COMBINATORIAL PROPERTIES AND DEPENDENT CHOICE IN SYMMETRIC EXTENSIONS BASED ON LÉVY COLLAPSE

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Abstract. We work with symmetric extensions based on Lévy Collapse and extend a few results of Arthur Apter. We prove a conjecture of Ioanna Dimitriou from her P.h.d. thesis. We also observe that if V is a model of ZFC, then $DC_{<\kappa}$ can be preserved in the symmetric extension of V in terms of symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, if $\mathbb{P}$ is κ-distributive and $\mathcal{F}$ is κ-complete. Further we observe that if V is a model of ZF + DC_κ, then $DC_{<\kappa}$ can be preserved in the symmetric extension of V in terms of symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, if $\mathbb{P}$ is κ-strategically closed and $\mathcal{F}$ is κ-complete.

1. introduction

Serge Grigorieff proved in [Gri75] that symmetric extensions in terms of *symmetric system* $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$[1] are intermediate models of the form $HOD(V[a])^{V[G]}$ as a varies over $V[G]$. Arthur Apter constructed several symmetric inner models based on Lévy Collapse in terms of hereditarily definable sets. The purpose of this note is to translate the arguments of a few of those symmetric inner model constructions to symmetric extensions in terms of symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ and extend a few published results. In particular, we prove the following.

- We observe an *infinitary Change conjecture* in the model constructed in **Theorem 11** of [AK06]. Moreover, we prove that $\aleph_{\omega_1}$ is an almost Ramsey cardinal in the model.
- We reduce the large cardinal assumption of **Theorem 2** and **Theorem 3** of [AC13].
- We prove the failure of AC_κ in the symmetric extension of **subsection 4.1** of [Kar19]. Moreover, we study a different argument to preserve the supercompactness of κ in the symmetric model.
- We observe the *mutually stationarity property* of a sequence of stationary sets in the symmetric model constructed in [Apt83a]. We also observe an alternating sequence of measurable and non-measurable cardinals in the model. Moreover, we observe that $\aleph_\omega$ is an almost Ramsey cardinal in the model.

Secondly, we prove a conjecture of Dimitriou related to the failure of Dependent choice–or DC–in a symmetric extension based on finite support products of collapsing functions, from [Dim11]. We also study new lemmas related to preserving DC in symmetric extensions inspired by **Lemma 1** of [Kar14]. In particular, we observe the following.

- Let V be a model of ZFC. If $\mathbb{P}$ is κ-distributive and $\mathcal{F}$ is κ-complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.
- Let V be a model of ZF + DC_κ where AC can consistently fail. If $\mathbb{P}$ is κ-strategically closed and $\mathcal{F}$ is κ-complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.

1.1. **Preserving Dependent choice.** Woodin asked in the context of ZFC, that if κ is strongly compact and GCH holds below κ, then must GCH hold everywhere? The problem is still open in the context of ZFC. One variant of this question is if GCH can fail at every limit cardinal less than or equal to a strongly compact cardinal κ where as GCH holds above κ^+. Apter answered

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[1] $\mathbb{P}$ is a forcing notion, $\mathcal{G}$ an automorphism group of $\mathbb{P}$, and $\mathcal{F}$ is a normal filter of subgroups over $\mathcal{G}$.

this in the context of ZF. In **Theorem 3** of [Apt12], Apter constructed a symmetric inner model based on Lévy collapse where κ is a regular limit cardinal and a supercompact cardinal, and GCH holds for a limit δ if and only if $\delta > \kappa$. In that model the Countable choice–or AC_ω– fails. At the end of [Apt12], Apter asked the following question.

Question 1.1. *Is it possible to construct analogous of Theorem 3 in which some weak version of AC holds ?*

In **Lemma 1** of [Kar14], Asaf Karagila proved that if $\mathbb{P}$ is κ-closed and $\mathcal{F}$ is κ-complete then $DC_{<\kappa}$ is preserved in the symmetric extension in terms of symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$. The author and Karagila both observe that "$\mathbb{P}$ is κ-closed" can be replaced by "$\mathbb{P}$ has κ-c.c." in **Lemma 1** of [Kar14].[2] We note that the natural assumption that $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a *tenacious system*[3] is required in the proof as written by Karagila in **Lemma 3.3** of [Kar19].

Observation 1.2. ***(Lemma 3.3 of*** [Kar19]***).*** *Let V be a model of ZFC. If $\mathbb{P}$ has κ-c.c. and $\mathcal{F}$ is κ-complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.*

Applying **Observation 1.2**, we construct a symmetric extension to answer **Question 1.1**.[4]

Theorem 1.3. *Let V be a model of ZFC + GCH with a supercompact cardinal κ. Then there is a symmetric extension with respect to a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ where $DC_{<\kappa}$ holds and AC_κ fails. Moreover, κ is a regular limit cardinal and a supercompact cardinal, and GCH holds for a limit δ if and only if $\delta > \kappa$.*

We observe that '$\mathbb{P}$ is κ-closed' can be replaced by '$\mathbb{P}$ is κ-distributive' in **Lemma 1** of [Kar14]. This slightly generalize **Lemma 1** of [Kar14], since there are κ-strategically closed forcing notions which are not κ-closed[5] and κ-distributivity is weaker than $< \kappa$-strategic closure.[6]

Observation 1.4. ***(Lemma 3.2).*** *Let V be a model of ZFC. If $\mathbb{P}$ is κ-distributive and $\mathcal{F}$ is κ-complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.*

We also observe that even if we start with a model V, which is a model of ZF + DC_κ where AC can consistently fail, we can still preserve $DC_{<\kappa}$ in a symmetric extension of V in certain cases. In particular, we observe the following.

Observation 1.5. ***(Lemma 3.3).*** *Let V be a model of $ZF + DC_\kappa$. If $\mathbb{P}$ is κ-strategically closed and $\mathcal{F}$ is κ-complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.*

1.2. **Proving Dimitriou's conjecture.** In **section 1.4** of [Dim11], Ioanna Dimitriou constructed a symmetric extension based on finite support products of collapsing functions. At the end of the section, Dimitriou conjectured that DC would fail in the symmetric extension (see **Question 1** of **Chapter 4** in [Dim11]). We prove the conjecture. In particular, we prove that AC_ω fails in the symmetric extension. For the sake of convenience, we call this model as Dimitriou's model and prove the following in **section 5**.

Theorem 1.6. *In Dimitriou's model AC_ω fails.*

[2] The author noticed this observation combining the role of κ-c.c. forcing notions from **Lemma 2.2** of [Apt01] and the role of κ-completeness of $\mathcal{F}$ from **Lemma 1** of [Kar14].

[3] Definition 4.6 of [Kar19a].

[4] The author would like to thank Asaf Karagila for helping to translate the arguments of Arthur Apter from **Theorem 1** of [Apt01] in terms of symmetric extension by a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$. We construct a similar symmetric extension to prove **Theorem 1.3**.

[5] As for an example, the forcing notion $\mathbb{P}(\kappa)$ which adds a non-reflecting stationary set of cofinality ω ordinals in κ is κ-strategically closed but not even ω_2-closed.

[6] As for an example, the forcing notion for *'killing a stationary subset of ω_1'* is ω_1-distributive, but not even $< \omega_1$-strategically closed.

1.3. **Reducing the assumption of supercompactness by strong compactness.** In **Theorem 2** of [AC13], Apter and Cody obtained a symmetric extension where $\aleph_1$ and $\aleph_2$ are both singular of cofinality ω, and there is a sequence of distinct subsets of $\aleph_1$ of length equal to any predefined ordinal, assuming a supercompact cardinal κ. In **section 6**, we observe that applying a recent result of Toshimichi Usuba, which is **Theorem 3.1** of [ADU19], followed by working on symmetric extensions based on *strongly compact Prikry forcing*[7], it is possible to reduce the assumption of a supercompact cardinal κ to a strongly compact cardinal κ.

Observation 1.7. *Suppose κ is a strongly compact cardinal, GCH holds, θ is an ordinal in a ground model V of ZFC. Then there is a symmetric inner model $\mathcal{M}$ in which $cf(\aleph_1) = cf(\aleph_2) = \omega$, and there is a sequence of distinct subsets of $\aleph_1$ of length θ. Consequently, AC_ω fails in $\mathcal{M}$.*

Similarly, we reduce the large cardinal assumption of **Theorem 3** of [AC13] from a supercompact cardinal to a strongly compact cardinal. In **Theorem 3** of [AC13], Apter and Cody obtained a symmetric extension where $\aleph_\omega$ and $\aleph_{\omega+1}$ are both singular with $\omega \leq cf(\aleph_{\omega+1}) < \aleph_\omega$, and there is a sequence of distinct subsets of $\aleph_\omega$ of length equal to any predefined ordinal, assuming a supercompact cardinal κ. We prove the following.

Observation 1.8. *Suppose κ is a strongly compact cardinal, GCH holds, θ is an ordinal in a ground model V of ZFC. Then there is a symmetric inner model $\mathcal{M}$ in which $\aleph_\omega$ and $\aleph_{\omega+1}$ are both singular with $\omega \leq cf(\aleph_{\omega+1}) < \aleph_\omega$, and there is a sequence of distinct subsets of $\aleph_\omega$ of length θ. Consequently, AC_ω fails in $\mathcal{M}$.*

1.4. **Infinitary Chang conjecture from a measurable cardinal.** Assuming a measurable cardinal, Apter and Koepke constructed a symmetric model $\mathcal{N}$ based on Lévy collapse in **Theorem 11** of [AK06]. In $\mathcal{N}$, ω_1 is singular, and $\aleph_{\omega_1}$ is a Rowbottom cardinal carrying a Rowbottom filter. They mentioned that in $\mathcal{N}$, AC_ω fails because of the singularity of ω_1. We first observe an *infinitary Chang conjecture* in a symmetric extension, which is very similar to $\mathcal{N}$, except we consider a finite support product construction. We use the observation that it is possible to force a *coherent* sequence of Ramsey cardinals after performing Prikry forcing on a normal measure over a measurable cardinal κ (see **Theorem 3**, [AK06]). We also use the observation that an infinitary Chang conjecture can be established in a symmetric model, assuming a coherent sequence of Ramsey cardinals. In the symmetric model, AC_ω fails because of the singularity of ω_1.

Theorem 1.9. *Let V be a model of ZFC where there is a measurable cardinal. Then there is a symmetric extension with respect to a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ where ω_1 is singular and thus AC_ω fails. Moreover in the symmetric extension, an infinitary Chang conjecture holds.*

Similarly, we also observe an infinitary Chang conjecture in the symmetric inner model $\mathcal{N}$. For the sake of convenience, we call the model $\mathcal{N}$ as Apter and Koepke's model and prove the following.

Theorem 1.10. *An infinite Chang conjecture holds in Apter and Koepke's model. Moreover, $\aleph_{\omega_1}$ is an almost Ramsey cardinal in the model.*

1.5. **Mutually stationary property from a sequence of measurable cardinals.** Foreman and Magidor asked whether it is consistent that $\langle S_n : n < \omega \rangle$ such that each S_n is stationary on $\aleph_n$ is mutually stationary in ZFC. In [Apt04], Apter answered the question in ZF assuming the consistency of ω-sequence of supercompact cardinals. In **Theorem 1** of [Apt83a], Apter further obtained a similar symmetric inner model based on Lévy collapse as constructed in [Apt04], where $\aleph_\omega$ carries a Rowbottom filter and $DC_{\aleph_{n_0}}$ holds for any arbitrary $n_0 \in \omega$, from a ω-sequence of measurable cardinals. We observe that in the symmetric model from **Theorem 1** of [Apt83a], if $\langle S_k : 1 \leq k < \omega \rangle$ is a sequence of stationary sets such that $S_k \subseteq \aleph_{n_0+2(k+1)}$ for every $1 \leq k < \omega$, then $\langle S_k : 1 \leq k < \omega \rangle$ is mutually stationary. We also observe that in the symmetric model each

[7] An exhibition of symmetric extension on strongly compact Prikry forcing can be found in [AH91].

$\aleph_{n_0+2(k+1)}$ is a measurable cardinal. For the sake of convenience, we fix an arbitrary $n_0 \in \omega$ in the ground model V, call the symmetric model from [Apt83a] as $\mathcal{N}_{n_0}$, and prove the following.

Observation 1.11. *The following holds in the symmetric inner model $\mathcal{N}_{n_0}$.*

(1) *For each $1 \leq k < \omega$, $\aleph_{n_0+2(k+1)}$ is a measurable cardinal and $\aleph_{n_0+2k}$ is not a measurable cardinal. In particular, for each $1 \leq k < \omega$, there are no unifrom ultrafilters on $\aleph_{n_0+2k}$.*
(2) *If $\langle S_k : 1 \leq k < \omega \rangle$ is a sequence of stationary sets such that $S_k \subseteq \aleph_{n_0+2(k+1)}$ for every $1 \leq k < \omega$, then $\langle S_k : 1 \leq k < \omega \rangle$ is mutually stationary.*
(3) *$\aleph_\omega$ is an almost Ramsey cardinal in the model.*

Structure of the paper.

- In section 2, we cover the basics.
- In section 3, we prove **Observation 1.4**, **Observation 1.5** and study a few lemmas related to preserving Dependent choice in symmetric extensions inspired by **Lemma 1** of [Kar14].
- In section 4, we prove **Theorem 1.3** applying **Observation 1.2**. In particular, we prove the failure of AC_κ in the symmetric extension of **subsection 4.1** of [Kar19]. Moreover, we study a different argument to preserve the supercompactness of κ in the symmetric model.
- In section 5, we prove **Theorem 1.6**. This answers the question of Dimitriou from her thesis.
- In section 6, we prove **Observation 1.7** and **Observation 1.8**. Consequently, we reduce the large cardinal assumption of **Theorem 2** and **Theorem 3** of [AC13], from a supercompact cardinal to a strongly compact cardinal.
- In section 7, we prove **Theorem 1.9** and **Theorem 1.10** and study an infinite Chang conjecture in Apter and Koepke's model from **Theorem 11** of [AK06].
- In section 8, we prove **Observation 1.11** and study the mutually stationary property of a sequence of stationary sets in Apter's model from **Theorem 1** of [Apt83a].

2. Basics

2.1. **Large Cardinals.** In this section, we recall the definition of inaccessible cardinals in the context of ZFC and other large cardinals in the context of ZF. In ZFC, we say κ is a strongly inaccessible cardinal if it is a regular strong limit cardinal where the definition of "strong limit" is that for all $\alpha < \kappa$, we have $2^\alpha < \kappa$. In the context of ZF, the above definition doesn't make sense, as 2^α may not be well-ordered. We refer the reader to [BDL07] for details concerning inaccessible cardinals in the context of ZF. We recall the other necessary large cardinal definitions in the context of ZF from *'The Higher Infinite'* [Kan03] of Akihiro Kanamori.

Definition 2.1. Given an uncountable cardinal κ, we recall the following definitions.

