Tight Logarithmic Approximations and Bounds for Generic Capacity Integrals and Their Applications to Statistical Analysis of Wireless Systems

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***Abstract*— We present tight yet tractable approximations and bounds for the ergodic capacity of any communication system in the form of a weighted sum of logarithmic functions, with the focus on the Nakagami and lognormal distributions that represent key building blocks for more complicated systems. A minimax optimization technique is developed to derive their coefficients resulting in uniform absolute or relative error. These approximations and bounds constitute a powerful tool for the statistical performance analysis as they enable the evaluation of the ergodic capacity of various communication systems that experience small-scale fading together with the lognormal shadowing effect and allow for simplifying the complicated integrals encountered when evaluating the ergodic capacity in different communication scenarios. Simple and tight closed-form solutions for the ergodic capacity of many classic and timely application examples are derived using the logarithmic approximations. The high accuracy of the proposed approximations is verified by numerical comparisons with existing approximations and with those obtained directly from numerical integration methods.**

***Index Terms*— Ergodic capacity, minimax approximation, bounds, performance analysis, fading distributions.**

I. Introduction

ERGODIC capacity is an important measure for analyzing the performance of different communication systems [1]. It specifies the maximum transmission rate of reliable communication that can be achieved over time-varying channels. Specific formulations of ergodic capacity can be referred to as *capacity integrals* based on the way how they are found by calculating the expectation of instantaneous channel capacity using probability density functions (PDFs) that model fading. Establishing closed-form expressions for ergodic capacity is of great importance in communication theory since they enable us to gain scientific understanding of the behavior of communication systems and the effect of their parameters on the performance. In this area, our research work aims at facilitating the statistical performance analysis of wireless systems by developing novel mathematical tools that build upon the following general result in this article.

Proposition 1: For any wireless system with instantaneous capacity $\mathcal{C} \triangleq \log_2(1+\gamma_{\text{eff}})$ conditioned on fading states, where $\gamma_{\text{eff}} \triangleq 2^{\mathcal{C}}-1$ denotes *effective* (not necessarily actual) signal-to-noise ratio (SNR) with average $\bar{\gamma}_{\text{eff}} \triangleq \mathrm{E}[\gamma_{\text{eff}}]$, the ergodic capacity can be approximated with arbitrary accuracy as

$$\bar{\mathcal{C}} \triangleq \mathrm{E}[\mathcal{C}] \approx \sum_{n=1}^{N} a_n \log_2(1+b_n\,\bar{\gamma}_{\text{eff}}) \tag{1}$$

by choosing the coefficients $\{(a_n,b_n)\}_{n=1}^{N}$ appropriately.

Proof: See Appendix A. ■

One will instantly notice that the generic approximation (1) is a weighted sum of the Shannon capacities of basic static additive white Gaussian noise (AWGN) channels. In other words, the greatness of Proposition 1 is that it proves that *any* system with fading channels is *in terms of capacity* equivalent to a system (cf. Fig. 1), wherein a scheduler employs randomly one of $N+1$ parallel static channels for the transmission of each data block:[1] Channel n, $n=1,2,\ldots,N$, having SNR of $b_n\bar{\gamma}_{\text{eff}}$ is chosen with probability a_n and Channel 0 represents a completely blocked channel ($b_0=0$), i.e., an outage event takes place with remaining probability $a_0 = 1-\sum_{n=1}^{N} a_n$.

While Proposition 1 is powerful in proving the general existence of the approximation (1) for the ergodic capacity of any wireless system at large, it is not so applicable as an actual approximation for any specific system. This is because, firstly, the coefficients a_n, $n=1,2,\ldots,N$, are in the direct application computed from the PDF of $\mathcal{C}$, which is typically not derived explicitly in statistical performance analysis, and it may be cumbersome or even impossible to express. Secondly and more importantly, when choosing the coefficients from the Riemann sum according to the proof, the resulting approximations are inefficient, because a very large number of logarithmic terms are needed for adequate accuracy.

In this paper, we aim to evolve Proposition 1 into a useful, efficient tool in two ways. Firstly, we develop a systematic methodology to optimize coefficients $\{(a_n,b_n)\}_{n=1}^{N}$ to approximate any communication system's ergodic capacity

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[1]An alternative interpretation is a scheduler that employs the parallel channels sequentially for data blocks with relative durations a_n, $n=0,1,\ldots,N$.

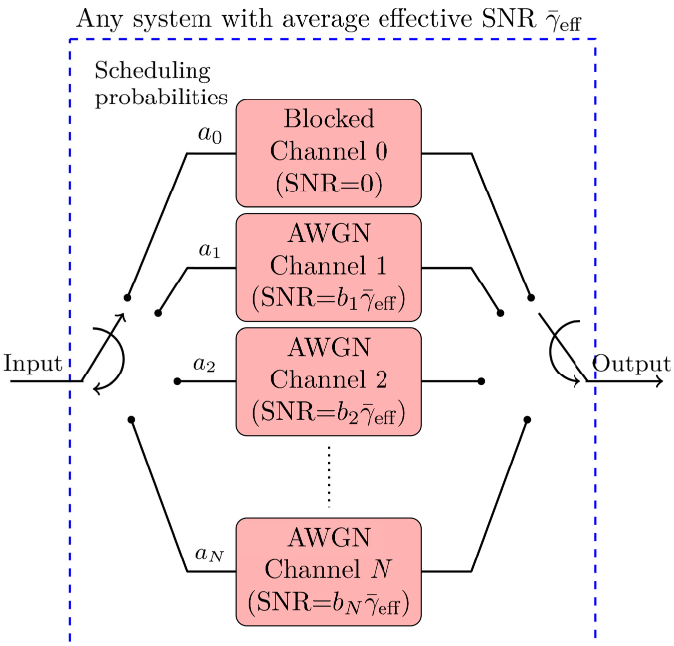


Fig. 1. Interpretation of the ergodic capacity of any communication system as a scheduler which randomly employs one of the parallel static channels when transmitting data blocks.

$\bar{C} = C(1/\bar{\gamma}_{\text{eff}})/\log_e(2)$ that can be expressed with the generic function $C(\cdot)$ of some open or closed form. Secondly, we implement the presented optimization methodology to find $\{(a_n, b_n)\}_{n=1}^{N}$ explicitly under Nakagami and lognormal fading (when $C(\cdot)$ becomes $C_m(\cdot)$, the 'Nakagami capacity integral', or $C_\sigma(\cdot)$, the 'lognormal capacity integral') and show how to use them as building blocks for the capacity analysis of complex systems that manifest them in intermediate steps.

A. Related Works

Capacity integrals have been investigated extensively in the literature for countless transmission systems under various assumptions on transmitter and receiver channel knowledge and over different fading distributions [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]. In [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], and [12], the ergodic capacity over Rayleigh fading is evaluated for single-antenna systems and multi-antenna systems — namely, multiple-input single-output (MISO), single-input multiple-output (SIMO), and multiple-input multiple-output (MIMO) — for correlated or non-correlated channels and different combining techniques at the receiver. Moreover, the ergodic capacity for single-antenna and multi-antenna systems with non-correlated channels is evaluated over Nakagami fading in [13] and [14] and over Rician fading in [15], [17], and [16]. The ergodic capacity under $\kappa - \mu$ fading is derived in [18].

Generally, the precise ergodic capacity expressions are difficult to express in analytical forms. This has motivated the work toward deriving approximations and bounds for capacity integrals [19], [20], [21], [22], [23], [24], [25]. They are also needed among many other purposes for optimal power allocation and network design. In particular, the authors in [19] present a lower bound for the capacity integral of MIMO Rayleigh channels with frequency-selective fading and/or channel correlation, together with an asymptotic approximation of the ergodic capacity over flat fading. Other asymptotic results are derived in [20] for specific multi-antenna scenarios with the channel knowledge at the receiver at first, and then at the transmitter as well.

In [21], more generic expressions for bounding the ergodic capacity are presented. In [22], two less accurate yet tractable approximations that enable the development of analytical resource allocation strategies in Rayleigh MIMO systems are derived. The authors in [23] propose two simple yet accurate approximations for the ergodic capacity in the low-SNR region. Closed-form bounds for the ergodic capacity in dual-hop fixed-gain amplify and forward relay networks are proposed in [24] over Rayleigh fading channels, and in [25] over Nakagami fading channels.

In addition to the small-scale fading, the ergodic capacity is also investigated under the shadowing effect that is usually modeled by the lognormal distribution. The ergodic capacity of communication systems under lognormal fading channels does not admit a closed-form expression. Therefore, several approximations and bounds have been proposed to express it in terms of analytical functions [26], [27], [28], [29], [30]. The very first lower and upper bounds for evaluating the ergodic capacity over lognormal fading channels were presented in [26], resulting in simple yet loose bounds for lower values of SNR.

Other approximations were later developed in [27] and [28] for single-input single-output (SISO) systems and the results were also generalized to approximate the capacity of diversity combining techniques with or without channel correlation, based on the fact that the sum of lognormal random variables can be well approximated by an equivalent lognormal one. In [29], a tight approximation for the lognormal capacity integral is presented and investigated for SISO and MIMO indoor ultra-wideband systems. The authors in [30] derive closed-form approximations for the capacity integral of various adaptive transmission schemes under lognormal distribution.

B. Contributions and Organization of the Paper

The unified fundamental tool, i.e., (1), contributed in this article enables the accurate evaluation of ergodic capacity in any communication system at large in the form of the weighted sum of logarithmic functions. It requires optimizing the corresponding coefficients so that they work as highly efficient replacements for those obtained from the numerical methods such as the Riemann sum in the proof of Proposition 1. Nevertheless, we also implement the proposed approach to offer novel logarithmic approximations and bounds with optimized coefficients specifically for the Nakagami and lognormal capacity integrals. Since these two integrals most frequently appear as building blocks for many more-complex communication systems, this often leads to logarithmic approximations and bounds in the same format of (1) for their capacity expressions. This avoids the need to formulate equivalent methodology and solve the coefficients specifically for every individual system despite the general tool facilitates that too.

We can summarize the contributions in this paper as follows.

- We propose a systematic methodology to optimize the approximations' coefficients and obtain the best logarithmic approximations in terms of the minimax absolute error for the capacity of any communication system. This requires redeveloping the related scheme that we previously presented in [31] for error probability analysis, which is inherently different from capacity analysis.
- We implement the optimization methodology on the Nakagami-m channel (and over Rayleigh fading as a special case thereof) to derive minimax approximations for it. Especially, the approximations are valid for any value of m, opposing to the exact closed-form expression in [13, Eq. 23], which is valid only for its integer values.
- We show how the optimized approximation of the Nakagami capacity integral can be used as a building block to derive the capacity integral of many complicated communication systems [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [32], [33], [34], [35], [36], [37] and can even often lead to the same logarithmic form as an end result.
- Likewise, we find the optimized coefficients for the approximation of the lognormal capacity integral which enables the evaluation of the ergodic capacity for various communication systems that experience small-scale fading together with the lognormal shadowing effect, in the form of a sum of logarithmic terms. In particular, for a composite lognormal channel, we apply the sum of logarithms with its optimized coefficients to approximate the ergodic capacity over the small-scale fading channel first. The resulting integral has exactly the same form as the lognormal capacity integral, which we approximate again by the sum of logarithmic functions.
- We extend the proposed minimax method to find the optimized parameters of the logarithmic approximation in terms of the relative error. We also extend it to find new logarithmic lower and upper bounds with optimized parameters in terms of both error measures.

