Characterisation Theorems for Weighted Tree Automaton Models

Dissertation

zur Erlangung des akademischen Grades
Doctor rerum naturalium (Dr. rer. nat.)

vorgelegt an der
Technischen Universität Dresden
Fakultät Informatik

eingereicht am
15. Februar 2022

von
Frederic Dörband

geboren am 25. April 1995
in Neustrelitz

Betreuer:

- Prof. Dr.-Ing. habil. Dr. h.c./Univ. Szeged Heiko Vogler,
  Technische Universität Dresden

Acknowledgements

My academic journey has given me the opportunity to connect with many extraordinarily talented and generally delightful people. I wish to thank everyone who contributed their wisdom, time, energy, love, and care to my path. Your kindness has been the strong support upon which I was able to pursue my goals.

Firstly, I wish to thank my supervisor Heiko Vogler for his fruitful guidance while letting me explore my own scientific path. Your influence has given me the structure I needed to turn ideas into results. Secondly, I wish to thank Ulrike Baumann, who has mentored me for many years and who always has my back when I want to do things my own way. I also wish to thank Laure Daviaud, who sparked my interest in the determinisation of weighted automata and who generously agreed to review this thesis.

Much of my research was done collaboratively and I wish to thank all my supremely capable colleagues and co-authors, in particular Richard Mörbitz, Heiko Vogler, Kevin Stier, Zoltán Fülöp, and Thomas Feller. Thank you for countless hours of brilliant discussions and for not going mad while working with me. I also wish to thank Kerstin Achtruth, who always has an open ear for other people's problems and who makes work a happier place.

I am extremely grateful to my professors, who made me fall in love with science. Stefan Siegmund, Andreas Thom, and Ulrich Krähmer, thank you for revealing the beauty and radiance of mathematics to me.

Moreover, I wish to thank my friends Robin, Chris, and Marleen who gave valuable comments on the thesis. I also wish to thank Paula for her continued encouragement and care. Lastly, I wish to thank my family who has given me unconditional love, even in times when I was not able to appreciate it.

Contents

1

Introduction

Virtually every aspect of modern societies relies on computers – electronic machines executing complex operations and programs in a step-by-step way. However, mathematical models for computing devices had already been studied before the invention of the digital computer. The first such model was the *Turing machine* [110], proposed by Alan Turing in the 1930s to study the boundaries of what is "computable". The high complexity of Turing machines quickly gave rise to manifold simpler and less expressive models of abstract machines. A very simple model, the *finite automaton* (short: FA) [93], became very prominent due to its wide range of applications, such as pattern matching on words [89, 108], lexical analysis of compilers [76, Chapter 3.3.2], and as a way of modelling nervous activity [81, 88] (cf. [72, 76] for further applications and the history of finite automata). Besides their applications in theoretical disciplines, finite automata can also be used to model the state behaviour of real-life processes and systems in an intuitive way. Some examples are vending machines, ticket reservation processes, and plant lifecycles.

A finite automaton is (up to some technicalities) a directed graph, where the vertices are called *states* and the edges are called *transitions*. Additionally, the edges are labelled with symbols from an input *alphabet*. When describing real-life systems, the states of an FA represent the possible configurations of the described system. The transitions are then regarded as changes of the system's configuration, which are triggered by an input signal. An input *word* – that is, a finite sequence of input symbols – is *recognised*

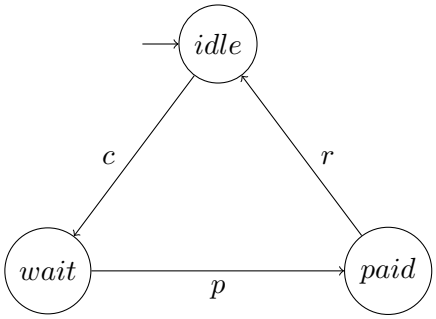


Figure 1.1: A simple vending machine FA with the three states *idle*, *wait*, and *paid*. Upon the input symbol c ("choose") in the *idle* state, the state changes to *wait*. A subsequent p ("pay") trigges the state to change to *paid*. Finally, the input symbol r ("retrieve") changes the state *paid* to *idle*. The state *idle* is moreover marked as the initial state.

by an FA if there exists a corresponding sequence of transitions of the FA such that the i-th transition in the sequence is labelled by the i-th symbol from the input word (for every position i of the input word). The *formal language recognised* by an FA is the set of all input words recognised by the FA and two FAs are *equivalent* if they recognise the same formal language. In Figure 1.1, we give an example for an FA that describes a simple vending machine. The input word *cpr* is recognised by the FA, but the input word *crcr* is not recognised (because all items need to be paid for before they can be retrieved).

The theory of finite automata has proven to be very fruitful and research has unveiled many desirable properties of this automata model. For example, the membership, emptiness, and equivalence problems have polynomial time complexity for finite automata [76, Chapters 4.3 and 4.4], one can minimise every finite automaton [76, Chapter 4.4], and one can determinise every finite automaton [93] (cf. also [76, Chapter 2.5.3]). Moreover, different characterisations of the class of formal languages recognised by FA (short: class of *recognisable languages*) exist. For example, the class of recognisable languages can be defined in formally different ways by the models of regular grammars [18], regular expressions [81] ("Kleene's Theorem"), monadic second-order logic formulas [13, 42, 109] ("Büchi's Theorem"), and Myhill-Nerode equivalence relations [91] ("Myhill-Nerode Theorem").

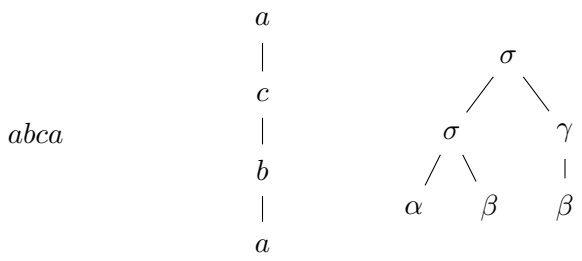


Figure 1.2: A word (left) and a tree (right). One can consider words as special cases of trees, where the last symbol of the word is the root of the tree (middle).

In attempts to model quantitative properties of processes, rather than mere qualitative properties, different *weighted* word automata (short: WA) models have been studied. Often, a WA consists of a finite automaton describing the state behaviour of the process together with a list of weight assignments mapping parts of the state behaviour to elements of a weight structure S. This can be used to model, for example, probabilities, costs of executions, and outputs of processes. Prominent cases of weight structures S are lattices [77, 95, 114], groups [21], the free monoid [5, 17], and the min-plus semiring [102, 103]. All of these weighted automata models are special cases of the general class of weighted automata over semirings [7, 36, 84, 98, 99]. In the class of weighted automata over semirings, some classical theorems from (unweighted) finite automata still hold. This includes Kleene's Theorem [99], the Myhill-Nerode Theorem [7], and Büchi's Theorem [32]. Other classical results, however, cannot be generalised to the weighted case. For example, a positive determinisation result for weighted automata is much harder to obtain [85]. In fact, there are weighted automata that cannot be determinised [12, Lemma 6.3].

Word automata models have been extended to handle more complex input structures like pictures, trees, and forests, both in the unweighted (cf. [96, 97]) and the weighted case (cf. [36, Chapters 9 and 10] and [87]). Trees in particular have a wide array of applications, especially in the field of natural language processing. A *tree* is a term represented by a finite, directed, acyclic graph with a designated unique root position, which can reach all other positions, see Figure 1.2. Trees can represent hierarchical data, for example, sentences in a natural language including their contextual

information and dependencies within the sentence, which makes them appealing for different disciplines [82, 100] (cf. also [19, 24, 25, 44] and [97, Chapter 1]).

This thesis evolves around models of *weighted tree automata* (short: WTA), which combine weighted automata models with trees as the input structure. As a weight structure, we primarily consider semirings but make an excursion to so-called *M-monoids* in Chapter 5. Some of the theorems that can be proven for weighted word automata can be lifted to the case of weighted tree automata, like Kleene's Theorem [38, 51], the Myhill-Nerode Theorem [11], and Büchi's Theorem [40, 52]. Similarly to the word case, positive determinisation results for weighted tree automata have only been found for restricted cases and oftentimes, these cases vary strongly between the literature [14, 29, 57]. Determinisation of weighted automata remains an important open research field, as non-deterministic behaviour cannot be simulated on computers and the state of research is theoretically unsatisfying.

A finite automaton is *determinisitic* if for every state q and every input symbol a there exists at most one transition starting in q and labelled by a. That is, a state change always results in a uniquely determined successor state. It already belongs to folklore that every finite automaton is *determinisable*, that is, there exists an equivalent deterministic finite automaton [76, Chapter 2.5.3]. This equivalent deterministic automaton is constructed from the original automaton using the well-known *power set construction*. For weighted (word or tree) automata, the notions of determinism and determinisability can be defined similarly, however, the determinisation construction in the weighted case is much more involved than in the unweighted case. Generally, a weighted alternative of the power set construction can be employed [90], which by itself yields infinite automata in many cases where the underlying semiring is not finite. Therefore, an additional step in the determinisation construction is required, called *factorisation*, where a lot of care is put into the choice of transition weights [14, 80]. We will provide a more detailed and formal investigation of the existing determinisation constructions in Chapter 3. One significant detail that spans the most influential determinisation approaches [5, 14, 21, 29, 57, 80] is that even the weighted power set construction with factorisation cannot be applied to every weighted automaton. An additional property of weighted automata called the *twinning property* is defined in

each approach, which is then proven as a sufficient condition for determinisability.

Even though significant research has been done and every individual approach is very much a valid theory, it is not optimal that the approaches have so much in common, yet each one applies to an "isolated" case. The first major focus of this thesis is to find a mathematical framework for the determinisation of weighted tree automata, which provides common ground for the existing determinisation approaches and yields determinisation results that cover as many of the existing results as possible.

The fact that some weighted automata cannot be determinised and some determinisable weighted automata do not satisfy the twinning property poses the question of what to do when our determinisation framework cannot be applied. One option is to turn away from this "exact" determinisation towards *approximated determinisation*. The task of approximated determinisation is still to find a deterministic alternative for a given weighted automaton. However, unlike in exact determinisation, the weighted languages recognised by the two weighted automata do not need to coincide – they only need to be approximately the same. Different approaches to this paradigm have been proposed [4, 8, 9] and our main focus lies on [4], which has applications in formal verification of quantitative properties [15, 16] and considers semiring-weighted automata (over the so-called *tropical* semiring). In [4], a modified weighted power set construction with factorisation is employed as an approximate determinisation construction. An approximate twinning property is then defined which is a sufficient condition for approximated determinisability and which is, maybe more importantly, less restrictive than the twinning property from exact determinisation.

Since [4] only considers weighted word automata, the question arises whether a similar theory can be employed for weighted tree automata. The second major focus of this thesis is to answer this question by generalising the ideas presented in [4] from the word case to the tree case.

Unfortunately, there exist weighted automata which are not approximately determinisable within this framework. A last resort in trying to understand recognisable weighted languages is to characterise recognisability itself. Two particularly interesting characterisations are Kleene's Theorem and Büchi's Theorem. Kleene's Theorem characterises the class of recognisable weighted languages by its algebraic closure prop-

erties using a formalism called *rational expressions*. Büchi's Theorem states that the class of recognisable weighted languages coincides with the class of languages defined by *monadic second-order logic formulas* (short: MSO-formulas). Since formal properties of processes are conveniently represented by rational expressions or formulas, both Kleene's Theorem and Büchi's Theorem are of special interest in applied disciplines like pattern matching or model checking (cf. [42, 81]). Both, Kleene- and Büchi-like theorems have been proven for many different weighted automata models, see for example [26, 33, 34, 38, 40, 51, 52, 70, 87, 104, 107] for the case of automata with tree-like input structures.

As a third major focus of this thesis, we investigate two more exotic models of recognisability and ways to characterise them with Kleene- or Büchi-like results. The first model is that of *weighted forest automata over M-monoids*, which considers tuples of trees as input and operations on monoids as weights (rather than semiring elements). Our goal is to characterise this weighted automaton model by both, rational expressions and MSO-formulas. For our second model, we consider the *rational weighted tree languages with storage* from [55], which are defined as rational weighted tree languages (without storage) composed with a storage map. Our goal is to show that this composition with a storage map preserves the algebraic closure properties of rational weighted tree languages and thereby equip rational weighted tree languages with storage with their own set of algebraic closure properties.

This theoretical voyage through determinisation, approximated determinisation, and characterisation theorems is reflected in the structure of this thesis. We begin by collecting the necessary mathematical basics and notational conventions in Chapter 2. In particular, we gather some notations for relations, hypergraphs, and multisets, recall the basics on monoids and semirings, and finally present the model of weighted tree automata over semirings. In Chapter 3, we develop our general determinisation framework that aims to capture and unify the determinisation approaches [5, 14, 29, 57]. We define our factorisation theory for monoids, provide a multi-step determinisation construction, prove results that allow us to apply our construction to general classes of weighted tree automata, and ultimately compare our approach to the existing literature. In Chapter 4, we provide a formal approximated determinisation construction

for weighted tree automata over the tropical semiring. The structure of this chapter follows [4], in that we consider approximation factors $t \geq 1$ and define a t-approximate determinisation construction based on a notion of t-approximation of weighted languages. Our construction is given in a formal mathematical way in contrast to the algorithmic description provided in [4]. In Chapter 5, we first recall the notions of forests and M-monoids and then introduce our model of weighted forest automata over M-monoids. We finally prove a Kleene- and a Büchi-like theorem for this weighted automaton model. Moreover, we make brief excursions to other potentially interesting weighted forest automata models and argue why our model yields the most robust theory. In Chapter 6, we recall storage types and rational weighted tree languages with storage (short: Rat) as introduced in [55]. We then recall the Kleene-Goldstine theorem from [55], which can be used to prove the closure of Rat under the rational operations (top concatenation, scalar multiplication, sum, α-concatenation, and α-Kleene star). Ultimately, we provide simpler proofs of these closure properties using only the definition of Rat, avoiding the detour through weighted tree grammars with storage. In Chapter 7, we conclude this thesis by summarising its main contributions.

This thesis is based on our research, which includes four publications and one manuscript.

- Chapter 3 is a generalisation of Dörband and Mörbitz [31] from the word case to the tree case and covers the results from Dörband, Feller, and Stier [29].
- Chapter 4 is an alternative presentation of Dörband, Feller, and Stier [28]. Here, we have replaced the algorithm provided in [28] by formal mathematical methods.
- Chapter 5 is based on Dörband [27].
- Chapter 6 is based on Dörband, Fülöp, and Vogler [30].

The scope of this thesis lies within the field of theoretical computer science and all of our research is motivated by very theoretical questions. This perspective poses the difficult task to illustrate our theories without defaulting to purely academic examples. Undeniably, our presentation takes this into account only sparsely and we consider practical applicability as a weak side of this thesis. Even though this is certainly

a limitation of our scientific research, it also gives us space to define very technical mathematical machinery, which allows us to distil precise assumptions in our theories and identify ingredients that leave room for improvement. We also want to note that significant parts of this thesis evolve around the standard model for weighted tree automata over semirings, which has already been studied extensively. This requires us to introduce highly non-trivial mathematical tools in order to see beyond the scope of the already existing literature. In return, Chapter 3 turns out to be quite lengthy. We provide detailed proofs for almost all claims made in the thesis and illustrate our new tools with examples, but still leave several open questions. This is partly due to the fact that we do not yet have a full understanding of the newly introduced machinery and how it compares to the previous approaches. We also provide no decidability or complexity results for most of our new properties and constructions.

