A NOTE ON DYADIC APPROXIMATION IN CANTOR'S SET

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ABSTRACT. We consider the convergence theory for dyadic approximation in the middle-third Cantor set, K, for approximation functions of the form $\psi_\tau(n) = n^{-\tau}$ $(\tau \geqslant 0)$. In particular, we show that for values of τ beyond a certain threshold we have that almost no point in K is dyadically ψ_τ-well approximable with respect to the natural probability measure on K. This refines a previous result in this direction obtained by the first, third, and fourth named authors.

1. Introduction

Throughout this note, we write K to denote the middle-third Cantor set and denote by μ the natural probability measure on K. We recall that K consists of the real numbers $x \in [0, 1]$ which have a ternary expansion consisting only of 0's and 2's, and that its Hausdorff dimension is

$$\dim_{\mathrm{H}} K = \frac{\log 2}{\log 3} =: \gamma.$$

The natural measure μ on K is the Hausdorff γ-measure restricted to K, which is a probability measure as $\mathcal{H}^\gamma(K) = 1$. For more information on Hausdorff dimension and Hausdorff measures, we refer the reader to [5].

The study of Diophantine approximation in the Cantor set was suggested by Mahler [13], and has since been an active subject of research — see, for example, [3, 4, 10, 12, 14, 15, 16, 17]. In [1], the first, third and fourth named authors discussed the problem of approximating elements of K by rationals with denominators that are a power of two: that is, *dyadic rationals*. Our methods realised the dyadic approximation problem as a manifestation of Furstenberg's "times two, times three" phenomenon [6, 7].

For $\psi : \mathbb{R} \to [0, \infty)$ and $y \in \mathbb{R}$, define

$$W_2(\psi, y) = \{x \in \mathbb{R} : \|2^n x - y\| < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Here, for $x \in \mathbb{R}$, we write $\|x\|$ to denote the Euclidean distance from x to the nearest integer. In analogy with Khintchine's theorem [11], Velani conjectured that if ψ is monotonic then

$$\mu(W_2(\psi, 0)) = \begin{cases} 0, & \text{if } \sum_{n=1}^{\infty} \psi(n) < \infty, \\ 1, & \text{if } \sum_{n=1}^{\infty} \psi(n) = \infty, \end{cases}$$

see [1, Conjecture 1.2]. The two parts of such a dichotomy are commonly referred to as the convergence and divergence theories of metric Diophantine approximation, respectively. The second named author [2] stated the following natural generalisation of Velani's conjecture, dropping the monotonicity condition and introducing an inhomogeneous shift. The latter

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relates the problem to distribution modulo 1, and also enables one to recast it in terms of shrinking targets [9].

Conjecture 1. ([2, Conjecture 1.2]) *If $y \in \mathbb{R}$, then*

$$\mu(W_2(\psi, y)) = \begin{cases} 0, & \text{if } \sum_{n=1}^{\infty} \psi(n) < \infty, \\ \\ 1, & \text{if } \sum_{n=1}^{\infty} \psi(n) = \infty. \end{cases}$$

Let us now consider the problem at the level of the exponent. For $\tau \geqslant 0$ and $n \in \mathbb{N}$, define $\psi_\tau(n) = n^{-\tau}$. Plainly $\mu(W_2(\psi_0, y)) = 1$ for any y. By [1, Theorem 1.5], we have

$$\mu(W_2(\psi_\tau, 0)) = 0 \qquad (\tau \geqslant 1/\gamma). \tag{1}$$

It follows from the recent work of the second named author [2] that if $y \in \mathbb{R}$ then

$$\mu(W_2(\psi_\tau, y)) = 1 \qquad (\tau \leqslant 0.01),$$

refining the progress on the divergence side made in [1]. The purpose of this note is to establish the following sharpening and generalisation of (1).

Theorem 2. *Let*

$$\tau > \frac{0.922(1-\gamma)+1}{\gamma(2-\gamma)},$$

and let $y \in \mathbb{R}$. Then $\mu(W_2(\psi_\tau, y)) = 0$.

