The notion of cusp forms for a class of reductive symmetric spaces of split rank one

Erik P. van den Ban, Job J. Kuit* and Henrik Schlichtkrull

Abstract

We study a notion of cusp forms for the symmetric spaces G/H with $G = \mathrm{SL}(n,\mathbb{R})$ and $H = \mathrm{S}\big(\mathrm{GL}(n-1,\mathbb{R})\times \mathrm{GL}(1,\mathbb{R})\big)$. We classify all minimal parabolic subgroups of G for which the associated cuspidal integrals are convergent and discuss the possible definitions of cusp forms. Finally, we show that the closure of the direct sum of the discrete series of representations of G/H coincides with the space of cusp forms.

Introduction

In this article we investigate the convergence of certain integrals that can be used to give a notion of cusp forms on the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{S}\big(\mathrm{GL}(n-1,\mathbb{R})\times \mathrm{GL}(1,\mathbb{R})\big)$, which we here denote by X_n. Furthermore, we determine the relation between the discrete series representations and the space of cusp forms for these spaces.

Harish-Chandra defined a notion of cusp forms for reductive Lie groups and proved that the space of cusp forms coincides with the closed span in the L^2-Schwartz space of the discrete series of representations. This fact plays an important role in his work on the Plancherel decomposition. In [1] a notion of cusp forms for real hyperbolic spaces was introduced, following a more general suggestion by M. Flensted-Jensen. Subsequently, in [4], the first and second author gave a definition of cusp forms for split rank 1 reductive symmetric spaces. However, the notion of [4] deviates from the general suggestion of Flensted–Jensen at an important point. The main purpose of the present article is to explore the necessity of this deviation.

In order to give a precise description of the purpose of the present article we first recall some background. Let G be semisimple and let G/H be a symmetric space of split rank 1; here H is an open subgroup of the group of fixed points of an involution σ of G. Every minimal parabolic subgroup P of G contains a σ-stable maximal split connected abelian subgroup A_P of G. The set of minimal parabolic subgroups decomposes into two disjoint sets: the set $\mathcal{P}$ of P such that $A_P/(A_P\cap H)$ is 1 dimensional and the set $\mathcal{Q}$ of P such that $A_P/(A_P\cap H)$ is 0 dimensional, i.e., $A_P\subseteq H$.

*Supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).

The main goal is now to identify a suitable class of minimal parabolic subgroups P with the property that for every ϕ in the Harish-Chandra Schwartz space $\mathcal{C}(G/H)$ the integral

$$\int_{N_P/(N_P\cap H)} \phi(n)\,dn \tag{I.1}$$

is absolutely convergent. Here N_P denotes the unipotent radical of P and dn is an N_P-invariant Radon measure on $N_P/(N_P \cap H)$. A cusp form is then defined to be a function $\phi \in \mathcal{C}(G/H)$ such that

$$\int_{N_P/(N_P\cap H)} \phi(gn)\,dn = 0 \tag{I.2}$$

for every such parabolic subgroup P and every $g \in G$.

Flensted-Jensen has suggested to use the set of parabolic subgroups that can be characterized as follows,

$$\mathcal{P}_* := \{P \in \mathcal{P} : \dim(N_P \cap H) = \max_{Q\in\mathcal{P}} \dim(N_Q \cap H)\}.$$

These parabolic subgroups are said to be $\mathfrak{h}$-extreme, see [4, Def. 1.1] for an equivalent characterization.

In [1] it was confirmed for real hyperbolic spaces that the integral is absolutely convergent for $P \in \mathcal{P}_*$ and for every $\phi \in \mathcal{C}(G/H)$.

In [4] a notion called $\mathfrak{h}$-compatibility was introduced for parabolic subgroups $P \in \mathcal{P}$ by imposing a condition on the roots that are positive for P. For the spaces X_n this condition is recalled in Definition 2.5. It was proved in [4] that for $\mathfrak{h}$-compatible parabolic subgroups $P \in \mathcal{P}$ the integrals (I.1) are absolutely convergent. Let

$$\mathcal{P}_\mathfrak{h} := \{P \in \mathcal{P} : P \text{ is } \mathfrak{h}\text{-compatible}\}.$$

For real hyperbolic spaces this set equals $\mathcal{P}_*$, but in general this need not be the case. The difference occurs for example for the symmetric spaces X_n. If $n \geq 4$ then $\mathcal{P}_*$ is not contained in $\mathcal{P}_\mathfrak{h}$, and if $n \geq 3$ is odd then $\mathcal{P}_\mathfrak{h}$ is not contained in $\mathcal{P}_*$. Therefore, the family X_n is a good test-case for determining whether or not the need for $\mathfrak{h}$-compatible parabolic subgroups in [4] is an artefact of the proof, and whether or not the dimension of $N_P \cap H$ is relevant for the convergence of the integrals.

In Section 1 we describe some generalities concerning parabolic subgroups. The results in this section hold for any reductive symmetric space. In Section 2 we describe the polar decomposition of X_n and the Harish-Chandra Schwartz space $\mathcal{C}(X_n)$ of X_n. Our main results are formulated in Section 3 and proved in the remaining sections.

The first main result (Theorem 3.1) is a classification of all minimal parabolic subgroups P such that the integral (I.1) is absolutely convergent for all $\phi \in \mathcal{C}(X_n)$. We extend the notion of $\mathfrak{h}$-compatibility to all minimal parabolic subgroups P (not just those in $\mathcal{P}$, but also the ones in $\mathcal{Q}$). We then show that (I.1) is absolutely convergent for all $\phi \in \mathcal{C}(X_n)$ if and only if P is $\mathfrak{h}$-compatible.

The second main result (Theorem 3.2) describes the behavior at infinity of the function

$$\mathcal{H}_P\phi : A_P \to \mathbb{C}; \qquad a \mapsto a^{\rho_P} \int_{N_P/(N_P\cap H)} \phi(an)\,dn, \tag{I.3}$$

for $P \in \mathcal{P}_{\mathfrak{h}} \cap \mathcal{P}_*$ and $\phi \in \mathcal{C}(X_n)$. In particular it is shown that if n is even, then $\mathcal{H}_P\phi$ is rapidly decreasing; if n is odd, then $\mathcal{H}_P\phi$ is rapidly decreasing in one direction, while it converges to a possibly non-zero limit in the other direction. Moreover, for every $\mathfrak{h}$-compatible $P \in \mathcal{Q}$ there exists a $P' \in \mathcal{P}_{\mathfrak{h}} \cap \mathcal{P}_*$ and a $g \in G$ such that the integral (I.1) equals a limit of $\mathcal{H}_{P'}\big(\phi(g\,\cdot\,)\big)$.

As a consequence of Theorem 3.2, we will show in Proposition 3.3 that if (I.2) holds for all $P \in \mathcal{P}_{\mathfrak{h}} \cap \mathcal{P}_*$, then (I.2) holds for all $\mathfrak{h}$-compatible parabolic subgroups. This and Theorem 3.1 justifies the use of $\mathcal{P}_{\mathfrak{h}} \cap \mathcal{P}_*$ in the definition of cusp forms for reductive symmetric space of split rank 1.

In [1] it was shown that there exist discrete series representations for certain real hyperbolic spaces for which the generating functions are not cusp forms. These discrete series representations are called non-cuspidal. Using the results of [4] and the estimates from Theorem 3.2 for the behavior at infinity of (I.3), we show that for X_n there are no non-cuspidal discrete series representations. Our final main result (Theorem 3.6) is thus that the space of cusp forms on X_n coincides with the closed span of the discrete series of representations for X_n.

For other papers concerning the particular symmetric space X_n, see for example [12], [11], [6], [7] and [16].

We would like to thank Mogens Flensted-Jensen for many fruitful discussions related to the present work.

1 Parabolic subgroups and split components

In this preliminary chapter we collect some properties which are valid for general reductive symmetric spaces G/H. Here G is a reductive Lie group of the Harish-Chandra class and H is an open subgroup of the group of fixed points for an involution σ of G. We are concerned with properties of σ-stable connected split abelian subgroups and parabolic subgroups of G. The main result is that every parabolic subgroup P of G contains a σ-stable maximal split abelian subgroup A of G, which is unique up to conjugation by an element of $N_P \cap H$.

1.1 Split abelian subalgebras

We write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ for the eigenspace decomposition for the infinitesimal involution σ. Here $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{q}$ is the -1 eigenspace. Recall that an abelian subspace $\mathfrak{a}$ of $\mathfrak{g}$ is called split if $\mathfrak{g}$ decomposes as a sum of joint $\mathfrak{a}$-weight spaces.

Lemma 1.1. *Let $\mathfrak{a}$ be a σ-stable maximal split abelian subalgebra of $\mathfrak{g}$. Then there exists a σ-stable Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{a} \subseteq \mathfrak{p}$.*

Proof. From the construction in [10, Thms. 6.10, 6.11] it follows that there exists a Cartan involution θ_0 such that $\theta_0(X) = -X$ for $X \in \mathfrak{a}$. Since $\mathfrak{a}$ is σ-stable, the construction in the proof of [8, Thm. III.7.1], applied to θ_0 and σ, gives a Cartan involution θ which in addition commutes with σ. □

Definition 1.2. We define $\mathcal{A}_{\mathfrak{q}}$ to be the set of all maximal split abelian subspaces $\mathfrak{b}$ of $\mathfrak{q}$, and $\mathcal{A}$ to be the set of all maximal split abelian subalgebras $\mathfrak{a}$ of $\mathfrak{g}$ for which $\mathfrak{a} \cap \mathfrak{q} \in \mathcal{A}_{\mathfrak{q}}$. The *split rank* of G/H is the dimension of any $\mathfrak{b} \in \mathcal{A}_{\mathfrak{q}}$ (it will follow from Corollary 1.5 below that this is well defined).

Note that H acts on $\mathcal{A}_{\mathfrak{q}}$ and $\mathcal{A}$ by conjugation.

Lemma 1.3. *If $\mathfrak{a} \in \mathcal{A}$, then $\mathfrak{a}$ is σ-stable.*

Proof. Let $Y \in \mathfrak{a}$. Then Y and σY both commute with $\mathfrak{a} \cap \mathfrak{q}$ and hence $Y - \sigma Y \in \mathfrak{a} \cap \mathfrak{q}$ by maximality. Hence, $\sigma Y \in \mathfrak{a}$. □

Proposition 1.4. *A subspace $\mathfrak{a} \subseteq \mathfrak{g}$ belongs to $\mathcal{A}$ if and only if there exists a σ-stable Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and $\mathfrak{a} \cap \mathfrak{q}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. The action of H on $\mathcal{A}$ by conjugation is transitive.*

Proof. Let $\mathcal{A}'$ denote the set of subspaces $\mathfrak{a} \subseteq \mathfrak{g}$ for which there exists a σ-stable Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and $\mathfrak{a} \cap \mathfrak{q}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. It follows from [14, Lemmas 4,7] that H acts transitively on $\mathcal{A}'$.

It follows from Lemma 1.1 and Lemma 1.3 that $\mathcal{A} \subseteq \mathcal{A}'$. Hence, $\mathcal{A} = \mathcal{A}'$. □

Corollary 1.5. *A subspace $\mathfrak{b} \subseteq \mathfrak{q}$ belongs to $\mathcal{A}_{\mathfrak{q}}$ if and only if there exists a σ-stable Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{b}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. The action of H on $\mathcal{A}_{\mathfrak{q}}$ by conjugation is transitive.*

Proof. Let $\mathfrak{b} \in \mathcal{A}_{\mathfrak{q}}$ and let $\mathfrak{a} \in \mathcal{A}$ with $\mathfrak{b} = \mathfrak{a} \cap \mathfrak{q}$. The asserted Cartan decomposition exists according to Proposition 1.4.

Conversely, let $\mathfrak{b} \subseteq \mathfrak{q}$ and assume that $\mathfrak{b}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$ for some σ-stable Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace with $\mathfrak{a} \supset \mathfrak{b}$. Then $\mathfrak{a} \cap \mathfrak{q} = \mathfrak{b}$ by maximality. It follows from Proposition 1.4 that $\mathfrak{a} \in \mathcal{A}$ and hence $\mathfrak{b} \in \mathcal{A}_{\mathfrak{q}}$.

The transitivity of the action follows from the corresponding statement in Proposition 1.4. □

1.2 Parabolic subgroups

We recall that if P and Q are parabolic subgroups, then Q is called opposite to P if $P \cap Q$ is a common Levi subgroup of P and Q. If P is a parabolic subgroup we write N_P for its unipotent radical. Recall also that a *split component* of P is a maximal connected split subgroup of the center of a Levi subgroup of P.

Note that a parabolic subgroup P is minimal if and only if the Lie algebras of its split components are maximal split in $\mathfrak{g}$. Note also that if $\mathfrak{a}$ is a maximal split abelian subalgebra of $\mathfrak{g}$ and $A = \exp(\mathfrak{a})$, then the normalizer $N_G(A)$ of A in G acts by conjugation on the set of minimal parabolic subgroups containing A. This action is transitive.

Lemma 1.6. *Let P be a minimal parabolic subgroup. If A and B are two split components of P, then there exists a unique $n \in N_P$ such that $B = nAn^{-1}$.*

Proof. There exists $g \in G$ such that $B = gAg^{-1}$. Then P and $g^{-1}Pg$ both contain A and hence $g^{-1}Pg = wPw^{-1}$ for some $w \in N_G(A)$. The product gw normalizes P and thus belongs to P. The existence of n now follows by decomposing this element according to the Levi decomposition $P = Z_G(A)N_P$.

Now assume that $n_1, n_2 \in N_P$ satisfy $B = n_{1,2}An_{1,2}^{-1}$. Then $n_1^{-1}n_2$ centralizes A and thus $n_1^{-1}n_2 \in Z_G(A) \cap N_P = \{e\}$. This proves uniqueness. □

Lemma 1.7. *Let P and Q be minimal parabolic subgroups. Assume that P and Q have common split components A and B. Then there exists a unique $n \in N_P \cap N_Q$ such that $B = nAn^{-1}$.*

Proof. By Lemma 1.6 there exists a unique $n \in N_P$ such that $B = nAn^{-1}$. It suffices to show that $n \in N_Q$. Let $a \in A$ be dominant with respect to P. Then for every $k \in \mathbb{N}$ we have $a^{-k}na^kn^{-1} \in a^{-k}B \subseteq Q$. Since Q is a closed subgroup and $a^{-k}na^kn^{-1}$ converges to n^{-1} for $k \to \infty$, it follows that $n \in Q$. Furthermore, since $n \in N_P$, the element $\log(n)$ is a sum of $\mathfrak{a}$-weight vectors with non-zero weights. This implies that $n \in N_Q$. □

Theorem 1.8. *Let P be a minimal parabolic subgroup.*

(i) There exists a σ-stable split component A of P.

(ii) If A and B are two σ-stable split components of P, then there exists a unique $n \in N_P \cap H$ such that $B = nAn^{-1}$.

Proof. See [17, Lemma 12] for a proof that uses [15] and only applies to algebraic groups. An alternative proof is given in [9, Lemma 2.4], also under the assumption of algebraic groups. The statement *(i)* is proved without this assumption in [14, Lemma 2], and hence we only need to prove *(ii)*.

If A and B are two σ-stable split components of P, then by Lemma 1.7 there exists a unique $n \in N_P \cap \sigma(N_P)$ such that $B = nAn^{-1}$. Observe that

$$B = \sigma(B) = \sigma(n)\sigma(A)\sigma(n)^{-1} = \sigma(n)A\sigma(n)^{-1}.$$

From the uniqueness of n we conclude that $\sigma(n) = n$, or equivalently, $n \in N_P \cap H$. □

The preceding theorem allows for the following definition.

Definition 1.9. Let P be a minimal parabolic subgroup and let A be a σ-stable split component of P. We define the σ-parabolic rank of P to be the dimension of $A/(A \cap H)$.

We write $\mathcal{P}$ for the set of minimal parabolic subgroups of maximal σ-parabolic rank. (Note that $P \in \mathcal{P}$ if and only if P admits a split component $A = \exp \mathfrak{a}$ with $\mathfrak{a} \in \mathcal{A}$.) Furthermore, we write $\mathcal{Q}$ for the set of all minimal parabolic subgroups of minimal σ-parabolic rank. If $\mathfrak{a} \in \mathcal{A}$, then we write $\mathcal{P}(\mathfrak{a})$ for the set of minimal parabolic subgroups containing $\exp \mathfrak{a}$. Note that $\mathcal{P}(\mathfrak{a}) \subseteq \mathcal{P}$. If $\mathfrak{b}$ is a σ-stable maximal split abelian subalgebra of $\mathfrak{g}$ such that $\mathfrak{b} \cap \mathfrak{h}$ has maximal dimension, then we write $\mathcal{Q}(\mathfrak{b})$ for the set of minimal parabolic subgroups containing $\exp \mathfrak{b}$. Note that $\mathcal{Q}(\mathfrak{b}) \subseteq \mathcal{Q}$.

Corollary 1.10.

(i) Let $\mathfrak{a} \in \mathcal{A}$ and let $P \in \mathcal{P}$. Then there exists an element $h \in H$ such that $hPh^{-1} \in \mathcal{P}(\mathfrak{a})$.

(ii) Let $\mathfrak{b}$ be a σ-stable maximal split abelian subalgebra of $\mathfrak{g}$ such that $\mathfrak{b} \cap \mathfrak{h}$ has maximal dimension. Then for every $Q \in \mathcal{Q}$ there exists an element $h \in H$ such that $hQh^{-1} \in \mathcal{Q}(\mathfrak{b})$.

Proof. Ad (i): There exists a $\mathfrak{b} \in \mathcal{A}$ such that $\exp \mathfrak{b} \subseteq P$. By Proposition 1.4 there exists an $h \in H$ such that $A = hBh^{-1}$. This implies that A is contained in hPh^{-1}.

Ad (ii): The claim follows from the fact that all such subalgebras are H-conjugate. See Lemma 1.1 and [14, Lemmas 4,7]. □

1.3 Positive systems of $\mathfrak{h}$-roots

Let A be a σ-stable connected maximal split abelian subgroup of G and let $\mathfrak{a}$ be its Lie algebra. We write $\Sigma(\mathfrak{a})$ for the root system of $\mathfrak{a}$ in $\mathfrak{g}$. Given a root $\alpha \in \Sigma(\mathfrak{a})$, we write $\mathfrak{g}_\alpha$ for the associated root space.