We validate the aforementioned contributions with an extensive set of application examples that demonstrate the wide range of applicability of the proposed approximations. We further illustrate their high accuracy by numerical comparisons with other existing approximations or those obtained by numerical integration methods. In fact, their accuracy is so high that they can be considered to be virtually exact in most applications while they allow deriving closed-form results in cases where exact analysis is considered to be impossible.

We organize the rest of this paper as follows. Section II introduces some needed background information to formulate and solve the research problem. The main contribution is presented in Section III, where we propose the new methodology to acquire tight logarithmic approximations and bounds for ergodic capacity at large. In Section IV, a wide range of applications are considered and their capacities are evaluated in terms of the proposed approximations. In Section V, the numerical results demonstrate the high accuracy of the proposed approximations compared to other existing and numerical ones. Finally, we conclude the paper in the last section.

II. Preliminaries

In this paper, we shall develop unified approximations and bounds in the format of (1) that apply for the ergodic capacity $\bar{\mathcal{C}} = C(1/\bar{\gamma}_{\text{eff}})/\log_e(2)$ of any communication system, where the *generic capacity function* $C(x)$ can be of any mathematical form. In most communication systems' analysis, $C(x)$ can be represented as a capacity integral that calculates the average of $\mathcal{C} \triangleq \log_2(1+\gamma_{\text{eff}})$ per the following definitions.

Definition 1: Given average effective SNR $\bar{\gamma}_{\text{eff}}$ with $G \triangleq \frac{2^{\mathcal{C}}-1}{\bar{\gamma}_{\text{eff}}} = \frac{\gamma_{\text{eff}}}{\bar{\gamma}_{\text{eff}}}$, whose PDF exists and is denoted by $f_G(\cdot)$, the ergodic capacity of the corresponding communication system is $\bar{\mathcal{C}} = C(1/\bar{\gamma}_{\text{eff}})/\log_e(2)$ [bit/s/Hz], where the *generic capacity integral* is defined as

$$C(x) \triangleq \int_0^\infty \log_e\left(1+\frac{t}{x}\right) f_G(t)\,\mathrm{d}t. \tag{2}$$

One should note that the generic capacity function $C(x)$ is not necessarily given by the above generic capacity integral when the presented tool is still applicable. Nevertheless, we shall focus on developing the approximations and applications for the following specific integrals, which originate from evaluating (2) for Nakagami (including Rayleigh) and lognormal fading channels. These integrals appear frequently as part of longer expressions or in intermediate calculation steps when analyzing the capacity of more complex wireless systems.

Definition 2: Given average SNR $\bar{\gamma}$, the ergodic capacity of a Nakagami-m fading channel is $\bar{\mathcal{C}} = C_m(1/\bar{\gamma})/\log_e(2)$ [bit/s/Hz], where the *Nakagami capacity integral* is defined as

$$\begin{aligned} C_m(x) &\triangleq \int_0^\infty \frac{m^m}{\Gamma(m)} \log_e\left(1+\frac{t}{x}\right) t^{m-1}\exp(-m\,t)\,\mathrm{d}t \\ &= \exp(m\,x) \sum_{k=0}^{m-1} \Gamma(-k, m\,x)\,(m\,x)^k, \end{aligned} \tag{3}$$

for $x > 0$ [13, Eqs. 21 and 23] with $\Gamma(\zeta, x) = \int_x^\infty t^{\zeta-1}\exp(-t)\,\mathrm{d}t$ denoting the upper incomplete gamma function [38, Eq. 6.5.3] and m being the fading parameter; the latter expression is valid for integer values of m only.

Substituting $m = 1$ in the above definition, we obtain the ergodic capacity of a Rayleigh fading channel as a special case as $\bar{\mathcal{C}} = C_1(1/\bar{\gamma})/\log_e(2)$ [bit/s/Hz], where the *Rayleigh capacity integral* is defined as

$$\begin{aligned} C_1(x) &= \int_0^\infty \log_e\left(1+\frac{t}{x}\right)\exp(-t)\,\mathrm{d}t \\ &= \exp(x)\,\mathrm{E}_1(x), \end{aligned} \tag{4}$$

for $x > 0$ [2, Eqs. 4 and 5] with $\mathrm{E}_1(x) = \int_x^\infty \exp(-t)/t\,\mathrm{d}t$ denoting the exponential integral [38, Eq. 5.1.1].

Definition 3: Given average SNR $\bar{\gamma} = \exp(\eta + \frac{\sigma^2}{2})$, in which η and σ are the mean and the standard deviation of the corresponding instantaneous SNR's natural logarithm, respectively, the ergodic capacity of a lognormal fading channel is $\bar{\mathcal{C}} = C_\sigma(1/\bar{\gamma})/\log_e(2)$ [bit/s/Hz], where the *lognormal capacity integral* is defined as

$$C_\sigma(x) \triangleq \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \log_e\left(1 + \frac{1}{x}\exp\left(\sqrt{2\,\sigma^2}\,t - \frac{\sigma^2}{2}\right)\right) \times \exp(-t^2)\,\mathrm{d}t, \tag{5}$$

for $x > 0$ [26, Eq. 29]; this integral does not admit a closed-form expression so its approximations are crucial to have.

The Rayleigh capacity integral in (4) admits a sandwich bound according to [38, Eq. 5.1.20] as

$$\frac{1}{2}\log_e\left(1 + \frac{2}{x}\right) < C_1(x) < \log_e\left(1 + \frac{1}{x}\right), \tag{6}$$

and any linear combination thereof could be used as an obvious, but loose, approximation for the ergodic capacity over a Rayleigh fading channel. Inspired by this fact and Proposition 1, we develop a family of *tractable* functions

$$\tilde{C}(x) \triangleq \sum_{n=1}^{N} a_n \log_e\left(1 + \frac{b_n}{x}\right) \tag{7}$$

for $x > 0$, that offer *tight approximations and bounds* for $C(x)$ as $\tilde{C}(x)$, for $C_m(x)$ in (3) as $\tilde{C}_m(x)$ and for $C_\sigma(x)$ in (5) as $\tilde{C}_\sigma(x)$ by proper parameter choice. They are directly related to Proposition 1 as $\tilde{C}(1/\bar{\gamma}_{\text{eff}})/\log_e(2)$ results in the logarithmic approximation given in (1). Furthermore, it should be noted that all $N!$ permutations of the parameter set $\{(a_n, b_n)\}_{n=1}^{N}$ yield an equivalent function, although we always choose the canonical (sorted) representation with $a_1 \leq a_2 \leq \ldots \leq a_N$.

The absolute and relative error functions $d(x)$ and $r(x)$, respectively, as well as their first-order derivatives $d'(x)$ and $r'(x)$, respectively, are needed in what follows. They are defined as

$$d(x) \triangleq \tilde{C}(x) - C(x), \tag{8}$$

$$r(x) \triangleq \frac{d(x)}{C(x)} = \frac{\tilde{C}(x)}{C(x)} - 1, \tag{9}$$

and their derivatives are given by

$$d'(x) = \tilde{C}'(x) - C'(x), \tag{10}$$

$$r'(x) = \frac{C(x)\tilde{C}'(x) - C'(x)\tilde{C}(x)}{[C(x)]^2}, \tag{11}$$

for which

$$\tilde{C}'(x) = -\sum_{n=1}^{N} \frac{a_n b_n}{(x+b_n)x}, \tag{12}$$

and generally, whenever $C(x)$ is given by (2),

$$C'(x) = -\int_0^{\infty} \frac{t}{(t+x)\,x}\, f_G(t)\,\mathrm{d}t. \tag{13}$$

For Nakagami-m and lognormal fading, (13) becomes

$$C'_m(x) = -\frac{m}{x} + m\,C_m(x) + \left[\frac{\exp(mx)}{x} \times \sum_{k=0}^{m-1} k\,(mx)^k \Gamma(-k, mx)\right] \tag{14}$$

and

$$C'_\sigma(x) = -\int_{-\infty}^{\infty} \frac{\exp\left(\sqrt{2\,\sigma^2}t - \frac{\sigma^2}{2} - t^2\right)}{\sqrt{\pi}\left(\exp\left(\sqrt{2\,\sigma^2}t - \frac{\sigma^2}{2}\right) + x\right)x}\,\mathrm{d}t, \tag{15}$$

respectively.

III. New Logarithmic Approximations and Bounds

Inspired by Proposition 1 and by the table-book bounds restated in (6) for the Rayleigh capacity integral defined in (4), we replace the generic capacity function $C(x)$ as well as the generic, Nakagami and lognormal capacity integrals in (2), (3) and (5), respectively, by a *weighted sum of logarithmic functions* and design appropriate values for the corresponding coefficients. A possible choice would be to use the numerical coefficients that result from applying the numerical integration rules. However, much higher accuracy can be achieved by optimizing these coefficients in the minimax sense to give the best logarithmic approximations and bounds as will be explained soon. To begin with, we can make two minor but useful observations.

Remark 1: An approximation for the exponential integral, $\mathrm{E}_1(x)$, is directly derived from approximating (4) by (7) as

$$\mathrm{E}_1(x) \approx \exp(-x)\sum_{n=1}^{N} a_n \log_e\left(1 + \frac{b_n}{x}\right). \tag{16}$$

Thus, the following results are applicable also beyond ergodic capacity analysis and in other fields of science than communication engineering, where the exponential integral occurs.

Remark 2: As originally reported in [39], the numerical evaluation of the latter form of the Rayleigh capacity integral in (4) is subject to a severe stability issue. In particular with double-precision floating-point arithmetic, $\exp(x)$ overflows and $\mathrm{E}_1(x)$ underflows whenever $x \geq 740$ although their product, $C_1(x)$, is finite and of the magnitude of $1/x$ as shown by [38, Eq. 5.1.19]: $1/(x+1) < C_1(x) < 1/x$ for all $x > 0$. On the other hand, all approximations and bounds according to (7) avoid this stability issue completely.

A. Approximations From Numerical Integration

As already mentioned, a possible choice for the parameters of (7) can be acquired by applying the Riemann sum method. However, slightly better parameter choice is achieved by applying the various quadrature numerical integration methods which are more direct and efficient to be used than the Riemann sum method. Therefore, the numerical coefficients can be easily found as given in the following three lemmas, for which the common proof given underneath holds for all, and where $\{t_n\}_{n=1}^{N}$ are the nodes and $\{w_n\}_{n=1}^{N}$ are the quadrature weights of the corresponding numerical integration rule [40].

