^{n-1}(A)$.

A_α will be given by

$$A_\alpha = \{x \in D^n \setminus \partial D^n \mid d\left(\Phi_\alpha^{-1}(A) \setminus \partial D^n, x\right) < \epsilon(\alpha)\}$$

an $\epsilon(\alpha)-$neighbourhood of $\Phi_\alpha^{-1}(A) \setminus \partial D^n$ in $D^n \setminus \partial D^{n-1}$.

To define B_α, we first consider the homeomorphism $r\colon (0, 1] \times \partial D^n \to D^n \setminus \{0\}$, where 0 denotes the centre of the disk, given by $r(t, s) = ts$. Observe that $\phi_\alpha^{-1}\big(N_\epsilon^{n-1}(A)\big) \subseteq \partial D^n$. Then,

$$B_\alpha = r\Big((1 - \epsilon(\alpha), 1] \times \phi_\alpha^{-1}\big(N_\epsilon^{n-1}(A)\big)\Big)$$

suffices. It is clear that $N_\epsilon(A) = \cup_n N_\epsilon^n(A)$ is an open neighbourhood of A in X.

To prove the normality of X, we prove that for small enough values of ϵ, $N_\epsilon(A) \cap N_\epsilon(B) = \emptyset$ for disjoint closed sets A, B. Assume $N^i_\epsilon(A) \cap N^i_\epsilon(B) = \emptyset$ for all $i \leq n$, after observing that the base case $n = 0$ is trivial due to the definition of N^0_ϵ and the fact that A, B are disjoint.

For any given $\Phi_\alpha \colon D^{n+1} \to X$, we observe that

$$d\left(\Phi_\alpha^{-1}\left(N^n_\epsilon(A)\right), \Phi_\alpha^{-1}(B)\right) > 0$$

as otherwise we would be able to find a sequence in $\Phi_\alpha^{-1}(B)$ of points arbitrarily close to $\Phi_\alpha^{-1}\left(N^n_\epsilon(A)\right)$, which by compactness of $\Phi_\alpha^{-1}(B)$ contains a converging subsequence to a point with 0 such distance in $\Phi_\alpha^{-1}(B)$, contradiction with the existence of $N^n_\alpha(B)$, as $\Phi_\alpha^{-1}\left(N^n_\alpha(B)\right)$ is a neighbourhood of $\Phi_\alpha^{-1}(B) \setminus \partial D^n$, disjoint with $\Phi_\alpha^{-1}\left(N^n_\epsilon(A)\right)$ by hypothesis. By symmetry,

$$d\left(\Phi_\alpha^{-1}\left(N^n_\epsilon(B)\right), \Phi_\alpha^{-1}(A)\right) > 0$$

implying $d\left(\Phi_\alpha^{-1}(A), \Phi_\alpha^{-1}(B)\right) > 0$, and so there exists some $\epsilon(\alpha)$ for which the thesis follows. □

In the last results we have developed inductions on the dimension of the cells of CW-complexes, which can be regarded as inductions on the dimension of their $n-$skeleta, relying on the inductive definition of the complexes themselves. This inductive nature is one of the reasons they are well-behaved with respect to general topological spaces, as it helps prove many interesting results on them.

For example, this kind of argument allows for inductive proofs of the continuity of maps:

Proposition 6.5. *Let $F \colon X \to Y$ be a map, with X a CW-complex, and Y an arbitrary topological space. Then, F is continuous if and only if $F_n = F|_{X^n}$ is continuous for each n.*

As homotopies are, in the end, continuous maps, these inductive proofs of continuity allow for inductive constructions of homotopies, which is one of the advantages of CW-complexes over general topological spaces for Homotopy Theory. One such homotopy is inductively described in the proof of next proposition:

Proposition 6.6. *Let X be a CW-complex, and $x \in X$ a point. Then, the family of the sets of the form $N_\epsilon(\{x\})$ as described in the proof of proposition 6.4, contains an open neighbourhood basis for x, such that every such neighbourhood is contractible.*

Proof. Let U be a neighbourhood of x in X. As U is a neighbourhood, $\exists B$ open set with $x \in B \subseteq U$. Considering B^C the complementary of B in X, which is closed, and the fact $\{x\}$ is closed because X is T_2, we have proven in the proof of proposition 6.4 that $\exists \epsilon$ such that $N_\epsilon(\{x\})$ and $N_\epsilon(B^C)$ are disjoint. This implies $N_\epsilon(\{x\}) \subseteq B$, and so the family of these $N_\epsilon(\{x\})$ is a neighbourhood basis. We will prove that this family suffices.

We denote $N^i = N^i_\epsilon(\{x\})$, $\forall i \in \mathbb{N} \cup \{0\}$. To prove contractibility, we will prove that $N_\epsilon(\{x\}) \simeq \{x\}$. For this, we construct a deformation retract of $N_\epsilon(\{x\})$ into $\{x\}$.

Assume $x \in X^m \setminus X^{m-1}$, and we construct a deformation retract of N^n into N^{n-1} $\forall n > m$. Let Φ_α be the characteristic map of a cell e^n_α. Observe that due to the assumption that $x \in X^m \setminus X^{m-1}$, no point in the interior of the cell is mapped to x. Then, Φ_α doesn't map $0 \in D^n$ to x. This means that we can assume $0 \notin \Phi_\alpha^{-1}\left(N_\epsilon(\{x\})\right)$, and it is contained in $D^n \setminus \{0\}$, which admits a deformation retract ∂D^n by example 3.13. Composing with this retract for the characteristic maps of all $n-$cells, yields by the definition of N^n a deformation retract of it into N^{n-1}.

In the case $n = m$, N^m is by definition an open ball around x, and so $N^m \simeq \{x\}$. Then, it suffices to show $N^m \simeq N_\epsilon(\{x\})$.

Now, define the homotopy f_t such that the homotopy from N^{m+i} to N^{m+i-1} happens during the interval $\left[\frac{1}{2^i}, \frac{1}{2^{i-1}}\right]$. This can be defined inductively, leaving the parts of the interval for which the homotopy is still undefined constant. We note that, for each N^j, at some point the image on its skeleton will be stationary, and so the resulting F associated map of f_t is continuous when restricted to every N^j, which implies it is continuous. Then, f_t is a homotopy from $N_\epsilon(\{x\})$ to N^m, completing the proof. □

Corollary 6.7. *CW-complexes are locally path-connected. This implies that a CW-complex X is path-connected if and only if it is connected.*

6.2 Relations between homotopy groups

This subsection mainly deals with two ways that we can define relations between the different homotopy groups of a given based topological space: the long exact sequences of homotopy groups for pairs, and that higher homotopy groups possess a structure of $\mathbb{Z}[\pi_1]-$module.

Definition 6.8 (Exact sequence)**.** When we have a collection of objects of **Grp**, G_i, and morphisms between them, connecting them in the following way

$$\cdots \xrightarrow{\varphi_{i+2}} X_{i+2} \xrightarrow{\varphi_{i+1}} X_{i+1} \xrightarrow{\varphi_i} X_i \xrightarrow{\varphi_{i-1}} \cdots$$

and such that $im\ \varphi_{i+1} = ker\ \varphi_i$ for all $i \in \mathbb{N}$, we say they conform an exact sequence.

We introduce a useful lemma, which helps characterise when a given continuous map $g\colon I^n \to X$ represents the trivial class in $\pi_n(X, A, x_0)$.

Lemma 6.9 (Compression criterion)**.** *Let X be a topological space, and $x_0 \in A \subseteq X$. Then, an element $[f] \in \pi_n(X, A, x_0)$ is the trivial element of the group if and only if f is homotopic relative to ∂I^n to a map $g\colon (I^n, J^{n-1}) \to (A, x_0)$.*

Proof. The first implication comes from the fact that, if $f \simeq_{\partial I^n} g$ for some such g, $[f] = [g]$, and we just have to prove $[g]$ is trivial. For this, we consider the homotopy $H\colon I^n \times I \to X$ as given by

$$H(\mathbf{x}, t) = g\left(x_1, x_2, \cdots, (1-t)x_n + t\right)$$

We have that $H(\mathbf{x}, 0) = g(\mathbf{x})$ and $H(\mathbf{x}, 1) = g\big((x_1, x_2, \cdots, x_{n-1}, 1)\big) = x_0$, as we have that $(x_1, x_2, \cdots, x_{n-1}, 1) \in \partial I^n$, so g is homotopic to the constant map, and H respect the requirements so that $[g] = [e_{x_0}]$ in $\pi_n(X, A, x_0)$, as $H(J^{n-1} \times I) = \{x_0\}$, $H(I^{n-1} \times I) \subseteq A$.

Now, assume that $[f]$ is the trivial element. Then, $H\colon e_{x_0} \simeq_{(\partial I^n, J^{n-1})} f$.

If we consider $r_t\colon I^n \times I \to I^n \times I$ to be a family of retracts as the one in the proof of lemma 5.10, but such that the intersection point of the rays is taken at $(I^n \times \{t\}) \cup (\partial I^n \times [t, 1])$ instead of $(I^n \times \{0\}) \cup (\partial I^n \times I)$, we have that this can be proven to be a homotopy by arguments similar to the ones sketched in the proof of the lemma, and $h_t\colon I^n \times I \to X$ given by $h_t(\mathbf{x}) = (H \circ r_t)(\mathbf{x}, 1)$ is a homotopy which suffices. □

Proposition 6.10 (Long exact sequence of relative homotopy)**.** *Let X be a topological space, and $x_0 \in B \subseteq A \subseteq X$. Then, there exists an exact sequence*

$$\cdots \to \pi_{n+1}(X, A, x_0) \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \to \cdots$$

where i_* and j_* are the homomorphisms induced by the inclusions $i\colon (A, x_0) \hookrightarrow (X, x_0)$ and $j\colon (X, \{x_0\}, x_0) \hookrightarrow (X, A, x_0)$, and $\partial\colon \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ is obtained by $\partial([f]) = [\partial f]$, where ∂f is the restriction of f to I^{n-1}.

In particular, taking $B = \{x_0\}$, there is an exact sequence

$$\cdots \to \pi_{n+1}(X, A, x_0) \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots$$

Proof. First, we observe that near the end of the sequence, objects like $\pi_0(X, x_0)$ do not have group structure. Nevertheless, exactness still makes sense if they are defined to be trivial when they consist of a single element.

We first observe that $j_* \circ i_* \equiv 0$, as maps $(I^n, \partial I^n, J^{n-1}) \to (A, B, x_0)$ represent zero elements in $\pi_n(X, A, x_0)$, due to lemma 6.9.

To prove $\ker j_* \subseteq im\, i_*$, we assume some $f\colon (I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$ to represent the zero element in $\pi_n(X, A, x_0)$, and then again by lemma 6.9, f is homotopic *rel* ∂I^n to a map with image in A, and $[f]$ lies in the image of i_*.

The exactness between j_* and ∂ is analogous. To prove $i_* \circ \partial \equiv 0$, we observe that the restriction of some map $f\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0)$ is homotopic *rel* ∂I^n to a map representing the zero element through f itself, as it is a homotopy by definition.

As we are only interested in the case where $B = \{x_0\}$, we will prove the converse for just this case. Assume a homotopy $f_t\colon (I^n, \partial I^n) \to (X, x_0)$ from $f\colon (I^n, \partial I^n) \to (A, x_0)$ to the constant map e_{x_0}. Here, f represents the zero element of $\pi_n(X, x_0)$. Then, if we consider $F\colon (I^n \times I, \partial(I^n \times I), J^n) \to (X, A, x_0)$ given by $F(\mathbf{x}, t) = f_t(\mathbf{x})$, we have $[f] = \partial([F])$, and so we are done.

It can be seen that the definitions given for the sets $\pi_1(X, A, x_0), \pi_0(X, A, x_0), \pi_0(X, x_0)$ can be made to fit nicely in the sequence, even though the maps are no longer group homomorphisms and are just maps between sets. There, the kernel of a map is considered to be the preimage of the base-point, and a based set is considered to be trivial when it has only one element. If we define the sets for values of $n \in \mathbb{Z}$, such that they are one point sets for all negative values of n, the sequence can be extended while remaining exact. □

This result allows us to see interesting relationships between homotopy groups. Imagine, for example, that we know the homotopy groups of some subspace A, and the relative homotopy groups of the $n-$connected pair (X, A). Then,

$$\pi_n(X) \cong \pi_n(A) \big/ ker\, i_* \cong \pi_n(A) \Big/ \partial\big(\pi_{n+1}(X, A)\big)$$

These kind of relationships allow us to convert information about homotopy groups we know, into information about those we don't, sometimes even allowing us to compute those unknown groups.