Definition 1.11. By an $\mathfrak{h}$-root in $\Sigma(\mathfrak{a})$ we mean a root $\alpha \in \Sigma(\mathfrak{a})$ such that $\mathfrak{g}_\alpha \cap \mathfrak{h} \neq \{0\}$. The set of such roots is denoted by $\Sigma_{\mathfrak{h}}(\mathfrak{a})$.

Proposition 1.12.

(i) If $\alpha \in \Sigma_{\mathfrak{h}}(\mathfrak{a})$, then $\alpha\big|_{\mathfrak{a}\cap\mathfrak{q}} = 0$.

(ii) $Z_{\mathfrak{h}}(\mathfrak{a} \cap \mathfrak{q})$ is a reductive Lie algebra of which $\mathfrak{a} \cap \mathfrak{h}$ is a maximal split abelian subalgebra. The root system of $\mathfrak{a} \cap \mathfrak{h}$ in $Z_{\mathfrak{h}}(\mathfrak{a} \cap \mathfrak{q})$ equals $\Sigma_{\mathfrak{h}}(\mathfrak{a})$, viewed as a subset of $(\mathfrak{a} \cap \mathfrak{h})^*$.

(iii) Assume that $\mathfrak{a} \in \mathcal{A}$. Then $\Sigma_{\mathfrak{h}}(\mathfrak{a}) = \Sigma(\mathfrak{a}) \cap (\mathfrak{a} \cap \mathfrak{h})^*$. Furthermore, if $\alpha \in \Sigma_{\mathfrak{h}}(\mathfrak{a})$, then $\mathfrak{g}_\alpha \subseteq \mathfrak{h}$.

Proof. Ad (i): If $\mathfrak{g}_\alpha \cap \mathfrak{h} \neq \{0\}$, then $\mathfrak{g}_{\sigma\alpha} \cap \mathfrak{g}_\alpha = \sigma(\mathfrak{g}_\alpha) \cap \mathfrak{g}_\alpha \neq \{0\}$. This implies $\sigma\alpha = \alpha$.

Ad (ii): Let θ be a Cartan involution giving rise to a Cartan decomposition as in Lemma 1.1. Since $Z_{\mathfrak{h}}(\mathfrak{a} \cap \mathfrak{q})$ is θ-stable, it follows from [10, Cor. 6.29] that $Z_{\mathfrak{h}}(\mathfrak{a} \cap \mathfrak{q})$ is reductive. The maximality of $\mathfrak{a}$ implies that $\mathfrak{a} \cap \mathfrak{h}$ is a maximal split abelian subalgebra of $Z_{\mathfrak{h}}(\mathfrak{a} \cap \mathfrak{q})$. Let Φ be the root system of $\mathfrak{a} \cap \mathfrak{h}$ in $Z_{\mathfrak{h}}(\mathfrak{a} \cap \mathfrak{q})$. It follows from *(i)* that $\alpha|_{\mathfrak{a}\cap\mathfrak{h}} \in \Phi$ for every $\alpha \in \Sigma_{\mathfrak{h}}(\mathfrak{a})$. Now let $\beta \in \Phi$ and let $\alpha \in \mathfrak{a}^*$ be given by $\alpha|_{\mathfrak{a}\cap\mathfrak{h}} = \beta$ and $\alpha|_{\mathfrak{a}\cap\mathfrak{q}} = 0$. Then $\mathfrak{g}_\alpha \cap \mathfrak{h}$ contains the root space of β, hence $\alpha \in \Sigma_{\mathfrak{h}}(\mathfrak{a})$. We conclude that $\Phi = \Sigma_{\mathfrak{h}}(\mathfrak{a})$.

Ad (iii): It suffices to prove that under the given assumption, $\mathfrak{g}_\alpha \subseteq \mathfrak{h}$ for every $\alpha \in \Sigma(\mathfrak{a})$ that vanishes on $\mathfrak{a} \cap \mathfrak{q}$. Let α be such a root and let $X \in \mathfrak{g}_\alpha \cap \mathfrak{q}$. We write θ for a Cartan involution giving rise to a Cartan decomposition as in Lemma 1.1. Since X is in the centralizer of $\mathfrak{a} \cap \mathfrak{q}$, it follows from the maximality of $\mathfrak{a} \cap \mathfrak{q}$ that $X - \theta(X) \in \mathfrak{a} \cap \mathfrak{q}$. Since $X - \theta(X) \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, it follows that $X - \theta(X) = 0$ and therefore $X = 0$. □

If $P \in \mathcal{P}(\mathfrak{a})$, then we write $\Sigma(\mathfrak{a}; P)$ for the positive system of $\Sigma(\mathfrak{a})$ consisting of roots that are positive with respect to P. In other words, a root $\alpha \in \Sigma(\mathfrak{a})$ is an element of $\Sigma(\mathfrak{a}; P)$ if and only if the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{n}_P$.

We fix a positive system $\Sigma^+_{\mathfrak{h}}(\mathfrak{a})$ of $\Sigma_{\mathfrak{h}}(\mathfrak{a})$.

Corollary 1.13. *Let Q be a minimal parabolic subgroup and let A be a σ-stable split component of Q. Then Q is $N_H(\mathfrak{a} \cap \mathfrak{h}) \cap Z_H(\mathfrak{a} \cap \mathfrak{q})$-conjugate to a minimal parabolic subgroup P such that*

$$\Sigma^+_{\mathfrak{h}}(\mathfrak{a}) \subseteq \Sigma(\mathfrak{a}; P).$$

Proof. The set $\Sigma_{\mathfrak{h}}(\mathfrak{a}) \cap \Sigma(\mathfrak{a}; Q)$ is a positive system for the root system $\Sigma_{\mathfrak{h}}(\mathfrak{a})$. It follows from Proposition 1.12 *(ii)* that each such positive system is conjugate to $\Sigma^+_{\mathfrak{h}}(\mathfrak{a})$ by an element of the normalizer of $\mathfrak{a} \cap \mathfrak{h}$ in $Z_H(\mathfrak{a} \cap \mathfrak{q})$. □

2 The symmetric space under consideration

2.1 The space X_n

For the remainder of this article n will be a natural number with $n \geq 3$ and G will be the real Lie group $\mathrm{SL}(n, \mathbb{R})$. Let σ be the involution on G given by

$$\sigma(g) = SgS^{-1},$$

where

$$S = S^{-1} = \begin{pmatrix} & & -1 \\ & I_{n-2} & \\ -1 & & \end{pmatrix}.$$

The fixed point subgroup H of σ is obtained from the subgroup $\mathrm{S}\big(\mathrm{GL}(n-1, \mathbb{R}) \times \mathrm{GL}(1, \mathbb{R})\big)$ of G (embedded in the usual manner) by conjugation with the orthogonal matrix

$$\kappa = \begin{pmatrix} \frac{1}{\sqrt{2}} & & \frac{1}{\sqrt{2}} \\ & I_{n-2} & \\ \frac{-1}{\sqrt{2}} & & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{2.1}$$

We denote the $2n-2$ dimensional reductive symmetric space G/H by X_n.

Let θ be the Cartan involution given by

$$\theta(g) = (g^{-1})^t,$$

and let K be the fixed point subgroup of θ, i.e., K is the maximal compact subgroup $\mathrm{SO}(n)$. Since S is orthogonal, the involutions σ and θ commute. The Lie algebra $\mathfrak{g}$ of G admits the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ into the eigenspaces for σ. Here

$$\mathfrak{h} = \Big\{ \begin{pmatrix} \frac{-\operatorname{tr} x}{2} & v^t & y \\ w & x & -w \\ y & -v^t & \frac{-\operatorname{tr} x}{2} \end{pmatrix} : x \in \mathrm{Mat}(n-2, \mathbb{R}), y \in \mathbb{R}, v \in \mathbb{R}^{n-2}, w \in \mathbb{R}^{n-2} \Big\}$$

is the Lie algebra of H and

$$\mathfrak{q} = \Big\{ \begin{pmatrix} z & v^t & y \\ w & & w \\ -y & v^t & -z \end{pmatrix} : y \in \mathbb{R}, z \in \mathbb{R}, v \in \mathbb{R}^{n-2}, w \in \mathbb{R}^{n-2} \Big\}.$$

Similarly, we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$; here $\mathfrak{k}$ equals $\mathfrak{so}(n)$ and $\mathfrak{p}$ equals the space of traceless symmetric $n \times n$ matrices. We note that

$$\mathfrak{a}_\mathfrak{q} := \mathbb{R}\,\mathrm{diag}(1, 0, \ldots, 0, -1)$$

is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. Hence, the split rank of X_n is equal to 1. We put $A_\mathfrak{q} := \exp(\mathfrak{a}_\mathfrak{q})$ and for $t \in \mathbb{R}$ we define $a_t \in A_\mathfrak{q}$ by

$$a_t := \exp\big(t\,\mathrm{diag}(1, 0, \ldots, 0, -1)\big). \tag{2.2}$$

2.2 Polar decomposition

The space X_n admits a polar decomposition: the map

$$K \times A_\mathfrak{q} \to X_n; \qquad (k, a) \mapsto ka \cdot H$$

is surjective. If $a \cdot H \in Ka' \cdot H$ for $a, a' \in A_\mathfrak{q}$, then there exists a k in $N_{K\cap H}(\mathfrak{a}_\mathfrak{q})$, the normalizer of $\mathfrak{a}_\mathfrak{q}$ in $K \cap H$, such that $a = ka'k^{-1}$. In fact, since the action of $N_{K\cap H}(\mathfrak{a}_\mathfrak{q})$ on $\mathfrak{a}_\mathfrak{q}$ is length-preserving (with respect to the Killing form) and

$$k_0 = \begin{pmatrix} 0 & 0 & & 1 \\ 0 & -1 & & 0 \\ & & I_{n-3} & \\ 1 & 0 & & 0 \end{pmatrix} \tag{2.3}$$

is an element in $K \cap H$ such that $k_0 a k_0^{-1} = a^{-1}$ for every $a \in A_\mathfrak{q}$, we have $a \cdot H \in Ka' \cdot H$ if and only if $a' \in \{a, a^{-1}\}$.

Lemma 2.1. *Let $g \in G$ and $t \in \mathbb{R}$. If $g \in Ka_t \cdot H$, then*

$$\|g\sigma(g)^{-1}\|_{HS}^2 = n - 2 + 2\cosh(4t),$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm on $\mathrm{Mat}(n, \mathbb{R})$.

Proof. A straightforward computation shows that

$$\|g\sigma(g)^{-1}\|_{HS}^2 = \mathrm{tr}\left(g\sigma(g)^{-1}\big(g\sigma(g)^{-1}\big)^t\right) = \mathrm{tr}\,(a_t^4).$$

The result now follows from the fact that a_t^4 equals the matrix $\mathrm{diag}(e^{4t}, 1, \ldots, 1, e^{-4t})$. □

2.3 Schwartz functions

In this section we give a definition of the space of Schwartz function on X_n. This definition differs from the one in [3, Sect. 17], but it is easily seen from [3, Thm. 17.1, Prop. 17.2] combined with Remark 4.5 of the present paper, that the definitions are equivalent.

Definition 2.2. A Schwartz function on X_n is a smooth function $\phi : X_n \to \mathbb{C}$, such that for every $u \in U(\mathfrak{g})$ and $m \geq 0$ the seminorm

$$\sup_{k\in K, t\in\mathbb{R}} \cosh^{n-1}(t)\big(1+|t|\big)^m \big|(u\phi)(ka_t \cdot H)\big|$$

is finite. Here $u\phi$ is obtained with the regular action of $U(\mathfrak{g})$ from the left. The vector space of Schwartz functions on X_n, equipped with the locally convex topology induced by these seminorms, is denoted by $\mathcal{C}(X_n)$.

Remark 2.3. To simplify computational expressions later on, it will be useful to work with the following seminorms instead, for $u \in U(\mathfrak{g})$ and $m \in \mathbb{N}$,

$$\mu_{u,m}(\phi) := \sup_{k\in K, t\in\mathbb{R}} (2\cosh 4t)^{\frac{n-1}{4}} \big(1+\log(2\cosh(4t))\big)^m \big|(u\phi)(ka_t \cdot H)\big|. \tag{2.4}$$

Obviously, a smooth function $\phi : X_n \to \mathbb{C}$ belongs to $\mathcal{C}(X_n)$ if and only if these seminorms are finite. Moreover, the seminorms $\mu_{u,m}$ detemine the Fréchet topology on $\mathcal{C}(X_n)$.

For future reference we shall now construct some specific Schwartz functions on X_n.

Lemma 2.4. *Let $\nu < \frac{1-n}{4}$. The function $\phi_\nu : X_n \to \mathbb{R}$, defined by*

$$\phi_\nu(ka_t \cdot H) = \cosh^\nu(4t),$$

is Schwartz.

Proof. We will show that $\mathcal{C}(X_n)$ contains every function $\phi \in C^\infty(X_n)$ such that

$$K \times (1,\infty) \to \mathbb{C}; \qquad (k,t) \mapsto \phi(ka_t \cdot H)$$

is a linear combination of functions of the form

$$K \times (1,\infty) \to \mathbb{C}; \qquad (k,t) \mapsto \varphi(k)\cosh^\lambda(t)\sinh^\mu(t), \tag{2.5}$$

with $\varphi \in C^\infty(K)$ and real numbers λ, μ with sum $\lambda + \mu = 4\nu < 1 - n$. Clearly our function ϕ_ν has this property. It is easily seen that $\mu_{1,m}(\phi) < \infty$ for such a function, and hence it suffices to show that this class of functions is invariant under the left action by $U(\mathfrak{g})$. For this it suffices to consider the action by $\mathfrak{g}$. Let $X \in \mathfrak{g}$, then we can write X as a linear combination of elements of the form $\mathrm{Ad}(k)Y$ with some fixed basis elements $Y \in \mathfrak{g}$ and with coefficients which are smooth functions of $k \in K$. We shall use a basis of weight vectors Y for the adjoint action of $\mathfrak{a}_\mathfrak{q}$ on $\mathfrak{g}$. If Y belongs to $\mathfrak{a}_\mathfrak{q}$ or $Z_\mathfrak{h}(\mathfrak{a}_\mathfrak{q})$, then

it is easily seen that $(k,t) \mapsto [\mathrm{Ad}(k)(Y)\phi](ka_t \cdot H)$ will again be of the form (2.5) with $\lambda + \nu = 4\nu$. It remains to consider the case where Y is a root vector for a root α of $\mathfrak{a}_\mathfrak{q}$.

Since $\mathfrak{a}_\mathfrak{q}$ is $\sigma\theta$-invariant, the associated root space in $\mathfrak{g}$ is $\sigma\theta$-invariant as well, hence we may assume that $\sigma\theta Y = \pm Y$. If $\sigma\theta Y = Y$, a simple computation shows that

$$Y = \frac{-a^{-\alpha}}{a^\alpha - a^{-\alpha}}(Y + \theta Y) + \frac{1}{a^\alpha - a^{-\alpha}}\mathrm{Ad}(a)(Y + \sigma Y)$$

for all $a \in A_\mathfrak{q}$, and if $\sigma\theta Y = -Y$, the corresponding formula reads

$$Y = \frac{a^{-\alpha}}{a^\alpha + a^{-\alpha}}(Y + \theta Y) + \frac{1}{a^\alpha + a^{-\alpha}}\mathrm{Ad}(a)(Y + \sigma Y).$$

Note that for $a = a_t$, in both cases the coefficient of $Y + \theta Y$ is a linear combination of functions $\cosh^p t \sinh^q t$ with $p + q = 0$. Hence, when we apply $\mathrm{Ad}(k)Y$ to ϕ as above, the term with $Y + \theta Y$ will produce a new function of the same kind, whereas the term with $\mathrm{Ad}(a_t)(Y + \sigma Y)$ will be annihilated because of the H-invariance from the right. This proves the claim. □

2.4 $\mathfrak{h}$-compatible parabolic subgroups

Let P be a minimal parabolic subgroup. By Theorem 1.8 there exists a σ-stable split component A of P, which is unique up to conjugation by $P \cap H$. We fix such a split component A and write $\mathfrak{a}$ for its Lie algebra. We recall the definition of the root system $\Sigma_\mathfrak{h}(\mathfrak{a})$ of $\mathfrak{h}$-roots from Definition 1.11. We write $\Sigma_\mathfrak{h}(\mathfrak{a}; P)$ for the positive system $\Sigma(\mathfrak{a}; P) \cap \Sigma_\mathfrak{h}(\mathfrak{a})$ of $\Sigma_\mathfrak{h}(\mathfrak{a})$ and define

$$\rho_{P,\mathfrak{h}} = \frac{1}{2}\sum_{\alpha \in \Sigma_\mathfrak{h}(\mathfrak{a};P)} \alpha.$$

Definition 2.5. The parabolic subgroup P is said to be $\mathfrak{h}$-compatible if one of the following conditions are fulfilled:

(a) P is of σ-parabolic rank 1 and $\langle \alpha, \rho_{P,\mathfrak{h}} \rangle \geq 0$ for all $\alpha \in \Sigma(\mathfrak{a}; P)$;

(b) P is of σ-parabolic rank 0 and $\langle \alpha, \rho_{P,\mathfrak{h}} \rangle > 0$ for all $\alpha \in \Sigma(\mathfrak{a}; P)$.

Remark 2.6. We note that this notion is independent of the choice of $\mathfrak{a}$, since any other choice would be $P \cap H$-conjugate to $\mathfrak{a}$. Furthermore, it is now readily seen that the property of $\mathfrak{h}$-compatibility is preserved under conjugation by H.

Since X_n is of split rank 1, every minimal parabolic subgroup is either of σ-parabolic rank 1 or of σ-parabolic rank 0. We recall that $\mathcal{P}$ denotes the set of minimal parabolic subgroups of σ-parabolic rank 1 and $\mathcal{Q}$ denotes the set of minimal parabolic subgroups of σ-parabolic rank 0. We write $\mathcal{P}_\mathfrak{h}$ and $\mathcal{Q}_\mathfrak{h}$ for the sets of $\mathfrak{h}$-compatible parabolic subgroups in $\mathcal{P}$ and $\mathcal{Q}$, respectively. Recall that $\mathcal{P}_*$ denotes the set of $P \in \mathcal{P}$ such that the dimension of $N_P/(N_P \cap H)$ is minimal, i.e., $\dim\big(N_P/(N_P \cap H)\big) = n - 1$; see Proposition 4.4 and Equation (4.10).