Lemma 1: The generic capacity integral can be numerically approximated by (7) with its numerical coefficients given as

$$\{(a_n, b_n)\}_{n=1}^{N} = \{(w_n\, f_G(t_n), t_n)\}_{n=1}^{N}. \tag{17}$$

Lemma 2: The Nakagami capacity integral can be numerically approximated as (7) with its numerical coefficients given as

$$\{(a_n, b_n)\}_{n=1}^N = \left\{ \left(w_n \frac{m^m}{\Gamma(m)} t_n^{m-1} \exp(-m\, t_n), t_n \right) \right\}_{n=1}^N. \tag{18}$$

Lemma 3: The lognormal capacity integral can be numerically approximated as (7) with its numerical coefficients given as

$$\{(a_n, b_n)\}_{n=1}^N = \left\{ \left(\frac{w_n}{\sqrt{\pi}} \exp(-t_n^2), \times \exp\left(\sqrt{2\,\sigma^2}\, t_n - \frac{\sigma^2}{2} \right) \right) \right\}_{n=1}^N. \tag{19}$$

Proof: Starting from the capacity integral expressions in Section II, we implement the quadrature numerical integration techniques, which approximate any integral of the form $\int_u^v f(t)\,\mathrm{d}t$ as a finite sum of the form $\sum_{n=1}^N w_n\, f(t_n)$ for which $f(t)$ is given in (2) for Lemma 1, in (3) for Lemma 2 and in (5) for Lemma 3. This yields the same logarithmic sum as in (7) with the numerical coefficients stated in the lemmas to approximate the respective generic, Nakagami and lognormal capacity integrals. ■

In particular, the capacity integral is an improper convergent integral that can be approximated directly by applying the Gauss–Laguerre or Gauss–Hermite quadrature rules or by considering a large yet finite integration interval with Newton–Cotes methods [38]. Another alternative way would be to use transformation of variables to limit the integration interval and thus enable the application of various other integration techniques. Nevertheless, the numerical approximations have relatively low accuracy in terms of global error and need a large number of logarithmic terms in order to achieve adequate accuracy. Therefore, we only consider the commonly used Gauss–Laguerre and Gauss–Hermite quadrature rules in the analysis of the proposed approximations in this paper.

B. Minimax Approximations

The adopted weighted sum of logarithmic functions in (7) can be optimized to establish best minimax approximations and bounds for the generic capacity function as well as the generic, Nakagami and lognormal capacity integrals. In particular, the *best* approximation or bound refers to the member of the function family (7) that is the tightest of them all for given N and always occur with optimal set of coefficients $\{(a_n^*, b_n^*)\}_{n=1}^N$ that minimizes the maximum error and is expressed as the solution to the following *minimax optimization problem*:

$$\{(a_n^*, b_n^*)\}_{n=1}^N = \underset{\{(a_n,b_n)\}_{n=1}^N}{\arg\min}\; e_{\max} \tag{20}$$

where $e \in \{d, r\}$ represents both the absolute and relative errors collectively in what follows, and $e_{\max}$ is the *maximum* error, which is defined as

$$\begin{aligned} e_{\max} &\triangleq \sup\{|e(x)| \,:\, x > 0\} \\ &= \max\{|e_0|, |e_1|, \ldots, |e_L|, |e_\infty|\}. \end{aligned} \tag{21}$$

The latter expression comes from Fermat's theorem, where $e_l = e(x_l)$, $l = 1, 2, \ldots, L$, are the error values at the stationary points x_l, $l = 1, 2, \ldots, L$, at which $e'(x_l) = 0$.

In the following proposition, we describe the expected shape of the solution to the minimax optimization problem in (20) that gives the best approximation or bound.

Proposition 2: The unique best logarithmic approximation or bound of the function family (7) with degree D for the capacity integral occurs when the corresponding error function $e(x)$ alternates D times between $D+1$ extrema points of the same value of error and alternating signs. Its extreme points are found at the roots of its derivatives or asymptotically at the endpoints of its open domain.

Proof: According to the theorem in [41], the proposed approximation defined in (7) with $\{(a_n^*, b_n^*)\}_{n=1}^N$ is the best minimax approximation to $C(x)$ (including $C_m(x)$ and $C_\sigma(x)$), if and only if $d(x)$ or $r(x)$ defined respectively in (8) and (9), alternate D times. Moreover, the uniqueness of the solution,$\{(a_n^*, b_n^*)\}_{n=1}^N$, is guaranteed since the set of functions $\{\log_e\left(1 + \frac{b_n}{x}\right), n = 1, 2, \ldots, N\}$ used in the approximation in (7) satisfies the Haar condition on $(0, \infty)$ with a null set $\{\infty\}$, since for every set of N distinct points $\{x_n\}_{n=1}^N, x > 0$, the determinant of the $N \times N$ matrix, whose (i, j)th entry is $\log_e\left(1 + \frac{b_i}{x_j}\right)$, is nonzero [42]. This condition is essential to establish a unique best Chebyshev approximation [43, Theorem 1]. ■

After characterizing the shape of the minimax error function, we need to find the solution which gives such an error function. This is achieved by formulating a set of nonlinear equations and solving them as explained next.

1) Optimization in Terms of Absolute Error: When considering the absolute error, the best logarithmic approximation for the ergodic capacity can be found by optimizing its corresponding parameters according to (20), which implies that we seek to minimize the maximum/global error. This problem can be solved by formulating a set of nonlinear equations that describe the best absolute error function which is proved to be uniform with all its extrema points alternating in sign with the same value of error per Proposition 2.

Corollary 1: The best approximation in terms of the absolute error is found as the solution to the following set of equations:

$$\begin{cases} d'(x_l) = 0, & \text{for } l = 1, 2, \ldots, L, \\ d(x_l) = (-1)^l\, d_{\max}, & \text{for } l = 1, 2, \ldots, L, \\ d_0 = \lim_{x \to 0} d(x) = d_{\max}, & \\ \sum_{n=1}^N a_n = 1, & \end{cases} \tag{22}$$

where $L = 2\,N - 1$.

The equation $\sum_{n=1}^N a_n = 1$ in (22) is actually a condition that is necessary to construct a bounded error function from the left, otherwise $d_0 = \pm\infty$. In particular, the first extrema point occurs asymptotically at zero, i.e., we choose x_0 to be a very small value near zero and assign $d_0 = d(x_0) = d_{\max}$. Thus, when meeting the condition, x_0 contributes only with a single equation that expresses the error value at that point, opposing to the other extrema points which contribute with two equations; one expresses its value and the other expresses the zero derivative of the error function at the corresponding stationary point. The absolute error is also bounded from the right, i.e., $d_\infty = \lim_{x\to\infty} d(x) = 0$. Therefore, with including the imposed condition, a total of $4N$ equations are constructed and their number is equal to the number of unknowns, namely, $\{(a_n^*, b_n^*)\}_{n=1}^N, \{x_l\}_{l=1}^L$ and $d_{\max}$.

It should be noted that $\tilde{C}(x)$ has a degree $D = 2N$ at the optimized set of coefficients $\{(a_n^*, b_n^*)\}_{n=1}^N$. However, the imposed condition $\sum_{n=1}^N a_n = 1$ decreases its degrees of freedom by one to be $D = 2N - 1$. Therefore, $\tilde{C}(x)$ with $\{(a_n^*, b_n^*)\}_{n=1}^N$ is the best Chebyshev approximation that alternates exactly $2N-1$ times between local maximum and minimum values of equal magnitude according to Proposition 2. This confirms exactly with the proposed approach in (22) which alternates $2N-1$ times and results in a total of $2N$ extrema points including x_0.

2) Optimization in Terms of Relative Error: Similar to optimizing the approximation's parameters in terms of the absolute error, the best approximation in terms of the relative error is derived by solving the minimax optimization problem in (20) through formulating a set of nonlinear equations describing the uniform minimax relative error function.

Corollary 2: The best approximation in terms of the relative error is found by the solution to the following set of equations:

$$\begin{cases} r'(x_l) = 0, & \text{for } l = 1, 2, \ldots, L, \\ r(x_l) = (-1)^{l+1}\, r_{\max}, & \text{for } l = 1, 2, \ldots, L, \\ r_0 = \lim_{x\to 0} r(x) = -r_{\max}, & \\ \sum_{n=1}^N a_n\, b_n = -r_{\max} + 1. & \end{cases} \tag{23}$$

In a similar way as for the absolute error, the extrema point x_0 is set to be a very small value near zero and it only contributes with a single equation ($r_0 = r(x_0) = -r_{\max}$). On the other hand, the relative error converges to a constant value when x tends to infinity opposing to $d(x)$ which converges to zero, i.e., $r_\infty = \lim_{x\to\infty} r(x) = \sum_{n=1}^N a_n\, b_n - 1$ and we assign $r_\infty = -r_{\max}$, which results in the last equation in (23). A solution to this system of equations yields the required optimized parameters $\{(a_n^*, b_n^*)\}_{n=1}^N$ that define the best approximation. Since no condition is imposed herein, $D = 2N$ and, hence, $r(x)$ alternates $2N$ times as seen in Fig. 4(a) which confirms with the proposed approach in (23).

3) Lower and Upper Bounds: The proposed minimax optimization method for the logarithmic approximation in (7) can also be extended to give upper and lower bounds in terms of both absolute and relative errors. They are additionally constrained in (21) by $e(x) \leq 0$ or $e(x) \geq 0$ when solving for the best lower or upper bound, respectively. Therefore, we construct the lower bound by shifting down the corresponding error function in such a way as to make it oscillate between zero and $-e_{\max}$ with $2N$ extrema and $e_0 = 0$ for the absolute error, and $2N+1$ extrema and $e_0 = -r_{\max}$ for the relative error. With these properties of the corresponding error function, the optimization problem can be easily formulated in the same manner as in (22) and (23) for both error measures.

Similarly, using the shifting approach, the error function is forced to oscillate between zero and $e_{\max}$ for the upper bound, resulting in an error function with $2N-1$ extrema and $e_0 = d_{\max}$ for the absolute error and with $2N+1$ extrema and $e_0 = 0$ for the relative error. It should be noted that for the upper bound in terms of absolute error, an extra equation $\sum_{n=1}^N a_n\, b_n - 1 = 0$ is added to the system of equations in order to get an equal number of equations and unknowns. In addition, for the absolute error, d_∞ is never counted as an extremum since it converges to zero when x tends to infinity, whereas for the relative error, r_∞ is counted as an extremum since it converges to a constant value when x tends to infinity.

C. Proof by Construction

We prove the existence of the proposed solution to (20) by construction. While the set of equations in (22) and (23) can be directly formulated and solved for any communication system in order to find the optimized sets of coefficients $\{(a_n^*, b_n^*)\}_{n=1}^N$ for the novel minimax approximations in (7), we have implemented the proposed methodology to find the optimized coefficients in terms of the absolute error for the Nakagami and lognormal capacity integrals which are to be used as building blocks in the capacity analysis of the more complicated systems as will be seen shortly. These coefficients are calculated by constructing (22) through substituting (7) with (3) for Nakagami capacity integral, or (5) for lognormal capacity integral, in (8) together with substituting (12) with (14) for Nakagami capacity integral, or (15) for lognormal capacity integral, in (10). Each formulated system of equations is then numerically solved using the `fsolve` command in Matlab with an equal number of equations and unknowns after using good initial guesses for the unknowns.