(1) κ is weakly compact if for all $f : [\kappa]^2 \to 2$, there is a homogeneous set $X \subseteq \kappa$ for f of order type κ.
(2) κ is Ramsey if for all $f : [\kappa]^{<\omega} \to 2$, there is a homogeneous set $X \subseteq \kappa$ for f of order type κ.
(3) κ is almost Ramsey if for all $\alpha < \kappa$ and $f : [\kappa]^{<\omega} \to 2$, there is a homogeneous set $X \subseteq \kappa$ for f having order type α.
(4) κ is μ-Rowbottom if for all $\alpha < \kappa$ and $f : [\kappa]^{<\omega} \to \alpha$, there is a homogeneous set $X \subseteq \kappa$ for f of order type κ such that $|f''[X]^{<\omega}| < \mu$. κ is Rowbottom if it is ω_1-Rowbottom. A filter $\mathcal{F}$ on κ is a Rowbottom filter on κ if for any $f : [\kappa]^{<\omega} \to \lambda$, where $\lambda < \kappa$ there is a set $X \in \mathcal{F}$ such that $|f''[X]^{<\omega}| \leq \omega$.
(5) κ is measurable if there is a κ-complete free ultrafilter on κ. A filter $\mathcal{F}$ on a cardinal κ is normal if it is closed under diagonal intersections:

$$\text{If } X_\alpha \in \mathcal{F} \text{ for all } \alpha < \kappa, \text{ then } \Delta_{\alpha<\kappa} X_\alpha \in \mathcal{F}.$$

In ZF we have the following lemma.

Lemma 2.2. *(Lemma 0.8 of* [Dim11]*). An ultrafilter $\mathcal{U}$ over κ is normal if and only if for every regressive $f : \kappa \to \kappa$ there is an $X \in \mathcal{U}$ such that f is constant on X.*

Thus, we say an ultrafilter $\mathcal{U}$ over κ is normal if for every regressive $f : \kappa \to \kappa$ there is an $X \in \mathcal{U}$ such that f is constant on X.

(6) For a set A we say $\mathcal{U}$ a fine measure on $\mathcal{P}_\kappa(A)$ if $\mathcal{U}$ is a κ-complete ultrafilter and for any $i \in A$, $\{x \in \mathcal{P}_\kappa(A) : i \in x\} \in \mathcal{U}$. We say $\mathcal{U}$ is a normal measure on $\mathcal{P}_\kappa(A)$, if $\mathcal{U}$ is a fine measure and if $f : \mathcal{P}_\kappa(A) \to A$ is such that $f(X) \in X$ for a set in $\mathcal{U}$, then f is constant on a set in $\mathcal{U}$. κ is λ-strongly compact if there is a fine measure on $\mathcal{P}_\kappa(\lambda)$, it is strongly compact if it is λ-strongly compact for all $\kappa \leq \lambda$.

(7) κ is λ-supercompact if there is a normal measure on $\mathcal{P}_\kappa(\lambda)$, it is supercompact if it is λ-supercompact for all $\kappa \leq \lambda$.

Remark 1. We note that the definition of supercompact (similarly strongly compact) is meant in the terms of ultrafilters, which is weaker than the definition of supercompact in terms of elementary embedding due to Woodin [**Definition 220,** [Wood10]] (e.g. $\aleph_1$ can be supercompact or strongly compact if we consider the definition of supercompact or strongly compact in terms of ultrafilters [Ina13], but $\aleph_1$ can not be the critical point of an elementary embedding).

Remark 2. In **section 2** of [IT19], Ikegami and Trang defined that an ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa X$ is normal if for any set $A \in \mathcal{U}$ and $f : A \to \mathcal{P}_\kappa X$ with $\emptyset \neq f(\sigma) \subseteq \sigma$ for all $\sigma \in A$, there is an $x_0 \in X$ such that for $\mathcal{U}$-measure one many σ in A, $x_0 \in f(\sigma)$. They note that their definition of normality is equivalent to the closure under diagonal intersections in ZF, while it may not be equivalent to the definition of normality in our sense without AC.

From now on, all our inaccessible cardinals are strongly inaccessible. We recall that a limit of Ramsey cardinals is an almost Ramsey cardinal in ZF (**Proposition 1** of [AK08]).

2.2. **Lévy–Solovay Theorem.** We state a part of Lévy–Solovay Theorem (**Theorem 21.2** of [Jec03]) in ZFC. By a *small forcing extension with respect to* κ we mean a forcing extension $V[G]$ obtained from V after forcing with a partially ordered set of size less than κ.

Theorem 2.3. *Let κ be an infinite cardinal, and let $\mathbb{P}$ be a partially ordered set of size less than κ. Let G be a $\mathbb{P}$-generic filter over V.*

- *If κ is Ramsey in V, then κ is Ramsey in $V[G]$.*
- *If κ is measurable with a κ-complete ultrafilter $\mathcal{U}$ in V then κ is measurable with a κ-complete ultrafilter $\mathcal{U}_1 = \{X \subseteq \kappa : X \in V[G], \exists Y \in \mathcal{U}[Y \subseteq X]\}$ defined in $V[G]$ generated by $\mathcal{U}$ in $V[G]$.*

Proof. Proof of preserving *Ramseyness* follows from **Theorem 21.2** of [Jec03] and proof of preserving *measurability* and the fact that κ-complete ultrafilters in the ground model generate κ-complete ultrafilters in the small forcing extensions with respect to κ follows from the **Lévy–Solovay Theorem** in [LS67]. □

2.3. **Symmetric extension.** Symmetric extensions are submodels of the generic extension containing the ground model, where the axiom of choice can consistently fail. Let $\mathbb{P}$ be a forcing notion, $\mathcal{G}$ be a group of automorphisms of $\mathbb{P}$ and $\mathcal{F}$ be a normal filter of subgroups over $\mathcal{G}$. We recall the following Symmetry Lemma from [Jec03].

Theorem 2.4. ***(Symmetry Lemma, Lemma 14.37 of*** [Jec03]***).** *Let $\mathbb{P}$ be a forcing notion, φ be a formula of the forcing language with n variables and let $\sigma_1, \sigma_2, ...\sigma_n \in V^{\mathbb{P}}$ be $\mathbb{P}$-names. If $a \in Aut(\mathbb{P})$, then $p \Vdash \varphi(\sigma_1, \sigma_2, ...\sigma_n) \Leftrightarrow a(p) \Vdash \varphi(a(\sigma_1), a(\sigma_2), ...a(\sigma_n))$.*

For $\tau \in V^{\mathbb{P}}$, we denote the symmetric group with respect to $\mathcal{G}$ by $sym^{\mathcal{G}}\tau = \{g \in \mathcal{G} : g\tau = \tau\}$ and say τ is symmetric with respect to $\mathcal{F}$ if $sym^{\mathcal{G}}\tau \in \mathcal{F}$. Let $HS^{\mathcal{F}}$ be the class of all hereditary symmetric names. That is, recursively for $\tau \in V^{\mathbb{P}}$,

$\tau \in HS^{\mathcal{F}}$ iff τ is symmetric with respect to $\mathcal{F}$, and for each $\sigma \in dom(\tau)$, $\sigma \in HS^{\mathcal{F}}$.

We define symmetric extension of V with respect to $\mathcal{F}$ as $V(G)^{\mathcal{F}} = \{\tau^G : \tau \in HS^{\mathcal{F}}\}$. For the sake of our convenience we omit the subscript $\mathcal{F}$ sometimes and call $V(G)^{\mathcal{F}}$ as $V(G)$.

Definition 2.5. ***(Symmetric System, Definition 2.1 of*** [KH19]***).*** *We say $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system if $\mathbb{P}$ is a forcing notion, $\mathcal{G}$ the automorphism group of $\mathbb{P}$ and $\mathcal{F}$ a normal filter of subgroups over $\mathcal{G}$.*

Definition 2.6. ***($\mathcal{F}$-Tenacious system, Definition 4.6 of*** [Kar19a]***).*** *Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system. A condition $p \in \mathbb{P}$ is $\mathcal{F}$-tenacious if $\{\pi \in \mathcal{G} : \pi(p) = p\} \in \mathcal{F}$. We say $\mathbb{P}$ is $\mathcal{F}$-tenacious if there is a dense subset of $\mathcal{F}$-tenacious conditions. We say $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a tenacious system if $\mathbb{P}$ is $\mathcal{F}$-tenacious.*

Karagila and Hayut proved in Appendix A of [Kar19a] that every symmetric system is equivalent to a tenacious system. Thus, it is natural to assume tenacity and work with tenacious system. We recall the following theorem which states that the symmetric extension $V(G)$ is a transitive model of ZF.

Theorem 2.7. ***(Lemma 15.51 of*** [Jec03]***).*** *If $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system and G is a V-generic filter, then $V(G)$ is a transitive model of ZF and $V \subseteq V(G) \subseteq V[G]$.*

2.4. **Terminologies from Dimitriou's thesis.** We recall the terminologies like *Approximation Lemma*, *Approximation property* and *($\mathcal{G}, \mathcal{I}$)-homogeneous forcing notion*, from [Dim11]. For $E \subseteq \mathbb{P}$, let us define the pointwise stabilizer group to be $\text{fix}_{\mathcal{G}} E = \{g \in \mathcal{G} : \forall p \in E, g(p) = p\}$ i.e. it is the set of automorphisms which fix E pointwise. We denote $\text{fix}_{\mathcal{G}} E$ by fix E for the sake of convenience. A subset $\mathcal{I} \subseteq \mathcal{P}(\mathbb{P})$ is called *$\mathcal{G}$-symmetry generator* if it is closed under unions and if for all $g \in \mathcal{G}$ and $E \in \mathcal{I}$, there is an $E' \in \mathcal{I}$ s.t. $g(\text{fix}E)g^{-1} \supseteq \text{fix}E'$. It is possible to see that if $\mathcal{I}$ is a $\mathcal{G}$-symmetry generator, then the set $\{\text{fix}E : E \in \mathcal{I}\}$ generates a normal filter over $\mathcal{G}$ (Proposition 1.23 of Chapter 1 in [Dim11]). Let $\mathcal{I}$ be the $\mathcal{G}$-symmetry generator generating a normal filter $\mathcal{F}$ over $\mathcal{G}$, we say $E \in \mathcal{I}$ supports a name $\sigma \in HS$ if $\text{fix}E \subseteq sym(\sigma)$. Since $\mathbb{P}, \mathcal{G}$ and $\mathcal{I}$ are enough to define a symmetric extension, we define a symmetric triple $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ and work with it.

Definition 2.8. ***(Symmetric Triple*** $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$***).*** *We say $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ is a symmetric triple if $\mathbb{P}$ is a forcing notion, $\mathcal{G}$ an automorphism group and $\mathcal{I}$ a $\mathcal{G}$-symmetry generator.*

Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ be a symmetric triple, then $\mathcal{I}$ is projectable for the pair $(\mathbb{P}, \mathcal{G})$ if for every $p \in \mathbb{P}$ and every $E \in \mathcal{I}$, there is a $p^* \in E$ that is minimal in the partial order and unique such that $p^* \geq p$. We call $p \restriction E = p^*$ the projection of p to E. We say that $\mathbb{P}$ is $(\mathcal{G}, \mathcal{I})$-homogeneous if for every $E \in \mathcal{I}$, every $p \in \mathbb{P}$ and every $q \leq p \restriction E$ there is an automorphism $a \in \text{fix}E$ s.t. $a(p) \parallel q$. $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ has the approximation property if for all formula φ with n free variables, names $\sigma_1, \sigma_2, ...\sigma_n \in HS$ all with support $E \in \mathcal{I}$ and for every $p \in \mathbb{P}$, $p \Vdash \varphi(\sigma_1, \sigma_2, ...\sigma_n)$ implies that $p \restriction E \Vdash \varphi(\sigma_1, \sigma_2, ...\sigma_n)$.

Lemma 2.9. ***(Lemma 1.27 of*** [Dim11]***).*** *Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ be a symmetric triple. If $\mathbb{P}$ is $(\mathcal{G}, \mathcal{I})$-homogeneous, then $(\mathbb{P}, \mathcal{G}, \mathcal{I})$ has the approximation property.*

Lemma 2.10. ***(Approximation Lemma, Lemma 1.29 of*** [Dim11]***).*** *Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ be a symmetric triple. If $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ has the approximation property then for all set of ordinals $X \in V(G)$, there exists an $E \in \mathcal{I}$ and an E name for X. Thus, $X \in V[G \cap E]$.*

2.5. **Homogeneity of forcing notions.** We recall the definition of *weakly homogeneous* and *cone homogeneous* forcing notions from [DF08].

Definition 2.11. ***(Definition 2 of*** [DF08]***).*** *Let $\mathbb{P}$ be a set forcing notion.*

- *We say $\mathbb{P}$ is weakly homogeneous if for any $p, q \in \mathbb{P}$, there is an automorphism $a : \mathbb{P} \to \mathbb{P}$ such that $a(p)$ and q are compatible.*[8]

[8] The Levy collapse $Col(\lambda, < \kappa)$ is weakly homogeneous, given an infinite cardinal κ and a regular cardinal λ.

- *For $p \in \mathbb{P}$, let $Cone(p)$ denote $\{r \in \mathbb{P} : r \leq p\}$. We say $\mathbb{P}$ is cone homogeneous if and only if for any $p, q \in \mathbb{P}$, there exist $p' \leq p$, $q' \leq q$, and an isomorphism $\pi : Cone(p') \to Cone(q')$.*

Following **Fact 1** of [DF08], if $\mathbb{P}$ is a weakly homogeneous forcing notion, then it is cone homogeneous too. Also, the finite support products of weakly (cone) homogeneous forcing notions are weakly (cone) homogeneous. A crucial feature of symmetric extensions using weakly (cone) homogeneous forcings are that they can be approximated by certain intermediate submodel where AC holds.

2.6. **Failure of a weaker form of the axiom of choice.** A weaker version of the axiom of choice is AC_κ for a cardinal κ. We use AC_κ to denote the statement "Every family of κ non-empty sets admits a choice function". We note that if κ^+ is singular, then AC_κ fails. This is due to the following well known fact.

Fact 2.12. $AC_\kappa \implies cf(\lambda) > \kappa$ for all successor cardinal λ.

We sketch another way of refuting AC_κ. One of the weaker forms of AC is $AC_A(B)$ which states that for each set X of non-empty subsets of B, if there is an injection from X to A then there is a choice function for X. We recall **Lemma 0.2**, **Lemma 0.3** and **Lemma 0.12** from [Dim11]. Under $AC_A(B)$, if there is a surjection from B to A, then there is an injection from A to B. We recall that in ZF if κ is measurable with a normal measure or weakly compact and $\alpha < \kappa$ then there is no injection $f : \kappa \to \mathcal{P}(\alpha)$ (This is **Proposition 0.1** of [Bul78]) and in ZF for every infinite cardinal κ, there is a surjection from $\mathcal{P}(\kappa)$ onto κ^+. The following lemma states that if a successor cardinal κ is either measurable with normal measure or weakly compact then AC_κ fails, which is **Corollary 0.3** from [Bul78].

Lemma 2.13. *Let $\kappa = \alpha^+$ be a successor cardinal. If κ is measurable with normal measure or weakly compact then $AC_{\alpha^+}(\mathcal{P}(\alpha))$ fails.*

Proof. Let $AC_{\alpha^+}(\mathcal{P}(\alpha))$ holds. We show $\kappa = \alpha^+$ is neither measurable with normal measure nor weakly compact. In ZF, there is a surjection from $\mathcal{P}(\alpha)$ onto α^+. Now $AC_{\alpha^+}(\mathcal{P}(\alpha))$ implies there is an injection $f^{'}$ from α^+ to $\mathcal{P}(\alpha)$ which states that $\kappa = \alpha^+$ is neither measurable with normal measure nor weakly compact. □

3. Preserving Dependent choice in symmetric extensions

Dependent Choice, denoted by DC or DC_ω, is a weaker version of the Axiom of choice (AC) which is strictly stronger[9] than the countable choice, denoted by AC_ω. This principle is strong enough to give the basis of analysis as it is equivalent to the Baire Category Theorem which is a fundamental theorem in functional analysis. Further, DC is equivalent to other important theorems like the countable version of the Downward Löweinheim–Skolem theorem and every tree of height ω without a maximum node has an infinite branch etc. On the other hand, AC has several controversial applications like the existence of a non-Lebesgue measurable set of real numbers, Banach–Tarski Paradox and the existence of a well-ordering of real numbers whereas DC does not have such counter-intuitive consequences. Thus it is desirable to preserve dependent choice in symmetric extensions.

