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Univariate and Multivariate Linear Profiles Using Max-Type Extended Exponentially Weighted Moving Average Schemes

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ABSTRACT Many studies have shown that industrial as well as non-industrial business organisations present a growing need of robust and more efficient multivariate monitoring schemes in order to be able to monitor several quality characteristics simultaneous. To monitor two or more parameters simultaneously, several monitoring schemes are used concurrently in most of the cases instead of using a single scheme. Thus, in this paper, the exponentially weighted moving average (EWMA), double EWMA (DEWMA) and the recent triple EWMA (TEWMA) procedures are used to develop new single univariate and multivariate Max-type monitoring schemes for linear profiles under the assumptions of fixed and random linear models to monitor the regression parameters and variance error simultaneously. It is observed that the newly proposed schemes are better alternatives of the classical univariate and multivariate EWMA, DEWMA and TEWMA schemes for linear profiles in terms of the average run-length (*ARL*) and expected *ARL* profiles. Numerical examples are presented using simulated and real-life data.

INDEX TERMS Quality control, fixed explanatory variable, linear profiles, Max-TEWMA, random explanatory variable.

NOMENCLATURE

ARL	Average run-length.
CUSUM	Cumulative Sum.
DEWMA	Double exponentially weighted moving average.
EARL	Expected average run-length.
ESDRL	Expected standard deviation of the run-length.
EWMA	Exponentially weighted moving average.
FEV	Fixed explanatory variable.
IC	In-control.
LCL	Lower control limit.
LPM	Linear profile monitoring.
MDEWMA	Multiple double exponentially weighted moving average.
MEWMA	Multiple exponentially weighted moving average.
MTEWMA	Multiple triple exponentially weighted moving average.
OOC	Out-of-control.
REV	Random explanatory variable.
SAS	Statistical Analysis Software.
SDRL	Standard deviation of the run-length.
SPM	Statistical Process Monitoring.
TEWMA	Triple exponentially weighted moving average.
UCL	Upper control limit.

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I. INTRODUCTION

Statistical techniques play a vital role in solving analytical and experimental problems covering all types of fields ranging from the domain of militarisation to peacemaking treaties, engineering to management services, manufacturing to production chains, just to cite a few; [1]–[3]. To achieve the goal in all these cases and stop worst scenarios, a well-organised monitoring system is needed. Since the 1920s to nowadays, researchers have been developing and improving procedures (or tools) for the well-being of the universe and all that it contains. The fundamental tools that opened ways to all modern monitoring tools was introduced by Walter A. Shewhart of Bell-laboratory in the 1920s and 1930s; [4]. These tools are called Shewhart control charts or monitoring schemes. Since then, researchers took the responsibility of developing more sophisticated control charts to separate between the causes of instability in various processes. Statistical process monitoring (SPM) literature distinguishes two causes of variation namely, natural and special causes of variation also knows as chance and assignable causes, respectively. A process that runs only under the former causes is considered to be in-control (IC). An assignable cause of variation is a sign of instability in the process and must be investigated in order to make required corrections. Instability in a process is determined in comparison to a standard target that is believed to yield better results. Shifts from the target may be caused by various problems including raw material, change of climate, poor maintenance, incompetency of the operators and many more. The size of the shift is very important and determine the magnitude of deviation from the target value. In many applications, moderate and large shifts in the process are signs of significant deviation from the process target. However, in other fields, like medicine, even small shifts are not tolerated. The choice of the type of control charts depends among others on the size of the shift to be detected. Thus, the literature distinguishes memoryless from memory-type control charts. The former is used when the detection of large shifts is of interest (e.g. Shewhart-type charts) and the latter for the detection of small and moderate shifts in the process parameters (e.g. the exponentially weighted moving average (EWMA) chart by [5] and the cumulative sum (CUSUM) chart by [6]).

Both memoryless and memory-type schemes are used to monitor the quality characteristic of various processes by controlling the deviations in the process parameters such as the mean and the standard deviation of the process. Univariate control charts are used to monitor one quality characteristic of the process (see for example, [7] and [8]) while multivariate control charts can be used to monitor several quality characteristics simultaneously; [9]–[11]. Most of the SPM procedures designed to monitor both the process mean and standard deviation use two separate control charts; see for example, [12] and [13]. Recently, authors in [14]–[21] have developed control procedures for monitoring both process parameters using a single control chart. In most of the cases, these classical control charts are designed to monitor one or more quality characteristics that are not functionally related; [22]–[24]. However, in many real-life applications, there is a functional relationship between one dependent variable and one, two or more explanatory variables. In this case, profile (or regression) monitoring schemes are used to control linear and nonlinear quality profiles; [25], [26]. These control charts are used to monitor the regression parameters and error variance simultaneously using several control charts for simple and general profiles. At the best knowledge of the authors, only few univariate Max-EWMA control charts for simple linear profile has been investigated using a fixed explanatory variable; see for example, [26]–[28]. Kim *et al.* [25] proposed a simple linear profile monitoring (LPM) chart where the explanatory variable is coded to set the average to zero and the authors of [29] proposed a method based on a F-test for Phase I monitoring with calibration applications and compared the resulting chart to the ones with the results of the papers [30], [25]. Mahmoud *et al.* [31] proposed control charts based on the change point linear profiles monitoring schemes when the parameters are estimated. More details on simple linear profiles can be found in [29] and [32]–[35].

Later on, Refs. [36]–[38] proposed the multivariate EWMA (MEWMA) chart for monitoring general linear profiles. Abbas *et al.* [37] proposed the Bayesian EWMA and MEWMA schemes to monitor linear profiles when the explanatory variable is random using separate and single monitoring schemes. Thus, there is a growing need to improve and extend the Max-EWMA and Max-MEWMA regression control charts proposed in [26] for monitoring the regression parameters and error variance simultaneously using single schemes.

Motivated by the above discussion, this paper develops univariate and multivariate Max-type EWMA, double EWMA (DEWMA) and triple EWMA (TEWMA) control charts for monitoring linear profiles using both fixed and random explanatory variables. The performance of the proposed profile (or regression) control charts are investigated in terms of the average run-length (*ARL*) and standard deviation of the run-length (*SDRL*) performance profiles as well as the expected *ARL* (*EARL*) and expected *SDRL* (*ESDRL*) profiles. Our main contributions are summarised as follows:

- Develop single control charts for monitoring the coefficients of the regression model and the error variance simultaneously.
- Investigate the effect of a fixed explanatory variable (FEV) on the sensitivity (or performance) of the proposed regression schemes.
- Investigate the random effect of the explanatory variable on the sensitivity of the proposed regression schemes.
- Develop new univariate and multivariate EWMA, DEWMA and TEWMA regression control charts to monitor linear profiles under the assumption of FEV and random explanatory variable (REV).
- Illustrate the implementation of the proposed control charts using simulated and real-life data.

The remainder of this paper is presented as follows: Section II introduces the univariate and multivariate EWMA, DEWMA and TEWMA monitoring schemes for linear profiles using fixed and random explanatory variables. Moreover, the operation procedures of the proposed linear profiles are also described. In Section III, the performances of the proposed schemes are discussed under the assumptions of FEV and REV. Section IV presents illustrative examples using simulated and real-life data to facilitate the implementation and application of the proposed schemes. Concluding remarks and future research works are provided in Section V.

II. THE PROPOSED UNIVARIATE AND MULTIVARIATE MEMORY-TYPE LPM TECHNIQUES USING FIXED AND RANDOM EXPLANATORY VARIABLES

In this section, a brief review of the existing univariate and multivariate Max-type EWMA schemes for linear profiles is presented and the new univariate and multivariate Max-type DEWMA and TEWMA monitoring schemes for simple linear profiles are also introduced in the literature.

A. MEMORY-TYPE SCHEMES FOR SIMPLE LINEAR PROFILES USING A FIXED EXPLANATORY VARIABLE

1) THE MAX-TYPE EWMA AND MEWMA SCHEMES

a: THE MAX-TYPE UNIVARIATE EWMA SCHEME

Assume that a random subgroup (x_i, y_{ij}) for the j^{th} profile is collected over time, where y_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots)$ represent the observations of the response variable y which is related to the explanatory variable x with observations x_i. Let α_0 and α_1 be the IC intercept and slope, respectively. Thus, for an IC process, the underlying mathematical model will be defined as

$$y_{ij} = \alpha_0 + \alpha_1 x_i + \epsilon_{ij}, \tag{1}$$

where ϵ_{ij} is a random error component (for all $i = 1, 2, \ldots, n; j = 1, 2, \ldots$). Here, we assume that the ϵ_{ij}'s are independently and identically normally distributed with mean 0 and variance σ^2. Based on the j^{th} profile data, the least square estimators of the regression coefficients of the model defined in (1) are:

$$\hat{\alpha}_{0j} = \bar{y}_j - \hat{\alpha}_{1j}\bar{x} \quad \text{and } \hat{\alpha}_{1j} = S_{xx}^{-1} S_{xy(j)}, \tag{2}$$

where

$$\bar{y}_j = \frac{1}{n}\sum_{i=1}^{n} y_{ij}, \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, S_{xy(j)} = \sum_{i=1}^{n} (x_i - \bar{x})\, y_{ij}$$
$$\text{and } S_{xx} = \sum_{i=0}^{n} (x_i - \bar{x})^2 .$$

The mean vector ($\boldsymbol{\mu}$) and covariance matrix ($\boldsymbol{\Sigma}$) of the estimators of the regression coefficients defined in (2) are given by

$$\boldsymbol{\mu} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{\alpha_0}^2 & \sigma_{\alpha_0\alpha_1} \\ \sigma_{\alpha_0\alpha_1} & \sigma_{\alpha_1}^2 \end{pmatrix}, \tag{3}$$

where

$$\sigma_{\alpha_0}^2 = \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right), \quad \sigma_{\alpha_0\alpha_1} = -\sigma^2\left(\frac{\bar{x}}{S_{xx}}\right) \text{ and } \sigma_{\alpha_1}^2 = \frac{\sigma^2}{S_{xx}}.$$

To simplify the analysis, we use the following model proposed in [25]:

$$y_{ij} = \beta_0 + \beta_1 x_i^{\tau} + \epsilon_{ij}, \tag{4}$$

where $x_i^{\tau} = x_i - \bar{x}$ and ε_{ij}'s are as defined in (1). Eq. (4) may be regarded as the reparametrised version of (1). The coefficients β_0 and β_1 are the intercept and slope of the new model, respectively, and are mathematically expressed as

$$\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 + \alpha_1\bar{x} \\ \alpha_1 \end{pmatrix}. \tag{5}$$

Based on the j^{th} profile data, the least square estimators of the regression parameters β_0 and β_1 are $\hat{\beta}_{0j} = \bar{y}_j$ and $\hat{\beta}_{1j} = \hat{\alpha}_{1j}$, respectively. The mean vector ($\boldsymbol{\mu}^{\tau}$) and covariance matrix ($\boldsymbol{\Sigma}^{\tau}$) of the estimators of the regression coefficients defined in (5) are given by

$$\boldsymbol{\mu}^{\tau} = \begin{pmatrix} \mu_{\hat{\beta}_{0j}} \\ \mu_{\hat{\beta}_{1j}} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \text{ and } \boldsymbol{\Sigma}^{\tau} = \begin{pmatrix} \sigma_{\hat{\beta}_{0j}}^2 & 0 \\ 0 & \sigma_{\hat{\beta}_{1j}}^2 \end{pmatrix}$$
$$= \begin{pmatrix} \dfrac{\sigma^2}{n} & 0 \\ 0 & \dfrac{\sigma^2}{S_{xx}} \end{pmatrix}. \tag{6}$$

The above properties are very important in the design of the Max-type EWMA scheme. Thus, the EWMA statistics for the intercept and slope are given as follows:

$$UEZ_{j1} = \lambda\hat{\beta}_{0j} + (1-\lambda)\, UEZ_{(j-1)1}$$
$$\text{and } UEZ_{j2} = \lambda\hat{\beta}_{1j} + (1-\lambda)\, UEZ_{(j-1)2}, \tag{7}$$

where $j = 1, 2, \ldots, \lambda (0 < \lambda \leq 1)$ is the weighing coefficient also known as smoothing parameter, $UEZ_{01} = \beta_0$ and $UEZ_{02} = \beta_1$. Using Lemma 1, Appendix A shows that the means and variances of the EWMA statistics for the intercept and slope are given as follows:

$$\mathrm{E}(UEZ_{jk}) = \beta_{k-1}$$
$$\text{and } \mathrm{Var}(UEZ_{jk}) = \frac{\lambda(1-(1-\lambda)^{2j})}{(2-\lambda)}\sigma_k^2, \tag{8}$$

where $k \in \{1, 2\}$, $\sigma_1^2 = \sigma^2/n$ and $\sigma_2^2 = \sigma^2/S_{xx}$.

When the process has been running for a very long time, as $j \to \infty$, then

$$\mathrm{Var}(UEZ_{jk}) \sim \frac{\lambda}{(2-\lambda)}\sigma_k^2, \quad k = 1 \textit{ and } 2.$$

Following (7) and (8), define

$$UEU_j = \sum_{k=1}^{2}\left(\frac{UEZ_{jk} - \beta_{k-1}}{\sqrt{\mathrm{Var}\left(UEZ_{jk}\right)}}\right)^2. \tag{9}$$

It is to be noted that for the IC process, UEU_j follows a χ^2 distribution with 2 degrees of freedom. Let us also define

$$UMSE_j = \frac{1}{n-2}\sum_{i=1}^{n}\left(y_{ij} - \hat{\beta}_{0j} - \hat{\beta}_{1j}x_i^{\tau}\right)^2$$
$$\textit{and } UEV_j = \Phi^{-1}\left\{F\left(\frac{(n-2)UMSE_j}{\sigma^2}; n-2\right)\right\}. \tag{10}$$

Here $UMSE_j$ provides an unbiased estimated of the error variance σ^2 and $(n-2)UMSE_j/\sigma^2$ follows a χ^2 distribution with $(n-2)$ degrees of freedom when the process is IC, $\Phi^{-1}(.)$ is the inverse of the c.d.f. (cumulative distribution function) of the standard normal distribution and $F(.,v)$ is the c.d.f. of a χ^2 distribution with v degrees of freedom. To control the error variance, let us define the statistic based on UEV_j as

$$UEZ_{j3} = \lambda UEV_j + (1-\lambda)\, UEZ_{(j-1)3}, \tag{11a}$$

where $UEZ_{03} = E(UEZ_{j3}) = 0$. Thus, using Lemma 1, Appendix A shows that the variance of the UEZ_{j3} statistic is then given by

$$\text{Var}(UEZ_{j3}) = \frac{\lambda(1-(1-\lambda)^{2j})}{(2-\lambda)}. \tag{11b}$$

When the process has been running for a very long time, as $j \to \infty$, then

$$\text{Var}(UEZ_{j3}) \sim \frac{\lambda}{(2-\lambda)}. \tag{11c}$$

Define

$$UEM_j = \text{Max}\left\{\left|\Phi^{-1}\left\{F\left(UEU_j;2\right)\right\}\right|, \left|\frac{UEZ_{j3}}{\sqrt{Var(UEZ_{j3})}}\right|\right\}. \tag{12}$$

Then, mean and variance of UEU_j are given by

$$\text{E}\left(UEU_j\right) = \frac{2}{\sqrt{\pi}} \approx 1.12838$$
$$\text{and Var}(UEU_j) = \left(1-\frac{2}{\pi}\right) \approx 0.3634, \tag{13}$$

respectively.