This has many effects, like allowing us to relate the homotopy groups of a space and general subspaces of it, by mediating with relative homotopy groups.

Another important consequence is that this result allows us to work with relative homotopy groups, for which we haven't done any real calculations, by using absolute homotopy groups. Observe the following result.

Corollary 6.11. *Let X be a path-connected space, and $A \subseteq X$ be a subspace, such that $\pi_1(X) = 1$, $\pi_2(X) = 0$. Then, $\pi_2(X, A) \cong \pi_1(A)$*

Proof. If $\pi_2(X) = \pi_1(X) = 1$, we have an exact subsequence of the long exact sequence of homotopy of the pair (X, A) given by

$$1 \longrightarrow \pi_2(X, A) \longrightarrow \pi_1(A) \longrightarrow 1$$

The fact that the image of the first arrow is trivial, proves the second one is monomorphism, and the fact that the third arrow has kernel equal to $\pi_1(A)$ shows the second is surjective, so the second arrow is an isomorphism in between $\pi_2(X, A)$ and $\pi_1(A)$. □

In particular, if $\pi_1(A)$ is not abelian, we have found our example of a non-abelian second relative homotopy group, finally proving that the condition $n \geq 3$ for relative homotopy groups to be abelian in general is optimal. In proposition 6.61 we will give a concrete example of this.

We will now show another interesting relationship between homotopy groups.

Definition 6.12 ($R-$modules. $G-$modules)**.** Let G be a group. A (left) $G-$module consists of an abelian group M, together with a group action of G on M given by $\Phi \colon G \times M \to M$, such that $\Phi(g, a + b) = \Phi(g, a) + \Phi(g, b)$ for all $a, b \in M$.

If R is a ring with multiplicative identity, a (left) $R-$module consists of an abelian group M, and an operation given by $\Phi \colon R \times M \to M$, such that, $\forall r, s \in R \; \forall a, b \in M$, we have:

1. $\Phi(r, a + b) = \Phi(r, a) + \Phi(r, b)$.
2. $\Phi(r + s, a) = \Phi(r, a) + \Phi(s, a)$.
3. $\Phi\left(r, \Phi(s, a)\right) = \Phi(rs, a)$.
4. $\Phi(1, a) = a$.

Analogous definitions can be given for right modules, and even modules that are both left and right modules. We will however just use left modules and call them modules.

The fact $\pi_n(X)$ are abelian for $n \geq 2$ implies that they naturally have a structure of $\mathbb{Z}-$modules, given by $\Phi(n, a) = \underbrace{a + a + \cdots + a}_{n \text{ times}}$. This structure can be strengthened through the fundamental group.

Definition 6.13 (Group ring)**.** Let G be a group. $\mathbb{Z}[G]$, the group ring of G over $\mathbb{Z}$, is the ring given by $\mathbb{Z}-$linear combinations of elements of G, with the product induced by the product in G.

Lemma 6.14. *Let G be a group, and M be an abelian group, such that M has a structure of $G-$module. Then, the action of G on M can be extended linearly to an operation of $\mathbb{Z}[G]$ on M, and M can be given structure of $\mathbb{Z}[G]-$module.*

Proof. Let M be a $G-$module and denote $\Phi(g, a)$ by $g \cdot a$ for $g \in G$, $a \in M$.

As M is an abelian group, it has a canonical structure of $\mathbb{Z}-$module. We will also denote the action of $\mathbb{Z}$ on M induced by this structure by $n \cdot a$ for $n \in \mathbb{Z}$, $a \in M$.

We then define the action of $\mathbb{Z}[G]$ on M by

$$\left(\sum_i n_i \cdot g_i \right) \cdot m = \sum_i n_i \cdot (g_i \cdot m)$$

We must prove it is indeed a ring action. We prove the items in definition 6.12.

$$\left(\sum_i n_i \cdot g_i\right) \cdot (a+b) = \sum_i n_i \cdot (g_i \cdot (a+b))) = \sum_i n_i \cdot (g_i \cdot a + g_i \cdot b) =$$
$$\sum_i n_i \cdot (g_i \cdot a) + \sum_i n_i \cdot (g_i \cdot b) = \left(\sum_i n_i \cdot g_i\right) \cdot a + \left(\sum_i n_i \cdot g_i\right) \cdot b$$

and so the first item is proved.

The second item is trivial, due to the definition of the sum in $\mathbb{Z}[G]$ and the action of G on M.

For the third one,

$$\left(\sum_i n_i \cdot g_i\right) \cdot \left(\sum_j n'_j \cdot g'_j\right) \cdot m = \left(\sum_i n_i \cdot g_i\right) \cdot \left(\sum_j n'_j \cdot (g'_j \cdot m)\right) =$$
$$= \sum_j \sum_i n_i \cdot g_i \cdot \big(n'_j \cdot (g'_j \cdot m))\big) = \sum_j \sum_i n_i n'_j \cdot (g_i g'_j \cdot m) =$$
$$= \left(\sum_i \sum_j n_i n'_j \cdot (g_i g'_j)\right) \cdot m$$

and the last term is equivalent to the definition of the product in $\mathbb{Z}[G]$, acting on m, which implies the third item. Lastly

$$(1 \cdot e) \cdot a = 1 \cdot (e \cdot a) = 1 \cdot a = a$$

and so the action of the trivial element is trivial, and we are done. With this, we have proven the structure of $\mathbb{Z}[G]-$module on M. If we are given a $\mathbb{Z}[G]-$module on M, we can recover the action of G on M by considering $g \cdot m = (1 \cdot g) \cdot m$ for each $g \in G$, $m \in M$ □

Corollary 6.15. *Let X be a path-connected topological space. Then, $\pi_n(X)$ has a structure of $\mathbb{Z}[\pi_1(X)]-$module, $\forall n \geq 2$.*

Proof. It follows from the last result by considering the action given by the homomorphism $\beta^n \colon \pi_1(X) \to Aut\big(\pi_n(X)\big)$ described in observation 5.14. □

The action of π_1 can be extended to act on itself by inner automorphisms, but it doesn't always have a module structure, as it is not abelian in general.

This structure is interesting in general, as the action of the fundamental group is not a mere curiosity, and plays a role in the theory distinguished from other actions π_1 might possess. A theorem due to Serre included in [1], for example, states that for topological spaces such that the action of π_1 on all π_n is trivial, their homotopy groups are finitely generated if and only if their homology groups are finitely generated, which is a result that doesn't follow in general. We will nevertheless not develop this further, as it lies beyond the scope of this essay.

6.3 Seifert-van Kampen's theorem

Seifert-van Kampen's theorem helps compute the fundamental group of some path-connected spaces, by decomposing them into simpler subspaces of them whose fundamental groups are already known.

Definition 6.16 (Free product of groups)**.** Let $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ be an indexed family of groups. Consider the set $*_\alpha G_\alpha$ of words of finite length $g_1g_2\cdots g_n$ such that the empty word is allowed, and each g_i belongs to some G_{α_i} for some $\alpha_i \in \mathcal{A}$.

If in this set, we identify words in which two consecutive elements belong to the same G_{α_i} with the word that replaces them with their product in G_{α_i}, it is a group, with the operation given by concatenation and identity given by the empty word. This group is called the free product of the family $\{G_\alpha\}_{\alpha\in\mathcal{A}}$.

Lemma 6.17. *Let $\{\phi_\alpha\colon G_\alpha \to H\}_{\alpha\in\mathcal{A}}$ be a family of group homomorphisms. Then, they extend uniquely to a homomorphism*

$$\begin{aligned}\phi\colon *_\alpha G_\alpha &\longrightarrow H\\ g_1g_2\cdots g_n &\longmapsto \varphi_{\alpha_1}(g_1)\varphi_{\alpha_2}(g_2)\cdots\varphi_{\alpha_n}(g_n)\end{aligned}$$

Observe that, while on the left $g_1g_2\cdots g_n$ represents a word, on the right each $\varphi_{\alpha_i}(g_i)$ is an element of H, and the result is the product in H of the images.

Theorem 6.18 (Seifert-van Kampen)**.** *Let (X, x_0) be a based topological space, such that X is the union of path-connected open sets A_α such that $x_0 \in A_\alpha, \forall\alpha \in \mathcal{A}$, and each intersection $A_\alpha \cap A_\beta$ is path-connected. Let*

$$j_\alpha\colon \pi_1(A_\alpha) \to \pi_1(X)$$

be the group homomorphism induced by the inclusion $A_\alpha \hookrightarrow X$. The family $\{j_\alpha\}_{\alpha\in\mathcal{A}}$ extends by lemma 6.17 to a group homomorphism

$$\Phi\colon *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$$

This homomorphism is surjective. Moreover, if each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then, if $i_{\alpha\beta}\colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$, the kernel of Φ is N, the normal subgroup generated by elements of the form $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$ for each $w \in \pi_1(A_\alpha \cap A_\beta)$, and we have

$$\pi_1(X) \cong *_\alpha\pi_1(A_\alpha)\,/N$$

Proof. Let $f\colon I \to X$ be a loop based at x_0. We claim that there exists a partition $0 = s_0 < s_1 < \cdots < s_n = 1$ of I such that for all s_i in the partition, $f([s_i, s_{i+1}]) \subseteq A_{\alpha_i}$ for some $\alpha_i \in \mathcal{A}$.

We have that, as f is continuous, $f^{-1}(A_\alpha)$ is open in I for all $\alpha \in \mathcal{A}$ because each A_α is open in X. This means that each $s \in I$ must have a neighbourhood mapped to the same A_α, which implies that, $\forall s \in I$, $\exists a > 0$ such that $f([s-a, s+a]) \subseteq A_\alpha$ for some $\alpha \in \mathcal{A}$. The family of the interiors of these intervals forms an open cover of I, and by compactness of I, we have that it is covered with a finite subcover of this family, and so it is also covered by a finite amount of the closed intervals. The existence of the partition follows from taking the endpoints of these intervals.

Let f_i be the restriction of f to $[s_i, s_{i+1}]$. We have $f \simeq f_1 * f_2 * \cdots * f_n$ product of paths, each of them lying in some A_{α_i}. We have that f is homotopic to $f_1 * g_1 * \overline{g_1} * f_2 * g_2 * \overline{g_2} * \cdots * f_n$, where g_i is a path in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$ from $f_i(1)$ to x_0, which exists because the intersection is path-connected and contains x_0. From this,

$$[f] = [f_1 * g_1] * [\overline{g_1} * f_2 * g_2] * \cdots * [\overline{g_{n-1}} * f_n]$$

where $f_1 * g_1, \overline{g_1} * f_2 * g_2, \cdots$ are loops based at x_0 lying in A_{α_i}, which shows that $[f] \in \Phi\left(*_\alpha \pi_1(A_\alpha)\right)$, and so Φ is surjective.

To prove the second assertion, we notice that we have proven $[f]$ can be factored into a product $[f_1][f_2]\cdots[f_n]$, with each f_i lying in some A_α, and the $*$ symbol for the product omitted for simplicity. We can see this, not as a product in $\pi_1(X)$, but as a word in $*_\alpha \pi_1(A_\alpha)$, mapped to the product by Φ.

We will say two such factorisations are equivalent if they are related by a sequence of these two operations:

- We combine $[f_i][f_{i+1}]$ into their product if they belong to the same group $\pi_1(A_\alpha)$.
- Regard some $[f_i]$ as an element of $\pi_1(A_\beta)$ instead of $\pi_1(A_\alpha)$ if f_i lies in $A_\alpha \cap A_\beta$.

We observe that the first operation doesn't change the element of $*_\alpha \pi_1(X_\alpha)$ we are considering, according to the definition of the operation in the free product, and, by definition of N, the second doesn't change the element of ${*_\alpha \pi_1(X_\alpha)} / {N}$ we consider, and so equivalent elements give the same element of the quotient.

If we can show that any two given factorisations of some $[f]$ are equivalent, we would deduce that the map from the quotient to $\pi_1(X)$ induced by Φ is injective, and so we have the isomorphism desired.