For every $n \geq 3$ the set $\mathcal{P}_\mathfrak{h}$ is non-empty. If n is even, then H acts transitively on $\mathcal{P}_\mathfrak{h}$ and $\mathcal{P}_\mathfrak{h} \subseteq \mathcal{P}_*$. If n is odd, then the H-action admits three orbits, see Proposition 4.4. One orbit is equal to $\mathcal{P}_\mathfrak{h} \setminus (\mathcal{P}_* \cap \mathcal{P}_\mathfrak{h})$; the other two orbits are contained in $\mathcal{P}_* \cap \mathcal{P}_\mathfrak{h}$.

If n is even, then $\mathcal{Q}_\mathfrak{h} = \emptyset$. If n is odd, then $\mathcal{Q}_\mathfrak{h}$ is non-empty and H acts transitively on it, see Proposition 5.3.

3 Main theorems

3.1 Convergence

Let P be a minimal parabolic subgroup of G and let dx be a non-zero N_P-invariant Radon measure on $N_P/(N_P \cap H)$.

Theorem 3.1. *The integral*

$$\int_{N_P/(N_P \cap H)} \phi(x \cdot H)\, dx$$

is absolutely convergent for every $\phi \in \mathcal{C}(X_n)$ if and only if P is $\mathfrak{h}$-compatible.

The proof of this theorem will be given in Sections 4 and 5.

3.2 Limit behavior

Assume that $P \in \mathcal{P}_\mathfrak{h} \cap \mathcal{P}_*$. Let A be a σ-stable split component of P and let L be the centralizer of A in G. Then $L = MA$, where M is the (unique) maximal compact subgroup of L. Now $P = LN_P$ and $P = MAN_P$ are a Levi and a Langlands decomposition respectively. Note that $\mathfrak{a} = \mathrm{Lie}(A) \in \mathcal{A}$. We define

$$\delta_P : L \to \mathbb{R}_{>0}; \qquad l \mapsto \left| \frac{\det \mathrm{Ad}(l)\big|_{\mathfrak{n}_P}}{\det \mathrm{Ad}(l)\big|_{\mathfrak{n}_P \cap Z_\mathfrak{g}(\mathfrak{a} \cap \mathfrak{q})}} \right|^{\frac{1}{2}}.$$

Let

$$\rho_P = \frac{1}{2} \sum_{\alpha \in \Sigma(\mathfrak{a}; P)} \alpha.$$

Then for $m \in M$ and $a \in A$

$$\delta_P(ma) = a^{\rho_P - \rho_{P,\mathfrak{h}}}.$$

For $\phi \in \mathcal{C}(X_n)$ we define its Harish-Chandra transform $\mathcal{H}_P\phi$ to be the function on $L/(L \cap H)$ given by

$$\mathcal{H}_P\phi(l) = \delta_P(l) \int_{N_P/(N_P \cap H)} \phi(ln)\, dn \qquad (l \in L).$$

Note that the integrals are absolutely convergent by Theorem 3.1 and define a right $(L \cap H)$-invariant function on L.

Theorem 3.2. *Let $P \in \mathcal{P}_\mathfrak{h} \cap \mathcal{P}_*$ be as above and let $v \in \mathfrak{a} \cap \mathfrak{q}$ be such that $\rho_P(v) > 0$.*

(i) Assume n is even. Then for every $\phi \in \mathcal{C}(X_n)$ and every $N \in \mathbb{N}$,

$$\sup_{m \in M, t \in \mathbb{R}} \left| t^N \mathcal{H}_P \phi\big(m \exp(tv)\big) \right| < \infty.$$

(ii) Assume n is odd. Then for every $\phi \in \mathcal{C}(X_n)$, every $r \in \mathbb{R}$ and every $N \in \mathbb{N}$,

$$\sup_{m \in M, t < r} \left| t^N \mathcal{H}_P \phi\big(m \exp(tv)\big) \right| < \infty.$$

Moreover, the limit

$$\mu_P(\phi) := \lim_{t \to \infty} \mathcal{H}_P \phi\big(\exp(tv)\big)$$

exists, and there exists an $\mathfrak{h}$-compatible $Q \in \mathcal{Q}$ and an element $g \in G$ such that for every $\phi \in \mathcal{C}(X_n)$,

$$\mu_P(\phi) = \int_{N_Q/(N_Q \cap H)} \phi(gn)\, dn.$$

Vice versa, if $Q \in \mathcal{Q}$, then there exists a $P' \in \mathcal{P}_\mathfrak{h} \cap \mathcal{P}_*$ and an element $g \in G$ such that for every $\phi \in \mathcal{C}(X_n)$,

$$\int_{N_Q/(N_Q \cap H)} \phi(gn)\, dn = \mu_{P'}(\phi).$$

The proof of the theorem will be given in Section 6.

3.3 Cusp forms

As explained in the introduction, the aim of the article is to explore which parabolic subgroups should be used in the definition of cusp forms for reductive symmetric spaces of split rank 1. In [4] it was proved that for such a symmetric space X the integral

$$\int_{N_P/(N_P \cap H)} \phi(n)\, dn$$

is absolutely convergent for all $\phi \in \mathcal{C}(X)$ and every $P \in \mathcal{P}_\mathfrak{h}$.

It follows from Theorem 3.1 that for the spaces X_n *only* the $\mathfrak{h}$-compatible parabolic subgroups provide integrals that are convergent for all Schwartz functions. We conclude from this that the condition that P is $\mathfrak{h}$-compatible, which was needed in [4], is not an artefact of the proof.

For n odd the set $\mathcal{Q}_\mathfrak{h}$ is non-empty. In [4] only the minimal parabolic subgroups from $\mathcal{P}_\mathfrak{h}$ were used. The remaining question that needs to be answered is whether the class of parabolic subgroups that is used for the definition of cusp forms should include any minimal parabolic subgroup from $\mathcal{Q}_\mathfrak{h}$.

Proposition 3.3. *Let $\phi \in \mathcal{C}(X_n)$. The following four conditions are equivalent.*

(i) For every $g \in G$ and every $\mathfrak{h}$-compatible parabolic subgroup P,

$$\int_{N_P/(N_P \cap H)} \phi(gn)\, dn = 0. \tag{3.1}$$

(ii) For every $g \in G$ and every $P \in \mathcal{P}_{\mathfrak{h}}$ equation (3.1) holds.

(iii) For every $g \in G$ and every $P \in \mathcal{P}_{\mathfrak{h}} \cap \mathcal{P}_*$ equation (3.1) holds.

(iv) There exists a P in each H-conjugacy class in $\mathcal{P}_{\mathfrak{h}} \cap \mathcal{P}_*$ such that for every $g \in G$ equation (3.1) holds.

Proof. The implications *(i)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(iv)* are trivial.

Conversely, if (3.1) holds for a given $\mathfrak{h}$-compatible parabolic subgroup P and every $g \in G$, then it also holds for every H-conjugate of P. This proves *(iv)* $\Rightarrow$ *(iii)*. The implication *(iii)* $\Rightarrow$ *(ii)* is proven in [4, Lemma 8.14].

Now assume $Q \in \mathcal{Q}_{\mathfrak{h}}$. By Theorem 3.2 the integrals over $N_Q/(N_Q \cap H)$ can be obtained as limits of integrals over $N_P/(N_P \cap H)$ for some $P \in \mathcal{P}_{\mathfrak{h}} \cap \mathcal{P}_*$. This establishes the implication *(ii)* $\Rightarrow$ *(i)*. □

Definition 3.4. A function $\phi \in \mathcal{C}(X_n)$ is said to be a cusp form if one of the equivalent conditions in Proposition 3.3 is satisfied.

We write $\mathcal{C}_{\mathrm{cusp}}(X_n)$ for the space of cusp forms on X_n and $\mathcal{C}_{\mathrm{ds}}(X_n)$ for the closed span of the discrete series representations of X_n. Given $g \in G$ and $\phi \in \mathcal{C}(X_n)$, we write $L_g\phi$ for the function given by $L_g\phi(x) = \phi(g^{-1}x)$, for $x \in X_n$. In [4] the following theorem, which we here only state for X_n, is proved for general reductive symmetric spaces of split rank 1.

Theorem 3.5.

(i) $\mathcal{C}_{\mathrm{cusp}}(X_n) \subseteq \mathcal{C}_{\mathrm{ds}}(X_n)$.

(ii) If $\mathcal{C}_{\mathrm{ds}}(X_n)^K \subseteq \mathcal{C}_{\mathrm{cusp}}(X_n)^K$, *then* $\mathcal{C}_{\mathrm{ds}}(X_n) = \mathcal{C}_{\mathrm{cusp}}(X_n)$.

(iii) Fix $\mathfrak{a} \in \mathcal{A}$ with $\mathfrak{a}_{\mathfrak{q}} \subseteq \mathfrak{a}$. Let $\phi \in \mathcal{C}(X_n)$ be K-finite. Then $\phi \in \mathcal{C}_{\mathrm{ds}}(X_n)$ if and only if for every $\mathfrak{h}$-compatible minimal parabolic $P \in \mathcal{P}(\mathfrak{a})$ and every $k \in K$ the function

$$\mathcal{H}_P\big(L_k\phi\big)\Big|_{A_{\mathfrak{q}}}$$

is a finite linear combination of exponential functions with non-zero exponents.

Theorem 3.5 and the estimates in Theorem 3.2 have the following corollary.

Theorem 3.6. $\mathcal{C}_{\mathrm{ds}}(X_n) = \mathcal{C}_{\mathrm{cusp}}(X_n)$.

Proof. By *(i)* and *(ii)* in Theorem 3.5 it suffices to prove that $\mathcal{C}_{\mathrm{ds}}(X_n)^K \subseteq \mathcal{C}_{\mathrm{cusp}}(X_n)^K$. Let $\phi \in \mathcal{C}_{\mathrm{ds}}(X_n)^K$. We need to show that ϕ is a cusp form. For this we will prove that condition *(iv)* in Proposition 3.3 is satisfied.

By *(iii)* in Theorem 3.5 the restriction of $\mathcal{H}_P\phi$ to $A_{\mathfrak{q}}$ is of exponential type with non-zero exponents. From Theorem 3.2 it follows that this function is bounded. The only function on $A_{\mathfrak{q}}$ that satisfies both conditions is the 0-function. This proves that

$$\int_{N_P/(N_P\cap H)} \phi(an)\,dn = 0$$

for every $a \in A_{\mathfrak{q}}$. Now, let $g \in G$. By the Iwasawa decomposition there exist $k \in K$, $a \in A_{\mathfrak{q}}$ and $a_H \in A \cap H$ such that $g \in kaa_H N_P$. Using that ϕ is K-invariant, we find

$$a_H^{\rho_P-\rho_{P,\mathfrak{h}}} \int_{N_P/(N_P\cap H)} \phi(gn)\,dn = \int_{N_P/(N_P\cap H)} \phi(an)\,dn = 0.$$

This proves the claim. □

4 Proof of Theorem 3.1 for σ-parabolic rank 1

In this section we deal with the proof for Theorem 3.1 under the assumption that P is of σ-parabolic rank 1. In 4.1 – 4.4 we first reduce the statement to a (seemingly) less general statement, which we then prove in 4.5 – 4.7.

4.1 Root systems.

We recall the definition of $\mathfrak{a}_{\mathfrak{q}}$, $A_{\mathfrak{q}}$ and a_t from Section 2.1, and define

$$\mathfrak{a}_{\mathfrak{h}} := \big\{\mathrm{diag}(x_1, x_2, \ldots, x_{n-1}, x_1) : x_i \in \mathbb{R}, 2x_1 + \sum_{i=2}^{n-1} x_i = 0\big\}.$$

Note that $\mathfrak{a}_{\mathfrak{q}}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ and $\mathfrak{a}_{\mathfrak{h}}$ is a subspace of $\mathfrak{p} \cap \mathfrak{h}$ such that $\mathfrak{a} = \mathfrak{a}_{\mathfrak{h}} \oplus \mathfrak{a}_{\mathfrak{q}}$ is a maximal abelian subspace of $\mathfrak{p}$. We write A for $\exp(\mathfrak{a})$

In the remainder of this section we shall describe the root system of $\mathfrak{a}$ in $\mathfrak{g}$. For $1 \leq k \leq n$ we define the functional

$$e_k : \mathfrak{a} \to \mathbb{R}; \qquad \mathrm{diag}(x_1, \ldots, x_n) \mapsto x_k.$$

The root system of $\mathfrak{a}$ in $\mathfrak{g}$ then equals

$$\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a}) = \big\{e_i - e_j : 1 \leq i, j \leq n, i \neq j\big\}.$$

The root spaces are $\mathfrak{g}_{e_i-e_j} = \mathbb{R}E_{i,j}$, where $E_{i,j}$ is the matrix whose entry on the i^{th} row and j^{th} column equals 1, whereas all remaining entries equal zero.

Note that

$$\Sigma_{\mathfrak{h}} := \Sigma_{\mathfrak{h}}(\mathfrak{a}) = \big\{e_i - e_j : 2 \leq i, j \leq n-1, i \neq j\big\}$$

is the root system in $\mathfrak{a}_\mathfrak{h}$ for $Z_\mathfrak{h}(\mathfrak{a}_\mathfrak{q})$; see Proposition 1.12 *(ii)*. We set

$$\Sigma^+_\mathfrak{h} = \Sigma^+_\mathfrak{h}(\mathfrak{a}) = \{e_i - e_j : 2 \le i < j \le n-1\} \tag{4.1}$$

and define

$$\rho_\mathfrak{h} = \frac{1}{2}\sum_{\alpha \in \Sigma^+_\mathfrak{h}} \alpha.$$

The set

$$\Sigma_\mathfrak{q} = \Sigma \cap \mathfrak{a}^*_\mathfrak{q} = \{\pm(e_1 - e_n)\}$$

forms a root system. We denote the associated positive system $\{e_1 - e_n\}$ by $\Sigma^+_\mathfrak{q}$.

4.2 Classification of parabolic subgroups

Recall that $\mathcal{P}$ denotes the set of minimal parabolic subgroups containing a σ-stable maximal connected split abelian subgroup B, such that $\mathfrak{b} \cap \mathfrak{q}$ has maximal dimension, i.e., $\dim(\mathfrak{b} \cap \mathfrak{q}) = 1$. Recall also that $\mathcal{P}(\mathfrak{a})$ denotes the subset of $\mathcal{P}$ consisting of minimal parabolic subgroups containing A. From now on we shall write $\Sigma(P) := \Sigma(\mathfrak{a}; P)$ for $P \in \mathcal{P}(\mathfrak{a})$.

Lemma 4.1. *Let $Q \in \mathcal{P}$. Then there exists a parabolic subgroup $P \in \mathcal{P}(\mathfrak{a})$ which is H-conjugate to Q and satisfies*

$$(\Sigma^+_\mathfrak{h} \cup \Sigma^+_\mathfrak{q}) \subseteq \Sigma(P). \tag{4.2}$$

Let P be any such parabolic subgroup. Then $\rho_{P,\mathfrak{h}} = \rho_\mathfrak{h}$. Moreover, Q is $\mathfrak{h}$-compatible if and only if P is $\mathfrak{h}$-compatible (see Definition 2.5).

Proof. By Corollaries 1.10 and 1.13, Q is H-conjugate to a parabolic subgroup $P' \in \mathcal{P}(\mathfrak{a})$ with $\Sigma^+_\mathfrak{h} \subseteq \Sigma(P')$. If $\Sigma(P') \cap \mathfrak{a}^*_\mathfrak{q} = \Sigma^+_\mathfrak{q}$, then (4.2) holds with $P = P'$. Otherwise, recall the element k_0 from (2.3). This element is in $N_{K\cap H}(\mathfrak{a}_\mathfrak{q}) \cap Z_{K\cap H}(\mathfrak{a}_\mathfrak{h})$ and acts by inversion on $\mathfrak{a}_\mathfrak{q}$. Hence, $P = k_0 P' k_0^{-1}$ satisfies (4.2). In particular, $\rho_{P,\mathfrak{h}} = \rho_\mathfrak{h}$. The final assertion follows from Remark 2.6. □

We will now classify the parabolic subgroups $P \in \mathcal{P}(\mathfrak{a})$ satisfying (4.2). Every parabolic subgroup $P \in \mathcal{P}(\mathfrak{a})$ is uniquely determined by the corresponding positive system $\Sigma(P)$. The set of these positive systems is in bijection with the symmetric group S_n: an element $\tau \in S_n$ corresponds to the positive system $\Sigma(P)$ given by

$$\Sigma(P) = \{e_{\tau^{-1}(i)} - e_{\tau^{-1}(j)} : 1 \le i < j \le n\}.$$

Equivalently, a root $e_i - e_j \in \Sigma$ belongs to $\Sigma(P)$ if and only if $\tau(i) < \tau(j)$. We thus see that the parabolic subgroups $P \in \mathcal{P}(\mathfrak{a})$ satisfying (4.2) correspond to the $\tau \in S_n$ such that

$$\tau(i) < \tau(j) \quad \text{for} \quad 2 \le i < j \le n-1 \qquad \text{and} \qquad \tau(1) < \tau(n). \tag{4.3}$$

Given such a permutation $\tau \in S_n$, there exists a unique k, with $2 \le k \le n$, such that

$$\tau(k-1) \le \tau(1) < \tau(k), \tag{4.4}$$

and a unique l, with $k \leq l \leq n$, such that

$$\tau(l-1) < \tau(n) \leq \tau(l). \tag{4.5}$$

Conversely, for each pair of integers (k,l) satisfying $2 \leq k \leq l \leq n$, there exists a unique $\tau \in S_n$ satisfying (4.3), (4.4) and (4.5). From now on we write $P_{k,l}$ for the corresponding minimal parabolic subgroup and $\Sigma^+_{k,l}$ for $\Sigma(P_{k,l})$. Furthermore, we write $P_{k,l} = MAN_{k,l}$ for the Langlands decomposition of $P_{k,l}$. For future reference we note that the positive system corresponding to (k,l) is given by the disjoint union

$$\begin{aligned}\Sigma^+_{k,l} = \Sigma^+_{\mathfrak{h}} \cup \Sigma^+_{\mathfrak{q}} \cup \big\{e_j - e_1 : 2 \leq j \leq k-1\big\} \cup \big\{e_1 - e_j : k \leq j \leq n-1\big\} \\ \cup \big\{e_j - e_n : 2 \leq j \leq l-1\big\} \cup \big\{e_n - e_j : l \leq j \leq n-1\big\}.\end{aligned} \tag{4.6}$$

In the following we assume that b is the ad-invariant bilinear form on $\mathfrak{sl}(n,\mathbb{R})$ given by $b(X,Y) = \mathrm{tr}\,(XY)$. It is well known that b is a positive multiple of the Killing form B, in fact, $b = \frac{1}{4n}B$. The restriction of b to $\mathfrak{a}$, denoted $\langle\,\cdot\,,\,\cdot\,\rangle$, is a positive definite inner product. We equip $\mathfrak{a}^*$ with the dual inner product, also denoted $\langle\,\cdot\,,\,\cdot\,\rangle$, which on two elements $\mu, \nu \in \mathfrak{a}^*$ is given as follows. The elements have unique expressions $\mu = \sum_j \mu_j e_j$ and $\nu = \sum_j \nu_j e_j$ provided we demand that $\sum_j \mu_j = 0$ and $\sum_j \nu_j = 0$. In terms of these expressions,

$$\langle \mu\,,\,\nu\rangle = \sum_j \mu_j \nu_j. \tag{4.7}$$

In particular, each root $e_i - e_j$ has length $\sqrt{2}$.