One computes that $\tau > 1/\gamma - 0.03$ is sufficient. This makes progress towards the convergence part of Velani's conjecture. In [1], it was shown conditionally that

$$\mu(W_2(\psi_\tau, 0)) = \begin{cases} 0, & \text{if } \tau > 1, \\ 1, & \text{if } \tau \leqslant 1, \end{cases} \tag{2}$$

which constitutes a conditional solution to Velani's conjecture at the level of the exponent. Specifically, the appendix of [1] contains empirical data supporting the assertion that

$$D_2(y) + D_3(y) \gg \log y \qquad (y \in \mathbb{N}),$$

where $D_b(y)$ denotes the number of digit changes of y in base b, and (2) was established subject to this hypothesis. We refer the reader to [1, Section 5] for further results of a similar flavour. Theorem 2 is unconditional.

We finish this section by briefly discussing the significance of the exponent $1/\gamma$. By a comparatively simple argument, one can see that if $\tau > 1/\gamma$ and $y \in \mathbb{R}$ then $\mu(W_2(\psi_\tau, y)) = 0$, see the proof of [1, Proposition 1.4]. In [1], we attained the exponent $1/\gamma$ in establishing (1). Thus, as explained in the introduction of that article, dyadic approximation in K behaves very differently to triadic approximation in K, the latter having been thoroughly investigated by Levesley, Salp and Velani [12]. Theorem 2 extends the admissible range for the exponent beyond this threshold.

Notation. For complex-valued functions f and g, we write $f \ll g$ or $f = O(g)$ if $|f| \leqslant C|g|$ pointwise, for some constant $C > 0$.

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2. Preliminaries

During our proof of Theorem 2 we will make use of a number of technical results from [1], [2] and [17]. These are detailed below. To this end, let us first recall the following constructive definition of K: let $K_0 := [0,1]$ and let $K_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ be the set obtained by removing the open middle third from K_0. Next, suppose the set K_{n-1} has been defined. Let K_n be the set obtained upon removing the open middle thirds from all the component intervals of K_{n-1}. With the sets K_n constructed in this way, we have

$$K = \bigcap_{n=0}^{\infty} K_n.$$

Note that for each $n \in \mathbb{N}$, the set K_n consists of 2^n closed intervals, each of length 3^{-n}. Let $\mathcal{C}_N$ denote the set of all (left and right) endpoints of the intervals comprising K_N.

The result we use from [1] estimates the μ-measure of a union of balls by counting nearby triadic rationals in $\mathcal{C}_N$ for a sufficiently large $N \in \mathbb{N}$. For $n \in \mathbb{N}$, $\sigma > 0$ and $y \in \mathbb{R}$, denote

$$A_n^y(\sigma) = \{x \in \mathbb{R} : \|2^n x - y\| < \sigma\}.$$

Lemma 3. ([1, Lemma 2.1], special case) *Let $n, N \in \mathbb{N}$ and $\sigma \in (0,1)$ with $3^{-N} \leqslant \frac{\sigma}{5 \cdot 2^n}$, and let $y \in \mathbb{R}$. Then*

$$2^{-(N+1)} \#(\mathcal{C}_N \cap A_n^y(\sigma/5)) \leqslant \mu(A_n^y(\sigma)) \leqslant 2^{-(N-1)} \#(\mathcal{C}_N \cap A_n^y(5\sigma)).$$

The results we use from [2] and [17] are formulated in terms of the Fourier transform of a measure. Recall that this quantity is defined as follows: given a Borel probability measure ν supported on $[0,1]$, let

$$\hat{\nu}(\xi) = \int e^{-2\pi i \xi x} \, \mathrm{d}\nu(x).$$

Lemma 4. ([2, Lemma 2.2]) *Let $N \in \mathbb{N}$, and let $t \in \mathbb{Z} \setminus \{0\}$. Then there exist constants $C_1, C_2 > 0$ independent of N and t such that*

$$\#\left\{0 \leqslant n < N : |\hat{\mu}(t 2^n)| > C_1 N^{-0.078}\right\} \leqslant C_2 N^{0.922}.$$

Lemma 5. ([17, Theorem 4.1]) *Let ν be a Borel probability measure on $[0,1]$. Let $\delta \in (0,1)$, let $Q \in \mathbb{N}$, and let $y \in [0,1]$. Then*

$$\nu(\{x \in [0,1] : \|Qx - y\| \leqslant \delta\}) \ll \delta \left(1 + \sum_{\substack{0 < |\xi| \leqslant 2Q/\delta \\ Q \mid \xi}} |\hat{\nu}(\xi)|\right).$$

The statement given in [17, Theorem 4.1] also provides a lower bound for $\nu(\{x \in [0,1] : \|Qx - y\| \leqslant \delta\})$ and applies in arbitrary dimensions, but we will only use this simpler statement.