Let $[f_1][f_2]\cdots[f_n]$ and $[f'_1][f'_2]\cdots[f'_m]$ be two factorisations of $[f]$. Then, there exists a homotopy $F\colon f_1 * f_2 * \cdots * f_n \simeq_{\partial I} f'_1 * f'_2 * \cdots * f'_m$.

Now, the open sets of the form $(a,b)\times(c,d)$ form a neighbourhood basis for $I \times I$. As each A_α is open, $F^{-1}(A_\alpha)$ is open, and so $\forall (x,y) \in I \times I$, $\exists (a,b)\times(c,d) \subseteq I \times I$ neighbourhood of (x,y) such that $F\left((a,b)\times(c,d)\right) \subseteq A_{\alpha_{(x,y)}}$ for some $\alpha_{(x,y)}$. This means that there is an open cover for $I \times I$ made of open rectangles, with each one mapped by F to a single A_α. As $I \times I$ is compact, we will have a finite cover, and noting that the sets such a cover must overlap, one can take a cover made of smaller rectangles $[a,b]\times[c,d]$. Moreover, we can refine this cover, making two partitions $0 = s_0 < \cdots < s_n = 1$ and $0 = t_0 < t_2 < \cdots < t_n = 1$ fine enough that the rectangles $[s_i \times s_{i+1}] \times [t_j, t_{j+1}]$ cover $I \times I$ and is each mapped to a single $A_{\alpha_{i,j}}$. We may even assume, by adding more points to these partitions, that these rectangles subdivide the rectangles induced by the f_i and f'_i maps.

As F maps an open neighbourhood of these closed rectangles to a given A_α, the "horizontal" lengths can be perturbed, shrinking some rectangles and enlarging some, as to leave the property that each be mapped to a single A_α, and at each vertex of the partition only three rectangles meet, as in figure 3.

We may assume that there are at least three rows, as in figure 3, and only the intermediate rows are perturbed, the top and bottom rows still induced by refining the factorisations of the f_i and f'_i maps. We then number them row by row as in figure 4, the numbering increasing one by one as it runs from left to right in the same row.

Observe that a path γ lying in $I \times I$ from the left edge to the right edge, after composed with F, yields a loop based at x_0, as the left and right edges are mapped to x_0 by the relative

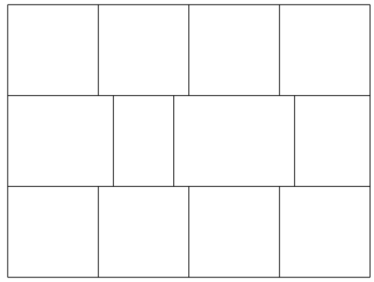


Figure 3: "Perturbed" partition of $I \times I$

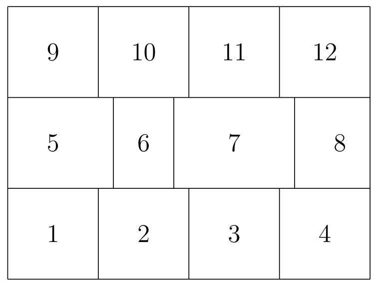


Figure 4: Numbered rectangles

homotopy hypothesis. We define a sequence $\{\gamma_r\}$ of paths in the following way: for a given r, γ_r is the path that leaves the first r rectangles "below" and the rest "above". Observe that γ_0 corresponds to one of the original factorisations, the last one to the other, and γ_{r+1} is computed by "pushing through the $r-$eth rectangle". If we are able to prove that γ_r and γ_{r+1} induce factorisations, and that they are equivalent for any r, we are done.

Let g_v be a path from x_0 to $F(v)$ for each vertex of the rectangles such that $F(v) \neq x_0$. As by the assumptions of the theorem, $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected for all α, β, γ and contains x_0, we can assume g_v to lie in the intersection of the A_{α_i} corresponding to the three rectangles that meet at v.

If we introduce $\overline{g_v} * g_v$ in $F \circ \gamma$ at these points as in the proof of surjectivity, we can induce a factorisation of $[F \circ \gamma]$, thinking of the segments and vertices of the rectangles as lying in either of the A_α the rectangles meeting at them map to. We observe that changes in the criteria for which rectangle a given edge belongs to, induces an equivalent factorisation by the second operation of the definition of the equivalence. Furthermore, "pushing through" a given rectangle corresponds to a homotopy contained inside the same A_α, and so the factor affected induces the same class as before in $\pi_1(A_\alpha)$, and the factorisations before and after the push remain equivalent.

Thus, there is a sequence of equivalent factorisations for $[f]$ from one given to any other, and the thesis follows. $\square$

There exist cases when van Kampen's theorem can hugely simplify the computing of certain fundamental groups. For example, when we can find the A_α such that each set is good enough on its own and the intersection is exactly $\{x_0\}$.

Definition 6.19. Let $(X, x_0), (Y, y_0)$ be based topological spaces. We define $(X, x_0) \vee (Y, y_0)$ the wedge sum or one-point sum of (X, x_0) and (Y, y_0) to be the space given from the disjoint union $X \sqcup Y$, by identifying $x_0 \sim y_0$.

This can be extended to arbitrary families $\{(X_\alpha, x_\alpha)\}_{\alpha\in\mathcal{A}}$, by taking the disjoint union $\bigsqcup_\alpha X_\alpha$, and taking quotient by identifying $x_\alpha \sim x_{\alpha'}$ $\forall \alpha, \alpha' \in \mathcal{A}$.

Example 6.20. *Let $X = \bigvee_\alpha (A_\alpha, x_\alpha)$ with each A_α path connected, and such that, for all α, $\exists U_\alpha \subseteq A_\alpha$ open neighbourhood of x_α such that $U_\alpha \simeq \{x_\alpha\}$. Then, X is path-connected, and $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)$.*

If we take each A_α to be some more concrete spaces, we can find some interesting results.

Example 6.21.

$$\pi_1\left(\bigvee_{\alpha\in\mathcal{A}} S^1\right) \cong *_{\alpha\in\mathcal{A}}\mathbb{Z}$$

Proof. We observe that for any point $x \in S^1$, it has a neighbourhood consisting of an open arc. This arc is homeomorphic to $\mathbb{R}$, and by proposition 3.12, $\mathbb{R} \simeq \{x\}$, so we can apply what we have seen in example 6.20 and conclude the proof using theorem 4.17. □

Example 6.22.

$$\pi_1(S^n) = 1, \ \forall n \geq 2$$

Proof. Consider some point $p \in S^n$, and some base-point $x_0 \in S^n \setminus \{p, -p\}$. If we consider $U = S^n \setminus \{p\}$ and $V = S^n \setminus \{-p\}$, we have $x_0 \in U \cap V$ and $X = U \cup V$. Moreover, $U, V, U \cap V$ are path-connected, and so we can apply Seifert-van Kampen's theorem. We have that there is a surjective homomorphism $\Phi\colon \pi_1(U) * \pi_1(V) \to \pi_1(S^n)$, but $\pi_1(U) = \pi_1(V) = 1$ because $S^n \setminus \{p\} \cong \mathbb{R}^n$ as given by the stereographic projection, and so $\Phi\colon 0 \to \pi_1(S^n)$ is surjective, implying $\pi_1(S^n) = 1$. □

There's another way to understand Seifert-van Kampen's theorem through the use of presentations, as quotients of free groups have a natural structure of presentation.

Definition 6.23. We say a group G has a presentation $\langle S|R\rangle$ if S is a set of elements, R is a set of words of the free group with the elements of S as generators, $Free(S) := *_{s\in S}\mathbb{Z}$, and

$$G \cong Free(S) \Big/ N_{Free(S)}(R)$$

where $N_{Free(S)}(R)$ is the normal closure of R in $Free(S)$ (the smallest normal subgroup of $Free(S)$ containing R).

Seifert-van Kampen's theorem can sometimes be naturally stated as yielding a presentation for $\pi_1(X)$, by "combining" presentations from each $\pi_1(A_\alpha)$. For this, S is induced by the set of generators of these presentations, and R is determined by the R_α from each presentation, together with the words of the group N from the statement of the theorem, which relate to how the combination is arranged.

However, this also shows the limitations of the techniques it induces on the computation of fundamental groups, as the problem of proving that a presentation corresponds to a given group, called the word problem for groups, has been proven to be undecidable (for more on this topic, see [18]). This means that Seifert-van Kampen's theorem can give a presentation for a group, that we cannot prove corresponds to said group.

Again, the computation of fundamental groups is hard in general, even in well-behaved cases, such as these, where we can apply Seifert-van Kampen's theorem.

Lemma 6.24. *Every group G has a presentation.*

Corollary 6.25. *For each group G, there exists a $2-$dimensional connected CW-complex X such that $\pi_1(X) \cong G$*

Proof. Let $\langle S|R\rangle$ be a presentation of G. Then, if we let $X^0 = \{x_0\}$, and $X^1 = \bigvee_{s\in S} S^1$, where x_0 is the base-point of each copy of S^1, $\pi_1(X^1) \cong Free(S)$. Let $N = N_{Free(S)}(R)$, and denote S_s the copy of S^1 associated to each $s \in S$.

Each element of N is a finite word $s_1 s_2 \cdots s_n$. Technically, $s_i \in \mathbb{Z}$ for some copy of $\mathbb{Z}$, but we expand the word so that either s_i or s_i^{-1} is in S, by representing the value n by n copies of the generator. If we understand s as a loop in the counter-clockwise direction around S_s, and $-s$ as a loop in the clockwise direction, each of these words represents a loop in X^1. If we attach a $2-$cell by identifying ∂D^2 to a parametrisation of this loop, the fundamental group of the resulting space is exactly the quotient of the fundamental group before by the group generated by the word, as this loop becomes homotopy equivalent to the constant map through the interior of the $2-$cell.

Doing this for each word in N results in a connected CW-complex with only one $0-$cell, as many $1-$cells as $|S|$, as many $2-$cells as $|R|$, and the desired fundamental group. □

This result is, in fact, a particular case for $n = 1$ of a more general result. There exist path-connected topological spaces such that any group appears as $\pi_n(X)$ (respecting the restriction that those of order higher than 1 be abelian).

Definition 6.26 (Eilenberg-MacLane spaces)**.** Let G be a group, and $n \in \mathbb{N}$. An Eilenberg-MacLane space $K(G, n)$ is a path-connected topological space such that $\pi_i(K(G, n)) = 0,\ \forall i \neq n$ and $\pi_n(K(G, n)) \cong G$.

Theorem 6.27. *$K(G, 1)$ exist for all G group, and $K(G, n)$ exist for all $n \geq 2$, and for all abelian groups G.*

The existence of Eilenberg-MacLane spaces is of great interest for the theory of Algebraic Topology, although we will not see instances of their application in this essay.

Although we will not give a proof of their existence here, it is a particular case of the constructions we will sketch in the context of CW-approximation for theorem 6.52. For the moment, we can observe that, for a choice of a group G_1 and a sequence $\{G_i\}_{i=2}^{\infty}$ of abelian groups, the space $\Pi_{i\in\mathbb{N}} K(G_i, i)$ is a path-connected topological space that has the desired sequence as its homotopy groups. This can help us visualise the true size of the category of homotopy types, $\pi(\mathbf{Top})$, as it has a class of objects at least as big as the class of these sequences. In fact, the class of homotopy types is bigger than this, as homotopy groups alone are not enough to characterise the homotopy type of a given topological space. We will discuss this in more detail in 6.5

6.4 Covering spaces

Another tactic for studying the fundamental group is generalising the map $p\colon \mathbb{R} \to S^1$ we employed in the proof of theorem 4.17, through the notion of covering spaces.

Definition 6.28 (Covering space)**.** Let X be a topological space. A covering space is a pair $(\tilde{X}, p)$, where $\tilde{X}$ is a topological space and $p\colon \tilde{X} \to X$ satisfies the condition that $\exists \{U_\alpha\}_{\alpha\in\mathcal{A}}$

open cover of X, such that for all $\alpha \in \mathcal{A}$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets of $\tilde{X}$, and if V is any such open set, $p|_V : V \to U_\alpha$ is homeomorphism.

Usually, we just say $p: \tilde{X} \to X$ is a covering space of X.