2

Preliminaries

2.1 Languages

Sets, Relations, and Maps We denote the *empty set* by $\emptyset$ and call a set A *non-empty* if $A \neq \emptyset$. Moreover, we denote the *size* of a set A by $\#A$. We call A *finite* if $\#A < \infty$ and we call A a *singleton set* if $\#A = 1$. We denote the *set of subsets of* A, the *set of finite subsets of* A, and the *set of singleton subsets of* A by $\mathcal{P}(A)$, $\mathcal{P}_{\text{fin}}(A)$, and $\mathcal{S}(A)$, respectively.

We denote the set of *non-negative integers* by $\mathbb{N}$ and the set of *positive integers* by $\mathbb{N}_+$. Moreover, we denote the set of *real numbers* by $\mathbb{R}$ and the set of *rational numbers* by $\mathbb{Q}$. For every $k, \ell \in \mathbb{N}$, we denote the set $\{i \in \mathbb{N} \mid k \leq i \leq \ell\}$ by $[k, \ell]$ and abbreviate $[1, k]$ by $[k]$. We note that $[0] = \emptyset$.

The *Cartesian product* of two sets A and B is the set $A \times B$ given by

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

and for every $k \in \mathbb{N}_+$ we denote the k-fold Cartesian product $\overbrace{A \times \cdots \times A}^{k \text{ times}}$ by A^k.

Let $k \in \mathbb{N}$ and $A_1, \ldots, A_k$ be sets. A *relation on* $A_1 \times \cdots \times A_k$ is a subset $R \subseteq A_1 \times \cdots \times A_k$. Let A and B be sets. We say that a relation on $A \times B$ is a *relation between* A *and* B. A *binary relation on* A is a relation $R \subseteq A \times A$. The *identity relation on* A is the binary relation $\mathrm{id}_A \subseteq A \times A$ given by $\mathrm{id}_A = \{(a, a) \mid a \in A\}$. Given a relation $R \subseteq A \times B$, the *inverse relation of* R is the relation $R^{-1} \subseteq B \times A$

given by

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}.$$

Given two relations $R \subseteq A \times B$ and $S \subseteq B \times C$, the *composition of R and S* is the relation $R; S \subseteq A \times C$ given by

$$R; S = \{(a, c) \mid (a, b) \in R, (b, c) \in S\}.$$

We sometimes denote $R; S$ by $S \circ R$.

A binary relation $R \subseteq A \times A$ is called

- *reflexive* if $\mathrm{id}_A \subseteq R$,
- *irreflexive* if $\mathrm{id}_A \cap R = \emptyset$,
- *symmetric* if $R^{-1} = R$,
- *anti-symmetric* if $R^{-1} \cap R \subseteq \mathrm{id}_A$,
- *linear* if $R^{-1} \cup R = A \times A$,
- *transitive* if $R \circ R \subseteq R$,
- a *(partial) order* if R is reflexive, anti-symmetric, and transitive,
- a *strict order* if R is irreflexive, anti-symmetric, and transitive, and
- an *equivalence relation* if R^{-1} is reflexive, symmetric, and transitive.

We sometimes write aRb rather than $(a, b) \in R$, especially if R is a partial order, strict order, or equivalence relation. Moreover, we use symbols like $\leq, \trianglelefteq, \subseteq$, and $\sqsubseteq$ for partial orders, symbols like $<, \subset$, and $\sqsubset$ for strict orders, and symbols like $\sim, \cong, \equiv$, and $\triangleq$ for equivalence relations.

Given a linear order $\leq$ on A, a set $A' \subseteq A$, and $a \in A$ such that $a \leq b$ for every $b \in A$, we denote a by $\min_{\leq}(A')$.

Given an equivalence relation $\sim$ on A and an element $a \in A$, the *equivalence class of a under* $\sim$ is the set $[a]_\sim \subseteq A$ given by $[a]_\sim = \{a' \in A \mid a \sim a'\}$. The *quotient set of A by* $\sim$ is the set $A/_\sim$ given by $A/_\sim = \{[a]_\sim \mid a \in A\}$.

Given a relation $R \subseteq A \times B$, the *domain of* R is the set $\mathrm{dom}(R) \subseteq A$ given by

$$\mathrm{dom}(R) = \{a \in A \mid \exists b \in B \colon (a, b) \in R\}.$$

Moreover, R is called

- *left-total* if $\mathrm{dom}(R) = A$,
- *left-unique* (or *injective*) if $a = a'$ for every $a, a' \in A$ and $b \in B$ such that $(a, b), (a', b) \in R$,
- *right-total* (or *surjective*) if R^{-1} is left-total,
- *right-unique* (or *partial map*) if R^{-1} is left-unique, and
- a *map* if R is a left-total partial map.

We use symbols like f, g, h to denote relations that are partial maps. A partial map $f \subseteq A \times B$ is also denoted by $f \colon A \dashrightarrow B$ and a map $f \subseteq A \times B$ is also denoted by $f \colon A \to B$. For a partial map $f \colon A \dashrightarrow B$ and an element $a \in A$ we denote the unique element $b \in B$ such that $(a, b) \in f$ by $f(a)$. Given a set $A' \subseteq A$, we define the set $f(A') = \{f(a) \mid a \in A'\}$. Moreover, the *restriction of* f *to* A' is the map $f|_{A'} \colon A' \to B$ given by $f|_{A'}(a) = f(a)$ for every $a \in A'$. The *image of* f is the set $\mathrm{im}(f) \subseteq B$ given by $\mathrm{im}(f) = f(A)$.

A map $f \subseteq A \times B$ is called *bijective* if f is injective and surjective. In this case, we also say that A and B are *bijective* and denote this fact by $A \cong B$.

Let $k \in \mathbb{N}$ and $A_1, \ldots, A_k$ be sets. For every $i \in [k]$ we define the map

$$\mathrm{proj}_i \colon A_1 \times \cdots \times A_k \to A_i,$$

where $\mathrm{proj}_i(a_1, \ldots, a_k) = a_i$ for every $(a_1, \ldots, a_k) \in A_1 \times \cdots \times A_k$. We call every proj_i a *projection map*.

Let A be a set and $f \colon A \to \mathbb{N}$. We define the *argument minimum of* f, denoted by $\mathrm{argmin}_{a \in A} f(a)$, as

$$\operatorname*{argmin}_{a \in A} f(a) = \{a \in A \mid f(a) = \min f(A)\}.$$

We note that if $A = \emptyset$, then $\mathrm{argmin}_{a \in A} f(a) = \emptyset$.

Let I be a set. An (I-)*family over* A is a map $f\colon I \to A$ and we denote such a family f by $(f(i) \mid i \in I)$. An (*infinite*) *sequence of elements in* A is an $\mathbb{N}$-family $(f(k) \mid k \in \mathbb{N})$. A *finite sequence over* A is an $[n]$-family $(f(k) \mid k \in [n])$ which we also denote by $f(1) \dots f(n)$.

Let $f\colon \mathbb{N} \to \mathbb{N}$. We call f *strongly monotone* if $f(n_1) < f(n_2)$ for every $n_1, n_2 \in \mathbb{N}$ such that $n_1 < n_2$. We note that if f is strongly monotone, then $f(n) \geq n$ for every $n \in \mathbb{N}$.

Graphs and Hypergraphs A *(directed) graph* is a pair $G = (V, E)$, where V is a set and $E \subseteq V \times V$ is a binary relation on V. The elements of V are called *vertices of* G and the elements of E are called *edges of* G.

Let A be a set. A *hypergraph with labels in* A is a pair $H = (V, E)$, where V is a set and $E \subseteq V^* \times A \times V$. The elements of V are called *vertices of* H and the elements of E are called *hyperedges of* H.

Let $H = (V, E)$ be a hypergraph with labels in A. We depict H graphically in the following way. Each vertex of H is represented by a circle labeled by the name of the vertex. An edge $(v_1 \dots v_n, a, v) \in E$ is represented by a box which is labeled by a, has n incoming lines, and has a single outgoing line. The outgoing line ends in v. The incoming lines originate from $v_1, \dots, v_n$, respectively, by counter-clockwise traversal starting to the left of the outgoing line. An exemplary graphical depiction of a hypergraph can be found in Figure 2.1.

Multisets Let A be a set. A *multiset* over A is a map $M\colon A \to \mathbb{N}$. A multiset N over A is called *submultiset* of M, denoted by $N \leq M$, if $N(a) \leq M(a)$ for every $a \in A$. The *support* of M is the set $\mathrm{supp}(M) \subseteq A$ given by $\mathrm{supp}(M) = \{a \in A \mid M(a) \geq 1\}$. The *size* of M is the cardinal number $\#M \in \mathbb{N} \cup \{\infty\}$ given by $\#M = \sum_{a \in \mathrm{supp}(M)} M(a)$ if $\mathrm{supp}(M)$ is finite and by $\#M = \infty$ otherwise. We call M *finite* if $\#M < \infty$. The *set of multisets over* A and the *set of finite multisets over* A are denoted by $\mathcal{M}(A)$ and $\mathcal{M}_{\mathrm{fin}}(A)$, respectively. For every $M, N \in \mathcal{M}(A)$ we define the *union of* M *and* N, denoted $M \cup N$, by $(M \cup N)(a) = M(a) + N(a)$ for every $a \in A$.

We make the following convention when working with multisets. If $\mathrm{supp}(M)$ has the

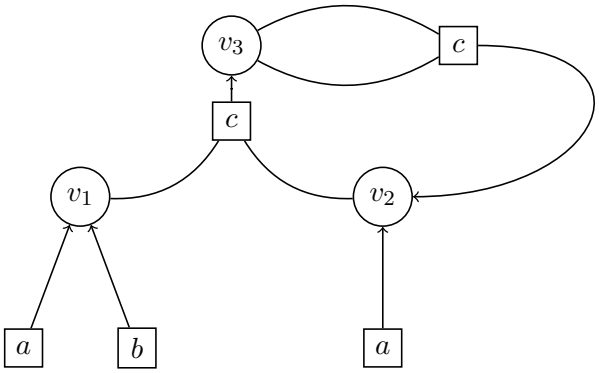


Figure 2.1: A hypergraph with labels in $A = \{a, b, c\}$. Circles represent vertices and boxes represent hyperedges. One hyperedge of this hypergraph is $(v_1 v_2, c, v_3)$.

form $\{a_1, \ldots, a_n\}$ for some pairwise distinct $a_1, \ldots, a_n \in A$, then we also denote M by

$$(a_1 \mapsto M(a_1), \ldots, a_n \mapsto M(a_n)) \qquad \text{or} \qquad \{\!\{\overbrace{a_1, \ldots, a_1}^{M(a_1)\text{ times}}, \ldots, \overbrace{a_n, \ldots, a_n}^{M(a_n)\text{ times}}\}\!\}.$$

Given a set B, $k \geq 1$, $M_1, \ldots, M_k \in \mathcal{M}_{\text{fin}}(A)$, a relation Φ on A^k and a map $f\colon A^k \to B$, we define the term

$$\{\!\{f(m_1, \ldots, m_k) \mid m_1 \in M_1, \ldots, m_k \in M_k, \Phi(m_1, \ldots, m_k)\}\!\} \tag{2.1}$$

as the multiset $M\colon B \to \mathbb{N}$ such that for every $b \in B$ we have

$$M(b) = \sum_{\substack{(m_1,\ldots,m_k) \in f^{-1}(b) \\ \text{s.th. } (m_1,\ldots,m_k) \in \Phi}} M_1(m_1) \cdot \ldots \cdot M_k(m_k).$$

Moreover, if $\Phi = A^k$, then we drop $\Phi(m_1, \ldots, m_k)$ from (2.1).

The *Cartesian product* of two finite multisets M_1 over A and M_2 over B (where B is an arbitrary set) is the multiset $M_1 \times M_2$ over $A \times B$ which is defined as follows:

$$M_1 \times M_2 = \{\!\{(m_1, m_2) \mid m_1 \in M_1, m_2 \in M_2\}\!\}.$$

We use the standard conventions for the Cartesian product. In particular, for every multiset M, we denote the k-fold Cartesian product $\underbrace{M \times \cdots \times M}_{k \text{ times}}$ by M^k.

Example 2.1. The multisets $M_1 = (1 \mapsto 1, 2 \mapsto 3)$ and $M_2 = (2 \mapsto 1, 4 \mapsto 2)$ over $\mathbb{N}$ can also be written as

$$M_1 = \{\!\{1, 2, 2, 2\}\!\} \qquad \text{and} \qquad M_2 = \{\!\{2, 4, 4\}\!\}.$$

Neither $M_1 \leq M_2$ nor $M_2 \leq M_1$ holds. Moreover, it holds that $\#M_1 = 4$, $\#M_2 = 3$, and hence, both M_1 and M_2 are finite. The Cartesian product of M_1 and M_2 is

$$M_1 \times M_2 = \Big((1, 2) \mapsto 1, (1, 4) \mapsto 2, (2, 2) \mapsto 3, (2, 4) \mapsto 6\Big)$$

and $\mathrm{supp}(M_1 \times M_2) = \{(1, 2), (1, 4), (2, 2), (2, 4)\} = \mathrm{supp}(M_1) \times \mathrm{supp}(M_2)$.

Next, we illustrate the notation introduced in (2.1). For this, we let $f \colon \mathbb{N}^2 \to \mathbb{N}$ such that $f(m_1, m_2) = m_1 \cdot m_2$ for every $m_1, m_2 \in \mathbb{N}$ and $\Phi = \{(m_1, m_2) \in \mathbb{N}^2 \mid m_1 + m_2 > 3\}$. We define the multiset M as

$$M = \{\!\{m_1 \cdot m_2 \mid m_1 \in M_1, m_2 \in M_2, m_1 + m_2 > 3\}\!\}$$

and obtain that $M = \{\!\{4, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8\}\!\}$. Note that two 4s in M are generated as $1 \cdot 4$ and the remaining three 4s in M are generated as $2 \cdot 2$. ◀

Word Languages An *alphabet* is a finite and non-empty set. Let A be an alphabet. We let $A^* = \bigcup_{k \in \mathbb{N}} A^k$ be the set of all (finite) *words* over A. Let $w \in A^*$ such that $w = (w_1, \ldots, w_k)$ for some $k \in \mathbb{N}$ and $w_1, \ldots, w_k \in A$. We also denote w by $w_1 \ldots w_k$. In particular, we denote $()$ by ε and call ε the *empty word*. The *length* of w, denoted $|w|$, is k. Moreover, for every $a \in A$ we denote by $|w|_a$ the number of occurrences of a in w. Now, let $v \in A^*$ such that $v = v_1 \ldots v_\ell$ for some $\ell \in \mathbb{N}$ and $v_1, \ldots, v_\ell \in A$. The *concatenation of w and v*, denoted $w \circ v$ or wv, is the word $w_1 \ldots w_k v_1 \ldots v_\ell$. We note that $\varepsilon \circ w = w = w \circ \varepsilon$.

Let $w \in A^*$ such that $w = w_1 \ldots w_k$ for some $k \in \mathbb{N}$ and $w_1, \ldots, w_k \in A$. Moreover, let $i, j \in [k]$ be indices such that $i \leq j$. We define $w[i : j] = w_i \ldots w_j$ and $w[i] = w[i : i]$. A word $v \in A^*$ is called *prefix* (or *suffix*) of w if there exists an $i \in [k]$ such that $v = w[1 : i]$ (or $v = w[i : k]$, respectively).

