Chapter

Existence Results for Boundary Value Problem of Nonlinear Fractional Differential Equation

Noureddine Bouteraa and Habib Djourdem

Abstract

In this chapter, we investigate the existence and uniqueness of solutions for class of nonlinear fractional differential equations with nonlocal boundary conditions. The existence results are obtained by using Leray-Schauder nonlinear alternative and Banach contraction principle. An illustrative example is presented at the end to illustrated the validity of our results.

Keywords: fractional differential equations, existence, nonlocal boundary, fixed-point theorem

1. Introduction

In this chapter, we are interested in the existence of solutions for nonlinear fractional difference equations

$$ {}^cD^{\alpha}_{0^+}u(t) - A\,{}^cD^{\beta}_{0^+}u(t) = f\left(t, u(t), {}^cD^{\beta}_{0^+}u(t), {}^cD^{\alpha}_{0^+}u(t)\right), \quad t \in J = [0, T], \tag{1} $$

subject to the three-point boundary conditions

$$ \begin{cases} \lambda u(0) - \mu u(T) - \gamma u(\eta) = d, \\ \lambda u'(0) - \mu u'(T) - \gamma u'(\eta) = l, \end{cases} \tag{2} $$

where $T > 0$, $0 \leq \eta \leq T$, $\lambda \neq \mu + \gamma$, d, l, λ, μ, $\gamma \in \mathbb{R}$, $\beta + 1 < \alpha$, A is an $\mathbb{R}^{n\times n}$ matrix and ${}^cD^{\alpha}_{0^+}$, ${}^cD^{\beta}_{0^+}$ are the Caputo fractional derivatives of order $1 < \alpha \leq 2$, $0 < \beta \leq 1$, respectively.

The first definition of fractional derivative was introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus, was mentioned already in 1695 by Leibniz and L'Hospital. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For more details we refer the reader to [1–5] and the references cited therein.

Fractional differential equation theory have recieved increasing attention. This theory has been developed very quickly and attracted a considerable interest from researches in this field, which revealed the flexibility of fractional calculus theory in designing various mathematical models. The main methods conducted in this articles are by terms of fixed point techniques [6]. Boundary value problems for nonlinear differential equations arise in a variety of areas of applied mathematics, physics and variational problems of control theory. A point of central importance in the study of nonlinear boundary value problems is to understand how the properties of nonlinearity in a problem influence the nature of the solutions to the boundary value problems. The multi-point boundary conditions are important in various physical problems of applied science when the controllers at the end points of the interval (under consideration) dissipate or add energy according to the sensors located, at intermediate points, see [7, 8] and the references therein. We quote also that realistic problems arising from economics, optimal control, stochastic analysis can be modeled as differential inclusion. The study of fractional differential inclusions was initiated by EL-Sayad and Ibrahim [9]. Also, recently, several qualitative results for fractional differential inclusion were obtained in [10–13] and the references therein.

The techniques of nonlinear analysis, as the main method to deal with the problems of nonlinear differential equations (DEs), nonlinear fractional differential equations (FDEs), nonlinear partial differential equations (PDEs), nonlinear fractional partial differential equations (FPDEs), nonlinear stochastic fractional partial differential equations (SFPDEs), plays an essential role in the research of this field, such as establishing the existence, uniqueness and multiplicity of solutions (or positive solutions) and mild solutions for nonlinear of different kinds of FPDEs, FPDEs, SFPDEs, inclusion differential equations and inclusion fractional differential equations with various boundary conditions, by using different techniques (approaches). For more details, see [14–36] and the references therein. For example, iterative method is an important tool for solving linear and nonlinear boundary value problems. It has been used in the research areas of mathematics and several branches of science and other fields. However, many authors showed the existence of positive solutions for a class of boundary value problem at resonance case. Some recent devolopment for resonant case can be found in [37, 38]. Let us cited few papers. In [39], the authors studied the boundary value problems of the fractional order differential equation:

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \;\; D^{\beta}_{0+}u(1) = aD^{\beta}_{0+}u(\eta), \end{cases}$$

where $1 < \alpha \le 2, \;\; 0 < \eta < 1, \;\; 0 < a, \; \beta < 1, \; f \in C\big([0,1] \times \mathbb{R}^2, \mathbb{R}\big)$ and $D^{\alpha}_{0+}, \;\; D^{\beta}_{0+}$ are the standard Riemann-Liouville fractional derivative of order α. They obtained the multiple positive solutions by the Leray-Schauder nonlinear alternative and the fixed point theorem on cones.

In 2017, Resapour et al. [40] investigated a Caputo fractional inclusion with integral boundary condition for the following problem

$$\begin{cases} {}^cD^{\alpha}u(t) \in F\left(t, u(t), {}^cD^{\beta}u(t), u'(t)\right), \\ u(0) + u'(0) + {}^cD^{\beta}u(0) = \int_0^{\eta} u(s)ds, \\ u(1) + u'(1) + {}^cD^{\beta}u(1) = \int_0^{\nu} u(s)ds, \end{cases}$$

where $1 < \alpha \leq 2,\ \ \eta, \nu, \beta \in (0, 1),\ F : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction and ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order α.

In 2017, Sheng and Jiang [41] studied the existence and uniqueness of the solutions for fractional damped dynamical systems

$$\begin{cases} {}^cD^{\alpha}_{0^+}u(t) - A\,{}^cD^{\beta}_{0^+}u(t) = f(t, u(t)), \quad t \in [0, T], \\ u(0) = u_0, \qquad u'(0) = u'_0, \end{cases}$$

where $0 < \beta \leq 1 < \alpha \leq 2,\ 0 < T < \infty,\ u \in \mathbb{R}^n,\ A$ is an $\mathbb{R}^{n \times n}$ matrix, $f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ jointly continuous function and ${}^cD^{\alpha}_{0^+}$, ${}^cD^{\beta}_{0^+}$ are the Caputo derivatives of order $\alpha,\ \beta$, respectively.

In 2018, Abbes et al. [42] studied the existence and uniqueness of the solutions for fractional damped dynamical systems

$$\begin{cases} {}^cD^{\alpha}_{0^+}u(t) - A\,{}^cD^{\beta}_{0^+}u(t) = f\left(t, u(t), {}^cD^{\beta}_{0^+}u(t), {}^cD^{\alpha}_{0^+}u(t)\right), \quad t \in [0, T], \\ u(0) = u(T), \qquad u'(0) = u'(T), \end{cases}$$

where $0 < \beta \leq 1 < \alpha \leq 2,\ 0 < T < \infty,\ u \in \mathbb{R}^n,\ A$ is an $\mathbb{R}^{n \times n}$ matrix and $f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ jointly continuous.