Let us now define the following Max-EWMA control chart for simultaneously monitoring the parameters and error variance of the model stated in (4) as

$$\begin{aligned} UEUCL &= E(UEU_j) + L^{UE}\sqrt{\text{Var}(UEU_j)} \\ &\approx 1.12838 + 0.6028L^{UE}, \end{aligned} \tag{14}$$

where L^{UE} is the control limit coefficient selected to yield a desired large nominal IC ARL. Therefore, the Max-EWMA scheme gives a signal on the j^{th} profile if the charting statistic defined in (12) plots beyond the control limit defined in (14).

b: MAX-TYPE MULTIVARIATE EWMA SCHEME

The multivariate EWMA (MEWMA) scheme for monitoring the regression parameters and the standard deviation of a general linear profile was introduced in [36]. Assume $(\boldsymbol{X}_j, \boldsymbol{Y}_j)$, $j = 1, 2, \ldots,$ represent the observations of the j^{th} profile where $\boldsymbol{Y}_j$ is a $n-$variate response vector and $\boldsymbol{X}_j$ (denoted as $\boldsymbol{X}$) is a $n \times p (n > p)$ matrix corresponding to the covariates. When the process is IC, the underlying process model for the j^{th} profile is mathematically defined by

$$\boldsymbol{Y}_j = \boldsymbol{X\beta} + \boldsymbol{\epsilon}_j, \tag{15}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)^T$ is the $p-$dimensional regression coefficient vector and ϵ_j is an $n-$dimentional vector of random errors which follows $N(\boldsymbol{0}, \sigma^2\boldsymbol{I}_n)$. Thus, the Max-type MEWMA (denoted as Max-MEWMA) scheme is used to monitor $(p+1)$ parameters of which p parameters are the regression coefficients of the model and one is the standard deviation σ. Let us suppose that the parameters of the model defined in (15) are unknown. The least square estimators of the parameters are

$$\hat{\boldsymbol{\beta}}_j = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}_j$$
$$\text{and } MMSE_j = \frac{1}{n-p}(\boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}_j)^T(\boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}_j). \tag{16}$$

It is then to be noted that $MMSE_j$ is an unbiased estimator of σ^2 and $(n-p)MMSE_j/\sigma^2$ follows a χ^2 distribution with $(n-p)$ degrees of freedom when the process is IC. Then, it follows that

$$\text{E}\left(\hat{\boldsymbol{\beta}}_j\right) = \boldsymbol{\beta} \text{ and Var}\left(\hat{\boldsymbol{\beta}}_j\right) = \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\sigma^2. \tag{17}$$

The MEWMA statistic for the parameters representing the regression coefficients of the model (15) is defined as

$$\boldsymbol{MEZ}_{j1} = \lambda\hat{\boldsymbol{\beta}}_j + (1-\lambda)\,\boldsymbol{MEZ}_{(j-1)1} \tag{18}$$

where $\boldsymbol{MEZ}_{01} = \text{E}(\boldsymbol{MEZ}_{j1}) = \boldsymbol{\beta}$. Using Lemma 1, it is shown in Appendix A that the variance of the MEWMA statistic for a simultaneous monitoring of the parameters representing the regression coefficients is given by

$$\text{Var}(\boldsymbol{MEZ}_{j1}) = \frac{\lambda(1-(1-\lambda)^{2j})\sigma^2}{2-\lambda}(\boldsymbol{X}^T\boldsymbol{X})^{-1}. \tag{19a}$$

When the process has been running for very long time, that is, in steady-state, the variance in (19a) becomes

$$\text{Var}\left(\boldsymbol{MEZ}_{j1}\right) \sim \frac{\lambda\sigma^2}{2-\lambda}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1} \; as\, j \to \infty. \tag{19b}$$

Define

$$MEU_j = (\boldsymbol{MEZ}_{j1} - \boldsymbol{\beta})^T(\text{Var}(\boldsymbol{MEZ}_{j1}))^{-1}(\boldsymbol{MEZ}_{j1} - \boldsymbol{\beta}). \tag{20}$$

It follows that, for an in-control process, MEU_j follows a χ^2 distribution with p degrees of freedom. Define

$$MEV_j = \Phi^{-1}\left\{F\left(\frac{(n-p)MMSE_j}{\sigma^2}; n-p\right)\right\}. \tag{21}$$

Here, it is to be noted that $MMSE_j$ is an unbiased estimator of σ^2 and $(n-p)MMSE_j/\sigma^2$ follows a χ^2 distribution with $(n-p)$ degrees of freedom when the process is IC. To control the error variance, let us define the statistic based on MEV_j as

$$MEZ_{j3} = \lambda MEV_j + (1-\lambda)\, MEZ_{(j-1)3}, \tag{22a}$$

where $MEZ_{03} = E(MEZ_{j3}) = 0$. It follows that the variance of the MEZ_{j3} statistic is given by

$$\text{Var}(MEZ_{j3}) = \frac{\lambda(1-(1-\lambda)^{2j})}{(2-\lambda)}. \tag{22b}$$

In steady-state, the variance of MEZ_{j3} becomes:

$$\mathrm{Var}\left(MEZ_{j3}\right) \sim \frac{\lambda}{(2-\lambda)} \; as\; j \to \infty. \tag{22c}$$

Define

$$MEM_j = \mathrm{Max}\left\{\left|\Phi^{-1}\left\{F\left(MEU_j; p\right)\right\}\right|, \left|\frac{MEZ_{j3}}{\sqrt{Var(MEZ_{j3})}}\right|\right\}. \tag{23}$$

Then, mean and variance of MEU_j are given by

$$\mathrm{E}\left(MEU_j\right) = \frac{2}{\sqrt{\pi}} \approx 1.12838$$
$$and\ \mathrm{Var}(MEU_j) = \left(1-\frac{2}{\pi}\right) \approx 0.3634, \tag{24}$$

respectively.

Let us now define the following Max-MEWMA control chart for simultaneously monitoring the parameters and error variance of the model stated in (15) as

$$\begin{aligned} MEUCL &= E(MEU_j) + L^{ME}\sqrt{\mathrm{Var}(MEU_j)} \\ &\approx 1.12838 + 0.6028L^{ME}, \end{aligned} \tag{25}$$

where L^{ME} is the control limit coefficient selected to yield a desired large nominal IC *ARL*. Therefore, the Max-MEWMA scheme gives a signal on the j^{th} profile if the charting statistic defined in (23) plots beyond the control limit defined in (25).

2) DESIGN OF THE MAX-TYPE DEWMA AND MDEWMA SCHEMES

a: THE MAX-TYPE UNIVARIATE DEWMA SCHEME

To develop the DEWMA statistics for the intercept and slope parameters, we define in continuation to (7) the following statistics:

$$\begin{aligned} UDEZ_{j1} &= \lambda UEZ_{j1} + (1-\lambda)\, UDEZ_{(j-1)1} \\ \text{and } UDEZ_{j2} &= \lambda UEZ_{j2} + (1-\lambda)\, UDEZ_{(j-1)2}, \end{aligned} \tag{26}$$

where $UDEZ_{01} = \beta_0$ and $UDEZ_{02} = \beta_1$. Using Lemma 1, it is shown in the Appendix B that the means and variances of the DEWMA statistics for the intercept and slope are given as follows:

$$\begin{aligned} \mathrm{E}(UDEZ_{jk}) &= \beta_{k-1} \\ \text{and } \mathrm{Var}\left(UDEZ_{jk}\right) &= \lambda^4\psi\sigma_k^2,\ k \in \{1,2\}, \end{aligned} \tag{27}$$

where σ_1^2 and σ_2^2 are defined in (8) and ψ, as shown at the bottom of the next page.

In steady-state case, the variances given in (27) become:

$$\mathrm{Var}\left(UDEZ_{jk}\right) \sim \frac{\lambda^4(1+\zeta^2)}{\left(1-\zeta^2\right)^3}\sigma_k^2 \text{ as } j \to \infty, \quad k \in \{1,2\}.$$

Following (26) and (27), define

$$UDEU_j = \sum_{k=1}^{2}\left(\frac{UDEZ_{jk}-\beta_{k-1}}{\sqrt{\mathrm{Var}\left(UDEZ_{jk}\right)}}\right)^2. \tag{28}$$

It is to be noted that for the IC process, $UDEU_j$ follows a χ^2 distribution with 2 degrees of freedom. To control the error variance, let us define the DEWMA statistic in continuation to (11a) as

$$UDEZ_{j3} = \lambda UEZ_{j3} + (1-\lambda)\, UDEZ_{(j-1)3}, \tag{29a}$$

where $UDEZ_{03} = \mathrm{E}(UDEZ_{j3}) = 0$. Thus, it is to be noted that the variance of the $UDEZ_{j3}$ statistic is then given by

$$\mathrm{Var}(UDEZ_{j3}) = \lambda^4\psi. \tag{29b}$$

In steady-state mode, the variance of $UDEZ_{j3}$ becomes:

$$\mathrm{Var}(UDEZ_{j3}) \sim \frac{\lambda^4(1+\zeta^2)}{\left(1-\zeta^2\right)^3} \; as\; j \to \infty. \tag{29c}$$

Define

$$UDEM_j = \mathrm{Max}\left\{\left|\Phi^{-1}\left\{F\left(UDEU_j; 2\right)\right\}\right|, \left|\frac{UDEZ_{j3}}{\sqrt{\mathrm{Var}(UDEZ_{j3})}}\right|\right\}. \tag{30}$$

Then, mean and variance of $UDEM_j$ are given by

$$\mathrm{E}\left(UDEM_j\right) = \frac{2}{\sqrt{\pi}} \approx 1.12838$$
$$and\ \mathrm{Var}(UDEM_j) = \left(1-\frac{2}{\pi}\right) \approx 0.3634, \tag{31}$$

respectively.

Let us now define the following Max-DEWMA control chart for simultaneously monitoring the parameters and error variance of the model stated in (4) as

$$\begin{aligned} UDEUCL &= E(UDEM_j) + L^{UDE}\sqrt{\mathrm{Var}(UDEM_j)} \\ &\approx 1.12838 + 0.6028L^{UDE}, \end{aligned} \tag{32}$$

where L^{UDE} is the control limit coefficient selected to yield a desired large nominal IC *ARL*. Therefore, the Max-DEWMA scheme gives a signal on the j^{th} profile if the charting statistic defined in (30) plots beyond the control limit defined in (32).

b: MAX-TYPE MULTIVARIATE DEWMA SCHEME

In continuation to the statistic defined in (18), let us define the following statistic to introduce the multivariate DEWMA (MDEWMA) statistic for monitoring the parameters representing the regression coefficients of the model given in (15):

$$\boldsymbol{MDEZ}_{j1} = \lambda \boldsymbol{MEZ}_{j1} + (1-\lambda)\, \boldsymbol{MDEZ}_{(j-1)1} \tag{33}$$

where $\boldsymbol{MDEZ}_{01} = \mathrm{E}(\boldsymbol{MDEZ}_{j1}) = \boldsymbol{\beta}$. Using Lemma 1, it can be shown that the variance of the MDEWMA statistic for a simultaneous monitoring of the parameters representing the regression coefficients is given by

$$Var(\boldsymbol{MDEZ}_{j1}) = \lambda^4\psi\sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}. \tag{34a}$$

In steady-state mode, the variance in Equation (34a) becomes

$$\mathrm{Var}\left(\boldsymbol{MDEZ}_{j1}\right)\sim\left[\frac{\sigma^2\lambda^4\left(1+\zeta^2\right)}{\left(1-\zeta^2\right)^3}\right]\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\ \text{as}\ j\rightarrow\infty. \tag{34b}$$

Define

$$MDEU_j=(\boldsymbol{MDEZ}_{j1}-\boldsymbol{\beta})^T(\mathrm{Var}(\boldsymbol{MDEZ}_{j1}))^{-1}\times(\boldsymbol{MDEZ}_{j1}-\boldsymbol{\beta}). \tag{35}$$

It follows that, for an IC process, $MDEU_j$ follows a χ^2 distribution with p degrees of freedom. To control the error variance, let us define the statistic based on MEZ_{j3} as

$$MDEZ_{j3}=\lambda MEZ_{j3}+(1-\lambda)\,MDEZ_{(j-1)3}, \tag{36a}$$

where $MDEZ_{03}=\mathrm{E}(MDEZ_{j3})=0$. It follows that the variance of the $MDEZ_{j3}$ statistic is given by

$$\mathrm{Var}(MDEZ_{j3})=\lambda^4\psi. \tag{36b}$$

In steady-state mode, the variance of $MDEZ_{j3}$ becomes:

$$\mathrm{Var}\left(MDEZ_{j3}\right)\sim\frac{\lambda^4\left(1+\zeta^2\right)}{\left(1-\zeta^2\right)^3}\ \ j\rightarrow\infty. \tag{36c}$$

Define

$$MDEM_j=\mathrm{Max}\left\{\left|\Phi^{-1}\left\{F\left(MDEU_j;p\right)\right\}\right|,\left|\frac{MDEZ_{j3}}{\sqrt{\mathrm{Var}(MDEZ_{j3})}}\right|\right\}. \tag{37}$$

Then, mean and variance of $MDEM_j$ are given by

$$\mathrm{E}\left(MDEM_j\right)=\frac{2}{\sqrt{\pi}}\approx 1.12838$$
$$\text{and}\ \mathrm{Var}(MDEM_j)=\left(1-\frac{2}{\pi}\right)\approx 0.3634, \tag{38}$$

respectively.

Let us now define the following Max-MDEWMA control chart for simultaneously monitoring the parameters and error variance of the model stated in (15) as

$$MDEUCL=\mathrm{E}(MDEM_j)+L^{MDE}\sqrt{Var(MDEM_j)}\approx 1.12838+0.6028L^{MDE}, \tag{39}$$

where L^{MDE} is the control limit coefficient selected to yield a desired large nominal IC ARL. Therefore, the Max-MDEWMA scheme gives a signal on the j^{th} profile if the charting statistic defined in (37) plots beyond the control limit defined in (39).