If A is linearly ordered by some strict order $<_A$, we define the relations $<_l$ and $\leq_{\mathrm{lex}}$ on A^* as follows. Let $w, v \in A^*$. It holds that $v <_l w$ if there exists $u, v_1, w_1 \in A^*$ and

$a, b \in A$ such that $a <_A b$, $v = uav_1$, and $w = ubw_1$. Moreover, it holds that $v \leq_{\text{lex}} w$ if $v <_l w$ or v is a prefix of w. In fact, the relation $<_l$ is a linear strict order and $\leq_{\text{lex}}$ is a linear partial order, called *lexicographic order*. We note that we can define $<_l$ and $\leq_{\text{lex}}$ analogously if A is an infinite linearly ordered set.

A *(word) language over* A is a set $L \subseteq A^*$. Given languages $L, L' \subseteq A^*$ over A, the *concatenation of* L *and* L' is the language $L \circ L'$ over A given by

$$L \circ L' = \{wv \mid w \in L, v \in L'\}.$$

We note that $\{\varepsilon\} \circ L = L = L \circ \{\varepsilon\}$ and $\emptyset \circ L = \emptyset = L \circ \emptyset$. Let $n \in \mathbb{N}$. The *n-th power of* L is the language L^n on A given recursively by $L^0 = \{\varepsilon\}$ and $L^{n+1} = L \circ L^n$. The *Kleene-star of* L is the language L^* on A given by $L^* = \bigcup_{n \in \mathbb{N}} L^n$.

Tree Languages A *ranked set* is a pair (Σ, rk), where Σ is a set and $\mathrm{rk}\colon \Sigma \to \mathbb{N}$ is a map. Given $\sigma \in \Sigma$, we call $\mathrm{rk}(\sigma)$ the *rank of* σ. For every $s \in \mathbb{N}$ we define the set $\Sigma^{(s)} = \{\sigma \in \Sigma \mid \mathrm{rk}(\sigma) = s\}$ and write $\sigma^{(s)}$ to denote that $\sigma \in \Sigma^{(s)}$ for every $\sigma \in \Sigma$. A *ranked alphabet* is a ranked set (Σ, rk) such that Σ is finite. Given a ranked alphabet (Σ, rk), the *maximal rank of* Σ is the number $\mathrm{maxrk}(\Sigma) \in \mathbb{N}$ given by $\mathrm{maxrk}(\Sigma) = \max\{s \in \mathbb{N} \mid \Sigma^{(s)} \neq \emptyset\}$. We will denote a ranked set (Σ, rk) only by its first component Σ. The map rk will be clear from the context or referred to as rk_Σ.

Let Σ be a ranked set and H be a set such that $\Sigma \cap H = \emptyset$. The *set of all trees over* Σ *indexed by* H, denoted by $\mathrm{T}_\Sigma(H)$, is the smallest set $T \subseteq (\Sigma \cup H \cup \Upsilon)^*$ (with Υ containing the comma, the opening parenthesis, and the closing parenthesis) such that (i) $H \subseteq T$ and (ii) for every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $\xi_1, \ldots, \xi_s \in T$ also $\sigma(\xi_1, \ldots, \xi_s) \in T$. We abbreviate $\mathrm{T}_\Sigma(\emptyset)$ by T_Σ. Also, for $\sigma \in \Sigma^{(1)}$ and $\xi \in \mathrm{T}_\Sigma$ we abbreviate $\sigma(\ldots\sigma(\xi)\ldots)$ with n occurrences of σ by $\sigma^n(\xi)$. Finally, for each $\alpha \in \Sigma^{(0)}$, we abbreviate $\alpha()$ by α.

> *Throughout this thesis, whenever we quantify a ranked alphabet Σ, we assume that $\Sigma^{(0)} \neq \emptyset$.*
>
> *Moreover, given a tree $\xi \in \mathrm{T}_\Sigma(H)$, we write "Assume that $\xi = \sigma(\xi_1, \ldots, \xi_s)$" to abbreviate "Assume that $\xi = \sigma(\xi_1, \ldots, \xi_s)$ for some $s \in \mathbb{N}, \sigma \in \Sigma^{(s)}$, and $\xi_1, \ldots, \xi_s \in \mathrm{T}_\Sigma(H)$".*

We fix the set $X = \{x_i \mid i \in \mathbb{N}_+\}$ of *variables* and denote the set $X_n = \{x_i \mid i \in [n]\}$ for every $n \in \mathbb{N}$. These sets are assumed to be disjoint from any other occuring set.

We define the maps

$$\begin{aligned} \mathrm{pos}\colon \mathrm{T}_\Sigma(H) &\to \mathcal{P}(\mathbb{N}_+^*), \\ \mathrm{size}\colon \mathrm{T}_\Sigma(H) &\to \mathbb{N}, \text{ and} \\ \mathrm{height}\colon \mathrm{T}_\Sigma(H) &\to \mathbb{N}, \end{aligned}$$

respectively, as follows. Let $\xi \in \mathrm{T}_\Sigma(H)$. If $\xi \in H$, then we define

$$\mathrm{pos}(\xi) = \{\varepsilon\}, \quad \mathrm{size}(\xi) = 0, \text{ and } \quad \mathrm{height}(\xi) = 0.$$

Otherwise, we assume that $\xi = \sigma(\xi_1, \ldots, \xi_s)$ and define

$$\begin{aligned} \mathrm{pos}(\xi) &= \{\varepsilon\} \cup \bigcup_{i \in [s]} \big(\{i\} \circ \mathrm{pos}(\xi_i)\big), \\ \mathrm{size}(\xi) &= 1 + \sum_{i \in [s]} \mathrm{size}(\xi_i), \text{ and} \\ \mathrm{height}(\xi) &= 1 + \max_{i \in [s]} \big(\mathrm{height}(\xi_i)\big). \end{aligned}$$

We call $\mathrm{pos}(\xi), \mathrm{size}(\xi)$, and $\mathrm{height}(\xi)$ the *set of positions of* ξ, *size of* ξ, and *height of* ξ, respectively. For every $\Delta \subseteq \Sigma \cup H$, we denote the set $\{w \in \mathrm{pos}(\xi) \mid \xi(w) \in \Delta\}$ by $\mathrm{pos}_\Delta(\xi)$ and abbreviate $\#\mathrm{pos}_\Delta(\xi)$ by $|\xi|_\Delta$. If $\Delta = \{\delta\}$ for some $\delta \in \Sigma \cup H$, then we also write $\mathrm{pos}_\delta(\xi)$ and $|\xi|_\delta$ for $\mathrm{pos}_\Delta(\xi)$ and $|\xi|_\Delta$, respectively. Moreover, we define the set of *leaves of* ξ as the set $\mathrm{leaf}(\xi) = \mathrm{pos}_{\Sigma^{(0)}}(\xi)$. We note that, in this thesis, the elements of H are not considered to be leaves in trees and they do not add to the height and size of trees. This distinguishes $\mathrm{T}_\Sigma(H)$ from $T_{\Sigma \cup H}$.

Now, let $\xi, \zeta \in \mathrm{T}_\Sigma(H)$ and $w \in \mathrm{pos}(\xi)$. The *label of* ξ *at* w, denoted by $\xi(w)$, the *subtree of* ξ *at* w, denoted by $\xi|_w$, and the *replacement of the subtree of* ξ *at* w *by* ζ, denoted by $\xi[\zeta]_w$, are defined as follows. If $\xi \in H$, then $w = \varepsilon$ and we define

$$\xi(w) = \xi, \quad \xi|_w = \xi, \text{ and } \quad \xi[\zeta]_w = \zeta.$$

Otherwise, assume that $\xi = \sigma(\xi_1, \ldots, \xi_s)$. Then, either $w = \varepsilon$ and we define

$$\xi(w) = \sigma, \quad \xi|_w = \xi, \text{ and } \quad \xi[\zeta]_w = \zeta,$$

or $w = iv$ for some $i \in [s]$ and $v \in \mathrm{pos}(\xi_i)$ and we define

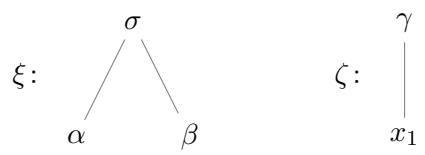
$$\xi(w) = \xi_i(v), \quad \xi|_w = \xi_i|_v, \text{ and } \quad \xi[\zeta]_w = \sigma(\xi_1, \ldots, \xi_{i-1}, \xi_i[\zeta]_v, \xi_{i+1}, \ldots, \xi_s).$$

Let $n \in \mathbb{N}$. For every $(w_1, \ldots, w_n) \in \mathrm{pos}(\xi)^n$ with $w_1 <_l \ldots <_l w_n$, and $\zeta_1, \ldots, \zeta_n \in \mathrm{T}_\Sigma$, we abbreviate $\xi[\zeta_1]_{w_1} \ldots [\zeta_n]_{w_n}$ by $\xi[\zeta_1, \ldots, \zeta_n]_{(w_1,\ldots,w_n)}$. If $\zeta_1 = \ldots = \zeta_n = \zeta$, then we abbreviate $\xi[\zeta, \ldots, \zeta]_{(w_1,\ldots,w_n)}$ by $\xi[\zeta]_{(w_1,\ldots,w_n)}$. Moreover, for every $i \in [n]$, $k = |\xi|_{x_i}$, $\xi_1, \ldots, \xi_k \in \mathrm{T}_\Sigma(X_n)$, and $w_1, \ldots, w_k \in \mathrm{pos}(\xi)$ such that $\mathrm{pos}_{x_i}(\xi) = \{w_1, \ldots, w_k\}$ and $w_1 <_l \cdots <_l w_k$, we denote $\xi[\xi_1, \ldots, \xi_k]_{(w_1,\ldots,w_k)}$ by $\xi[x_i \leftarrow \xi_1, \ldots, \xi_k]$. Furthermore, for every $\xi \in \mathrm{T}_\Sigma(X_n)$ and $\xi_1, \ldots, \xi_n \in \mathrm{T}_\Sigma$, we define the tree $\xi[\xi_1, \ldots, \xi_n] \in \mathrm{T}_\Sigma$ as the tree obtained from ξ by replacing every occurrence of x_i by ξ_i for every $i \in [n]$.

Let $\xi \in \mathrm{T}_\Sigma(X_1)$. We call ξ a *context (tree) over* Σ if there exists a unique position $w \in \mathrm{pos}(\xi)$ such that $\xi(w) = x_1$. In this case, we define $\mathrm{pos}_{\mathrm{var}}(\xi) = w$. The set of contexts over Σ is denoted by C_Σ.

Given a context $\zeta \in \mathrm{C}_\Sigma$ and a tree $\xi \in \mathrm{T}_\Sigma(H)$, the *substitution of* ξ *into* ζ, denoted by $\zeta[\xi]$, is the tree $\zeta[\xi]_{\mathrm{pos}_{\mathrm{var}}(\zeta)}$. Note that, given $\zeta, \zeta' \in \mathrm{C}_\Sigma$, also $\zeta[\zeta'] \in \mathrm{C}_\Sigma$. We write ζ^k for $\zeta[\zeta[\cdots[\zeta]\cdots]]$ containing the context ζ a total of k times.

Example 2.2. We consider the ranked alphabet $\Sigma = \{\alpha^{(0)}, \beta^{(0)}, \gamma^{(1)}, \sigma^{(2)}\}$. Two trees in $\mathrm{T}_\Sigma(X_1)$ are $\xi = \sigma(\alpha, \beta)$ and $\zeta = \gamma(x_1)$, which can be visualised as follows



It holds that $\mathrm{pos}(\xi) = \{\varepsilon, 1, 2\}$, $\mathrm{pos}(\zeta) = \{\varepsilon, 1\}$, $\mathrm{size}(\xi) = 3$, $\mathrm{size}(\zeta) = 1$, $\mathrm{height}(\xi) = 2$, and $\mathrm{height}(\zeta) = 1$. Furthermore, ζ is a context over Σ such that $\mathrm{pos}_{\mathrm{var}}(\zeta) = 1$ and $\mathrm{pos}_\Sigma(\zeta) = \{\varepsilon\}$. Moreover, $\zeta[\xi] = \gamma(\sigma(\alpha, \beta))$ and $\zeta^n = \gamma^n(x_1)$. ◀

A *tree language over* Σ *and* H is a set $L \subseteq \mathrm{T}_\Sigma(H)$.

Let A be an alphabet. We define the ranked alphabet Σ_A, where $\Sigma_A^{(0)} = \{\#\}$ contains solely the hashtag symbol, $\Sigma^{(1)} = A$, and $\Sigma^{(s)} = \emptyset$ for every $s \geq 2$. It holds that $A^* \cong \mathrm{T}_{\Sigma_A}$, which shows that word languages are special cases of tree languages. In order to see this, consider the map $\varphi\colon A^* \to \mathrm{T}_{\Sigma_A}$ given for every $w \in A^*$ by

$$\varphi(w) = w[k](\ldots w[1](\#)\ldots).$$

It can easily be shown that φ is indeed a bijection and hence it holds that $A^* \cong \mathrm{T}_{\Sigma_A}$.

2.2 Weighted Languages

Operations and Monoids Let A be a non-empty set. For $k \in \mathbb{N}$, a *k-ary operation on A* is a map $\omega\colon A^k \to A$ and we denote the set of all k-ary operations on A by $\mathrm{Ops}^k(A)$. Moreover, we denote the set $\bigcup_{k\in\mathbb{N}} \mathrm{Ops}^k(A)$ by $\mathrm{Ops}(A)$. We write *binary*, *unary*, and *nonary* rather than 2-ary, 1-ary, and 0-ary, respectively.

We note that every set of operations $\Omega \subseteq \mathrm{Ops}(A)$ is naturally a ranked set, where the rank of an operator is its arity. Formally, $\Omega^{(k)} = \Omega \cap \mathrm{Ops}^k(A)$ for every $k \in \mathbb{N}$.

Let $\odot\colon A \times A \to A$ be a binary operation on A. $\odot$ is called

- *associative* if $(a \odot b) \odot c = a \odot (b \odot c)$ for every $a, b, c \in A$,
- *commutative* if $a \odot b = b \odot a$ for every $a, b \in A$, and
- *cancellative* if $a \odot b = a \odot c$ implies $b = c$ for every $a, b, c \in A$.

An element $1 \in A$ is called *neutral with respect to* $\odot$ if $1 \odot a = a = a \odot 1$ for every $a \in A$.

A *semigroup* is a tuple $(A, \odot)$, where A is a set and $\odot\colon A \times A \to A$ is an associative binary operation on A. A semigroup $(A, \odot)$ is called *commutative* (or *cancellative*) if $\odot$ is commutative (or *cancellative*, respectively). A *monoid* is a tuple $(\mathbb{M}, \odot, 1)$, where $(\mathbb{M}, \odot)$ is a semigroup and $1 \in \mathbb{M}$ is neutral with respect to $\odot$.

Let $(\mathbb{M}, \odot, 1)$ be a monoid and $m, n \in \mathbb{M}$. We call n the *left inverse* (or *right inverse*) of m if $n \odot m = 1$ (or $m \odot n = 1$, respectively) and we call n the *inverse* of m if n is both the left inverse and right inverse of m. If n is the inverse of m, then we denote n by m^{-1}. A *group* is a monoid $(\mathbb{M}, \odot, 1)$ such that every $m \in \mathbb{M}$ has an inverse.