In 2019, Tao Zhu [43] studied the existence and uniqueness of positive solutions of the following fractional differential equations

$$\begin{cases} D^{\alpha}_{0^+}u(t) - A\,{}^cD^{\beta}_{0^+}u(t) = f(t, u(t)), \quad t \in [0, T), \quad 0 < \beta < \alpha < 1, \\ u(0) = u_0. \end{cases}$$

Inspired and motivated by the works mentioned above, we establish the existence results for the nonlocal boundary value problem (1.1)–(1.2) by using Leray-Schauder nonlinear alternative and the Banach fixed point theorem. Note that our work generalized the three works cited above [41–43]. The chapter is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel. In Section 3, deals with main results and we give an example to illustrate our results.

2. Existence and uniqueness results for our problem

2.1 Preliminaries

Let as introduce notations, definitions and preliminary facts that will be need in the sequel. For more details, see for example [44–46].

Definition 2.1. The Caputo fractional derivative of order α for the function $u \in C^n([0, \infty), \mathbb{R})$ is defined by

$$ {}^{c}D_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds. $$

where $\Gamma(\cdot)$ is the Eleur gamma function and $\alpha > 0$, $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α.

Definition 2.2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \to \mathbb{R}$ is given by

$$ I_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds, \quad t > 0. $$

where $\Gamma(\cdot)$ is the Eleur gamma function, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.1. *Let* $u \in AC^n[0, T]$, $n \in \mathbb{N}$ *and* $u(\cdot) \in C[0, T]$. *Then, we have*

$$ {}^{c}D_{0^+}^{\beta}\left(I_{0^+}^{\alpha}u(t)\right) = I_{0^+}^{\alpha-\beta}u(t), $$

$$ I_{0^+}^{\alpha}\left({}^{c}D_{0^+}^{\alpha}u(t)\right) = u(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}u^{(k)}(0), \quad t > 0, \quad n-1 < \alpha < n, $$

Especially, when $1 < \alpha < 2$, *then we have*

$$ I_{0^+}^{\alpha}\left({}^{c}D_{0^+}^{\alpha}u(t)\right) = u(t) - u(0) - tu'(0). $$

Lemma 2.2. ([10]) *Let* $0 < \beta < 1 < \alpha < 2$, *then we have*

$$ I_{0^+}^{\alpha}\left({}^{c}D_{0^+}^{\beta}u(t)\right) = I_{0^+}^{\alpha-\beta}u(t) - \frac{u(0)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}. $$

2.2 Existence results

Let $C(J, \mathbb{R}^n)$ be the Banach space for all continuous function from J into $\mathbb{R}^n$ equipped with the norm

$$ \|u\|_{\infty} = \sup\{\|u(t)\| : t \in J\}, $$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$. Denote $L^1(J, \mathbb{R}^n)$ the Banach space of the measurable functions $u : J \to \mathbb{R}^n$ that are Lebesgue integrable with norm

$$ \|u\|_{L^1} = \int_0^T \|u(t)\|dt. $$

Let $AC(J, \mathbb{R}^n)$ be the Banach space of absolutely continuous valued functions on J and set

$$ AC^n(J) = \left\{u : J \to \mathbb{R}^n : u, u', u'', \ldots, u^{(n-1)} \in C(J, \mathbb{R}^n)\right\} \ \text{and} \ \ u^{(n-1)} \in AC(J, \mathbb{R}^n). $$

By

$$C^1(J) = \{u : J \to \mathbb{R}^n \ where \ \ u' \in C(J, \mathbb{R}^n)\},$$

we denote the Banach space equipped with the norm

$$\|u\|_1 = \max\left\{\|u\|_\infty, \|u'\|_\infty\right\}.$$

For the sake of brevity, we set

$$\begin{aligned}
&\delta = (\lambda - \mu - \gamma)\Gamma(\alpha - \beta + 1) + A\left(\mu T^{\alpha-\beta} + \gamma\eta^{\alpha-\beta}\right), \quad \Delta = \frac{\Gamma(\alpha-\beta+1)}{\delta} \\
&\sigma = A(\alpha - \beta)\left(\mu T^{\alpha-\beta-1} + \gamma\eta^{\alpha-\beta-1}\right), \quad \Lambda = (\lambda - \mu - \gamma) - (\mu T + \gamma\eta)\left(\frac{\sigma}{\delta}\right), \\
&R_1 = \left(1 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)M_1 + TM_2 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \qquad , \\
&R_2 = M_2 + \frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_1 + \frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \\
&M_1 = \Delta\left\{\left[\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T + \gamma\eta) + 1\right]\Phi + \Lambda^{-1}(\mu T + \gamma\eta)\Theta\right\}
\end{aligned}$$

and

$$M_2 = \Lambda^{-1}\left\{\left(\frac{\sigma}{\delta}\right)\Phi + \Theta\right\},$$

with

$$\Phi = \frac{\|A\|}{\Gamma(\alpha-\beta+1)}\left(\mu T^{\alpha-\beta} + \gamma\eta^{\alpha-\beta}\right) + \frac{\left(\mu T^\alpha + \gamma\eta^\alpha\right)\left(L_1\Gamma(2-\beta) + T^{1-\beta}(L_3\|A\| + L_2)\right)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_3)},$$

$$\Theta = \frac{\|A\|}{\Gamma(\alpha-\beta)}\left(\mu T^{\alpha-\beta-1} + \gamma\eta^{\alpha-\beta-1}\right) + \frac{\left(\mu T^{\alpha-1} + \gamma\eta^{\alpha-1}\right)\left(L_1\Gamma(2-\beta) + T^{1-\beta}(L_3\|A\| + L_2)\right)}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_3)}.$$

Lemma 2.3. *Let $y(\cdot) \in C(J, \mathbb{R}^n)$. The function $u(\cdot) \in C^1(J, \mathbb{R}^n)$ is a solution of the fractional differential problem*

$$\begin{cases}
{}^cD^\alpha_{0^+}u(t) - A\,{}^cD^\beta_{0^+}u(t) = y(t), \quad t \in J = [0, T], \\
\lambda u(0) - \mu u(T) - \gamma u(\eta) = d, \\
\lambda u'(0) - \mu u'(T) - \gamma u'(\eta) = l,
\end{cases} \tag{3}$$

if and only if, u is a solution of the fractional integral equation

$$\begin{aligned}
u(t) = \left(1 - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)u(0) + tu'(0) + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}u(s)ds \\
+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,
\end{aligned} \tag{4}$$

with

$$u(0) = \Delta\Big\{\Big[\Lambda^{-1}\Big(\frac{\sigma}{\delta}\Big)(\mu T + \gamma\eta) + 1\Big]\Big[AI_{0^+}^{\alpha-\beta}(\mu u(T) + \gamma u(\eta)) + I_{0^+}^{\alpha}(\mu y(T) + \gamma y(\eta)) + d\Big]$$
$$+\Lambda^{-1}(\mu T + \gamma\eta)\Big[AI_{0^+}^{\alpha-\beta-1}(\mu u(T) + \gamma u(\eta)) + I_{0^+}^{\alpha-1}(\mu y(T) + \gamma y(\eta)) + l\Big]\Big\}, \tag{5}$$