We denote the principle of Dependent Choice for κ by DC_κ for a cardinal κ. This principle states that for every non-empty set X, if R is a binary relation such that for each ordinal $\alpha < \kappa$, and each $f : \alpha \to X$ there is some $y \in X$ such that $f \ R \ y$, then there is $f : \kappa \to X$ such that for each $\alpha < \kappa$, $f \restriction \alpha \ R \ f(\alpha)$. We denote the assertion $(\forall \lambda < \kappa) DC_\lambda$ by $DC_{<\kappa}$. The axiom of choice is equivalent to $\forall \kappa (DC_\kappa)$ and DC_κ implies AC_κ.

[9]In Howard–Rubin's first model (N38 in [HR98]), AC_ω holds but DC_ω fails.

We recall the definition of a κ-c.c. forcing notion, κ-closed forcing notion, and a κ-distributive forcing notion from **Definition 5.8** of [Cum10] and the definition of a κ-strategically closed forcing notion from **Definition 5.14** and **Definition 5.15** of [Cum10].

Asaf Karagila proved in **Lemma 1** of [Kar14], that $DC_{<\kappa}$ can be preserved in the symmetric extension in terms of the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$, if $\mathbb{P}$ is κ-closed and $\mathcal{F}$ is κ-complete. In **Lemma 3.3** of [Kar19], Karagila and the author both observed independently that '$\mathbb{P}$ is κ-closed' can be replaced by '$\mathbb{P}$ has κ-c.c.' in **Lemma 1** of [Kar14]. The author independently observed this by combining the role of κ-c.c. forcing notions from **Lemma 2.2** of [Apt01], and the role of κ-completeness of $\mathcal{F}$ from **Lemma 1** of [Kar14].

The idea was the following. If $\mathbb{P}$ has κ-c.c., then any antichain is of size less than κ. So by Zorn's Lemma in the ground model, there is a maximal antichain of conditions $\mathcal{A} = \{p_\alpha : \alpha < \gamma < \kappa\}$ extending p such that for all $\alpha < \gamma$, $p_\alpha \Vdash \dot{f}(\hat{\alpha}) = \dot{t}_\alpha$ where $\dot{t}_\alpha \in HS$. Then we can follow **Lemma 1** of [Kar14] to finish the proof.

In a private conversation with Karagila, the author came to know that they independently observed the same fact. We note that there was a gap in the above observation. Specifically, the author was not aware of the fact that every symmetric system is equivalent to a tenacious system. Karagila fixed this gap. In particular, in **Lemma 3.3** of [Kar19], Karagila wrote that the natural assumption that $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ is a tenacious system is also required in the proof.

Lemma 3.1. ***(Lemma 3.3 of*** [Kar19]***).*** *Let V be a model of ZFC. If $\mathbb{P}$ has κ-c.c. and $\mathcal{F}$ is κ-complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$.*

We can slightly generalize **Lemma 1** of [Kar14] and observe that '$\mathbb{P}$ is κ-closed' can be replaced by '$\mathbb{P}$ is κ-distributive'.

Lemma 3.2. ***(Observation 1.4).*** *Let V be a model of ZFC. If $\mathbb{P}$ is κ-distributive and $\mathcal{F}$ is κ-complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$.*

Proof. Let G be a $\mathbb{P}$-generic filter over V. Let $\delta < \kappa$, we show DC_δ holds in $V(G)$. Let X and R are elements of $V(G)$ as in the assumptions of DC_δ. Since AC is equivalent to $\forall\kappa(DC_\kappa)$ and $V[G]$ a model of AC, using $\forall\kappa(DC_\kappa)$ in $V[G]$, we can find a $f : \delta \to X$ in $V[G]$. We show this $f : \delta \to X$ is in $V(G)$. Let $p_0 \Vdash \dot{f}$ is a function whose domain is δ and range is X which is a subset of $V(G)$. For each $\alpha < \delta$, $D_\alpha = \{p \leq p_0 : (\exists x \in X) p \Vdash \dot{f}(\check{\alpha}) = \dot{x}$ where $\dot{x} \in HS\}$ is open dense below p_0. Consequently by δ-distributivity of $\mathbb{P}$, $D = \cap_{\alpha<\delta} D_\alpha$ is dense below p_0. So, there is some $p \in D \cap G$. We can see that for each $\alpha < \delta$, there is a x_α such that $p \Vdash \dot{f}(\check{\alpha}) = \dot{x}_\alpha$ where $\dot{x}_\alpha \in HS$. Define $\dot{g} = \{\langle \check{\alpha}, \dot{x}_\alpha\rangle : \alpha < \delta\}$. Now, since each $\dot{x}_\alpha \in HS$, $sym(\dot{x}_\alpha) \in \mathcal{F}$. By κ-completeness of $\mathcal{F}$, $H = \cap_{\alpha<\kappa} sym(\dot{x}_\alpha) \in \mathcal{F}$. Next, since $H \subseteq sym(\dot{g})$ and $\mathcal{F}$ is a filter, $\dot{g} \in HS$. We can see that $p \Vdash \dot{g} = \dot{f}$. Thus, there is a dense open set of conditions $q \leq p$, such that for some $\dot{g} \in HS$, $q \Vdash \dot{g} = \dot{f}$. By genericity, $\dot{f}^G = f \in V(G)$. □

Remark. If κ is either a supercompact cardinal or a strongly compact cardinal and $\lambda > \kappa$ is a regular cardinal, there are certain forcing notions like supercompact Prikry forcing [Apt85] and strongly compact Prikry forcing [AH91] which are known to be non-κ-closed, but still can preserve DC_κ in the symmetric extension based on such forcings. In particular, Apter communicated to us that, assuming the consistency of a 2^λ-supercompact cardinal κ and a regular cardinal $\lambda > \kappa$, Kofkoulis proved in [Kof90], that in a symmetric extension based on supercompact Prikry forcing, DC_κ was preserved. In particular, DC_κ holds in the symmetric inner model constructed in **Theorem 1** of [Apt85]. Further applying the methods of Kofkoulis, assuming the consistency of a λ-strongly compact cardinal κ and a measurable cardinal $\lambda > \kappa$, a symmetric extension based on strongly compact Prikry forcing was constructed in [AH91] where κ became a singular cardinal of cofinality ω, κ^+ remained a measurable cardinal and DC_κ was preserved. We can also find another exhibition of Kofkoulis's method with certain modifications in [AM95].

We may observe that even if we start with a model V, which is a model of ZF + DC_κ where AC can consistently fail, we can still preserve $DC_{<\kappa}$ in a symmetric extension of V in certain cases. Specifically, we observe the following lemma.

Lemma 3.3. ***(Observation 1.5).*** *Let V be a model of $ZF + DC_\kappa$. If $\mathbb{P}$ is κ-strategically closed and $\mathcal{F}$ is κ-complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V with respect to the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$.*

Proof. Let G be a $\mathbb{P}$-generic filter over V. By **Theorem 2.2** of [GJ14], DC_κ is preserved in $V[G]$. Let $\delta < \kappa$, we show DC_δ holds in $V(G)$. Let X and R are elements of $V(G)$ as in the assumptions of DC_δ. Since DC_κ is preserved in $V[G]$, we can find a $f : \delta \to X$ in $V[G]$. We show this $f : \delta \to X$ is in $V(G)$. Let $p \Vdash \dot{f}$ is a function whose domain is δ and range a subset of $V(G)$. Consider a game of length κ, between two players I and II who play at odd stages and even stages respectively such that initially II chooses a trivial condition and I chooses a condition extending p and at non-limit even stages $2\alpha > 0$, II chooses a condition extending the condition of the previous stage deciding $\dot{f}(\check{\alpha}) = \dot{t}_\alpha$ where $\dot{t}_\alpha$ is in HS. By κ-strategic closure of $\mathbb{P}$, II has winning strategy. Thus, we can assume the existence of an increasing sequence of conditions $\langle p_\alpha : \alpha < \delta\rangle$ extending p such that $p_\alpha \Vdash \dot{f}(\check{\alpha}) = \dot{t}_\alpha$ where $\dot{t}_\alpha$ is in HS for each $\alpha < \delta$. It is enough to show that $\dot{f} = \{\dot{t}_\beta : \beta < \delta\}$ is in HS which follows using κ-completeness of $\mathcal{F}$ as done in **Lemma 1** of [Kar14]. □

Remark. Let V be a model of $ZF + DC_\kappa$. We can also observe that if $\mathbb{P}$ is well-orderable of order type at most κ and κ-c.c. at the same time and $\mathcal{F}$ is κ-complete, then $DC_{<\kappa}$ is preserved in the symmetric extension of V in terms of the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$. Let G be a $\mathbb{P}$-generic filter over V. By **Theorem 2.1** of [GJ14], DC_κ is preserved in $V[G]$. Rest follows from the proof of **Lemma 3.3** of [Kar19].

Question 3.4. *Suppose V be a model of $ZF + DC_\kappa$ and $\mathbb{P}$ is κ-distributive. Can we preserve DC_κ in every forcing extension $V[G]$ by $\mathbb{P}$?*

If the answer is in the affirmative, we can say that if V is a model of $ZF + DC_\kappa$, $\mathbb{P}$ is κ-distributive and $\mathcal{F}$ is κ-complete, then $DC_{<\kappa}$ is preserved in the symmetric extension in terms of the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ following **Lemma 3.2**.

3.1. **Number of normal measures a successor cardinal can carry and DC.** Takeuti [Tak70] and Jech [Jec68] independently proved that if we assume the consistency of "ZFC + there is a measurable cardinal" then the theory "ZF + DC + $\aleph_1$ is a measurable cardinal" is consistent. In **section 1.33** of [Dim11], Dimitriou modified Jech's construction and proved that if we assume the consistency of "ZFC + there is a measurable cardinal κ and $\gamma < \kappa$ is a regular cardinal" then the theory "ZF + the cardinality of γ is preserved + γ^+ is a measurable cardinal" is consistent. Apter, Dimitriou, and Koepke [ADK14] constructed symmetric models in which for an arbitrary ordinal ρ, $\aleph_{\rho+1}$ can be the least measurable as well as the least regular uncountable cardinal. Bilinsky and Gitik [BG12] proved that if we assume the consistency of "ZFC + GCH + there is a measurable cardinal κ" then we can obtain a symmetric extension where κ is a measurable cardinal without a normal measure. Assuming the consistency of "ZFC + GCH + there is a measurable cardinal", we observe that a successor of regular cardinals like $\aleph_1$, $\aleph_2$, $\aleph_{\omega+2}$, as well as $\aleph_{\omega_1+2}$, can carry an arbitrary (non-zero) number of normal measures in ZF + DC.

In **Theorem 1** of [MF09], Friedman and Magidor proved that a measurable cardinal can be forced to carry arbitrary number of normal measures in ZFC.

Lemma 3.5. ***(Theorem 1 of*** [MF09]***).*** *Assume GCH. Suppose that κ is a measurable cardinal and let α be a cardinal at most κ^{++}. In a cofinality preserving forcing extension, then κ carries exactly α normal measures.*

We recall the definition of a *symmetric collapse* from [KH19].

Definition 3.6. ***(Symmetric Collapse, Definition 4.1 of*** [KH19]***).*** *Let $\kappa \leq \lambda$ be two infinite cardinals. The symmetric collapse is the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ defined as follows.*

- $\mathbb{P} = Col(\kappa, < \lambda)$.
- *$\mathcal{G}$ is the group of automorphisms π such that there is a sequence of permutations $\overrightarrow{\pi} = \langle \pi_\alpha : \kappa < \alpha < \lambda \rangle$ such that π_α is a permutation of α satisfying $\pi p(\alpha, \beta) = \pi_\alpha p(\alpha, \beta)$.*
- *$\mathcal{F}$ is the normal filter of subgroups generated by fix(E) for bounded $E \subseteq \lambda$, where fix(E) is the group $\{\pi : \forall \alpha \in E, \pi p(\alpha, \beta) = p(\alpha, \beta)\}$.*

Lemma 3.7. *Let $\kappa \leq \lambda$ be two infinite cardinals such that $cf(\lambda) \geq \kappa$ and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is the symmetric collapse where $\mathbb{P} = Col(\kappa, < \lambda)$. Then, $\mathcal{F}$ is κ-complete.*

Proof. Fix $\gamma < \kappa$ and let, for each $\beta < \gamma$, $K_\beta \in \mathcal{F}$. There must be bounded $E_\beta \subseteq \lambda$ for each $\beta < \gamma$ such that fix$E_\beta \subseteq K_\beta$. Next, fix$(\cup_{\beta<\gamma} E_\beta) \subseteq \cap_{\beta<\gamma}$ fix$E_\beta \subseteq \cap_{\beta<\gamma} K_\beta$. Since $cf(\lambda) \geq \kappa$, $\cup_{\beta<\gamma} E_\beta$ is a bounded subset of λ. Consequently, $\cap_{\beta<\gamma} K_\beta \in \mathcal{F}$. □

We observe that after a symmetric collapse, the successor of a regular cardinal can be a measurable cardinal carrying an arbitrary (non-zero) number of normal measures assuming the consistency of a measurable cardinal. Further we can preserve Dependent choice in certain cases.

Theorem 3.8. *Let V be a model of ZFC + GCH with a measurable cardinal κ. Let λ be any non-zero cardinal at most κ^{++} and let $\eta \leq \kappa$ be regular. Then, there is a symmetric extension where $\kappa = \eta^+$ is a measurable cardinal carrying λ normal measures. Moreover, AC_κ fails and $DC_{<\eta}$ holds*[10] *in the symmetric model.*

Proof. Applying **Lemma 3.5**, we obtain a cofinality preserving forcing extension V' of V where κ is a measurable cardinal with λ many normal measures. Let $V'(G)$ be the symmetric extension of V' obtained by the symmetric collapse $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ where $\mathbb{P} = Col(\eta, < \kappa)$ and G a $\mathbb{P}$-generic filter over V'. In $V'(G)$, $\kappa = \eta^+$. We can also have the following in $V'(G)$.

- By **Lemma 2.4** and **Lemma 2.5** of [Apt01], κ remains a measurable cardinal with λ many normal measures.
- Since κ is a successor as well as a measurable cardinal, AC_κ fails using **Lemma 2.13**.
- Since $\mathbb{P}$ is η-closed and the filter $\mathcal{F}$ is η-complete by **Lemma 3.7**, $DC_{<\eta}$ holds using **Lemma 1** of [Kar14].

□

Remark. The referee pointed out that $DC_{<\kappa}$ is preserved in $V'(G)$. Assuming that λ is regular, the proof of **Lemma 3.7** gives that $\mathcal{F}$ is λ-complete. Consequently, since κ is a regular cardinal in V', $\mathcal{F}$ is κ-complete. Since $\mathbb{P}$ is κ-c.c., by **Lemma 3.1**, $DC_{<\kappa}$ is preserved in $V'(G)$.

Question 3.9. *Can $\aleph_{\omega+1}$, $\aleph_{\omega_1+1}$ carry any number of normal measures in ZF?*

In [Apt06] and [Apt10], Apter proved that $\aleph_{\omega+1}$ can carry $\geq \aleph_{\omega+2}$ number of normal measures and $\aleph_{\omega_1+1}$ can carry $\geq \aleph_{\omega_1+2}$ number of normal measures respectively. If it is consistent that κ is supercompact and $\lambda > \kappa$ carry arbitrary number of normal measures then we can prove the consistency of successor of singular cardinals like $\aleph_{\omega+1}$ and $\aleph_{\omega_1+1}$ being measurable cardinals with arbitrary normal measures by methods of [Apt06] and [Apt10].