Lemma 4.2. *Let i and j be integers such that $1 \leq i < j \leq n$. Then*

$$\langle e_i - e_j, \rho_{\mathfrak{h}}\rangle = \begin{cases} j-i & \text{if} \quad 2 \leq i < j \leq n-1, \\ j - \frac{n+1}{2} & \text{if} \quad i = 1 \quad \text{and} \quad 2 \leq j \leq n-1, \\ \frac{n+1}{2} - i & \text{if} \quad 2 \leq i \leq n-1 \quad \text{and} \quad j = n, \\ 0 & \text{if} \quad i = 1 \quad \text{and} \quad j = n. \end{cases}$$

In particular, $\langle \alpha, \rho_{\mathfrak{h}}\rangle \geq 0$ for every $\alpha \in \Sigma(P_{k,l})$ if and only if $\frac{n+1}{2} \leq k \leq l \leq \frac{n+3}{2}$.

Proof. Using the definition of $\rho_{\mathfrak{h}}$, we find

$$2\rho_{\mathfrak{h}} = \sum_{2 \leq i < j \leq n-1} (e_i - e_j) = \sum_{i=2}^{n-1} (n+1-2i)e_i.$$

The first statement in the lemma is a direct consequence of this formula; the second follows from comparison with (4.6). □

Lemma 4.3. *Let $2 \leq k \leq l \leq n$ and let $\mathfrak{n}_{k,l}$ be the Lie algebra of $N_{k,l}$. Then*

$$\mathfrak{n}_{k,l} \cap \mathfrak{h} = \bigoplus_{\alpha \in \Sigma^+_{\mathfrak{h}}} \mathfrak{g}_\alpha \oplus \bigoplus_{2 \leq j \leq k-1} (1+\sigma)\mathfrak{g}_{e_j - e_1} \oplus \bigoplus_{l \leq j \leq n-1} (1+\sigma)\mathfrak{g}_{e_n - e_j}. \tag{4.8}$$

Proof. Since $\sigma(e_1) = e_n$ and $\sigma(e_j) = e_j$ for $2 \leq j \leq n-1$, we find

$$\begin{aligned}\Sigma^+_{k,l} \cap \sigma(\Sigma^+_{k,l}) = \Sigma^+_{\mathfrak{h}} &\cup \big\{e_j - e_1 : 2 \leq j \leq k-1\big\} \cup \big\{e_1 - e_j : l \leq j \leq n-1\big\} \\ &\cup \big\{e_j - e_n : 2 \leq j \leq k-1\big\} \cup \big\{e_n - e_j : l \leq j \leq n-1\big\},\end{aligned}$$

the union being disjoint. The root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma^+_{\mathfrak{h}}$ are contained in $\mathfrak{h}$. Furthermore, if both α and $\sigma(\alpha)$ are roots in $\Sigma^+_{k,l} \setminus \Sigma^+_{\mathfrak{h}}$, then

$$\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma(\alpha)} = (1+\sigma)\mathfrak{g}_\alpha \oplus (1-\sigma)\mathfrak{g}_\alpha,$$

where the first term in the right-hand side is a subspace of $\mathfrak{n}_{k,l} \cap \mathfrak{h}$ and the second term is a subspace in $\mathfrak{n}_{k,l} \cap \mathfrak{q}$. This proves that the right-hand side of (4.8) is contained in $\mathfrak{n}_{k,l} \cap \mathfrak{h}$.

To prove the converse, assume that $Y \in \mathfrak{n}_{k,l} \cap \mathfrak{h}$. Then $Y = \sum_{\alpha \in \Sigma^+_{k,l}} Y_\alpha$, where $Y_\alpha \in \mathfrak{g}_\alpha$. Since $Y \in \mathfrak{h}$, we have $\sigma(Y) = Y$. This implies that $\sigma(Y_\alpha) = Y_{\sigma(\alpha)}$ if both α and $\sigma(\alpha)$ are elements of $\Sigma^+_{k,l}$ and $Y_\alpha = 0$ otherwise. Let α be a root such that $Y_\alpha \neq 0$. If $\alpha = \sigma(\alpha)$ then $\alpha \in \Sigma^+_{\mathfrak{h}}$. If $\alpha \neq \sigma(\alpha)$, then there exist a $2 \leq j \leq k-1$ such that either $\alpha = e_j - e_1$ or $\sigma(\alpha) = e_j - e_1$, or there exists a $l \leq j \leq n-1$ such that either $\alpha = e_n - e_j$ or $\sigma(\alpha) = e_n - e_j$. Therefore, Y is contained in the right-hand side of (4.8). This proves the lemma. □

Proposition 4.4. *Let $P \in \mathcal{P}$. Then there exist unique integers k and l, with $2 \leq k \leq l \leq n$, such that P is H-conjugate to $P_{k,l}$. Moreover, $P \in \mathcal{P}_{\mathfrak{h}}$ if and only if*

$$\frac{n+1}{2} \leq k \leq l \leq \frac{n+3}{2}$$

and $P \in \mathcal{P}_*$ if and only if $k = l$.

Proof. The existence of the integers k and l follows directly from Lemma 4.1. To prove uniqueness of these integers, let $2 \leq k \leq l \leq n$ and $2 \leq k' \leq l' \leq n$, and assume that there exists an $h \in H$ with $hP_{k,l}h^{-1} = P_{k',l'}$. Note that A and hAh^{-1} are both σ-stable split components of $P_{k',l'}$. From Theorem 1.8 it follows that there exists a unique $n \in N_{k',l'} \cap H$ such that $nhAh^{-1}n^{-1} = A$. We write h' for nh. Note that $h' \in N_H(\mathfrak{a}) = N_H(\mathfrak{a}_{\mathfrak{q}}) \cap N_H(\mathfrak{a}_{\mathfrak{h}})$ and $h'P_{k,l}h'^{-1} = P_{k',l'}$. It follows that h' acts trivially on $\Sigma^+_{\mathfrak{q}}$ and therefore $h' \in Z_H(\mathfrak{a}_{\mathfrak{q}})$. From Corollary 1.12 we see that $\Sigma_{\mathfrak{h}}$ is the root system of $\mathfrak{a}_{\mathfrak{h}}$ in $Z_{\mathfrak{h}}(\mathfrak{a}_{\mathfrak{q}})$. Since $h' \in N_H(\mathfrak{a}_{\mathfrak{h}}) \cap Z_H(\mathfrak{a}_{\mathfrak{q}})$, it induces an element w in the Weyl group of this root system. As the positive system $\Sigma^+_{\mathfrak{h}}$ is contained in both $\Sigma^+_{k,l}$ and $\Sigma^+_{k',l'}$, it follows that w acts trivially on $\Sigma^+_{\mathfrak{h}}$, and thus we conclude that h' acts trivially on $\mathfrak{a}_{\mathfrak{h}}$. This proves that $h'P_{k,l}h'^{-1} = P_{k,l}$ and hence $k = k'$ and $l = l'$.

From Lemma 4.2 it is easily seen that $P \in \mathcal{P}_{\mathfrak{h}}$ if and only if $\frac{n+1}{2} \leq k \leq l \leq \frac{n+3}{2}$. The final claim, that $P \in \mathcal{P}_*$ if and only if $k = l$, follows from Lemma 4.3. □

Remark 4.5. We recall from [4, Def. 1.1] that a parabolic subgroup $P \in \mathcal{P}(\mathfrak{a})$ is said to be $\mathfrak{q}$-extreme if $\Sigma(P) \cap \sigma\theta\Sigma(P) = \Sigma(P) \setminus \Sigma_{\mathfrak{h}}$. Since $\sigma\theta e_j = -e_j$ for $2 \leq j \leq n-1$, whereas $\sigma\theta e_1 = -e_n$, it follows from the characterization of $\Sigma_{k,l}$ above that $P_{k,l}$ is $\mathfrak{q}$-extreme if and only if $k = 2$ and $l = n-1$. Let Σ^+_0 be the positive system for the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathfrak{q}})$ obtained by restricting the roots from $\Sigma_{2,n-1} \setminus \Sigma_{\mathfrak{h}}$ to $\mathfrak{a}_{\mathfrak{q}}$. Then $\Sigma^+_0 =$

$\{\alpha, 2\alpha\}$, where $2\alpha = e_1 - e_n$. It is now readily checked that α has multiplicity $2(n-2)$, whereas 2α has multiplicity 1. Accordingly, $\rho_0 := \rho(\Sigma_0^+)$ is given by $\rho_0 = (n-1)\alpha$, so that

$$a_t^{\rho_0} = e^{(n-1)t} \qquad (t \in \mathbb{R}).$$

The element $\rho_0 \in \mathfrak{a}_{\mathrm{q}}^*$ defined above corresponds to the element ρ_Q appearing in [3, Prop. 17.2]. Accordingly, it follows that Definition 2.2 of the Schwartz space $\mathcal{C}(X_n)$ is consistent with the characterisation given in [3, Thm. 17.1].

4.3 Decomposition and invariant measures

Let k and l be integers such that $2 \leq k \leq l \leq n$. We recall that $P_{k,l} = Z_K(\mathfrak{a})AN_{k,l}$ is the minimal parabolic subgroup containing A such that $\Sigma(P_{k,l}) = \Sigma_{k,l}^+$, and that the latter root system is given by (4.6).

We define

$$\mathfrak{u}_{k,l} = \mathfrak{g}_{e_1-e_n} \oplus \bigoplus_{2\leq j\leq l-1} \mathfrak{g}_{e_j-e_n} \oplus \bigoplus_{k\leq j\leq n-1} \mathfrak{g}_{e_1-e_j}.$$

Note that $\mathfrak{u}_{k,l}$ is a Lie subalgebra of $\mathfrak{n}_{k,l}$. We write $U_{k,l}$ for the Lie subgroup $\exp(\mathfrak{u}_{k,l})$. For $x, y \in \mathbb{R}^{n-2}$ and $z \in \mathbb{R}$, we define

$$u_{x,y,z} = \begin{pmatrix} 1 & x^t & z \\ & I_{n-2} & y \\ & & 1 \end{pmatrix}. \tag{4.9}$$

A straightforward computation shows that

$$U_{k,l} = \big\{u_{x,y,z} : x \in \{0\}^{k-2} \times \mathbb{R}^{n-k}, y \in \mathbb{R}^{l-2} \times \{0\}^{n-l}, z \in \mathbb{R}\big\}.$$

By Lemma 4.3, we have

$$\mathfrak{n}_{k,l} = (\mathfrak{n}_{k,l} \cap \mathfrak{h}) \oplus \mathfrak{u}_{k,l}. \tag{4.10}$$

Lemma 4.6. *The map*

$$U_{k,l} \times (N_{k,l} \cap H) \to N_{k,l}; \qquad (u, n) \mapsto un$$

is a diffeomorphism. There exist normalizations for the invariant measure dx on $N_{k,l}/N_{k,l}\cap H$ and the Haar measure du of $U_{k,l}$ such that for every $\phi \in C_c\big(N_{k,l}/(N_{k,l} \cap H)\big)$

$$\int_{N_{k,l}/(N_{k,l}\cap H)} \phi(x)\, dx = \int_{U_{k,l}} \phi\big(u \cdot (N_{k,l} \cap H)\big)\, du. \tag{4.11}$$

Finally, the normalizations may be chosen such that, in addition, the above integrals equal

$$\int_{z\in\mathbb{R}} \int_{y\in\mathbb{R}^{l-2}\times\{0\}^{n-l}} \int_{x\in\{0\}^{k-2}\times\mathbb{R}^{n-k}} \phi\big(u_{x,y,z} \cdot (N_{k,l} \cap H)\big)\, dx\, dy\, dz. \tag{4.12}$$

Proof. The first two assertions follow from [2, Prop. 2.16]. For the final assertion, we note that

$$u_{x,y,z+\langle x\,,\,y\rangle/2} = \exp\left(\begin{array}{c|c|c} 0 & x^t & z \\ \hline & 0_{n-2} & y \\ \hline & & 0 \end{array}\right).$$

It follows that, up to suitable normalization of measures, the second integral in (4.11) equals

$$\int_{z\in\mathbb{R}} \int_{y\in\mathbb{R}^{l-2}\times\{0\}^{n-l}} \int_{x\in\{0\}^{k-2}\times\mathbb{R}^{n-k}} \phi\big(u_{x,y,z+\langle x\,,\,y\rangle/2}\cdot(N_{k,l}\cap H)\big)\,dx\,dy\,dz.$$

The equality with (4.12) now follows from a simple substitution of variables. □

To conclude this section we state one more lemma.

Lemma 4.7. *Let w be the longest Weyl group element for $\Sigma_{\mathfrak{h}} = \Sigma\big(Z_{\mathfrak{h}}(\mathfrak{a}_{\mathfrak{q}});\mathfrak{a}_{\mathfrak{h}}\big)$, relative to the positive system (4.1), and let $w_0 \in N_{K\cap H}(\mathfrak{a})\cap Z_{K\cap H}(\mathfrak{a}_{\mathfrak{q}})$ be a representative for w. Then*

$$\sigma\theta(w_0 P_{k,l} w_0^{-1}) = P_{n+2-l,n+2-k}.$$

Moreover, if $\phi\in C_c^\infty(X_n)$, then

$$\int_{U_{n+2-l,n+2-k}} \phi(u\cdot H)\,du = \int_{U_{k,l}} \phi\Big(\sigma\theta(w_0 u)\cdot H\Big)\,du.$$

Proof. From the identities

$$w\cdot e_i = \begin{cases} e_1 & i=1 \\ e_{n+1-i} & 2\le i\le n-1 \\ e_n & i=n \end{cases} \qquad\text{and}\qquad \sigma\theta e_i = \begin{cases} -e_n & i=1 \\ -e_i & 2\le i\le n-1 \\ -e_1 & i=n, \end{cases}$$

we obtain that $\sigma\theta(w\cdot\Sigma_{k,l}^+) = \Sigma_{n+2-l,n+2-k}^+$. This proves the first statement. The second statement follows from the first as

$$U_{k,l}\to U_{n+2-l,n+2-k};\qquad u\mapsto \sigma\theta(w_0 u w_0^{-1})$$

is a diffeomorphism with Jacobian 1. □

4.4 Reduction to an equivalent theorem

The space $\mathcal{C}(X_n)$ is stable under translation by G and pull-back by $\sigma\theta$. Under the assumption that P is of σ-parabolic rank 1, we conclude from Proposition 4.4, Lemma 4.6 and Lemma 4.7 that Theorem 3.1 is equivalent to the following theorem.

Theorem 4.8. *Let k and l be integers such that $2\le k\le l\le n+2-k$. Then the integral*

$$\int_{U_{k,l}} \phi(u\cdot H)\,du$$

is absolutely convergent for every $\phi\in\mathcal{C}(X_n)$ if and only if $\frac{n+1}{2}\le k\le l\le\frac{n+3}{2}$.

From now on we assume that k and l are integers such that $2\le k\le l\le n+2-k$.