A covering space $p: \tilde{X} \to X$ is a continuous map. We observe, though, that the condition $p^{-1}(U_\alpha)$ be a disjoint union of open sets allows it to be empty, and so p may not be surjective. All results exposed here are nevertheless also true if p is required to be surjective, as some texts do also assume surjectivity in their definitions of covering maps, but we will stick to the one given by Hatcher, who gives a non-surjective definition in [1].

Example 6.29. *$p: \mathbb{R} \to S^1$ with $p(t) = \big(\cos(2\pi t), \sin(2\pi t)\big)$ is a covering space of S^1.*

Example 6.30. *$p: S^1 \to S^1$ with $p(z) = z^n$ (considering z as a complex number) is a covering space of S^1, $\forall n \in \mathbb{N}$.*

Definition 6.31. The real projective plane $\mathbb{R}P^2$ is the quotient space on S^2, $\frac{S^2}{\sim}$, that identifies together antipodal points, $x \sim -x$, $\forall x \in S^2$. Observe that the classes $[x] \in \mathbb{R}P^2$ are given by sets $\{x, -x\}$.

Example 6.32. *The quotient map $p: S^2 \to \mathbb{R}P^2$ given by $p(x) = [x]$ is a covering space of $\mathbb{R}P^2$.*

Proof. We observe that, for any open set $U \subseteq S^2$ such that $\forall x \in U$ we have $-x \notin U$ (equivalently, any open set contained in just one hemisphere), p maps U to $p(U)$ homeomorphically. Trivially, we can find a cover for $\mathbb{R}P^2$ as desired by constructing an open cover $\{\tilde{U}_\alpha\}_{\alpha\in\mathcal{A}}$ of S^2 such that each $\tilde{U}_\alpha$ is contained in just one hemisphere, and taking $\{p(\tilde{U}_\alpha)\}_{\alpha\in\mathcal{A}}$ open cover of $\mathbb{R}P^2$. □

Note that, through the first two examples, we have shown that covering spaces are not unique. Their classification is an interesting enough problem, and we will later indulge in showing some of its aspects.

Definition 6.33. Let $p: \tilde{X} \to X$ be a covering space, and $f: Y \to X$ be a continuous map. We say a continuous map $\tilde{f}: Y \to \tilde{X}$ is a lift of f if $f = p \circ \tilde{f}$.

We will now prove a proposition that generalises lemma 4.16.

Proposition 6.34 (Homotopy lifting property)**.** *Let $p: \tilde{X} \to X$ be a covering space, a homotopy $f_t: Y \to X$, and a map $\tilde{f}_0: Y \to \tilde{X}$ lifting f_0. Then, $\tilde{f}_0$ extends to a unique homotopy $\tilde{f}_t: Y \to \tilde{X}$ such that $\tilde{f}_t$ lifts f_t for all $t \in I$.*

Proof. We will first prove that, for any fixed $y \in Y$, $\exists N \subseteq Y$ open neighbourhood of y such that $\exists \tilde{f}_t: N \to \tilde{X}$ lift of $f_t|_N$, and such that it agrees with $\tilde{f}_0|_N$ for $t = 0$.
As $p: \tilde{X} \to X$ is a covering space, we have an associated open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of X. As it covers X, it means that, $\forall t \in I$, $\exists \alpha_t \in \mathcal{A}$ with $f_t(y) \in U_{\alpha_t}$. Because $F: Y \times I \to X$, the associated map of f_t, is continuous, we have that, for all $t \in I$, $F^{-1}(U_{\alpha_t})$ is open in Y and contains (y, t). This means that there is some open neighbourhood of (y, t) in Y, B, with $F(B) \subseteq U_{\alpha_t}$. In particular, as $Y \times I$ has a basis which is formed by products of open sets, we can take a neighbourhood of the form $N_t \times A_t$, for some basic open set $A_t \subseteq I$, which can be assumed to be of the form $[0, a)$, (a, b) or $(a, 1]$ for $a, b \in I$.

Now, as $\{y_0\} \times I$ is compact, and $\{N_t \times A_t\}_{t\in I}$ covers it, $\{y_0\} \times I$ can be covered with a finite amount of such neighbourhoods, for $0 = t_0 < t_1 < \cdots < t_n = 1$. We will denote A_i, U_i, N_i for simplicity.

Taking the non-empty intersection of the N_i, which is open due to the intersection being of finitely many open sets, we have an open neighbourhood N of y in Y such that there exist a finite family of open sets $\{N \times A_i\}_{i=0}^n$ that cover $N \times I$, and such that $F\big(N \times A_i\big) \subseteq U_i$ for some $U_i \in \{U_\alpha\}_{\alpha \in \mathcal{A}}$.

Moreover, $\{A_i\}_{i=1}^n$ cover I, which means that, because I is connected, and remembering each A_i is an interval, every such interval must intersect at least one other. This means we can take a partition $0 = t_0' < t_1' < \cdots < t_m = 1$ using points in the intersection of those intervals, such that, for each $[t_i, t_{i+1}]$, it is contained in at least one of the original intervals. This means we can take $\{N \times [t_i, t_{i+1}]\}_{i=0}^{m-1}$ to be the cover, and we still have that, for all i, $F\big(N \times [t_i, t_{i+1}]\big) \subseteq U_i$ for some $U_i \in \{U_\alpha\}_{\alpha \in \mathcal{A}}$.

Assume inductively that we have the desired lift in $[0, t_i]$, with associated map $\tilde{F}\colon N \times [0, t_i] \to \tilde{X}$. The base case is trivial, as $t_0 = 0$, and so the lifting corresponds to $\tilde{f}_t\colon N \to \tilde{X}$ for $t \in \{0\}$, that is, the original $\tilde{f}_0$.

Consider now $N \times [t_i, t_{i+1}]$. Remember that we have $F(N \times [t_i, t_{i+1}]) \subseteq U_i$ for some U_i. By definition of covering space, there exists a family of open sets $\{\tilde{U}_i^\alpha\}_\alpha$ with each $\tilde{U}_i^\alpha \subseteq \tilde{X}$ such that, for each α, $p|_{\tilde{U}_i^\alpha}\colon \tilde{U}_i^\alpha \to U_i$ is a homeomorphism. We know $p^{-1}(U_i)$ is non-empty, because by the inductive step we have already defined $\tilde{f}_{t_i}(y)$, which must be in this set, and by the same argument, we must have a unique $\tilde{U}_i^\alpha$, we will denote $\tilde{U}_i$, which contains $\tilde{f}_{t_i}(y)$, as the sets in $\{\tilde{U}_i^\alpha\}_\alpha$ are disjoint by definition.

This means that we can replace N with a new neighbourhood of y given by the intersection of N and $(F|_{N \times \{t_i\}})^{-1}(\tilde{U}_i)$. This is not a problem, as we will only repeat this process a finite number of times, and guarantees that $\tilde{F}(N \times \{t_i\}) \subseteq \tilde{U}_i$. Now, we can define $\tilde{F}\colon N \times [t_i, t_{i+1}] \to \tilde{X}$ as $(p_i^{-1} \circ F)|_{N \times [t_i, t_{i+1}]}$, where $p_i = p|_{\tilde{U}_i}$. Then, $\tilde{F}$ is defined for $N \times [0, t_{i+1}]$, because $[0, t_i], [t_i, t_{i+1}]$ are closed sets that cover $[0, t_{i+1}]$ and such that, by construction, our maps $\tilde{F}$ agree in $N \times \{t_i\}$, so by the lemma 3.2 the resulting map is continuous. This means that, for all $y \in Y$, we can find $N_y \subseteq Y$ open neighbourhood of y such that we can define the lift in the neighbourhood, $\tilde{f}_t\colon N_y \to \tilde{X}$.

To complete the proof, we will first assume $Y = \{y\}$, and prove the uniqueness of the lift. Assume there are two lifts of the given homotopy, $\tilde{f}_t, \tilde{g}_t\colon \{y\} \times I \to \tilde{X}$. We choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ as before such that $F(\{y\} \times [t_i, t_{i+1}]) \subseteq U_i$ for some $U_i \in \{U_\alpha\}_{\alpha \in \mathcal{A}}$. Assume inductively that $\tilde{f}_t = \tilde{g}_t$, $\forall t \in [0, t_i]$. The base case is trivial again, as $t_0 = 0$, and both coincide with the $\tilde{f}_0$ already given to us. As $[t_i, t_{i+1}]$ is connected, $\tilde{F}(\{y\} \times [t_i, t_{i+1}]) \subseteq \tilde{U}_i$ for $\tilde{F}$ the associated map of $\tilde{f}_t$, because the sets that map to U_i are disjoint by definition of covering space, and the continuous image of a connected space is connected. By the same argument, $\tilde{G}(\{y\} \times [t_i, t_{i+1}]) \subseteq \tilde{U}_i'$. But, as $t_i \in [t_i, t_{i+1}]$, and we have assumed in the inductive step that $\tilde{f}_{t_i} = \tilde{g}_{t_i}$, both images must lie in some $\tilde{U}_i$, which we have already shown before is unique, and so both images must lie in the same set. Now, $p \circ \tilde{F} = F = p \circ \tilde{G}$ by definition of the lift, and as p is injective because $p|_{\tilde{U}_i}\colon \tilde{U}_i \to U_i$ is homeomorphism, we have $\tilde{f}_t = \tilde{g}_t$ for all $t \in [t_i, t_{i+1}]$.

Now, as we have shown that the lift is unique for each point, and restrictions of continuous maps are continuous, we must have that any lift $\tilde{F}\colon N_y \times I \to \tilde{X}$ must coincide at every point with any other lift with the same domain, meaning the lift we found is unique for each $y \in Y$. As $\{N_y\}_{y \in Y}$ is trivially an open cover of Y, and we have that the map $\tilde{F}\colon N_y \times I \to \tilde{X}$ is unique for each y, including points in the intersection of two of these neighbourhoods. Trivially, by applying lemma 3.2, we can uniquely extend $\tilde{F}$ to the whole space, and we are done. □

We have already seen how this property can be used for our purposes in computing fundamental groups. If one were to observe it, they are to appreciate the sophistication in even the most basic computations of fundamental groups, as we remember we used a particular case of this result for theorem 4.17. Once again, we observe that the computations of fundamental groups are a hard affair in general.

A useful corollary of this theorem is the case when Y is a one-point space, which yields a path-lifting property.

Corollary 6.35. *Let $p\colon \tilde{X} \to X$ be a covering space, and $\alpha\colon I \to X$ be a continuous path. Then, $\forall \tilde{x} \in \tilde{X}$ such that $p(\tilde{x}) = \alpha(0)$, $\exists! \tilde{\alpha}\colon I \to \tilde{X}$ continuous path such that $\alpha = p \circ \tilde{\alpha}$ and $\tilde{\alpha}(0) = \tilde{x}$.*

Proof. It follows from proposition 6.34, by taking $Y = \{y\}$ a one-point space, $f_t\colon Y \to X$ given by $f_t(y) = \alpha(t)$ and $\tilde{\alpha}_0(y) = \tilde{x}$, as the homotopy lift guaranteed by the proposition, $\tilde{f}_t\colon Y \to \tilde{X}$, yields the path lift $\tilde{\alpha}\colon I \to \tilde{X}$ given by $\tilde{\alpha}(t) = \tilde{f}_t(y)$. □

We can give other properties that relate the fundamental group and covering spaces.

Proposition 6.36. *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space. Then, $p_*\colon \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is a monomorphism. The image $p_*\big(\pi_1(\tilde{X}, \tilde{x}_0)\big)$ in $\pi_1(X, x_0)$ consists of the homotopy classes of loops in X based at x_0 such that their lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are also loops.*

Proof. An element of *ker* p_* is a loop $\tilde{f}_0\colon I \to \tilde{X}$ such that $[p \circ \tilde{f}_0] = [e_{x_0}]$ in $\pi_1(X, x_0)$. This means that there is a homotopy $f_t\colon I \to X$ from $p \circ \tilde{f}_0$ to e_{x_0}, relative to ∂I. Now, by the homotopy lifting property, this means there is a lift of this homotopy to $\tilde{X}$, $\tilde{f}_t\colon I \to \tilde{X}$, such that $\tilde{f}_1(s) = \tilde{x}_0$ for all $s \in I$. This is a homotopy from $\tilde{f}_0$ to $e_{\tilde{x}_0}$. We can see that, as f_t is relative to ∂I, $f_t(0)$ and $f_t(1)$ are constant for all $t \in I$, and so when lifting as in the proof of proposition 6.34, the lift will be constant at $\tilde{f}_t(0)$ and $\tilde{f}_t(1)$ for each $t \in I$, implying that it is relative to ∂I as a homotopy, and so the kernel of p_* is trivial.