3) DESIGN OF THE MAX-TYPE TEWMA AND MTEWMA SCHEMES

a: THE MAX-TYPE UNIVARIATE TEWMA SCHEME

To develop the TEWMA statistics for the intercept and slope parameters, we define in continuation to (26) the following statistics:

$$UTEZ_{j1}=\lambda UDEZ_{j1}+(1-\lambda)\,UTEZ_{(j-1)1}$$
$$\text{and}\ UTEZ_{j2}=\lambda UDEZ_{j2}+(1-\lambda)\,UTEZ_{(j-1)2}, \tag{40}$$

where $UTEZ_{01}=\beta_0$ and $UTEZ_{02}=\beta_1$. Using Lemma 1, Appendix C shows that the means and variances of the TEWMA statistics for the intercept and slope are given as follows:

$$\mathrm{E}(UTEZ_{jk})=\beta_{k-1}$$
$$\text{and}\ \mathrm{Var}\left(UTEZ_{jk}\right)=\lambda^6\varphi\sigma_k^2,\quad k\in\{1,2\}, \tag{41}$$

where σ_1^2 and σ_2^2 are defined in (8) and

$$\begin{aligned}\varphi=\frac{\zeta^6}{4}&\left[-\left[\frac{j\left(j^2-1\right)(j-2)\,\zeta^{2j-6}}{1-\zeta^2}\right]\right.\\&-4\left[\frac{j\left(j^2-1\right)\zeta^{2j-4}}{\left(1-\zeta^2\right)^2}\right]\\&-12\left[\frac{j\,(j+1)\,\zeta^{2j-2}}{\left(1-\zeta^2\right)^3}\right]-24\left[\frac{(j+1)\,\zeta^{2j}}{\left(1-\zeta^2\right)^4}\right]\\&\left.+24\left[\frac{1-\zeta^{2j+2}}{\left(1-\zeta^2\right)^5}\right]\right]+2\zeta^4\left[-\left[\frac{j\left(j^2-1\right)\zeta^{2j-4}}{1-\zeta^2}\right]\right.\\&-3\left[\frac{j\,(j+1)\,\zeta^{2j-2}}{\left(1-\zeta^2\right)^2}\right]-6\left[\frac{(j+1)\,\zeta^{2j}}{\left(1-\zeta^2\right)^3}\right]\\&\left.+6\left[\frac{1-\zeta^{2j+2}}{\left(1-\zeta^2\right)^4}\right]\right]+\frac{7\zeta^2}{2}\left[-\left[\frac{j\,(j+1)\,\zeta^{2j-2}}{1-\zeta^2}\right]\right.\\&\left.-\left[\frac{2\,(j+1)\,\zeta^{2j}}{\left(1-\zeta^2\right)^2}\right]+\left[\frac{2\left(1-\zeta^{2j+2}\right)}{\left(1-\zeta^2\right)^3}\right]\right]\\&+\left[\left(\frac{1-\zeta^{2j+2}}{\left(1-\zeta^2\right)^2}\right)-\left(\frac{(j+1)\,\zeta^{2j}}{1-\zeta^2}\right)\right]\ \textit{and}\ \zeta=1-\lambda.\end{aligned}$$

In steady-state mode, the variances given in (27) become:

$$\begin{aligned}\mathrm{Var}\left(UTEZ_{jk}\right)\sim&\left[\frac{6\lambda\zeta^6}{(2-\lambda)^5}+\frac{12\lambda^2\zeta^4}{(2-\lambda)^4}\right.\\&\left.+\frac{7\lambda^3\zeta^2}{(2-\lambda)^3}+\frac{\lambda^4}{(2-\lambda)^2}\right]\sigma_k^2\ \text{as}\ j\rightarrow\infty,\\&k\in\{1,2\}.\end{aligned}$$

$$\psi=\left[\frac{1+\zeta^2-\left(j^2+2j+1\right)\zeta^{2j}+\left(2j^2+2j-1\right)\zeta^{2j+2}-j^2\zeta^{2j+4}}{\left(1-\zeta^2\right)^3}\right]$$
$$\text{and}\ \zeta=1-\lambda.$$

Following (40) and (41), define

$$UTEU_j = \sum_{k=1}^{2}\left(\frac{UTEZ_{jk} - \beta_{k-1}}{\sqrt{\mathrm{Var}\left(UTEZ_{jk}\right)}}\right)^2. \quad (42)$$

It is to be noted that for the IC process, $UTEU_j$ follows a χ^2 distribution with 2 degrees of freedom. To control the error variance, let us define the TEWMA statistic in continuation to (29a) as

$$UTEZ_{j3} = \lambda UDEZ_{j3} + (1-\lambda)\,UTEZ_{(j-1)3}, \quad (43a)$$

where $UTEZ_{03} = \mathrm{E}(UTEZ_{j3}) = 0$. Thus, it is to be noted that the variance of the $UTEZ_{j3}$ statistic is then given by

$$\mathrm{Var}(UTEZ_{j3}) = \lambda^6\varphi. \quad (43b)$$

In steady-state mode, the variance of $UDEZ_{j3}$ becomes:

$$\mathrm{Var}\left(UTEZ_{j3}\right) \sim \left[\frac{6\lambda\zeta^6}{(2-\lambda)^5} + \frac{12\lambda^2\zeta^4}{(2-\lambda)^4} + \frac{7\lambda^3\zeta^2}{(2-\lambda)^3} + \frac{\lambda^4}{(2-\lambda)^2}\right] \text{ as } j \to \infty. \quad (43c)$$

Define

$$UTEM_j = \mathrm{Max}\left\{\left|\Phi^{-1}\left\{F\left(UTEU_j;2\right)\right\}\right|, \left|\frac{UTEZ_{j3}}{\sqrt{\mathrm{Var}(UTEZ_{j3})}}\right|\right\}. \quad (44)$$

Then, mean and variance of $UTEM_j$ are given by

$$\mathrm{E}\left(UTEM_j\right) = \frac{2}{\sqrt{\pi}} \approx 1.12838$$
$$\text{and } \mathrm{Var}(UTEM_j) = \left(1-\frac{2}{\pi}\right) \approx 0.3634, \quad (45)$$

respectively.

Let us now define the following Max-TEWMA control chart for simultaneously monitoring the parameters and error variance of the model stated in (4) as

$$UTEUCL = \mathrm{E}(UTEM_j) + L^{UTE}\sqrt{\mathrm{Var}(UTEM_j)} \approx 1.12838 + 0.6028L^{UTE}, \quad (46)$$

where L^{UTE} is the control limit coefficient selected to yield a desired large nominal IC *ARL*. Therefore, the Max-TEWMA scheme gives a signal on the j^{th} profile if the charting statistic defined in (44) plots beyond the control limit defined in (46).

b: MAX-TYPE MULTIVARIATE TEWMA SCHEME

In continuation to the statistic defined in (33), let us define the following statistic to introduce the multivariate TEWMA (MTEWMA) statistic for monitoring the parameters representing the regression coefficients of the model given in (15):

$$MTEZ_{j1} = \lambda MDEZ_{j1} + (1-\lambda)\,MTEZ_{(j-1)1} \quad (47)$$

where $MTEZ_{01} = \mathrm{E}(MTEZ_{j1}) = \boldsymbol{\beta}$. Using Lemma 1 it can be shown that the variance of the MDEWMA statistic for a simultaneous monitoring of the parameters representing the regression coefficients is given by

$$\mathrm{Var}(MEZ_{j1}) = \lambda^6\varphi\sigma^2(X^TX)^{-1}. \quad (48a)$$

In steady-state mode, the variance in (48a) becomes

$$Var\left(MTEZ_{j1}\right) \sim \sigma^2\left[\frac{6\lambda\zeta^6}{(2-\lambda)^5} + \frac{12\lambda^2\zeta^4}{(2-\lambda)^4} + \frac{7\lambda^3\zeta^2}{(2-\lambda)^3} + \frac{\lambda^4}{(2-\lambda)^2}\right]\left(X^TX\right)^{-1} \text{ as } j \to \infty. \quad (48b)$$

Define

$$MTEU_j = (MTEZ_{j1} - \beta)^T(\mathrm{Var}(MTEZ_{j1}))^{-1} \times (MTEZ_{j1} - \beta). \quad (49)$$

It follows that, for an IC process, $MTEU_j$ follows a χ^2 distribution with p degrees of freedom. To control the error variance, let us define the statistic based on $MTEZ_{j3}$ as

$$MTEZ_{j3} = \lambda MDEZ_{j3} + (1-\lambda)\,MTEZ_{(j-1)3}, \quad (50a)$$

where $MTEZ_{03} = \mathrm{E}(MTEZ_{j3}) = 0$. It follows that the variance of the $MTEZ_{j3}$ statistic is given by

$$\mathrm{Var}(MTEZ_{j3}) = \lambda^6\varphi. \quad (50b)$$

In steady-state mode, the variance of $MTEZ_{j3}$ becomes:

$$\mathrm{Var}\left(MTEZ_{j3}\right) \sim \left[\frac{6\lambda\zeta^6}{(2-\lambda)^5} + \frac{12\lambda^2\zeta^4}{(2-\lambda)^4} + \frac{7\lambda^3\zeta^2}{(2-\lambda)^3} + \frac{\lambda^4}{(2-\lambda)^2}\right] \text{ as } j \to \infty. \quad (50c)$$

Define

$$MTEM_j = \mathrm{Max}\left\{\left|\Phi^{-1}\left\{F\left(MTEU_j;p\right)\right\}\right|, \left|\frac{MTEZ_{j3}}{\sqrt{\mathrm{Var}(MTEZ_{j3})}}\right|\right\}. \quad (51)$$

Then, mean and variance of $MTEM_j$ are given by

$$\mathrm{E}\left(MTEM_j\right) = \frac{2}{\sqrt{\pi}} \approx 1.12838$$
$$\text{and } \mathrm{Var}(MTEM_j) = \left(1-\frac{2}{\pi}\right) \approx 0.3634, \quad (52)$$

respectively.

Let us now define the following Max-MTEWMA control chart for simultaneously monitoring the parameters and error variance of the model stated in (15) as

$$MTEUCL = \mathrm{E}(MTEM_j) + L^{MTE}\sqrt{\mathrm{Var}(MTEM_j)} \approx 1.12838 + 0.6028L^{MTE}, \quad (53)$$

where L^{MTE} is the control limit coefficient selected to yield a desired large nominal IC *ARL*. Therefore, the Max-MTEWMA scheme gives a signal on the j^{th} profile if the charting statistic defined in (51) plots beyond the control limit defined in (53).

B. THE PROPOSED UNIVARIATE AND MULTIVARIATE MEMORY-TYPE LPM TECHNIQUES USING RANDOM EXPLANATORY VARIABLE

1) SIMPLE LINEAR PROFILE

In Section 2, the proposed schemes are designed using fixed random variables (i.e. x_i represents a FEV). However, if x_i represents a random explanatory variable (with $x_i \sim N(\mu_x, \sigma_x^2)$), the mean vector and covariance matrix defined in (6) become:

$$\mu^{\tau} = \beta_0 + \beta_1 \mu_x$$
$$\text{and } \boldsymbol{\Sigma}^{\tau} = \begin{pmatrix} \dfrac{\sigma^2 + \sigma_x^2 \beta_1^2}{n} & 0 \\ 0 & \dfrac{\sigma^2}{S_{xx}} \end{pmatrix}, \quad (54)$$

where σ_x^2 is the variance of the explanatory variable x_i and β_1 is the slope of the regression model. Therefore, when the proposed schemes are implemented using REV, the equations (or formulas) developed in Section 2 remain applicable with the only difference that the mean vector and variance of the estimate of the intercept is replaced with $\beta_0 + \beta_1 x$ and $(\sigma^2 + \sigma_x^2 \beta_1^2)/n$, respectively.

2) GENERAL LINEAR PROFILE

For the general linear profile model, the mean vector and covariance matrix of $\hat{\beta}_j$ defined in (17) become:

$$\mathrm{E}\left(\hat{\boldsymbol{\beta}}_j\right) = \frac{\sigma^2 \boldsymbol{\beta} + \left(X_j^T Y_j\right) \boldsymbol{\Sigma}}{\sigma^2 + X_j^T X_j \Sigma} \text{ and } \Sigma_{\left(\hat{\beta}_j\right)} = \frac{\sigma^2 \boldsymbol{\Sigma}}{\sigma^2 + X_j^T X_j \boldsymbol{\Sigma}}, \quad (55)$$

respectively, where $\boldsymbol{\beta}$ represents p-dimensional coefficient vector of the hyperparameters and $\boldsymbol{\Sigma}$ is the variance-covariance matrix. The formulas for the general profile monitoring described in Section 2 are still valid here with the only difference that the mean vector and covariance matrix are computed using (55).

C. OPERATION OF THE PROPOSED MEMORY-TYPE LPM SCHEMES

In this section, the important steps in the design and implementation of the proposed scheme are provided in Figure 1. Figure 1 presents the flow chart of the operations of the proposed Max-EWMA, Max-DEWMA and Max-TEWMA schemes using fixed and random explanatory variables where UM_j and UCL represent the j^{th} charting statistic and upper control limit of the corresponding scheme. That is, $UM_j = UEM_j$, $UDEM_j$ and $UTEM_j$ for the Max-EWMA, Max-DEWMA and Max-TEWMA, respectively; and $UCL = UEUCL$, $UDEUCL$ and $UTEUCL$ for the Max-EWMA, Max-DEWMA and Max-TEWMA, respectively. The operations of the proposed Max-MEWMA, Max-MDEWMA and Max-MTEWMA schemes for general linear profiles using FEV and REV are similar to the ones described in Figure 1 with the only difference that the matrix $\boldsymbol{X}_j$ is find by generating n vectors from a $\boldsymbol{X}_j \sim \mathrm{N}\left(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0\right)$.

III. PERFORMANCE ANALYSIS

This section investigates the IC and OOC performances of the proposed univariate and multivariate Max-EWMA, Max-DEWM and Max-TEWMA control charts when $n = 4$, $\alpha_0 = 3$ and $\alpha_0 = 2$ for a nominal $ARL_0 = 200$. For FEV, $x_i =$ 2,4,6 and 8. However, for the REV, the explanatory variable is generated for a normal distribution with a mean $\mu_x = 0$ and standard deviation $\sigma_x = 1$.

A. PERFORMANCE MEASURES

The performance a control chart is mostly evaluated using the characteristics of its run-length distribution. The most popular measures used in the SPM literature are the *ARL*, median run-length and *SDRL*. In this paper, we make use of the *ARL* and *SDRL* profiles to investigate the specific performances of the proposed control charts for different shifts in the parameters and error variance of a linear regression. Therefore, the change in the intercept of the regression model is observed when β_0 has shifted from β_0 to $\beta_0 + \delta\sigma_{\beta_0}$. A shift in the slope of a regression model is observed if the slope β_1 has shifted to $\beta_1 + \Delta\sigma_{\beta_1}$. When $\delta = 0$ and $\Delta = 0$, the intercept and slope of the regression model are considered to be IC. However, a shift in the error variance has occurred if the error standard deviations has shifted from σ to $\gamma\sigma$ which means that the error variance is IC when $\gamma = 1$. In this paper, the model parameters and the error variance are monitored simultaneously. To investigate the overall performance of the propose control charts we use the *EARL* and *ESDRL* profiles which are mathematically defined by

$$EARL_{(\vartheta_{\min},\vartheta_{\max}]} = \frac{1}{\eta} \sum_{\vartheta=\vartheta_{\min}}^{\vartheta_{\max}} ARL(\vartheta),$$
$$\text{and } ESDRL_{(\vartheta_{min},\vartheta_{max}]} = \frac{1}{\eta} \sum\nolimits_{\vartheta=\vartheta_{min}}^{\vartheta_{max}} SDRL(\vartheta), \quad (56)$$

respectively, where $ARL(\vartheta)$ and $SDRL(\vartheta)$ are the values of the *ARL* and *SDRL* for a specific shift ϑ in standard deviation unit (with $\vartheta = \delta$, Δ or γ) and η represents the number of increments between the lower and upper bound shifts, that is, ϑ_{min} and ϑ_{max} (with $(\vartheta_{min,} \vartheta_{max}] = (\delta_{min}, \delta_{\max}], (\Delta_{min}, \Delta_{max}]$ or $(\gamma_{min}, \gamma_{max}]$).