3. Proof of Theorem 2

Set $C > 0$ to be the constant C_1 arising from Lemma 4. Define

$$\beta_1 = 0.078, \qquad \beta_2 = 0.922, \qquad \alpha = \frac{1 - \beta_2}{2 - \gamma}.$$

Observe that the assumption of the theorem can be rewritten as

$$\tau\gamma > \beta_2 + \alpha = 1 - \alpha(1-\gamma), \tag{3}$$

and that $\tau > \alpha$. Let $N \in \mathbb{N}$ be sufficiently large so that $N^{\tau-\alpha} \geqslant 150$. For $n \in [N, 2N] \cap \mathbb{Z}$, put

$$\sigma_n = n^{-\tau}, \qquad \delta_n = n^{-\alpha}.$$

Write G_N for the set of integers $n \in [N, 2N]$ such that

$$\max\{|\hat{\mu}(t2^n)| : t \in \mathbb{Z},\ 1 \leqslant |t| \leqslant 2/\delta_{2N}\} \leqslant CN^{-\beta_1},$$

and let B_N be its complement in $[N, 2N] \cap \mathbb{Z}$. Applying the union bound, and then Lemma 4 with $2N+1$ in place of N, we have

$$\begin{aligned} \#B_N &\leqslant \sum_{1 \leqslant |t| \leqslant 2/\delta_{2N}} \#\{n \in [N, 2N] \cap \mathbb{Z} : |\hat{\mu}(t2^n)| > CN^{-\beta_1}\} \\ &\leqslant \sum_{1 \leqslant |t| \leqslant 2/\delta_{2N}} \#\{n \in [0, 2N+1) \cap \mathbb{Z} : |\hat{\mu}(t2^n)| > C(2N+1)^{-\beta_1}\} \\ &\ll \sum_{1 \leqslant |t| \leqslant 2/\delta_{2N}} (2N+1)^{\beta_2} \\ &\ll N^{\beta_2+\alpha}. \end{aligned}$$

Observe that

$$W_2(\psi_\tau, y) = \limsup_{n\to\infty} A_n^y(\sigma_n).$$

By the first Borel–Cantelli lemma [8, Lemma 1.2], it suffices to prove that

$$\sum_{n=1}^{\infty} \mu(A_n^y(\sigma_n)) < \infty. \tag{4}$$

For $n \in B_N$, we use the following estimate, the proof of which follows straightforwardly from the argument in [1, §2.1].

Lemma 6. *Let $y \in \mathbb{R}$. Then*

$$\mu(A_n^y(\sigma_n)) \ll \sigma_n^\gamma \qquad (n \in \mathbb{N}).$$

In the case that $n \in G_N$, we are able to obtain a stronger estimate by transferring data from the coarse scale δ_n to the fine scale σ_n. By Lemma 5, we have

$$\mu(A_n^y(\delta_n)) \ll \delta_n \left(1 + \sum_{1 \leqslant |t| \leqslant 2/\delta_n} |\hat{\mu}(t2^n)|\right) \qquad (n \in \mathbb{N}).$$

As $\alpha < \beta_1$, we find that if $n \in G_N$, then

$$\mu(A_n^y(\delta_n)) \ll \delta_n. \tag{5}$$

To pass between the two scales δ_n and σ_n, we require an inhomogeneous analogue of [1, Lemma 2.2]. Its statement and proof are based upon the iterative construction of K, which we now briefly recall, see [1, §2] for further details. For $\mathcal{N} \in \mathbb{N}$, recall that the $\mathcal{N}^{\text{th}}$ level in the construction of the Cantor set, which we denote by $K_{\mathcal{N}}$, comprises $2^{\mathcal{N}}$ intervals of length $3^{-\mathcal{N}}$. The left endpoints of these intervals form the set $L_{\mathcal{N}}$ of rationals $a/3^{\mathcal{N}}$ such that $a \in [0, 3^{\mathcal{N}}]$ is an integer whose ternary expansion contains only the digits 0 and 2, and the right endpoints form the set $R_{\mathcal{N}} = \{1 - x : x \in L_{\mathcal{N}}\}$. Note that $\mathcal{C}_{\mathcal{N}} = L_{\mathcal{N}} \cup R_{\mathcal{N}}$. The following is an inhomogeneous analogue of [1, Lemma 2.2].