The coefficients $\{(a_n^*, b_n^*)\}_{n=1}^N$ are calculated for up to $N = 10$ or when the order of accuracy is 10^{-9}, and are released to public domain in a supplementary digital file.[2] Likewise, we prove the existence of the solutions to (23) and the bounds by finding them for two example cases in Section V. Together with the released data sets, we also provide a basic Matlab code that implements solving (22) to calculate the optimized coefficients of (7) for any communication system in terms of the absolute error.[2]

Despite the simplicity of implementing this numerical approach, the challenge is to find heuristic initial guesses for the unknowns: $\{(a_n, b_n)\}_{n=1}^N, \{x_l\}_{l=1}^L$ and $e_{\max}$. In this work, we have used iteratively random values for the lower values of N and then used curve fitting techniques to draw some relationships that indicate their successive values for higher values of N. We followed this procedure to find the initial guesses for one certain value for both m and σ and found the optimized values of the corresponding unknowns which are then used as initial guesses for the same optimization problem but with shifted values of m and σ with small steps; the new optimized values are then used for the next shifted values and so on. It is worth mentioning that the numerical coefficients of $\{(a_n, b_n)\}_{n=1}^N$ in Lemma 1 can be a very good choice as initial guesses too, especially for the lower values of N, to converge

[2]Available at `https://doi.org/10.5281/zenodo.6641977` for download.

to the optimized values or at least to work as mean values around which small random variance is introduced.

IV. Applications of the Proposed Approximations and Bounds

As discussed earlier, one can straightforwardly apply the methodology of Section III-B in order to solve the coefficients of (1) for any communication system. Alternatively, one can use $\tilde{C}_m(x)$ and $\tilde{C}_\sigma(x)$ as building blocks to derive the ergodic capacity whenever possible. Particularly in this section, we mainly focus on the second approach for which we study its important role in simplifying the complicated integrals encountered when evaluating the ergodic capacity in different communication scenarios.

A frequently seen integral in the intermediate steps when analyzing the performance of many wireless communication systems with respect to their ergodic capacity [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [32], [33], [34], [35], [36], [37] has a similar form to that of the Nakagami capacity integral as

$$\begin{aligned} I_{m,\phi}(x) &\triangleq \int_0^\infty \log_e(1+x\,t)\; t^{m-1}\exp(-\phi\,t)\,\mathrm{d}t \\ &= \phi^{-m}\,\Gamma(m)\,C_m(\phi/(x\,m)) \\ &\approx \phi^{-m}\,\Gamma(m)\,\tilde{C}_m(\phi/(x\,m)). \end{aligned} \tag{24}$$

Above, the second line has been written in terms of (3) and the third line is correspondingly approximated in terms of (7).

In particular, $\tilde{C}_m(x)$ can be used to directly approximate the ergodic capacity of a Nakagami-m channel as $\bar{\mathcal{C}} \approx \tilde{C}_m(1/\bar{\gamma})/\log_e(2)$ including Rayleigh fading as a special case with $m = 1$, and $\tilde{C}_\sigma(x)$ can be used to directly approximate the ergodic capacity of a lognormal channel as $\bar{\mathcal{C}} \approx \tilde{C}_\sigma(1/\bar{\gamma})/\log_e(2)$. Next, we illustrate the use of $\tilde{C}_m(x)$ to evaluate the ergodic capacity in different communications systems under small-scale fading, and then the use of $\tilde{C}_\sigma(x)$ to approximate the ergodic capacity when the lognormal shadowing is introduced to the system. One can also use $\tilde{C}_m(x)$ to evaluate the ergodic capacity of the more complicated systems that encounter a similar integral as $I_{m,\phi}(x)$ in (24) and do not eventually result in the logarithmic expression (1); such a case is illustrated in Section IV-D.

A. Ergodic Capacity Under Small-Scale Fading

In addition to the Nakagami-m distribution (and Rayleigh distribution thereof), $\tilde{C}_m$ is used to approximate the capacity integral of the single-antenna systems over the more complicated distributions as

$$\bar{\mathcal{C}} \approx \tfrac{1}{\log_e(2)} \textstyle\sum_{j=0}^{\infty} \Phi_j\, \tilde{C}_{m_j}\left(\frac{\theta_j}{\bar{\gamma}}\right) \approx \sum_{n=1}^{N} a_n \log_2(1+b_n\,\bar{\gamma}), \tag{25}$$

where Φ_j, $j = 0, 1, \ldots$, are constants. Table I lists the values of Φ_j, m_j and θ_j for the ergodic capacities of SISO systems under different fading distributions. It should be mentioned that the infinite series in (25) results from expanding the modified Bessel function of the first kind as a power series (which is included in the PDF of many of the fading distributions) [38, Eq. 9.6.12], and it can be truncated up to several terms that are adequate to obtain the required accuracy. The double-summation logarithmic terms (when including the approximation sum) can be rearranged into a single summation, yielding the same logarithmic approximation as in (1)

TABLE I

Values of Φ_j, m_j and θ_j for the Ergodic Capacities of SISO Systems Under Different Fading Distributions

Fading	Φ_j	m_j	θ_j
Rice	$\frac{K^j}{j!}\exp(-K)$	$j+1$	$\frac{1+K}{1+j}$
Nakagami-q (Hoyt)	$\frac{(2j)!q(1+q^2)^{-2j-1}(1-q^2)^{2j}}{j!^2\,2^{2j-1}}$	$2j+1$	$\frac{(1+q^2)^2}{4(2j+1)q^2}$
$\eta-\mu$	$\frac{\sqrt{\pi}\Gamma(2\mu+2j)2^{-2\mu-2j+1}h^{-\mu-2j}H^{2j}}{\Gamma(\mu+j+\frac{1}{2})\Gamma(\mu)\Gamma(j+1)}$	$2\mu+2j$	$\frac{2\mu h}{2\mu+2j}$
$\kappa-\mu$	$\frac{(\mu\,\kappa)^j}{j!\exp(\mu\kappa)}$	$\mu+j$	$\frac{\mu(1+\kappa)}{(\mu+j)}$

*notes: K is the Rician factor, $h=(2+\eta^{-1}+\eta)/4$ and $H=(\eta^{-1}-\eta)/4$ for Format 1 of the $\eta-\mu$ distribution and $h=\frac{1}{(1-\eta^2)}$ and $H=\eta/(1-\eta^2)$ for Format 2

Table II lists closed-form expressions for the ergodic capacity of various point-to-point multi-antenna systems in terms of $\tilde{C}_m(x)$, where they usually encounter similar integrals as $I_{m,\phi}(x)$ in (24). In particular, we consider two diversity combining techniques for SIMO, namely, maximum ratio combining (MRC) and selection combining (SC) at the receiver (RX). We also consider some MISO schemes including beamforming (BF) or distributed MISO systems with channel distribution information (CDIT) at the transmitter (TX), in addition to space time block codes (STBCs). Finally, some combined transmit–receive diversity and spatial multiplexing schemes are considered for MIMO channels.

B. Ergodic Capacity Under Small-Scale Fading Channels With Lognormal Shadowing

Another side of novelty is that this tool enables the evaluation of the ergodic capacity for different communication systems in the presence of shadowing and results in the same logarithmic approximation as in (1). In particular, the capacity integral of a composite fading channel with $\gamma_{\text{eff}} = \psi\, s$, where ψ and s are two independent random variables representing the respective small-scale and lognormal fading, is calculated by averaging the small-scale distributed SNR over the conditional density of the lognormal-distributed conditional SNR, i.e., the average SNR of the small-scale fading is lognormally distributed, thus

$$\bar{\mathcal{C}} = \mathrm{E}_{\gamma_{\text{eff}}}[\log_2(1+\gamma_{\text{eff}})] = \mathrm{E}_s[\mathrm{E}_{\gamma_{\text{eff}}|s}[\log_2(1+\gamma_{\text{eff}})]]. \tag{26}$$

The inner expectation which refers to the small-scale fading can be directly evaluated in terms of $\tilde{C}_m(x)$ and it results in a similar expression as in (5) when considering the outer expectation which refers to the shadowing effect, for which we apply $\tilde{C}_\sigma(x)$.

Next, we calculate the ergodic capacity for some single-antenna and multi-antenna systems, for which we use $\{(a_{n_1,m}, b_{n_1,m})\}_{n_1=1}^{N_1}$ to refer to the optimized coefficients of $\tilde{C}_m$ of the Nakagami capacity integral in (3). The ergodic capacity of a Nakagami–lognormal composite fading channel can be approximated as a function of the lognormal average SNR ($\bar{\gamma}_s$) as $\bar{\mathcal{C}} \approx \sum_{n_1=1}^{N_1} a_{n_1,m}\, \tilde{C}_\sigma(1/(b_{n_1,m}\bar{\gamma}_s))/\log_e(2)$ including Rayleigh fading as a special case with $m = 1$.