and

$$u'(0) = \Lambda^{-1}\Big\{\Big(\frac{\sigma}{\delta}\Big)\Big[AI_{0^+}^{\alpha-\beta}(\mu u(T) + \gamma u(\eta)) + I_{0^+}^{\alpha}(\mu y(T) + \gamma y(\eta)) + d\Big]$$
$$+\Big[AI_{0^+}^{\alpha-\beta-1}(\mu u(T) + \gamma u(\eta)) + I_{0^+}^{\alpha-1}(\mu y(T) + \gamma y(\eta))\Big] + l\Big\}, \tag{6}$$

where

$$I_{0^+}^{\alpha}u(T) = \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}u(s)ds,$$

$$I_{0^+}^{\alpha}u(\eta) = \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}u(s)ds,$$

$$I_{0^+}^{\alpha-\beta}u(T) = \frac{1}{\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}u(s)ds,$$

$$I_{0^+}^{\alpha-\beta}u(\eta) = \frac{1}{\Gamma(\alpha-\beta)}\int_0^\eta (\eta-s)^{\alpha-\beta-1}u(s)ds.$$

Proof. From Lemmas 2.1 and 2.2, we have

$$u(t) = u(0) + tu'(0) - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}u(0) + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}u(s)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds$$

Applying conditions (2), we obtain (5) and (6).

Conversely, assume that u satisfies the fractional integral (4), and using the facts that ${}^cD_{0^+}^{\alpha}$ is the left inverse of $I_{0^+}^{\alpha}$ and the fact that ${}^cD_{0^+}^{\alpha}C = 0$, where C is a constant, we get

$${}^cD_{0^+}^{\alpha}u(t) - A{}^cD_{0^+}^{\beta}u(t) = f(t, u(t), u'(t)), \quad t \in J = [0, T].$$

Also, we can easily show that

$$\begin{cases} \lambda u(0) - \mu u(T) - \gamma u(\eta) = d, \\ \lambda u'(0) - \mu u'(T) - \gamma u'(\eta) = l. \end{cases}$$

The proof is complete.

To simplify the proofs in the forthcoming theorem, we etablish the bounds for the integrals and the bounds for the term arising in the sequel.

Lemma 2.4. *For $y(\cdot) \in C(J, \mathbb{R}^n)$, we have*

$$\left|I_{0^+}^{\alpha} y(\eta)\right| = \left|\int_0^{\eta} \frac{(\eta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau\right| \leq \frac{\eta^{\alpha}}{\Gamma(\alpha+1)} \|y\|.$$

Proof. Obviously,

$$\int_0^{\eta} \frac{(\eta-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau = \left[-\frac{(\eta-\tau)^{\alpha}}{\alpha\Gamma(\alpha)}\right]_0^{\eta} = \frac{\eta^{\alpha}}{\alpha\Gamma(\alpha)} = \frac{s^{\alpha}}{\Gamma(\alpha+1)}.$$

Hence

$$\left|\int_0^{\eta} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau) d\tau\right| \leq \frac{\eta^{\alpha}}{\Gamma(\alpha+1)} \|y\|.$$

Lemma 2.5. *For $u(\cdot) \in C^1(J, \mathbb{R}^n)$ and $0 < \beta \leq 1$, we have*

$$\left\|{}^c D_{0^+}^{\beta} u(t)\right\|_{\infty} \leq \frac{T^{1-\beta}}{\Gamma(2-\beta)} \|u'\|_{\infty},$$

and, so

$$\left\|{}^c D_{0^+}^{\beta} u(t)\right\|_{\infty} \leq \frac{T^{1-\beta}}{\Gamma(2-\beta)} \|u\|_1.$$

Proof.

Clearly, when $\beta = 1$, the conclusion are true. So, consider the case $0 < \beta < 1$. By Definition 2.1, for each $u(\cdot) \in C^1(J, \mathbb{R}^n)$ and $t \in J$, we have

$$\begin{aligned}
\left|D_{0^+}^{\beta} u(t)\right| &= \frac{1}{\Gamma(1-\beta)} \left|\int_0^t (t-s)^{-\beta} u'(s) ds\right| \\
&\leq \|u'\|_{\infty} \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} ds \\
&= \|u'\|_{\infty} \frac{t^{1-\beta}}{\Gamma(1-\beta)} \\
&\leq \frac{T^{1-\beta}}{\Gamma(1-\beta)} \|u'\|_{\infty} \\
&\leq \frac{T^{1-\beta}}{\Gamma(1-\beta)} \|u'\|_1.
\end{aligned}$$

We need to give the following hypothesis:

(H_1) there existe a constants $L_1,\ L_2 > 0$ and $0 < L_3 < 1$ such that

$$|f(t,u,v,w) - f(t,\overline{u},\overline{v},\overline{w})| \leq L_1\|u-\overline{u}\| + L_2\|v-\overline{v}\| + L_3\|w-\overline{w}\|,$$

for any $u, v, \overline{u}, \overline{v}, \overline{w} \in \mathbb{R}^n$ and $t \in J$.

Now we are in a position to present the first main result of this paper. The existence results is based on Banach contraction principle.

Theorem 1.1. ([47] Banach's fixed point theorem). Let C be a non-empty closed subset of a Banach space E, then any contraction mapping T of C into itself has a unique fixed point.

Theorem 1.2. Assume that (H_1) holds. If

$$\max(R_1, R_2) < 1, \tag{7}$$

then the boundary value problem (1.1)–(1.2) has a unique solution on J.

Proof. We transform the problem (1.1)–(1.2) into fixed point problem. Let $N : C^1(J,\mathbb{R}^n) \to C^1(J,\mathbb{R}^n)$ the operator defined by

$$\begin{aligned}(Nu)(t) = {} & \left(1 - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)B + tD + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}u(s)ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds,\end{aligned} \tag{8}$$

with

$$\begin{aligned}B = \Delta\Big\{ & \Big[\Lambda^{-1}\Big(\frac{\sigma}{\delta}\Big)(\mu T + \gamma\eta) + 1\Big]\Big[AI_{0^+}^{\alpha-\beta}(\mu u(T) + \gamma u(\eta)) + I_{0^+}^{\alpha}(\mu y(T) + \gamma y(\eta)) + d\Big] \\ & + \Lambda^{-1}(\mu T + \gamma\eta)\Big[AI_{0^+}^{\alpha-\beta-1}(\mu u(T) + \gamma u(\eta)) + I_{0^+}^{\alpha-1}(\mu y(T) + \gamma y(\eta)) + l\Big]\Big\},\end{aligned}$$

and

$$\begin{aligned}D = \Lambda^{-1}\Big\{ & \Big(\frac{\sigma}{\delta}\Big)\Big[AI_{0^+}^{\alpha-\beta}(\mu u(T) + \gamma u(\eta)) + I_{0^+}^{\alpha}(\mu g(T) + \gamma g(\eta)) + d\Big] \\ & + \Big[AI_{0^+}^{\alpha-\beta-1}(\mu u(T) + \gamma u(\eta)) + I_{0^+}^{\alpha-1}(\mu g(T) + \gamma g(\eta))\Big] + l\Big\},\end{aligned}$$

where $g \in C(J,\mathbb{R}^n)$ be such that

$$g(t) = f\left(t, u(t), {}^cD_{0^+}^{\beta}u(t), g(t) + A\,{}^cD_{0^+}^{\beta}u(t)\right)$$