Lemma 7. *Fix an absolute constant $c > 0$. Let $n, \mathcal{N}, \mathcal{M} \in \mathbb{N}$ and $\sigma, \delta \in \mathbb{R}$ be such that $\mathcal{N} \geqslant \mathcal{M}$ and*

$$0 < \sigma < \delta \leqslant 1, \qquad 3^{-\mathcal{N}} \geqslant \frac{c\sigma}{2^n}, \qquad \frac{\sigma}{2^n} \leqslant 3^{-\mathcal{M}} \leqslant \frac{\delta}{2^n},$$

and let $y \in \mathbb{R}$. Then

$$\#(\mathcal{C}_{\mathcal{N}} \cap A_n^y(\sigma)) \ll \#(\mathcal{C}_{\mathcal{M}} \cap A_n^y(2\delta)).$$

Proof. We imitate the proof of [1, Lemma 2.2]. By symmetry, it suffices to prove that

$$\#(L_{\mathcal{N}} \cap A_n^y(\sigma)) \ll \#(L_{\mathcal{M}} \cap A_n^y(2\delta)). \tag{6}$$

Suppose $x \in L_{\mathcal{N}} \cap A_n^y(\sigma)$. Then $x = a/3^{\mathcal{N}}$ for some integer $a \in [0, 3^{\mathcal{N}})$ whose ternary expansion contains only the digits 0 and 2. Further, there exists an integer $b \in [0, 2^n]$ such that

$$\left| x - \frac{b+y}{2^n} \right| < \frac{\sigma}{2^n}.$$

Therefore $\#(L_{\mathcal{N}} \cap A_n^y(\sigma))$ is bounded above by the number of integer solutions (a, b) to the inequality

$$\left| \frac{a}{3^{\mathcal{N}}} - \frac{b+y}{2^n} \right| < \frac{\sigma}{2^n}$$

such that $a \in [0, 3^{\mathcal{N}})$, $b \in [0, 2^n]$, and each ternary digit of a is 0 or 2.

We write

$$a = 3^{\mathcal{N}-\mathcal{M}} a_1 + a_2, \qquad a_1, a_2 \in \mathbb{Z}, \qquad 0 \leqslant a_1 < 3^{\mathcal{M}}, \qquad 0 \leqslant a_2 < 3^{\mathcal{N}-\mathcal{M}}.$$

This reveals that $\#(L_{\mathcal{N}} \cap A_n^y(\sigma))$ is bounded above by the number of integer solutions (a_1, a_2, b) to

$$\left| \frac{3^{\mathcal{N}-\mathcal{M}} a_1 + a_2}{3^{\mathcal{N}}} - \frac{b+y}{2^n} \right| < \frac{\sigma}{2^n} \tag{7}$$

such that

$$0 \leqslant a_1 < 3^{\mathcal{M}}, \qquad 0 \leqslant a_2 < 3^{\mathcal{N}-\mathcal{M}}, \qquad 0 \leqslant b \leqslant 2^n,$$

and the ternary digits of a_1, a_2 are all 0 or 2. As

$$\left| \frac{a_1}{3^{\mathcal{M}}} - \frac{b+y}{2^n} \right| \leqslant \left| \frac{a_1}{3^{\mathcal{M}}} + \frac{a_2}{3^{\mathcal{N}}} - \frac{b+y}{2^n} \right| + \frac{a_2}{3^{\mathcal{N}}} < \frac{\sigma}{2^n} + \frac{1}{3^{\mathcal{M}}} \leqslant \frac{2}{3^{\mathcal{M}}}, \tag{8}$$

we must have $a_1/3^{\mathcal{M}} \in A_n^y(2\delta)$ for any such solution.