TABLE II

THE ERGODIC CAPACITY OF SOME MULTI-ANTENNA SYSTEMS IN TERMS OF $\tilde{C}_m(x)$

Communication system	Fading	$\bar{\mathcal{C}} \cdot \log_e(2)$
Receiver spatial diversity (SIMO) with optimal rate adaptation to channel fading with constant transmit power	Rayleigh	MRC at RX [10]: $\tilde{C}_{N_r}\left(\frac{1}{N_r\bar{\gamma}}\right)$ SC at RX [10]: $N_r\sum_{i=0}^{N_r-1}\frac{(-1)^i}{i+1}\binom{N_r-1}{i}\tilde{C}_1\left(\frac{i+1}{\bar{\gamma}}\right)$
	Nakagami-m	MRC at receiver [13]: $\tilde{C}_{N_r\times m}\left(\frac{1}{N_r\bar{\gamma}}\right)$
Transmitter spatial diversity (MISO)	Rayleigh	STBC for uncorrelated channels: $\tilde{C}_{N_t}\left(\frac{1}{N_t\bar{\gamma}}\right)$ Distributed MISO system [6, Eq. 4]: $\sum_{i=1}^{M}\sum_{n=1}^{K_i} a_{in}\,\bar{\gamma}_i^n\tilde{C}_n\left(\frac{1}{n\bar{\gamma}_i}\right)$
	Nakagami-m	STBC for uncorrelated channels: $R\,\tilde{C}_{m\times N_t}(\frac{1}{mN_t\bar{\gamma}})$
	Rice	STBC for uncorrelated channels: $R\sum_{i=0}^{\infty}\frac{(N_tK)^i\,\exp(-N_tK)}{\Gamma(i+1)}\tilde{C}_{N_t+i}\left(\frac{1}{(N_t+i)\bar{\gamma}}\right)$ Optimum BF with CDIT at TX [17]: $\exp(-\frac{m_{\mathbf{V}}^2}{\sigma_{\mathbf{V}}})\sum_{i=0}^{\infty}\frac{m_{\mathbf{V}}^{2i}}{i!\sigma_{\mathbf{V}}^i}\tilde{C}_{i+1}\left(\frac{1}{(i+1)\sigma_{\mathbf{V}}\,\rho}\right)\vert_{\mathbf{V}=\mathbf{V}_\theta}$
Combined transmit–receive diversity (MIMO)	Rayleigh	Maximum ratio transmission with MRC at RX [12]: $\sum_{k=1}^{m}\sum_{l=n-m}^{(n+m-2k)k} a'_{k,l}\tilde{C}_{l+1}\left(\frac{k}{(l+1)\bar{\gamma}}\right)$ STBC for uncorrelated channels [3], [4]: $R\,\tilde{C}_{N_r\times N_t}\left(\frac{1}{N_rN_t\bar{\gamma}}\right)$ STBC for correlated channels [5]: $R\sum_{i=1}^{g}\sum_{j=1}^{\nu_g}K_{i,j}\tilde{C}_j\left(\frac{1}{j\,a\,\tilde{\lambda}_i}\right)$
	Nakagami-m	STBC for uncorrelated channels [3], [4]: $R\,\tilde{C}_{m\times N_r\times N_t}(\frac{1}{mN_rN_t\bar{\gamma}})$
	Rice	STBC for uncorrelated channels [3], [4]: $R\sum_{i=0}^{\infty}\frac{(N_rN_tK)^i\,\exp(-N_rN_tK)}{\Gamma(i+1)}\tilde{C}_{N_r\times N_t+i}\left(\frac{1}{(N_rN_t+i)\bar{\gamma}}\right)$
Spatial multiplexing (MIMO)	Rayleigh	i.i.d. channels [8], [9]: $\sum_{z=0}^{\alpha-1}\sum_{j=0}^{z}\sum_{i=0}^{2j}\frac{(-1)^i(2j)!(\beta-\alpha+i)!}{2^{2z-i}j!i!(\beta-\alpha+j)!}\binom{2z-2j}{z-j}\binom{2\beta-2\alpha+2j}{2j-i}\tilde{C}_{\beta-\alpha+i+1}\left(\frac{N_t}{(\beta-\alpha+i+1)\rho}\right)$ Correlated channels without CSI at TX: [11, Eq. 25] with $\left\{\mathbf{\Psi}_1(k)\right\}_{i,j}=(\frac{1}{\phi_j})^{t-i+1}\Gamma(t-i+1)\tilde{C}_{t-i+1}(\frac{1}{\rho(t-i+1)\phi_j})$, if $i=k$ Correlated channels with partial CSI at TX: [11, Eqs. 27 and 28] with $\left\{\mathbf{\Psi}_{2B}(k)\right\}_{i,j}=(\frac{1}{\phi_j})^{s-i+1}\Gamma(s-i+1)\tilde{C}_{s-i+1}(\frac{1}{\rho(s-i+1)\phi_j})$, if $i=k$

*notes: N_t is the number of transmit antennas, N_r is the number of receive antennas, $\alpha=\min\{N_t,N_r\}$, $\beta=\max\{N_t,N_r\}$, ρ is the transmit SNR, a_{in}, M and K_i are defined in [6], $a'_{k,l}$ is derived in [12], R is the code rate of the STBC, K is the Rician factor, $K_{i,j}$, a, g and $\tilde{\lambda}_i$ are defined in [5], $m_{\mathbf{V}}$, $\sigma_{\mathbf{V}}$ and $\mathbf{V}_\theta$ are defined in [17].

Moreover, it is calculated using Table I for the more complicated small-scale distributions with lognormal shadowing as

$$\bar{\mathcal{C}} \approx \frac{1}{\log_e(2)}\sum_{j=0}^{\infty}\sum_{n_1=1}^{N_1} a_{n_1,m_j}\,\Phi_j\,\tilde{C}_\sigma\left(\frac{\theta_j}{b_{n_1,m_j}\bar{\gamma}_s}\right)$$
$$\approx \sum_{n=1}^{N} a_n \log_2\left(1+b_n\,\bar{\gamma}_s\right) = \frac{1}{\log_e(2)}\,\tilde{C}\left(\frac{1}{\bar{\gamma}_s}\right), \quad (27)$$

where the latter form occurs after applying $\tilde{C}(x)$ twice and rearranging the triple summation into a single one with truncating the outer summation to sufficient number of terms.

In the same way as above, the ergodic capacity of some multi-antenna systems under small-scale fading and lognormal shadowing can also be approximated using $\tilde{C}_m(x)$ and $\tilde{C}_\sigma(x)$. In particular, the ergodic capacity of MIMO spatial multiplexing over Rayleigh fading channels with lognormal shadowing [32], [33] is calculated as

$$\bar{\mathcal{C}} \approx \frac{1}{\log_e(2)}\sum_{z=0}^{\alpha-1}\sum_{j=0}^{z}\sum_{i=0}^{2j}\sum_{n_1=1}^{N_1} a_{n_1,\beta-\alpha+i+1}$$
$$\times\frac{(-1)^i(2j)!(\beta-\alpha+i)!}{2^{2z-i}j!i!(\beta-\alpha+j)!}\binom{2z-2j}{z-j}\binom{2\beta-2\alpha+2j}{2j-i}$$
$$\times\tilde{C}_\sigma\left(\frac{N_t}{(\beta-\alpha+i+1)\,\rho\,b_{n_1,\beta-\alpha+i+1}\,\bar{\gamma}_s}\right). \quad (28)$$

Moreover, the ergodic capacity of cooperative spatial multiplexing systems with Rayleigh fading and lognormal shadowing [34] is calculated as

$$\bar{\mathcal{C}} \approx \frac{1}{\log_e(2)}\sum_{k=1}^{\varrho}\sum_{n_1}^{N_1}\frac{a_{n_1,N_r-\varrho+1}}{2}$$
$$\times\tilde{C}_\sigma\left(\frac{1}{(N_r-\varrho+1)\rho_0\,\Omega_{RD,k}\,b_{n_1,N_r-\varrho+1}}\right), \quad (29)$$

where $\Omega_{RD,k}$ is the channel mean power for the link from the kth relay to the destination, ϱ is the number of relays and ρ_0 is the average SNR per symbol.

C. Ergodic Capacity in Recent Research Directions

After the above wide range of fundamental applications for the proposed approximations/bounds, let us proceed to illustrate their applicability and usefulness in timely wireless systems with specific applications from the recent literature.[3]

In particular, the ergodic capacity (2) of downlink non-orthogonal multiple access (NOMA) system over the $\alpha-\mu$ fading distribution [44] does not admit a similar integral as $I_{m,\phi}(x)$ in (24) as intermediate step and, thus, we cannot use $\tilde{C}_m(x)$ to calculate its ergodic capacity. For that, we implement the first proposed approach which means directly approximating the ergodic capacity (2) by (7). We have used the openly released Matlab code[2] which we have modified to make it comply with the studied system in order to find the optimized

[3]We had to use some notations and symbols herein which are the same as in the original publications to preserve comparability, due to which some unavoidable overloading exists in this subsection compared to the rest of the article.

coefficients for $\alpha = \mu = N = 2$ with two users, $L = 2$, (U_1 and U_2) in terms of the absolute error to approximate the ergodic capacity for both users respectively as

$$\bar{\mathcal{C}}_{U_1} \approx \frac{1}{\log(2)}\Big[\sum_{n=1}^{2} a_n \log_2(1+b_n\,\bar{\gamma}) - \sum_{n=1}^{2} a_n \log_2(1+b_n\,\beta_2\,\bar{\gamma})\Big], \tag{30}$$

with $\{(a_n,b_n)\}_{n=1}^{2} = \{(0.336, 0.172), (0.664, 0.835)\}$, and

$$\bar{\mathcal{C}}_{U_2} \approx \tfrac{1}{\log(2)} \textstyle\sum_{n=1}^{2} a_n \log_2(1+b_n\,\beta_2\,\bar{\gamma}), \tag{31}$$

with $\{(a_n,b_n)\}_{n=1}^{2} = \{(0.409, 0.610), (0.591, 1.887)\}$. The parameter $\beta_l, l = 1, 2, \ldots, L$ is the power allocation coefficient. In particular, $\{(a_n,b_n)\}_{n=1}^{N}$ can be calculated for the logarithmic approximation of $\bar{\mathcal{C}}_l = C_l(1/(\beta_l\,\bar{\gamma}))/\log_e(2)$ in [44, Eq. 46] by formulating (22) through substituting (7) and (2) to (8) together with substituting (12) and (13) to (10). The PDF $f_G(t)$ in (2) corresponds herein to $f_\gamma(\frac{\gamma}{\bar{\gamma}})$ in [44, Eq. 8]. These equations are then solved using the `fsolve` command in Matlab. The openly released code[2] can be used after modification to find the optimized coefficients for any values of α, μ and L.

On the other hand, we can derive the ergodic capacity in terms of $\tilde{C}_m(x)$, if the system encounters similar integral as $I_{m,\phi}(x)$ in (24). For example, the ergodic capacity for a system with coordinated multipoint reception for mm-wave uplink with blockages and Nakagami-m fading [35] can be calculated as

$$\bar{\mathcal{C}} \approx \sum_{n=1}^{N}\sum_{i=1}^{n}\sum_{k=1}^{m_i} \frac{k^k}{\log(2)} q_n \Lambda_{n,i,k} \tilde{C}_k\left(\frac{N}{k\,\bar{\gamma}_i}\right), \tag{32}$$

where m_i is the Nakagami parameter of the ith link, N is the number of base stations, q_n is defined in [35, Eq. 8] and $\Lambda_{n,i,k}$ is recursively obtained using [35, Eq. 9 and Eq. 10].

Likewise, the ergodic capacity is calculated for a mm-wave downlink NOMA system over fluctuating two-ray channels under general power allocation in [36] as

$$\bar{\mathcal{C}} \approx \frac{1}{\log(2)}\Bigg[\sum_{j_p=0}^{\infty} H_p\,\tilde{C}_{j_p+1}\left(\frac{1}{2\sigma_p^2(j_p+1)\,a\,Q_p\,\bar{\gamma}}\right) + \sum_{j_p=0}^{\infty} H_p\,\tilde{C}_{j_q+1}\left(\frac{1}{2\sigma_q^2(j_q+1)\,Q_q\,\bar{\gamma}}\right) - \sum_{j_q=0}^{\infty} H_q\,\tilde{C}_{j_q+1}\left(\frac{1}{2\sigma_q^2(j_q+1)\,a\,Q_q\,\bar{\gamma}}\right)\Bigg], \tag{33}$$

where its parameters are defined in [36]. In addition, the ergodic capacity for reflecting intelligent surface-assisted SISO system with correlated channels [37] can be approximated as

$$\bar{\mathcal{C}} \approx \frac{1}{\log(2)}\,\tilde{C}_{k_a}\left(\frac{1}{k_a\,w_a\,\rho_0}\right), \tag{34}$$

where k_a and w_a are defined in [37, Eqs. 5 and 6], respectively.