B. DISCUSSION OF THE RESULTS

1) IC PERFORMANCES OF THE PROPOSED CONTROL CHARTS

The performances of the proposed regression Max-type control charts are investigated using SAS® 9.4 for a nominal ARL_0 of 200 when $n = 4$, $\lambda \in \{0.05, 0.5, 0.95\}$, $\alpha_0 = 3$, $\alpha_1 = 2$, $\mu_x = 0$ and $\sigma_x^2 = 1$. From Table 1, it is observed that the control limit constants of the Max-TEWMA, Max-DEWMA and Max-TEWMA charts under FEV are smaller compared to the ones of the corresponding control charts under the REV. This means that the control limits under REV

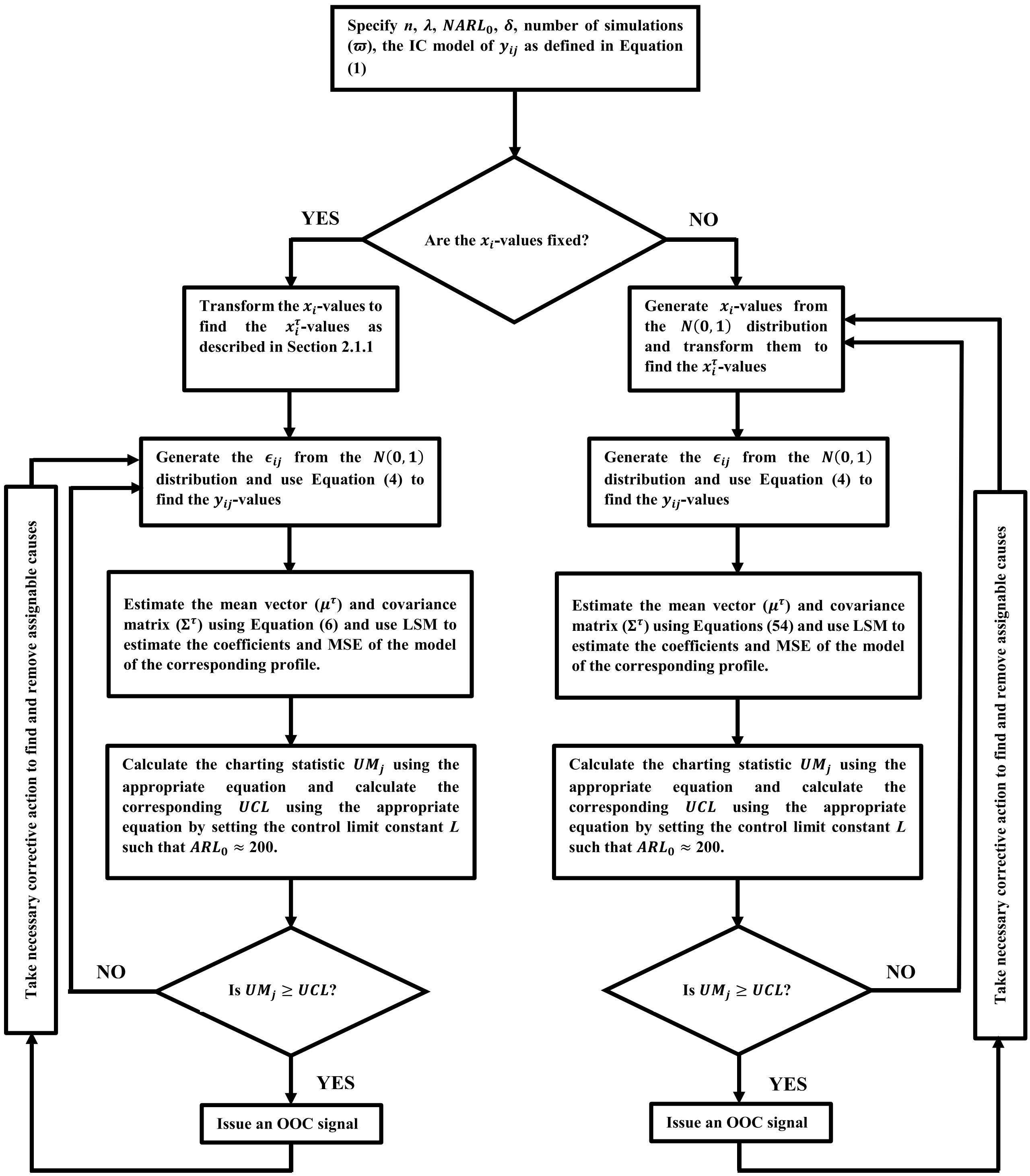


FIGURE 1. Flow chart of the proposed monitoring schemes using FEV and REV.

are wider than the corresponding ones under FEV. The IC *SDRL* ($SDRL_0$) are larger under REV which means that the probability of false alarm is larger under REV and smaller under FEV. Since both univariate and their corresponding multivariate counterparts are designed to monitor the same intercept and slope parameters as well as the error variance using single control charts under FEV, the distances from the centerline to the control limits of these control charts are equal to the ones of their corresponding counterparts. For instance, it is found that $L^{UE} = L^{ME} = 2.724$ so that both the Max-EWMA and Max-MEWMA charts yield an attained IC ARL(ARL_0) value of 200 and $SDRL_0 = 200.6$. However,

TABLE 1. Control limit constants along with the attained ARL_0 and $SDRL_0$ values for a nominal $ARL_0 = 200$.

		Univariate processes						Multivariate processes					
Scheme	λ	L^{UE}	ARL $SDRL$	L^{UDE}	ARL $SDRL$	L^{UTE}	ARL $SDRL$	L^{ME}	ARL $SDRL$	L^{MDE}	ARL $SDRL$	L^{MTE}	ARL $SDRL$
TV Max-EWMA (FEV)	0.05	2.724	200.0 206.1	2.022	200.1 233.0	1.635	200.1 242.6	2.724	200.0 206.1	2.022	200.1 233.0	1.635	200.1 242.6
	0.5	3.120	200.1 197.4	3.032	200.3 197.4	2.925	200.4 201.1	3.120	200.1 197.4	3.032	200.3 197.4	2.925	200.4 201.1
	0.95	3.146	200.0 198.4	3.147	200.2 201.6	3.150	200.0 201.7	3.146	200.0 198.4	3.147	200.2 201.6	3.150	200.0 201.7
TV Max-EWMA (REV)	0.05	3.474	200.5 690.8	2.964	200.1 837.3	2.759	202.1 913.3	2.727	200.2 204.1	2.005	199.9 229.0	1.634	200.6 245.8
	0.5	3.480	200.9 397.1	3.470	199.7 429.8	3.427	200.4 470.6	3.126	199.8 199.8	3.034	200.2 198.6	2.926	200.8 201.3
	0.95	3.320	201.3 322.3	3.344	200.5 329.3	3.359	200.4 336.4	3.143	200.3 200.1	3.150	200.4 201.6	3.137	200.4 200.8

TABLE 2. IC and OOC ARL (1st row) and $SDRL$ (2nd row) profiles of the proposed univariate Max-type control charts for simple linear profiles for FEV and REV with an intercept shift of size δ for a nominal $ARL_0 = 200$.

λ	Scheme	Exploratory variable	δ								
			0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
0.05	**Max-EWMA**	**FEV**	200.0 206.1	33.8 24.2	9.5 5.1	4.5 2.1	2.7 1.0	1.9 0.6	1.5 0.5	1.0 0.0	1.0 0.0
		REV	201.0 690.8	144.6 505.1	70.0 303.2	23.8 80.2	11.5 24.7	6.5 10.8	4.3 6.0	3.1 4.0	2.3 2.8
	Max-DEWMA	**FEV**	200.1 233.0	30.3 21.6	8.9 5.3	4.0 2.1	2.3 1.0	1.5 0.5	1.1 0.3	1.0 0.0	1.0 0.0
		REV	200.1 837.3	122.1 574.2	45.0 189.7	16.7 42.0	9.1 15.4	6.0 8.9	4.2 5.9	2.9 4.0	2.3 2.9
	Max-TEWMA	**FEV**	199.7 243.7	30.2 22.2	9.1 6.1	4.0 2.4	2.1 1.0	1.3 0.5	1.0 0.0	1.0 0.0	1.0 0.0
		REV	197.9 878.9	120.5 611.0	40.4 144.1	16.3 32.6	9.9 15.6	6.6 9.9	4.6 6.9	3.2 4.7	2.5 3.4
0.5	**Max-EWMA**	**FEV**	200.1 197.4	121.3 118.5	26.4 24.2	8.2 6.2	3.8 2.1	2.3 0.9	1.6 0.5	1.2 0.4	1.0 0.0
		REV	201.3 397.5	182.2 363.8	153.4 338.8	106.2 262.2	71.3 206.1	39.8 137.2	18.5 68.4	8.6 33.1	4.9 19.6
	Max-DEWMA	**FEV**	200.3 197.4	89.5 86.4	16.4 13.7	5.8 3.4	3.1 1.2	2.0 0.7	1.6 0.5	1.1 0.4	1.0 0.0
		REV	199.4 429.8	182.0 422.8	138.4 357.3	89.7 258.5	50.1 168.7	23.3 85.1	10.4 39.0	5.2 17.4	3.1 7.7
	Max-TEWMA	**FEV**	200.3 200.3	78.5 74.6	13.9 10.7	5.2 2.6	3.0 1.1	2.0 0.7	1.5 0.5	1.1 0.2	1.0 0.0
		REV	200.4 470.6	185.5 451.8	136.5 387.3	80.8 262.9	43.3 153.7	17.0 60.8	8.0 27.6	4.3 11.3	2.6 4.8
0.95	**Max-EWMA**	**FEV**	200.0 198.4	193.2 192.3	82.8 81.9	28.6 27.5	10.8 10.2	4.7 4.0	2.3 1.6	1.2 0.4	1.0 0.0
		REV	201.1 322.3	195.2 310.9	173.2 286.9	146.5 273.1	109.9 218.1	75.6 170.2	48.9 139.2	26.8 86.1	12.3 50.0
	Max-DEWMA	**FEV**	200.2 201.6	187.4 189.2	74.7 73.7	25.0 24.1	9.3 8.6	4.1 3.4	2.1 1.3	1.2 0.4	1.0 0.0
		REV	200.5 329.3	197.5 324.7	174.6 301.7	140.5 261.9	107.3 218.8	72.4 173.8	44.1 120.8	23.6 81.6	12.3 52.9
	Max-TEWMA	**FEV**	201.0 202.8	181.9 181.7	68.3 67.3	21.7 20.9	8.3 7.4	3.7 2.8	2.0 1.1	1.2 0.4	1.0 0.0
		REV	200.4 336.4	197.6 337.3	172.9 306.3	140.0 265.9	103.1 216.7	69.6 171.0	41.6 126.9	21.6 81.4	10.0 39.7

under REV, the control limits for multivariate control charts are narrower compared to the ones of their corresponding univariate counterparts.

2) OOC PERFORMANCES OF THE PROPOSED CONTROL CHARTS

In Tables 2-5, the performances of the proposed schemes are investigated in terms of the *ARL* (1st row) and *SDRL* (2nd row). Table 2 displays the performances of univariate Max-EWMA, Max-DEWMA and Max-TEWMA control charts for simple linear profile with fixed and random explanatory variables for a shift in the intercept regression parameter when $n = 4$, $\lambda \in \{0.05, 0.5, 0.95\}$, $\alpha_0 = 3$, $\alpha_1 = 2$, $\mu_x = 0$ and $\sigma_x^2 = 1$ for a nominal $ARL_0 = 200$. For the FEV case, we used $x_i = 2$, 4, 6 and 8 where $i = 1,2,3$ and 4; while for the REV case, the explanatory random variable is generated from standard normal distribution as explained earlier in this section. The results in Table 2 can be summarised as followed:

- The proposed control charts are faster in detecting shifts in the intercept when the explanatory variable is fixed. Under a random effect model, the performance of the proposed charts deteriorates considerably. For instance, for a small shift of 0.25 standard deviation in the intercept of the regression model, the Max-EWMA chart for

TABLE 3. IC and OOC *ARL* (1st row) and *SDRL* (2nd row) profiles of the proposed multivariate Max-type control charts for linear profiles for FEV and REV with an intercept shift of size δ for a nominal $ARL_0 = 200$.

			δ								
λ	Scheme	Exploratory variable	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
0.05	Max-MEWMA	Fixe xi	200.0	35.7	10.5	5.2	3.3	2.3	1.8	1.5	1.3
			206.1	26.7	6.2	2.8	1.6	1.1	0.8	0.6	0.5
		Random xi	200.3	36.0	10.4	5.3	3.3	2.3	1.8	1.5	1.2
			205.3	26.8	6.2	2.9	1.7	1.1	0.8	0.6	0.5
	Max-MDEWMA	Fixe xi	200.1	31.9	10.2	5.1	3.1	2.1	1.6	1.3	1.1
			233.0	24.1	6.7	3.2	1.8	1.2	0.8	0.5	0.4
		Random xi	199.9	31.7	10.1	5.0	3.0	2.1	1.6	1.3	1.14
			229.0	24.0	6.7	3.2	1.8	1.2	0.8	0.54	0.37
	Max-MTEWMA	Fixe xi	199.7	31.2	10.8	5.3	3.1	2.1	1.5	1.2	1.0
			243.7	24.6	7.9	3.8	2.1	1.3	0.8	0.5	0.3
		Random xi	200.6	31.1	10.7	5.3	3.1	2.1	1.5	1.25	1.1
			245.8	24.5	7.9	3.8	2.2	1.3	0.8	0.55	0.33
0.5	Max-MEWMA	Fixe xi	200.1	119.8	28.8	9.2	4.5	2.8	2.1	1.7	1.4
			197.4	112.3	26.5	7.3	2.9	1.5	1.0	0.7	0.6
		Random xi	199.8	115.8	29.2	9.4	4.5	2.8	2.1	1.6	1.3
			199.8	113.8	26.9	7.5	3.0	1.5	1.0	0.7	0.5
	Max-MDEWMA	Fixe xi	200.3	88.6	18.1	6.7	3.7	2.5	1.9	1.6	1.3
			197.4	86.4	15.5	4.5	2.0	1.2	0.8	0.6	0.5
		Random xi	200.2	89.2	18.1	6.7	3.7	2.5	1.9	1.55	1.3
			198.6	88.3	15.6	4.5	2.0	1.2	0.8	0.63	0.49
	Max-MTEWMA	Fixe xi	200.3	79.7	15.5	6.1	3.7	2.6	1.9	1.5	1.2
			200.3	76.4	12.4	3.7	1.8	1.2	0.9	0.6	0.4
		Random xi	200.8	80.2	15.5	6.2	3.7	2.5	1.9	1.5	1.3
			201.3	77.4	12.3	3.8	1.8	1.2	0.9	0.7	0.5
0.95	Max-MEWMA	Fixe xi	200.0	160.9	82.0	30.4	12.0	5.4	3.0	1.9	1.5
			198.4	159.9	80.6	29.9	11.3	4.8	2.3	1.3	0.8
		Random xi	199.8	158.7	81.2	30.1	11.9	5.4	3.0	1.93	1.43
			201.8	158.7	80.2	29.9	11.2	4.8	2.3	1.27	0.74
	Max-MDEWMA	Fixe xi	200.2	158.4	75.2	26.7	10.3	4.9	2.8	1.9	1.5
			201.6	157.1	73.9	26.3	9.5	4.1	2.1	1.2	0.8
		Random xi	200.4	157.9	76.2	26.6	10.3	4.9	2.8	1.9	1.4
			201.6	156.6	75.1	26.2	9.6	4.1	2.0	1.1	0.7
	Max-MTEWMA	Fixe xi	201.0	155.7	69.2	23.8	9.1	4.4	2.6	1.8	1.4
			202.8	154.2	67.9	22.9	8.3	3.6	1.8	1.0	0.7
		Random xi	200.4	150.6	67.9	23.4	9.0	4.4	2.6	1.8	1.4
			200.8	151.0	67.3	22.5	8.3	3.5	1.7	1.1	0.7

TABLE 4. OOC *ARL* (1st row) and *SDRL* (2nd row) profiles of the proposed multivariate Max-type control charts for linear profiles for FEV and REV with a shift of size ς for a nominal $ARL_0 = 200$.