We refer to a monoid $(\mathbb{M}, \odot, 1)$ by the set $\mathbb{M}$, whenever the operation $\odot$ and the element 1 are clear from the context.

Let $\mathbb{M}$ be a monoid. A set $\mathbb{U} \subseteq \mathbb{M}$ is called a *submonoid* of $\mathbb{M}$, in symbols $\mathbb{U} \leq \mathbb{M}$, if $(\mathbb{U}, \odot|_{\mathbb{U}\times\mathbb{U}}, 1)$ is a monoid. Given a set $\Gamma \subseteq \mathbb{M}$, we define the *submonoid of* $\mathbb{M}$ *generated by* Γ, denoted by $\langle \Gamma \rangle_\odot$, as the smallest submonoid $\mathbb{U}$ of $\mathbb{M}$ such that $\Gamma \subseteq \mathbb{U}$. We note

that $1 \in \langle \Gamma \rangle_\odot$ even if $1 \notin \Gamma$. If $\langle \Gamma \rangle_\odot = \mathbb{M}$, then we call Γ a *generating set of* $\mathbb{M}$ and if Γ is moreover finite, then we say that $\mathbb{M}$ is *finitely generated (by Γ).*

Let $(\mathbb{M}, \odot_\mathbb{M}, 1_\mathbb{M})$ and $(\mathbb{L}, \odot_\mathbb{L}, 1_\mathbb{L})$ be monoids and $f \colon \mathbb{M} \to \mathbb{L}$. We say that f is a *(monoid) homomorphism* if (a) $f(1_\mathbb{M}) = 1_\mathbb{L}$ and (b) $f(m_1 \odot_\mathbb{M} m_2) = f(m_1) \odot_\mathbb{L} f(m_2)$ for every $m_1, m_2 \in \mathbb{M}$. Furthermore, we say that f is a *(monoid) isomorphism* if f is a bijective homomorphism. In this case, we say that $\mathbb{M}$ and $\mathbb{L}$ are *isomorphic (as monoids)* and denote this fact by $\mathbb{M} \cong \mathbb{L}$.

Example 2.3. We list some monoids that are either well-known or will become relevant throughout the thesis.

- We consider the monoid $(\mathbb{N}, +, 0)$ which is finitely generated by $\Gamma = \{1\}$.
- We let $k \in \mathbb{N}$ and consider the monoid $\mathbb{N}_{\leq k} = (\{n \in \mathbb{N} \mid n \leq k\}, +_k, 0)$ where $+_k \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is given by

  $$n_1 +_k n_2 = \min(n_1 + n_2, k)$$

  for every $n_1, n_2 \in \mathbb{N}$. We note that $\mathbb{N}_{\leq k}$ is finitely generated by $\Gamma = \{1\}$.
- We consider the monoid $\mathbb{BF} = \big(\mathbb{N}^2, \circ_\mathbb{BF}, (0,0)\big)$ where $\circ_\mathbb{BF} \colon \mathbb{N}^2 \times \mathbb{N}^2 \to \mathbb{N}^2$ is given by

  $$(n_\mathrm{a}, n_\mathrm{b}) \circ_\mathbb{BF} (m_\mathrm{a}, m_\mathrm{b}) = (n_\mathrm{a} + m_\mathrm{a}, 2^{m_\mathrm{a}} \cdot n_\mathrm{b} + m_\mathrm{b})$$

  for every $(n_\mathrm{a}, n_\mathrm{b}), (m_\mathrm{a}, m_\mathrm{b}) \in \mathbb{N}^2$. We call $\mathbb{BF}$ the *bifunctional monoid.* One can show that $\mathbb{BF}$ is finitely generated by $\Gamma_\mathbb{BF} = \big\{(1,0), (0,1)\big\}$.

  Next we show that $\mathbb{BF}$ is cancellative. Let $(n_\mathrm{a}, n_\mathrm{b}), (m_\mathrm{a}, m_\mathrm{b}), (m'_\mathrm{a}, m'_\mathrm{b}) \in \mathbb{BF}$ such that $(n_\mathrm{a}, n_\mathrm{b}) \circ_\mathbb{BF} (m_\mathrm{a}, m_\mathrm{b}) = (n_\mathrm{a}, n_\mathrm{b}) \circ_\mathbb{BF} (m'_\mathrm{a}, m'_\mathrm{b})$. By the definition of $\circ_\mathbb{BF}$ we obtain

  $$n_\mathrm{a} + m_\mathrm{a} = n_\mathrm{a} + m'_\mathrm{a} \qquad \text{and} \qquad 2^{m_\mathrm{a}} \cdot n_\mathrm{b} + m_\mathrm{b} = 2^{m'_\mathrm{a}} \cdot n_\mathrm{b} + m'_\mathrm{b}. \tag{2.2}$$

  The first equation in (2.2) implies $m_\mathrm{a} = m'_\mathrm{a}$ and hence the second equation in (2.2) degenerates into $2^{m_\mathrm{a}} \cdot n_\mathrm{b} + m_\mathrm{b} = 2^{m_\mathrm{a}} \cdot n_\mathrm{b} + m'_\mathrm{b}$, which yields $m_\mathrm{b} = m'_\mathrm{b}$. In total we have $(m_\mathrm{a}, m_\mathrm{b}) = (m'_\mathrm{a}, m'_\mathrm{b})$, which concludes the proof that $\mathbb{BF}$ is cancellative.

The word "bifunctional" in the name of $\mathbb{BF}$ is inspired by the following connection. Let $f, g\colon \mathbb{N} \to \mathbb{N}$ such that $f(n) = 2 \cdot n$ and $g(n) = n + 1$ for every $n \in \mathbb{N}$. We consider the closure $\langle \{f, g\} \rangle_;$ of f and g under composition (recall that $(f; g)(n) = g(f(n))$ for every $n \in \mathbb{N}$). One can show that $(\langle \{f, g\} \rangle_;, ;, \mathrm{id}_{\mathbb{N}})$ is isomorphic to $\mathbb{BF}$. The isomorphism $\varphi\colon \langle \{f, g\} \rangle_; \to \mathbb{BF}$ maps f to $(1, 0)$ and g to $(0, 1)$.

- We let A be an alphabet and consider the monoid $(A^*, \circ, \varepsilon)$ which is finitely generated by $\Gamma = A$. This monoid is called the *free monoid over* A.

  Let $\mathbb{M}$ be a finitely generated monoid with finite generating set $\Gamma \neq \emptyset$. We consider the monoid $(\Gamma^*, \circ, \varepsilon)$ and note that there exists a unique homomorphism $h\colon \Gamma^* \to \mathbb{M}$ such that $h(\gamma) = \gamma$ for every $\gamma \in \Gamma$. This fact belongs to folklore and can easily be proven using the fact that $\mathbb{M}$ is finitely generated by Γ. ◀

Let $\mathbb{M}$ be a monoid, $m \in \mathbb{M}$, and $M \in \mathcal{M}_{\mathrm{fin}}(\mathbb{M})$. We define

$$m \odot M = \{\!\{ m \odot n \mid n \in M \}\!\}$$

and note that $\#(m \odot M) = \#M$.

Semirings Let $\oplus, \odot\colon A \times A \to A$ be two binary operations on A. We say that $\odot$ *distributes over* $\oplus$ if $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$ for every $a, b, c \in A$.

A *semiring* (cf. [65] and [36, Chapter 1]) is a tuple $(S, \oplus, \odot, 0, 1)$ such that $(S, \oplus, 0)$ is a commutative monoid, $(S, \odot, 1)$ is a monoid, $\odot$ distributes over $\oplus$, and $a \odot 0 = 0 \odot a = 0$ for every $a \in S$. The operations $\oplus$ and $\odot$ are referred to as *addition* and *multiplication*, respectively. A semiring $(S, \oplus, \odot, 0, 1)$ is called

- *commutative* if $\odot$ is commutative,
- *extremal* if $a \oplus b \in \{a, b\}$ for every $a, b \in S$,
- *(additively) idempotent* if $a \oplus a = a$ for every $a \in S$,
- *additively locally finite* if $\langle \Gamma \rangle_\oplus$ is finite for every $\Gamma \in \mathcal{P}_{\mathrm{fin}}(S)$,
- *multiplicatively locally finite* if $\langle \Gamma \rangle_\odot$ is finite for every $\Gamma \in \mathcal{P}_{\mathrm{fin}}(S)$, and
- *locally finite* if S is additively and multiplicatively locally finite.

Clearly, "extremal" implies "additively idempotent" and "commutative and additively idempotent" implies "additively locally finite".

Moreover, a semiring $(S, \oplus, \odot, 0, 1)$ is called *complete* if for every countable index set I there exists an operation $\bigoplus_I \colon S^I \to S$, called *infinitary sum operation*, such that the following axioms hold ([41, p.124], see also [66]). For every I-family $(a_i \mid i \in I)$ over S, countable set J, J-family $(I_j \mid j \in J)$ over $\mathcal{P}(I)$, and $a \in S$, we have that

$$\bigoplus_{i \in I} a_i = a_{i_1} \oplus \cdots \oplus a_{i_k}, \text{ if } \#I = k \text{ and } I = \{i_1, \ldots, i_k\},$$

$$\bigoplus_{i \in I} a_i = \bigoplus_{j \in J} \left(\bigoplus_{i \in I_j} a_i \right), \text{ if } \bigcup_{j \in J} I_j = I \text{ and } I_j \cap I_{j'} = \emptyset \text{ for } j \neq j', \tag{2.3}$$

$$\bigoplus_{i \in I} (a \odot a_i) = a \odot (\bigoplus_{i \in I} a_i)\,, \text{ and } \bigoplus_{i \in I} (a_i \odot a) = (\bigoplus_{i \in I} a_i) \odot a, \tag{2.4}$$

where $\bigoplus_{i \in I} a_i$ is an abbreviation for $\bigoplus_I (a_i \mid i \in I)$.

We refer to a semiring $(S, \oplus, \odot, 0, 1)$ by the set S, whenever the operations $\oplus$ and $\odot$ and the elements 0 and 1 are clear from the context. We call $a \in S$ *vanishing* if $a = 0$ and *non-vanishing* if $a \neq 0$.

We refer the reader to [35, 65, 73] for the theory and more examples of semirings.

Example 2.4. We list some semirings that are either well-known or will become relevant in upcoming examples of this thesis.

- We consider the semiring $\mathbb{B} = (\{\bot, \top\}, \vee, \wedge, \bot, \top)$, where $\vee$ and $\wedge$ are the logical operations "or" and "and", respectively. $\mathbb{B}$ is called the *Boolean semiring* and is commmutative, extremal, locally finite, and complete.

- We consider the semiring $\mathrm{Arct} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$, where for every $n_1, n_2 \in \mathbb{N} \cup \{-\infty\}$ we define

$$\max(n_1, n_2) = \begin{cases} n_{3-i} & \text{if } n_i = -\infty \text{ for some } i \in \{1, 2\} \\ \max(n_1, n_2) & \text{otherwise} \end{cases}$$

$$n_1 + n_2 = \begin{cases} -\infty & \text{if } n_i = -\infty \text{ for some } i \in \{1, 2\} \\ n_1 + n_2 & \text{otherwise.} \end{cases}$$

Arct is called the *arctic semiring (over* $\mathbb{N}$*)*. We note that Arct is commutative and extremal. Similarly, one can introduce tropical semirings, which we will do in Chapter 4.

- Let $n \in \mathbb{N}_+$ and define $\mathbb{Z}_n = [0, n]$. We consider the semiring $(\mathbb{Z}_n, \oplus_n, \odot_n, 0, 1)$, where $\oplus_n$ and $\odot_n$ denote addition and multiplication modulo n, respectively. We note that $\mathbb{Z}_n$ is commutative and locally finite.

- We consider the semiring $\mathbb{X} = (\mathbb{N} \cup \{\bot, \top\}, \vee, +, \bot, 0)$, where $\vee$ and $+$ are given for every $n_1, n_2 \in \mathbb{N} \cup \{\bot, \top\}$ by

$$n_1 \vee n_2 = \begin{cases} n_{3-i} & \text{if } n_i = \bot \text{ for some } i \in \{1, 2\} \\ \top & \text{otherwise} \end{cases}$$

$$n_1 + n_2 = \begin{cases} \bot & \text{if } n_i = \bot \text{ for some } i \in \{1, 2\} \\ n_1 + n_2 & \text{if } n_1, n_2 \in \mathbb{N} \\ \top & \text{otherwise} \end{cases}$$

  We note that $\mathbb{X}$ is commutative but not additively idempotent, since $n \vee n = \top$ for every $n \in \mathbb{N}$. We have constructed $\mathbb{X}$ as a "simple" semiring that is not additively idempotent in order to provide examples for upcoming theorems that were classically only applicable in idempotent settings.

- Let $(\mathbb{M}, \odot, 1)$ be a monoid. We consider the semiring $(\mathcal{M}_{\text{fin}}(\mathbb{M}), \cup, \odot, \emptyset, \{\!\{1\}\!\})$, where $\cup$ is the multiset union and for every $M, N \in \mathcal{M}_{\text{fin}}(\mathbb{M})$ we define

$$M \odot N = \{\!\{m \odot n \mid m \in M, n \in N\}\!\}.$$

  We call $\mathcal{M}_{\text{fin}}(\mathbb{M})$ the *semiring of finite multisets over* $\mathbb{M}$. We note that $\mathcal{M}_{\text{fin}}(\mathbb{M})$ is commutative if and only if $\mathbb{M}$ is commutative. Moreover, $\mathcal{M}_{\text{fin}}(\mathbb{M})$ is not idempotent, as $\{\!\{1\}\!\} \cup \{\!\{1\}\!\} = \{\!\{1, 1\}\!\}$.

- Let $(\mathbb{M}, \odot, 1)$ be a monoid. Analogously to the semiring $\mathcal{M}_{\text{fin}}(\mathbb{M})$ we can consider the semiring $(\mathcal{P}_{\text{fin}}(\mathbb{M}), \cup, \odot, \emptyset, \{1\})$, called the *semiring of finite sets over* $\mathbb{M}$. In contrast to $\mathcal{M}_{\text{fin}}(\mathbb{M})$, the semiring $\mathcal{P}_{\text{fin}}(\mathbb{M})$ is idempotent. ◀

Let M be a finite multiset over a set A and $f\colon \mathrm{supp}(M) \to S$. We define

$$\bigoplus_{m \in M} f(m) = \bigoplus_{m \in \mathrm{supp}(M)} \Big(\bigoplus_{i=1}^{M(m)} f(m) \Big).$$

If $S = (\mathcal{M}(\mathbb{M}), \cup, \odot, \emptyset, \{\!\{1\}\!\})$, then we also write $\bigcup_{m \in M} f(m)$ for $\bigoplus_{m \in M} f(m)$.

Weighted Word Languages Let S be a semiring and Σ be an alphabet. A map $\varphi\colon \Sigma^* \to S$ is called *(Σ, S)-weighted (word) language*. We drop the parameter (Σ, S) whenever it is clear from the context. For every set A and map $\varphi\colon A \to S$, we define the *support* of φ as

$$\mathrm{supp}(\varphi) = \{a \in A \mid \varphi(a) \neq 0\}$$

and for every $a \in S$ we denote the constant map to a of type $A \to S$ by a as well. For every $w \in \Sigma^*$ we define the weighted language $1.w\colon \Sigma^* \to S$ by

$$(1.w)(v) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

for every $v \in \Sigma^*$. Let $\varphi\colon \Sigma^* \to S$ be a weighted language. We say that φ is *proper* if $\varphi(\varepsilon) = 0$.