For every $u \in C^1(J,\mathbb{R}^n)$ and any $t \in J$, we have

$$\begin{aligned}(Nu)(t) = {} & D - \frac{(\alpha-\beta)At^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}B + \frac{A}{\Gamma(\alpha-\beta-1)}\int_0^t (t-s)^{\alpha-\beta-2}u(s)ds \\ & + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}g(s)ds.\end{aligned} \tag{9}$$

Clearly, the fixed points of operator N are solutions of problem (1.1)–(1.2).
It is clear that $(Nu) \in C(J, \mathbb{R}^n)$, consequently, N is well defined.
Let $u, v \in C(J, \mathbb{R}^n)$. Then for $t \in J$, we have

$$\begin{aligned}\|(Nu)(t) - (Nv)(t)\| \leq & \left(1 + \frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)}\right)\|B - B_1\| + T\|D - D_1\| \\ & + \frac{\|A\|}{\Gamma(\alpha - \beta)} \int_0^T (T - s)^{\alpha-\beta-1} \|u(s) - v(s)\| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} \|g(s) - h(s)\| ds,\end{aligned}$$

with

$$\begin{aligned}B_1 = \Delta\Big\{ & \left[\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T + \gamma\eta) + 1\right]\left[A I_{0^+}^{\alpha-\beta}(\mu v(T) + \gamma v(\eta)) + I_{0^+}^{\alpha}(\mu h(T) + \gamma h(\eta)) + d\right] \\ & + \Lambda^{-1}(\mu T + \gamma\eta)\left[A I_{0^+}^{\alpha-\beta-1}(\mu v(T) + \gamma v(\eta)) + I_{0^+}^{\alpha-1}(\mu h(T) + \gamma h(\eta)) + l\right]\Big\},\end{aligned}$$

and

$$\begin{aligned}D_1 = \Lambda^{-1}\Big\{ & \left(\frac{\sigma}{\delta}\right)\left[A I_{0^+}^{\alpha-\beta}(\mu v(T) + \gamma v(\eta)) + I_{0^+}^{\alpha}(\mu h(T) + \gamma h(\eta)) + d\right] \\ & + \left[A I_{0^+}^{\alpha-\beta-1}(\mu v(T) + \gamma v(\eta)) + I_{0^+}^{\alpha-1}(\mu h(T) + \gamma h(\eta))\right] + l\Big\},\end{aligned}$$

From (H), for any $t \in J$, we have

$$\begin{aligned}\|g(t) - h(t)\| = & L_1\|u(t) - v(t)\| + L_2\left\|{}^cD_{0^+}^{\beta}u(t) - {}^cD_{0^+}^{\beta}v(t)\right\| \\ & + L_3\left\|g(t) + A\,{}^cD_{0^+}^{\beta}u(t) - h(t) - A\,{}^cD_{0^+}^{\beta}v(t)\right\| \\ \leq & L_1\|u(t) - v(t)\| + L_2\left\|{}^cD_{0^+}^{\beta}u(t) - {}^cD_{0^+}^{\beta}v(t)\right\| \\ & + L_3\|g(t) - h(t)\| + L_3\|A\|\left\|{}^cD_{0^+}^{\beta}u(t) - {}^cD_{0^+}^{\beta}v(t)\right\| \\ \leq & L_1\|u(t) - v(t)\| + L_3\|g(t) - h(t)\| + (L_3\|A\| + L_2)\left\|{}^cD_{0^+}^{\beta}(u(t) - v(t))\right\|.\end{aligned}$$

Thus

$$\begin{aligned}\|g(t) - h(t)\| & \leq \frac{L_1}{1 - L_3}\|u(t) - v(t)\| + \frac{L_3\|A\| + L_2}{1 - L_3}\left\|{}^cD_{0^+}^{\beta}(u(t) - v(t))\right\| \\ & \leq \frac{L_1}{1 - L_3}\|u - v\|_{\infty} + \frac{L_3\|A\| + L_2}{1 - L_3}\left\|{}^cD_{0^+}^{\beta}(u - v)\right\|_{\infty}.\end{aligned}$$

Then, according to the Lemma 3.2, we get

$$\begin{aligned}\|g(t) - h(t)\| & \leq \frac{L_1}{1 - L_3}\|u - v\|_1 + \frac{T^{1-\beta}(L_3\|A\| + L_2)}{\Gamma(2 - \beta)(1 - L_3)}\|u - v\|_1 \\ & = \frac{L_1\Gamma(2 - \beta) + T^{1-\beta}(L_3\|A\| + L_2)}{\Gamma(2 - \beta)(1 - L_3)}\|u - v\|_1.\end{aligned} \tag{10}$$

By employing (10) and Lemma 3.1, we get

$$\begin{aligned}\|B_1 - B_2\| &\le \Delta\left\{\left[\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T + \gamma\eta) + 1\right]\Phi + \Lambda^{-1}(\mu T + \gamma\eta)\Theta\right\}\|u - v\|_1 \\ &= M_1\|u - v\|_1.\end{aligned}$$

and

$$\begin{aligned}\|D_1 - D_2\| &\le \Lambda^{-1}\left\{\left(\frac{\sigma}{\delta}\right)\Phi + \Theta\right\}\|u - v\|_1 \\ &= M_2\|u - v\|_1,\end{aligned}$$

where Φ and Θ defined above.
Thus, for $t \in J$, we have

$$\begin{aligned}\|(Nu)(t) - (Nv)(t)\| &\le \left[\left(1 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)M_1 + TM_2 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right. \\ &\left.+ \frac{T^{\alpha}L_1\Gamma(2-\beta) + T^{1-\beta+\alpha}(L_3\|A\| + L_2)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_2)}\right]\|u - v\|_1 \\ &= R_1\|u - v\|_1.\end{aligned}$$

Also

$$\begin{aligned}\|(Nu)(t) - (Nv)(t)\| &\le \|D_2 - D_1\| + \frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}\|B_1 - B_2\| \\ &+ \frac{\|A\|}{\Gamma(\alpha-\beta-1)}\int_0^T (T-s)^{\alpha-\beta-2}\|u(s) - v(s)\|ds \\ &+ \frac{1}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}\|g(s) - h(s)\|ds.\end{aligned}$$

By employing (10) and Lemma 3.2, we get

$$\begin{aligned}\|(Nu)(t) - (Nv)(t)\| &\le \left[M_2 + \frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_1 + \frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right. \\ &\left.+ \frac{T^{\alpha-1}L_1\Gamma(2-\beta) + T^{\alpha-\beta}(L_3\|A\| + L_2)}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_2)}\right]\|u - v\|_1 \\ &= R_2\|u - v\|_1.\end{aligned}$$

Therefore

$$\|(Nu)(t) - (Nv)(t)\| \le \max\{R_1, R_2\}\|u - v\|_1.$$

Thus, by (10) the operator N is a contraction. Hence it follows by Banach's contraction principle that the boundary value problem (1)–(12) has a unique solution on J.