		Univariate schemes						Multivariate schemes					
		Max-EWMA		Max-DEWMA		Max-TEWMA		Max-MEWMA		Max-MDEWMA		Max-MTEWMA	
λ	ς	FEV	REV	FEV	REV	FEV	REV	FEV	REV	FEV	REV	FEV	REV
0.05	0.50	3.3	16.7	2.8	15.2	2.7	41.0	1.1	25.2	1.1	23.5	1.1	22.9
		1.3	49.5	1.3	46.2	1.5	123.9	0.4	36.8	0.3	36.8	0.2	33.1
	1.00	1.0	6.7	1.0	6.3	1.0	11.4	1.0	8.0	1.0	7.4	1.0	7.7
		0.0	20.8	0.0	15.6	0.0	48.3	0.0	13.1	0.0	11.6	0.0	12.6
	1.50	1.0	3.8	1.0	3.6	1.0	5.8	1.0	4.1	1.0	3.8	1.0	3.8
		0.0	7.2	0.0	5.4	0.0	32.1	0.0	6.8	0.0	6.0	0.0	6.1
	2.00	1.0	2.6	1.0	2.6	1.0	3.2	1.0	2.6	1.0	2.4	1.0	2.4
		0.0	4.7	0.0	5.9	0.0	13.9	0.0	3.7	0.0	3.8	0.0	5.3
0.5	0.50	5.1	67.9	3.8	47.5	3.6	41.0	1.2	71.9	1.1	52.7	1.2	48.1
		3.3	159.1	1.8	135.0	1.5	123.9	0.4	92.8	0.0	75.2	0.4	72.3
	1.00	1.1	20.8	1.0	13.1	1.0	11.4	1.0	20.3	1.0	14.0	1.0	12.1
		0.3	67.85	0.2	48.4	0.0	48.3	0.0	42.6	0.0	29.8	0.0	26.7
	1.50	1.0	8.9	1.0	6.1	1.0	5.8	1.0	8.0	1.0	6.0	1.0	5.4
		0.0	35.3	0.0	28.6	0.0	32.1	0.0	24.1	0.0	16.8	0.0	13.5
	2.00	1.0	5.0	1.0	3.6	1.0	3.2	1.0	4.0	1.0	3.5	1.0	3.1
		0.0	23.7	0.0	14.9	0.0	13.9	0.0	9.8	0.0	10.4	0.0	6.4
0.95	0.50	15.9	113.3	13.8	110.4	12.0	106.8	1.2	121.0	1.2	117.9	1.2	112.0
		15.3	192.3	13.1	200.0	11.2	195.7	0.5	133.2	0.0	132.0	0.0	126.5
	1.00	1.1	46.6	1.1	44.5	1.1	41.2	1.0	45.8	1.0	43.2	1.0	38.8
		0.3	100.5	0.3	99.3	0.3	98.1	0.0	72.9	0.0	71.3	0.0	65.6
	1.50	1.0	21.1	1.0	19.7	1.0	17.5	1.0	19.5	1.0	18.1	1.0	15.6
		0.0	60.0	0.0	62.4	0.0	54.0	0.0	43.9	0.0	41.8	0.0	37.4
	2.00	1.0	6.9 37.5	1.0	10.5	1.0	9.5	1.0	9.6	1.0	8.8	1.0	7.6
		0.0		0.0	41.3	0.0	38.7	0.0	28.0	0.0	28.1	0.0	21.5

simple linear profile with FEV gives a signal on the 34th sample. However, the one with REV give a signal on the 145th sample. For a moderate shift of 1 standard deviation and a large shift of 2 standard deviation, the FEV Max-EWMA chart gives a signal on the 3rd and 1st samples, while the REV Max-EWMA chart gives

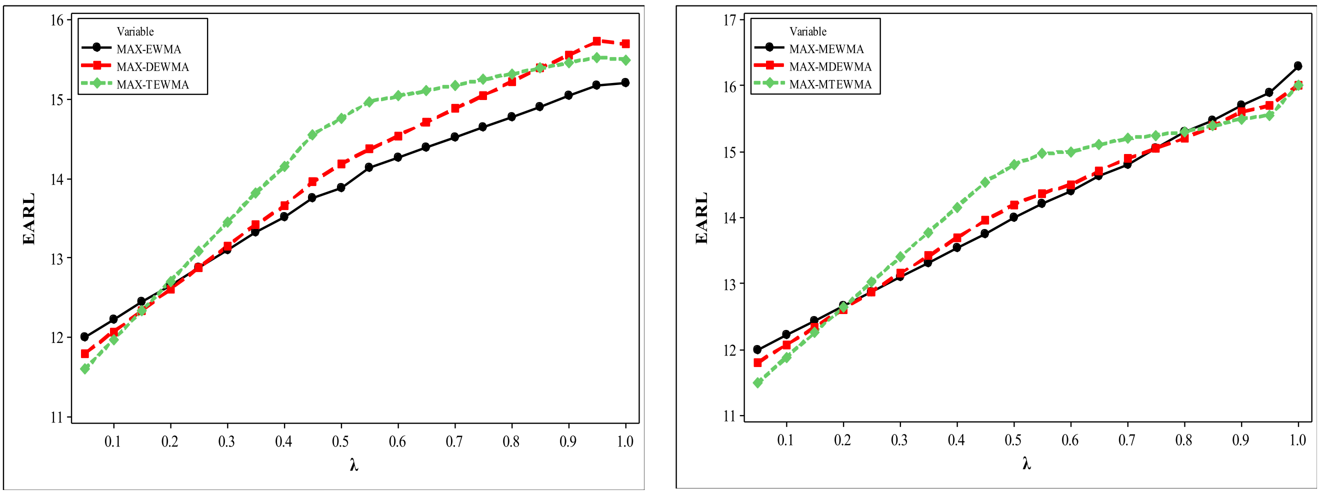
TABLE 5. OOC *ARL* (1st row) and *SDRL* (2nd row) profiles of the proposed univariate and multivariate Max-type control charts for linear profiles with FEV and REV with a shift of size γ in the error standard deviation for a nominal $ARL_0 = 200$.

		Univariate scheme						Multivariate scheme					
		Max-EWMA		Max-DEWMA		Max-TEWMA		Max-MEWMA		Max-MDEWMA		Max-MTEWMA	
λ	γ	FEV	REV	FEV	REV	FEV	REV	FEV	REV	FEV	REV	FEV	REV
0.05	1.20	34.1	15.2	33.3	15.2	32.1	16.8	34.1	34.3	33.3	33.3	32.0	32.02
		32.0	25.6	33.6	25.6	33.6	25.9	32.0	32.1	33.6	33.4	33.6	33.36
	1.50	7.9	5.4	8.0	5.7	8.1	6.6	7.9	7.8	8.0	7.8	8.1	8.0
		7.1	5.7	8.3	6.7	9.3	7.9	7.2	7.1	8.3	8.1	9.3	9.2
	1.75	3.7	3.2	3.7	3.4	3.7	3.8	3.7	3.7	3.7	3.6	3.7	3.7
		3.3	2.9	3.9	3.5	4.4	4.2	3.3	3.3	3.9	3.8	4.4	4.4
	2.00	2.4	2.3	2.3	2.4	2.3	2.6	2.4	2.4	2.3	2.3	2.3	2.3
		1.9	1.8	2.3	2.2	2.6	2.7	1.9	1.9	2.3	2.2	2.6	2.6
0.5	1.25	41.4	28.6	41.4	25.9	43.5	24.6	41.4	42.5	41.4	42.4	43.5	72.9
		40.8	50.6	40.6	49.2	42.8	48.4	40.8	40.3	40.6	41.4	42.8	55.2
	1.50	8.4	6.5	8.7	6.0	9.3	5.9	8.4	8.43	8.7	8.6	9.3	16.5
		7.6	8.1	7.8	7.4	8.4	7.2	7.6	7.7	7.8	7.9	8.4	13.5
	1.75	3.9	3.3	4.0	3.3	4.2	3.4	3.9	3.9	4.0	4.0	4.2	7.1
		3.1	3.1	3.4	3.1	3.6	3.2	3.1	3.1	3.4	3.4	3.6	6.9
	2.00	2.4	2.3	2.5	2.4	2.7	2.4	2.4	2.5	2.5	2.6	2.7	4.0
		1.8	1.8	2.0	1.8	2.2	2.0	1.8	1.8	2.0	2.0	2.2	4.2
0.95	1.25	47.0	40.8	46.4	39.1	45.9	38.2	47.0	46.6	46.4	46.3	45.9	45.1
		46.8	62.0	46.2	60.6	45.6	60.5	46.8	46.1	46.4	45.7	45.6	44.9
	1.50	9.8	8.6	9.6	8.4	9.5	8.1	9.8	9.7	9.6	9.6	9.5	9.4
		9.2	11.1	9.1	10.9	9.0	10.5	9.2	9.1	9.1	9.1	9.0	8.8
	1.75	4.1	3.9	4.1	4.0	4.0	3.9	4.1	4.1	4.1	4.1	4.0	4.0
		3.6	4.2	3.5	4.2	3.4	4.1	3.6	3.5	3.5	3.5	3.4	3.4
	2.00	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5
		1.9	2.3	1.9	2.2	1.9	2.7	1.9	2.0	1.9	2.0	1.9	1.9

a signal on the 12th and 2nd samples, respectively. Similar findings in the patterns of the *ARL* profiles of the Max-DEWMA and Max-TEWMA control charts is observed for simple linear profiles with FEV and REV.

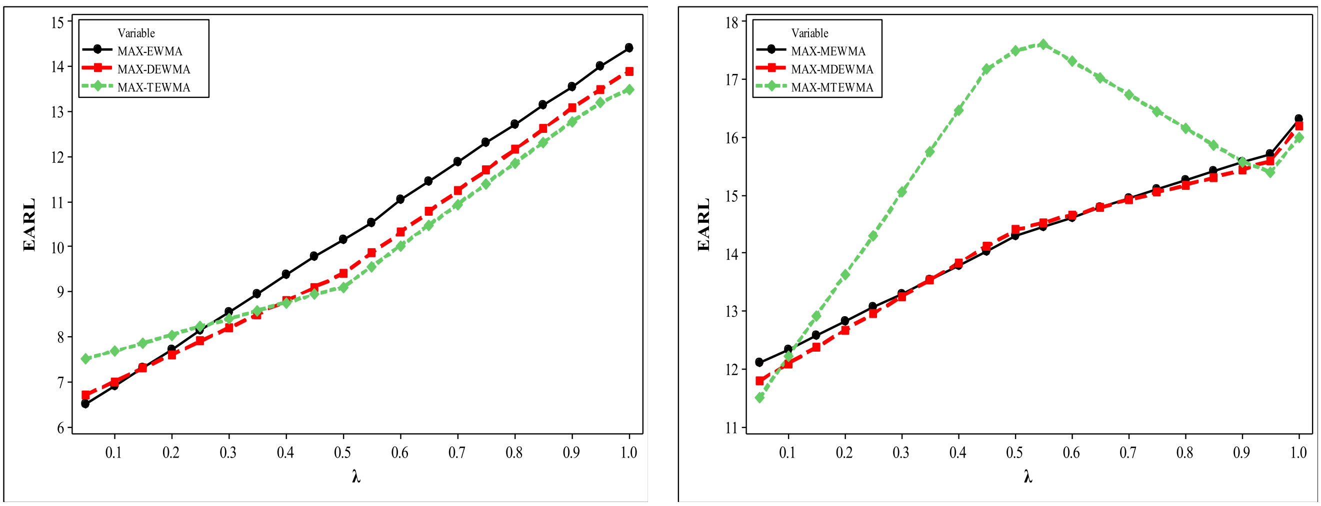
- Regardless of the size of the shift in the intercept of the regression model, the Max-DEWMA and Max-TEWMA control charts are similar in performance. However, these two schemes outperform the Max-EWMA control chart for small and moderate shifts in the intercept.
- The smaller the smoothing parameter λ, the more efficient the proposed Max-EWMA, Max-DEWMA and Max-TEWMA control chart are. As λ increases, the performance of these monitoring schemes deteriorates dramatically.
- Similar findings are also observed in terms of the *SDRL* profile; that is, the *SDRL* values increase as λ increases. This indicates that the probability of giving a false alarm signal is directly proportional to the values of λ. Therefore, it is recommended to use smaller values of λ to get reliable monitoring scheme.
- For very small shift the Max-EWMA with REV yields smaller *SDRL* compared to the Max-DEWMA and Max-TEWMA charts with REV. The latter is less reliable than the Max-DEWMA control chart in terms of the *SDRL* profile for very small shifts. As the magnitude of the shift increases, the Max-TEWMA scheme with REV becomes more reliable than the Max-EWMA and Max-DEWMA control charts in similar conditions.
- The random effect deteriorates considerably the performances of the proposed Max-EWMA, Max-DEWMA and Max-TEWMA control chart regardless of λ. This means, a preliminary study is necessary in order to find the appropriate explanatory variable that has a strong (or nearly perfect) functional relationship with the profile.

Table 3 displays the performances of Max-MEWMA, Max-MDEWMA and Max-MTEWMA control charts for linear profiles with fixed and random explanatory variables for a shift in the intercept regression parameter when $n = 4$, $\lambda \in \{0.05,0.5,0.95\}$, $b_0 = 3$, $b_1 = 2$, $\mu_x = 0$ and $\sigma_x^2 = 1$ for a nominal $ARL_0 = 200$. Table 3 shows that for small smoothing parameters (say, $\lambda = 0.05$), the Max-MTEWMA charts outperforms both the Max-MEWMA and Max-MDEWMA charts in terms of the *ARL* profile for very small shifts (e.g., $\delta \in(0,0.25)$) in the intercept under the assumptions of both FEV and REV. As the magnitude of the shift in the intercept increases, the three multivariate charts perform almost similarly. For moderate values of λ (say, $\lambda = 0.5$), the Max-MTEWMA chart is faster than the Max-MEWMA and Max-MDEWMA charts in detecting small shifts in the intercept of the regression model for both assumptions. For moderate and large shifts, the proposed monitoring schemes are almost equivalent in terms of the *ARL* profile. For large values of λ, the Max-MTEWMA chart remains superior over the Max-MEWMA and Max-MDEWMA charts for small and moderate shifts in the intercept and for large shifts, the three multivariate Max-type charts are similar in performance. Similar to the findings for univariate processes, the performance of Max-MEWMA, Max-MDEWMA and Max-MTEWMA charts deteriorate significantly as the value of λ increases. In terms of the *SDRL* profile, for small and moderate shifts in the intercept, the Max-MEWMA chart is likely to give a false OOC signal than the Max-MDEWMA



(a) (b)

FIGURE 2. Overall performance of the proposed Max-type control chart for linear profiles with FEV when $n = 4$, $\alpha_0 = 3$, $\alpha_1 = 2$ with $\gamma_{min} = 1.2$ and $\gamma_{max} = 2$ for a nominal $ARL_0 = 200$: (a) Univariate monitoring schemes; (b) Multivariate monitoring schemes.