D. Ergodic Capacity of Dual-Hop Fixed-Gain Relay Networks in Nakagami-m Fading

This subsection gives an application example on the use of $\tilde{C}_m$ in the intermediate steps when analyzing the capacity of the more complex wireless systems without necessarily resulting in the same logarithmic expression as in (1). In particular, we study the performance of a dual-hop fixed-gain relay network under Nakagami-m fading [25]. Its ergodic capacity can be approximated as

$$\begin{aligned}\bar{\mathcal{C}} \approx\ & \frac{1}{\log_e(2)} \exp\left(\frac{m_1}{\bar{\gamma}_1}\right) \frac{{m_1}^{m_1}}{\Gamma(m_1)} \sum_{n_1=1}^{N_1} a_{n_1,m_2} \\ & \times\Bigg[\sum_{j=0}^{m_1-1}\sum_{n_2=1}^{N_2} a_{n_2,j+1}\binom{m_1-1}{j} m_1^{-j-1}\, j!\,\bar{\gamma}_1^{-m_1+j+1} \\ & \times(-1)^{m_1-j-1}\log_e\left(1+\frac{(j+1)\bar{\gamma}_1\bar{\gamma}_2 b_{n_1,m_2} b_{n_2,j+1}}{m_1\,\varkappa}\right) \\ & - \sum_{i=0}^{\infty}\sum_{j=0}^{m_1-1}\binom{m_1-1}{j}\frac{m_1^i}{i!}\frac{(-1)^{m_1+j+i-1}}{j+i+1}\frac{1}{\bar{\gamma}_1^{m_1+i}} \\ & \times\Bigg[\left(1-\left(\frac{-\varkappa}{\bar{\gamma}_2\,b_{n_1,m_2}}\right)^{j+i+1}\right)\log_e\left(1+\frac{\bar{\gamma}_2\,b_{n_1,m_2}}{\varkappa}\right) \\ & + \sum_{q=1}^{j+i+1}\frac{(-1)^q\,\varkappa^{q-1}}{(j+i-q+2)\,(\bar{\gamma}_2\,b_{n_1,m_2})^{q-1}}\Bigg]\Bigg] \\ & - \frac{1}{\log_e(2)}\tilde{C}_{m_2}\left(\frac{\varkappa}{\bar{\gamma}_2}\right). \end{aligned} \tag{35}$$

where $\varkappa$ is a constant defined in [45, Eq. 16], $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are the statistical averages of the instantaneous SNRs γ_1 and γ_2 of the first and second hop, respectively, whose fading parameters are m_1 and m_2. This expression is valid for any value of m_1 opposing to [25] which is valid only for integer values of m_1. It is worth mentioning that the same expression in (35) is also obtained when evaluating the ergodic capacity under Rician, Nakagami-q (Hoyt), $\eta - \mu$ and $\kappa - \mu$ distributions without performing individual analysis for each fading distribution. The detailed derivation of (35) is available in Appendix B.

E. Tractability Comparison

In this section, we illustrate the mathematical tractability of the proposed approximations and bounds and the insightful observations gained from using them for calculating the ergodic capacity of the different communication systems. For that, we consider some of the previous example applications and compare the novel analytical expressions derived herein with the corresponding expressions in the literature.

In Sections IV-A and IV-B, the capacity of the single-antenna and multi-antenna systems under small-scale fading or when combined with lognormal shadowing is evaluated using the proposed tool into the elegant simple logarithmic form in (1) which is unified for all these systems. On the contrary, it is evaluated in the literature as different complicated expressions that are unique always to the specific system under study so that a complete study and analysis are required for each system independently and using different mathematical steps.

In particular, the ergodic capacity is written in terms of the exponential integral and the incomplete gamma function in [15] for Rician fading, and in terms of the Meijer G-function in [18] for $\kappa - \mu$ fading. In [26], [29], [27], and [28], the ergodic capacity is written in terms of the Gaussian Q-function or the multiplication of the complementary error function by the exponential function. On the other hand, to the best of our knowledge, there is no available ergodic capacity analysis for the SISO system in the literature on the composite lognormal fading models and only asymptotic analysis is available in [46]. Thus, the proposed tool renders new analytical solutions that were previously deemed unattainable.

Table II evaluates the ergodic capacity of different multi-antenna system models and various fading distributions as a summation of logarithmic functions, whereas the corresponding references write them using complex functions such as the exponential integral and incomplete gamma functions. Moreover, the ergodic capacity of the multi-antenna systems under combined fading in (28) and (29) is written respectively in terms of the exponential function together with exponential integral functions in [32] and in terms of the incomplete gamma function together with the power function in [34].

The impressive advantage in terms of analytical complexity is best seen in the timely applications of Section IV-C. In particular, the ergodic capacity of the downlink NOMA over the $\alpha - \mu$ fading in [44] and of the NOMA-based mm-wave communications in [36] are written respectively in terms of the complicated Fox H-function and the Meijer G-function which are themselves unsolvable integrals, whereas they are written in (30), (31) and (33) in the unified logarithmic form.

The importance and elegance of the proposed tool are demonstrated by its ability to provide direct or even visual insights into the system's performance opposing to expressions that comprise special functions. For example, from Table II, we can immediately see that the ergodic capacity improves with increasing N_r and $\bar{\gamma}$ for the SIMO systems, whereas it improves with increasing N_t, $\bar{\gamma}$, m under Nakagami-m fading, K under Rician fading and R of the STBC, all for the MISO systems. On the other hand, none of these observations can be concluded from the corresponding complicated expressions in the literature.

V. Numerical Results and Discussion

This section demonstrates the accuracy of the proposed approximations and bounds while the actual behavior of the corresponding systems has already been analyzed extensively in the references. In particular, we compare them with previously derived ones, in addition to the numerical approximations obtained by applying Gauss–Laguerre and Gauss–Hermite quadrature rules for the Nakagami and the lognormal capacity integrals, respectively. Furthermore, we validate and compare some of the application examples presented in Section IV with those obtained from the numerical and existing approximations and bounds.

Let us begin with plotting the global absolute error, $d_{\max}$, for the Nakagami capacity integral in Fig. 2(a) for different

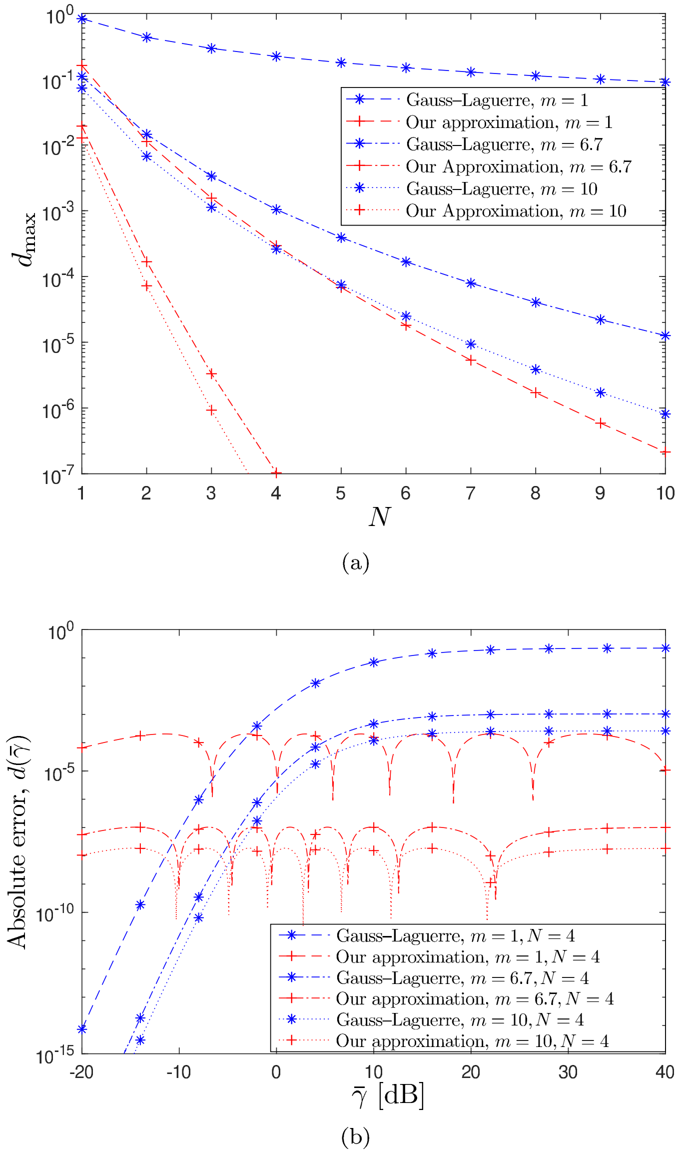
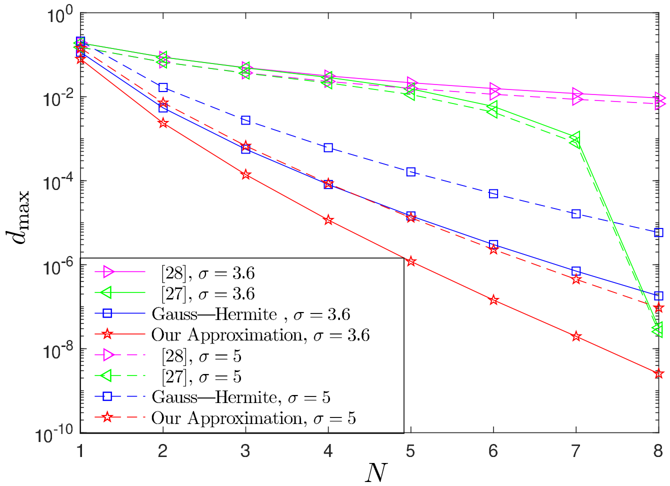


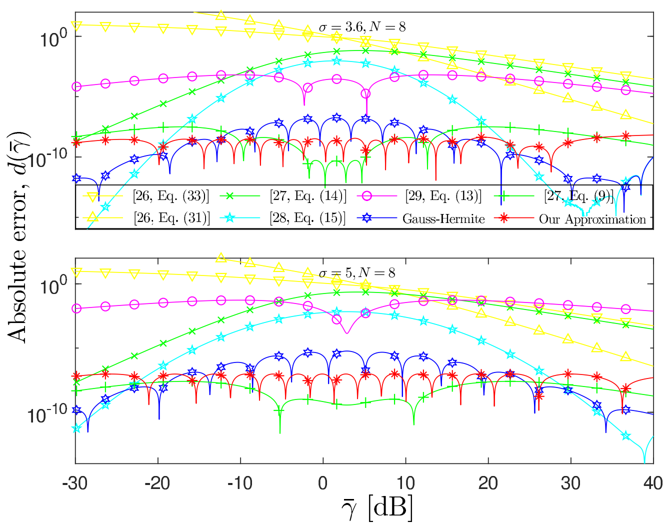
Fig. 2. (a) Comparison between our approximations and those obtained using Gauss–Laguerre for the Nakagami capacity integral with different values of m in terms of global absolute error. (b) Same as (a) but in terms of the absolute error function over the whole considered range of the argument with $N = 4$.

values of m, using our approximations and the numerical approximation resulting from applying the Gauss–Laguerre quadrature rule. It is clearly realized from the figure that our minimax coefficients result in much more accurate logarithmic approximations in terms of the global error than those resulting from numerical integration. Moreover, as the number of terms increases, the accuracy increases substantially, especially for higher values of m. We further verify the accuracy of the proposed approximation by comparing its absolute error with that of the Gauss–Laguerre approximation for the whole considered range of the argument in Fig. 2(b). Obviously, our optimized coefficients not only have the least global error, but they also achieve higher accuracy for most of the considered range of the argument for the different values of m.