Now we are in a position to present the second main result of this paper. The existence results is based on Leray-Schauder nonlinear alternative.

Theorem 1.3. ([6] Nonlinear alternative for single valued maps). Let E be a Banach space, C a closed, convex subset of E and U an open subset of C with $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous and compact (that is $F(\overline{U})$ is relatively compact subset of C) map. Then either

i. F has a fixed point in $\overline{U}$, or

ii. there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

Set

$$l_1 = M_3 + TM_4 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}M_3 + TM_4 + \frac{\|A\|rT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)}M,$$

and

$$l_2 = M_4 + \frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_3 + \frac{\|A\|rT^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{T^\alpha M}{\Gamma(\alpha)}.$$

Theorem 1.4. Assume that (H_1) holds and there exists a positive constant $M > 0$ such that $\max\{l_1, l_2\} = l < M$. Then the boundary value problem (1.1)–(1.2) has at least one solution on J.

Proof. Let N be the operator defined in (8).

N is continuous. Let (u_n) be a sequence such that $u_n \to u$ in $C(J, \mathbb{R}^n)$. Then for $t \in J$, we have

$$\begin{aligned}\|(Nu)(t) - (Nu_n)(t)\| \le & \left(1 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\|B_1 - B_{n2}\| + T\|D_1 - D_{n2}\| \\ & + \frac{\|A\|}{\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}\|u(s) - u_n(s)\|ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|g(s) - g_n(s)\|ds,\end{aligned}$$

where $B_{n2}, D_{n2} \in \mathbb{R}^n$, with

$$\begin{aligned}B_{n2} = \Delta\Big\{ & \left[\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T + \gamma\eta) + 1\right]\left[AI_{0^+}^{\alpha-\beta}(\mu u_n(T) + \gamma u_n(\eta)) + I_{0^+}^{\alpha}(\mu g_n(T) + \gamma g_n(\eta)) + d\right] \\ & + \Lambda^{-1}(\mu T + \gamma\eta)\left[AI_{0^+}^{\alpha-\beta-1}(\mu u_n(T) + \gamma u_n(\eta)) + I_{0^+}^{\alpha-1}(\mu g_n(T) + \gamma g_n(\eta)) + l\right]\Big\},\\ D_{n2} = \Lambda^{-1}\Big\{ & \left(\frac{\sigma}{\delta}\right)\left[AI_{0^+}^{\alpha-\beta}(\mu u_n(T) + \gamma u_n(\eta)) + I_{0^+}^{\alpha}(\mu g_n(T) + \gamma g_n(\eta)) + d\right] \\ & + AI_{0^+}^{\alpha-\beta-1}(\mu u_n(T) + \gamma u_n(\eta)) + I_{0^+}^{\alpha-1}(\mu g_n(T) + \gamma g_n(\eta)) + l\Big\},\end{aligned}$$

and

$$g_n(t) = f\left(t, u_n(t), {}^cD_{0^+}^{\beta}u_n(t), g_n(t) + A\,{}^cD_{0^+}^{\beta}u_n(t)\right).$$

From (H), for any $t \in J$, we have

$$\begin{aligned}\left\|g(t)-g_n(t)\right\| &= L_1\|u(t)-u_n(t)\| + L_3\left\|g(t)+A^cD^{\beta}_{0^+}u(t)-g_n(t)-A^cD^{\beta}_{0^+}u_n(t)\right\|\\ &\quad + L_2\left\|{}^cD^{\beta}_{0^+}u(t) - {}^cD^{\beta}_{0^+}u_n(t)\right\|\\ &\le L_1\|u(t)-u_n(t)\| + L_2\left\|{}^cD^{\beta}_{0^+}u(t) - {}^cD^{\beta}_{0^+}u_n(t)\right\|\\ &\quad + L_3\left\|g(t)-g_n(t)\right\| + L_3\|A\|\left\|{}^cD^{\beta}_{0^+}u(t) - {}^cD^{\beta}_{0^+}u_n(t)\right\|\\ &\le L_1\|u(t)-u_n(t)\| + L_3\left\|g(t)-g_n(t)\right\| + (L_3\|A\|+L_2)\left\|{}^cD^{\beta}_{0^+}(u(t)-u_n(t))\right\|.\end{aligned}$$

Thus

$$\begin{aligned}\left\|g(t)-g_n(t)\right\| &\le \frac{L_1}{1-L_3}\|u(t)-u_n(t)\| + \frac{L_3\|A\|+L_2}{1-L_3}\left\|{}^cD^{\beta}_{0^+}(u(t)-u_n(t))\right\|\\ &\le \frac{L_1}{1-L_3}\|u-u_n\|_{\infty} + \frac{L_3\|A\|+L_2}{1-L_3}\left\|{}^cD^{\beta}_{0^+}(u-u_n)\right\|_{\infty}.\end{aligned}$$

Then, according to the Lemma 3.2, we get

$$\begin{aligned}\left\|g(t)-g_n(t)\right\| &\le \frac{L_1}{1-L_3}\|u-u_n\|_1 + \frac{T^{1-\beta}(L_3\|A\|+L_2)}{(1-L_3)\Gamma(2-\beta)}\|u-u_n\|_1\\ &= \frac{L_1\Gamma(2-\beta)+T^{1-\beta}(L_3\|A\|+L_2)}{(1-L_3)\Gamma(2-\beta)}\|u-u_n\|_1.\end{aligned}$$

By employing (10) and Lemma 3.1, we get

$$\begin{aligned}\|B_1-B_{n2}\| &\le \Delta\left\{\left[\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T+\gamma\eta)+1\right]\Phi + \Lambda^{-1}(\mu T+\gamma\eta)\Theta\right\}\|u-u_n\|_1,\\ &= M_1\|u-u_n\|_1.\end{aligned}$$

and

$$\begin{aligned}\|D_1-D_{n2}\| &\le \Lambda^{-1}\left\{\left(\frac{\sigma}{\delta}\right)\Phi+\Theta\right\}\|u-u_n\|_1\\ &= M_2\|u-u_n\|_1,\end{aligned}$$