4.5 The integral

Lemma 4.9. *Let $x, y \in \mathbb{R}^{n-2}$ and $z \in \mathbb{R}$. If $u_{x,y,z} \cdot H \in Ka_t \cdot H$, then*

$$\begin{aligned} 2\cosh(4t) =&(1 - z + \langle x, y\rangle)^2(1 + z)^2 + (1 - z + \langle x, y\rangle)^2\|y\|^2 + (\langle x, y\rangle - z)^2 \\ &+ (1 + z)^2\|x\|^2 + 2\langle x, y\rangle + \|x\|^2\|y\|^2 + \|x\|^2 + z^2 + \|y\|^2 + 1. \end{aligned} \tag{4.13}$$

We denote the orthogonal projection $\mathbb{R}^{n-2} \to \{0\}^{k-2} \times \mathbb{R}^{l-k} \times \{0\}^{n-l}$ by π. Let $x \in \{0\}^{k-2} \times \mathbb{R}^{n-k}$, $y \in \mathbb{R}^{l-2} \times \{0\}^{n-l}$ and $z \in \mathbb{R}$. If $u_{x,y,z} \cdot H \in Ka_t \cdot H$, then

$$2\cosh(4t) = \langle x, A_{y,z}x\rangle + \langle b_{y,z}, x\rangle + c_{y,z}, \tag{4.14}$$

where

$$\begin{aligned} A_{y,z} &= \big(\|y\|^2 + (1 + z)^2 + 1\big)\Big(I_{n-2} + \pi(y)\pi(y)^t\Big) \in \mathrm{Aut}(\mathbb{R}^{n-2}), \\ b_{y,z} &= 2(1 - z)\big(\|y\|^2 + (1 + z)^2 + 1\big)\pi(y) \in \{0\}^{k-2} \times \mathbb{R}^{l-k} \times \{0\}^{n-l}, \\ c_{y,z} &= (1 - z)^2\big(\|y\|^2 + (1 + z)^2 + 1\big) + (z^2 + 2z + \|y\|^2) \in \mathbb{R}_{\geq 2}. \end{aligned}$$

Proof. Straight-forward computations show that

$$u_{x,y,z}\sigma(u_{x,y,z})^{-1} = u_{x,y,z}Su_{x,y,z}^{-1}S^{-1} = \left(\begin{array}{c:c:c} (\langle x, y\rangle - z + 1)(1 + z) & (1 + z)x^t & z \\ \hdashline (\langle x, y\rangle - z + 1)y & I_{n-2} + yx^t & y \\ \hdashline \langle x, y\rangle - z & x^t & 1 \end{array}\right)$$

and

$$\|I_{n-2} + yx^t\|_{HS}^2 = n - 2 + 2\langle x, y\rangle + \|x\|^2\|y\|^2.$$

Equation (4.13) now follows from Lemma 2.1. Equation (4.14) is a direct consequence of (4.13). □

Corollary 4.10. *Let $\phi \in C(X_n)$ be K-invariant and non-negative. Let $\bar{\phi} : \mathbb{R}_{\geq 2} \to \mathbb{R}_{\geq 0}$ be given by*

$$\bar{\phi}(2\cosh 4t) = \phi(a_t \cdot H).$$

(Note that the function $\mathbb{R} \ni t \mapsto \phi(a_t \cdot H)$ is even since ϕ is K-invariant.) Furthermore, let $J_k : \mathbb{R}^{n-2} \times \mathbb{R} \to \mathbb{R}_{>0}$ be given by

$$J_k(y, z) = \big(\|y\|^2 + (1 + z)^2 + 1\big)^{\frac{k-n}{2}}(1 + \|\pi(y)\|^2)^{-\frac{1}{2}}$$

and let $c' : \mathbb{R}^{n-2} \times \mathbb{R} \to \mathbb{R}_{\geq 2}$ be given by

$$c'(y, z) = (1 - z)^2\Big(\|y\|^2 + (1 + z)^2 + 1\Big)\frac{1}{1 + \|\pi(y)\|^2} + \big(z^2 + 2z + \|y\|^2\big).$$

Then

$$\int_{U_{k,l}} \phi(u\cdot H)\, du = \int_{z\in\mathbb{R}} \int_{y\in\mathbb{R}^{l-2}\times\{0\}^{n-l}} J_k(y, z) \int_{x\in\{0\}^{k-2}\times\mathbb{R}^{n-k}} \bar{\phi}\big(\|x\|^2 + c'(y, z)\big)\, dx\, dy\, dz.$$

Proof. First, note that, by Lemma 4.6,

$$\int_{U_{k,l}} \phi(u \cdot H)\, du = \int_{z\in\mathbb{R}} \int_{y\in\mathbb{R}^{l-2}\times\{0\}^{n-l}} \int_{x\in\{0\}^{k-2}\times\mathbb{R}^{n-k}} \phi(u_{x,y,z} \cdot H)\, dx\, dy\, dz.$$

We will use Lemma 4.9 to rewrite this integral. Note that the restriction $B_{y,z}$ of $A_{y,z}$ to $\{0\}^{k-2} \times \mathbb{R}^{n-k}$ is a positive definite symmetric automorphism of $\{0\}^{k-2} \times \mathbb{R}^{n-k}$. We define $B_{y,z}^{1/2}$ and $B_{y,z}^{-1/2}$ to be the square root (defined in the usual way) and the inverse square root of $B_{y,z}$, respectively. Now we apply the substitution of variables $x' = B_{y,z}^{1/2}x + \frac{1}{2}B_{y,z}^{-1/2}b_{y,z}$ to the inner integral. When $u_{x,y,z} \cdot H \in Ka_t \cdot H$ we obtain from (4.14) that

$$2\cosh(4t) = \|x'\|^2 + c'_{y,z},$$

where

$$2 \le c'_{y,z} = c_{y,z} - \frac{1}{4}\|B_{y,z}^{-\frac{1}{2}} b_{y,z}\|^2.$$

The Jacobian of the substitution equals the determinant of $B_{y,z}^{-1/2}$. In turn, this determinant equals $(\det B_{y,z})^{-1/2} = J_k(y,z)$. Observing that $b_{y,z}$ is an eigenvector of $B_{y,z}$ with eigenvalue $\big(\|y\|^2 + (1+z)^2 + 1\big)(1 + \|\pi(y)\|^2)$, we see that

$$\frac{1}{4}\|B_{y,z}^{-\frac{1}{2}} b_{y,z}\|^2 = (1-z)^2\big(\|y\|^2 + (1+z)^2 + 1\big)\frac{\|\pi(y)\|^2}{1+\|\pi(y)\|^2}.$$

Hence $c'_{y,z} = c'(y,z)$, and the corollary is proved. □

4.6 The case of convergence

In view of Remark 2.3, the following proposition implies the 'if' part of Theorem 4.8.

Proposition 4.11. *Assume $\frac{n+1}{2} \le k \le l \le \frac{n+3}{2}$. For $m \in \mathbb{R}$, let $\phi_m : X_n \to \mathbb{R}$ be defined by*

$$\phi_m(k'a_t \cdot H) := (2\cosh 4t)^{\frac{1-n}{4}}\big(1 + \log(2\cosh(4t))\big)^{-m} \qquad (k' \in K, t \in \mathbb{R}).$$

Then there exists $m \ge 0$ such that the integral $\int_{U_{k,l}} \phi_m(u \cdot H)\, du$ is convergent.

Proof. Let $f_m(u) = u^{\frac{1-n}{4}}(1 + \log u)^{-m}$ for $u \ge 1$. Then f equals $\bar{\phi}_m$ as defined in Corollary 4.10, and we see that it suffices to show that

$$I_m := \int_{z\in\mathbb{R}} \int_{y\in\mathbb{R}^{l-2}\times\{0\}^{n-l}} J_k(y,z) \int_{x\in\{0\}^{k-2}\times\mathbb{R}^{n-k}} f_m\big(\|x\|^2 + c'(y,z)\big)\, dx\, dy\, dz < \infty,$$

for $m \ge 0$ sufficiently large. We substitute $x = c'(y,z)^{1/2}\xi$ in the inner integral and obtain that I_m is equal to

$$\int_{z\in\mathbb{R}} \int_{y\in\mathbb{R}^{l-2}\times\{0\}^{n-l}} J_k(y,z)c'(y,z)^{\frac{n-k}{2}} \int_{\xi\in\{0\}^{k-2}\times\mathbb{R}^{n-k}} f_m\big(c'(y,z)(\|\xi\|^2+1)\big)\, d\xi\, dy\, dz.$$

Observe that if $m = m_1 + m_2$ with $m_1, m_2 \geq 0$, then

$$\big(1 + \log\big[c'(y,z)(\|\xi\|^2+1)\big]\big)^{-m} \leq \big(1+\log c'(y,z)\big)^{-m_2}\big(1+\log(\|\xi\|^2+1)\big)^{-m_1}.$$

Hence,

$$\begin{aligned} I_m \leq \int_{z\in\mathbb{R}} \int_{y\in\mathbb{R}^{l-2}\times\{0\}^{n-l}} & J_k(y,z)c'(y,z)^{\frac{n+1}{4}-\frac{k}{2}}\big(1+\log c'(y,z)\big)^{-m_2}\,dy\,dz \\ & \times \int_{\xi\in\{0\}^{k-2}\times\mathbb{R}^{n-k}} (\|\xi\|^2+1)^{\frac{1-n}{4}}\big(1+\log(\|\xi\|^2+1)\big)^{-m_1}\,d\xi. \end{aligned} \tag{4.15}$$

Since $k \geq \frac{n+1}{2}$, we have $\frac{n-1}{2} \geq n-k$ and see that the integral over ξ converges for $m_1 \geq 2$ (use polar coordinates).

For the integral over (y,z) we shall need the following estimate of $c'(y,z)$. We write $y = (v,w,0)$, where $v \in \mathbb{R}^{k-2}$ and $w = \pi(y) \in \mathbb{R}^{l-k}$, and claim that

$$c'(y,z) \geq \tfrac{1}{4}(\|w\|^2+1)^{-1}\big((1-z)^2+1\big)\big(\|v\|^2+(1+z)^2+1\big) + \|w\|^2 + 1. \tag{4.16}$$

To verify the claim, we note that

$$\begin{aligned} c'(y,z) &= (\|w\|^2+1)^{-1}(1-z)^2\big(\|v\|^2+\|w\|^2+(1+z)^2+1\big) + z^2 + 2z + \|v\|^2 + \|w\|^2 \\ &= (\|w\|^2+1)^{-1}\Big(\big((1-z)^2+1+\|w\|^2\big)\|v\|^2 + z^4 + 1 + 2z^2\|w\|^2\Big) + \|w\|^2 + 1 \\ &\geq (\|w\|^2+1)^{-1}\Big(\big((1-z)^2+1\big)\|v\|^2 + z^4 + 1\Big) + \|w\|^2 + 1 \\ &\geq \tfrac{1}{4}(\|w\|^2+1)^{-1}\Big(\big((1-z)^2+1\big)\|v\|^2 + z^4 + 4\Big) + \|w\|^2 + 1. \end{aligned}$$

Using

$$\big((1-z)^2+1\big)\big((1+z)^2+1\big) = z^4 + 4,$$

we obtain the validity of claim (4.16).

We first assume that $k = l$. Then $w = 0$ and we obtain

$$c'(y,z) \geq \tfrac{1}{4}\big((1-z)^2+1\big)\big(\|v\|^2+(1+z)^2+1\big) + 1$$

and

$$J_k(y,z) = (\|v\|^2+(1+z)^2+1)^{\frac{k-n}{2}}.$$

Hence, there exists a constant $C > 0$, independent of m, such that

$$\begin{aligned} I_m \leq C \int_{z\in\mathbb{R}} \int_{v\in\mathbb{R}^{k-2}} & \big(\|v\|^2+(1+z)^2+1\big)^{\frac{1-n}{4}}\big((1-z)^2+1\big)^{\frac{n+1}{4}-\frac{k}{2}} \\ & \times \Big(1+\log\big[1+\tfrac{1}{4}\big((1-z)^2+1\big)\big(\|v\|^2+(1+z)^2+1\big)\big]\Big)^{-m_2}\,dv\,dz. \end{aligned}$$

The substitution of $v = ((1+z)^2+1)^{1/2}\eta$ now allows us to estimate I_m by a constant times the product of the integrals

$$\int_{\eta\in\mathbb{R}^{k-2}} (\|\eta\|^2+1)^{\frac{1-n}{4}}\big(1+\log(\|\eta\|^2+1)\big)^{-m_3}\,dy \tag{4.17}$$

and

$$\int_{z\in\mathbb{R}} \big((1+z)^2+1\big)^{\frac{k}{2}-\frac{n+3}{4}} \big((1-z)^2+1\big)^{\frac{n+1}{4}-\frac{k}{2}} \big(1+\log(\tfrac{1}{4}z^4+2)\big)^{-m_4}\,dz$$

where $m_2 = m_3 + m_4$. In the above we already saw that (4.17) converges for $m_3 \geq 2$. The remaining integral is easily seen to converge for $m_4 \geq 2$.

Next we assume $k < l$. Then n is odd and $k = \frac{n+1}{2}$, $l - k = 1$. Hence, the power of $c'(y,z)$ in the first integral in (4.15) equals zero. We thus see that I_m is bounded by a constant times

$$\int_{z\in\mathbb{R}} \int_{v\in\mathbb{R}^{k-2}} \int_{w\in\mathbb{R}} \big(\|v\|^2+w^2+(1+z)^2+1\big)^{\frac{1-n}{4}} (w^2+1)^{-\frac{1}{2}} \big(1+\log c'(y,z)\big)^{-m_2}\,dw\,dv\,dz.$$

Furthermore,

$$c'(y,z) \geq \tfrac{1}{4}(w^2+1)^{-1}\big(\|v\|^2+(1+z)^2+1\big)+w^2+1.$$

The substitutions $v = (w^2+1)^{1/2}\eta$ and $1+z = (w^2+1)^{1/2}\zeta$ then allow us to estimate by the product of

$$\int_{(\eta,\zeta)\in\mathbb{R}^{k-2}\times\mathbb{R}} (\|\eta\|^2+\zeta^2+1)^{\frac{1-n}{4}} \big(1+\log(\|\eta\|^2+\zeta^2+1)\big)^{-m_3}\,d(\eta,\zeta)$$

and

$$\int_{w\in\mathbb{R}} (w^2+1)^{-\frac{1}{2}} \big(1+\log(w^2+1)\big)^{-m_4}\,dw,$$

where $m_2 = m_3 + m_4$. Both integrals are easily seen to converge for $m_3, m_4 \geq 2$. □

4.7 The case of divergence

The following proposition and its corollary imply the 'only if' part of Theorem 4.8.

Proposition 4.12. *Let $\nu \in \mathbb{R}$ and let ϕ_ν be the function $X_n \to \mathbb{R}$ given by*

$$\phi_\nu(k'a_t \cdot H) = (2\cosh 4t)^\nu \qquad (k' \in K, t \in \mathbb{R}).$$

Then the integral $\int_{U_{k,l}} \phi_\nu(u\cdot H)\,du$ is divergent for $\nu > \min\{\frac{k-n}{2}, \frac{2-l}{2}\}$.

Proof. Let $\bar{\phi}_\nu$ be the function associated to ϕ_ν as in Corollary 4.10. Then $\bar{\phi}_\nu(r) = r^\nu$ for $r \geq 1$. Clearly, the integral

$$\int_{x\in\mathbb{R}^{n-k}} \big(\|x\|^2+c\big)^\nu\,dx$$

is divergent for all positive constants c if $\nu > \frac{k-n}{2}$. Hence, it follows from Corollary 4.10 that $\int_{U_{k,l}} \phi_\nu(u\cdot H)\,du$ is divergent for such ν. Note that ϕ_ν satisfies

$$\phi_\nu\Big(\sigma\theta(kg)\cdot H\Big) = \phi_\nu(g\cdot H) \qquad (g \in G, k \in K).$$

Combining this with Lemma 4.7, we infer that

$$\int_{U_{n+2-l,n+2-k}} \phi_\nu(u \cdot H)\, du = \int_{U_{k,l}} \phi_\nu(u \cdot H)\, du.$$

By the previous argument, we obtain that the first, hence also the second integral is divergent for $\nu > \frac{(n+2-l)-n}{2} = \frac{2-l}{2}$. The assertion now follows. □

Corollary 4.13. *Assume $k < \frac{n+1}{2}$ or $l > \frac{n+3}{2}$. Then there exists a function $\phi \in \mathcal{C}(X_n)$ such that $\int_{U_{k,l}} \phi(u \cdot H)\, du$ is divergent.*

Proof. Assume $k < \frac{n+1}{2}$ or $l > \frac{n+3}{2}$. Then $\min\{\frac{k-n}{2}, \frac{2-l}{2}\} < \frac{1-n}{4}$. We may therefore take ν such that $\min\{\frac{k-n}{2}, \frac{2-l}{2}\} < \nu < \frac{1-n}{4}$. Then $\phi_\nu \in \mathcal{C}(X_n)$ by Lemma 2.4. The result now follows by application of Proposition 4.12. □

5 Proof of Theorem 3.1 for σ-parabolic rank 0

We now turn to the proof of Theorem 3.1 under the assumption that P is of σ-parabolic rank 0. In 5.1 – 5.4 we first reduce the statement to a (seemingly) less general statement, which we then prove in 5.5 – 5.7.

5.1 Root systems

Recall the element κ from (2.1). We define $\mathfrak{b} = \mathrm{Ad}(\kappa)\mathfrak{a}$. Note that $\mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}$ and that $\mathfrak{a}_\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{h}$. Furthermore, $\mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{h}$ as well. We write B for $\exp(\mathfrak{b})$.

Recall the functionals e_k from Section 4.1. For $1 \leq k \leq n$ we define $f_k : \mathfrak{b} \to \mathbb{R}$ by $f_k = e_k \circ \mathrm{Ad}(\kappa)^{-1}$. The root system of $\mathfrak{b}$ in $\mathfrak{g}$ then equals

$$\overline{\Sigma} := \Sigma(\mathfrak{g}, \mathfrak{b}) = \big\{ f_i - f_j : 1 \leq i, j \leq n, i \neq j \big\}.$$

The associated root spaces are given by $\mathfrak{g}_{f_i - f_j} = \mathbb{R}\big(\mathrm{Ad}(\kappa) E_{i,j}\big)$.

Note that

$$\overline{\Sigma}_\mathfrak{h} := \Sigma_\mathfrak{h}(\mathfrak{b}) = \big\{ f_i - f_j : 1 \leq i, j \leq n-1, i \neq j \big\}$$

is both the root system of $\mathfrak{b}$ in $\mathfrak{h}$ and the set of $\mathfrak{h}$-roots in $\overline{\Sigma}$. Let

$$\overline{\Sigma}_\mathfrak{h}^+ = \big\{ f_i - f_j : 1 \leq i < j \leq n-1 \big\}. \tag{5.1}$$

Then $\overline{\Sigma}_\mathfrak{h}^+$ is a positive system for $\overline{\Sigma}_\mathfrak{h}$. Finally, we define

$$\bar{\rho}_\mathfrak{h} := \frac{1}{2} \sum_{\alpha \in \overline{\Sigma}_\mathfrak{h}^+} \alpha.$$

5.2 Classification of parabolic subgroups

Recall that $\mathcal{Q}$ denotes the set of minimal parabolic subgroups containing a maximal connected split abelian subgroup that is contained in H. Furthermore, $\mathcal{Q}(\mathfrak{b})$ denotes the subset of $\mathcal{Q}$ consisting of minimal parabolic subgroups containing B. Given $Q \in \mathcal{P}(\mathfrak{b})$, we agree to use the abbreviation $\overline{\Sigma}(Q) := \Sigma(\mathfrak{b}; Q)$.

Lemma 5.1. *Let $P \in \mathcal{Q}$. Then there exists a parabolic subgroup $Q \in \mathcal{Q}(\mathfrak{b})$ which is H-conjugate to P and satisfies*

$$\overline{\Sigma}_{\mathfrak{h}}^{+} \subseteq \overline{\Sigma}(Q). \tag{5.2}$$

Let Q be any such parabolic subgroup. Then $\rho_{Q,\mathfrak{h}} = \bar{\rho}_{\mathfrak{h}}$. Moreover, P is $\mathfrak{h}$-compatible if and only if Q is $\mathfrak{h}$-compatible (see Definition 2.5).