Moreover, the same comparisons are made for the lognormal capacity integral for different values of σ. Our approximations are compared with those obtained using the Gauss–Hermite

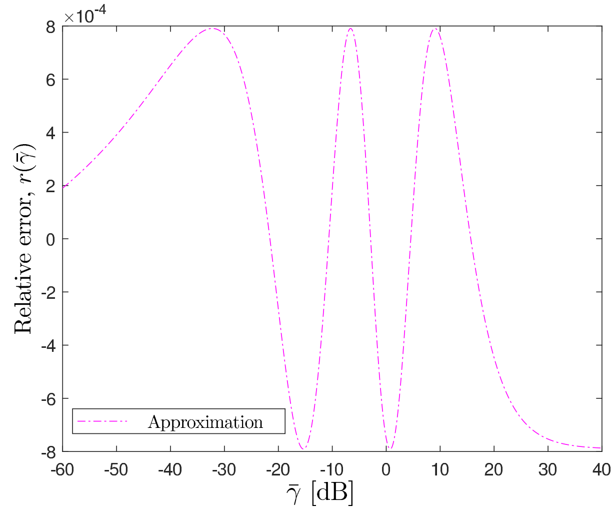


(a)

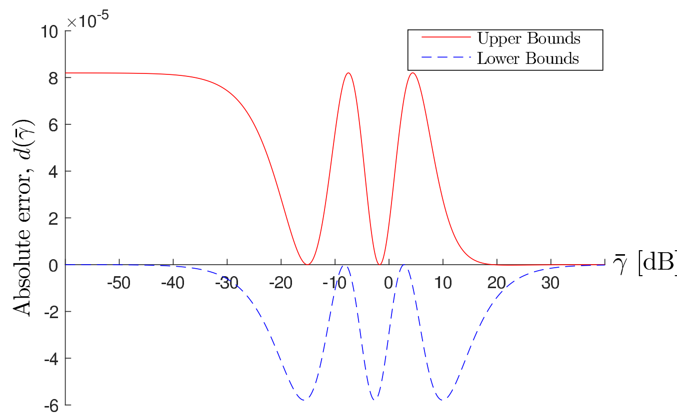


(b)

Fig. 3. (a) Comparison between our approximations and those obtained using Gauss–Hermite and the existing approximations for the lognormal capacity integral with different values of σ in terms of global absolute error. (b) Same as (a) but in terms of the absolute error function over the whole considered range of the argument with $N = 8$.



(a)



(b)

Fig. 4. (a) The optimized approximation in terms of relative error for Rayleigh capacity integral with $N = 3$. (b) The optimized upper and lower bounds in terms of the absolute error for Nakagami capacity integral with $m = 3$ and $N = 3$.

quadrature rule which has the same logarithmic form, in addition to the existing approximations which encounter very complicated functions such as the complementary Gaussian error function and the trigonometric functions [26], [27], [28], [29]. The proposed approximations mostly outperform all the other ones in terms of the global error as depicted in Fig. 3(a). They also have comparable or even better accuracy than those with the very complex form over the whole considered range of the argument as seen in Fig. 3(b) despite their significantly simpler form.

The minimax optimization method is not only used for constructing the approximations in terms of the absolute error but also for the approximations in terms of relative error as explained in Section III-B.2, and for the lower and upper bounds in terms of both error measures as explained in Section III-B.3. The approximation for the special case of Rayleigh capacity integral is optimized in terms of the relative error for $N = 3$ and the corresponding relative error function is plotted in Fig 4(a), whereas in Fig. 4(b), we plot the uniform absolute error functions resulting from the optimized upper and lower bounds of the Nakagami capacity integral for $m = 3$ and $N = 3$. As expected, the resulting error functions oscillate uniformly and achieve high accuracy. We can conclude from Figs. 2, 3 and 4 that the proposed approximations with the optimized coefficients achieve significant improvement in accuracy by several orders of magnitude when compared to the numerical and existing approximation. The absolute and relative errors are so small that they are virtually exact with the actual capacity measures.

Next, we numerically investigate some of the applications of the proposed approximations which are included in Section IV. In Fig. 5, the ergodic capacity for Rician fading channel with lognormal shadowing is studied and its absolute error is plotted for different values of the Rician factor using three approaches, namely, (i) Gauss–Laguerre and Gauss–Hermite

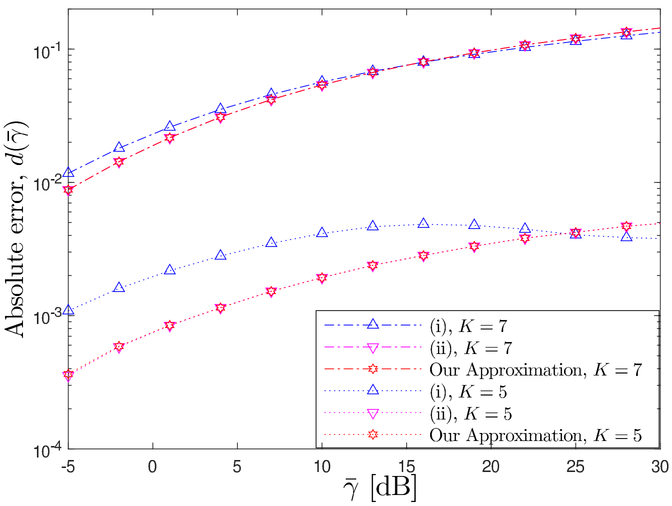


Fig. 5. Comparison of the absolute error function of the proposed approximations and the numerical ones for shadowed-Rician network with $N_1 = 4$, $N_2 = 6$, $\sigma = 4$, and different values of K.

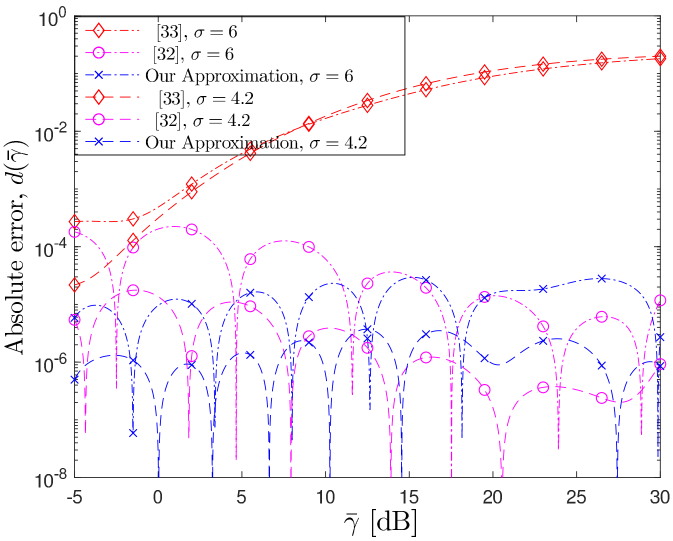


Fig. 6. Comparison of the absolute error function of the proposed approximations and the existing ones for 2×2 MIMO network with $N_1 = 7$, $N_2 = 5$, $\rho = 1$ dB and different levels of shadowing.

rules respectively, (ii) using (3) for the small-scale stage and then Gauss–Hermite rule for the shadowing stage and finally (iii) using (27) with the necessary coefficients from Table I. We can observe that approach (iii) results in a tighter approximation than that of approach (i) which has exactly the same analytical form. It also has the same accuracy as that of approach (ii).

Figure 6 illustrates the error resulting from applying our approximation to evaluate the ergodic capacity in 2×2 MIMO network over shadowed-Rayleigh channel as in (28), and compares it with the theoretical results presented in [32] and [33]. Our optimized coefficients yield significantly higher accuracy than those of [33], having exactly the same logarithmic form and number of terms. Despite the simplicity of our approximation's analytical form compared to that of [32], it achieves higher accuracy over a wide range of the argument. The ergodic capacity of both users, (30) and (31), is plotted in Fig. 7 along with the exact capacity derived in [44]

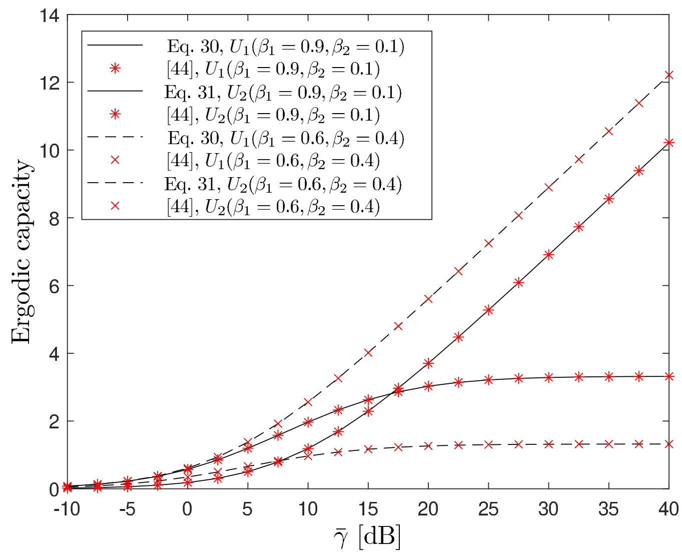


Fig. 7. Ergodic capacity for two NOMA users with $\alpha = \mu = 2$ and for different values of β_2.

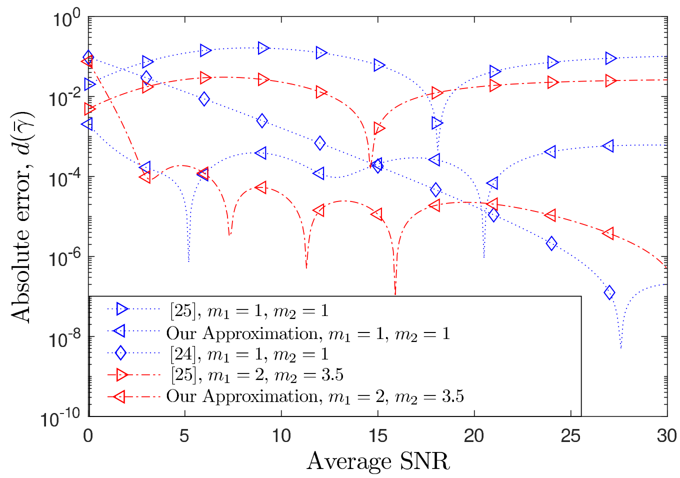


Fig. 8. The absolute error of the ergodic capacity in a dual-hop cooperative system with $N_1 = N_2 = 4$.

for different selections of power allocation coefficients. The figure shows virtually exact match between the logarithmic approximation and the exact results with only two logarithmic terms ($N = 2$).

In Fig. 8, the absolute error of the ergodic capacity in a dual-hop cooperative system is plotted as a function of the average SNR of each hop, where we considered $\bar{\gamma}_1 = \bar{\gamma}_2$. It is clear from the figure that the ergodic capacity resulting from applying our approximation is extremely accurate. In particular, the mathematical form of the ergodic capacity in (35) is not only much more tractable than that in [24] and [25], but also its accuracy outperforms [24] for the lower and moderate values, when considering Rayleigh fading channels, and outperforms [25] over the whole range of the argument when considering Nakagami-m fading channels, with less error by three orders of magnitude.