(a) (b)

FIGURE 3. Overall performance of the proposed Max-type control chart for simple and general linear profiles with REV when $n = 4$, $\alpha_0 = 3$, $\alpha_1 = 2$ with $\gamma_{min} = 1.2$ and $\gamma_{max} = 2$ for a nominal $ARL_0 = 200$: (a) Univariate monitoring schemes; (b) Multivariate monitoring schemes.

and Max-MTEWMA control charts for moderate and large values of λ. However, for small values of λ, the three monitoring schemes are all reliable.

Table 4 presents the performances of both the univariate and multivariate Max-EWMA, Max-DEWMA and Max-TEWMA control charts for linear profiles with fixed and random explanatory variables for a shift in the slope (ς) of the regression model when $n = 4$, $\lambda \in \{0.05, 0.5, 0.95\}$, $b_0 = 3$, $b_1 = 2$, $\mu_x = 0$ and $\sigma_x^2 = 1$ for a nominal $ARL_0 = 200$. It can be clearly observed that the proposed control charts are very sensible under the assumption of FEV as compared to REV. The sensitivities of the proposed charts increase as the magnitude of the shift in standard deviation unit increases. The smaller the value of λ, the more sensitive the proposed charts are. The univariate schemes are faster in detecting shifts in the slope of the regression model as compared to multivariate schemes. In terms of the *SDRL* profile, it can be seen that the proposed charts are less reliable under the assumption of REV.

Table 5 presents the IC and OOC performances in terms of the *ARL* and *SDRL* profiles of both univariate and multivariate Max-EWMA, Max-DEWMA and Max-TEWMA control charts for linear profiles with FEV and REV for a shift in the error variance (γ) of the regression model when $n = 4$, $\lambda \in \{0.05, 0.5, 0.95\}$, $b_0 = 3$, $b_1 = 2$, $\mu_x = 0$ and $\sigma_x^2 = 1$ for a nominal $ARL_0 = 200$. Table 5 reveals that the three univariate monitoring schemes as whole are quite similar in terms of the *ARL* and *SDRL* profiles under the assumptions of FEV and REV. This finding is also true for the multivariate schemes as a whole. However, the univariate monitoring schemes are

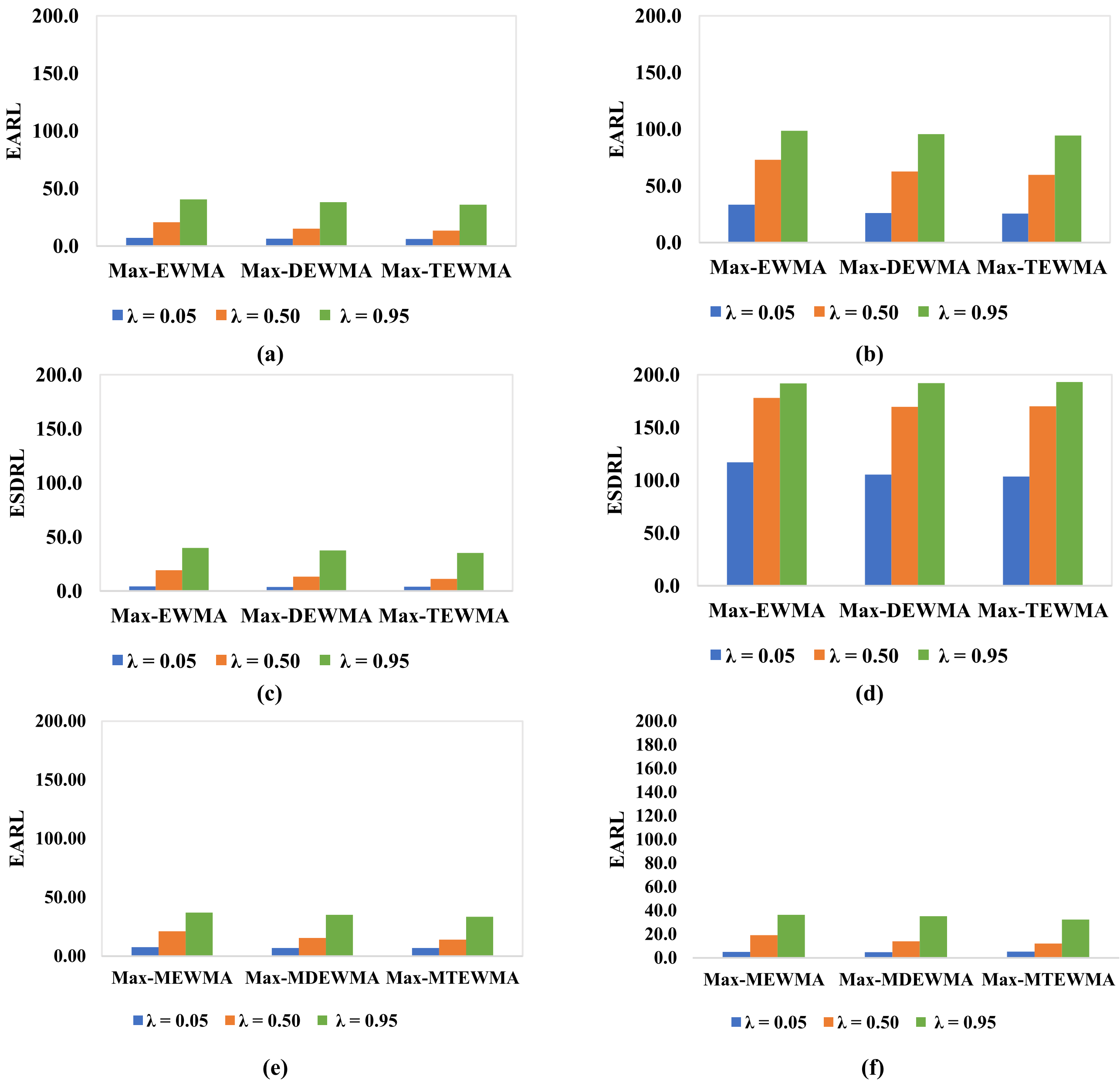


FIGURE 4. Overall performances of the proposed Max-type control charts for linear profiles under FEV and REV when $n = 4$, $\alpha_0 = 3$, $\alpha_1 = 2$ with $\delta_{min} = 0.25$ and $\delta_{max} = 2$ for a nominal $ARL_0 = 200$: (a) *EARL* profile comparison under FEV; (b) *EARL* profile comparison under REV; (c) *ESDRL* profile comparison under FEV; (d) ESDRL profile comparison under REV; (e) *EARL* profile comparison under FEV; (f) *SDRL* profile comparison under REV.

faster than the multivariate ones in detecting shifts in the error variance (or standard deviation) of the regression model. Under the assumption of REV, the univariate Max-EWMA, Max-DEWMA and Max-TEWMA control charts are more sensitive to the shifts in the error variance as compared to the ones under the assumption of FEV.

3) OVERALL PERFORMANCE ANALYSIS

Figures 2 and 3 depict the patterns of the overall performances of the proposed univariate and multivariate monitoring schemes, respectively, when there is a shift in the error variance with $\gamma_{max} = 1.2$ and $\gamma_{max} = 2$ with an increment of 0.25 standard deviation. The following findings are observed:

1) Under the assumption of FEV:
- For both univariate and multivariate processes, the overall performances of the proposed Max-TEWMA, Max-DEWMA and Max-TEWMA control charts decrease as λ increases; see Figures 2 (a) and (b).
- The Max-EWMA control chart outperforms the Max-DEWMA and Max-TEWMA control charts except for small shifts where the Max-TEWMA chart performs better; see Figure 2 (a). The Max-TEWMA chart outperforms the Max-DEWMA chart except for moderate shifts in the error variance.
- The Max-MTEWMA control chart outperforms both the Max-MEWMA and Max-MDEWMA control

charts except for moderate shift where the Max-EWMA chart performs better; see Figure 2 (b).

2) Under the assumption of REV:

- For univariate processes, the overall performances of the proposed Max-TEWMA, Max-DEWMA and Max-TEWMA control charts decrease as λ increases; see Figures 3 (a).
- For multivariate processes, the overall performances of the Max-MEWMA and Max-MDEWMA charts deteriorate as λ increases. However, the performance of the Max-MTEWMA chart decreases for small to moderate values of λ and increases for large values of λ; Figure 3 (b).
- The Max-TEWMA control chart outperforms the Max-EWMA and Max-DEWMA control charts except for small shifts where the Max-EWMA chart performs better. The Max-DEWMA chart outperforms the Max-TEWMA chart except for moderate shifts in the error variance; Figure 3 (a).
- The Max-MTEWMA control chart outperforms both the Max-MEWMA and Max-MDEWMA control charts except for moderate shifts where the Max-EWMA chart performs better; see Figure 3 (b).

Figure 4 displays the overall comparisons of the proposed regression Max-type control charts under FEV and REV in terms of the *EARL* and *ESDRL* profiles when $n = 4$, $\alpha_0 = 3$, $\alpha_1 = 2$ with $\delta_{min} = 0.25$ and $\delta_{max} = 2$ for a nominal $ARL_0 = 200$. From Figure 4, it can be seen that the proposed regression control charts perform better for small values of λ. As λ increases, the overall performances of these charts decrease dramatically. It can also be noticed that the performances of the proposed charts degrade significantly under the assumption of REV. For small and moderate values of λ, the Max-TEWMA charts outperform the Max-EWMA and Max-DEWMA charts, respectively. For large values of λ, the Max-EWMA chart outperforms slightly the Max-DEWMA and Max-TEWMA charts in terms of the *EARL* and *ESDRL* profiles. Similar findings are observed for the multivariate case.

4) THE PROPOSED SCHEMES VERSUS THE EXISTING MEMORY-TYPE SCHEMES

In this section, the performances (or sensitivities) of the proposed Max-EWMA, Max-DEWMA and Max-TEWMA schemes are compared to the ones of the existing EWMA, DEWMA and TEWMA schemes for monitoring of the regression parameters and error variance concurrently. The latter three schemes are denoted as EWMA3, DEWMA3 and TEWMA3 schemes, respectively. The comparison is done under the assumption of FEV when $n = 4$, $\alpha_0 = 3$, $\alpha_1 = 2$, $\mu_x = 0$ and $\sigma_x = 1$ for a nominal $ARL_0 = 200$. In this comparison, it is assumed that the slope and error variance remain IC while the shift in the intercept varies from 0.25 to 2 with an increment of 0.25. From Figure 5, it can be seen that the proposed Max-EWMA, Max-DEWMA and Max-TEWMA schemes outperforms their respective counterparts except for very small shifts (i.e. $0 < \delta \leq 0.25$). Similar findings are observed for the multivariate processes (this is not shown here to preserve space). The proposed schemes are also preferred over the existing schemes because of their simplicity and interesting overall properties.

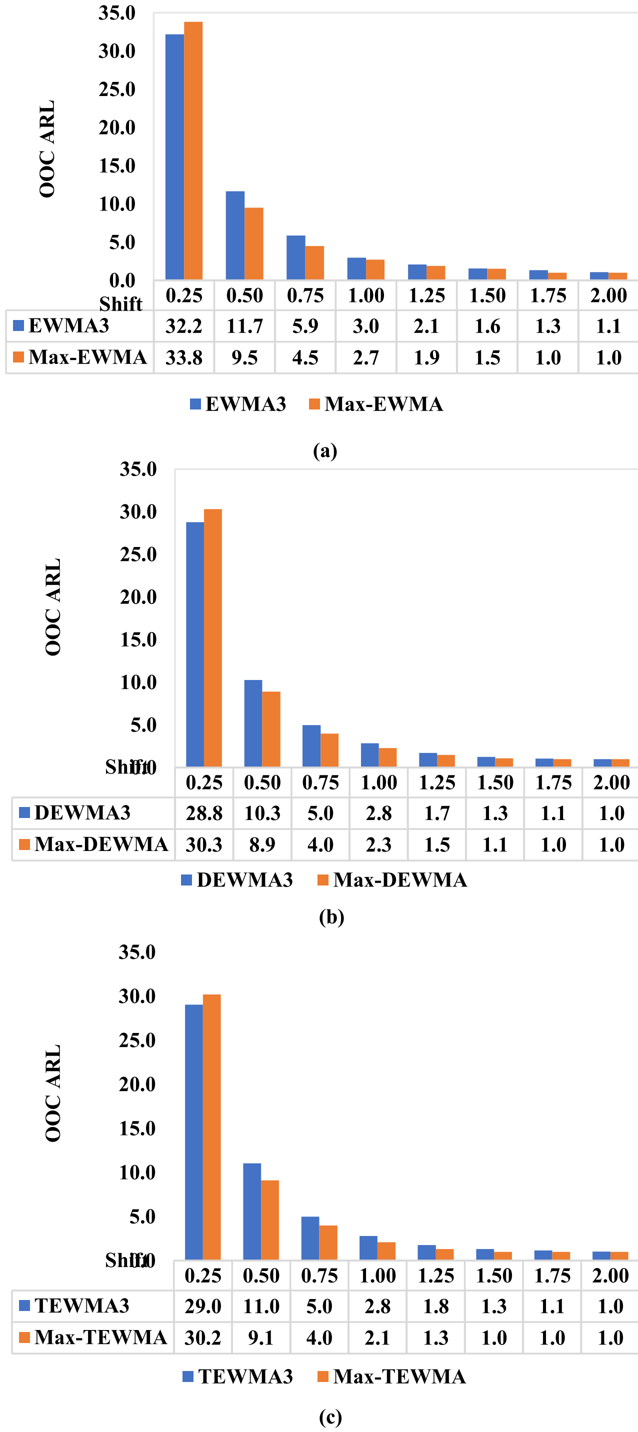


FIGURE 5. Performance comparison of the proposed schemes and the existing counterparts: (a) The Max-EWMA scheme versus the EWMA3 scheme; (b) The Max-DEWMA scheme versus the DEWMA3 scheme; (c) The Max-TEWMA scheme versus the TEWMA3 scheme.