Proof. Since $P \in \mathcal{Q}$, it follows from Corollaries 1.10 and 1.13 that P is H-conjugate to a minimal parabolic subgroup $Q \in \mathcal{Q}(\mathfrak{b})$ satisfying (5.2). The latter condition implies that $\rho_{Q,\mathfrak{h}} = \bar{\rho}_{\mathfrak{h}}$. The final statement follows from Remark 2.6. □

We will now classify the parabolic subgroups $Q \in \mathcal{Q}(\mathfrak{b})$ satisfying (5.2). The assignment $Q \mapsto \overline{\Sigma}(Q)$ defines a bijection from the set $\mathcal{P}(\mathfrak{b})$ onto the set of positive systems for $\overline{\Sigma}$. In turn, the latter set is in bijective correspondence with the permutation group S_n. For a given $\tau \in S_n$, the associated positive system is given by

$$\overline{\Sigma}(Q) = \{f_{\tau^{-1}(i)} - f_{\tau^{-1}(j)} : 1 \leq i < j \leq n\}.$$

Equivalently, a root $f_i - f_j \in \overline{\Sigma}$ belongs to $\overline{\Sigma}(Q)$ if and only if $\tau(i) < \tau(j)$. We infer that the parabolic subgroups $Q \in \mathcal{Q}(\mathfrak{b})$ satisfying (5.2) correspond to the permutations $\tau \in S_n$ satisfying

$$\tau(i) < \tau(j) \quad \text{for} \quad 1 \leq i < j \leq n-1. \tag{5.3}$$

Given such a permutation $\tau \in S_n$, there exists a unique k, with $1 \leq k \leq n$, such that $\tau(n) = k$. Conversely, for each integer k with $1 \leq k \leq n$, there exists a unique $\tau \in S_n$ satisfying (5.3) and $\tau(n) = k$. From now on we write Q_k for the corresponding minimal parabolic subgroup and $\overline{\Sigma}_k^{+}$ for $\overline{\Sigma}(Q_k)$. Moreover, we write N_k for N_{Q_k}. For future reference we note that the positive system determined by k is given by the disjoint union

$$\overline{\Sigma}_k^{+} = \overline{\Sigma}_{\mathfrak{h}}^{+} \cup \big\{f_i - f_n : 1 \leq i \leq k-1\big\} \cup \big\{f_n - f_i : k \leq i \leq n-1\big\}. \tag{5.4}$$

We now provide $\mathfrak{b}^*$ with the inner product that turns $\mathrm{Ad}(\kappa)^* : \mathfrak{b}^* \to \mathfrak{a}^*$ into an isometry; see (4.7) for the description of the inner product on $\mathfrak{a}^*$.

Lemma 5.2. *Let i and j be integers such that $1 \leq i < j \leq n$. Then*

$$\langle f_i - f_j, \bar{\rho}_{\mathfrak{h}} \rangle = \begin{cases} j - i & \text{if} \quad 1 \leq i < j \leq n-1, \\ \frac{n}{2} - i & \text{if} \quad 1 \leq i \leq n-1 \quad \text{and} \quad j = n. \end{cases}$$

In particular $\langle \alpha, \bar{\rho}_{\mathfrak{h}} \rangle > 0$ for every $\alpha \in \overline{\Sigma}_k^{+}$ if and only if $\frac{n}{2} < k < \frac{n}{2} + 1$, i.e., if and only if n is odd and $k = \frac{n+1}{2}$.

Proof. Using the definition of $\bar{\rho}_{\mathfrak{h}}$, we find

$$2\bar{\rho}_{\mathfrak{h}} = \sum_{i=1}^{n-1}(n-2i)f_i.$$

The first statement follows directly from this formula and the second follows from comparison with (5.4). □

Combining the previous lemmas, we now arrive at the following proposition.

Proposition 5.3. *Let $Q \in \mathcal{Q}$. Then there exist a unique integer k, with $1 \leq k \leq n$, such that Q is H-conjugate to Q_k. Moreover, Q is $\mathfrak{h}$-compatible if and only if n is odd and $k = \frac{n+1}{2}$.*

Proof. Only the uniqueness remains to be proved. Let $1 \leq k, k' \leq n$ and assume that there exists an $h \in H$ such that $hQ_kh^{-1} = Q_{k'}$. Then B and hBh^{-1} are both σ-stable split components of $Q_{k'}$. From Theorem 1.8 it follows that there exists a unique $n \in N_{k'} \cap H$ such that $nhBh^{-1}n^{-1} = B$. Let $h' = nh$. Then $h' \in N_H(\mathfrak{b})$ and $h'Q_kh'^{-1} = Q_{k'}$. Now h' induces an element w in the Weyl group of the root system $\overline{\Sigma}_{\mathfrak{h}}$. Since $\overline{\Sigma}^+_{\mathfrak{h}}$ is a positive system for $\overline{\Sigma}_{\mathfrak{h}}$ and $\overline{\Sigma}^+_{\mathfrak{h}}$ is contained in both $\overline{\Sigma}^+_k$ and $\overline{\Sigma}^+_{k'}$, it follows that w acts trivially on $\overline{\Sigma}^+_{\mathfrak{h}}$ and hence that h' acts trivially on $\mathfrak{b}$. We conclude that $hQ_kh^{-1} = Q_{k'}$ and therefore $k = k'$. This proves uniqueness. □

5.3 Decomposition and invariant measures

Let k be an integer such that $1 \leq k \leq n$.

Lemma 5.4. *Let $\mathfrak{n}_k$ be the Lie algebra of N_k. Then*

$$\mathfrak{n}_k \cap \mathfrak{h} = \bigoplus_{\alpha \in \overline{\Sigma}^+_{\mathfrak{h}}} \mathfrak{g}_\alpha = \bigoplus_{1 \leq i < j \leq n-1} \mathbb{R}\big(\mathrm{Ad}(\kappa)E_{i,j}\big),$$

$$\mathfrak{n}_k \cap \mathfrak{q} = \bigoplus_{\alpha \in \overline{\Sigma} \setminus \overline{\Sigma}^+_{\mathfrak{h}}} \mathfrak{g}_\alpha = \bigoplus_{1 \leq i \leq k-1} \mathbb{R}\big(\mathrm{Ad}(\kappa)E_{i,n}\big) \oplus \bigoplus_{k \leq i \leq n-1} \mathbb{R}\big(\mathrm{Ad}(\kappa)E_{n,i}\big).$$

Proof. Let $\alpha \in \overline{\Sigma}$. Since $\sigma\alpha = \alpha$ and $\mathfrak{g}_\alpha$ is 1-dimensional, we have either $\mathfrak{g}_\alpha \subseteq \mathfrak{h}$ or $\mathfrak{g}_\alpha \subseteq \mathfrak{q}$. By definition, the first is the case for $\alpha \in \overline{\Sigma}_{\mathfrak{h}}$ and the latter for $\alpha \in \overline{\Sigma} \setminus \overline{\Sigma}_{\mathfrak{h}}$. The lemma now follows from (5.4). □

We write V_k for the submanifold $\exp(\mathfrak{n}_k \cap \mathfrak{q})$ of N_k. For $x, y \in \mathbb{R}^{n-1}$ with $\langle x, y\rangle = 0$ we define

$$v_{x,y} = \kappa \exp\left(\begin{array}{c|c} 0 & x \\ \hline y^t & 0 \end{array}\right)\kappa^{-1} = \kappa\left(\begin{array}{c|c} I_{n-1} + \frac{1}{2}xy^t & x \\ \hline y^t & 1 \end{array}\right)\kappa^{-1}. \tag{5.5}$$

A straightforward computation shows that

$$V_k = \big\{v_{x,y} : x \in \mathbb{R}^{k-1} \times \{0\}^{n-k}, y \in \{0\}^{k-1} \times \mathbb{R}^{n-k}\big\}.$$

We equip V_k with the push-forward along $\exp$ of the Lebesgue measure on $\mathfrak{n}_k \cap \mathfrak{q}$. Then the following lemma is a direct consequence of [5, Prop. 1.1] and Lemma 5.4.

Lemma 5.5. *The map*

$$V_k \times (N_k \cap H) \to N_k; \qquad (v, n) \mapsto vn$$

is a diffeomorphism. Moreover, there exists a normalization for the invariant measure dx on $N_k/(N_k \cap H)$ such that for every $\phi \in C_c^\infty\big(N_k/(N_k \cap H)\big)$,

$$\int_{N_k/(N_k \cap H)} \phi(x)\, dx = \int_{V_k} \phi\big(v \cdot (N_k \cap H)\big)\, dv.$$

5.4 Reduction to an equivalent theorem

Under the assumption that P is of σ-parabolic rank 0, it follows from Proposition 5.3 and Lemma 5.5 that Theorem 3.1 is equivalent to the following theorem.

Theorem 5.6. *Let k be an integer with $1 \le k \le n$. Then the integral*

$$\int_{V_k} \phi(v \cdot H)\, dv$$

is absolutely convergent for every $\phi \in \mathcal{C}(X_n)$ if and only if $k = \frac{n+1}{2}$. In particular, if n is even, then for every k there exists $\phi \in \mathcal{C}(X_n)$ such that the integral is divergent.

5.5 The integral

Recall (5.5) for the definition of $v_{x,y}$.

Lemma 5.7. *Let $x, y \in \mathbb{R}^{n-1}$ with $\langle x, y\rangle = 0$. If $v_{x,y} \cdot H \in Ka_t \cdot H$, then*

$$2\cosh(4t) = 2 + 4\|x\|^2 + 4\|y\|^2 + 4\|x\|^2\|y\|^2.$$

Proof. By straightforward computations we see that

$$v_{x,y}\sigma(v_{x,y})^{-1} = v_{x,y}^2 = v_{2x,2y} = \kappa \left(\begin{array}{c:c} I_{n-1} + 2xy^t & 2x \\ \hdashline 2y^t & 1 \end{array}\right) \kappa^{-1} \tag{5.6}$$

and

$$\|I_{n-1} + 2yx^t\|_{HS}^2 = n - 1 + 4\|x\|^2\|y\|^2.$$

Therefore,

$$\|v_{x,y}\sigma(v_{x,y})^{-1}\|_{HS}^2 = n + 4\|x\|^2 + 4\|y\|^2 + 4\|x\|^2\|y\|^2.$$

The lemma now follows from Lemma 2.1. □

Corollary 5.8. *Let $\phi \in C(X_n)$ be K-invariant and non-negative. As in Corollary 4.10, let $\bar\phi : \mathbb{R}_{\ge 2} \to \mathbb{R}_{\ge 0}$ be defined by*

$$\bar\phi(2\cosh 4t) = \phi(a_t \cdot H).$$

Then

$$\int_{V_k} \phi(v \cdot H)\, dv = \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}^{n-k}} \bar\phi\big(2 + 4\|x\|^2 + 4\|y\|^2 + 4\|x\|^2\|y\|^2\big)\, dy\, dx.$$

Proof. Let $x \in \mathbb{R}^{k-1} \times \{0\}^{n-k}$, $y \in \{0\}^{k-1} \times \mathbb{R}^{n-k}$ and assume that $v_{x,y} \in k' a_t H$, for $k' \in K$ and $t \in \mathbb{R}$. Then it follows that

$$\phi(v_{x,y} \cdot H) = \bar{\phi}(2\cosh 4t) = \bar{\phi}(2 + 4\|x\|^2 + 4\|y\|^2 + 4\|x\|^2\|y\|^2),$$

by Lemma 5.7. □

5.6 The case of convergence

In view of Remark 2.3, the following proposition implies the 'if' part of Theorem 5.6.

Proposition 5.9. *Assume $k = \frac{n+1}{2}$. For $m \in \mathbb{R}$, let $\phi_m : X_n \to \mathbb{R}$ be given by*

$$\phi_m(k' a_t \cdot H) = (2\cosh 4t)^{\frac{1-n}{4}} \big(1 + \log(2\cosh 4t)\big)^{-m} \qquad (k' \in K, t \in \mathbb{R}).$$

Then there exists $m \geq 0$ such that the integral $\displaystyle\int_{V_k} \phi_m(v \cdot H)\, dv$ is absolutely convergent.

Proof. Let $\bar{\phi}_m$ be defined in terms of $\bar{\phi}$ as in Corollary 5.8. Then $\bar{\phi}_m(z) = z^{\frac{1-n}{4}}(1 + \log z)^{-m}$ from which we see that $\bar{\phi}_m$ is a decrasing function of $z \geq 1$. Hence,

$$\bar{\phi}_m(2 + 4\|x\|^2 + 4\|y\|^2 + 4\|x\|^2\|y\|^2)) \leq \bar{\phi}_m(1 + \|x\|^2 + \|y\|^2 + \|x\|^2\|y\|^2),$$

for $x \in \mathbb{R}^{k-1} = \mathbb{R}^{\frac{n-1}{2}}$, $y \in \mathbb{R}^{n-k} = \mathbb{R}^{\frac{n-1}{2}}$ and by the mentioned corollary we see that

$$\begin{aligned}
&\int_{V_{\frac{n+1}{2}}} \phi_m(v \cdot H)\, dv \\
&\quad \leq \int_{\mathbb{R}^{\frac{n-1}{2}}} \int_{\mathbb{R}^{\frac{n-1}{2}}} \bar{\phi}_m(1 + \|x\|^2 + \|y\|^2 + \|x\|^2\|y\|^2)\, dy\, dx \\
&\quad = \int_{\mathbb{R}^{\frac{n-1}{2}}} \int_{\mathbb{R}^{\frac{n-1}{2}}} \frac{\big(1 + \|x\|^2\big)^{\frac{1-n}{4}} \big(1 + \|y\|^2\big)^{\frac{1-n}{4}}}{\big(1 + \log(1 + \|x\|^2 + \|y\|^2 + \|x\|^2\|y\|^2)\big)^m}\, dy\, dx.
\end{aligned}$$

Since

$$\big(1 + \log(1 + \|x\|^2 + \|y\|^2 + \|x\|^2\|y\|^2)\big)^2 \geq \big(1 + \log(1 + \|x\|^2)\big)\big(1 + \log(1 + \|y\|^2)\big),$$

the last double integral is at most

$$\left(\int_{\mathbb{R}^{\frac{n-1}{2}}} \frac{\big(1 + \|x\|^2\big)^{\frac{1-n}{4}}}{\big(1 + \log(1 + \|x\|^2)\big)^{\frac{m}{2}}}\, dr \right)^2.$$

By using polar coordinates, one readily verifies that the integral in this expression is absolutely convergent for $m > 2$. □

5.7 The case of divergence

The following proposition and its corollary imply the 'only if' part of Theorem 5.6.

Proposition 5.10. *Let $\nu \in \mathbb{R}$ and let $\phi_\nu : X_n \to \mathbb{R}$ be given by*

$$\phi_\nu(k' a_t \cdot H) = (2\cosh 4t)^\nu \qquad (k' \in K, t \in \mathbb{R}).$$

Then the integral $\int_{V_k} \phi_\nu(v \cdot H)\, dv$ is divergent for $\nu \geq \min\left\{\frac{1-k}{2}, \frac{k-n}{2}\right\}$.

Proof. The function $\bar{\phi}_\nu$ associated to ϕ_ν as in Corollary 5.8 is given by $z \mapsto z^\nu$. By the mentioned corollary we obtain

$$\int_{V_k} \phi_\nu(v \cdot H)\, dv = \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}^{n-k}} \left(2 + 4\|x\|^2 + 4\|y\|^2 + 4\|x\|^2\|y\|^2\right)^\nu dy\, dx.$$

Clearly the integral on the right-hand side is divergent if $\nu \geq 0$. We assume that $\nu < 0$. Then the integral on the right-hand side is larger than

$$\begin{aligned} &4^\nu \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}^{n-k}} \left(1 + \|x\|^2 + \|y\|^2 + \|x\|^2\|y\|^2\right)^\nu dy\, dx \\ &\qquad = 4^\nu \int_{\mathbb{R}^{k-1}} \left(1 + \|x\|^2\right)^\nu dx \int_{\mathbb{R}^{n-k}} \left(1 + \|y\|^2\right)^\nu dy. \end{aligned} \tag{5.7}$$

The condition on ν implies that $2\nu \geq 1 - k$ or $2\nu \geq k - n$. By using polar coordinates, we see that one of the integrals in (5.7) diverges. This completes the proof. □

Corollary 5.11. *Assume $k \neq \frac{n+1}{2}$. Then there exists a function $\phi \in \mathcal{C}(X_n)$ such that $\int_{V_k} \phi(v \cdot H)\, dv$ is divergent.*

Proof. Assume $k \neq \frac{n+1}{2}$. Then $\min\left\{\frac{1-k}{2}, \frac{k-n}{2}\right\} < \frac{1-n}{4}$ and we may select ν such that

$$\min\left\{\tfrac{1-k}{2}, \tfrac{k-n}{2}\right\} < \nu < \tfrac{1-n}{4}.$$

Then $\phi_\nu \in \mathcal{C}(X_n)$ by Lemma 2.4. The claim now follows from Proposition 5.10. □

6 Proof of Theorem 3.2

6.1 Reduction to an equivalent theorem

By Proposition 4.4 and Proposition 5.3 it suffices to prove the claims in Theorem 3.2 for $P = P_{k,l}$ with $\frac{n+1}{2} \leq k = l \leq \frac{n+3}{2}$ and (for n odd) $Q = Q_{\frac{n+1}{2}}$ only.

We recall the definition of a_t from (2.2). An easy computation shows that

$$\delta_{P_{k,k}}(a_t) = e^t.$$

It follows from Theorem 4.8 that for all $\phi \in \mathcal{C}(X_n)$ the integral

$$\int_{U_{k,k}} \phi(a_s u \cdot H)\, du$$

yields a well-defined function of $s \in \mathbb{R}$. We are interested in the decay of this function, or more precisely, of the modified function

$$s \mapsto e^s \int_{U_{k,k}} \phi(a_s u \cdot H)\, du,$$

and will prove the following result, which implies Theorem 3.2.