4. Failure of GCH at limit cardinals below a supercompact cardinal

In this section we prove **Theorem 1.3** applying **Lemma 3.1**. Consequently, we answer **Question 1.1** asked by Apter. We note that **Theorem 1.3** was already observed by the author and written by Karagila in **subsection 4.1** of [Kar19]. In this section we write the proof in more details. We further observe the failure of AC_κ in the symmetric model and provide a different argument to prove that the supercompactness of κ is preserved in $V(G)$ from [Ina13].

[10] If we assume $\eta > \omega$.

Proof. ***(Theorem 1.3)**.*

(1) **Defining ground model (V):** At the beginning of the proof of **Theorem 3** of [Apt12], from the given requirements, Apter constructed a model V where there is an enumeration $\langle\kappa_i : i < \kappa\rangle$ of $C \cup \{\omega\}$ where $C \subseteq \kappa$ is a club of inaccessible and limit cardinals below a supercompact cardinal κ such that $2^{\kappa_i} = \kappa_i^{++}$ holds. We consider V to be our ground model. For reader's convenience we recall the steps from the proof of **Theorem 3** of [Apt12] as follows.
   - Let V be a model of ZFC + GCH with a supercompact cardinal κ.
   - Let $\mathbb{Q}_1$ be Lavers partial ordering which makes κs supercompactness indestructible under κ-directed closed forcing. Since $\mathbb{Q}_1$ may be defined so that $|\mathbb{Q}_1| = \kappa$, we have $V^{\mathbb{Q}_1 * \dot{Add}(\kappa,\kappa^{++})} = V_2$ is a model of 'ZFC + κ is supercompact + $2^\kappa = \kappa^{++}$ + $2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa^+$'.
   - Let $\mathbb{Q}_3$ be the Radin forcing defined over κ. Taking a suitable measure sequence will enable one to preserve the supercompactness of κ (c.f. [Git10]). Consequently, $V_2^{\mathbb{Q}_3} = \bar{V}$ is a model of 'ZFC + κ is supercompact + $2^\kappa = \kappa^{++}$ + $2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa$ + There is a club $C \subseteq \kappa$ composed of inaccessible cardinals and their limits with $2^\delta = 2^{\delta^+} = \delta^{++}$ for every $\delta \in C$'.
   - With an abuse the notion for the sake of convenience we consider the ground model to be $\bar{V} = V$. Let $\langle\kappa_i : i < \kappa\rangle \in V$ be the continuous, increasing enumeration of $C \cup \{\omega\}$.

(2) **Defining symmetric system $\langle\mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$:**
   - Let $\mathbb{P}$ be the Easton support product of $\mathbb{P}_\alpha = Col(\kappa_\alpha^{++}, <\kappa_{\alpha+1})$ where $\alpha < \kappa$.
   - Let $\mathcal{G}$ be the Easton support product of the automorphism groups of each $\mathbb{P}_\alpha$.
   - Let $\mathcal{F}$ be the filter generated by fix(α) groups for $\alpha < \kappa$, where fix$(\alpha) = \{\pi \in \Pi_{\alpha<\kappa} Aut(\mathbb{P}_\alpha) : \pi \restriction \alpha = id\}$.

(3) **Defining symmetric extension of V:** Let G be a $\mathbb{P}$-generic filter. We construct a model $V(G)^{\mathcal{F}}$ by the symmetric system $\langle\mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ defined above in (2) and call it as $V(G)$ for the sake of our convenience.

Since each $\mathbb{P}_\alpha$ is weakly homogeneous, the following holds.

Lemma 4.1. *If $A \in V(G)$ is a set of ordinals, then $A \in V[G \restriction \alpha]$ for some $\alpha < \kappa$.*

Proof. Without loss of generality we may assume that $\dot{A} = \{\langle p, \check{\epsilon}\rangle : p \Vdash \check{\epsilon} \in \dot{A}\} \in HS$ is a name for A. Let $q \Vdash \check{\epsilon} \in \dot{A}$ and let β support $\check{\epsilon}$ and $\dot{A}$. Let, for the sake of contradiction $q \restriction \beta \nVdash \check{\epsilon} \in \dot{A}$. Then, there is a q' such that $q' \leq q \restriction \beta$ where $q' \Vdash \neg(\check{\epsilon} \in \dot{A})$. Since each $\mathbb{P}_\alpha$ is weakly homogeneous, the Easton support product is weakly homogeneous too. Thus there is a $a \in$ fixβ such that $a(q) \mid\mid q'$. By **Lemma 2.4**, $a(q) \Vdash a(\check{\epsilon}) \in a(\dot{A})$. Since β supports $\check{\epsilon}$ and $\dot{A}$, and $a \in$ fixβ we get $a(q) \Vdash \check{\epsilon} \in \dot{A}$ which is a contradiction to the fact that $a(q) \mid\mid q'$ and $q' \Vdash \neg(\check{\epsilon} \in \dot{A})$. Thus, $q \restriction \beta \Vdash \check{\epsilon} \in \dot{A}$. If $\alpha = sup\beta$ then we get that $\{\langle\langle q \restriction \alpha\rangle, \check{\epsilon}\rangle : q \Vdash \check{\epsilon} \in \dot{A}\}$ is a name for A. □

We apply **Lemma 4.1** to prove that κ remains supercompact in our symmetric extension $V(G)$. Inamder [Ina13] proved that if we assume the consistency of "ZFC + there is a supercompact cardinal κ, and $\gamma < \kappa$ is a regular cardinal" then the theory "ZF + the cardinality of γ is preserved + γ^+ is a supercompact cardinal" is consistent. We incorporate the arguments from [Ina13] in order to show that κ remains supercompact in our symmetric extension $V(G)$. We recall **Lemma 26** of [Ina13], **Lévy–Solovay Lemma** (**Lemma 27** of [Ina13]) and **Theorem 29** of [Ina13].

Lemma 4.2. *(Lemma 26 of* [Ina13]*). Let κ be a regular cardinal, $\gamma \geq \kappa$ and $\mathbb{P}$ be a partial order of size less than κ. Then for every $C \in \mathcal{P}_\kappa(\gamma)^{V[G]}$, there is a $D \in \mathcal{P}_\kappa(\gamma)^V$ such that in $V[G]$, $C \subseteq D$.*

Lemma 4.3. *(Lévy–Solovay Lemma, Lemma 27 of* [Ina13]*). In V, let κ be a regular cardinal, D be a set and $\mathcal{U}$ a κ-complete ultrafilter on D. Let $\mathbb{P}$ be a poset of size less than κ and G a V-*

generic filter on $\mathbb{P}$. Suppose $V[G] \models f : D \to V$. Then there is $S \in \mathcal{U}$ and $g : S \to V$ in V s.t. $V[G] \models f \restriction S = g$.

Applying **Lemma 4.1** and **Lemma 4.3** we obtain the following lemma, which is analogous to **Lemma 33** of [Ina13].

Lemma 4.4. *Let D be a set and $\mathcal{U}$ a κ-complete ultrafilter on D in V. Suppose $V(G) \models f : D \to V$. Then there is $S \in \mathcal{U}$ and $g : S \to V$ in V s.t. $V(G) \models f \restriction S = g$.*

Proof. By **Lemma 4.1**, for some $\alpha < \kappa$ we get $f \in V[G \restriction \alpha]$. Now we can say $G \restriction \alpha$ is $\mathbb{P}'$-generic over V where $|\mathbb{P}'| < \kappa$. By **Lemma 4.3** we get a $S \in \mathcal{U}$ and $g : S \to V$ in V such that $V[G \restriction \alpha] \models f \restriction S = g$. So, $V(G) \models f \restriction S = g$. □

Similarly **Lemma 34** of [Ina13], we obtain the following lemma by applying **Lemma 4.4**.

Lemma 4.5. *In V, let D be a set and $\mathcal{U}$ a κ-complete ultrafilter on D. Let $\mathcal{W}$ be the filter on D generated by $\mathcal{U}$ in $V(G)$. Then $\mathcal{W}$ is a κ-complete ultrafilter.*

We follow the proof of **Theorem 35** from [Ina13] and refer the reader to [Ina13] for further details.

Lemma 4.6. *In $V(G)$, κ is supercompact.*

Proof. Let $\gamma \geq \kappa$ be arbitrary. Since κ is supercompact in V, there is a normal measure $\mathcal{U}$ on $\mathcal{P}_\kappa(\gamma)$ in V. Let $\mathcal{V}$ be the κ-complete measure it generates on $\mathcal{P}_\kappa(\gamma)^V$ in $V(G)$. Let $\mathcal{W}$ be the filter generated by $\mathcal{V}$ on $\mathcal{P}_\kappa(\gamma)$ in $V(G)$. Since $\mathcal{W}$ is generated by a κ-complete ultrafilter on $\mathcal{P}_\kappa(\gamma)^V \subseteq \mathcal{P}_\kappa(\gamma)$, $\mathcal{W}$ is a κ-complete ultrafilter by **Lemma 4.5**.

Fineness: Let $X \in \mathcal{P}_\kappa(\gamma)^{V(G)}$. By **Lemma 4.1**, for some $\alpha < \kappa$ we have $X \in V[G \restriction \alpha]$. Since κ is not collapsed while going from V to $V[G \restriction \alpha]$, $X \in \mathcal{P}_\kappa(\gamma)^{V[G\restriction\alpha]}$. By **Lemma 4.2** and following the arguments in the last three lines from (ii) of **Theorem 35**, [Ina13], $\hat{X} \in \mathcal{V}'$, where $\mathcal{V}'$ is the fine measure that $\mathcal{U}$ generates on $\mathcal{P}_\kappa(\gamma)^{V[G\restriction\alpha]}$. Now $\mathcal{U} \subseteq \mathcal{V}' \subseteq \mathcal{W}$ since $\mathcal{P}_\kappa(\gamma)^{V[G\restriction\alpha]} \subseteq \mathcal{P}_\kappa(\gamma)^{V(G)}$. Consequently $\mathcal{W}$ is fine.

Choice function: Let $V(G) \models f : \mathcal{P}_\kappa(\gamma) \to \gamma$ and $V(G) \models \forall X \in \mathcal{P}_\kappa(\gamma)(f(X) \in X)$. By **Lemma 4.1**, for some $\alpha < \kappa$ we get $h = f \restriction \mathcal{P}_\kappa(\gamma)^V \in V[G \restriction \alpha]$. By **Lemma 4.3**, we get $Y \in \mathcal{U}$ and $(g : Y \to \gamma)^V$ such that $V[G \restriction \alpha] \models h \restriction Y = g$. Now by normality of $\mathcal{U}$ in V we get a set x in $\mathcal{U}$ such that g is constant on x, and so h is constant on a set in $\mathcal{U}$. Hence, we will get a set y in $\mathcal{W}$ such that f is constant on y. □

Lemma 4.7. *In $V(G)$, $DC_{<\kappa}$ holds.*

Proof. We see that $\mathcal{F}$ is κ-complete. Fix $\gamma < \kappa$ and let, for each $\beta < \gamma$, $K_\beta \in \mathcal{F}$. There must be fixβ for each $\beta < \gamma$ such that fix$\beta \subseteq K_\beta$. Next, fix$(\max\{\beta : \beta < \gamma\}) \subseteq \cap_{\beta<\gamma}$ fix$\beta \subseteq \cap_{\beta<\gamma} K_\beta$ implies $\cap_{\beta<\gamma} K_\beta \in \mathcal{F}$. Since $\mathbb{P}$ is the Easton-support product of the appropriate Lévy collapse, $\mathbb{P}$ has κ-c.c. Since $\mathcal{F}$ is κ-complete and $\mathbb{P}$ has κ-c.c., we obtain $DC_{<\kappa}$ in $V(G)$ by **Lemma 3.1**. □

Lemma 4.8. In $V(G)$, AC_κ fails.

Proof. Since the cardinality of κ_α^{++} is preserved in $V(G)$ for $\alpha < \kappa$, we can define in $V(G)$ the set $X_\alpha = \{x \subseteq \kappa_\alpha^{++} : x$ codes a well ordering of $(\kappa_\alpha^{+++})^V$ of order type $\kappa_\alpha^{++}\}$. We claim that $\langle X_\alpha : \alpha < \kappa\rangle \in V(G)$. Let β be a support of X_α for each $\alpha < \kappa$. Since $X_\alpha \in V(G)$, let $\dot{X}_\alpha \in HS$ be a name for X_α. We define the collection $\{\dot{X}_\alpha : \alpha < \kappa\}$ in V and let $\dot{X} = \{\dot{X}_\alpha : \alpha < \kappa\}$. Since β is a support of X_α for each $\alpha < \kappa$, fix$\beta \subset sym(\dot{X}_\alpha)$ for each $\alpha < \kappa$. Consequently, fix$\beta \subset \bigcap_{\alpha<\kappa} sym(\dot{X}_\alpha)$. Now we have that for every $\pi \in \bigcap_{\alpha<\kappa} sym(\dot{X}_\alpha)$, $\pi(\dot{X}) = \dot{X}$, and so $\pi \in sym(\dot{X})$. Thus fix$\beta \subset sym(\dot{X})$. Consequently, $sym(\dot{X}) \in \mathcal{F}$, i.e., $\dot{X}$ is symmetric with respect to $\mathcal{F}$, since we define $\mathcal{F}$ to be the filter generated by fix(α) groups for $\alpha < \kappa$. Since all the names appearing in $\dot{X}$ are from HS, $\dot{X} \in HS$. Consequently, $\langle X_\alpha : \alpha < \kappa\rangle \in V(G)$.

Although each $X_\alpha \neq \emptyset$, we claim that $(\Pi_{\alpha<\kappa}X_\alpha)^{V(G)} = \emptyset$. Otherwise let $y \in (\Pi_{\alpha<\kappa}X_\alpha)^{V(G)}$. Since y is a sequence of sets of ordinals, so can be coded as a set of ordinals. There is then a $\gamma < \kappa$ such that $y \in V[G \restriction \gamma]$ by **Lemma 4.1** and $V[G \restriction \gamma]$ is V-generic over $\mathbb{P}$ such that $|\mathbb{P}| < \kappa$. There is then a final segment of the sequence $\langle(\kappa_\alpha^{+++}) : \alpha < \kappa\rangle$ which remains a sequence of cardinals in $V[G \restriction \gamma]$ which is a contradiction. □

We prove that in $V(G)$, GCH holds at a limit cardinal δ if and only if $\delta > \kappa$. Since GCH implies AC, GCH is weakened to a form which states that *there is no injection from δ^{++} into $\mathcal{P}(\delta)$* in Theorem 3 of [Apt12]. We follow this weakened version of GCH in our following lemma. We follow the explanation given in **subsection 4.1** of [Kar19] by Karagila, to observe that in $V(G)$, GCH holds for a limit cardinal δ if and only if $\delta > \kappa$.

Lemma 4.9. *In $V(G)$, GCH holds for a limit cardinal δ if and only if $\delta > \kappa$.*

Proof. Since $\mathbb{P}$ has κ-c.c., cardinals above κ are preserved in $V[G]$. Also, $\mathbb{P} \subseteq V_\kappa$. Thus for any limit cardinal $\delta > \kappa$ there is no injection from δ^{++} into $\mathcal{P}(\delta)$ in $V[G]$, since GCH holds above κ in V. Consequently, there is no injection from δ^{++} into $\mathcal{P}(\delta)$ in $V(G)$.