Thus, for $t \in J$, we have

$$\begin{aligned}\|(Nu)(t)-(Nu_n)(t)\| &\le \left[\left(1+\frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)M_1 + TM_2 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right.\\ &\quad \left.+\frac{T^{\alpha}L_1\Gamma(2-\beta)+T^{1-\beta+\alpha}(L_3\|A\|+L_2)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_2)}\right]\|u-u_n\|_1\\ &= R_1\|u-u_n\|_1.\end{aligned}$$

Also

$$\|(Nu)(t) - (Nu_n)(t)\| \leq \|D_{n2} - D_1\| + \frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}\|B_1 - B_{n2}\|$$

$$+\frac{\|A\|}{\Gamma(\alpha-\beta-1)}\int_0^T (T-s)^{\alpha-\beta-2}\|u(s) - u_n(s)\|ds$$

$$+\frac{1}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}\|g(s) - g_n(s)\|ds.$$

By employing (10), we get

$$\|(Nu)(t) - (Nu_n)(t)\| \leq \left[M_2 + \frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_1 + \frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right.$$
$$\left.+\frac{T^{\alpha-1}L_1\Gamma(2-\beta)(L_3\|A\|+L_2)T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_1 + \frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right]\|u-u_n\|_1.$$
$$= R_2\|u-u_n\|_1.$$

Thus $\|Nu - Nu_n\|_1 \to 0$ as $n \to \infty$, which implies that the operator N is continuous.

Now, we show N maps bounded sets into bounded sets in $C(J, \mathbb{R}^n)$. For a positive number r, let $B_r = \{u \in C^1(J, \mathbb{R}^n) : \|u\|_1 \leq r\}$ be a bounded set in $C(J, \mathbb{R}^n)$. Then we have

$$\|g(t)\| \leq \left\| f\left(t, u(t), g(t) + A\,{}^cD^\beta_{0^+}u(t), D^\beta_{0^+}u(t)\right) - f(t,0,0,0)\right\| + \|f(t,0,0,0)\|$$

$$\leq L_1\|u(t)\| + L_3\left\|g(t) + A\,{}^cD^\beta_{0^+}u(t)\right\| + L_2\left\|D^\beta_{0^+}u(t)\right\| + \|f(t,0,0,0)\|$$

$$\leq L_1\|u\|_\infty + L_3\|g(t)\| + (L_3\|A\| + L_2)\left\|D^\beta_{0^+}u\right\|_\infty + f^*,$$

where $\sup_{t\in J}|f(t,0,0,0)| = f^* < \infty$. Thus

$$\|g(t)\| \leq \frac{L_1}{1-L_3}\|u\|_\infty + \frac{L_3\|A\|+L_2}{1-L_3}\left\|D^\beta_{0^+}u\right\|_\infty + \frac{f^*}{1-L_3}.$$

Then, By Lemma 3.2, we have

$$\|g(t)\| \leq \frac{L_1}{1-L_3}\|u\|_\infty + \frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\|u'\|_\infty + \frac{f^*}{1-L_3}$$

$$\leq \frac{L_1}{1-L_3}\|u\|_1 + \frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\|u\|_1 + \frac{f^*}{1-L_3} \tag{11}$$

$$\leq \frac{L_1 r}{1-L_3} + \frac{(L_3\|A\|+L_2)rT^{1-\beta}}{(1-L_3)\Gamma(2-\beta)} + \frac{f^*}{1-L_3} = M,$$

which implies that

$$\|B\| \leq r\|A\|\Delta\left[\left(\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T+\gamma\eta)+1\right)\left(\frac{\mu T^{\alpha-\beta}+\gamma\eta^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\right.$$
$$\left.+\Lambda^{-1}(\mu T+\gamma\eta)\left(\frac{\mu T^{\alpha-\beta-1}+\gamma\eta^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right)\right]$$
$$+M\Delta\left[\left(\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)(\mu T+\gamma\eta)+1\right)\left(\frac{\mu T^{\alpha}+\gamma\eta^{\alpha}}{\Gamma(\alpha+1)}\right)+\Lambda^{-1}(\mu T+\gamma\eta)\left(\frac{\mu T^{\alpha-1}+\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right]$$
$$+\Delta\Lambda^{-1}(\mu T+\gamma\eta)\left[l+d\left(\left(\frac{\sigma}{\delta}\right)+1\right)\right]=M_3,$$

and

$$\|D\| \leq r\|A\|\left[\Lambda^{-1}\left(\frac{\sigma}{\delta}\right)\frac{\mu T^{\alpha-\beta}+\gamma\eta^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\mu T^{\alpha-\beta-1}+\gamma\eta^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right]$$
$$+M\Lambda^{-1}\left[\left(\frac{\sigma}{\delta}\right)\left(\frac{\mu T^{\alpha}+\gamma\eta^{\alpha}}{\Gamma(\alpha+1)}\right)+\left(\frac{\mu T^{\alpha-1}+\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right]+\Lambda^{-1}\left[\left(\frac{\sigma}{\delta}\right)d+l\right]=M_4.$$

Thus (8) implies

$$\|(Nu)(t)\| \leq M_3+\frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}M_3+TM_4+\frac{\|A\|rT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}M=l_1,$$

and

$$\|(Nu)(t)\| \leq M_4+\frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_3+\frac{\|A\|rT^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+\frac{T^{\alpha}M}{\Gamma(\alpha)}=l_2.$$

Therefore

$$\|(Nu)\|_1 \leq \max\{l_1,l_2\}=l. \tag{12}$$

Now, we show that N maps bounded sets into equicontinuous sets of $C^1(J,\mathbb{R}^n)$. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $u\in B_r$ is bounded sets of $C^1(J,\mathbb{R}^n)$. Then

$$\|(Nu)(t_2)-(Nu)(t_1)\| \leq M_4(t_2-t_1)+\left(1+\frac{\|A\|M_3}{\Gamma(\alpha-\beta+1)}\right)\left(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}\right)$$
$$+\frac{\|A\|r}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1}ds+\frac{\|A\|r}{\Gamma(\alpha-\beta)}\int_0^{t_1}\left[(t_2-s)^{\alpha-\beta-1}-(t_1-s)^{\alpha-\beta-1}\right]ds$$
$$+\frac{M_1}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds+\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]ds\right]$$

Obviously, the right-hand side of the above inequality tends to zero as $t_2\to t_1$. Similarly, we have

$$\|(Nu)(t_2) - (Nu)(t_1)\| \leq \frac{(\alpha-\beta)\|A\|M_3}{\Gamma(\alpha-\beta+1)}\left(t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}\right)$$

$$+\frac{\|A\|r}{\Gamma(\alpha-\beta-1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-2}ds + \frac{\|A\|r}{\Gamma(\alpha-\beta-2)}\int_0^{t_1}\left[(t_2-s)^{\alpha-\beta-2} - (t_1-s)^{\alpha-\beta-2}\right]ds$$

$$+\frac{M}{\Gamma(\alpha-1)}\left[\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}ds + \int_0^{t_1}\left[(t_2-s)^{\alpha-2} - (t_1-s)^{\alpha-2}\right]ds\right]$$

Again, it is seen that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. Thus, $\|(Nu)(t_2) - (Nu)(t_1)\| \to 0$, as $t_2 \to t_1$. This shows that the operator N is completely continuous, by the Ascoli-Arzela theorem. Thus, the operator N satisfies all the conditions of Theorem 3.4, and hence by its conclusion, either condition (i) or condition (ii) holds. We show that the condition (ii) is not possible.