To check the second assertion, it is obvious that loops in (X, x_0) lifting to loops based at $\tilde{x}_0$, are representatives of the class we get from applying p_* to their lift by the definition of p. We get the other implication by just noting that an element of $p_*\big(\pi_1(\tilde{X}, \tilde{x}_0)\big)$ is represented by a loop such that it is the result of applying p to a loop in $(\tilde{X}, \tilde{x}_0)$, which is the desired lift. □

The relationship between these two fundamental groups can also encode other geometric aspects of the relationship between a space and a covering space of it. For example, in connected spaces, it can measure the "relative size" of both spaces, in the sense of the cardinality of $p^{-1}(x)$ for all $x \in X$.

Definition 6.37. Let $p\colon \tilde{X} \to X$ be a covering space, with $X, \tilde{X}$ path-connected spaces. Then, we define $|p^{-1}(x)|$ the number of sheets of the covering (the reason why we attribute this number to the covering, and not the particular choice of x, will become apparent shortly).

Proposition 6.38. *Let $p\colon \tilde{X} \to X$ be a covering space, with $X, \tilde{X}$ path-connected spaces. Then, the number of sheets of the covering is equal to the index $\Big[\pi_1(X)\colon p_*\big(\pi_1(\tilde{X})\big)\Big]$. In particular, it is constant as x ranges over X.*

Proof. Let α be a loop based at x_0 in X. Consider its lift to the covering space, $\tilde{\alpha}$, based at $\tilde{x}_0$ in $\tilde{X}$, as given by corollary 6.35. We observe that, although the lift starts at $\tilde{x}_0$, it is not guaranteed to be a loop, as the endpoint might be another point $\tilde{x} \in \tilde{X}$ with $p(\tilde{x}) = x_0$. If we take some $[h] \in p_*\big(\pi_1(\tilde{X}, \tilde{x}_0)\big) = H$, we can consider a representative h of this class. Observe that the fact that $[h]$ is in the image of p_* implies that we can choose this representative so that it has a lift $\tilde{h}\colon I \to \tilde{X}$ which is a loop.

We can now consider $h * \alpha$. We know that, by corollary 6.35, there exists a path lift of α, $\tilde{\alpha}$, such that $\tilde{\alpha}(0) = \tilde{x}_0$, which is also the endpoint of $\tilde{h}$, as it is a loop, and so $\tilde{h} * \tilde{\alpha}$ suffices as the unique lift of $h * \alpha$ with starting point $\tilde{x}_0$.

This means that $\tilde{\alpha}$ and $\tilde{h} * \tilde{\alpha}$ have the same end point. If we now consider the map

$$\begin{aligned} \Phi\colon \pi_1(X, x_0) \big/ H &\longrightarrow p^{-1}(x_0) \\ H[g] &\longmapsto \tilde{g}(1) \end{aligned}$$

where $\tilde{g}$ is the unique path lifting g with starting point $\tilde{x}_0$, we have that it is well defined, as $\Phi([\alpha]) = \Phi([h] * [\alpha])$ for all $[h] \in H$.

As $\tilde{X}$ is path-connected, there will be a path $\tilde{g}$ such that $\tilde{g}(0) = \tilde{x}_0$ and $\tilde{g}(1) = y$ for all $y \in p^{-1}(x_0)$, which means there is some $g = p \circ \tilde{g}$ loop in X based at x_0 with $\Phi(H[g]) = y$, and so Φ is surjective. To show injectivity, we observe that $\Phi(H[g_1]) = \Phi(H[g_2])$ implies $\tilde{g_2}(1) = \tilde{g_1}(1)$. Taking $\tilde{g_1} * \overline{\tilde{g_2}}$, it is a loop in $\tilde{X}$, and so $g_1 * \overline{g_2}$ lifts to a loop, which means $[g_1] * [g_2]^{-1} \in H$ and $H[g_1] = H[g_2]$. □

Example 6.39.

$$\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$$

Proof. Let $p\colon S^2 \to \mathbb{R}P^2$ be a covering space of $\mathbb{R}P^2$, given by the quotient map in example 6.32. We observe that, for some $[x] \in \mathbb{R}P^2$, $p^{-1}([x]) = \{x, -x\}$. This means $|p^{-1}(\{x, -x\})| = 2$, $\forall [x] \in \mathbb{R}P^2$.

Then, the cover has 2 sheets, which means that $[\pi_1(\mathbb{R}P^2)\colon p_*(\pi_1(S^2))] = 2$. Now, $\pi_1(S^2)$ is trivial by example 6.22, which means $p_*(\pi_1(S^2))$ is also trivial, and so $|\pi_1(\mathbb{R}P^2)| = 2$. But there is only one group of cardinal 2, $\mathbb{Z}_2$, completing the proof. □

The fact that the number of sheets of a covering space is related to the index of a subgroup of π_1, could induce us to raise some questions about the nature of the relationship between covering spaces $p\colon \tilde{X} \to X$, and subgroups of $\pi_1(X)$ of the form $p_*\left(\pi_1(\tilde{X})\right)$. For example, is every subgroup of $\pi_1(X)$ of this form?

As it turns out, there is a rich theory on the classification of covering spaces that gives a positive answer to this question for certain sufficiently nice spaces.

Definition 6.40. We say that a topological space X is semilocally simply-connected if, $\forall x \in X$, $\exists U \subseteq X$ neighbourhood of x, such that the map $i_*\colon \pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion $U \hookrightarrow X$ is trivial.

Lemma 6.41 (Lifting criterion)**.** *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space, and a map $f\colon (Y, y_0) \to (X, x_0)$ with Y a path-connected and locally path-connected topological space. Then, $\exists \tilde{f}\colon (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ that lifts f if and only if $f_*\Big(\pi_1(Y, y_0)\Big) \subseteq p_*\Big(\pi_1(\tilde{X}, \tilde{x}_0)\Big)$.*

Proof. Considering that $f = p \circ \tilde{f}$, it is trivial that, if such a lift exists,

$$f_*\Big(\pi_1(Y, y_0)\Big) = (p_* \circ \tilde{f}_*)\Big(\pi_1(Y, y_0)\Big) \subseteq p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right) \subseteq \pi_1(X, x_0)$$

To prove the other implication, for each $y \in Y$, let $\alpha\colon I \to Y$ be a path between y_0 and y. Then, $f \circ \alpha\colon I \to X$ is a path in X. By corollary 6.35, there exists a unique lift $\widetilde{f \circ \alpha}$ such that $\widetilde{f \circ \alpha}(0) = \tilde{x}_0$. We define the map $\tilde{f}\colon (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ by $\tilde{f}(y) = \widetilde{f \circ \alpha}(1)$.

We have to prove that this assignment is independent of the choice of α. Let α' be another path between y_0 and y. Then, $h_0 = f \circ (\alpha * \overline{\alpha'})$ is a loop in (X, x_0). As $[h_0] \in f_*(\pi_1(Y, y_0)) \subseteq p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$, $\exists \tilde{h}_1 \colon (I, \partial I) \to (\tilde{X}, \tilde{x}_0)$ a loop on $\tilde{X}$ such that $p_*([\tilde{h}_1]) = [h_0]$ in $\pi_1(X, x_0)$. This implies that, taking $h_1 = p \circ \tilde{h}_1$, there exists a relative homotopy $H \colon h_0 \simeq_{\partial I} h_1$. By the homotopy lifting property, we have a homotopy lift of H, $\tilde{H}$, and since $\tilde{h}_1$ is a loop in $\tilde{X}$ based at $\tilde{x}_0$, and the homotopy $\tilde{H}$ preserves endpoints, $\tilde{h}_0 \colon I \to \tilde{X}$ given by $\tilde{h}_0(0) = \tilde{H}(x, 0)$ is also a loop, which is a lift of h_0. As the lifts of $f \circ \alpha, f \circ \alpha'$ are unique, this implies $\tilde{h}_0$ is the unique lift of $f \circ (\alpha * \overline{\alpha})$, and in particular, the end-points of the lifts of $f \circ \alpha, f \circ \alpha'$ are equal to the midpoint of $\tilde{h}_0$, so $\tilde{f}$ is well defined.

To prove the continuity, let $y \in Y$ be a point, and take a neighbourhood $U \subseteq X$ of $f(y)$. As $p \colon \tilde{X} \to X$ is a covering space, can take it such that there is an open set $\tilde{U} \subseteq \tilde{X}$ such that $p|_{\tilde{U}} \colon \tilde{U} \to U$ is homeomorphism, and $\tilde{f}(y) \in \tilde{U}$. As Y is locally path-connected, we can take an open path-connected neighbourhood of y, $V \subseteq Y$, such that $f(V) \subseteq U$.

We fix α a path between y_0 and y, and for each $y' \in V$, with a path α' from y_0 to y', we find a path γ from y to y'. As γ is contained in V, $f \circ \gamma$, which is a path from $f(y)$ to $f(y')$, is contained in $f(V) \subseteq U$. If we consider the fact that $p|_{\tilde{U}} \colon \tilde{U} \to U$ is a homeomorphism, it admits a continuous inverse, $p_U^{-1} \colon U \to \tilde{U}$. This allows us to consider $p_U^{-1} \circ f \circ \gamma \colon I \to \tilde{U}$, which is trivially a path lift of $f \circ \gamma$. As by corollary 6.35 we have uniqueness of the lift, this implies that $\tilde{f}(y') \in \tilde{U}$, and so $\tilde{f}(V) \subseteq \tilde{U}$, as the choice of point y' was arbitrary. This means that $\tilde{f}|_V = p_U^{-1} \circ f$, which is continuous.

If we consider each choice of $y \in Y$, this means that for each one there is an open neighbourhood V_y such that $\tilde{f}_y = \tilde{f}|_{V_y}$ is continuous. Observe that this family of open sets covers Y, and so, by lemma 3.2, we have that $\tilde{f}$ is continuous. □

Proposition 6.42. *Let X be a path-connected, locally path-connected and semilocally simply-connected topological space. Then, $\exists p \colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ covering space such that $\tilde{X}$ is simple connected.*

Proof. Before we begin to construct this covering space, assume that it exists to study its form, let $p \colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space with $\tilde{X}$ simple connected. Observe that, for any $\tilde{x} \in \tilde{X}$, if we take two paths $\alpha, \alpha' \colon I \to \tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}$, $\alpha * \overline{\alpha'}$ is a loop, which is homotopy equivalent to $e_{\tilde{x}_0}$ relative to ∂I by the condition that $\tilde{X}$ is simple connected. This means that $\alpha * \overline{\alpha'} \simeq_{\partial I} e_{\tilde{x}_0} \implies \alpha * \overline{\alpha'} * \alpha' \simeq_{\partial I} e_{\tilde{x}_0} * \alpha' \implies \alpha \simeq_{\partial I} \alpha'$, and so it is clear that there is a single class in $[I, \tilde{X}]_{\partial I}$ for the paths between $\tilde{x}_0$ and some given point $\tilde{x}$. This means that the points in $\tilde{X}$ are in correspondence with the classes in $[I, \tilde{X}]_{\partial I}$ of paths starting at $\tilde{x}_0$, and $\tilde{X}$ can be purely described by these classes.

Now, if this covering space exists, we expect by corollary 6.35 that each path on X from x_0 to some point x be lifted to a path on $\tilde{X}$ from $\tilde{x}_0$ to some point $\tilde{x}$, and we now know that this point $\tilde{x}$ can be described solely by the homotopy class of this path. Consider

$$\tilde{X} = \{[\gamma]_{\partial I} \mid \gamma \colon I \to X \text{ path with } \gamma(0) = x_0\}$$

Consider also the map $p \colon \tilde{X} \to X$ given by $p([\gamma]_{\partial I}) = \gamma(1)$. This map is clearly well defined. We will give a topology on $\tilde{X}$ that makes p a covering space and $\tilde{X}$ simple connected. We will from now on write $[\gamma]$ instead of $[\gamma]_{\partial I}$ for the sake of simplicity.