We show that if $\delta \leq \kappa$ is a limit cardinal then δ^{++} can be injected into $\mathcal{P}(\delta)$ in $V(G)$. Since $V_\kappa^{V(G)} = V_\kappa^{V[G]}$, we note that it is enough to prove this phenomenon in $V[G]$. If $\delta < \kappa$ is a limit cardinal in $V[G]$, then $\delta = \kappa_i$ for some $i < \kappa$. Since $\delta \in C$, we have $2^\delta = \delta^{++}$. We note that the Easton support product up to i is δ^+-c.c., so it does not collapse δ^{++} and the Easton support Product above i is δ^{++}-closed, so it does not collapse δ^{++} also. Thus, δ^{++} is not collapsed in $V[G]$. By similar arguments, κ^{++} injects into $\mathcal{P}(\kappa)$ in $V(G)$ since $2^\kappa = \kappa^{++}$ holds in V and cardinals at and above κ are same in V, $V(G)$ and $V[G]$. □

□

Remark 1. The referee suggested us to remark the following. In the context of ZF, there are two reasonable definitions for the statement GCH at μ.

(1) There is no injection $\mu^{++} \to^{inj} \mathcal{P}(\mu)$.
(2) There is no surjection $\mathcal{P}(\mu) \to^{sur} \mu^{++}$.

In ZF, it is possible that there is no $\mu^+ \to^{inj} \mathcal{P}(\mu)$, but there is always a surjection $\mathcal{P}(\mu) \to^{sur} \mu^+$. In our case the above two definitions behave the same, so the referee suggested us to remark that both definitions (1) and (2) work, by the same proof.

Remark 2. In **Theorem 1** of [Apt01], assuming '$o(\kappa) = \delta^*$ for $\delta^* \leq \kappa^+$ any finite or infinite cardinal' Apter constructed an analogous symmetric extension where $DC_{<\kappa}$ holds and where κ can carry an arbitrary number of normal measures regardless of the specified behavior of the continuum function on sets having measure one with respect to every normal measure over κ. We observe that we can obtain the result of **Theorem 1** of [Apt01] starting from just one measurable cardinal κ if we use **Theorem 1** of [MF09] by Friedman and Magidor instead of passing to an inner model of Mitchell from [Mit74].[11]

Corollary 4.10. ***(of Theorem 1 of*** [Apt01]***).*** *Let V be a model of ZFC + GCH with a measurable cardinal κ and let λ be a cardinal at most κ^{++}. There is then a symmetric extension with respect to a symmetric system $\langle\mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ where κ is a measurable cardinal carrying λ many normal measures $\langle\mathcal{U}_\alpha^* : \alpha < \lambda\rangle$. Moreover for each $\alpha < \lambda$, the set $\{\delta : 2^\delta = \delta^{++}$ and δ is inaccessible$\} \in \mathcal{U}_\alpha^*$[12] and $DC_{<\kappa}$ holds.*

Remark 3. Apter used analogous arguments in **Lemma 2.2** of [Apt01], similar to **Lemma 4.1** to preserve a certain amount of dependent choice in some symmetric models (e.g. symmetric models from **Theorem 1** of [Apt01], **Theorem** of [Apt00], and **Theorem 2** of [Apt12]).

[11] as done in the proof of Theorem 1 of [Apt01].
[12] There is nothing specific about δ^{++}, the continuum function can take any value.

5. Proving Dimitriou's Conjecture

In **Theorem 1** of [Apt83a], Apter obtained a symmetric inner model where $\aleph_\omega$ carries a Rowbottom filter and $DC_{\aleph_{n_0}}$ holds for any arbitrary $n_0 \in \omega$ from a ω-sequence of measurable cardinals. In **section 8**, we observe that there is an alternating sequence of measurable and non-measurable cardinals in the symmetric model. Apter constructed the model based on Easton support products of Lévy collapse. Consequently, $DC_{\aleph_{n_0}}$ was preserved (see **Lemma 1.4** of [Apt83a]). In **section 1.4** of [Dim11], Dimitriou constructed a similar symmetric extension with an alternating sequence of measurable and non-measurable cardinals, excluding the singular limits. She constructed the model based on finite support products of collapsing functions, unlike the model from [Apt83a]. In [Dim11], Dimitriou claimed that by using such a finite support product construction, a lot of arguments could be made easier. In particular, she used finite support products of *injective tree-Prikry forcings*, in several constructions from **Chapter 2** of [Dim11]. There are many symmetric extensions based on finite support products of Lévy Collapse. In **Theorem 5.6** of [KH19], Karagila and Hayut considered a symmetric extension based on finite support product of Lévy Collapse. In **section 6**, we encounter two symmetric extensions based on the finite support products of Lévy Collapse due to Apter and Cody from [AC13] (see **Theorem 2** of [AH91] also). On the other hand, there is a downside to this method. Specifically, Dimitriou conjectured that DC_ω would fail in the model. In this section, we prove that AC_ω fails in the model and thus prove the conjecture of Dimitriou. In other words, we prove **Theorem 1.6**. We recall the terminologies from **section 2.4**.

Proof. ***(Theorem 1.6)***. Firstly, we give a description of the symmetric extension constructed in **section 1.4** of [Dim11] as follows.

(1) **Defining ground model (V):** Let V be a model of ZFC, ρ is an ordinal, and $\mathcal{K} = \langle \kappa_\epsilon : 0 < \epsilon < \rho \rangle$ is a sequence of measurable cardinals with a regular cardinal κ_0 below all the regular cardinals in $\mathcal{K}$.
(2) **Defining a triple** $(\mathbb{P}, \mathcal{G}, \mathcal{I})$**:**
   - For each $\epsilon \in (0, \rho)$ we define the following cardinals,
     $\kappa'_1 = \kappa_0$,
     $\kappa'_\epsilon = \kappa^+_{\epsilon-1}$ if ϵ is a successor ordinal,
     $\kappa'_\epsilon = ((\cup_{\zeta<\epsilon}\kappa_\zeta)^+)^V$ if ϵ is a limit ordinal and $\cup_{\zeta<\epsilon}\kappa_\zeta$ is singular,
     $\kappa'_\epsilon = (\cup_{\zeta<\epsilon}\kappa_\zeta)^{++}$ if ϵ is a limit ordinal and $\cup_{\zeta<\epsilon}\kappa_\zeta = \kappa_\epsilon$ is regular,
     $\kappa'_\epsilon = \cup_{\zeta<\epsilon}\kappa_\zeta$ if ϵ is a limit ordinal and $\cup_{\zeta<\epsilon}\kappa_\zeta < \kappa_\epsilon$ is regular.
     Let $\mathbb{P} = \Pi_{0<i<\rho}\mathbb{P}_i$ be the Easton support product of $\mathbb{P}_i = Fn(\kappa'_i, \kappa_i, \kappa'_i)$ ordered componentwise where for each $0 < i < \rho$, $Fn(\kappa'_i, \kappa_i, \kappa'_i)$= $\{p : \kappa'_i \rightharpoonup \kappa_i : |p| < \kappa'_i$ and p is an injection$\}$ ordered by reverse inclusion. Also $p : \kappa'_i \rightharpoonup \kappa_i$ is denoted as a partial function from κ'_i to κ_i.
   - $\mathcal{G} = \Pi_{0<i<\rho}\mathcal{G}_i$ where for each $0 < i < \rho$, $\mathcal{G}_i$ is the full permutation group of κ_i that can be extended to $\mathbb{P}_i$ by permuting the range of its conditions, i.e., for all $a \in \mathcal{G}_i$ and $p \in \mathbb{P}_i$, $a(p) = \{(\psi, a(\beta)) : (\psi, \beta) \in p\}$.
   - For $m < \omega$ and $e = \{\alpha_i : i \leq m\}$ is a sequence of ordinals such that for each $1 \leq i \leq m$, there is a distinct $\epsilon_i \in (0, \rho)$ such that $\alpha_i \in (\kappa'_{\epsilon_i}, \kappa_{\epsilon_i})$. We define $E_e = \{\langle \emptyset, ..., p_{\epsilon_1} \cap (\kappa'_{\epsilon_1} \times \alpha_1), \emptyset, ..., p_{\epsilon_2} \cap (\kappa'_{\epsilon_2} \times \alpha_2), \emptyset, ..., p_{\epsilon_i} \cap (\kappa'_{\epsilon_i} \times \alpha_i), \emptyset, ...p_{\epsilon_m} \cap (\kappa'_{\epsilon_m} \times \alpha_m), \emptyset, ...\rangle; \overrightarrow{p} \in \mathbb{P}\}$ and $\mathcal{I} = \{E_e : e \in \Pi^{fin}_{0<i<\rho}(\kappa'_i, \kappa_i)\}$, where $\Pi^{fin}_{0<i<\rho}(\kappa'_i, \kappa_i)$ is the finite support product.
(3) **Defining symmetric extension of V:** Clearly, $\mathcal{I}$ is a projectable symmetry generator with projections $\overrightarrow{p} \restriction E_e = \langle \emptyset, ..., p_{\epsilon_1} \cap (\kappa'_{\epsilon_1} \times \alpha_1), \emptyset, ...p_{\epsilon_2} \cap (\kappa'_{\epsilon_2} \times \alpha_2), \emptyset, ...p_{\epsilon_m} \cap (\kappa'_{\epsilon_m} \times \alpha_m), \emptyset, ...\rangle$. Let $\mathcal{I}$ generate a normal filter $\mathcal{F}_\mathcal{I}$ over $\mathcal{G}$. Let G be a $\mathbb{P}$-generic filter. We consider the symmetric model $V(G)^{\mathcal{F}_\mathcal{I}}$ as our desired symmetric extension.

It is possible to see that $\mathbb{P}$ is $(\mathcal{G}, \mathcal{I})$-homogeneous and so $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ has the approximation property. Consequently, by **Lemma 2.10** for all set of ordinals $X \in V(G)^{\mathcal{F}_\mathcal{I}}$, there exists an $E \in \mathcal{I}$ such that $X \in V[G \cap E]$. Following **Lemma 1.35** of [Dim11], in $V(G)^{\mathcal{F}_\mathcal{I}}$ for every $\epsilon \in (0, \rho)$,

$(\kappa'_\epsilon)^+ = \kappa_\epsilon$. We prove that AC_ω fails in $V(G)^{\mathcal{F}_\mathcal{I}}$. For the sake of convenience we define $V(G)^{\mathcal{F}_\mathcal{I}}$ as $V(G)$, $HS^{\mathcal{F}_\mathcal{I}}$ as HS, and $\mathcal{F}_\mathcal{I}$ as $\mathcal{F}$.

Lemma 5.1. In $V(G)$, AC_ω fails.

Proof. Since the cardinality of κ'_n is preserved in $V(G)$ for $n < \omega$, we can define in $V(G)$ the set $X_n = \{x \subseteq \kappa'_n : x$ codes a well ordering of $((\kappa'_n)^+)^V$ of order type $\kappa'_n\}$. We claim that $\langle X_n : n < \omega\rangle \in V(G)$. Let $E \in \mathcal{I}$ be a support of X_n for each $n \in \omega$. Since $X_n \in V(G)$, let $\dot{X}_n \in HS$ be a name for X_n. We define the collection $\{\dot{X}_n : n < \omega\}$ in V and let $\dot{X} = \{\dot{X}_n : n < \omega\}$. Since E is a support of X_n for each $n \in \omega$, fix$E \subset sym(\dot{X}_n)$ for each $n \in \omega$. Consequently, fix$E \subset \bigcap_{n\in\omega} sym(\dot{X}_n)$. Now we have that for every $\pi \in \bigcap_{n<\omega} sym(\dot{X}_n)$, $\pi(\dot{X}) = \dot{X}$, and so $\pi \in sym(\dot{X})$. Thus fix$E \subset sym(\dot{X})$. Consequently, $sym(\dot{X}) \in \mathcal{F}$, i.e., $\dot{X}$ is symmetric with respect to $\mathcal{F}$, since $E \in \mathcal{I}$ and the symmetry generator $\mathcal{I}$ generates $\mathcal{F}$. Since all the names appearing in $\dot{X}$ are from HS, $\dot{X} \in HS$. Consequently, $\langle X_n : n < \omega\rangle \in V(G)$.

Although $X_n \neq \emptyset$, we claim that $(\Pi_{n<\omega} X_n)^{V(G)} = \emptyset$. Otherwise let $y \in (\Pi_{n<\omega} X_n)^{V(G)}$. Since y is a sequence of sets of ordinals, so can be coded as a set of ordinals. Thus, there is an $e = \{\alpha_1, ...\alpha_m\}$ such that $y \in V[G \cap E_e]$ by **Lemma 2.10**. There are distinct ϵ_i such that $\alpha_i \in (\kappa'_{\epsilon_i}, \kappa_{\epsilon_i})$ and let l be $\max\{\epsilon_i : \alpha_i \in e\}$ such that l is an integer. Next let $M = \{i : \epsilon_i \leq l\}$ and $M' = \{i : \epsilon_i > l\}$. Then $V[G \cap E_e]$ is $\Pi_{i\in M} Fn(\kappa'_{\epsilon_i}, \alpha_i, \kappa'_{\epsilon_i}) \times \Pi_{i\in M'} Fn(\kappa'_{\epsilon_i}, \alpha_i, \kappa'_{\epsilon_i})$-generic over V. By closure properties of $\Pi_{i\in M'} Fn(\kappa'_{\epsilon_i}, \alpha_i, \kappa'_{\epsilon_i})$, all elements of the sequence $\langle(\kappa'_n)^+ : n < \omega\rangle$ remain cardinals after forcing with $\Pi_{i\in M'} Fn(\kappa'_{\epsilon_i}, \alpha_i, \kappa'_{\epsilon_i})$. Next, since M is finite we can find $j < \omega$ such that for all $r \geq j$, $|\Pi_{i\in M} Fn(\kappa'_{\epsilon_i}, \alpha_i, \kappa'_{\epsilon_i})| < \kappa_r$. Thus, a final segment of the sequence $\langle(\kappa'_n)^+ : n < \omega\rangle$ remains a sequence of cardinals in $V[G \cap E_e]$ which is a contradiction. □

□

Remark. In **Theorem 5.6** of [KH19], Karagila and Hayut proved the following.

- Assuming the existence of countably many measurable cardinals, it is consistent that there is a uniform ultrafilter on $\aleph_\omega$ but for all $0 < n < \omega$, there are no uniform ultrafilters on $\aleph_n$.

They considered a symmetric extension M based on finite support product of the symmetric collapses $Col(\kappa_n, < \kappa_{n+1})$. Following the proof of **Lemma 5.1**, we can say that AC_ω fails in the symmetric extension M. We consider another similar symmetric extension. Let V_1 be a model of ZFC where $\langle\kappa_n : 1 \leq n < \omega\rangle$ is a countable sequence of supercompact cardinals. Let $\mathbb{Q}$ be the forcing notion (see [Apt83], [Apt04]) which makes the supercompactness of each κ_n indestructible under κ_n-directed closed forcing notions. Let H be a $\mathbb{Q}$-generic filter over V_1 and $V = V_1[H]$ be our ground model. Let $\kappa_0 = \omega$ in V. Consider the symmetric extension $\mathcal{N}$ obtained by taking the finite support product of the symmetric collapses $Col(\kappa_n, < \kappa_{n+1})$. In the resulting model $\mathcal{N}$ the following hold:

(1) Since the forcing notions involved are weakly homogeneous, if A is a set of ordinals in $\mathcal{N}$, then A was added by an intermediate submodel where AC holds.
(2) For $n > 0$, each κ_n becomes $\aleph_n$ in $\mathcal{N}$.