Given a_1, the inequality (8) forces $b/2^n$ to lie in an interval of length $4/3^{\mathcal{M}}$, and so there are at most $O(1)$ possibilities for b. Next, suppose we are given a_1 and b. Then, by (7), the integer a_2 is forced to lie in the interval of length $3^{\mathcal{N}}\sigma 2^{1-n}$ centred at $3^{\mathcal{N}}(b+y)2^{-n} - 3^{\mathcal{N}-\mathcal{M}} a_1$. Consequently, as $3^{-\mathcal{N}} \geqslant c\sigma/2^n$, there are at most $O(1)$ solutions a_2 to (7). Finally, since $a_1/3^{\mathcal{M}} \in L_{\mathcal{M}} \cap A_n^y(2\delta)$, we conclude that there are $O(\#(L_{\mathcal{M}} \cap A_n^y(2\delta)))$ solutions in total. This confirms (6) and completes the proof of the lemma. □

Let $n \in [N, 2N] \cap \mathbb{Z}$. Let $\mathcal{N}, \mathcal{M}$ be positive integers such that

$$\frac{\sigma_n}{15 \cdot 2^n} < 3^{-\mathcal{N}} \leqslant \frac{\sigma_n}{5 \cdot 2^n} \quad \text{and} \quad \frac{\delta_n}{30 \cdot 2^n} < 3^{-\mathcal{M}} \leqslant \frac{\delta_n}{10 \cdot 2^n}.$$

We apply Lemma 3 with $\sigma = \sigma_n$ and $\mathcal{N}$ in place of N therein, giving

$$\mu(A_n^y(\sigma_n)) \ll 2^{-\mathcal{N}} \#(\mathcal{C}_{\mathcal{N}} \cap A_n^y(5\sigma_n)).$$

As $\delta_n/\sigma_n = n^{\tau-\alpha} \geqslant N^{\tau-\alpha} \geqslant 150$, we may apply Lemma 7 with $\sigma = 5\sigma_n$ and $\delta = \delta_n/10$, giving

$$\#(\mathcal{C}_{\mathcal{N}} \cap A_n^y(5\sigma_n)) \ll \#(\mathcal{C}_{\mathcal{M}} \cap A_n^y(\delta_n/5)).$$

Next we apply Lemma 3 again, now with $\sigma = \delta_n$ and $\mathcal{M}$ in place of N therein, giving

$$\#(\mathcal{C}_{\mathcal{M}} \cap A_n^y(\delta_n/5)) \ll 2^{\mathcal{M}} \mu(A_n^y(\delta_n)).$$

Note that we have

$$2^{-\mathcal{N}} \ll (\sigma_n/2^n)^\gamma \quad \text{and} \quad 2^{-\mathcal{M}} \gg (\delta_n/2^n)^\gamma,$$

and that, combined with the above, these inequalities furnish

$$\mu(A_n^y(\sigma_n)) \ll \frac{(\sigma_n/2^n)^\gamma}{(\delta_n/2^n)^\gamma} \mu(A_n^y(\delta_n)).$$

Thus, by (5), for $n \in G_N$ we have

$$\mu(A_n^y(\sigma_n)) \ll \delta_n^{1-\gamma} \sigma_n^\gamma.$$

Hence, by Lemma 6 and our earlier observation that $\# B_N \ll N^{\beta_2+\alpha}$, we have

$$\sum_{n=N}^{2N} \mu(A_n^y(\sigma_n)) \ll \sum_{n=N}^{2N} \delta_n^{1-\gamma}\sigma_n^\gamma + \sum_{n \in B_N} \sigma_n^\gamma$$
$$\ll \sum_{n=N}^{2N} \frac{1}{n^{\tau\gamma+\alpha(1-\gamma)}} + N^{\beta_2+\alpha-\tau\gamma}.$$

In view of (3), and noting that we can write

$$\sum_{n=1}^{\infty} \mu(A_n^y(\sigma_n)) \leqslant \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}} \mu(A_n^y(\sigma_n))$$
$$\ll \sum_{k=0}^{\infty} \left(\sum_{n=2^k}^{2^{k+1}} \frac{1}{n^{\tau\gamma+\alpha(1-\gamma)}} + 2^{k(\beta_2+\alpha-\tau\gamma)} \right),$$

we finally have (4), which completes the proof of Theorem 2.

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