Let $U = \{u \in C^1(J, \mathbb{R}^n) : \|u\| < M\}$ with $\max\{l_1, l_2\} = l < M$. In view of condition $l < M$ and by (12), we have

$$\|Nu\| \leq \max\{l_1, l_2\} < M.$$

Now, suppose there exists $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda Nu$. Then for such a choice of u and the constant λ, we have

$$M = \|u\| = \lambda\|Nu\| < \max\{l_1, l_2\} < M,$$

which is a contradiction. Consequently, by the Leray-Schauder alternative, we deduce that F has a fixed point $u \in \overline{U}$ which is a solution of the boundary value problem (1)–(12). The proof is completed.

We construct an example to illustrate the applicability of the results presented.

Example 2.1. *Consider the following fractional differential equation*

$$^cD_{0^+}^{\alpha}u(t) - A\,^cD_{0^+}^{\beta}u(t) = f\left(t, u(t), ^cD_{0^+}^{\beta}u(t), ^cD_{0^+}^{\alpha}u(t)\right), \quad t \in J = [0,1], \tag{13}$$

subject to the three-point boundary conditions

$$\begin{cases} u(0) - u(1) - u\left(\frac{1}{2}\right) = 1, \\ u'(0) - u'(1) - u'\left(\frac{1}{2}\right) = 1, \end{cases} \tag{14}$$

where $\alpha = 2, \quad \beta = 1, \quad \lambda = \mu = d = l = 1, \quad A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and

$$f_i(t, u, v, w) = \frac{c_i t}{8}\, arctan\,(|u| + |v| + |w|),\ i = 1,\ 2,$$

such that $f = (f_1, f_2)$ with $0 < c_i < 1,\ i = 1, 2$.

For every $u_i,\ v_i \in \mathbb{R}^2,\ i = 1,\ 2,\ 3$, *we have*

$$\left| f_i(t,u_1,u_2,u_3) - f_i(t,v_1,v_2,v_3) \right| \le \frac{c_i}{8}(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),\ i = 1,\ 2,$$

where $L_1 = L_2 = L_3 = \frac{c_i}{8}$ for appropriate choice of constants $c_i,\ i = 1,\ 2$. we check the condition of Theorem 2.2. Clearly, assumption (H_1) holds. A simple computations of $R_1,\ R_2,\ l_1$ and l_2 shows tha the second condition of Theorems 3.3 and 3.5 is satisfied. Thus the conclusion of Theorems 3.3 and 3.5 applies, and hence the problem (13)–(14) has a unique solution and at least one solution on $[0, 1]$.

3. Conclusions

This chapter concerns the boundary value problem of a class of fractional differential equations involving the Riemann-Liouville fractional derivative with nonlocal boundary conditions. By using Leray-Schauder nonlinear alternative and the Banach fixed point theorem, we shows the existence and uniqueness of positive solutions of our problem. In addition, an example is provided to demonstrate the effectiveness of the main results. The results of the present chapter are significantly contribute to the existing literature on the topic.

Acknowledgements

The authors want to thank the anonymous referee for the thorough reading of the manuscript and several suggestions that help us improve the presentation of the chapter.

Conflict of interest

The authors declare no conflict of interest.

Author details

Noureddine Bouteraa[1,2*†] and Habib Djourdem[1,3*†]

1 Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran, Ahmed Benbella, Algeria

2 Oran Graduate School of Economics, Bir El Djir, Algeria

3 University of Ahmed Zabbana, Relizane, Algeria

*Address all correspondence to: bouteraa-27@hotmail.fr and djourdem.habib7@gmail.com

† These authors contributed equally.

References

[1] Kac V, Cheung P. Quantum Calculus. New York: Springer; 2002

[2] Lakshmikantham V, Vatsala AS. General uniqueness and monotone iterative technique for fractional differential equations. Applied Mathematics Letters. 2008;**21**(8): 828-834

[3] Miller S, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations. New York: John Wiley and Sons, Inc.; 1993

[4] Rudin W. Functional analysis. In: International Series in Pure and Applied Mathematics. 2nd ed. New York: McGraw Hill; 1991

[5] Samko SG, Kilbas AA, Marichev OI. Fractional Integrals and Derivatives: Theory and Applications. Yverdon: Gordon & Breach; 1993

[6] Deimling K. Functional Analysis. Berlin: Springer; 1985

[7] Jarad F, Abdeljaw T, Baleanu D. On the generalized fractional derivatiives and their Caputo modification. Journal of Nonlinear Sciences and Applications. 2017;**10**(5):2607-2619

[8] Tian Y. Positive solutions to m-point boundary value problem of fractional differential equation. Acta Mathematicae Applicatae Sinica, English Series. 2013; **29**:661-672

[9] EL-Sayed AMA, EL-Haddad FM. Existence of integrable solutions for a functional integral inclusion. Differential Equations & Control Processes. 2017;**3**: 15-25

[10] Bouteraa N, Benaicha S. Existence of solutions for nonlocal boundary value problem for Caputo nonlinear fractional differential inclusion. Journal of Mathematical Sciences and Modelling. 2018;**1**(1):45-55

[11] Bouteraa N, Benaicha S. Existence results for fractional differential inclusion with nonlocal boundary conditions. Rivista di Matematica della Università di Parma. 2020;**11**: 181-206

[12] Cernia A. Existence of solutions for a certain boundary value problem associated to a fourth order differential inclusion. International Journal of Analysis and Applications. 2017;**14**:27-33

[13] Ntouyas SK, Etemad S, Tariboon J, Sutsutad W. Boundary value problems for Riemann-Liouville nonlinear fractional diffrential inclusions with nonlocal Hadamard fractional integral conditions. Mediterranean Journal of Mathematics. 2015;**2015**:16

[14] Bouteraa N, Benaicha S. Triple positive solutions of higher-order nonlinear boundary value problems. Journal of Computer Science and Computational Mathematics. June 2017; **7**(2):25-31