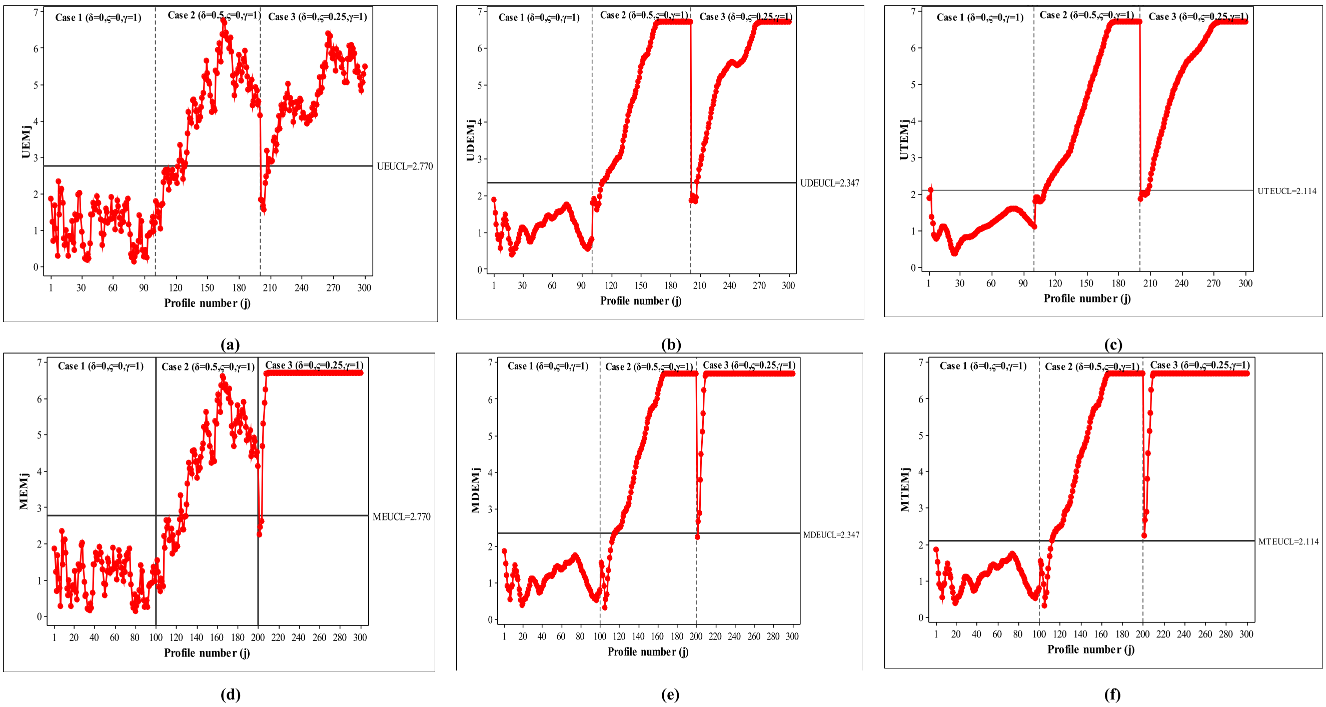


FIGURE 6. Proposed univariate and multivariate Max-type control charts under the assumption of FEV using simulated data when $n = 4$, $\alpha_0 = 3$ and $\alpha_1 = 2$ and $x_i \in \{2, 4, 6, 8\}$: (a) Max-EWMA; (b) Max-DEWMA; (c) Max-TEWMA; (d) Max-MEWMA; (e) Max-MDEWMA; (f) Max-MTEWMA.

TABLE 6. Truncated VDP data and charting statistics of the proposed regression max-type control charts for a nominal $ARL_0 = 200$.

	X_i	0	0.002	0.004	0.006	0.008	0.01	0.012	0.014	0.016	0.018	0.02	Max-EWMA/ Max-MEWMA	Max-DEWMA/ Max-MDEWMA	Max-TEWMA/ Max-MTEWMA
	profile 1	60.26	59.62	59.53	59.08	58.64	57.85	57.34	56.74	56.49	55.82	55.58	6.71	6.71	6.71
	profile 2	55.62	55.13	54.72	54.08	53.60	53.84	53.44	52.82	52.97	52.89	52.20	6.28	6.71	6.71
	profile 3	60.97	60.41	59.77	58.96	59.23	58.96	58.25	58.51	57.81	57.12	56.95	6.78	6.71	6.71
	profile 4	59.54	59.63	59.26	58.72	58.36	57.91	57.82	57.20	56.34	55.91	55.57	7.81	7.47	6.93
	profile 5	58.51	57.71	57.19	57.01	55.98	55.65	55.73	54.98	54.65	54.40	53.75	8.75	8.24	7.55
	profile 6	52.80	52.33	51.86	51.63	51.10	50.50	50.28	49.98	49.32	48.89	48.60	10.31	9.16	8.20
	profile 7	56.96	56.27	55.94	55.35	54.93	54.36	53.87	53.31	52.83	52.76	51.98	11.51	10.09	8.87
	profile 8	59.39	58.41	58.15	57.63	57.28	56.43	56.18	55.85	55.60	55.27	54.46	12.20	10.91	9.53
	profile 9	55.38	54.56	53.92	53.61	53.06	52.59	52.75	52.13	51.90	51.07	51.00	12.72	11.64	10.16
	profile 10	59.48	59.30	58.86	58.07	57.29	57.38	56.37	55.87	55.29	55.54	54.81	13.12	12.27	10.74
	profile 11	53.19	52.57	52.26	51.88	51.35	51.05	50.83	50.46	49.80	49.66	49.37	14.11	12.95	11.31
Y_j	profile 12	57.36	56.76	56.25	54.93	54.85	54.12	53.65	53.72	52.86	52.17	51.71	14.43	13.56	11.86
	profile 13	58.51	58.69	58.25	57.81	57.81	57.47	56.96	56.79	56.53	56.20	55.70	15.17	14.17	12.39
	profile 14	58.65	58.22	57.70	57.36	56.76	56.43	56.09	55.60	54.95	54.55	54.39	16.13	14.83	12.92
	profile 15	59.50	59.14	58.78	58.61	57.74	58.08	57.57	57.14	56.71	56.05	55.71	16.56	15.44	13.44
	profile 16	52.00	51.62	51.24	50.86	50.63	50.04	49.75	49.23	49.16	48.73	48.73	17.22	16.06	13.95
	profile 17	57.00	55.82	55.57	55.24	54.75	54.35	54.03	53.39	53.00	52.69	52.61	17.56	16.63	14.46
	profile 18	56.80	56.04	55.71	55.30	54.72	53.92	53.52	53.37	52.98	52.36	52.36	17.92	17.19	14.95
	profile 19	57.97	58.41	57.71	57.28	57.03	56.43	56.35	55.60	55.35	55.03	54.38	18.23	17.71	15.44
	profile 20	58.47	57.68	57.43	57.08	56.15	55.40	54.91	54.67	54.75	54.43	54.11	18.18	18.17	15.91
	profile 21	61.08	60.53	59.98	59.35	59.26	58.81	58.20	57.59	57.25	56.83	56.24	18.87	18.66	16.36
	profile 22	60.03	59.29	58.75	58.12	58.03	57.59	57.24	56.29	55.78	55.53	55.11	19.26	19.14	16.81
	profile 23	59.64	59.55	58.57	57.79	57.53	56.76	56.51	56.01	55.59	55.51	55.27	19.30	19.58	17.25
	profile 24	56.50	55.83	55.58	55.08	54.84	54.43	53.79	53.16	52.77	52.54	52.16	19.87	20.04	17.69
												Control limit constant	20.126	25.694	28.663
												Upper control limit	13.260	16.617	18.406

IV. ILLUSTRATIVE EXAMPLES

A. ILLUSTRATIVE EXAMPLE USING SIMULATED DATA

In this section, we illustrate the implementation and application of the proposed regression control chart using simulated data (100 subgroups i.e. $j = 1, 2, \ldots, 100$) under both the assumptions of fixed and random explanatory variables for a nominal $ARL_0 = 200$. For the FEV, the explanatory variable $n = 4$, $x_i \in \{2, 4, 6, 8\}$, $\alpha_0 = 3$ and $\alpha_1 = 2$. However, for the REV, the explanatory variable is generated from a normal distribution with mean 0 and variance 1, $n = 4$, $\alpha_0 = 3$ and $\alpha_1 = 2$. The control constant of the regression control charts under consideration are given in Table 1. For instance, with the regression Max-EWMA control chart we found that $L^{UE} = 2.724$ yields an attained $ARL_0 = 200$ under the assumption of FEV. However, under the assumption of REV, we found that $L^{UE} = 3.474$ yields an attained

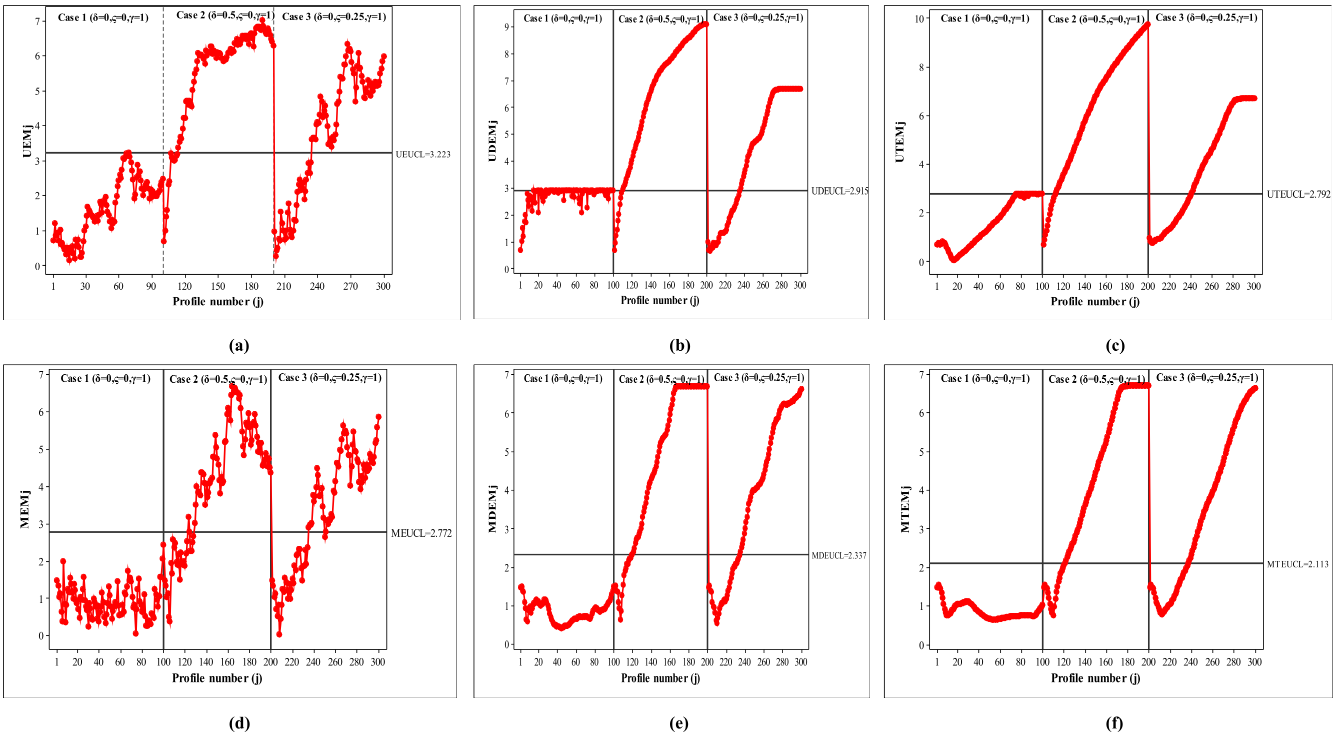


FIGURE 7. **Proposed univariate and multivariate Max-type control charts under the assumption of REV using simulated data when $n = 4$, $\alpha_0 = 3$ and $\alpha_1 = 2$ and $x_i \sim N(0, 1)$: (a) Max-EWMA; (b) Max-DEWMA; (c) Max-TEWMA; (d) Max-MEWMA; (e) Max-MDEWMA; (f) Max-MTEWMA.**

$ARL_0 = 200.5$. The investigation is done when: (i) the process is assumed to be IC, (ii) a shift of 0.5 standard deviation has occurred in the intercept of the regression model assuming the slope and error variance both remain IC and (iii) there is only a shift of 0.25 standard deviation in the slope of the regression model assuming that the intercept and error variance are IC. The plots of the charting statistics of the Max-EWMA, Max-DEWMA, Max-TEWMA, Max-MEWMA, Max-MDEWMA and Max-MTEWMA control charts for FEV are shown in Figures 6 (a)-(f), respectively; the ones for the REV are shown in Figures 7 (a)-(f), respectively.

From Figures 6 (a)-(f), the findings are summarised as follows (FEV):

- When the process is IC, the proposed univariate and multivariate regression control charts do not give a signal (See Case 1 in Figures 6 (a)-(f)).
- For univariate processes, for a shift of 0.5 standard deviation in the intercept, the Max-EWMA chart gives a signal on the 22$^{\text{nd}}$ profile; the Max-DEWMA and Max-TEWMA charts give a signal on the 11$^{\text{th}}$ profile. In addition, for a shift of 0.25 standard deviation, assuming no shift occurred in the intercept and error variance, the proposed Max-EWMA chart gives a signal on the 7$^{\text{th}}$ profile while the Max-DEWMA and Max-TEWMA charts give signals on the 7$^{\text{th}}$ and 9$^{\text{th}}$ profiles, respectively.
- For multivariate processes, for a shift of 0.5 standard deviation in the intercept, the Max-MEWMA chart gives a signal on the 24$^{\text{th}}$ profile; the Max-DEWMA and Max-MTEWMA charts give a signal on the 15$^{\text{th}}$ and 18$^{\text{th}}$ profiles, respectively. In addition, for a shift of 0.25 standard deviation, assuming no shift occurred in the intercept and error variance, the proposed Max-MEWMA chart gives a signal on the 4$^{\text{th}}$ profile while both the Max-MDEWMA and Max-MTEWMA charts give signals on the 2$^{\text{nd}}$ profile.

The findings in Figures 7 (a)-(f) can be summarised as follows (REV):

- When the process is IC, the proposed univariate and multivariate regression control charts do not give a signal (See Case 1 in Figures 7 (a)-(f)).
- For univariate processes, for a shift of 0.5 standard deviation in the intercept, the Max-EWMA chart gives a signal on the 14$^{\text{th}}$ profile; the Max-DEWMA and Max-TEWMA charts give signals on the 10$^{\text{th}}$ and 14$^{\text{th}}$ profiles, respectively. However, for a shift of 0.25 standard deviation, assuming no shift occurred in the intercept and error variance, the proposed Max-EWMA chart gives a signal on the 35$^{\text{th}}$ profile while the Max-DEWMA and Max-TEWMA charts give signals on the 36$^{\text{th}}$ and 41$^{\text{st}}$ profiles, respectively.
- For multivariate processes, for a shift of 0.5 standard deviation in the intercept, the Max-EWMA chart gives a signal on the 23$^{\text{rd}}$ profile; the Max-MDEWMA and Max-MTEWMA charts give signals on the 21$^{\text{st}}$ profile. In addition, for a shift of 0.25 standard deviation, assuming no shift occurred in the intercept and error variance, both the proposed Max-MEWMA and

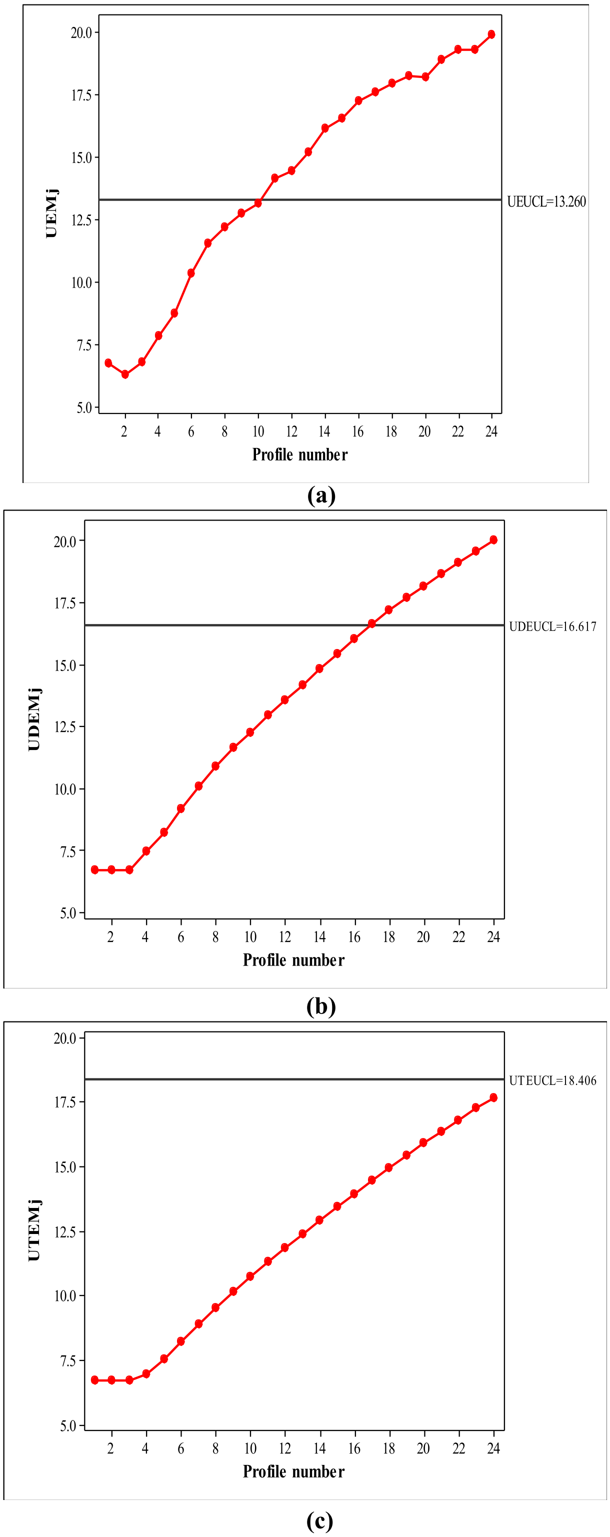


FIGURE 8. **The proposed monitoring schemes for the truncated vertical density profile data using FEV with a nominal $ARL_0 = 200$: (a) Max-EWMA/ Max-MEWMA chart; (b) Max-DEWMA/ Max-MDEWMA chart; (c) Max-TEWMA/ Max-MTEWMA chart.**

Max-MDEWMA charts give signals on the 35$^{\text{th}}$ profile while the Max-MTEWMA chart gives a signal on the 38$^{\text{th}}$ profiles.