VI. Conclusion

This paper presented an accurate and efficient tool for facilitating statistical performance analysis in different wireless communication systems in terms of ergodic capacity. A novel systematic methodology was also developed in order to optimize its accuracy in the minimax sense. This tool was applied to a wide range of fundamental and recent applications, including single-antenna and multi-antenna systems under small-scale fading and with or without lognormal shadowing in order to derive tractable closed-form expressions for the ergodic capacity. We validated the tightness of the proposed tool by numerical comparisons with existing and numerical ones, in which our tool showed significant improvement in the accuracy by several orders of magnitude.

Appendix A
Proof of Proposition 1

Denoting that the PDF of instantaneous capacity $\mathcal{C}$ is given by $f_{\mathcal{C}}(c)$, the ergodic capacity is calculated as

$$\begin{aligned}\mathrm{E}[\mathcal{C}] &= \int_0^\infty c\, f_{\mathcal{C}}(c)\,\mathrm{d}c \\ &= \int_0^\infty \underbrace{\frac{\bar{\gamma}_{\text{eff}}\, f_{\mathcal{C}}(\log_2(1+\bar{\gamma}_{\text{eff}}\, g))}{\log_e(2)\,(1+\bar{\gamma}_{\text{eff}}\, g)}}_{\triangleq f_G(g)} \log_2(1+\bar{\gamma}_{\text{eff}}\, g)\,\mathrm{d}g,\end{aligned} \tag{36}$$

where the second expression is obtained by changing the integration variable to $g \triangleq (2^c-1)/\bar{\gamma}_{\text{eff}}$. Next, we implement the Riemann sum method to approximate the above integral by truncating it and dividing the integration interval into N partitions, each of length δ. Therefore, the ergodic capacity can be approximated by a finite sum of logarithmic functions according to (1) by choosing

$$a_n \triangleq \frac{\delta\,\bar{\gamma}_{\text{eff}}\, f_{\mathcal{C}}(\log_2(1+\bar{\gamma}_{\text{eff}}\, n\delta))}{\log_e(2)\,(1+\bar{\gamma}_{\text{eff}}\, n\delta)} \quad \text{and} \quad b_n \triangleq n\,\delta, \tag{37}$$

while arbitrary accuracy can be achieved when $\delta \to 0$ and $N \to \infty$. Furthermore, by applying appropriately the left, intermediate or the right rule for each partition, one can always guarantee that $\sum_{n=1}^{N} a_n \le 1$, since each a_n represents a part of the total probability mass of random variable $G \triangleq (2^{\mathcal{C}}-1)/\bar{\gamma}_{\text{eff}}$, whose PDF is denoted by $f_G(g)$.

In the general case without making specific assumptions about the distribution of $\mathcal{C}$, coefficients a_n, $n = 1, 2, \ldots, N$, will depend on $\bar{\gamma}_{\text{eff}}$, which would make (1) an inconvenient approximation for the statistical analysis of specific systems. However, we can express $f_{\mathcal{C}}(c)$ in terms of $f_G(g)$ as

$$f_{\mathcal{C}}(c) = \frac{2^c\,\log_e(2)}{\bar{\gamma}_{\text{eff}}}\, f_G\left(\frac{2^c-1}{\bar{\gamma}_{\text{eff}}}\right), \tag{38}$$

which results, by substitution into (37), in $a_n = \delta\, f_G(n\,\delta)$. Thus, whenever $f_G(g)$ is independent of $\bar{\gamma}_{\text{eff}}$, a_n becomes independent of $\bar{\gamma}_{\text{eff}}$ too, and the same approximation (i.e., the same coefficients) can be conveniently applied with any value of $\bar{\gamma}_{\text{eff}}$. This condition is not very restrictive in practice, and it is satisfied in the applications discussed in this article.

Appendix B
Derivation of (35) for Dual-Hop Fixed-Gain Relay Networks Under Nakagami Fading

From [24], the end-to-end SNR herein is $\gamma_e \triangleq \frac{\gamma_1\,\gamma_2}{\varkappa+\gamma_2}$ and the ergodic capacity is calculated as

$$\begin{aligned}\bar{\mathcal{C}} = &\overbrace{\frac{1}{\log_e(2)}\mathrm{E}\left[\log_e\left(1+\frac{(1+\gamma_1)\,\gamma_2}{\varkappa}\right)\right]}^{A} \\ &- \underbrace{\frac{1}{\log_e(2)}\mathrm{E}\left[\log_e\left(1+\frac{\gamma_2}{\varkappa}\right)\right]}_{B}.\end{aligned} \tag{39}$$

We will consider Nakagami-m fading channels. Part B of (39) can be directly approximated using our logarithmic approximation with the optimized parameters $\{(a_{n_1,m_2}, b_{n_1,m_2})\}_{n_1=1}^{N_1}$ as

$$B = \tfrac{1}{\log_e(2)}\tilde{C}_{m_2}\left(\tfrac{\varkappa}{\bar{\gamma}_2}\right), \tag{40}$$

whereas part A is evaluated as

$$\begin{aligned}A = &\frac{1}{\log_e(2)}\int_0^\infty \left(\frac{m_1}{\bar{\gamma}_1}\right)^{m_1} \frac{\gamma_1^{m_1-1}}{\Gamma(m_1)}\exp\left(-m_1\frac{\gamma_1}{\bar{\gamma}_1}\right) \\ &\times \int_0^\infty \log_e\left(1+\frac{(1+\gamma_1)\,\gamma_2}{\varkappa}\right)\left(\frac{m_2}{\bar{\gamma}_2}\right)^{m_2}\frac{\gamma_2^{m_2-1}}{\Gamma(m_2)} \\ &\times \exp\left(-m_2\frac{\gamma_2}{\bar{\gamma}_2}\right)\mathrm{d}\gamma_2\,\mathrm{d}\gamma_1.\end{aligned} \tag{41}$$

We approximate the inner integral which is of the form $I_{m_2,\frac{m_2}{\bar{\gamma}_2}}\left(\frac{1+\gamma_1}{\varkappa}\right)$ using (24). Therefore, (41) becomes

$$\begin{aligned}A = &\frac{1}{\log_e(2)}\sum_{n_1=1}^{N_1} a_{n_1,m_2}\int_0^\infty \log_e\left(1+\frac{(1+\gamma_1)\,\bar{\gamma}_2\, b_{n_1,m_2}}{\varkappa}\right) \\ &\times \left(\frac{m_1}{\bar{\gamma}_1}\right)^{m_1}\frac{\gamma_1^{m_1-1}}{\Gamma(m_1)}\exp\left(-m_1\frac{\gamma_1}{\bar{\gamma}_1}\right)\mathrm{d}\gamma_1.\end{aligned} \tag{42}$$

Using change of variables $z = \frac{1+\gamma_1}{\bar{\gamma}_1}$, we obtain

$$\begin{aligned}A = &\frac{1}{\log_e(2)}\frac{{m_1}^{m_1}}{\Gamma(m_1)}\sum_{n_1=1}^{N_1} a_{n_1,m_2} \\ &\times\left[\int_0^\infty P_1(z)\,\mathrm{d}z - \int_0^{1/\bar{\gamma}_1} P_1(z)\,\mathrm{d}z\right],\end{aligned} \tag{43}$$

where

$$\begin{aligned}P_1(z) = &\log_e\left(1+\frac{\bar{\gamma}_1\,\bar{\gamma}_2\, b_{n_1,m_2}\, z}{\varkappa}\right)\bar{\gamma}_1^{-m_1+1} \\ &\times(\bar{\gamma}_1\, z-1)^{m_1-1}\exp\left(-m_1\, z+\frac{m_1}{\bar{\gamma}_1}\right).\end{aligned} \tag{44}$$

Next, we expand $(\bar{\gamma}_1\, z-1)^{m_1-1}$ using the binomial theorem, and approximate the resulting expression which contains $I_{j+1,m_1}\left(\frac{\bar{\gamma}_1\bar{\gamma}_2\, b_{n_1,m_2}}{\varkappa}\right)$ using (24) with

$\{(a_{n_2,j+1}, b_{n_2,j+1})\}_{n_2=1}^{N_2}$ to evaluate $A_1 = \int_0^\infty P_1(z)\,\mathrm{d}z$ as

$$A_1 = \exp\left(\frac{m_1}{\bar{\gamma}_1}\right) \sum_{j=0}^{m_1-1} \sum_{n_2=1}^{N_2} a_{n_2,j+1} \binom{m_1-1}{j} \times m_1^{-j-1}\, j!\, \bar{\gamma}_1^{-m_1+j+1}\, (-1)^{m_1-j-1} \times \log_e\left(1 + \frac{(j+1)\bar{\gamma}_1\bar{\gamma}_2 b_{n_1,m_2} b_{n_2,j+1}}{m_1\,\varkappa}\right), \quad (45)$$

whereas for $A_2 = \int_0^{1/\bar{\gamma}_1} P_1(z)\,\mathrm{d}z$, we apply [47, Eqs. 1.110, 1.211.1, 2.729] as

$$\begin{aligned} A_2 &= \exp\left(\frac{m_1}{\bar{\gamma}_1}\right) \sum_{i=0}^{\infty} \sum_{j=0}^{m_1-1} \binom{m_1-1}{j} \frac{m_1^i}{i!} (-1)^{m_1+j+i-1} \\ &\quad \times \bar{\gamma}_1^{-m_1+j+1} \int_0^{1/\gamma_1} z^{j+i} \log_e\left(1 + \frac{\bar{\gamma}_1\bar{\gamma}_2 b_{n_1,m_2} z}{\varkappa}\right) \mathrm{d}z \\ &= \exp\left(\frac{m_1}{\bar{\gamma}_1}\right) \sum_{i=0}^{\infty} \sum_{j=0}^{m_1-1} \binom{m_1-1}{j} \frac{(m_1)^i}{i!} \frac{(-1)^{m_1+j+i-1}}{j+i+1} \\ &\quad \times \frac{1}{\bar{\gamma}_1^{m_1+i}} \left[\left(1 - \left(\frac{-\varkappa}{\bar{\gamma}_2\, b_{n_1,m_2}}\right)^{j+i+1}\right) \log_e \right. \\ &\quad \times \left(1 + \frac{\bar{\gamma}_2 b_{n_1,m_2}}{\varkappa}\right) \\ &\quad \left. + \sum_{q=1}^{j+i+1} \frac{(-1)^q\, \varkappa^{q-1}}{(j+i-q+2)\,(\bar{\gamma}_2\, b_{n_1,m_2})^{q-1}}\right]. \quad (46) \end{aligned}$$

We substitute (45) and (46) back into (43) which then are substituted together with (40) into (39) to obtain a closed-form approximation for the ergodic capacity in a dual-hop fixed-gain relay networks over Nakagami-m fading channels according to (35).

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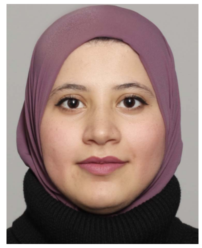

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