Let $\varphi_1, \varphi_2\colon \Sigma^* \to S$ be weighted languages. We define the *concatenation of φ_1 and φ_2* (cf. [99, Chapter III.]) as the weighted language $\varphi_1 \circ \varphi_2\colon \Sigma^* \to S$ defined for every $w \in \Sigma^*$ by

$$(\varphi_1 \circ \varphi_2)(w) = \bigoplus_{\substack{w_1, w_2 \in \Sigma^* \\ \text{s.th. } w = w_1 w_2}} \varphi_1(w_1) \odot \varphi_2(w_2).$$

We note that for every $w \in \Sigma^*$ the set $\{(w_1, w_2) \in (\Sigma^*)^2 \mid w = w_1 w_2\}$ is finite and hence, $\varphi_1 \circ \varphi_2$ is well-defined.

Let $\varphi\colon \Sigma^* \to S$. For every $n \in \mathbb{N}$, the *n-th power of φ* is the weighted language $\varphi^n\colon \Sigma^* \to S$ defined by induction on n as follows:

$$\varphi^0 = 1.\varepsilon \quad \text{and}$$
$$\varphi^{n+1} = (\varphi \circ \varphi^n) + 1.\varepsilon \quad \text{for every } n \geq 0.$$

Let us assume that φ is proper. It holds that, for every $w \in \Sigma^*$ and $n \in \mathbb{N}$, if $n \geq |w| + 1$, then $\varphi^{n+1}(w) = \varphi^n(w)$ (see e.g. [37, Chapter 4]). This justifies

the following definition (cf. [99, Chapter III]). The *Kleene star of* φ is the weighted language $\varphi^*\colon \Sigma^* \to S$ given for every $w \in \Sigma^*$ by

$$\varphi^*(w) = \varphi^{|w|+1}(w).$$

Weighted Tree Languages Let S be a semiring, Σ be a ranked alphabet, and H be a set such that $\Sigma \cap H = \emptyset$. A map $\varphi\colon \mathrm{T}_\Sigma(H) \to S$ is called (Σ, H, S)*-weighted tree language*. We abbreviate the parameter $(\Sigma, \emptyset, S)$ by (Σ, S) and drop the parameter (Σ, H, S) whenever it is clear from the context. For every $a \in S$ and $\xi \in \mathrm{T}_\Sigma$ we define the weighted tree language $a.\xi\colon \mathrm{T}_\Sigma \to S$ by

$$(a.\xi)(\zeta) = \begin{cases} a & \text{if } \zeta = \xi \\ 0 & \text{otherwise} \end{cases}$$

for every $\zeta \in \mathrm{T}_\Sigma$. We call a weighted tree language $\varphi\colon \mathrm{T}_\Sigma \to S$ α*-proper* if $\varphi(\alpha) = 0$.

We now recall the usual rational operations on weighted tree languages [38, 53].

Let $\varphi\colon \mathrm{T}_\Sigma \to S$ and $a \in S$. The *scalar multiplication of* φ *with* a is the (Σ, S)-weighted tree language $a \odot \varphi$ defined for every $\xi \in \mathrm{T}_\Sigma$ by $(a \odot \varphi)(\xi) = a \odot \varphi(\xi)$.

Let $\varphi_1, \varphi_2\colon \mathrm{T}_\Sigma \to S$. The *sum of* φ_1 *and* φ_2 is the (Σ, S)-weighted tree language $(\varphi_1 \oplus \varphi_2)$ defined for every $\xi \in \mathrm{T}_\Sigma$ by $(\varphi_1 \oplus \varphi_2)(\xi) = \varphi_1(\xi) \oplus \varphi_2(\xi)$.

Let $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $\varphi_1, \ldots, \varphi_s\colon \mathrm{T}_\Sigma \to S$. The *top-concatenation of* $\varphi_1, \ldots, \varphi_s$ *with* σ is the (Σ, S)-weighted tree language $\mathrm{top}_\sigma(\varphi_1, \ldots, \varphi_s)$ defined for every $\xi \in \mathrm{T}_\Sigma$ by $\mathrm{top}_\sigma(\varphi_1, \ldots, \varphi_s)(\xi) = \varphi_1(\xi_1) \odot \ldots \odot \varphi_s(\xi_s)$ if $\xi = \sigma(\xi_1, \ldots, \xi_s)$ for some $\xi_1, \ldots, \xi_s \in \mathrm{T}_\Sigma$, and 0 otherwise. In particular, for $s = 0$, we have $\mathrm{top}_\sigma(\,) = 1.\sigma$.

For the definition of concatenation of weighted tree languages, we need the concept of α*-cuts*. Let $\xi \in \mathrm{T}_\Sigma$ and $\alpha \in \Sigma^{(0)}$. Intuitively, an α-cut through ξ is a tuple $(w_1, \ldots, w_n)$ of positions of ξ such that $w_1 <_l \ldots <_l w_n$, and each α-labeled position is covered by some w_i. Formally, we define the set of α*-cuts through* ξ, denoted by $\mathrm{cut}_\alpha(\xi)$, by

$$\begin{aligned} \mathrm{cut}_\alpha(\xi) = \big\{(w_1, \ldots, w_n) \mid\ & n \in \mathbb{N}, w_1, \ldots, w_n \in \mathrm{pos}(\xi), \text{ such that} \\ & w_1 <_l \ldots <_l w_n, \text{ and} \\ & \forall w \in \mathrm{pos}_\alpha(\xi)\colon \exists i \in [n]\colon w_i \text{ is a prefix of } w\big\}. \end{aligned}$$

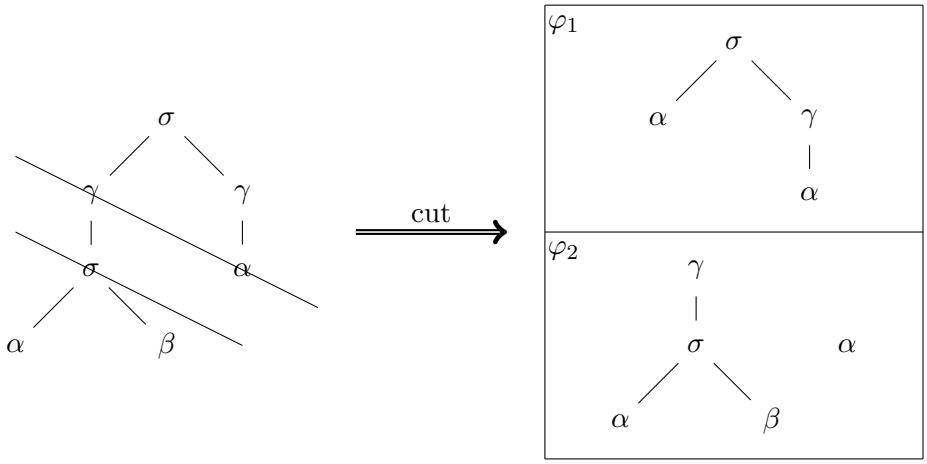


Figure 2.2: Left: The tree $\xi = \sigma(\gamma(\sigma(\alpha, \beta), \gamma(\alpha)))$ with two lines representing the tuples $(1, 21)$ and (11), respectively. $(1, 21)$ is an α-cut through ξ and (11) is not an α-cut through ξ since the rightmost α in ξ is not covered by (11). Right: ξ divided by $(1, 21)$ into $\xi[\alpha]_{(1,21)}$ (top), $\xi|_1$, and $\xi|_{21}$ (both bottom). The α-concatenation of two weighted tree languages φ_1 and φ_2 evaluates φ_1 on the top part and φ_2 on the bottom part for every α-cut.

Let $\varphi_1, \varphi_2 \colon \mathrm{T}_\Sigma \to S$ be weighted tree languages and let $\alpha \in \Sigma^{(0)}$. We define the *α-concatenation of φ_1 and φ_2* (cf. [38, Section 3]) as the weighted tree language $\varphi_1 \circ_\alpha \varphi_2 \colon \mathrm{T}_\Sigma \to S$ defined for every $\xi \in \mathrm{T}_\Sigma$ by

$$(\varphi_1 \circ_\alpha \varphi_2)(\xi) = \bigoplus_{(w_1,\ldots,w_n) \in \mathrm{cut}_\alpha(\xi)} \varphi_1(\xi[\alpha]_{(w_1,\ldots,w_n)}) \odot \varphi_2(\xi|_{w_1}) \odot \ldots \odot \varphi_2(\xi|_{w_n}).$$

We note that for every $\xi \in \mathrm{T}_\Sigma$ the index set $\mathrm{cut}_\alpha(\xi)$ is finite. An illustration of α-cuts and α-concatenation can be found in Figure 2.2.

Let $\varphi \colon \mathrm{T}_\Sigma \to S$ and $\alpha \in \Sigma^{(0)}$. We define the *n-th iteration of φ at α* as the weighted tree language $\varphi_\alpha^n \colon \mathrm{T}_\Sigma \to S$ for every $n \in \mathbb{N}$ inductively as follows (cf. [38, Def. 3.9] and [47]):

$$\varphi_\alpha^0 = 0 \quad \text{and}$$
$$\varphi_\alpha^{n+1} = (\varphi \circ_\alpha \varphi_\alpha^n) + 1.\alpha \quad \text{for every } n \geq 0.$$

Let us assume that φ is α-proper. It holds that, for every $\xi \in \mathrm{T}_\Sigma$ and $n \in \mathbb{N}$, if $n \geq \mathrm{height}(\xi) + 1$, then $\varphi_\alpha^{n+1}(\xi) = \varphi_\alpha^n(\xi)$ (see [38, Lm. 3.10]). This justifies the

following definition (cf. [38, Def. 3.11] and [47]). We define the *α-Kleene star of φ* as the weighted tree language $\varphi_\alpha^*\colon \mathrm{T}_\Sigma \to S$ given for every $\xi \in \mathrm{T}_\Sigma$ by

$$\varphi_\alpha^*(\xi) = \varphi_\alpha^{\mathrm{height}(\xi)+1}(\xi).$$

2.3 Weighted Tree Automata

We now present the prevalent automaton model for this thesis. Since we use the standard weighted tree automaton model, we do not give individual examples for each concept, but rather collect these into one big example at the end of Chapter 2.3.

> *Throughout the rest of Chapter 2.3, we let Σ be an arbitrary ranked alphabet and S be an arbitrary semiring.*

A *weighted tree automaton* (short: WTA) *over Σ and S* is a tuple $\mathscr{A} = (Q, T, \mathrm{final})$, where

- Q is a non-empty set of *states*,
- $T = (T_\sigma\colon Q^s \times Q \to S \mid s \in \mathbb{N}, \sigma \in \Sigma^{(s)})$ is a family of *transition weight* maps, and
- $\mathrm{final}\colon Q \to S$ is the *final weight* map.

If Q is finite, then we call $\mathscr{A}$ *finite*.

Let $\mathscr{A} = (Q, T, \mathrm{final})$ be a WTA over Σ and S and let $\xi \in \mathrm{T}_\Sigma \cup \mathrm{C}_\Sigma$ be a tree or a context. A *run of $\mathscr{A}$ on ξ* is a map $\rho\colon \mathrm{pos}(\xi) \to Q$. The set of all runs of $\mathscr{A}$ on ξ is denoted by $\mathrm{Runs}_{\mathscr{A}}(\xi)$. If ξ is a context and $\rho \in \mathrm{Runs}_{\mathscr{A}}(\xi)$, then we define $\mathrm{in}(\rho) = \rho(\mathrm{pos}_{\mathrm{var}}(\xi))$. We define the maps

$$\mathrm{locwt}_{\mathscr{A}}(\xi, \_, \_)\colon \mathrm{Runs}_{\mathscr{A}}(\xi) \times \mathrm{pos}(\xi) \to S \text{ and}$$
$$\mathrm{wt}_{\mathscr{A}}(\xi, \_, \_)\colon \mathrm{Runs}_{\mathscr{A}}(\xi) \times \mathrm{pos}(\xi) \to S$$

inductively as follows. Let $\rho \in \mathrm{Runs}_{\mathscr{A}}(\xi)$ and $w \in \mathrm{pos}(\xi)$. If $\xi(w) = x_1$, then we define

$$\mathrm{locwt}_{\mathscr{A}}(\xi, \rho, w) = 1 \qquad \text{and} \qquad \mathrm{wt}_{\mathscr{A}}(\xi, \rho, w) = 1.$$

Otherwise, there exists $s \in \mathbb{N}$ and $\sigma \in \Sigma^{(s)}$ such that $\xi(w) = \sigma$ and we define

$$\begin{aligned}\mathrm{locwt}_{\mathscr{A}}(\xi, \rho, w) &= T_\sigma(\rho(w1), \ldots, \rho(ws), \rho(w)) \text{ and} \\ \mathrm{wt}_{\mathscr{A}}(\xi, \rho, w) &= \mathrm{wt}_{\mathscr{A}}(\xi, \rho, w1) \odot \cdots \odot \mathrm{wt}_{\mathscr{A}}(\xi, \rho, ws) \odot T_\sigma(\rho(w1), \ldots, \rho(ws), \rho(w)).\end{aligned}$$

Whenever the automaton $\mathscr{A}$ is clear from the context, we will omit the subscript $\mathscr{A}$ from $\mathrm{locwt}_{\mathscr{A}}$ and $\mathrm{wt}_{\mathscr{A}}$ and simply write locwt and wt, respectively.

Let $\rho \in \mathrm{Runs}_{\mathscr{A}}(\xi)$. We abbreviate $\mathrm{wt}(\xi, \rho) = \mathrm{wt}(\xi, \rho, \varepsilon)$ and call $\mathrm{wt}(\xi, \rho)$ the *weight of* ρ. Moreover, we call ρ *valid* if $\mathrm{locwt}(\xi, \rho, w) \neq 0$ for every $w \in \mathrm{pos}(\xi)$ and we call ρ *non-vanishing* if $\mathrm{wt}(\xi, \rho) \neq 0$. We denote the *set of valid runs of* $\mathscr{A}$ *on* ξ by $\mathrm{Runs}^v_{\mathscr{A}}(\xi)$ and define the sets

$$\begin{aligned}\mathrm{Runs}_{\mathscr{A}}(\xi, q) &= \{\rho' \in \mathrm{Runs}_{\mathscr{A}}(\xi) \mid \rho'(\varepsilon) = q\} \text{ and} \\ \mathrm{Runs}^v_{\mathscr{A}}(\xi, q) &= \{\rho' \in \mathrm{Runs}^v_{\mathscr{A}}(\xi) \mid \rho'(\varepsilon) = q\}\end{aligned}$$

for every $q \in Q$. Moreover, if ξ is a context, then we define the sets

$$\begin{aligned}\mathrm{Runs}_{\mathscr{A}}(p, \xi, q) &= \{\rho' \in \mathrm{Runs}_{\mathscr{A}}(\xi, q) \mid \mathrm{in}(\rho') = p\} \text{ and} \\ \mathrm{Runs}^v_{\mathscr{A}}(p, \xi, q) &= \{\rho' \in \mathrm{Runs}^v_{\mathscr{A}}(\xi, q) \mid \mathrm{in}(\rho') = p\}\end{aligned}$$

for every $p, q \in Q$.