[15] Bouteraa N, Benaicha S. Existence of solutions for three-point boundary value problem for nonlinear fractional equations. Analele Universitatii din Oradea. Fascicola Matematica. Tom 2017;**XXIV**(2):109-119

[16] Benaicha S, Bouteraa N. Existence of solutions for three-point boundary value problem for nonlinear fractional differential equations. Bulltin of the Transilvania University of Brasov, Series III: Mathematics, Informtics, Physics. 2017;**10**(2):59

[17] Bouteraa N, Benaicha S. Existence of solutions for third-order three-point boundary value problem. Mathematica. 2018;**60**(1):21-31

[18] Bouteraa N, Benaicha S. The uniqueness of positive solution for higher-order nonlinear fractional differential equation with nonlocal boundary conditions. Advances in the Theory of Nonlinear and Its Application. 2018;**2**(2):74-84

[19] Bouteraa N, Benaicha S, Djourdem H. Positive solutions for nonlinear fractional differential equation with nonlocal boundary conditions. Universal Journal of Mathematics and Applications. 2018;**1**(1):39-45

[20] Bouteraa N, Benaicha S. The uniqueness of positive solution for nonlinear fractional differential equation with nonlocal boundary conditions. Analele Universitatii din Oradea. Fascicola Matematica. Tom 2018; **XXV**(2):53-65

[21] Bouteraa N, Benaicha S, Djourdem H, Benatia N. Positive solutions of nonlinear fourth-order two-point boundary value problem with a parameter. Romanian Journal of Mathematics and Computer Science. 2018;**8**(1):17-30

[22] Bouteraa N, Benaicha S. Positive periodic solutions for a class of fourth-order nonlinear differential equations. Siberian Journal of Numerical Mathematics. 2019;**22**(1):1-14

[23] Bouteraa N. Existence of solutions for some nonlinear boundary value problems [thesis]. Ahmed Benbella, Algeria: University of Oran; 2018

[24] Bouteraa N, Benaicha S, Djourdem H. On the existence and multiplicity of positive radial solutions for nonlinear elliptic equation on bounded annular domains via fixed point index. Maltepe Journal of Mathematics. 2019;**I**(1):30-47

[25] Bouteraa N, Benaicha S, Djourdem H. Positive solutions for systems of fourth-order two-point boundary value problems with parameter. Journal of Mathematical Sciences and Modeling. 2019;**2**(1):30-38

[26] Bouteraa N, Benaicha S. Existence and multiplicity of positive radial solutions to the Dirichlet problem for the nonlinear elliptic equations on annular domains. Studia Universitatis Babeș-Bolyai Mathematica. 2020;**65**(1):109-125

[27] Benaicha S, Bouteraa N, Djourdem H. Triple positive solutions for a class of boundary value problems with integral boundary conditions. Bulletin of Transilvania University of Brasov, Series III: Mathematics, Informatics, Physics. 2020;**13**(1):51-68

[28] Bouteraa N, Djourdem H, Benaicha S. Existence of solution for a system of coupled fractional boundary value problem. Proceedings of International Mathematical Sciences. 2020;**II**(1):48-59

[29] Bouteraa N, Benaicha S. Existence results for second-order nonlinear differential inclusion with nonlocal boundary conditions. Numerical Analysis and Applications. 2021;**14**(1):30-39

[30] Bouteraa N, Inc M, Akinlar MA, Almohsen B. Mild solutions of fractional PDE with noise. Mathematical Methods in the Applied Sciences. 2021:1-15

[31] Bouteraa N, Benaicha S. A study of existence and multiplicity of positive solutions for nonlinear fractional differential equations with nonlocal boundary conditions. Studia

Universitatis Babeș-Bolyai Mathematica. 2021;**66**(2):361-380

[32] Djourdem H, Benaicha S, Bouteraa N. Existence and iteration of monotone positive solution for a fourth-order nonlinear boundary value problem. Fundamental Journal of Mathematics and Applications. 2018;**1**(2):205-211

[33] Djourdem H, Benaicha S, Bouteraa N. Two positive solutions for a fourth-order three-point BVP with sign-changing green's function. Communications in Advanced Mathematical Sciences. 2019;**II**(1):60-68

[34] Djourdem H, Bouteraa N. Mild solution for a stochastic partial differential equation with noise. WSEAS Transactions on Systems. 2020;**19**:246-256

[35] Ghorbanian R, Hedayati V, Postolache M, Rezapour SH. On a fractional differential inclusion via a new integral boundary condition. Journal of Inequalities and Applications. 2014; **2014**:20

[36] Inc M, Bouteraa N, Akinlar MA, Chu YM, Weber GW, Almohsen B. New positive solutions of nonlinear elliptic PDEs. Applied Sciences. 2020;**10**:4863. DOI: 10.3390/app10144863

[37] Bouteraa N, Benaicha S. Nonlinear boundary value problems for higher-order ordinary differential equation at resonance. Romanian Journal of Mathematic and Computer Science. 2018;**8**(2):83-91

[38] Bouteraa N, Benaicha S. A class of third-order boundary value problem with integral condition at resonance. Maltepe Journal of Mathematics. 2020; **II**(2):43-54

[39] Lin X, Zhao Z, Guan Y. Iterative technology in a singular fractional boundary value problem with q-difference. Applied Mathematics. 2016;**7**:91-97

[40] Rezapour SH, Hedayati V. On a Caputo fractional differential inclusion with integral boundary condition for convex-compact and nonconvex-compact valued multifunctions. Kragujevac Journal of Mathematics. 2017;**41**:143-158

[41] Sheng S, Jiang J. Existence and uniqueness of the solutions for fractional damped dynamical systems. Advances in Difference Equations. 2017;**2017**:16

[42] Abbas S, Benchohra M, Bouriah S, Nieto JJ. Periodic solution for nonlinear fractional differential systems. Differential Equations Applications. 2018;**10**(3):299-316

[43] Zhu T. Existence and uniqueness of positive solutions for fractional differential equations. Boundary Value Problems. 2019;**22**:11

[44] Annaby MH, Mansour ZS. q-Fractional calculus and equations. In: Lecture Notes in Mathematics. Vol. 2056. Berlin: Springer-Verlag; 2012

[45] Agrawal O. Some generalized fractional calculus operators and their applications in integral equations. Fractional Calculus and Applied Analysis. 2012;**15**:700-711

[46] Kilbas AA, Srivastava HM, Trijull JJ. Theory and Applications of Fractional Differential Equations. Amsterdam: Elsevier Science B.V.; 2006

[47] Ahmad B, Ntouyas SK, Alsaedi A. Coupled systems of fractional differential inclusions with coupled boundary conditions. Electronic Journal of Differential Equations. 2019; **2019**(69):1-21