Let $\mathcal{U}$ be the collection of path-connected open sets $U \subseteq X$ such that $i_* \colon \pi_1(U) \to \pi_1(X)$ induced by the inclusion is trivial. We see that any path-connected open subset of U, $V \subseteq U$, is also in $\mathcal{U}$, as $i_* \colon \pi_1(V) \to \pi_1(X)$ factors as $i_* \colon \pi_1(V) \to \pi_1(U) \to \pi_1(X)$. This means that, if X is locally path-connected and semilocally simple connected, $\mathcal{U}$ is a basis for the topology on X. We will use it to define a basis on $\tilde{X}$.

Let $U \in \mathcal{U}$, and $\gamma\colon I \to X$ be a path from x_0 to some point in U. We define $U_{[\gamma]} = \{[\gamma * \eta] \mid \eta\colon I \to U$ a path such that $\eta(0) = \gamma(1)\}$. We see that $p|_{U_{[\gamma]}}\colon U_{[\gamma]} \to U$ is surjective, as U is path-connected, and injective, as $i_*\colon \pi_1(U) \to \pi_1(X)$ is trivial, implying all paths $\eta, \eta'\colon I \to U$ joining $\gamma(1)$ and $\eta(1)$ are homotopy equivalent relative to ∂I in X.

We also observe that, if $[\gamma'] \in U_{[\gamma]}$, then $U_{[\gamma']} = U_{[\gamma]}$, as if $\gamma' = \gamma * \eta$, with η lying in U, then the elements of $U_{[\gamma']}$ are of the form $[\gamma * \eta * \nu] = [\gamma * (\eta * \nu)]$, implying $U_{[\gamma']} \subseteq U_{[\gamma]}$, and elements of $U_{[\gamma]}$ are of the form $[\gamma * \nu] = [\gamma * \eta * \overline{\eta} * \nu] = [\gamma' * (\overline{\eta} * \nu)]$, and so $U_{[\gamma]} \subseteq U_{[\gamma']}$.

Finally, we are able to describe the topology, after we see that, if we consider

$$\mathcal{U}' = \{U_{[\gamma]} \mid U \in \mathcal{U}, [\gamma] \in [I, X] \text{ with } \gamma(0) = x_0\}$$

we have that $\mathcal{U}'$ is a basis for a topology on $\tilde{X}$. Indeed, it covers $\tilde{X}$, and if we consider two $U_{[\gamma]}, V_{[\gamma']} \in \mathcal{U}'$, and an element $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, we have $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$ by what we have already seen, and so if $W \in \mathcal{U}$ is contained in $U \cap V$ and contains $\gamma''(1)$, then $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']} = U_{[\gamma]} \cap V_{[\gamma']}$ and $[\gamma''] \in W_{[\gamma'']}$.

We can see that p is continuous under the topology given by $\mathcal{U}'$ as a basis. Indeed, we just have to prove that $p|_{U_{[\gamma]}}\colon U_{[\gamma]} \to U$ is a continuous map and apply lemma 3.2. But we can see that $p|_{U_{[\gamma]}}$ is a homeomorphism, as it induces a bijection between the subsets $V_{[\gamma']} \subseteq U_{[\gamma]}$ and the sets $V \in \mathcal{U}$ with $V \subseteq U$. Indeed, $p(V_{[\gamma']}) = V$, and in the other direction, $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$ for any $[\gamma'] \in U_{[\gamma]}$ with endpoint in V.

We can also verify that p is a covering space, as the sets $U_{[\gamma]}$ for varying γ partition $p^{-1}(U)$, and if they are not disjoint, they are equal, as previously seen, so it only remains to see that $\tilde{X}$ is simple connected.

We first show that $\tilde{X}$ is path-connected. Let $[\alpha] \in \tilde{X}$ be any point in $\tilde{X}$. We must show there is a path between $[e_{x_0}]$ and $[\alpha]$.

For this, consider the following map $F\colon I \times I \to X$, as given by

$$F(s,t) = \begin{cases} \alpha(s) & \text{if } s \in [0,t] \\ \alpha(t) & \text{if } s \in [t,1] \end{cases}$$

We have that, for a fixed t, $f_t\colon I \to X$, given by $f_t(s) = F(s,t)$ is a path on X starting at x_0, and so $[f_t]$ is a point in $\tilde{X}$. We have that $f_0 = e_{x_0}$ and $f_1 = \alpha$, and so the map $g\colon I \to \tilde{X}$ given by $g(t) = [f_t]$ is the desired path on $\tilde{X}$, as it can be shown to be continuous, so it suffices.

To prove that $\pi_1(\tilde{X}) = 1$, we just show that $p_*(\pi_1(\tilde{X})) = 1$, as p_* is injective because p is a covering space. Elements in $p_*(\pi_1(\tilde{X}))$ are loops γ on X that lift to loops in $\tilde{X}$. If we substitute α with γ in the definition of the path g used before, this path lifts γ, and it is a loop if and only if $[\gamma] = g(1) = g(0) = [e_{x_0}]$, which implies $[\gamma] = [e_{x_0}]$, and so p_* is trivial. □

Theorem 6.43. *Let (X, x_0) be a path-connected, locally path-connected and semilocally simply-connected based topological space. Then, there is a bijective correspondence between path-connected based covering spaces $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ up to pointed homeomorphism of covering spaces (that is, homeomorphisms $f\colon (\tilde{X}, \tilde{x}_0) \to (\tilde{X}', \tilde{x}_0')$ such that $p = p' \circ f$) and subgroups of $\pi_1(X)$, given by $(\tilde{X}, p) \to p_*\Big(\pi_1(\tilde{X})\Big)$.*

Moreover, if we consider a topological space X with the same conditions, and a point $x_0 \in X$, although not necessarily a based topological space, there is a correspondence between *covering spaces* $p\colon \tilde{X} \to X$ *(up to homeomorphisms such* $f\colon \tilde{X} \to \tilde{X}'$ *such that* $p = p' \circ f$*) and the conjugacy classes of subgroups of* $\pi_1(X, x_0)$.

We will not give a full proof, but we will at least sketch how these covering spaces are. If we define $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma * \overline{\gamma'}] \in H$ for some $H \subseteq \pi_1(X)$, we see that this is an equivalence relation, as H is a group. It can be seen that $\tilde{X}_H$ the quotient of $\tilde{X}$ given by this equivalence relation, and $p\colon \tilde{X}_H \to X$ given by $p([\gamma]) = \gamma(1)$, is a covering space, and $p_*(\pi_1(\tilde{X})) = H$, by an argument similar to that showing the simple-connection of $\tilde{X}$, but now noting that the loops that admit a lift to a loop are those for which $[\gamma] \sim [e_{x_0}]$, that is, those for which $[\gamma] \in H$.

We can observe some distinguished covering spaces with respect to this result. For example, normal subgroups H of a given group G have only one conjugacy class. The covering spaces associated with these classes are denoted normal covering spaces. These covering spaces are interesting because, if we consider their deck transformations, that is, homeomorphisms $f\colon \tilde{X} \to \tilde{X}$ such that $p = p \circ f$, these are precisely the spaces such that for any two lifts $\tilde{x}, \tilde{x}'$ of a point $x \in X$, there is a deck transformation with $f(\tilde{x}) = \tilde{x}'$.

Ommiting the map p for brevity, the deck transformations of a covering space $\tilde{X}$ form a group, $G(\tilde{X})$, which, although we will not prove this here, is isomorphic to $N(H)/H$, where H is the subgroup of $\pi_1(X, x_0)$ associated to the covering space, and $N(H)$ is the normaliser of H in $\pi_1(X, x_0)$. If we understand $G(\tilde{X})$ as a group of symmetries of $\tilde{X}$, we see that for normal covering spaces, $G(\tilde{X}) \cong \pi_1(X, x_0)/H$, and so the study of the symmetries of these covering spaces is interesting for the study of the fundamental group.

It is also of interest to observe the simple connected covering spaces. By the previous results, for a sufficiently nice topological space X, a simple connected covering space exists, and is unique up to deck transformations. It is called the *universal covering space* of X, and is universal in the sense that it is a covering space of every other path-connected covering space of X. It is also closely related to the fundamental group of X. Observe that its group of deck transformations is isomorphic to the fundamental group of X. This implies that the computation of $\pi_1(X)$ is equivalent to the study of the symmetries of the universal covering space of X.

Although we will not fully prove this theorem, or develop the classification of covering spaces any further (see [1] for more detail on these results), we can observe how reminiscent this is of the Galois connection between subextensions of a Galois extension, and subgroups of its Galois group. We can even observe the connection between the notion of normal covering spaces, and normal field extensions. This relationship stems from the fact that both theories can be generalised through the concept of Galois categories, and in the context of Algebraic Geometry and schemes, the Galois group becomes a nice analogue for the fundamental group (see [2] for more on this).

Applying what we have seen on classification of covering spaces to the examples we know, for S^1, $\mathbb{R}$ is its universal covering space, as is S^2 for $\mathbb{R}P^2$.

Moreover, as the subgroups of $\mathbb{Z}$ are the trivial subgroups and $n\mathbb{Z}$ for $n \geq 2$, we know that the path-connected covering spaces of S^1 are those we have already shown, being $p\colon \mathbb{R} \to S^1$ the one corresponding to the trivial subgroup, the identity $id\colon S^1 \to S^1$ the one corresponding to $\mathbb{Z}$, and corresponding to $n\mathbb{Z}$ the covering spaces $p_n\colon S^1 \to S^1$ given by $p_n(z) = z^n$.

As for $\mathbb{R}P^2$, as $\mathbb{Z}_2$ only has the trivial subgroups, there are only two path-connected covering spaces up to homeomorphism: S^2 and itself.

If we ask ourselves how covering spaces relate to higher homotopy groups, they work nicely with them.

Proposition 6.44. *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space. It induces isomorphisms $p_*\colon \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$, $\forall n \geq 2$.*

Proof. Surjectivity follows from the lifts guaranteed by lemma 6.41, as by the universal property of the quotient space each continuous map $f\colon (I^n, \partial I^n) \to (X, x_0)$ factors through S^n, which is the quotient obtained from I^n by identifying together the points of ∂I^n. This means that, if $k\colon (I^n, \partial I^n) \to (S^n, *)$ is the quotient map, each such continuous map f factors as $(I^n, \partial I^n) \xrightarrow{k} (S^n, *) \xrightarrow{f'} (X, x_0)$ for some continuous map $f'\colon (S^n, *) \to (X, x_0)$.

As $\pi_1(S^n) = 1$, $\forall n \geq 2$ as seen in example 6.22, for all $f'\colon (S^n, *) \to (X, x_0)$, we have that $f'_*(\pi_1(S^n)) = f'_*(1) = 1 \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$, which implies that f' admits a lift $\tilde{f}'\colon (S^n, *) \to (\tilde{X}, \tilde{x}_0)$, and so $[f] = p_*([\tilde{f} \circ k])$, proving surjectivity.

Injectivity of p_* is proven in a way analogous to the injectivity in the case $n = 1$. □

6.5 Whitehead's theorem and CW approximation

Computing higher homotopy groups is a hard problem in general, as is computing fundamental groups. But, while there are results that are analogous to those used for computing fundamental groups, these turn out to be weaker in general than their $n = 1$ counterparts. One example is the excision theorem for homotopy groups.

The excision theorem references the namesake excision property, which is a general result in the case of homology groups. In the case of homotopy groups, the excision theorem is a partial analogue for CW-complexes.

Theorem 6.45 (Excision theorem for homotopy groups)**.** *Let X be a CW-complex, decomposed as the union of A, B subcomplexes, such that $C = A \cap B \neq \emptyset$. If (A, C) is $m-$connected, and (X, B) is $n-$connected, $m, n \geq 0$, then the map $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < m + n$, and an epimorphism for $i = m + n$.*

Theorem 6.45 will be left unproven, as the proof found in Hatcher's *Algebraic Topology*, together with the proof of some of the lemmas employed in it, results too technical for the use we will give it. (For the interested reader, see theorem 4.23 in [1]).

This result can be thought of as an analogue for a Seifert-van Kampen's theorem for $n > 1$, but is weaker in a number of ways. For example, it only applies to CW-complexes, allows for only finite indexing in the components in which you decompose the space, and the relative homotopy groups you compute are determined by the decomposition you make.