We define the *image of* $\mathscr{A}$, denoted by $\mathrm{im}(\mathscr{A})$, by

$$\mathrm{im}(\mathscr{A}) = \Big(\bigcup_{\sigma \in \Sigma} \mathrm{im}(T_\sigma)\Big) \cup \mathrm{im}(\mathrm{final}).$$

We call $\mathscr{A}$ *finite-run* if for every $\xi \in \mathrm{T}_\Sigma$ the set $\mathrm{Runs}^v_{\mathscr{A}}(\xi)$ is finite. Clearly, if $\mathscr{A}$ is finite, then $\mathscr{A}$ is finite-run. We call a state $q \in Q$ *reachable* if there exists a tree $\xi \in \mathrm{T}_\Sigma$ such that at least one run in $\mathrm{Runs}_{\mathscr{A}}(\xi, q)$ is valid. For $\xi \in \mathrm{C}_\Sigma$ we say that $\rho \in \mathrm{Runs}_{\mathscr{A}}(\xi)$ is a *loop (on* ξ*)* if there exists $q \in Q$ such that $\rho \in \mathrm{Runs}^v_{\mathscr{A}}(q, \xi, q)$. Moreover, for $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q$ we call $t = (q_1, \ldots, q_s, \sigma, q)$ a *transition of* $\mathscr{A}$ and say that t is a *transition in a loop of* $\mathscr{A}$ if there exists a loop ρ on some $\xi \in \mathrm{C}_\Sigma$ and a position $w \in \mathrm{pos}(\xi)$ such that $\rho(wi) = q_i$ for every $i \in [s]$, $\xi(w) = \sigma$, and $\rho(w) = q$. We sometimes enrich a transition t by its transition weight $x = T_\sigma(q_1, \ldots, q_s, q)$ and say that $(q_1, \ldots, q_s, \sigma, x, q)$ is a transition of $\mathscr{A}$.

Let $\xi \in \mathrm{T}_\Sigma \cup \mathrm{C}_\Sigma$, $\zeta \in \mathrm{C}_\Sigma$, $\rho \in \mathrm{Runs}_{\mathscr{A}}(\xi)$, and $\rho' \in \mathrm{Runs}_{\mathscr{A}}(\zeta)$ such that $\mathrm{in}(\rho') = \rho(\varepsilon)$. We define the run $\rho'[\rho] \in \mathrm{Runs}_{\mathscr{A}}(\zeta[\xi])$ in a natural way for every $w \in \mathrm{pos}(\zeta[\xi])$ by

$$\rho'[\rho](w) = \begin{cases} \rho(v) & \text{if } w = \mathrm{pos}_{\mathrm{var}}(\zeta)v \\ \rho'(w) & \text{otherwise.} \end{cases}$$

We note that this also defines $(\rho')^k$ for every $k \in \mathbb{N}_+$, if ρ' is a loop.

Let $\mathscr{A}$ be finite-run. The *weighted tree language recognised by* $\mathscr{A}$ is the weighted tree language $[\![\mathscr{A}]\!] \colon \mathrm{T}_\Sigma \to S$, where for every $\xi \in \mathrm{T}_\Sigma$ we define

$$[\![\mathscr{A}]\!](\xi) = \bigoplus_{\rho \in \mathrm{Runs}^v_{\mathscr{A}}(\xi)} \mathrm{wt}(\xi, \rho) \odot \mathrm{final}(\rho(\varepsilon)).$$

Two WTA $\mathscr{A}$ and $\mathscr{A}'$ over Σ and S are *equivalent* if $[\![\mathscr{A}]\!] = [\![\mathscr{A}']\!]$, that is, if $\mathscr{A}$ and $\mathscr{A}'$ recognise the same weighted tree language. The *class of weighted tree languages recognised by finite* WTA *over* Σ *and* S is denoted by $\mathrm{Rec}(\Sigma, S)$.

Let $\mathscr{A} = (Q, T, \mathrm{final})$ be a WTA over Σ and S. We call $\mathscr{A}$ *deterministic* if for every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s \in Q$ there exists at most one $q \in Q$ such that $T_\sigma(q_1, \ldots, q_s, \sigma, q) \neq 0$. Clearly, every deterministic WTA is finite-run. We call $\mathscr{A}$ *determinisable* if there exists a deterministic WTA $\mathscr{A}'$ such that $\mathscr{A}'$ is equivalent to $\mathscr{A}$. The *class of weighted tree languages recognised by deterministic finite* WTA *over* Σ *and* S is denoted by $\mathrm{dRec}(\Sigma, S)$.

In order to depict WTA easily, we use the concept of hypergraphs. More precisely, we associate to a WTA $\mathscr{A}$ a hypergraph $H_{\mathscr{A}}$ which contains the state behaviour of $\mathscr{A}$ in the natural way. We represent the final weights of $\mathscr{A}$ by using a fresh final vertex q_f and a fresh hyperedge label $\top$ and having a hyperedge into q_f labeled by $\top$ for every state in Q.

Formally, let $\top$ and q_f be fresh symbols such that $\top \notin \Sigma$ and $q_f \notin Q$. The *hypergraph of* $\mathscr{A}$ is the hypergraph $H_{\mathscr{A}} = (Q \cup \{q_f\}, E)$ with labels in $(\Sigma \cup \{\top^{(1)}\}) \times S$ such that a tuple $(q_1 \ldots q_s, (\sigma, x), q)$ is in E if and only if (a) $q_1, \ldots, q_s, q \in Q$, $\mathrm{rk}(\sigma) = s$, and $x = T_\sigma(q_1, \ldots, q_s, q)$ or (b) $s = 1$, $q_1 \in Q$, $\sigma = \top$, $q = q_f$, and $x = \mathrm{final}(q_1)$.

We make the following conventions for graphical depictions of hypergraphs of WTA. First, instead of drawing the entire label (σ, x) of a hyperedge inside of the square of the respective hyperedge, we draw the symbol σ inside of the square and the weight x

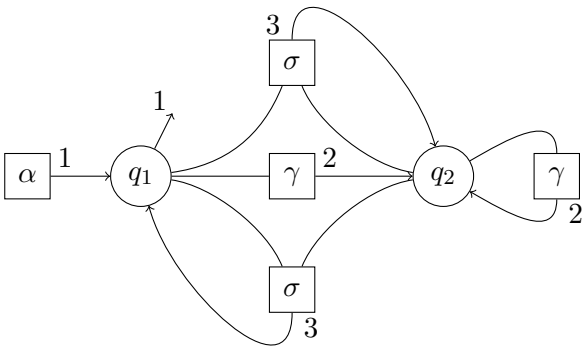


Figure 2.3: The hypergraph $H_{\mathscr{A}}$ for the WTA $\mathscr{A}$ from Example 2.5. We apply the conventions explained above Example 2.5 to keep the depiction of $H_{\mathscr{A}}$ simple and readable.

next to the outgoing arrow in counterclockwise direction. Second, to aid readability, we do not draw hyperedges where x vanishes. Third, we abbreviate edges of the form $(q, (\top, x), q_f)$, that is, the final weights of $\mathscr{A}$, by small outgoing arrows starting in q, pointing up- and rightwards, and labeled by x.

Example 2.5. Let $\Sigma = \{\alpha^{(0)}, \gamma^{(1)}, \sigma^{(2)}\}$. We consider the WTA $\mathscr{A} = (Q, T, \text{final})$ over Σ and $(\mathbb{N}, +, \cdot, 0, 1)$, where $Q = \{q_1, q_2\}$ and every transition weight and final weight of $\mathscr{A}$ is 0 except $\text{final}(q_1) = 1$ and

$$T_\alpha(q_1) = 1,\ T_\gamma(q_1, q_2) = T_\gamma(q_2, q_2) = 2,\ \text{and } T_\sigma(q_1, q_2, q_2) = T_\sigma(q_2, q_1, q_1) = 3.$$

The hypergraph of $\mathscr{A}$ is $H_{\mathscr{A}} = (Q \cup \{q_f\}, E)$, where

$$\begin{aligned} E = \{&(q_1, (\top, 1), q_f), (q_2, (\top, 0), q_f),\\ &((\alpha, 1), q_1), ((\alpha, 0), q_2),\\ &(q_1, (\gamma, 0), q_1), (q_2, (\gamma, 0), q_1), (q_1, (\gamma, 2), q_2), (q_2, (\gamma, 2), q_2),\\ &(q_1, q_1, (\sigma, 0), q_1), (q_1, q_2, (\sigma, 0), q_1), (q_2, q_1, (\sigma, 3), q_1), (q_2, q_2, (\sigma, 0), q_1),\\ &(q_1, q_1, (\sigma, 0), q_2), (q_1, q_2, (\sigma, 3), q_2), (q_2, q_1, (\sigma, 0), q_2), (q_2, q_2, (\sigma, 0), q_2)\} \end{aligned}$$

We depict $H_{\mathscr{A}}$ in Figure 2.3. ◀

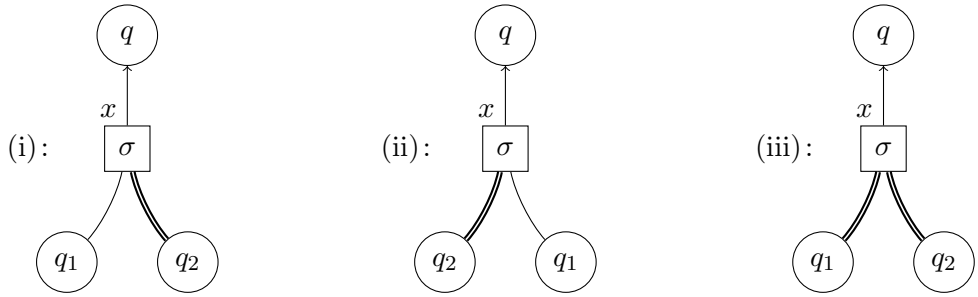


Figure 2.4: Our three options to collect hyperedges. This aims to keep illustrations of hypergraphs readable.

We make another convention to simplify illustrations of hypergraphs of WTA. Let $\mathscr{A} = (Q, T, \text{final})$ be a WTA and $\sigma \in \Sigma^{(2)}$. If T_σ maps many state combinations to non-vanishing weights, it quickly becomes unreadable to draw every hyperedge of $H_{\mathscr{A}}$ separately. Therefore, we collect hyperedges in the following way.

Let $q_1, q_2, q \in Q$. If $T_\sigma(q_1, q_1, q) = T_\sigma(q_1, q_2, q) = T_\sigma(q_2, q_1, q) = T_\sigma(q_2, q_2, q)$, then instead of drawing all four corresponding hyperedges, we draw structure (iii) from Figure 2.4. If $T_\sigma(q_1, q_2, q) = T_\sigma(q_2, q_1, q) = T_\sigma(q_2, q_2, q)$, then we draw the structure (i) or (ii) from Figure 2.4.

We now present some special cases of our automaton model. First note that our automaton model is the "standard model" for weighted tree automata, where we additionally allow the set of states Q to be infinite. In fact, requiring that Q is finite results in a syntactical variant of the weighted tree automaton models from [3, 6, 40, 53].

An *(unweighted) tree automaton over* Σ is a WTA over Σ and the Boolean semiring $\mathbb{B}$. Our model of finite (unweighted) tree automata is a syntactical variant of the tree automaton models in [58, 59].

Let A be an alphabet and recall the ranked alphabet Σ_A from Chapter 2.1. A *weighted (word) automaton* (short: WA) *over* A *and* S is a WTA over Σ_A and S. Let $\mathscr{A} = (Q, T, \text{final})$ be a WA over A and S. Since $T = (T_\#, T_a \mid a \in A)$, one can also represent $\mathscr{A}$ by the tuple $(Q, \text{init}, T_A, \text{final})$, where

- $\text{init}\colon Q \to S$ is given by $\text{init}(q) = T_\#(q)$ and
- $T_A\colon Q \times A \times Q \to S$ is given by $T_A(p, a, q) = T_a(p, q)$

for every $p, q \in Q$ and $a \in A$. In this sense, our model of weighted automata coincides with the automaton model from [31] and is a syntactical variant of the automaton model from [36, Chapter 3] without ε-transitions.

An *(unweighted) (word) automaton over* A is a WA over A and the Boolean semiring $\mathbb{B}$. Our model of finite automata is the classical finite automaton model (cf. [76, Chapter 2.2]).

Since weighted word automata and unweighted (tree or word) automata do not play a major role in this thesis, we waive examples and refer the interested reader to [36, Chapter 3], [58, 59], and [76, Chapter 2.2], respectively.

3

A Unifying Framework for the Determinisation of Weighted Tree Automata

This chapter is a canonical generalisation of Dörband and Mörbitz [31] from weighted (word) automata to weighted tree automata. Besides a higher notational complexity, this chapter closely follows [31]. We note that [31] and Chapter 3 are unpublished. The results from this chapter also cover the results from Dörband, Feller, and Stier [29].

Throughout Chapter 3, we assume Σ to be a ranked alphabet.

3.1 Introduction

In the world of unweighted automata, one classical result states that every finite automaton $\mathscr{A}$ can be determinised ([93], cf. also [76, Chapter 2.5.3]). That is, there exists a deterministic finite automaton $D_{\mathscr{A}}$ such that $D_{\mathscr{A}}$ is equivalent to $\mathscr{A}$. This can be shown using the well-known power set construction, where each state of $D_{\mathscr{A}}$ is a subset of the state set Q of $\mathscr{A}$. While a generalisation of the power set construction to the weighted setting can be attempted in a straightforward way, a positive determinisation result is much harder to obtain [85] and not every weighted automaton $\mathscr{A}$ is determinisable [12, Lemma 6.3]. As a result, the research on determinisation has

shifted towards finding conditions under which determinisation is possible. Different approaches have emerged to find weighted determinisation constructions, both, in the word case [5, 17, 21, 39, 80, 90] and the tree case [14, 29, 57]. These approaches have in common that they adapt the classical power set construction to the weighted setting and show that the respective construction returns a finite deterministic weighted (word or tree) automaton $D_{\mathscr{A}}$ whenever $\mathscr{A}$ satisfies a so-called twinning property. However, the notions of the twinning property and determinism vary between the approaches. Moreover, the respective determinisation results can only be applied to very restricted classes of semirings (sometimes only individual semirings), which demonstrates the fact that the research on weighted determinisation is scattered and misses a unified theory.

In [90], a weighted power set construction for determinisation was introduced for the min-plus semiring. This weighted power set construction is a natural extension of the unweighted power set construction, where the states of $D_{\mathscr{A}}$ are now maps of the type $Q \to S$. This approach was later generalised by [80] (cf. [14] for the tree case) who introduced a general notion of factorisations, which allowed for a determinisation result over extremal, commutative semirings. In a weighted power set construction with factorisation, each state of $D_{\mathscr{A}}$ is still a map $X \colon Q \to S$. However, whenever the weighted power set construction calculates a transition $X \xrightarrow{\sigma|1} Y$, the factorisation chooses a decomposition $Y = y \odot Y'$ such that y is a common factor of the elements in Y and replaces the transition $X \xrightarrow{\sigma|1} Y$ by $X \xrightarrow{\sigma|y} Y'$. By using factorisations, one tries to counterweigh the fact that the state set of $D_{\mathscr{A}}$ is potentially infinite.