Following **Lemma 4.3** of [KH19], we can observe that for each $1 \leq n < \omega$, there are no uniform ultrafilters on $\aleph_n$ in $\mathcal{N}$. Consequently for each $1 \leq n < \omega$, $\aleph_n$ can not be a measurable cardinal in $\mathcal{N}$. Since we are considering symmetric extension based on finite support products, AC_ω fails following the proof of **Lemma 5.1**. We can see that each $\aleph_n$ remains a Ramsey cardinal for $1 \leq n < \omega$ in $\mathcal{N}$. Fix $1 \leq n < \omega$. Let $f : [\kappa_n]^{<\omega} \to 2$ is in $\mathcal{N}$. Since f can be coded by a set of ordinals, f was added by an intermediate submodel (say V') where AC holds. Without loss of generality, we can say that $V' = V[G_1][G_2]$ where G_1 is $\mathbb{Q}_1$-generic over V such that $\mathbb{Q}_1$ is κ_n-directed closed and G_2 is $\mathbb{Q}_2$-generic over $V[G_1]$ such that $|\mathbb{Q}_2| < \kappa_n$. Since $\mathbb{Q}_1$ is κ_n-directed closed, κ_n remains supercompact in $V[G_1]$ as the supercompactness of κ_n was indestructible under κ_n-directed closed forcing notions in V. Consequently, κ_n remains a Ramsey cardinal in $V[G_1]$. Since $\mathbb{Q}_2$ is a small forcing with respect to κ_n, κ_n remains Ramsey in $V[G_1][G_2]$ by **Theorem 2.3**. There is then a set $X \in [\kappa_n]^{\kappa_n}$ homogeneous for f in V', and since $V' \subseteq \mathcal{N}$, $X \in [\kappa_n]^{\kappa_n}$ is homogeneous for f in $\mathcal{N}$. Consequently, for $1 \leq n < \omega$, each κ_n is Ramsey in $\mathcal{N}$.

6. Reducing the assumption of supercompactness by strong compactness

In this section, we prove **Observation 1.7** and **Observation 1.8**. Consequently, we reduce the large cardinal assumption of **Theorem 2** and **Theorem 3** of [AC13], from a supercompact cardinal to a strongly compact cardinal.

6.1. **Strongly compact Prikry forcing.** Suppose $\lambda > \kappa$ and κ be a λ-strongly compact cardinal in the ground model V. Let $\mathcal{U}$ be a κ-complete fine ultrafilter over $\mathcal{P}_\kappa(\lambda)$.

Definition 6.1. *(Definition 1.51,* [Git10]*). A set T is called a $\mathcal{U}$-tree with trunk t if and only if the following holds.*

1. *T consists of finite sequences $\langle P_1, ..., P_n\rangle$ of elements of $\mathcal{P}_\kappa(\lambda)$ so that $P_1 \subseteq P_2 \subseteq ...P_n$.*
2. *$\langle T, \trianglelefteq\rangle$ is a tree, where $\trianglelefteq$ is the order of the end extension of finite sequences.*
3. *t is a trunk of T, i.e., $t \in T$ and for every $\eta \in T$, $\eta \trianglelefteq t$ or $t \trianglelefteq \eta$.*
4. *For every $t \trianglelefteq \eta$, $Suc_T(\eta) = \{Q \in \mathcal{P}_\kappa(\lambda) : \eta \frown \langle Q\rangle \in T\} \in \mathcal{U}$.*

The set $\mathbb{P}_{\mathcal{U}}$ consists of all pairs $\langle t, T\rangle$ such that T is a $\mathcal{U}$-tree with trunk t. If $\langle t, T\rangle, \langle s, S\rangle \in \mathbb{P}_{\mathcal{U}}$, we say that $\langle t, T\rangle$ is stronger than $\langle s, S\rangle$, and denote this by $\langle t, T\rangle \geq \langle s, S\rangle$, if and only if $T \subseteq S$. Let G be V-generic over $\mathbb{P}_{\mathcal{U}}$.[13] Following a Prikry like lemma (c.f. **Theorem 1.52** of [Git10], **Lemma 1.1** of [AH91]), $\mathbb{P}_{\mathcal{U}}$ does not add bounded subsets to κ. Also, $(\lambda)^V$ is collapsed to κ in $V[G]$. Again, $\mathbb{P}_{\mathcal{U}}$ is $(\lambda^{<\kappa})^+$-c.c. Let $\delta \in [\kappa, \lambda)$ be an inaccessible cardinal. If $x \subseteq \mathcal{P}_\kappa(\lambda)$, let $x \restriction \delta = \{Z \cap \delta : Z \in x\}$ and $\mathcal{U} \restriction \delta = \{x \restriction \delta : x \in \mathcal{U}\}$. Since, $\mathcal{U}$ is a κ-complete, fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$, $\mathcal{U} \restriction \delta$ is a κ-complete, fine ultrafilter on $\mathcal{P}_\kappa(\delta)$. Consequently, we can consider the strongly compact Prikry forcing $\mathbb{P}_{\mathcal{U}\restriction\delta}$ like $\mathbb{P}_{\mathcal{U}}$.

Proof. ***(Observation 1.7).*** We perform the construction in two stages. In the first stage, we consider a symmetric inner model of a forcing extension based on strongly comact Prikry forcing as done in [AH91], instead of supercompact Prikry forcing as done in **Theorem 1** of [AC13].

1. **Defining ground model(V):** We start with a model V_0 of ZFC where κ is a strongly compact cardinal, θ an ordinal and GCH holds. By **Theorem 3.1** of [ADU19] we can obtain a forcing extension V where $2^\kappa = \theta$ and strong compactness of κ is preserved. We assume $\lambda > \kappa$ in V such that $(cf(\lambda))^V < \kappa$.
2. **Defining a symmetric inner model of the forcing extension of V:**
   - Let $\mathcal{U}$ be a fine measure on $\mathcal{P}_\kappa(\lambda)$ and $\mathbb{P} = \mathbb{P}_{\mathcal{U}}$ be the strongly compact Prikry forcing. Let G be V-generic over $\mathbb{P}_{\mathcal{U}}$.

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[13] Alternatively, we also recall the definition of a strongly compact Prikry forcing $\mathbb{P}_{\mathcal{U}}$ from [AH91]. Let $\mathcal{U}$ be a fine measure on $\mathcal{P}_\kappa(\lambda)$ and $\mathcal{F} = \{f : f \text{ is a function from } [\mathcal{P}_\kappa(\lambda)]^{<\omega} \text{ to } \mathcal{U}\}$. In particular, $\mathbb{P}_{\mathcal{U}}$ is the set of all finite sequences of the form $\langle p_1, ...p_n, f\rangle$ satisfying the following properties.

- $\langle p_1, ...p_n\rangle \in [\mathcal{P}_\kappa(\lambda)]^{<\omega}$.
- for $0 \leq i < j \leq n$, $p_i \cap \kappa \neq p_j \cap \kappa$.
- $f \in \mathcal{F}$.

The ordering on $\mathbb{P}_{\mathcal{U}}$ is given by $\langle q_1, ...q_m, g\rangle \leq \langle p_1, ..., p_n, f\rangle$ if and only if we have the following.

- $n \leq m$.
- $\langle p_1, ..., p_n\rangle$ is the initial segment of $\langle q_1, ..., q_m\rangle$.
- For $i = n+1, ..., m$, $q_i \in f(\langle p_1, ..., p_n, q_{n+1}, ..., q_{i-1}\rangle)$.
- For $\overrightarrow{s} \in [\mathcal{P}_\kappa(\lambda)]^{<\omega}$, $g(\overrightarrow{s}) \subseteq f(\overrightarrow{s})$.

For any regular $\delta \in [\kappa, \lambda]$, we denote $r \restriction \delta = \{\langle p_0 \cap \delta, ...p_n \cap \delta\rangle : \exists f \in \mathcal{F}\,[\langle p_0, ...p_n, f\rangle \in G]\}$. In $V[r \restriction \kappa] \subseteq V[G]$, κ is a singular cardinal having cofinality ω. Since any two conditions having the same stems are compatible, i.e. any two conditions of the form $\langle p_1, ..., p_n, f\rangle$ and $\langle p_1, ..., p_n, g\rangle$ are compatible., $\mathbb{P}_{\mathcal{U}}$ **is** $(\lambda^{<\kappa})^+$**-c.c.**

- We consider the model constructed in **section 2** of [AH91]. In particular, we consider our symmetric inner model $\mathcal{N}$ to be the least model of ZF extending V and containing $r \restriction \delta$ for each inaccessible $\delta \in [\kappa, \lambda)$ where $r \restriction \delta = \{\langle p_0 \cap \delta, ...p_n \cap \delta\rangle : \exists f \in \mathcal{F}\,[\langle p_0, ...p_n, f\rangle \in G]\}$ but not the λ-sequence of $r \restriction \delta$'s.

We follow the homogeneity of strongly compact Prikry forcing mentioned in **Lemma 2.1** of [AH91] to observe the following lemma.

Lemma 6.2. *If $A \in \mathcal{N}$ is a set of ordinals, then $A \in V[r \restriction \delta]$ for some inaccessible $\delta \in [\kappa, \lambda)$.*

Lemma 6.3. *In $\mathcal{N}$, κ is a strong limit cardinal.*

Proof. Since, $V \subseteq \mathcal{N} \subseteq V[G]$ and $\mathbb{P}$ does not add bounded subsets to κ, V and $\mathcal{N}$ have same bounded subsets of κ.[14] Consequently, in $\mathcal{N}$, κ is a limit of inaccessible cardinals and thus a strong limit cardinal as well. □

As explained in the introduction, our definitions of strong limit cardinal and inaccessible cardinal generally do not make sense in choiceless models. In spite of that, we can see that the assertion in **Lemma 6.3** makes sense (see the paragraph after **Theorem 1** of [AC13]). Since $\mathcal{N}$ and V have the same bounded subsets of κ, the usual definitions of κ is a strong limit cardinal and $\delta < \kappa$ is an inaccessible cardinal make sense in $\mathcal{N}$.

Lemma 6.4. *If $\gamma \geq \lambda$ is a cardinal in V, then γ remains a cardinal in $\mathcal{N}$.*

Proof. For the sake of contradiction, let γ is not a cardinal in $\mathcal{N}$. There is then a bijection $f : \alpha \to \gamma$ for some $\alpha < \gamma$ in $\mathcal{N}$. Since f can be coded by a set of ordinals, by **Lemma 6.2** $f \in V[r \restriction \delta]$ for some inaccessible $\delta \in [\kappa, \lambda)$. Since GCH is assumed in V_0 we have $(\delta^{<\kappa})^{V_0} = \delta$, and since $Add(\kappa, \theta)$ preserves cardinals and adds no sequences of ordinals of length less than κ, we conclude that $(\delta^{<\kappa})^V = (\delta^{<\kappa})^{V_0} = \delta$. Now $\mathbb{P}_{\mathcal{U}\restriction\delta}$ is $(\delta^{<\kappa})^+$-c.c. in V and hence δ^+-c.c. in V. Consequently, γ is a cardinal in $V[r \restriction \delta]$ which is a contradiction. □

Lemma 6.5. *In $\mathcal{N}$, $cf(\kappa) = \omega$. Moreover, $(\kappa^+)^{\mathcal{N}} = \lambda$ and $cf(\lambda)^{\mathcal{N}} = cf(\lambda)^V$.*

Proof. For each $\delta \in [\kappa, \lambda)$, we have $V[r \restriction \delta] \subseteq \mathcal{N}$. Consequently, $cf(\kappa)^{\mathcal{N}} = \omega$ since $cf(\kappa)^{V[r\restriction\kappa]} = \omega$. Following **Lemma 2.4** of [AH91], every ordinal in (κ, λ) which is a cardinal in V collapses to have size κ in $\mathcal{N}$, and so $(\kappa^+)^{\mathcal{N}} = \lambda$. Since V and $\mathcal{N}$ have same bounded subsets of κ, we see that $cf(\lambda)^{\mathcal{N}} = cf(\lambda)^V < \kappa$. □

We can see that since, $V \subseteq \mathcal{N}$ and $(2^\kappa = \theta)^V$, there is a θ-sequence of distinct subsets of κ in $\mathcal{N}$. Since $cf(\kappa^+)^{\mathcal{N}} < \kappa$ we can also see that AC_κ fails in $\mathcal{N}$.

In the second stage, we consider a symmetric inner model of a forcing extension of $\mathcal{N}$ based on product of Lévy collapse as done in the proof of **Theorem 2** of [AC13].

(1) **Defining ground model:** Consider the ground model to be $\mathcal{N}$. Let $\langle \kappa_n : n < \omega\rangle$ be a sequence of inaccessible cardinals less than κ which is cofinal in κ.
(2) **Defining a symmetric inner model of a forcing extension of $\mathcal{N}$:**
    - Let $\mathbb{P} = Col(\omega, <\kappa)$ and G be $\mathcal{N}$-generic for $\mathbb{P}$. Let $\mathbb{P}_n = Col(\omega, <\kappa_n)$. Following the proof of **Theorem 2** of [AC13], $G_n = G \cap \mathbb{P}_n$ is $\mathcal{N}$-generic for $\mathbb{P}_n$.
    - Let $\mathcal{M}$ be the least model of ZF extending $\mathcal{N}$ containing each G_n, but not G as constructed in **Theorem 2** of [AC13].

Following the proof of **Theorem 2** of [AC13], we have the following in $\mathcal{M}$.

(1) Since $\mathcal{M}$ contains G_n for each n, cardinals in $[\omega, \kappa)$ are collapsed to have size ω and so $\aleph_1^{\mathcal{M}} \geq \kappa$.
(2) If $x \in \mathcal{M}$ is a set of ordinals, then $x \in \mathcal{N}[G_n]$ for some $n < \omega$.

[14]We can observe another argument from Lemma 2.2 of [AH91].

(3) Since $Col(\omega, < \kappa_n)$ is canonically well-orderable in $\mathcal{N}$ with order type κ_n, cardinals and cofinalities greater than or equal to κ are preserved to $\mathcal{N}[G_n]$.
(4) Since κ is not collapsed, $\kappa = \aleph_1^{\mathcal{M}}$, $cf(\aleph_1)^{\mathcal{M}} = cf(\aleph_2)^{\mathcal{M}} = \omega$. Consequently, AC_ω fails in $\mathcal{M}$.
(5) There is a sequence of distinct subsets of $\aleph_1$ of length θ.

□

Proof. ***(Observation 1.8).*** We recall the symmetric inner model $\mathcal{N}$ from the previous proof. We consider a symmetric inner model of the forcing extension of $\mathcal{N}$ as done in the proof of **Theorem 3** of [AC13].

(1) **Defining ground model:** Consider the ground model to be $\mathcal{N}$ as in the previous proof. Let $\langle \kappa_n : n < \omega \rangle$ be a sequence of inaccessible cardinals less than κ which is cofinal in κ.
(2) **Defining a symmetric inner model of the forcing extension of $\mathcal{N}$:**
   - Let $\mathbb{P}_0 = Col(\omega, < \kappa_0)$, $\mathbb{P}_i = Col(\kappa_{i-1}, < \kappa_i)$ for $i \in [1, \omega)$. Let $\mathbb{P} = \Pi^{fin}_{i<\omega}\mathbb{P}_i$. For each $n < \omega$, we can factor $\mathbb{P}$ as $\mathbb{P} \cong \mathbb{P}^*_n \times \mathbb{P}^n$ where $\mathbb{P}^*_n = \Pi^{fin}_{0\leq i\leq n}\mathbb{P}_i$ and $\mathbb{P}^n = \Pi^{fin}_{n+1\leq i<\omega}\mathbb{P}_i$. Let $G \cong G^*_n \times G^n$ be $\mathcal{N}$-generic for $\mathbb{P}$. Following **Theorem 3** of [AC13], each G^*_n is $\mathcal{N}$-generic for $\mathbb{P}^*_n$.
   - Let $\mathcal{M}$ be the least model of ZF extending $\mathcal{N}$ containing each G^*_n, but not $\langle G^*_n : n < \omega \rangle$ as constructed in **Theorem 3** of [AC13].