Theorem 6.1. *Assume $\frac{n+1}{2} \le k \le \frac{n+3}{2}$. Let $\phi \in \mathcal{C}(X_n)$.*

(i) If n is even, then for every $N \in \mathbb{N}$ there exist $c > 0$ and $m \in \mathbb{N}$ such that for every $s \in \mathbb{R}$,

$$\left| e^s \int_{U_{k,k}} \phi(a_s u \cdot H)\, du \right| \le c(1+|s|)^{-N} \mu_{1,m}(\phi). \tag{6.1}$$

(ii) If n is odd, then for every $R \in \mathbb{R}$ and $N \in \mathbb{N}$ there exist a $c > 0$ and $m \in \mathbb{N}$ such that for every $s \in \mathbb{R}$ with $s < R$,

$$\left| e^s \int_{U_{k,k}} \phi(a_s u \cdot H)\, du \right| \le c(1+|s|)^{-N} \mu_{1,m}(\phi). \tag{6.2}$$

Furthermore, for s moving in the other direction, there exists an element $\kappa_0 \in K$, independent of ϕ, such that

$$\lim_{s\to\infty} e^s \int_{U_{k,k}} \phi(a_s u \cdot H)\, du = \int_{V_{\frac{n+1}{2}}} \phi(\kappa_0 v \cdot H)\, dv. \tag{6.3}$$

In particular, the limit exists, and is non-zero as a function of ϕ.

6.2 Proof of Theorem 6.1

We recall the definition of $u_{x,y,z}$ from (4.9) and start with a few lemmas.

Lemma 6.2. *Let $s \in \mathbb{R}$, $x \in \mathbb{R}^{n-k}$, $y \in \mathbb{R}^{k-2}$ and $z \in \mathbb{R}$. If $a_s u_{x,y,z} \cdot H \in K a_t \cdot H$, then*

$$2\cosh(4t) = f_1 + f_2\|x\|^2 + f_3\|y\|^2 + \|x\|^2\|y\|^2, \tag{6.4}$$

where

$$f_1 = f_1(s,z) = e^{4s}(1-z)^2(1+z)^2 + e^{-4s} + 2z^2, \tag{6.5}$$
$$f_2 = f_2(s,z) = e^{2s}(1+z)^2 + e^{-2s}, \tag{6.6}$$
$$f_3 = f_3(s,z) = e^{2s}(1-z)^2 + e^{-2s}. \tag{6.7}$$

Proof. From (4.9) we find

$$a_s u_{x,y,z} = \left(\begin{array}{c|c|c} e^s & e^s x^t & e^s z \\ \hline & I_{n-2} & y \\ \hline & & e^{-s} \end{array}\right),$$

hence

$$a_s u_{x,y,z} \sigma(a_s u_{x,y,z})^{-1} = \left(\begin{array}{c|c|c} e^{2s}(1-z^2) & e^s(1+z)x^t & z \\ \hline e^s(1-z)y & I_{n-2} + yx^t & e^{-s}y \\ \hline -z & e^{-s}x^t & e^{-2s} \end{array}\right).$$

The proof is completed by combining Lemma 2.1 with a straightforward computation of the squared Hilbert-Schmid norm of the last matrix, analogous to the computation in the proof of Lemma 4.9 (note that now $\langle x, y\rangle = 0$ since $k = l$). □

We shall need some estimates for f_1, f_2 and f_3.

Lemma 6.3. *Let $s, z \in \mathbb{R}$ and let $f_i = f_i(s, z)$ be as above for $i = 1, 2, 3$. Then*

$$f_2 f_3 = f_1 + 2, \quad 2 \le f_1, \quad f_1 \le f_2 f_3 \le 2 f_1 \tag{6.8}$$

and

$$1 + z^2 \le f_1. \tag{6.9}$$

Proof. The equality in (6.8) is easily verified, and the lower bound for f_1 follows from (6.4) with $x = y = 0$. Then, $f_1 \le f_1 + 2 \le 2f_1$ implies the final statement in (6.8). Finally we observe that in addition to $2 \le f_1$ we also have $2z^2 \le f_1$, whence (6.9). □

Lemma 6.4. *Let $R \in \mathbb{R}$. Then there exists a constant $A > 0$ such that*

$$f_i(s, z) \ge A(e^{2s} z^2 + e^{-2s}), \quad i = 2, 3, \tag{6.10}$$

for all $z \in \mathbb{R}$ and all $s \le R$.

Proof. We may assume $i = 2$ since $f_3(s, z) = f_2(s, -z)$. Let $b \in \mathbb{R}$ be the solution to $b^2 - b = e^{-4R}$ that is larger than 1. We shall establish (6.10) for all $s \le R$ with

$$A = \frac{b-1}{b}.$$

Inserting the definition of f_2 we see that with this value of A, (6.10) is equivalent to

$$e^{2s}(z+1)^2 + e^{-2s} \ge \frac{b-1}{b}(e^{2s} z^2 + e^{-2s})$$

and hence also to

$$e^{2s}(z^2 + 2bz + b) + e^{-2s} \ge 0.$$

This last inequality is valid for all $z \in \mathbb{R}$ and $s \le R$ since the minimum of $z^2 + 2bz + b$ as a function of z is $-b^2 + b = -e^{-4R}$. □

Proof of Theorem 6.1. Let $\phi \in \mathcal{C}(X_n)$. Throughout the proof we will use the notation

$$I_\phi(s) := \int_{U_{k,k}} \phi(a_s u)\, du.$$

The proof consists of three parts. In part (a) we will address the rapid decay of $e^s I_\phi(s)$ for $s \to -\infty$ both for n odd and even. In part (b) we will address the similar decay for $s \to \infty$ in case n is even. Finally, in part (c) we will address the limit behavior for $s \to \infty$ in case n is odd.

Fix $N \in \mathbb{N}$. Then by (2.4), the function ϕ satisfies the estimate

$$|\phi(x)| \le \mu_{1,N}(\phi) f(x) \qquad (x \in X_n), \tag{6.11}$$

where $f : X_n \to \mathbb{R}_{>0}$ is given by

$$f(k' a_t \cdot H) = \big(2\cosh(4t)\big)^{-\frac{n-1}{4}} \big(1 + \log\big(2\cosh(4t)\big)\big)^{-N} \qquad (k' \in K, t \in \mathbb{R}). \tag{6.12}$$

Let

$$I_f(s) := \int_{U_{k,k}} f(a_s u \cdot H)\, du,$$

then we have the estimate

$$|I_\phi(s)| \le \mu_{1,N}(\phi) I_f(s),$$

so that for parts (a) and (b) it suffices to show that (6.1) and (6.2) are satisfied with ϕ replaced by f. Define $\Psi : \mathbb{R}_{\ge 1} \to \mathbb{R}$ by

$$\Psi(r) = r^{-\frac{n-1}{4}} (1 + \log(r))^{-N}, \quad (r \ge 1, N \ge 0). \tag{6.13}$$

Then it follows that

$$\Psi(2\cosh(4t)) = f(k' a_t \cdot H)$$

for $k' \in K$ and $t \in \mathbb{R}$. Moreover, using (6.4) we see that

$$f(a_s u_{x,y,z}) = \Psi(f_1 + f_2\|x\|^2 + f_3\|y\|^2 + \|x\|^2\|y\|^2). \tag{6.14}$$

Hence,

$$I_f(s) = \int_{z\in\mathbb{R}} \int_{y\in\mathbb{R}^{k-2}} \int_{x\in\mathbb{R}^{n-k}} \Psi(f_1 + f_2\|x\|^2 + f_3\|y\|^2 + \|x\|^2\|y\|^2)\, dx\, dy\, dz.$$

Part (a). Performing the following substitutions on the inner integrals,

$$x = (f_1/f_2)^{1/2}\xi, \quad y = (f_1/f_3)^{1/2}\eta,$$

we obtain

$$I_f(s) = \int_{z\in\mathbb{R}} \int_{\eta\in\mathbb{R}^{k-2}} \int_{\xi\in\mathbb{R}^{n-k}} \Psi(F(\xi,\eta) f_1) \big(\frac{f_1}{f_2}\big)^{\frac{n-k}{2}} \big(\frac{f_1}{f_3}\big)^{\frac{k-2}{2}} d\xi\, d\eta\, dz, \tag{6.15}$$

where

$$F(\xi,\eta) := 1 + \|\xi\|^2 + \|\eta\|^2 + \frac{f_1}{f_2 f_3}\|\xi\|^2\|\eta\|^2.$$

In the following we assume that $N \geq 6$. Using (6.13) we now see that the integrand in (6.15) is absolutely bounded by

$$F(\xi,\eta)^{-\frac{n-1}{4}} f_1^{\frac{n-3}{4}} f_2^{-\frac{n-k}{2}} f_3^{-\frac{k-2}{2}} \big(1 + \log F(\xi,\eta) + \log(f_1)\big)^{-N}. \tag{6.16}$$

Let

$$\epsilon = \tfrac{n+3}{2} - k;$$

then $0 \leq \epsilon \leq 1$ by our assumption on k. Moreover,

$$n - k = \tfrac{n-3}{2} + \epsilon, \qquad k - 2 = \tfrac{n-3}{2} + 1 - \epsilon. \tag{6.17}$$

Since $f_1 \leq f_2 f_3$, it follows that the expression (6.16) is bounded from above by

$$F(\xi,\eta)^{-\frac{n-1}{4}} f_2^{-\frac{\epsilon}{2}} f_3^{-\frac{1-\epsilon}{2}} \big(1 + \log F(\xi,\eta) + \log(f_1)\big)^{-N}$$

for all s.

From (6.8) we infer that

$$F(\xi,\eta) \geq (1 + \tfrac{1}{2}\|\xi\|^2)(1 + \tfrac{1}{2}\|\eta\|^2).$$

It thus finally follows that

$$I_f(s) \leq I_1 I_2 I_3(s), \tag{6.18}$$

where

$$I_1 = \int_{\mathbb{R}^{n-k}} g_1(\xi)\, d\xi, \quad I_2 = \int_{\mathbb{R}^{k-2}} g_2(\eta)\, d\xi, \quad I_3(s) = \int_{\mathbb{R}} g_3(s,z)\, dz, \tag{6.19}$$

with

$$\begin{aligned} g_1(\xi) &= (1 + \tfrac{1}{2}\|\xi\|^2)^{-\frac{n-1}{4}} \big(1 + \log(1 + \tfrac{1}{2}\|\xi\|^2)\big)^{-2}, \\ g_2(\eta) &= (1 + \tfrac{1}{2}\|\eta\|^2)^{-\frac{n-1}{4}} \big(1 + \log(1 + \tfrac{1}{2}\|\eta\|^2)\big)^{-2}, \\ g_3(s,z) &= f_2(s,z)^{-\frac{\epsilon}{2}} f_3(s,z)^{-\frac{1-\epsilon}{2}} \big(1 + \log(f_1(s,z))\big)^{-(N-4)}. \end{aligned} \tag{6.20}$$

It follows from (6.17) that the dimensions $n-k$ and $k-2$ are at most $\frac{n-1}{2}$ so that I_1 and I_2 are finite, thanks to the logarithmic terms (which in fact are needed in at most one of the integrals). Thus it only remains to estimate the third integral in (6.19).

We first assume $s \leq R$ for some given $R \in \mathbb{R}$. Using (6.10) for f_2 and f_3, and estimating two of the logarithmic factors in $I_3(s)$ by (6.9) and the remaining ones by $f_1 \geq \max\{2, e^{-4s}\}$, we find

$$I_3(s) \leq C \int_{\mathbb{R}} (e^{2s} z^2 + e^{-2s})^{-1/2} \big(1 + \log(1 + z^2)\big)^{-2}\, dz \, \big(1 + \max\{\log 2, -4s\}\big)^{-(N-6)}$$

for all $s \le R$, with $C > 0$ a constant depending on N. By substitution of $z = e^{-2s}\zeta$, and using that $s \le R$, we find

$$I_3(s) \le Ce^{-s} \int_{\mathbb{R}} (\zeta^2+1)^{-1/2} \big(1+\log(1+e^{-4R}\zeta^2)\big)^{-2}\, d\zeta \, \big(1+|s|\big)^{-(N-6)},$$

with a new constant $C > 0$. The integral converges, and since N was arbitrary we conclude from (6.18) that (6.2) holds, regardless of the parity of n. This completes part (a) of the proof.

Part (b). We assume that $s \ge 0$ and that n is even. Then $k = \frac{n+2}{2}$ and $\epsilon = \frac{1}{2}$, hence

$$I_3(s) = \int_{\mathbb{R}} f_2(s,z)^{-\frac{1}{4}} f_3(s,z)^{-\frac{1}{4}} \big(1+\log(f_1(s,z))\big)^{-(N-4)}\, dz.$$

The integral over $\mathbb{R}$ can be replaced by an integral over $\mathbb{R}_{>0}$, because $f_2(s,-z) = f_3(s,z)$ and $f_1(s,-z) = f_1(s,z)$. We split the integration into two parts, and integrate separately over the interval $[1-\delta, 1+\delta]$ and its complement in $\mathbb{R}_{>0}$, with $\delta \in (0,1)$ to be fixed later (it will depend on s). Let us write $J_\delta(s)$ for the integral over $[1-\delta, 1+\delta]$ and ${}^cJ_\delta(s)$ for the integral over the complement of this set in $\mathbb{R}_{>0}$.

For $J_\delta(s)$ we use the estimates

$$f_1 \ge 2, \quad f_2(s,z) \ge e^{2s}, \quad f_3(s,z) \ge e^{2s}(z-1)^2,$$

and for $z \ge 0$ and obtain

$$J_\delta(s) \le e^{-s} \int_{1-\delta}^{1+\delta} |z-1|^{-1/2}\, dz = 4e^{-s}\delta^{1/2}. \tag{6.21}$$

For ${}^cJ_\delta(s)$ we estimate f_1 by (6.9) in two of the logarithmic factors and by $f_1 \ge e^{4s}\delta^2$ in the remaining factors. Furthermore, we estimate

$$f_2(s,z) \ge e^{2s}(1+z)^2, \quad f_3(s,z) \ge e^{2s}(z-1)^2,$$

and obtain

$${}^cJ_\delta(s) \le Ce^{-s} \int_0^\infty (1+z)^{-1/2} |z-1|^{-1/2} \big(1+\log(1+z^2)\big)^{-2}\, dz \big(1+\log(e^{4s}\delta^2)\big)^{-(N-6)},$$

with a constant $C > 0$ depending on N but independent of s and δ. The integral over z converges and we obtain (with a new constant $C > 0$ of the same (in)dependency),

$${}^cJ_\delta(s) \le Ce^{-s} \big(1+\log(e^{4s}\delta^2)\big)^{-(N-6)}. \tag{6.22}$$

By adding (6.21) and (6.22), we see that by choosing $\delta = e^{-\frac{3}{2}s}$ we can ensure that

$$I_3(s) \le Ce^{-s}(1+s)^{-(N-6)},$$

with yet another constant $C > 0$. This implies (6.1) for the remaining case $s \ge 0$.

Part (c). We now turn to the statements about the limit in Theorem 6.1. Assume that n is odd. We shall first deal with the case $k = l = \frac{n+1}{2}$ and consider the integral

$$\begin{aligned} I_\phi(s) &= \int_{U_{\frac{n+1}{2},\frac{n+1}{2}}} \phi(a_s u \cdot H)\, du \\ &= \int_{z\in\mathbb{R}} \int_{y\in\mathbb{R}^{\frac{n-3}{2}}\times\{0\}^{\frac{n-1}{2}}} \int_{x\in\{0\}^{\frac{n-3}{2}}\times\mathbb{R}^{\frac{n-1}{2}}} \phi(a_s u_{x,y,z})\, dx\, dy\, dz. \end{aligned} \tag{6.23}$$

This time, we perform the substitution of variables

$$x = e^s\xi, \quad y = e^{-s}\eta, \quad \text{and} \quad z = e^{-2s}\omega - 1, \tag{6.24}$$

and obtain from (6.23) that

$$e^s I_\phi(s) = \int_{\omega\in\mathbb{R}} \int_{\eta\in\mathbb{R}^{\frac{n-3}{2}}\times\{0\}^{\frac{n-1}{2}}} \int_{\xi\in\{0\}^{\frac{n-3}{2}}\times\mathbb{R}^{\frac{n-1}{2}}} \Phi_s(\xi,\eta,\omega)\, d\xi\, d\eta\, d\omega, \tag{6.25}$$

where

$$\Phi_s(\xi,\eta,\omega) = \phi(a_s u_{e^s\xi, e^{-s}\eta, e^{-2s}\omega-1}).$$

Recall the definition of κ from (2.1). From Lemma 6.5 below we see that

$$\lim_{s\to\infty} \Phi_s(\xi,\eta,\omega) = \phi(\kappa^{-1} v_{(\omega,\eta),\frac{2}{3}\xi}),$$

for all $(\omega,\eta,\xi) \in \mathbb{R}^{n-1}$. Assuming that we may interchange the limit for $s\to\infty$ with the integral on the right-hand side of (6.25) we obtain

$$\begin{aligned} \lim_{s\to\infty} e^s I_\phi(s) &= \int_{\mathbb{R}} \int_{\mathbb{R}^{\frac{n-3}{2}}} \int_{\mathbb{R}^{\frac{n-1}{2}}} \phi(\kappa^{-1} v_{(\omega,\eta),\frac{2}{3}\xi})\, d\xi\, d\eta\, d\omega \\ &= \int_{V_{\frac{n+1}{2}}} \phi(\kappa^{-1} v)\, dv, \end{aligned}$$

for the choice of Lebesgue measure dv corresponding to $(2/3)^{\frac{n-1}{2}} d\xi\, d\eta\, d\omega$.

Thus, for the proof of (6.3) it remains to be shown that we may interchange limit and integral in (6.25). To prove this, we adopt the following strategy.

For $0 < \delta < 1$ and $s > 0$ we define the set

$$A_{\delta,s} := \mathbb{R}^{\frac{n-1}{2}} \times \mathbb{R}^{\frac{n-3}{2}} \times [(2-\delta)e^{2s}, (2+\delta)e^{2s}]$$

and denote by $B_{\delta,s}$ its complement in $\mathbb{R}^{n-1} \simeq \mathbb{R}^{\frac{n-1}{2}} \times \mathbb{R}^{\frac{n-3}{2}} \times \mathbb{R}$. We observe that for every $v \in \mathbb{R}^{n-1}$ there exists $s_0 \in \mathbb{R}$ such that $v \in B_{\delta,s}$ for all $0 < \delta < 1$ and $s \geq s_0$. Accordingly, the characteristic function $1_{B_{\delta,s}}$ converges to the constant function 1, pointwise on $\mathbb{R}^{n-1}$, for $s\to\infty$.