However, the fact that results like theorem 6.45 only work for CW-complexes doesn't mean they are not of interest, as CW-complexes are important spaces in the context of Homotopy Theory. This is not only due to the many interesting topological spaces (like spheres, tori and projective planes) that admit a CW-complex structure, and due to the fact we have previously stated that they have some good properties over general topological spaces, but also because of some interesting results involving their homotopy types and their homotopy equivalences.

Definition 6.46. Let $f\colon (X, x_0) \to (Y, y_0)$ be a pointed map. We say f is a weak homotopy equivalence if $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ are isomorphisms for all $n \in \mathbb{N}$.

Lemma 6.47 (Compression lemma)**.** *Let* $(X, A), (Y, B)$ *be CW-pairs with* $B \neq \emptyset$*. Assume that* $\forall n \in \mathbb{N}$ *such that* $X \setminus A$ *has cells of size* n*, we have* $\pi_n(Y, B, y_0) = 0$ $\forall y_0 \in B$*. Then,* $\forall f : (X, A) \to (Y, B)$ *continuous map,* $f \simeq_A g$ *for some continuous map* $g : X \to B$*.*

Proof. We will do a proof by induction by proving that if we have some continuous map $f_n : (X, A) \to (Y, B)$ such that $f_n(X^n) \subseteq B$, then, $f_n \simeq_{A \cup X^n} f_{n+1}$ for some $f_{n+1}(X, A) \to (Y, B)$ such that $f_{n+1}(X^{n+1}) \subseteq B$. Then, we would be done, as a global homotopy can be constructed inductively as the one in the proof of proposition 6.6, and the base case is trivial.

Assume that $f_n : (X, A) \to (Y, B)$ is such that $f_n(X^n) \subseteq B$. For some cell e^{n+1}_α of $X \setminus A$, if $\Phi_\alpha : (D^{n+1}, \partial D^{n+1}) \to (X^{n+1}, X^n)$ is its characteristic map, we can consider the composition $f \circ \Phi_\alpha : (D^{n+1}, \partial D^{n+1}) \to (Y, B)$.

As there is a homeomorphism $p : (I^{n+1}, \partial I^{n+1}) \to (D^{n+1}, \partial D^{n+1})$, we have a continuous map $f \circ \Phi_\alpha \circ p : (I^{n+1}, \partial I^{n+1}) \to (Y, B)$. As $\pi_{n+1}(Y, B, y_0) = 0$ by hypothesis for all y_0, $[f \circ \Phi_\alpha \circ p] = 0$, and by lemma 6.9, this means $H : f \circ \Phi_\alpha \circ p \simeq_{\partial I^{n+1}} g$ with $g : (I^{n+1}, J^n) \to (B, y_0)$, and in particular $g(I^{n+1}) \subseteq B$. We then have $H \circ p^{-1} : f \circ \Phi_\alpha \simeq_{\partial D^{n+1}} g'$ with $g'(D^{n+1}) \subseteq B$.

Now, the map $G : (X^n \sqcup D^{n+1}) \times I \to Y$ given by

$$G(x, t) = \begin{cases} f_n(x) & \text{if } x \in X^n \\ (H \circ p^{-1})(x, t) & \text{if } x \in D^{n+1} \end{cases}$$

is a homotopy, as it is continuous on two disjoint open sets that cover the space and applying lemma 3.2 in its version for open sets, and induces a homotopy on the quotient $X^n \cup e^{n+1}_\alpha$ as the homotopy was relative to the borders, and so it is constant with respect to t on every pair of points identified. Doing this for each $n+1-$cell of $X \setminus A$ simultaneously, and taking the constant homotopy on those in A, we have that $f_n \simeq_{A \cup X^n} g$ with $g(X^{n+1}) \subseteq B$, and by taking $f_{n+1} = g$ we are done. □

Theorem 6.48 (Whitehead's theorem)**.** *If* X, Y *are CW-complexes, and* $f : X \to Y$ *is a weak homotopy equivalence, then* f *is a homotopy equivalence. Moreover, if* $f : X \hookrightarrow Y$ *is the inclusion map of a subcomplex and it is a weak homotopy equivalence,* X *is a deformation retract of* Y*.*

Proof. In the special case when $f : X \hookrightarrow Y$ is the inclusion of a subcomplex, by applying proposition 6.10, we have an exact sequence

$$\cdots \to \pi_{n+1}(X, A, x_0) \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots$$

We observe that i_* is the map induced by the inclusion, which we know is isomorphism, and so $im\ i_* = \pi_n(A, x_0)$, which implies $ker\ j_* = \pi_n(A, x_0) \cong \pi_n(X, x_0)$, and so $j_* \equiv 0$. This means that $ker\ \partial = im\ j_* = 0$, and so ∂ is monomorphism, but $im\ \partial = ker\ i_* = 0$, and so $\pi_n(X, A, x_0) = 0$ for all $n \in \mathbb{N}$. Applying lemma 6.47 to $id_Y : (Y, X) \to (Y, X)$, we have the deformation retract.

For the general case, we use spaces called mapping cylinders. For a given continuous map $f : X \to Y$, the mapping cylinder M_f is defined as the quotient of the space $(X \times I) \sqcup Y$ by identifying $(x, 1) \sim f(x)$, for all $x \in X$. M_f deformation retracts into Y in the natural way, and so as $f = r \circ i$ with $i : X \cong X \times \{0\} \hookrightarrow M_f$ the inclusion, it suffices to show M_f deformation retracts into X.

First, we show that $\pi_n(M_f, X, x_0) = 0$ for all $n \in \mathbb{N}$ and all $x_0 \in X$. Observe that for $[f] \in \pi_n(M_f, X, x_0)$, the map $H\colon I^{n+1} \times I \to M_f$ given by

$$H(x,t) = ((x_1, x_2, \cdots, x_n), tx_{n+1})$$

is a homotopy relative to ∂I^{n+1} such that $H(I^{n+1}, 1) \subseteq X$, and so by lemma 6.9, $[f] = 0$.

Now, if we consider $X \cup Y \subseteq M_f$ by taking $(X \times \{0\}) \cup (Y \times \{1\})$, applying lemma 6.47 to the inclusion $i\colon (X \cup Y, X) \hookrightarrow (M_f, X)$, we have a homotopy $G\colon i \simeq_X g$ for some $g\colon X \cup Y \to M_f$ such that $g(X \cup Y) \subseteq X$. Now, if we prove that the pair $(M_f, X \cup Y)$ has the homotopy extension property, we are done, as if we take $\tilde{g}_0\colon M_f \to M_f$ to be equal to the identity, and $g_t\colon X \cup Y \to M_f$ given by $g_t(x) = G(x,t)$, there would be a homotopy extension $\tilde{g}_t\colon M_f \to M_f$ with $\tilde{g}_0 = id_{M_f}$ and $\tilde{g}_1(M_f) \subseteq X$, which would render the required deformation retract from M_f into X.

The homotopy extension property, by lemma 5.9, is equivalent to $((X \cup Y) \times I) \cup (M_f \times \{0\})$ being a retract of $M_f \times I$. We have that

$$M_f \times I = ((X \times I) \cup (Y \times \{1\})) \times I = (X \times I^2) \cup (Y \times \{1\} \times I)$$

$$(M_f \times \{0\}) \cup (X \cup Y) \times I = (X \times I \times \{0\}) \cup (Y \times \{1\} \times \{0\}) \cup (X \times \{0\} \times I) \cup (Y \times \{1\} \times I) =$$
$$= \big(X \times ((I \times \{0\}) \cup (\{0\} \times I))\big) \cup \big(Y \times \{1\} \times I\big)$$

after the required identifications of points. Trivially, we just have to prove $(I \times \{0\}) \cup (\{0\} \times I)$ is a retract of I^2. By lemma 5.10, $(I \times \{0\}) \cup (\{0,1\} \times I)$ is a retract of I^2, and trivially a second retract from it into the subspace $(I \times \{0\}) \cup (\{0\} \times I)$ can be found. □

Observe that theorem 6.48 does not imply that two CW-complexes are homotopy equivalent if their homotopy groups are equivalent. The condition that there exists a map $f\colon X \to Y$ that induces the isomorphisms cannot be dropped in general.

Example 6.49. *Both $S^3 \times \mathbb{R}P^2$ and $S^2 \times \mathbb{R}P^3$ (the construction of $\mathbb{R}P^3$ is analogous to that of $\mathbb{R}P^2$, by identifying antipodes in S^3) have isomorphic homotopy groups. However, they are not homotopy equivalent, as the isomorphisms between the homotopy groups are not induced by a single continuous map.*

Proof. This can be shown by showing $\pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$ by an argument equal to that of the proof of example 6.39, as S^3 is a simply-connected covering space of $\mathbb{R}P^3$ as given by the quotient map, and seeing that both products have covering space $S^2 \times S^3$, which implies that all higher homotopy groups are isomorphic by proposition 6.44.

Through homology groups, one could build an argument that these spaces are not homotopy equivalent. We will not do so in this essay, as homology groups fall out of its scope. □

Whitehead's theorem can be related to an interesting property of CW-complexes called CW approximation.

Definition 6.50 ($n-$connected CW-pair)**.** We say a CW-pair is $n-$connected if it is $n-$connected as a pair of topological spaces.

Definition 6.51 (CW model)**.** Let (X, A) be a pair of topological spaces, with the subspace $\emptyset \neq A \subseteq X$ being a CW-complex. An $n-$connected CW model for (X, A) is a continuous map $f\colon (Z, A) \to (X, A)$ with (Z, A) an $n-$connected CW-pair such that $f\colon Z \to X$ follows that $f|_A = id_A$ and induces isomorphisms $f_*\colon \pi_i(Z) \to \pi_i(X)$, $\forall i > n$, and a monomorphism $f_*\colon \pi_n(Z) \to \pi_n(X)$.

Theorem 6.52 (CW approximation)**.** *$n-$connected CW-models exist for every pair (X, A) of a topological space and a non-empty CW-complex, and every $n \geq 0$. Furthermore, it can be assumed that Z is obtained from A by attaching cells of dimension greater that n.*

The term CW approximation comes from the fact that, if $n = 0$ and A is chosen to be a set composed of a single point for each path-component of X, an $n-$connected CW model becomes a CW-complex together with a weak homotopy equivalence from Z to X. Paired with Whitehead's theorem, we observe that a CW approximation of a CW-complex is a CW-complex of the same homotopy type.

Corollary 6.53. *Let (X, A) be an $n-$connected CW-pair. Then, $\exists$ (Z, A) $n-$connected CW-pair with $Z \simeq_A X$ and $Z \setminus A$ having only cells of size greater than n.*

Although we will not prove the CW approximation theorem, a sketch of the proof is interesting in its own right, as it includes interesting ideas about how Homotopy Theory is done on CW-complexes. It relies heavily on cellular maps and cellular approximations, and the main idea is that, for CW-complexes, adding cells of enough dimension doesn't affect the homotopy groups for low values of n.

Definition 6.54 (Cellular map)**.** Let $f\colon X \to Y$ be a continuous map between CW-complexes. We say f is cellular if $f(X^n) \subseteq Y^n$, $\forall n \in \mathbb{N}$.

Theorem 6.55 (Cellular approximation)**.** *Every continuous map $f\colon X \to Y$ between CW-complexes is homotopic to a cellular map. Moreover, if $A \subseteq X$ is a subcomplex such that $f|_A$ is cellular, the homotopy can be assumed to be relative to A.*

The observation of cellular approximation is that, when considering classes of maps up to homotopy between CW-complexes, large cells can be ignored. Indeed, when considering maps like those used in the construction of homotopy groups $f\colon (I^n, \partial I^n) \to (X, x_0)$, we observe that, if X is a CW-complex, these maps are homotopic to maps $f\colon (I^n, \partial I^n) \to (X^n, x_0)$ relative to ∂I^n. This means that adding cells of size n to a CW-complex won't make the groups $\pi_i(X)$ any bigger for $i < n$.