Following the proof of **Theorem 3** of [AC13], we have the following in $\mathcal{M}$.

(1) Since $G^*_n \in \mathcal{M}$ for each $n < \omega$, we have $\aleph_\omega \geq \kappa$ and hence $\aleph_{\omega+1} \geq (\kappa^+)^{\mathcal{N}}$ in $\mathcal{M}$.
(2) If x is a set of ordinals in $\mathcal{M}$, then $x \in \mathcal{N}[G^*_n]$ for some $n < \omega$ (see **Lemma 6** of [AC13]).
(3) Since $\mathcal{N}$ and V contain the same bounded subsets of κ, and $V \subseteq \mathcal{N}$, $\mathbb{P}^*_n$ can be well-ordered in both V and $\mathcal{N}$ with order type less than κ. Therefore, cardinals and cofinalities greater than or equal to κ are preserved.
(4) $\kappa = \aleph_\omega$ and $(\kappa^+)^{\mathcal{N}} = \aleph_{\omega+1}$ are both singular with $\omega \leq cf(\aleph_{\omega+1}) < \aleph_\omega$.
(5) There is a sequence of distinct subsets of $\aleph_\omega$ of length θ.

□

7. Infinitary Chang conjecture from a measurable cardinal

In this section, we prove **Theorem 1.9** and **Theorem 1.10**. In particular, first we observe an infinitary Chang conjecture in a symmetric extension in terms of $\langle \mathbb{P}, \mathcal{G}, \mathcal{I} \rangle$ triple, which is similar to the model constructed in **Theorem 11** of [AK06], except we construct a finite support product construction as in **section 5**. Secondly, we observe an infinite Chang conjecture in Apter and Koepke's model from **Theorem 11** of [AK06].

7.1. Infinitary Chang Conjecture.

We define a set of good indiscernibles, Erdős like partition property, infinitary Chang conjecture and state the relevant lemmas. We recall the required definitions and Lemmas from **Chapter 3** of [Dim11]. For the sake of our convenience we denote a structure $\mathcal{A}$ on domain A as $\mathcal{A} = \langle A, ... \rangle$.

Definition 7.1. ***(Set of good indiscernibles, Definition 3.2 of*** [Dim11]***).*** *For a structure $\mathcal{A} = \langle A, ... \rangle$ with $A \subseteq Ord$, a set $I \subseteq A$ is a set of indiscernibles if for all $n < \omega$, all n-ary formula ϕ in the language for $\mathcal{A}$ and every $\alpha_1, ..., \alpha_n, \alpha'_1, ..., \alpha'_n$ in I, if $\alpha_1 < ... < \alpha_n$ and $\alpha'_1 < ... < \alpha'_n$ then*

$$\mathcal{A} \models \phi(\alpha_1, ...\alpha_n) \text{ if and only if } \mathcal{A} \models \phi(\alpha'_1, ...\alpha'_n).$$

The set I is a set of good indiscernibles if and only if it is a set of indiscernibles and we allow parameters that lie below $min\{\alpha_1, ..., \alpha_n, \alpha'_1, ...\alpha'_n\}$ i.e., if for all $x_1, ...x_m \in A$ such that $x_1, ...x_m \leq min\{\alpha_1, ...\alpha_n, \alpha'_1, ...\alpha'_n\}$ and every $(n + m)$-ary formula, then

$$\mathcal{A} \models \phi(x_1, ...x_m, \alpha_1, ...\alpha_n) \text{ if and only if } \mathcal{A} \models \phi(x_1, ..., x_m, \alpha'_1, ...\alpha'_n).$$

Definition 7.2. ***(α-Erdős cardinal and Erdős-like Partition Property, Definition 3.7 of*** [Dim11]***).*** *The partition relation $\alpha \to (\beta)^\gamma_\delta$ for ordinals $\alpha, \beta, \gamma, \delta$ means for all $f : [\alpha]^\gamma \to \delta$ there is a $X \in [\alpha]^\beta$ such that X is homogeneous for f. For infinite ordinal α, the α-Erdős cardinal $\kappa(\alpha)$ is the least κ such that $\kappa \to (\alpha)^{<\omega}_2$. For cardinals $\kappa > \lambda$ and ordinal $\theta < \kappa$ we mean $\kappa \to^\theta (\lambda)^{<\omega}_2$ if for every first order structure $\mathcal{A} = \langle \kappa, ...\rangle$ with a countable language, there is a set $I \in [\kappa\backslash\theta]^\lambda$ of good indiscernibles for $\mathcal{A}$.*

Definition 7.3. ***(Infinitary Chang conjecture, Definition 3.10 of*** [Dim11]***).*** *Infinitary Chang conjecture is the statement $(\kappa_n)_{n\in\omega} \twoheadrightarrow (\lambda_n)_{n\in\omega}$ which means for every structure $\mathcal{A} = \langle \cup\kappa_n, ...\rangle$ there is an elementary substructure $\mathcal{B} \prec \mathcal{A}$ with domain B and cardinality $\cup\lambda_i$ such that for every $n \in \omega$, $|B \cap \kappa_n| = \lambda_n$.*

Definition 7.4. ***(Definition 3.14 of*** [Dim11]***).*** *Let $\langle \kappa_i : i < \omega\rangle$ and $\langle \lambda_i : 0 < i < \omega\rangle$ be two increasing sequence of cardinals such that $\kappa = \cup_{i<\omega}\kappa_i$. We say $\langle \kappa_i : i < \omega\rangle$ is a coherent sequence of cardinals with the property $\kappa_{i+1} \to^{\kappa_i} (\lambda_{i+1})^{<\omega}_2$ if and only if for every structure $\mathcal{A} = \langle \kappa, ...\rangle$ with a countable language there is a $\langle \lambda_i : 0 < i < \omega\rangle$-coherent sequence of good indiscernibles for $\mathcal{A}$ with respect to $\langle \kappa_i : i < \omega\rangle$.*

Lemma 7.5. ***(Corollary 3.15 of*** [Dim11]***).*** *(ZF) Let $\langle \kappa_i : i < \omega\rangle$ and $\langle \lambda_i : 0 < i < \omega\rangle$ be two increasing sequence of cardinals such that $\kappa = \cup_{i<\omega}\kappa_i$. If $\{\kappa_i : i < \omega\}$ is a coherent sequence of cardinals with the property $\kappa_{i+1} \to^{\kappa_i} (\lambda_{i+1})^{<\omega}_2$ then the Chang Conjecture $(\kappa_n)_{n\in\omega} \twoheadrightarrow (\lambda_n)_{n\in\omega}$ holds.*

Lemma 7.6. ***(Proposition 3.50 of*** [Dim11]***).*** *Let us assume that $V \models$ ZFC+ '$\kappa = \kappa(\lambda)$ exists', $\mathbb{P}$ is a partial order such that $|\mathbb{P}| < \kappa$ and $\mathbb{Q}$ is a partial order that doesn't add subsets to κ. If G is $\mathbb{P} \times \mathbb{Q}$ generic then for every $\theta < \kappa$, $V[G] \models \kappa \to^\theta (\lambda)^{<\omega}_2$.*

Lemma 7.7. ***(Lemma 3.52 of*** [Dim11]***).*** *Let $\langle \kappa_i : i < \omega\rangle$ and $\langle \lambda_i : 0 < i < \omega\rangle$ be two increasing sequence of cardinals such that $\langle \kappa_i : 0 < i < \omega\rangle$ is a coherent sequence of Erdős cardinals with respect to $\langle \lambda_i : 0 < i < \omega\rangle$. If $\mathbb{P}_1$ is a partial order of cardinality $< \kappa_1$ and G is V-generic over $\mathbb{P}_1$, then in $V[G]$, $\langle \kappa_n : n < \omega\rangle$ is a coherent sequence of cardinals with the property $\kappa_{n+1} \to^{\kappa_n} (\lambda_{n+1})^{<\omega}_2$.*

Proof. ***(Theorem 1.9).***

(1) **Defining ground model (V).** Let κ be a measurable cardinal in a model V' of ZFC. By Prikry forcing it is possible to make κ singular with cofinality ω where an end segment $\langle \kappa_i : 1 \leq i < \omega\rangle$ of the Prikry sequence $\langle \delta_i : 1 \leq i < \omega\rangle$ is a *coherent* sequence of Ramsey cardinals by **Theorem 3** of [AK06]. Now Ramsey cardinals κ_i are exactly the κ_i-Erdős cardinals. Thus we obtain a generic extension (say V) where $\langle \kappa_i : 1 \leq i < \omega\rangle$ is a coherent sequence of cardinals with supremum κ such that for all $1 \leq i < \omega$, $\kappa_i = \kappa(\kappa_i)$. We define the following cardinals.
   (a) $\kappa'_0 = \omega$ and $\kappa_0 = \aleph_\omega$.
   (b) $\kappa'_1 = \aleph_{\omega+1}$.
   (c) $\kappa'_i = \kappa^{+\omega_{i-1}+1}_{i-1}$for each $1 < i < \omega$.

(2) **Defining a triple $\langle \mathbb{P}, \mathcal{G}, \mathcal{I}\rangle$.** We consider a triple similar to the one constructed in **section 5**.
   - Let $\mathbb{P} = \Pi_{i<\omega}\mathbb{P}_i$ be the Easton support product of $\mathbb{P}_i = Fn(\kappa'_i, \kappa_i, \kappa'_i)$ ordered componentwise where for each $0 < i < \omega$, $Fn(\kappa'_i, \kappa_i, \kappa'_i)$= $\{p : \kappa'_i \rightharpoonup \kappa_i : |p| < \kappa'_i$ and p is an injection$\}$ ordered by reverse inclusion. Also $p : \kappa'_i \rightharpoonup \kappa_i$ is denoted as a partial function from κ'_i to κ_i.
   - $\mathcal{G} = \Pi_{i<\omega}\mathcal{G}_i$ where for each $i < \omega$, $\mathcal{G}_i$ is the full permutation group of κ_i that can be extended to $\mathbb{P}_i$ by permuting the range of its conditions, i.e., for all $a \in \mathcal{G}_i$ and $p \in \mathbb{P}_i$, $a(p) = \{(\psi, a(\beta)) : (\psi, \beta) \in p\}$.
   - For $m \in \omega$ and $e = \{\alpha_1, ...\alpha_m\}$ a sequence of ordinals such that for each $1 \leq i \leq m$, there is a distinct $\epsilon_i < \omega$ such that $\alpha_i \in (\kappa'_{\epsilon_i}, \kappa_{\epsilon_i})$, we define $E_e = \{\langle \emptyset, ..., p_{\epsilon_1} \cap (\kappa'_{\epsilon_1} \times \alpha_1), \emptyset, ...p_{\epsilon_2} \cap (\kappa'_{\epsilon_2} \times \alpha_2), \emptyset, ...p_{\epsilon_m} \cap (\kappa'_{\epsilon_m} \times \alpha_m), \emptyset, ...\rangle; \overrightarrow{p} \in \mathbb{P}\}$ and $\mathcal{I} = \{E_e : e \in \Pi^{fin}_{i<\omega}(\kappa'_i, \kappa_i)\}$.

(3) **Defining symmetric extension of V.** Let $\mathcal{I}$ generate a normal filter $\mathcal{F}_{\mathcal{I}}$ over $\mathcal{G}$. Let G be a $\mathbb{P}$-generic filter. We consider the symmetric model $V(G)^{\mathcal{F}_I}$. We denote $V(G)^{\mathcal{F}_I}$ by $V(G)$ for the sake of convenience.

Since the forcing notions involved are weakly homogeneous, the following holds.

Lemma 7.8. *If $A \in V(G)$ is a set of ordinals, then $A \in V[G \cap E_e]$ for some $E_e \in \mathcal{I}$.*

Following the arguments in **Lemma 1.35** of [Dim11], we can see that in $V(G)$, $(\kappa'_i)^+ = \kappa_i$ for every $i < \omega$. Similar to the arguments from the proof of **Theorem 11** of [AK06], it is possible to see that in $V(G)$, $\kappa = \aleph_{(\aleph_\omega)^V}$ and $(\aleph_\omega)^V = \aleph_1$. Consequently $\kappa = \aleph_{\omega_1}$ and $cf(\kappa) = \omega$ in $V(G)$. Further ω_1 is singular in $V(G)$. Following **Fact 2.12**, AC_ω fails in $V(G)$. We prove that an infinitary Chang conjecture holds in $V(G)$.

Lemma 7.9. *In $V(G)$, an infinitary Chang conjecture holds.*

Proof. Let $\mathcal{A} = \langle \kappa, ...\rangle$ be a structure in a countable language in $V(G)$. Let $\{\phi_n : n < \omega\}$ be an enumeration of the formulas of the language of $\mathcal{A}$ such that each ϕ_n has $k(n) \leq n$ many free variables. Define $f : [\kappa]^{<\omega} \to 2$ by,

$f(\epsilon_1, ...\epsilon_n) = 1$ if and only if $\mathcal{A} \models \phi_n(\epsilon_1, ..., \epsilon_{k(n)})$ and $f(\epsilon_1, ...\epsilon_n) = 0$ otherwise.

By **Lemma 7.8**, there is a $E_e \in \mathcal{I}$ such that $f \in V[G \cap E_e]$. Fix an arbitrary $1 \leq i < \omega$. We can write $V[G \cap E_e]$=$V[G_1][G_2]$ where G_1 is $\mathbb{Q}_1$-generic over V such that $|\mathbb{Q}_1| < \kappa_i$, and G_2 is $\mathbb{Q}_2$-generic over $V[G_1]$ such that G_2 adds no subsets of κ_i. Consequently, by **Lemma 7.6**, $\kappa_i \to^{\kappa_{i-1}} (\kappa_i)^{<\omega}_2$ in $V[G \cap E_e]$. So, for all $1 \leq i < \omega$, $\kappa_i \to^{\kappa_{i-1}} (\kappa_i)^{<\omega}_2$ in $V[G \cap E_e]$.

Let $e = \{\alpha_1, ..., \alpha_m\}$ where for each $i \in \{1, ..., m\}$, there is a dictinct ϵ_i such that $\alpha_i \in (\kappa'_{\epsilon_i - 1}, \kappa_{\epsilon_i})$. Consider j to be $max\{\epsilon_i : \alpha_i \in e\}$. If $G \cap E_e$ is $\mathbb{P}$-generic over V then since $|\mathbb{P}| < \kappa_j$, by **Lemma 7.7**, $\langle \kappa_i : j \leq i < \omega\rangle$ is a coherent sequence of cardinals with the property $\kappa_i \to^{\kappa_{i-1}} (\kappa_i)^{<\omega}_2$ for all $j \leq i < \omega$. By **Definition 7.4**, there is a $\langle \kappa_i : j \leq i < \omega\rangle$-coherent sequence $\langle A_n : j \leq n < \omega\rangle$ of good indiscernibles for $\mathcal{A}$ with respect to $\langle \kappa_i : j - 1 \leq i < \omega\rangle$. We obtain a $\langle \kappa_i : j - 1 \leq i < \omega\rangle$-coherent sequence $\langle A_n : j - 1 \leq n < \omega\rangle$ of good indiscernibles for $\mathcal{A}$ with respect to $\langle \kappa_i : j - 2 \leq i < \omega\rangle$ as follows.