In the text below, we will show that

$$\int_{A_{\delta,s}} |\Phi_s(\xi,\eta,\omega)|\, d\xi\, d\eta\, d\omega \leq C\delta \tag{6.26}$$

for a suitable constant $C > 0$, independent of s and δ. On the other hand, we will show that for each $0 < \delta < 1$ there exists an integrable function $F_\delta : \mathbb{R}^{n-1} \to \mathbb{R}_{\geq 0}$, such that

$$1_{B(\delta,s)}|\Phi_s| \leq F_\delta \quad \text{on } \mathbb{R}^{n-1} \tag{6.27}$$

for all $s > 0$. By application of Lebesgue's convergence theorem it then follows that

$$\lim_{s\to\infty} \int_{B_{\delta,s}} \Phi_s \, d\xi \, d\eta \, d\omega = \int_{\mathbb{R}^n} \lim_{s\to\infty} \Phi_s \, d\xi \, d\eta \, d\omega.$$

Combining this with (6.26) we readily see that the interchange of limit and integral is allowed.

To achieve the goals mentioned above, we recall the definition of f_1, f_2 and f_3 from (6.5) - (6.7), but now considered as functions of (s, ω). Then by virtue of the substitution (6.24), if follows from (6.6) that

$$f_2 = e^{-2s}(\omega^2 + 1). \tag{6.28}$$

Furthermore, we define f and Ψ as in (6.12) and (6.13). Then from (6.11) and (6.14) we infer that

$$|\Phi_s(\xi, \eta, \omega)| \leq \mu_{1,N}(\phi)\, \Psi(f_1 + e^{2s} f_2 \|\xi\|^2 + e^{-2s} f_3 \|\eta\|^2 + \|\xi\|^2\|\eta\|^2).$$

Since Ψ is decreasing on $\mathbb{R}_{\geq 1}$, whereas $f_3 \geq f_1/f_2$ by (6.8), it follows that

$$|\Phi_s(\xi, \eta, \omega)| \leq \mu_{1,N}(\phi)\, \Psi_s(\xi, \eta, \omega), \tag{6.29}$$

where

$$\Psi_s(\xi, \eta, \omega) := \Psi(f_1 + e^{2s} f_2 \|\xi\|^2 + \frac{f_1}{e^{2s} f_2}\|\eta\|^2 + \|\xi\|^2\|\eta\|^2).$$

This estimate, combined with (6.28), motivates the use of a final substitution

$$\xi = (1 + \omega^2)^{-\frac{1}{2}}\chi, \quad \eta = (1 + \omega^2)^{\frac{1}{2}}\psi.$$

Note that this substitution does not effect the subsets $A(\delta, s)$ and $B(\delta, s)$ of $\mathbb{R}^{n-1}$, defined above. For any measurable subset $S \subseteq \mathbb{R}$, the function Ψ_s is integrable over $\mathbb{R}^{n-2} \times S$ if and only if the function

$$\widetilde{\Psi}_s(\chi, \psi, \omega) := (1 + \omega^2)^{-\frac{1}{2}}\ \Psi(f_1 + \|\chi\|^2 + f_1\|\psi\|^2 + \|\chi\|^2\|\psi\|^2)$$

is integrable over this set, and accordingly,

$$\int_{\mathbb{R}^{n-2}\times S} \Psi_s(\xi, \eta, \omega)\ d\xi\, d\eta\, d\omega = \int_{\mathbb{R}^{n-2}\times S} \widetilde{\Psi}_s(\chi, \psi, \omega)\ d\chi\, d\psi\, d\omega.$$

We observe that

$$1 + \|\chi\|^2 + \|\psi\|^2 + \|\chi\|\|\psi\| \geq (1 + \frac{1}{2}\|\chi\|)(1 + \frac{1}{2}\|\psi\|).$$

Since $\Psi : r \mapsto r^{-\frac{n-1}{4}}(1+\log r)^{-N}$ is decreasing and $f_1 \geq 2$, we now obtain the estimate

$$\begin{aligned}\widetilde{\Psi}(\chi,\psi,\omega) &\leq (1+\omega^2)^{-1/2}\Psi(f_1+\|\chi\|^2+\|\psi\|^2+\|\chi\|^2\|\psi\|^2)\\ &\leq g_1(\chi)\, g_2(\psi)\, h_s(\omega),\end{aligned}$$

where g_1, g_2 are defined as in (6.20) and where

$$h_s(\omega) := (1+\omega)^{-\frac{1}{2}}(1+\log f_1)^{-(N-4)}.$$

We observed already that the functions g_1 and g_2 are integrable over $\mathbb{R}^{\frac{n-1}{2}}$ and $\mathbb{R}^{\frac{n-3}{2}}$, with integrals I_1 and I_2, respectively. Let us therefore focus on the function h_s. On the interval $[(2-\delta)e^{2s}, (2+\delta)e^{2s}]$ we have the estimates $1+\omega^2 \geq e^{4s}$ and $f_1 \geq 1$. Hence,

$$\int_{I_{\delta,s}} h_s(\omega)\, d\omega \leq \int_{(2-\delta)e^{2s}}^{(2+\delta)e^{2s}} e^{-2s}\, d\omega = 2\delta.$$

It follows that

$$\int_{A_{\delta,s}} \Psi_s(\xi,\eta,\omega)\, d\xi\, d\eta\, d\omega \leq 2I_1I_2\delta$$

and by applying (6.29) we obtain the estimate (6.26).

It remains to prove the claimed majorization of Φ_s on $B_{\delta,s}$. Here we shall use the following lower bound on f_1,

$$f_1 \geq 1+\frac{1}{2}\delta^2\omega^2 \qquad (\omega \notin [(2-\delta)e^{2s}, (2+\delta)e^{2s}]). \tag{6.30}$$

To see this, note that the condition on ω is equivalent to $|2-e^{-2s}\omega| \geq \delta$, and hence (6.30) follows immediately from the estimates $f_1 \geq \frac{1}{2}f_2f_3$ and $f_1 \geq 2$ (see (6.8)).

From (6.30) we obtain the estimate

$$h_s(\omega) \leq (1+\omega^2)^{-\frac{1}{2}}(1+\frac{1}{2}\delta^2\omega^2)^{-(N-4)} \qquad (\omega \in \mathbb{R} \setminus [(2-\delta)e^{2s}, (2+\delta)e^{2s}])$$

for every $s > 0$. We now make the additional assumption that $N \geq 5$ to ensure that the function on the right-hand side is integrable over $\mathbb{R}$.

Define the function $\widetilde{G}_\delta : \mathbb{R}^{n-1} \to \mathbb{R}_{\geq 0}$ by

$$\widetilde{G}_\delta(\chi,\psi,\omega) = g_1(\chi)\, g_2(\psi)\, (\omega^2+1)^{-\frac{1}{2}}\big(1+\log(1+\frac{1}{2}\delta^2\omega^2)\big)^{-(N-4)}.$$

Then $\widetilde{G}_\delta$ is integrable on $\mathbb{R}^{n-1}$ and for every $s > 0$ we have the estimate

$$\widetilde{\Psi}_s \leq \widetilde{G}_\delta \qquad \text{on } B_{\delta,s}.$$

Define $G_\delta : \mathbb{R}^{n-1} \to \mathbb{R}_{\geq 0}$ by

$$G_\delta(\xi,\eta,\omega) = (1+\omega^2)^{\frac{1}{2}}\widetilde{G}_\delta((1+\omega^2)^{\frac{1}{2}}\xi, (1+\omega^2)^{-\frac{1}{2}}\eta, \omega).$$

Then G_δ is integrable on $\mathbb{R}^{n-1}$, and it follows that

$$\Psi_s \le G_\delta \qquad \text{on } \ B_{\delta,s},$$

for every $s > 0$. In view of (6.29) this proves (6.27) with $F_\delta := \mu_{1,N}(\phi)G_\delta$. We have thus established the limit formula (6.3) for the case $k = l = \frac{n+1}{2}$.

We will complete the proof of Theorem 6.1 by proving (6.3) for the remaining case $k = l = \frac{n+3}{2}$. Let w be the longest Weyl group element for the root system $\Sigma_\mathfrak{h} = \Sigma\big(Z_\mathfrak{h}(\mathfrak{a}_\mathfrak{q}); \mathfrak{a}_\mathfrak{h}\big)$, relative to the positive system (4.1), and let $w_0 \in N_{K\cap H}(\mathfrak{a}) \cap Z_{K\cap H}(\mathfrak{a}_\mathfrak{q})$ be a representative for w. By Lemma 4.7,

$$\begin{aligned}\lim_{s\to\infty} e^s \int_{U_{\frac{n+3}{2},\frac{n+3}{2}}} \phi(a_s u)\,du &= \lim_{s\to\infty} e^s \int_{U_{\frac{n+1}{2},\frac{n+1}{2}}} \phi\big(w_0\,\sigma\theta(a_s u)\big)\,du \\ &= \int_{V_{\frac{n+1}{2}}} \phi\big(w_0\,\sigma\theta(\kappa^{-1}v)\cdot H\big)\,dv \\ &= \int_{\theta V_{\frac{n+1}{2}}} \phi\big(w_0\,\kappa\bar v\cdot H\big)\,d\bar v. \qquad (6.31)\end{aligned}$$

Let w_1 be a representative in $N_{K\cap H}(\mathfrak{b})$ for the longest Weyl group element of $\Sigma_\mathfrak{h}(\mathfrak{b})$, relative to the positive system (5.1). Then, in view of Proposition 5.3,

$$w_1\theta V_{\frac{n+1}{2}} w_1^{-1} = V_{\frac{n+1}{2}}.$$

Hence, the integral in (6.31) is equal to

$$\int_{V_{\frac{n+1}{2}}} \phi\big(w_0\kappa w_1^{-1}v\cdot H\big)\,dv.$$

This completes the proof of Theorem 6.1. □

Lemma 6.5. *Let $k = l = \frac{n+1}{2}$. Then with notation as in (4.9) and (5.5) and writing $u(x,y,z) = u_{x,y,z}$, we have the following limit in X_n,*

$$\lim_{s\to\infty} a_s\, u(e^s\xi, e^{-s}\eta, e^{-2s}\omega - 1)\cdot H = \kappa^{-1} v_{(\omega,\eta),\frac{2}{3}\xi}\cdot H.$$

Proof. We start with a computation in $\mathrm{SL}(2,\mathbb{R})$, whose Lie algebra has the standard Iwasawa decomposition $\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}U \oplus \mathbb{R}Y \oplus \mathbb{R}V$, with

$$U := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We agree to write $k_\varphi := \exp \varphi U$, $a_s := \exp sY$ and $n_z := \exp zV$. In particular,

$$\kappa := k_{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad n_{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Let σ be the involution of $\mathrm{SL}(2,\mathbb{R})$ given by switching the diagonal entries as well as the off diagonal entries, and let H_0 denote the associated group of fixed points.

We will first show that in the quotient $\mathrm{SL}(2,\mathbb{R})/H_0$ we have

$$\lim_{s\to\infty} a_s n_{-1}\cdot H_0 = \kappa^{-1}\cdot H_0. \tag{6.32}$$

According to [13, Thm. 1.3] the map $G\to G$, $g\mapsto g\sigma(g)^{-1}$ induces a diffeomorphism from G/G^σ onto a submanifold of G. Applying this general fact to the situation at hand, we see that for (6.32) to be valid, it suffices to show that

$$a_s n_{-1}\sigma(a_s n_{-1})^{-1}\to \kappa\sigma(\kappa)^{-1} = k_{\frac{\pi}{2}} \qquad (s\to\infty).$$

Now this follows by a straightforward calculation.

In turn, it follows from (6.32) that there exists a function $q:\mathbb{R}\to\mathrm{SL}(2,\mathbb{R})$ with $\lim_{s\to\infty} q(s)=e$ and

$$n_1 a_{-s}\kappa^{-1}\, q(s)\in H_0 \qquad (s\in\mathbb{R}).$$

We agree to identify $\mathrm{SL}(2,\mathbb{R})$ with a closed subgroup of $\mathrm{SL}(n,\mathbb{R})$ via the embedding

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} \;\mapsto\; \left(\begin{array}{c:c:c} a & & b\\ \hdashline & I_{n-2} & \\ \hdashline c & & d\end{array}\right).$$

Then κ and a_s in $\mathrm{SL}(2,\mathbb{R})$ correspond with the similarly denoted elements in $\mathrm{SL}(n,\mathbb{R})$. Furthermore, H_0 equals the intersection of $\mathrm{SL}(2,\mathbb{R})$ with the subgroup H of $\mathrm{SL}(n,\mathbb{R})$.

For $x\in\{0\}^{\frac{n-3}{2}}\times\mathbb{R}^{\frac{n-1}{2}}$, $y\in\mathbb{R}^{\frac{n-3}{2}}\times\{0\}^{\frac{n-1}{2}}$ and $z\in\mathbb{R}$ we define

$$w(x,y,z) := \exp\left(\begin{array}{c:c:c} & & z\\ \hdashline & & y\\ \hdashline & x^t & \end{array}\right) = \left(\begin{array}{c:c:c} 1 & \frac{1}{2}zx^t & z\\ \hdashline & I_{n-2}+\frac{1}{2}yx^t & y\\ \hdashline & x^t & 1\end{array}\right). \tag{6.33}$$

Then by a straightforward calculation, one checks that the matrix

$$A = w\big(x,y,z\big)^{-1} u\big((1-\tfrac{1}{2}z)x,y,z\big)$$

satisfies $SAS^{-1}=A$, hence belongs to H. Thus, if $z\neq 2$, then

$$\begin{aligned} a_s\, u(x,y,z)\cdot H &= a^s w\big((1-\tfrac{1}{2}z)^{-1}x,y,z\big)\cdot H\\ &= a^s\, w\big((1-\tfrac{1}{2}z)^{-1}x,y,z\big)\, n_1\, a_{-s}\, \kappa^{-1}\, q(s)\cdot H\\ &= a^s\, w\big((1-\tfrac{1}{2}z)^{-1}x,y,z+1\big) a_s^{-1}\, \kappa^{-1}\, q(s)\cdot H\\ &= w\big((1-\tfrac{1}{2}z)^{-1}e^{-s}x, e^s y, e^{2s}(z+1)\big)\, \kappa^{-1}\, q(s)\cdot H. \end{aligned}$$

Through the substitution (6.24) the last expression becomes

$$w\big(2(3-e^{-2s}\omega)^{-1}\xi,\eta,\omega\big)\ \kappa^{-1}\ q(s)\cdot H.$$

For $s\to\infty$ this expression tends to

$$w(\tfrac{2}{3}\xi,\eta,\omega)\,\kappa^{-1}\cdot H = \kappa^{-1} v_{(\omega,\eta),\frac{2}{3}\xi}\cdot H,$$

see (6.33) and (5.5). The result follows. □

References

[1] N. B. Andersen, M. Flensted-Jensen, and H. Schlichtkrull. Cuspidal discrete series for semisimple symmetric spaces. *J. Funct. Anal.*, 263(8):2384–2408, 2012.

[2] D. Bălibanu and E. P. van den Ban. Convexity theorems for semisimple symmetric spaces. arXiv:1401.1093, *Forum Mathemicum* (to appear), 2014.

[3] E. P. van den Ban. The principal series for a reductive symmetric space. II. Eisenstein integrals. *J. Funct. Anal.*, 109(2):331–441, 1992.

[4] E. P. van den Ban and J. J. Kuit. Cusp forms for reductive symmetric spaces of split rank one. arXiv:1511.05793, 2015.

[5] Y. Benoist. Analyse harmonique sur les espaces symétriques nilpotents. *J. Funct. Anal.*, 59(2):211–253, 1984.

[6] G. van Dijk and M. Poel. The Plancherel formula for the pseudo-Riemannian space $\mathrm{SL}(n,\mathbf{R})/\mathrm{GL}(n-1,\mathbf{R})$. *Compositio Math.*, 58(3):371–397, 1986.

[7] G. van Dijk and M. Poel. The irreducible unitary $\mathrm{GL}(n-1,\mathbf{R})$-spherical representations of $\mathrm{SL}(n,\mathbf{R})$. *Compositio Math.*, 73(1):1–30, 1990.

[8] S. Helgason. *Differential geometry and symmetric spaces*. Pure and Applied Mathematics, Vol. XII. Academic Press, New York, 1962.

[9] A. G. Helminck and S. P. Wang. On rationality properties of involutions of reductive groups. *Adv. Math.*, 99(1):26–96, 1993.

[10] A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 2002.

[11] M. T. Kosters and G. van Dijk. Spherical distributions on the pseudo-Riemannian space $\mathrm{SL}(n,\mathbf{R})/\mathrm{GL}(n-1,\mathbf{R})$. *J. Funct. Anal.*, 68(2):168–213, 1986.

[12] W. A. Kosters. Eigenspaces of the Laplace-Beltrami-operator on $\mathrm{SL}(n,\mathbf{R})/\mathrm{S}(\mathrm{GL}(1)\times\mathrm{GL}(n-1))$. I-II. *Nederl. Akad. Wetensch. Indag. Math.*, 47(1):99–123, 125–145, 1985.

[13] O. Loos. *Symmetric spaces. I: General theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[14] T. Matsuki. The orbits of affine symmetric spaces under the action of minimal parabolic subgroups. *J. Math. Soc. Japan*, 31(2):331–357, 1979.

[15] G. D. Mostow. Fully reducible subgroups of algebraic groups. *Amer. J. Math.*, 78:200–221, 1956.

[16] H. Ochiai. Invariant distributions on a non-isotropic pseudo-Riemannian symmetric space of rank one. *Indag. Math. (N.S.)*, 16(3-4):631–638, 2005.

[17] W. Rossmann. The structure of semisimple symmetric spaces. *Canad. J. Math.*, 31(1):157–180, 1979.

E. P. van den Ban
Mathematical Institute
Utrecht University
PO Box 80 010
3508 TA Utrecht
The Netherlands
E-mail: E.P.vandenBan@uu.nl

J. J. Kuit
Dep. of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 København Ø
Denmark
E-mail: j.j.kuit@gmail.com

H. Schlichtkrull
Dep. of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 København Ø
Denmark
E-mail: schlicht@math.ku.dk