Another approach to weighted determinisation is called sequentialisation[1] [5, 21, 29] (cf. also [17]). Sequentialisation considers just one operation given by a monoid $(\mathbb{M}, \odot, 1)$ and covers only semirings of the form $(\mathcal{P}_{\text{fin}}(\mathbb{M}), \cup, \odot, \emptyset, \{1\})$. A WTA $\mathscr{A}$ is called sequential if (1) $\mathscr{A}$ is deterministic and (2) all transition weights occurring in $\mathscr{A}$ are either $\emptyset$ or singleton sets over $\mathbb{M}$. Now, a sequentialisation construction takes as input a WTA $\mathscr{A}$ over $\mathcal{P}_{\text{fin}}(\mathbb{M})$ and returns a sequential WTA $D_{\mathscr{A}}$ over $\mathcal{P}_{\text{fin}}(\mathbb{M})$ such that $D_{\mathscr{A}}$ is equivalent to $\mathscr{A}$. That is, a sequentialisation construction needs to be more

[1] We note that the term "sequentiality" is used ambiguously in the literature. In some papers [85, 92], sequentiality and determinism are used interchangeably, whereas in other papers [5, 17, 21, 29], sequentiality is strictly more restrictive than determinism. We use the term "sequentiality" as given in [29].

elaborate than a classical weighted determinisation construction, as it needs to make an additional effort to keep the transition weights simple in order to satisfy condition (2) from the definition of sequentiality. The sequentialisation constructions in [5, 21, 29] are weighted power set constructions with factorisations. The additional requirement that $D_{\mathscr{A}}$ may only use singleton sets over $\mathbb{M}$ is met by considering special factorisations, which, given a state $Y\colon Q \to \mathcal{P}_{\text{fin}}(\mathbb{M})$, choose common factors in $\mathbb{M}$, rather than in $\mathcal{P}_{\text{fin}}(\mathbb{M})$. Moreover, [5, 21, 29] implicitly use a distance function $d\colon \mathbb{M} \times \mathbb{M} \to \mathbb{N}$ and ensure that factorisations keep the states of $D_{\mathscr{A}}$ "close to 1" with respect to d. This aims to make $D_{\mathscr{A}}$ finite. However, the actual class of semirings covered by these sequentialisation results is quite restricted. In [5], it is assumed that $\mathbb{M}$ is the free monoid Γ^* for an alphabet Γ and in [21, 29], it is assumed that $\mathbb{M}$ is a finitely generated group.

A third kind of determinisation is called crisp-determinisation [39, 57]. An automaton $\mathscr{A}$ is called crisp-deterministic if (1) $\mathscr{A}$ is deterministic and (2) all transition weights occurring in $\mathscr{A}$ are either 0 or 1. Now, a crisp-determinisation construction takes as input a WTA $\mathscr{A}$ over S and returns a crisp-deterministic WTA $D_{\mathscr{A}}$ such that $D_{\mathscr{A}}$ is equivalent to $\mathscr{A}$. Hence, a crisp-determinisation construction needs to be even more elaborate than a sequentialisation construction, as the neutral elements in $\mathcal{P}_{\text{fin}}(\mathbb{M})$ are $\emptyset$ and $\{1\}$. In [39, 57], it is required that S is an (additively and multiplicatively) locally finite semiring. Under this condition, the authors prove that every WTA over S can be crisp-determinised. The crisp-determinisation construction in [57, Algorithm 1] is a weighted power set construction without factorisations.

In the present chapter, we unify these three determinisation approaches as follows. First, we let $\mathbb{M}$ be a submonoid of the multiplicative monoid $(S, \odot, 1)$ of the semiring. Next, we say that an automaton $\mathscr{A}$ over S is $\mathbb{M}$-*sequential* if (1) $\mathscr{A}$ is deterministic and (2) all transition weights occurring in $\mathscr{A}$ are either 0 or in $\mathbb{M}$. Now, an $\mathbb{M}$-sequentialisation construction takes as input a finite WTA $\mathscr{A}$ over S and returns an $\mathbb{M}$-sequential WTA $D_{\mathscr{A}}$ such that $D_{\mathscr{A}}$ is equivalent to $\mathscr{A}$.

In this general framework, $\mathbb{M}$ is a parameter for our determinisation approach that determines how "regular" the resulting automata should be. This "regularity" ranges between the case $\mathbb{M} = S$, which yields classical determinisation, and the case that $\mathbb{M}$ is the trivial monoid $(\{1\}, \odot, 1)$, which yields crisp-determinisation.

3. A UNIFYING FRAMEWORK FOR THE DETERMINISATION OF WEIGHTED TREE AUTOMATA

The goal of Chapter 3 is to (1) introduce the mathematical tools necessary for our $\mathbb{M}$-sequentialisation framework and (2) provide an $\mathbb{M}$-sequentialisation construction for a large class of semirings. In the following paragraphs, we will explain our $\mathbb{M}$-sequentialisation construction in a slightly more technical way.

Let S be a semiring, $\mathbb{M} \leq (S, \odot, 1)$, and $\mathscr{A} = (Q, T, \mathrm{final})$ be a finite WTA over Σ and S. Our $\mathbb{M}$-sequentialisation construction works in three steps, which can be outlined as follows.

Step (I): We decompose every weight occurring in $\mathscr{A}$ into finite sums over $\mathbb{M}$ and rewrite these finite sums as finite multisets over $\mathbb{M}$. This yields a finite WTA $\mathscr{B}$ over Σ and $(\mathcal{M}_{\mathrm{fin}}(\mathbb{M}), \cup, \odot, \emptyset, \{1\})$ (cf. Lemma 3.55). This step can only be done if all weights occurring in $\mathscr{A}$ can be written as finite sums over $\mathbb{M}$, which is why we require that $\mathscr{A}$ is a WTA over Σ and $\langle \mathbb{M} \rangle_{\oplus}$, the additive closure of $\mathbb{M}$, in Lemma 3.55. We consider multisets over $\mathbb{M}$ rather than sets over $\mathbb{M}$ since we want to cover the cases where S is not idempotent. Moreover, we note that this step is trivial for classical determinisation, where $\mathbb{M} = S$.

Step (II): Recall that $\mathcal{S}(\mathbb{M})$ is the set of singleton sets over $\mathbb{M}$. We give an infinite $\mathcal{S}(\mathbb{M})$-sequentialisation construction that takes $\mathscr{B}$ as input and returns an $\mathcal{S}(\mathbb{M})$-sequential WTA $D_{\mathscr{B}}$ over $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$ that is not necessarily finite (cf. Definition 3.39). This infinite $\mathcal{S}(\mathbb{M})$-sequentialisation construction is similar to [29] where our weighted power set construction deals with multisets rather than sets. Formally, a state X of $D_{\mathscr{B}}$ is a map $X \colon Q \to \mathcal{M}_{\mathrm{fin}}(\mathbb{M})$.

Moreover, our approach to factorisations is more elaborate than the one in [5, 21, 29] since we deal with a general class of monoids rather than only groups [21, 29] or only free monoids [5]. In fact, our factorisations are based on a theory of minimising divisors in monoids. We assume that $\mathbb{M}$ is finitely generated by a set Γ and obtain the so-called *Cayley-distance* $d_\Gamma \colon \mathbb{M} \times \mathbb{M} \to \mathbb{N}$ on $\mathbb{M}$. Given a map $X \colon Q \to \mathcal{M}_{\mathrm{fin}}(\mathbb{M})$, we can determine all factorisations $y \odot Y = X$ such that $|Y|_\Gamma = \max_{q \in Q, y' \in Y(q)} d_\Gamma(y', 1)$ is minimal. Let (y, Y) be such a factorisation of X. We say that y is a *minimising divisor of* X and Y is a *minimal quotient of* X *divided by* y. Furthermore, we say that (y, Y) is a *centering factorisation of* X if $|Y|_\Gamma$ can be bounded by a function f depending only on the diameter of Y. That is, all values occurring in Y are close to 1 (up to f).

Our factorisation approach is to take a transition $(X_1, \ldots, X_s) \xrightarrow{\sigma|1} X$ generated by the weighted power set construction, choose a centering factorisation (y, Y) of X and replace the transition $(X_1, \ldots, X_s) \xrightarrow{\sigma|1} X$ by $(X_1, \ldots, X_s) \xrightarrow{\sigma|y} Y$.

In order for the factorisation to be well-defined and to have desirable properties, we require $\mathbb{M}$ to satisfy two properties. The first property is that $\mathbb{M}$ *divides Γ-monotone*, which provides that the diameter of states does not increase during our factorisation. The second property is that $\mathbb{M}$ *admits centering factorisations*, which yields that centering factorisations always exist. We combine these two properties to obtain the following result for $D_{\mathscr{B}}$: if $\mathscr{B}$ satisfies our *extended twinning property* (short: $\mathscr{B} \models \mathrm{ETP}$), then all values occurring in reachable states of $D_{\mathscr{B}}$ are "close to 1" (cf. Lemma 3.45). The ETP is our analogon to the twinning properties from the literature and essentially states that loops of the WTA $\mathscr{B}$ can be neglected during the calculation of distances of run weights. We discuss how the ETP compares to other twinning properties later in Chapter 3.1.

Step (III): We introduce a class of equivalence relations $\sim$ on the set of reachable states of $D_{\mathscr{B}}$ that preserve the transition weights and final weights. We call such an equivalence relation $\sim$ an *accumulator of $D_{\mathscr{B}}$* (cf. Definition 3.63). Next, we define the *accumulation of $D_{\mathscr{B}}$ via $\sim$*, denoted $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$, as the WTA over Σ and S obtained from $D_{\mathscr{B}}$ by (1) identifying all equivalent states under $\sim$ and (2) evaluating every multiset weight via $\oplus$ in S (cf. Definition 3.64). The WTA $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ inherits $\mathbb{M}$-sequentiality from the $\mathcal{S}(\mathbb{M})$-sequentiality of $D_{\mathscr{B}}$ and is equivalent to $\mathscr{A}$ (cf. Lemmas 3.66 and 3.67).

Ultimately, we show that we can find an accumulator $\sim$ of $D_{\mathscr{B}}$ such that $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ is finite, whenever S is additively idempotent or $\mathscr{B}$ is finitely ambiguous. In the case that S is additively idempotent, the accumulator $\sim$ simply identifies all states with the same support. In the case that $\mathscr{B}$ is finitely ambiguous, $D_{\mathscr{B}}$ is already finite (cf. Corollary 3.47) and hence $\sim$ is the identity relation. This yields a positive $\mathbb{M}$-sequentialisation result for the case that S is additively idempotent or $\mathscr{B}$ is finitely ambiguous (cf. Theorem 3.78). In fact, we obtain a positive $\mathbb{M}$-sequentialisation result, whenever an accumulator $\sim$ of $D_{\mathscr{B}}$ exists such that $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ is finite (cf. Theorem 3.77). This provides a blueprint result which can be used to find $\mathbb{M}$-sequentialisation results for further classes of semirings and weighted tree automata.

We summarize the three steps of our $\mathbb{M}$-sequentialisation construction. In Step (I) we shift the focus from S to $\mathbb{M}$ by decomposing all values occurring in $\mathscr{A}$ into finite multisets over $\mathbb{M}$. In Step (II) we obtain $\mathcal{S}(\mathbb{M})$-sequentiality without the finiteness condition by employing an infinite $\mathcal{S}(\mathbb{M})$-sequentialisation construction. In Step (III) we evaluate the multiset weights back into S and obtain finiteness by factoring out states with identical behaviours. An example of this three step process is given in Example 3.73 and illustrated in Figure 3.12.

$$\begin{array}{ccccccc} \mathscr{A} & & \mathscr{B} & & D_{\mathscr{B}} & & \mathrm{acc}_{\sim}(D_{\mathscr{B}}) \\ \text{WTA} & \xrightarrow{\text{(I)}} & \text{WTA} & \xrightarrow{\text{(II)}} & \mathcal{S}(\mathbb{M})\text{-seq. WTA} & \xrightarrow{\text{(III)}} & \mathbb{M}\text{-seq. WTA} \\ \text{over } \Sigma \text{ and } \langle\mathbb{M}\rangle_{\oplus} & & \text{over } \Sigma \text{ and } \mathcal{M}_{\mathrm{fin}}(\mathbb{M}) & & \text{over } \Sigma \text{ and } \mathcal{M}_{\mathrm{fin}}(\mathbb{M}) & & \text{over } \Sigma \text{ and } S \end{array}$$

Our main contribution is the following theorem.

Theorem 3.78. Let S be a semiring and $\mathbb{M} \leq (S, \odot, 1)$ such that $\mathbb{M}$ is finitely generated by some Γ, divides Γ-monotone, and admits centering factorisations. Let $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ be commutative and let $\mathscr{A}$ be a finite WTA over Σ and $\langle\mathbb{M}\rangle_{\oplus}$. Moreover, let one of the following conditions hold.

1. $\mathscr{A}$ is finitely $\mathbb{M}$-ambiguous
2. $\langle\mathbb{M}\rangle_{\oplus}$ is additively idempotent

If $\mathscr{A} \models \mathrm{ETP}$, then $\mathscr{A}$ is $\mathbb{M}$-sequentialisable.

Next, we briefly compare our Theorem 3.78 to the determinisation results from the literature. We note that the entire Chapter 3.8 is dedicated to an in-depth and formal literature comparison.

In [5, 21, 29], only the semiring of finite sets $\mathcal{P}_{\mathrm{fin}}(\mathbb{M})$ is considered, where $\mathbb{M}$ is a free monoid Γ^* or a finitely generated group. One can show that free monoids and finitely generated groups divide Γ-monotone and admit centering factorisations. Moreover, $\mathcal{P}_{\mathrm{fin}}(\mathbb{M})$ is additively idempotent. Furthermore, the twinning properties from [5, 21, 29] imply our ETP. Therefore, Theorem 3.78 covers the sequentialisation results from [5, 21]. However, both [5, 21] prove that $\mathscr{A}$ is sequentialisable if and only if $\mathscr{A}$ satisfies the twinning property, whereas we only prove the "if" direction of the equivalence.

In [14, Theorem 5.2], only extremal, commutative semirings with so-called maximal factorisations are considered (besides some trivial cases). There exist semirings covered by Theorem 3.78 that are not covered by [14]. At the time of writing, it remains an open problem whether there exist semirings covered by [14] that are not covered by Theorem 3.78. We conjecture that such semirings do not exist. However, the twinning property from [14] is incomparable to our ETP, which (unfortunately) implies that [14] is incomparable to Theorem 3.78.

The rest of this chapter is structured as follows. In Chapter 3.3, we introduce our theory of minimising divisors in monoids. In Chapters 3.4 and 3.5, we investigate weighted tree automata over $\mathcal{M}_{\text{fin}}(\mathbb{M})$ and provide Step (II) from our $\mathbb{M}$-sequentialisation construction. In Chapters 3.6 and 3.7, we provide Steps (I) and (III) from our $\mathbb{M}$-sequentialisation construction, respectively. In Chapter 3.8, we compare Theorem 3.78 to the determinisation results from the literature. We conclude Chapter 3 in Chapter 3.9 by presenting some open questions and research directions. We provide Step (II) before Step (I) because Step (II) is deeply connected to Chapters 3.3 and 3.4 and this order avoids jumping back and forth between the semiring perspective and the monoid perspective.

3.2 Preliminaries

Let $\mathbb{M}$ be a monoid and S be a semiring. We call S an $\mathbb{M}$-*semiring* if $\mathbb{M} \leq (S, \odot, 1)$. In this case, we use the operation symbol $\odot$ for multiplication in $\mathbb{M}$ (as well as in S). We note that the monoid $\langle \mathbb{M} \rangle_{\oplus}$ is also a semiring.