- Since $\kappa_{j-1} \to^{\kappa_{j-2}} (\kappa_{j-1})^{<\omega}_2$, we obtain a set $A_{j-1} \in [\kappa_{j-1} \backslash \kappa_{j-2}]^{\kappa_{j-1}}$ of indiscernibles for $\mathcal{A}$ with respect to parameters below κ_{j-2}. Consequently, we obtain a $\langle \kappa_i : j-1 \leq i < \omega\rangle$-coherent sequence $\langle A_n : j - 1 \leq n < \omega\rangle$ of good indiscernibles for $\mathcal{A}$ with respect to $\langle \kappa_i : j - 2 \leq i < \omega\rangle$.

If we continue in this manner step by step for the remaining cardinals $\kappa_1, ...\kappa_{j-2}$, then since $\kappa_i \to^{\kappa_{i-1}} (\kappa_i)^{<\omega}_2$ for each $1 \leq i \leq j - 2$, we can obtain a $\langle \kappa_i : 0 < i < \omega\rangle$-coherent sequence $A = \langle A_n : 0 < n < \omega\rangle$ of good indiscernibles for $\mathcal{A}$ with respect to $\langle \kappa_i : i < \omega\rangle$ and $A \in V[G \cap E_e] \subseteq V(G)$. Therefore for all $1 \leq i < \omega$, $\kappa_i \to^{\kappa_{i-1}} (\kappa_i)^{<\omega}_2$ and $\langle \kappa_i : 1 \leq i < \omega\rangle$ is a coherent sequence of cardinals in $V(G)$ by **Definition 7.4**. Using **Lemma 7.5**, we can obtain an infinitary Chang conjecture in $V(G)$ as **Lemma 7.5** can be proved in ZF. □

□

*Proof. **(Theorem 1.10).*** Let $\mathcal{N}$ be the symmetric inner model constructed in **Theorem 11** of [AK06]. We first translate the arguments in terms of a symmetric extension based on a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$.

- Consider $\mathbb{P}$ and $\mathcal{G}$ as mentioned in the previous construction (used for proving **Theorem 1.9**).
- Let $\mathcal{I} = \{E_e : e \in \Pi_{i<\omega}(\kappa'_i, \kappa_i)\}$ where for every $e = \{\alpha_i : i < \omega\} \in \Pi_{i\in\omega}(\kappa'_i, \kappa_i)$, $E_e = \{\langle p_i \cap (\kappa'_i \times \alpha_i) : i < \omega\rangle : \overrightarrow{p} \in \mathbb{P}\}$. Let $\mathcal{I}$ generate a normal filter $\mathcal{F}_{\mathcal{I}}$ over $\mathcal{G}$. We define $\mathcal{F}$ to be $\mathcal{F}_{\mathcal{I}}$.

Let G be a $\mathbb{P}$-generic filter. We consider the symmetric model $V(G)^{\mathcal{F}}$. We denote $V(G)^{\mathcal{F}}$ by $V(G)$ for the sake of convenience. The model $V(G)$ is analogous to the symmetric inner model $\mathcal{N}$ constructed in **Theorem 11** of [AK06]. Since the forcing notions involved are weakly homogeneous, the following holds.

Lemma 7.10. *If $A \in V(G)$ is a set of ordinals, then $A \in V[G \cap E_e]$ for some $E_e \in \mathcal{I}$.*

Similar to **Lemma 7.9**, we observe an infinite Chang conjecture in $V(G)$.

Lemma 7.11. *In $V(G)$, an infinite Chang conjecture holds .*

Proof. Let $\mathcal{A} = \langle \kappa, ... \rangle$ be a structure in a countable language in $V(G)$. Let $\{\phi_n : n < \omega\}$ be an enumeration of the formulas of the language of $\mathcal{A}$ such that each ϕ_n has $k(n) \leq n$ many free variables. Define $f : [\kappa]^{<\omega} \to 2$ by,

$f(\epsilon_1, ...\epsilon_n) = 1$ if and only if $\mathcal{A} \models \phi_n(\epsilon_1, ..., \epsilon_{k(n)})$ and $f(\epsilon_1, ...\epsilon_n) = 0$ otherwise.

By **Lemma 7.10**, there is a $E_e \in \mathcal{I}$ such that $f \in V[G \cap E_e]$. Fix an arbitrary $1 \leq i < \omega$. We can write $V[G \cap E_e]$=$V[G_1][G_2]$ where G_1 is $\mathbb{Q}_1$-generic over V such that $|\mathbb{Q}_1| < \kappa_i$, and G_2 is $\mathbb{Q}_2$-generic over $V[G_1]$ such that G_2 adds no subsets of κ_i. Consequently, by **Lemma 7.6**, $\kappa_i \to^{\kappa_{i-1}} (\kappa_i)^{<\omega}_2$ in $V[G \cap E_e]$. So, for all $1 \leq i < \omega$, $\kappa_i \to^{\kappa_{i-1}} (\kappa_i)^{<\omega}_2$ in $V[G \cap E_e]$. Thus by **Definition 7.2**, we obtain a set $A_i \in [\kappa_i \backslash \kappa_{i-1}]^{\kappa_i}$ of good indiscernibles for $\mathcal{A}$ for each $1 \leq i < \omega$, in $V[G \cap E_e]$. Consequently, we obtain a $\langle \kappa_i : 0 < i < \omega \rangle$-coherent sequence $A = \langle A_i : 0 < i < \omega \rangle$ of good indiscernibles for $\mathcal{A}$ with respect to $\langle \kappa_i : i < \omega \rangle$ and $A \in V[G \cap E_e] \subseteq V(G)$. Therefore for all $1 \leq i < \omega$, $\kappa_i \to^{\kappa_{i-1}} (\kappa_i)^{<\omega}_2$ and $\langle \kappa_i : 1 \leq i < \omega \rangle$ is a coherent sequence of cardinals in $V(G)$ by **Definition 7.4**. Using **Lemma 7.5**, we can obtain an infinitary Chang conjecture in $V(G)$ as **Lemma 7.5** can be proved in ZF. □

Applying **Theorem 4** of [AK06] and **Proposition 1** of [AK08], we prove that $\aleph_{\omega_1}$ is an almost Ramsey cardinal in $V(G)$.

Lemma 7.12. *In $V(G)$, $\aleph_{\omega_1}$ is an almost Ramsey cardinal.*

Proof. Following the terminologies from the proof of **Theorem 11** of [AK06], $\kappa = \aleph_{\omega_1}$ in $V(G)$. We show κ is an almost Ramsey cardinal in $V(G)$. Let $f : [\kappa]^{<\omega} \to 2$ be in $V(G)$. Since f can be coded by a subset of κ, $f \in V[G \cap E_e]$ for some $E_e \in \mathcal{I}$ by **Lemma 7.10**. Now, in V, κ is the supremum of a coherent sequence of Ramsey cardinals $\langle \kappa_i : i < \omega \rangle$. By **Theorem 4** of [AK06], we can see that $\langle \kappa_i : i < \omega \rangle$ stays a coherent sequence of Ramsey cardinals in $V[G \cap E_e]$. Also κ is the supremum of $\langle \kappa_i : i < \omega \rangle$ in $V[G \cap E_e]$. Thus κ is an almost Ramsey cardinal in $V[G \cap E_e]$ by **Proposition 1** of [AK08]. Thus for all $\beta < \kappa$, there is a set $X_\beta \in V[G \cap E_e] \subseteq V(G)$ which is homogeneous for f and has order type at least β. Hence, κ is almost Ramsey in $V(G)$ since f was arbitrary. □

□

8. Mutually stationary property from a sequence of measurable cardinals

Let κ be a cardinal. $C \subseteq \kappa$ is a *club set* if it is closed and unbounded. $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for every club C. We recall the definition of *mutually stationary sets* from Foreman–Magidor [MF01] and a theorem due to Foreman and Magidor.

Definition 8.1. ***(Mutually Stationary Sets, Definition 1.1 of*** [Apt04]***).*** *Let $\mathcal{K}$ be a set of regular cardinals with supremum λ. Suppose $S_\kappa \subseteq \kappa$ for all $\kappa \in \mathcal{K}$. Then $\langle S_\kappa : \kappa \in \mathcal{K} \rangle$ is mutually stationary if and only if for all algebras $\mathcal{A}$ on λ, there is an elementary substructure $\mathcal{B} \prec \mathcal{A}$ such that for all $\kappa \in \mathcal{B} \cap \mathcal{K}$, sup($\mathcal{B} \cap \kappa$) $\in S_\kappa$.*

Theorem 8.2. ***(Theorem 5.2 of*** [CFM06]***).*** *Let $\langle \kappa_i : i < \delta \rangle$ be an increasing sequence of measurable cardinals, where $\delta < \kappa_0$ is a regular cardinal. Let $S_i \subseteq \kappa_i$ be stationary for each $i < \delta$. It is then the case that $\langle S_i : i < \delta \rangle$ is mutually stationary.*

8.1. **Mutually Stationary property from a sequence of measurable cardinals.** It is not a theorem in ZFC, that if $\mathcal{K}$ consists of an increasing sequence of regular cardinals and for each $\kappa \in \mathcal{K}$, $S_\kappa \subseteq \kappa$ is stationary in κ, then $\langle S_\kappa : \kappa \in \mathcal{K}\rangle$ is mutually stationary. In particular, in L, by **Theorem 24** of [MF01], there is a sequence of stationary sets $\langle S_n : 1 < n < \omega\rangle$ such that $S_n \subseteq \aleph_n$, S_n is stationary and consists of points having cofinality $\aleph_1$, yet $\langle S_n : 1 < n < \omega\rangle$ is not mutually stationary. Foreman and Magidor asked[15] whether it is possible to construct a model of ZFC where if $\langle S_n : 1 \leq n < \omega\rangle$ is such that each S_n is stationary on $\aleph_n$, then $\langle S_n : 1 \leq n < \omega\rangle$ is mutually stationary. Starting from an ω-sequence of supercompact cardinals, Shelah constructed a model of ZFC in **section 6** of [CFM06], where if we define the sequence of stationary sets as follows,

$$S^f_n = \{\alpha < \aleph_n : cf(\alpha) = \aleph_{f(n)}\} \text{ if } n > 1 \text{ and } f : \omega \rightarrow 2 \text{ is an arbitrary function.}$$

then the sequence $\langle S^f_n : 1 < n < \omega\rangle$ is mutually stationary. In [Apt04], Apter gave a complete answer to the aforementioned question of Foreman and Magidor in a choiceless context. Specifically, Apter constructed a symmetric inner model preserving DC_ω, from a ω-sequence of supercompact cardinals where if $\langle S_n : 1 \leq n < \omega\rangle$ is a sequence of stationary sets such that $S_n \subseteq \aleph_n$, then $\langle S_n : 1 \leq n < \omega\rangle$ is mutually stationary.

We recall the symmetric inner model from **Theorem 1** of [Apt83a] and recall the terminologies from [Apt83a]. In particular we fix an arbitrary $n_0 \in \omega$ and assume an increasing sequence of measurable cardinals $\langle \chi_k : k < \omega\rangle$ in a ground model V of ZFC. Then we consider the symmetric inner model constructed in **Theorem 1** of [Apt83a]. For the sake of convenience we call the symmetric model $\mathcal{N}_{n_0}$.

Proof. ***(Observation 1.11).*** We note that in $\mathcal{N}_{n_0}$, $\chi_k = \aleph_{n_0+2(k+1)}$ for each $k < \omega$.

(1) Following **Lemma 1.36** of [Dim11], each $\aleph_{n_0+2(k+1)}$ is a measurable cardinal in $\mathcal{N}_{n_0}$, for each $1 \leq k < \omega$. Following **Lemma 4.3** of [KH19], for each $1 \leq k < \omega$, there are no uniform ultrafilters on $\aleph_{n_0+2k}$ in $\mathcal{N}_{n_0}$. Consequently for each $1 \leq k < \omega$, $\aleph_{n_0+2k}$ can not be a measurable cardinal in $\mathcal{N}_{n_0}$.

(2) We observe that in the symmetric model $\mathcal{N}_{n_0}$ from **Theorem 1** of [Apt83a], if $\langle S_k : 1 \leq k < \omega\rangle$ is a sequence of stationary sets such that $S_k \subseteq \chi_k$ for every $1 \leq k < \omega$, then $\langle S_k : 1 \leq k < \omega\rangle$ is mutually stationary. Suppose $\mathcal{N}_{n_0} \models \langle S_k : 1 \leq k < \omega\rangle$ is a sequence of stationary sets such that $S_k \subseteq \chi_k$ for every $1 \leq k < \omega$. Since $\langle S_k : 1 \leq k < \omega\rangle$ can be coded by set of ordinals, by **Lemma 1.1** of [Apt83a], there exists some $f \in K$ for which $\langle S_k : 1 \leq k < \omega\rangle \in V[G \restriction f]$.

Following **Lemma 1.3** of [Apt83a], χ_k remains measurable in $V[G \restriction f]$ for every $1 \leq k < \omega$. We can observe that S_k is a stationary subset of χ_k in $V[G \restriction f]$. Let C be any club set of χ_k in $V[G \restriction f]$. Since the notion of club subset of χ_k is upward absolute and $V[G \restriction f] \subseteq \mathcal{N}_{n_0}$, C is also a club set of χ_k in $\mathcal{N}_{n_0}$. Since in $\mathcal{N}_{n_0}, S_k$ is a stationary subset of χ_k we have $S_k \cap C \neq \emptyset$. By **Theorem 8.2**, $\langle S_k : 1 \leq k < \omega\rangle$ is mutually stationary in $V[G \restriction f]$.

We note that following **Lemma 1.2** of [Apt83a], if $\lambda = \cup_{k\in\omega}\chi_k$, then $\lambda = \aleph_\omega$ in $\mathcal{N}_{n_0}$. Thus for algebras $\mathcal{A}$ on λ, there is an elementary substructure $\mathcal{B} \prec \mathcal{A}$ in $V[G \restriction f]$ such that for all $k < \omega, sup(\mathcal{B} \cap \chi_k) \in S_k$. Thus there is an elementary substructure $\mathcal{B} \prec \mathcal{A}$ in $\mathcal{N}_{n_0}$ such that for all $k < \omega, sup(\mathcal{B} \cap \chi_k) \in S_k$. Hence in $\mathcal{N}_{n_0}$, $\langle S_k : 1 \leq k < \omega\rangle$ is mutually stationary.

(3) By **Lemma 1.2** of [Apt83a], if $\lambda = \cup_{n<\omega}\chi_n$, then $\lambda = \aleph_\omega$ in $\mathcal{N}_{n_0}$. We can see that λ is an almost Ramsey cardinal in $\mathcal{N}_{n_0}$ by a well-known argument from **Lemma 2.5** of [ADK16]. For reader's convenience, we provide a sketch of the proof. Let $f : [\lambda]^{<\omega} \rightarrow 2$ be in $\mathcal{N}_{n_0}$. Since f can be coded by a set of ordinals, $f \in V[G \restriction f]$ for some $f \in K$ by **Lemma 1.1** of [Apt83a]. Following **Lemma 1.3** of [Apt83a], χ_k remains measurable in $V[G \restriction f]$ for every $1 \leq k < \omega$. Consequently, χ_k is Ramsey in $V[G \restriction f]$ for every

[15]in page 290 of [MF01].

$1 \leq k < \omega$. Now, in $V[G \restriction f]$, λ is the supremum of Ramsey cardinals $\langle \chi_i : 1 \leq i < \omega \rangle$. Thus λ is an almost Ramsey cardinal in $V[G \restriction f]$ by **Proposition 1** of [AK08]. Thus for all $\beta < \lambda$, there is a set $X_\beta \in V[G \restriction f] \subseteq \mathcal{N}_{n_0}$ which is homogeneous for f and has order type at least β. Hence, λ is almost Ramsey in $\mathcal{N}_{n_0}$ since f was arbitrary.

□

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