Moreover, CW-complexes allow for a "fine tuning" of their homotopy groups. The last observation implies that (X, X^n) is an $n-$connected CW-pair, by applying lemma 6.9. If we consider the long exact sequence of the pair (X, X^n),

$$\cdots \to \pi_{i+1}(X, X^n, x_0) \to \pi_i(X^n, x_0) \xrightarrow{i_*} \pi_i(X, x_0) \xrightarrow{j_*} \pi_i(X, X^n, x_0) \xrightarrow{\partial} \pi_{i-1}(X^n, x_0) \to \cdots$$

We observe that $im\, \partial = 0$ for small enough i, and so $ker\, i_* = 0$, implying it is a monomorphism. Moreover, as $im\, j_* = 0$, $ker\, j_* = im\, i_* = \pi_i(X, x_0)$, and so i_* is isomorphism.

In fact, $\pi_i(X, x_0) \cong \pi_i(X^n, x_0)$, $\forall i < n$, and $i_*\colon \pi_n(X^n, x_0) \to \pi_n(X, x_0)$ is an epimorphism.

All of these observations together imply that adding cells of size $n+1$ to a CW-complex leaves the homotopy groups $\pi_i(X, x_0)$ constant for $i < n$, and can only make $\pi_n(X, x_0)$ smaller than it already is.

CW approximation consists of inductively fixing the homotopy groups, by assuming that for $n \in \mathbb{N}$ we have a map $f_n\colon Z^n \to X$ such that f_* is isomorphism on each π_i for $i < n$ and epimorphism for π_n, adding cells that correspond to the elements of the kernel of f_* for $i = n$ as in the proof of proposition 6.25, and guaranteeing surjectivity for the next step while keeping π_n constant. The map f can be extended in a natural way using the characteristic maps of the CW-complex, and an inductive proof of its continuity follows in a natural way, as its restriction to each $n-$skeleton is clearly continuous.

CW approximation shows in an indirect way that the condition of X, Y being CW-complexes in Whitehead's theorem cannot be dropped. Indeed, if it was so, any topological space would be homotopy equivalent to a CW-complex, but there are topological spaces that are not. For example, the Alexandroff line, sometimes called the long line (topological space given by the product $\mathbb{R} \times [0,1)$, together with the order topology given by lexicographic order), has trivial homotopy groups, and would need to be contractible, as the inclusion $i\colon * \hookrightarrow \mathbb{R} \times [0,1)$ is a weak homotopy equivalence, but its non-contractibility is a classical example in the literature.

Proposition 6.56. *Let X, X' be topological spaces, and $A \subseteq X$,$A' \subseteq X'$ CW-complexes. Assume we are given the following:*

- *An $n-$connected CW-model $f\colon (Z, A) \to (X, A)$.*
- *An $n'-$connected CW-model $f'\colon (Z', A') \to (X', A')$.*
- *A continuous map $g\colon (X, A) \to (X', A')$.*

Then, if $n \geq n'$, there exists a continuous map $h\colon Z \to Z'$ such that $h|_A = g|_A$ and such that $g \circ f \simeq_A f' \circ h$. Furthermore, h is unique up to homotopy relative to A.

This proposition implies that, when $n = n'$, there exists a unique $n-$connected CW-model for a pair (X, A) up to homotopy type.

This means, in particular, that the homotopy types of CW-complexes model the classes of topological spaces up to weak homotopy equivalence through CW-approximation. In fact, many authors prefer to use the category $Ho(\mathbf{Top}^*)$ instead of $\pi(\mathbf{Top}^*)$, usually also called homotopy category, which is analogous to our category in that, while the projection $\eta\colon \mathbf{Top}^* \to \pi(\mathbf{Top}^*)$ sends homotopy equivalences to isomorphisms, weak homotopy equivalences correspond to isomorphisms in $Ho(\mathbf{Top}^*)$. This category is obtained through what's called a process of localisation, by selectively adding morphisms so that a morphism has inverse if it is a weak homotopy equivalence. In this category, the classes modulo isomorphism correspond to the classes modulo homotopy equivalence of CW-complexes, as expected in light of Whitehead's theorem.

This is, for example, the motivation behind Grothendieck's Homotopy Hypothesis, which asserted that the definition of the category $\infty - \mathbf{Grpd}$ of $\infty-$groupoids, which hadn't been formally defined up to that moment, needed to account for it being equivalent to this homotopy category as an $(\infty, 1)-$category, in the context of Higher Category Theory (see [7]).

6.6 Freudenthal suspension theorem. Stable Homotopy Theory

An interesting topic to study is how certain operations on topological spaces affect their homotopy groups. We have seen that the homotopy groups "respond well" to products of spaces, but have unpredictable behaviours when presented with subspaces and quotients in general. There are, however, particular quotients that have interesting properties.

Definition 6.57. Let X be a topological space. Its suspension, SX, is defined as the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ and $X \times \{1\}$ each to one point.

If X is a CW-complex, we have the following result.

Theorem 6.58 (Freudenthal suspension theorem)**.** *If X is an $(n-1)-$connected CW-complex, then $\pi_i(X) \cong \pi_{i+1}(SX)$ for all $i < 2n-1$, and there exists a surjection $\pi_{2n-1}(X) \to \pi_{2n}(SX)$. These maps are called suspension maps.*

Proof. We observe that $SX \cong C_+X \cup C_-X$, where C_+X is the quotient of $X \times [\frac{1}{2}, 1]$ by collapsing the points $(x, 1)$, and C_-X is analogous by collapsing the points $(x, 0)$ in $X \times [0, \frac{1}{2}]$. These spaces are called cones over X.

Then, the map is given by

$$\pi_i(X) \cong \pi_{i+1}(C_+X, X) \to \pi_{i+1}(SX, C_-X) \cong \pi_{i+1}(SX)$$

where the two isomorphisms come from the long exact sequence of homotopy groups (observe that $i_*(\pi_i(X)) = 0$, as a homotopy from any continuous map $f\colon Y \to X \hookrightarrow C_+X$ (analogously for C_-X) to a constant map can be built by "sliding" the image towards a vertex). The proof is completed by showing the middle map is isomorphism for $i+1 < 2n$ and epimorphism for $i+1 = 2n$ by applying theorem 6.45 to the decomposition in C_+X and C_-X. □

Freudenthal suspension theorem has an interesting consequence, which follows from this lemma.

Lemma 6.59. *If X is a path-connected topological space, $\pi_1(SX) = 0$*

Proof. If we consider C_-X, C_+X path-connected open sets, and $x_0 \in C_-X \cap C_+X$, we have that Seifert-van Kampen's theorem guarantees an epimorphism $\pi_1(C_-X) * \pi_1(C_+X) \to \pi_1(SX)$. As $C_-X \cong C_+X$, we just have to prove $\pi_1(C_-X) = 0$.

Let $f\colon I \to C_-X$ be a path. Then, it factors through a map $\tilde{f}\colon I \to X \times [0, \frac{1}{2}]$. Let f_1, f_2 be such that $\tilde{f}(x) = (f_1(x), f_2(x))$. Then, the homotopy $H\colon I \times I \to X \times [0, \frac{1}{2}]$ given by

$$H(x, t) = (f_1(x), (1-t)f_2(x))$$

is such that $H(x, 0) = \tilde{f}(x)$, $\forall x \in X$, and $H(x, 1) = (f_1(x), 0)$, which is then mapped by the quotient map to the vertex for all $x \in X$, and so the thesis follows. □

This will have, together with Freudenthal suspension theorem, the consequence that if X is an $(n-1)-$connected CW-complex, SX is an $n-$connected CW-complex. Indeed, if X is at least $0-$connected, by the previous lemma, SX will be $1-$connected, and Freudenthal suspension theorem guarantees that all homotopy groups up to n of SX will be 0, as were the homotopy groups up to $n-1$ of X.

Now, this leads to an interesting concept known as stable homotopy. We have that each successive suspension of a topological space X will be more connected, and the suspension theorem guarantees isomorphisms of homotopy groups up to $2n-1$. As the second grows faster than the first, this implies that, at some point, iterative applications of the suspension map stabilise.

This means that

$$\lim_{k \to +\infty} \pi_{i+k}(S^k X) = \pi_i^S(X)$$

is well-defined, and reached in a finite number of steps, after which the sequence becomes constant. These groups π_i^S are called the stable homotopy groups of X, and their study induces the field of Stable Homotopy Theory, an interesting subfield of Homotopy Theory.

Lemma 6.60.

$$SS^n \cong S^{n+1},\ \forall n \geq 0$$

As we have hinted through this essay, the homotopy groups of spheres are, contrary to what could be expected, not trivial in general. Their computation is a difficult task in general, and one of the first results of Homotopy Theory proper was, as stated in the introduction, Hopf's proof that $\pi_3(S^2)$ was infinite, which was an unexpected result at the time.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	$\cdots$
S^0	0	0	0	0	0	0	0	0	0	
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	
$\cdots$										

Figure 5: Some homotopy groups for spheres of small dimension. Coloured, those groups that are isomorphic with those in the same diagonal due to Freudenthal suspension theorem.

However, Freudenthal suspension theorem can simplify the task somewhat, as spheres of dimension $n+1$ are obtained as suspensions of spheres of dimension n. This means that, when put in a table like in figure 5, Freudenthal suspension theorem guarantees the existence of diagonals composed of isomorphic groups, as coloured in the chart.

With Freudenthal suspension theorem at hand, we are finally able to prove one last proposition we have carried since we first introduced the relative homotopy groups.

Proposition 6.61. *There exist $x_0 \in A \subseteq X$ such that $\pi_2(X, A, x_0)$ is not abelian.*

Proof. By example 6.21, we know $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$. This group is not abelian, as $ab \neq ba$ by the definition of the free product. Consider $X = S^3$ and A any subspace of X homeomorphic to $S^1 \vee S^1$. If we prove $\pi_1(X) = \pi_2(X) = 0$, by corollary 6.11, we would have $\pi_2(X, A) \cong \pi_1(A)$, which is not abelian.

We have that, by theorem 6.58 and example 6.3, $\pi_2(S^3) \cong \pi_1(S^2) = 0$ and $\pi_1(S^3) = 0$, so we are done. $\square$

7 Conclusion

As we hope to have shown, Homotopy Theory is a rich field, with a variety of applications to more general Topology, and various other fields of Mathematics. In this essay, however, we have been limited to giving only a superficial approach to the subject, more specially when referring to specific subfields of it, like the classification of covering spaces, or the field of stable homotopy.

If the interested reader would want to read more on the subject, multiple avenues are available to them. For example, basic results in Algebraic Topology, like Hurewicz theorem, have been left out, as they would have required to also develop the field of Singular Homology along that of Homotopy. The relationships between Homotopy, Singular Homology and Singular Cohomology are an interesting topic on their own, and some of the sources cited for this essay (in particular [10] and [1]) develop to some extent this theory. It is also interesting to cite some works that, although have not been used for this essay, are classical references when studying the topic (see [12] and [13]).

Another more abstract road for expanding the topic of Algebraic Topology, is the study of more general theories, as Algebraic Topology has been widely generalised. For example, the study of general theories of homology was firstly axiomatised by Eilenberg and Steenrod (see [3]) and it is now common to find analogous theories for more general categories (see [6], [5], [19]). If one wanted to lean on the abstraction even harder, Homotopy Type Theory is a field that studies the internal logic of higher categories through homotopical notions, and offers an intuitionistic alternative to usual Set Theory and Propositional Logic as foundations for Mathematics, in a way that is very comfortable to implement programmatically and has been widely used as a setting to implement programming languages for automatic formal verification, like Agda (see [15]).

Again in more abstract terms, the correspondence between the subgroups of the fundamental group and covering spaces develops a connection which can be studied through a lens analogous to that of the study of Galois Theory, inducing very general categorical approaches to both theories (see [4], [2]).

Other path one could take, is a further study of the homotopy of CW-complexes. The theory of Postnikov towers and Whitehead towers is developed in Hatcher's *Algebraic Topology* ([1]) to some extent. CW-complexes are also crucial for more general Algebraic Topology, as their homology, for example, is also of great interest, and they are important for our understanding of weaker definitions of homotopy types, like that given by weak homotopy equivalences, that have shaped the theory of categories with weak equivalences (see [16]).

In conclusion, Homotopy Theory, though based in some very simple definitions and notions, is a massive field of study, with many approaches, applications and really hard and technical arguments and computations. Algebraic Topology, and Homotopy Theory in particular, are without a doubt some of the most interesting fields of contemporary Mathematics.

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