B. ILLUSTRATIVE EXAMPLE USING REAL-LIFE DATA

To illustrate the application and implementation of the proposed regression Max-type control charts, the data from [39] on the truncated vertical density profile are used under the assumption of FEV. The profile data ($Y_j = y_{ij}$, $i = 1, 2, \ldots, 11$ and $j = 1, 2, \ldots, 24$) represent the density of the wood board that takes measurements at a series of fixed depths ($X = x_i$, $i = 1, 2, \ldots, 11$; i.e. $n = 11$) (see Table 6). The purpose of this application is to monitor the intercept, slope and error variance of the simple regression model of the density of the wood board's surface simultaneously using a single control chart. The control limits constants are determined such that the nominal $ARL_0 = 200$ when $n = 11$ and $X = 0$ (0.002) 0.02. For instance, we find that the control limit constants of the Max-EWMA, Max-DEWMA and Max-TEWMA control charts as well as their corresponding multivariate counterparts are given by 20.126, 25.694 and 28.663 so that they yield attained ARL_0 values of 200.4, 200.2 and 200, respectively. The plots of the charting statistics of the Max-EWMA, Max-DEWMA and Max-TEWMA charts are shown in Figures 8 (a)-(c), respectively (see also Table 6). From Figure 8 and Table 6, it can be seen that the Max-EWMA and the Max-DEWMA charts give a signal on the 11$^{\text{th}}$ and 17$^{\text{th}}$ profiles, respective, while the Max-TEWMA chart does not give a signal in the prospective phase. In this particular case, the Max-MEWMA, Max-MDEWMA and Max-MDEWMA charts are equivalent to the Max-EWMA, Max-DEWMA and Max-TEWMA charts, respectively. This example demonstrates the superiority of the Max-EWMA chart over the Max-DEWMA and Max-TEWMA charts under the assumption of FEV.

V. CONCLUSION AND REMARKS

In this paper, we proposed new univariate and multivariate Max-EWMA, Max-DEWMA and Max-TEWMA control charts for simple linear profiles to monitor the coefficients of a regression model and error variance simultaneously under the assumptions of FEV and REV.

The results of this study reveal that the proposed control charts perform better under the assumption of FEV as compared to the one based on REV. The use of small smoothing parameters provides reliable and more efficient monitoring schemes. The use of large smoothing parameters is not recommended since large values of λ inflate the *ARL* and *SDRL* values of the proposed control charts especially under REV. When a process is susceptible to shifts in the error variance only, the Max-EWMA control chart is recommended because of its simplicity and interesting properties which are as attractive as the ones of the Max-DEWMA and Max-TEWMA control charts.

In this study, we assumed that the REV is a standard normal random variable. When the explanatory variable departures from the assumed distribution, the results and properties of the proposed control charts need to be revisited under new settings. Thus, researchers who are interested in this topic can consider the investigation of the proposed control charts under the assumption of REV with unknown underlying distribution. In addition, researchers can also investigate the performances the proposed control charts for nonlinear profiles. In addition, researchers can also look at the Bayesian profiles scheme proposed in [40] to build new max-type profile monitoring schemes. Following the multivariate homogeneously weighted moving average (MHWMA) design developed by [41], researchers can also look at the design of the max-type MHWMA scheme.

APPENDICES

The following Lemma will be helpful in deriving the expectations and the variances used in this paper

Lemma 1: For any $k \geq 1$ and $0 < d \leq 1$, we have

$$\sum_{l=1}^{k} l d^{l-1} = \frac{1-d^{k+1}}{(1-d)^2} - \frac{(k+1)\,d^k}{1-d}, \tag{A.1}$$

$$\sum_{l=1}^{k} l\,(l-1)\,d^{l-2} = -\frac{k\,(k+1)\,d^{k-1}}{1-d} - \frac{2\,(k+1)\,d^k}{(1-d)^2} + \frac{2\left(1-d^{k+1}\right)}{(1-d)^3}, \tag{A.2}$$

$$\sum_{l=1}^{k} l\,(l-1)\,(l-2) d^{l-3} = -\frac{k\left(k^2-1\right)d^{k-2}}{1-d} - 3\frac{k\,(k+1)\,d^{k-1}}{(1-d)^2} - 6\frac{(k+1)\,d^k}{(1-d)^3} + 6\frac{1-d^{k+1}}{(1-d)^4} \tag{A.3}$$

and

$$\sum_{l=1}^{k} l\,(l-1)\,(l-2)\,(l-3)\,d^{l-4} = -\frac{k\left(k^2-1\right)(k-2)\,d^{k-3}}{1-d} - 4\frac{k\left(k^2-1\right)d^{k-2}}{(1-d)^2} - 12\frac{k\,(k+1)\,d^{k-1}}{(1-d)^3} - 24\frac{(k+1)\,d^k}{(1-d)^4} + 24\frac{1-d^{k+1}}{(1-d)^5}. \tag{A.4}$$

APPENDIX A
PROPERTIES OF THE UEZ_{jk} AND MEZ_{jk} STATISTICS

Thus, Eq. (7) can be re-written as

$$\begin{aligned} UEZ_{j1} &= \lambda \sum_{i=1}^{j} (1-\lambda)^{j-i}\,\hat{\beta}_{0i} + (1-\lambda)^j\,UEZ_{01} \\ \text{and } UEZ_{j2} &= \lambda \sum_{i=1}^{j} (1-\lambda)^{j-i}\,\hat{\beta}_{1i} + (1-\lambda)^j\,UEZ_{02}. \end{aligned} \tag{A.5}$$

Hence,

$$\begin{aligned} \mathrm{E}\left(UEZ_{j1}\right) &= \left[\lambda \sum_{i=1}^{j} (1-\lambda)^{j-i} + (1-\lambda)^j\right]\beta_0 = \beta_0 \\ \mathrm{E}\left(UEZ_{j2}\right) &= \left[\lambda \sum_{i=1}^{j} (1-\lambda)^{j-i} + (1-\lambda)^j\right]\beta_1 = \beta_1. \end{aligned} \tag{A.6}$$

In addition, from Lemma 1, we get

$$\begin{aligned} \mathrm{Var}\left(UEZ_{j1}\right) &= \frac{\left[\lambda^2 \sum_{i=1}^{j} (1-\lambda)^{2(j-i)}\right]\sigma^2}{n} \\ &= \frac{\lambda\left(1-(1-\lambda)^{2j}\right)\sigma^2}{(2-\lambda)\,n} \\ \mathrm{Var}\left(UEZ_{j2}\right) &= \frac{\left[\lambda^2 \sum_{i=1}^{j} (1-\lambda)^{2(j-i)}\right]\sigma^2}{S_{xx}} \\ &= \frac{\lambda\left(1-(1-\lambda)^{2j}\right)\sigma^2}{(2-\lambda)\,S_{xx}}. \end{aligned} \tag{A.7}$$

The derivations of Var(UEZ_{j3}), E(MEZ_{j1}), Var(MEZ_{j1}) and Var(MEZ_{j3}) of Eqs. (11b), (18), (19a) and (22b) are done in a similar way.

APPENDIX B
PROPERTIES OF THE $UDEZ_{jk}$ AND $MDEZ_{jk}$ STATISTICS

Eq. (26) can be simplified to

$$\begin{aligned} UDEZ_{j1} &= \lambda^2 \sum_{i=1}^{j} (j-i+1)\,(1-\lambda)^{j-i}\,\hat{\beta}_{0i} \\ &\quad + 2\,(1+\lambda i)\,(1-\lambda)^i\,UDEZ_{01}, \\ \text{and } UDEZ_{j2} &= \lambda^2 \sum_{i=1}^{j} (j-i+1)\,(1-\lambda)^{j-i}\,\hat{\beta}_{1i} \\ &\quad + 2\,(1+\lambda i)\,(1-\lambda)^i\,UDEZ_{02}. \end{aligned} \tag{B.1}$$

Using Lemma 1, we get

$$\begin{aligned} &\mathrm{E}\left(UDEZ_{j1}\right) = \beta_0 \text{ and } E\left(UDEZ_{j2}\right) = \beta_1 \\ &\text{and } \mathrm{Var}\left(UDEZ_{j1}\right) = \lambda^4 \left(\sum_{l=1}^{j} l^2 d^{l-1}\right)\frac{\sigma^2}{n}, \end{aligned} \tag{B.2}$$

where $d = (1-\lambda)^2$. Then, using Lemma 1, we get

$$\mathrm{Var}\left(UDEZ_{j1}\right) = \lambda^4 \psi \sigma_1^2, \tag{B.3}$$

where σ_1^2 is as defined in Eq. (8) and ψ, as shown at the top of the next page.

In a similar manner, we have

$$\mathrm{Var}\left(UDEZ_{j2}\right) = \lambda^4 \psi \sigma_2^2, \tag{B.4}$$

where σ_2^2 is as defined in Eq. (8).

The derivations of $\mathrm{Var}\left(UDEZ_{j3}\right)$, E($MDEZ_{j1}$), $\mathrm{Var}\left(MDEZ_{j1}\right)$ and $\mathrm{Var}\left(MDEZ_{j3}\right)$ of Eqs. (29b), (33), (34a) and (36b) are done in a similar way.

$$\psi = \left[\frac{1+\zeta^2 - (j^2+2j+1)\,\zeta^{2j} + (2j^2+2j-1)\,\zeta^{2j+2} - j^2\zeta^{2j+4}}{(1-\zeta^2)^3} \right], \quad \text{and } \zeta = 1-\lambda.$$

APPENDIX C
PROPERTIES OF THE $UTEZ_{jk}$ AND $MTEZ_{jk}$ STATISTICS

Eq. (40) can be simplified to

$$\begin{aligned} UTEZ_{j1} &= \frac{\lambda^3}{2}\sum_{i=1}^{j}(j-i+1)(j-i+2)(1-\lambda)^{j-i}\hat{\beta}_{0i} \\ &\quad + \frac{(1-\lambda)^i}{2}[\lambda i(\lambda i+\lambda+2)+2]UTEZ_{01}, \\ \text{and } UTEZ_{j2} &= \frac{\lambda^3}{2}\sum_{i=1}^{j}(j-i+1)(1-\lambda)^{j-i}\hat{\beta}_{1i} \\ &\quad + \frac{(1-\lambda)^i}{2}[\lambda i(\lambda i+\lambda+2)+2]\,UTEZ_{02}. \end{aligned} \tag{C.1}$$

Using Lemma 1, we get

$$\begin{aligned} &\mathrm{E}\left(UTEZ_{j1}\right) = \beta_0 \text{ and } \mathrm{E}\left(UTEZ_{j2}\right) = \beta_1 \\ &\text{and } \mathrm{Var}\left(UTEZ_{j1}\right) = \frac{\lambda^6}{4}\left(\sum_{l=1}^{j} l^2(l+1)\wedge 2d^{l-1}\right)\frac{\sigma^2}{n}, \end{aligned} \tag{C.2}$$

where $d=(1-\lambda)^2$. Using Lemma 1, we get

$$\mathrm{Var}\left(UTEZ_{j1}\right) = \lambda^6\varphi\sigma_1^2, \tag{C.3}$$

where σ_1^2 is as defined in Eq. (8) and

$$\begin{aligned} \varphi = \frac{\zeta^6}{4}\Bigg[&-\left[\frac{j(j^2-1)(j-2)\zeta^{2j-6}}{1-\zeta^2}\right] \\ &-4\left[\frac{j(j^2-1)\zeta^{2j-4}}{(1-\zeta^2)^2}\right] - 12\left[\frac{j(j+1)\zeta^{2j-2}}{(1-\zeta^2)^3}\right] \\ &+24\left[\frac{(j+1)\zeta^{2j}}{(1-\zeta^2)^4}\right] + 24\left[\frac{1-\zeta^{2j+2}}{(1-\zeta^2)^5}\right]\Bigg] \\ &+2\zeta^4\Bigg[-\left[\frac{j(j^2-1)\zeta^{2j-4}}{1-\zeta^2}\right] - 3\left[\frac{j(j+1)\zeta^{2j-2}}{(1-\zeta^2)^2}\right] \\ &-6\left[\frac{(j+1)\zeta^{2j}}{(1-\zeta^2)^3}\right] + 6\left[\frac{1-\zeta^{2j+2}}{(1-\zeta^2)^4}\right]\Bigg] \\ &+\frac{7\zeta^2}{2}\Bigg[-\left[\frac{j(j+1)\zeta^{2j-2}}{1-\zeta^2}\right] - \left[\frac{2(j+1)\zeta^{2j}}{(1-\zeta^2)^2}\right] \\ &+\left[\frac{2(1-\zeta^{2j+2})}{(1-\zeta^2)^3}\right]\Bigg] + \Bigg[\left(\frac{1-\zeta^{2j+2}}{(1-\zeta^2)^2}\right) \\ &-\left(\frac{(j+1)\zeta^{2j}}{1-\zeta^2}\right)\Bigg] \quad \text{and } \zeta = 1-\lambda. \end{aligned}$$

In a similar manner, we have

$$\mathrm{Var}\left(UTEZ_{j2}\right) = \lambda^6\varphi\sigma_2^2, \tag{C.4}$$

where σ_2^2 is as defined in Eq. (8).

The derivations of $\mathrm{Var}\left(UTEZ_{j3}\right)$, $\mathrm{E}(MTEZ_{j1})$, $\mathrm{Var}\left(MTEZ_{j1}\right)$ and $\mathrm{Var}\left(MTEZ_{j3}\right)$ of Eqs. (43b), (47), (48a) and (50b) are done in a similar way.

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