Let $\mathbb{M}$ be a monoid, S be an $\mathbb{M}$-semiring, and $\mathscr{A} = (Q, T, \text{final})$ be a WTA over Σ and S. We call $\mathscr{A}$ $\mathbb{M}$-*sequential* if $\mathscr{A}$ is deterministic and all non-vanishing transition weights are elements of $\mathbb{M}$. We call $\mathscr{A}$ $\mathbb{M}$-*sequentialisable* if there exists an $\mathbb{M}$-sequential finite WTA $\mathscr{A}'$ such that $\mathscr{A}'$ is equivalent to $\mathscr{A}$. The *class of weighted languages recognised by* $\mathbb{M}$-*sequential finite* WTA *over* Σ *and* S is denoted by $\text{sRec}(\Sigma, S, \mathbb{M})$.

We note that if $\mathscr{A}$ is $\mathbb{M}$-sequentialisable, then $\mathscr{A}$ is determinisable. Moreover, $\mathscr{A}$ is $(S, \odot, 1)$-sequentialisable if and only if $\mathscr{A}$ is determinisable. This follows directly from the definition of $\mathbb{M}$-sequentiality where $\mathbb{M} = (S, \odot, 1)$. Similarly, $\mathscr{A}$ is

$(\{1\}, \odot, 1)$-sequentialisable if and only if $\mathscr{A}$ is a crisp-determinisable[1] weighted tree automaton over the semiring $\langle 1 \rangle_{\oplus}$.

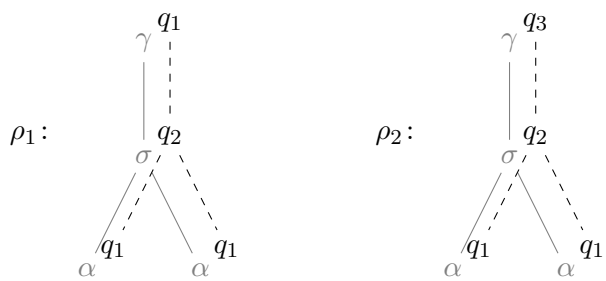
Hence, our concept of $\mathbb{M}$-sequentialisability introduces a spectrum of sequentialisability ranging between the extreme cases of crisp-determinisability and classical determinisability.

Example 3.1. Let $\Sigma = \{\alpha^{(0)}, \gamma^{(1)}, \sigma^{(2)}\}$. We consider the WTA $\mathscr{A} = (Q_{\mathscr{A}}, T_{\mathscr{A}}, \mathrm{final}_{\mathscr{A}})$ over Σ and $\mathbb{X}$ (see Example 2.4), where $Q_{\mathscr{A}} = \{q_1, q_2, q_3\}$ and every transition weight and final weight of $\mathscr{A}$ is $\bot$ except

$$\begin{aligned}
&\mathrm{final}_{\mathscr{A}}(q_1) = \mathrm{final}_{\mathscr{A}}(q_2) = \mathrm{final}_{\mathscr{A}}(q_3) = 0,\\
&(T_{\mathscr{A}})_{\alpha}(q_1) = 1, (T_{\mathscr{A}})_{\gamma}(q_2, q_1) = \top,\\
&(T_{\mathscr{A}})_{\gamma}(q_1, q_1) = (T_{\mathscr{A}})_{\gamma}(q_2, q_3) = (T_{\mathscr{A}})_{\gamma}(q_3, q_3) = 1, \text{ and}\\
&(T_{\mathscr{A}})_{\sigma}(q_i, q_j, q_2) = 1 \text{ for every } i, j \in \{1, 2\}.
\end{aligned}$$

A graphical representation of $\mathscr{A}$ is given in Figure 3.1.

We consider the tree $\xi = \gamma(\sigma(\alpha, \alpha)) \in \mathrm{T}_{\Sigma}$. There exist exactly two valid runs of $\mathscr{A}$ on ξ, namely ρ_1 and ρ_2 given as follows.



We obtain $\mathrm{wt}(\xi, \rho_1) = \top$ and $\mathrm{wt}(\xi, \rho_2) = 4$ and hence $[\![\mathscr{A}]\!](\xi) = \top \vee 4 = \top$.

It is easy to see (and we will prove in Example 3.54) that $[\![\mathscr{A}]\!]$ maps every input tree that contains a γ directly above a σ to $\top$; all other trees are mapped to their size. Formally, we conjecture that the weighted tree language recognised by $\mathscr{A}$ is given by

$$[\![\mathscr{A}]\!](\xi) = \begin{cases} \top & \text{if } \exists w \in \mathrm{pos}(\xi) \colon \xi(w) = \gamma \wedge \xi(w1) = \sigma \\ \mathrm{size}(\xi) & \text{otherwise.} \end{cases}$$

[1] We call $\mathscr{A}$ *crisp-deterministic* if $\mathscr{A}$ is deterministic and all transition weights are elements of $\{0, 1\}$. We refer the interested reader to [57].

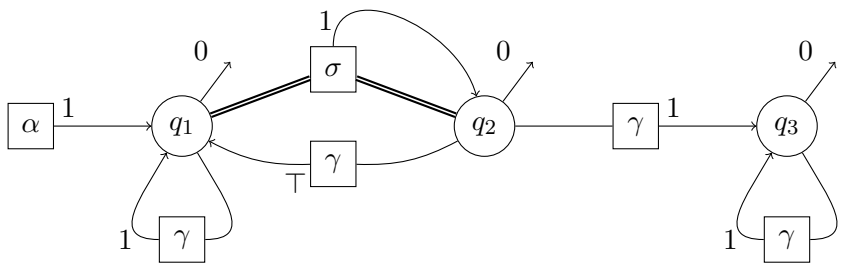


Figure 3.1: Illustration of the WTA $\mathscr{A}$ over $\{\alpha^{(0)}, \gamma^{(1)}, \sigma^{(2)}\}$ and $\mathbb{X}$ from Example 3.1. In order to save space, we use the conventions introduced on page 30.

The WTA $\mathscr{A}$ shall serve as our running example for investigating determinisability and $\mathbb{M}$-sequentialisability. For the latter, we consider the submonoid $(\mathbb{N}, +, 0)$ of $(\mathbb{X}, +, 0)$ which is finitely generated by $\{1\}$. The desire to $\mathbb{N}$-sequentialise $\mathscr{A}$ (rather than just determinise it) may be motivated by the fact that automata over $\mathbb{N}$ are well-studied and can be modeled in standard programming languages (like C or python) without the need to introduce a custom data type.

Since both $(T_{\mathscr{A}})_\gamma(q_2, q_1)$ and $(T_{\mathscr{A}})_\gamma(q_2, q_3)$ are not equal to $\bot$, $\mathscr{A}$ is not deterministic and thus not $\mathbb{N}$-sequential either. We observe that removing the state q_3 from $\mathscr{A}$ does not change $[\![\mathscr{A}]\!]$ and, moreover, results in a deterministic automaton which is not $\mathbb{N}$-sequential. On the other hand, letting $(T_{\mathscr{A}})_\gamma(q_2, q_1) = 1$ does not change $[\![\mathscr{A}]\!]$ and ensures that all non-vanishing transition weights are elements of $\mathbb{N}$, but the resulting automaton is not deterministic.

Throughout the course of Chapter 3, we will uncover that $\mathscr{A}$ is $\mathbb{N}$-sequentialisable nevertheless and that this $\mathbb{N}$-sequentialisation involves moving the weight $\top$ from the transitions to the final weights. ◀

3.3 Factorisation in Monoids

As we have discussed in the introduction to Chapter 3, our $\mathbb{M}$-sequentialisation construction applies a factorisation technique for multisets over $\mathbb{M}$. In this chapter, we set up the necessary theory for multisets over $\mathbb{M}$ and introduce two properties of monoids that will be used in Chapter 3.5 to make our factorisation approach successful.

3.3.1 Ordering Multisets over Monoids

We start by presenting a way to linearly order multisets over monoids. This will come in handy whenever our factorisation approach admits multiple equally good factorisations of a single multiset.

> *Throughout the rest of Chapter 3.3, we assume* $(\mathbb{M}, \odot, 1)$ *to be a finitely generated monoid with finite generating set* Γ*. Moreover, we assume that* Γ *is linearly ordered.*

Remark 3.2. Since Γ is linearly ordered, Γ^* is linearly ordered by the lexicographic order $\leq_{\text{lex}}$. This induces a linear order on $\mathbb{M}$ as follows. Let $h\colon \Gamma^* \to \mathbb{M}$ be the unique monoid homomorphism such that $h(\gamma) = \gamma$ for every $\gamma \in \Gamma$. For every $m \in \mathbb{M}$ there exists a unique $\min_{\leq_{\text{lex}}} h^{-1}(m)$ which we denote by $\Gamma^{-1}(m)$. We define the order $\leq_\Gamma$ on $\mathbb{M}$ by $m_1 \leq_\Gamma m_2$ if and only if $\Gamma^{-1}(m_1) \leq_{\text{lex}} \Gamma^{-1}(m_2)$ for every $m_1, m_2 \in \mathbb{M}$. It is easy to see that $\leq_\Gamma$ is a linear order on $\mathbb{M}$.

Moreover, the order $\leq_\Gamma$ induces a linear order on $\mathcal{M}_{\text{fin}}(\mathbb{M})$ as follows. First, we note that $\leq_\Gamma$ induces the lexicographic order on $\mathbb{M}^*$, which we denote by $\leq_{\text{lex}}^{\mathbb{M}}$. Second, there exists a unique map $\text{sort}\colon \mathcal{M}_{\text{fin}}(\mathbb{M}) \to \mathbb{M}^*$ such that for every $M \in \mathcal{M}_{\text{fin}}(\mathbb{M})$ it holds that $|\text{sort}(M)| = \#M$, $|\text{sort}(M)|_m = M(m)$ for every $m \in \mathbb{M}$, and $\text{sort}(M)[i] \leq_\Gamma \text{sort}(M)[i+1]$ for every $i \in [\#M - 1]$. We define the order $\sqsubseteq$ on $\mathcal{M}_{\text{fin}}(\mathbb{M})$ by $M_1 \sqsubseteq M_2$ if and only if $\text{sort}(M_1) \leq_{\text{lex}}^{\mathbb{M}} \text{sort}(M_2)$. It is easy to see that $\sqsubseteq$ is a linear order on $\mathcal{M}_{\text{fin}}(\mathbb{M})$. We write $M_1 \sqsubset M_2$ if $M_1 \sqsubseteq M_2$ and $M_1 \neq M_2$. We will use $\sqsubseteq$ in order to have a consistent way to choose unique successor states for transitions in our $\mathcal{S}(\mathbb{M})$-sequentialisation construction. ◀

Example 3.3. We have seen in Example 2.3 that the monoid $(\mathbb{N}, +, 0)$ is finitely generated by $\Gamma = \{1\}$.

Let $h_{\mathbb{N}}\colon \Gamma^* \to \mathbb{N}$ be the unique homomorphism such that $h_{\mathbb{N}}(1) = 1$. For every $n \in \mathbb{N}$ it holds that $h_{\mathbb{N}}^{-1}(n) = \{1^n\}$ and hence also $\Gamma^{-1}(n) = 1^n$. Therefore, for every $n_1, n_2 \in \mathbb{N}$ it holds that $n_1 \leq_\Gamma n_2$ if and only if $n_1 \leq n_2$.

We consider the multisets $M_1 = \{\!\{15, 3, 7, 10\}\!\}$ and $M_2 = \{\!\{4, 12, 4, 8\}\!\}$. An easy calculation shows that $\text{sort}(M_1) = 3\ 7\ 10\ 15$ and $\text{sort}(M_2) = 4\ 4\ 8\ 12$. It surely holds that

$\mathrm{sort}(M_1)$ is lexicographically smaller than $\mathrm{sort}(M_2)$ (that is, $\mathrm{sort}(M_1) \leq_{\mathrm{lex}}^{\mathbb{N}} \mathrm{sort}(M_2)$) and hence $M_1 \sqsubseteq M_2$. ◀

3.3.2 Cayley Graph and Cayley Distance

Since our $\mathbb{M}$-sequentialisation construction is heavily inspired by the sequentialisation construction from [5, 21, 29], we also use the concepts of a Cayley graph and Cayley distance. However, we note that Cayley graphs for monoids are directed graphs (rather than undirected graphs as in the group case) and the Cayley distance is not necessarily a metric.

Definition 3.4. The *(directed) Cayley graph for* $\mathbb{M}$ *and* Γ is the directed graph $\mathrm{Cay}_{\mathbb{M},\Gamma} = (\mathbb{M}, E)$, where $E = \{(m, m\gamma) \mid m \in \mathbb{M}, \gamma \in \Gamma\}$.

Let $m_1, m_2 \in \mathbb{M}$ and $w \in E^*$. We call w a *path from* m_1 *to* m_2 if either $w = \varepsilon$ and $m_1 = m_2$ or the following conditions are satisfied:

- $w[1] = (m_1, m')$ for some $m' \in \mathbb{M}$,
- $w[|w|] = (m', m_2)$ for some $m' \in \mathbb{M}$, and
- for every $i \in [|w| - 1]$, there are $n, m', n' \in \mathbb{M}$ such that $w[i] = (n, m')$ and $w[i+1] = (m', n')$.

We will sometimes denote a non-empty path only by the sequence of vertices it traverses. That is, we will abbreviate w by $\mathrm{proj}_1(w[1]) \ldots \mathrm{proj}_1(w[|w|])\mathrm{proj}_2(w[|w|])$ if $w \neq \varepsilon$.

If w is a path from m_1 to m_2, then we abbreviate this fact by $m_1 \overset{w}{\rightsquigarrow} m_2$. The set of paths from m_1 to m_2 is denoted by $\mathrm{Paths}(m_1, m_2)$. A *fork-path connecting* m_1 *and* m_2 is a pair (w, v) of paths such that $n \overset{w}{\rightsquigarrow} m_1$ and $n \overset{v}{\rightsquigarrow} m_2$ for some $n \in \mathbb{M}$. The set of fork-paths connecting m_1 and m_2 is denoted by $\mathrm{FPaths}(m_1, m_2)$. The *length* of a fork-path $(w, v) \in \mathrm{FPaths}(m_1, m_2)$, denoted by $|(w, v)|$, is $|w| + |v|$.

The *Cayley-distance between* m_1 *and* m_2, denoted by $d_\Gamma(m_1, m_2)$, is defined as

$$d_\Gamma(m_1, m_2) = \min_{\omega \in \mathrm{FPaths}(m_1, m_2)} |\omega|$$

and the Γ*-length of* m_1, denoted by $|m_1|_\Gamma$, is defined as

$$|m_1|_\Gamma = \min_{w \in \mathrm{Paths}(1, m_1)} |w|.$$

For $M \in \mathcal{M}_{\text{fin}}(\mathbb{M})$ we define the Γ-length of M, denoted by $|M|_\Gamma$, as $\max_{m \in \text{supp}(M)} |m|_\Gamma$.

Let $m \in \mathbb{M}$ and $r \in \mathbb{N}$. The *ball of radius r around m*, denoted by $\mathcal{B}_r(m)$, is defined as $\{m' \in \mathbb{M} \mid \exists w \in \text{Paths}(m, m') \colon |w| \leq r\}$, which is a finite set. Note that $\mathcal{B}_r(1) = \{m \in \mathbb{M} \mid |m|_\Gamma \leq r\}$. ◀

Example 3.5. We continue Example 2.3 and illustrate the Cayley graphs and Cayley-distances of some monoids. For the sake of clarity, we label every edge in each Cayley graph with the corresponding element γ of the generating set.

- The Cayley graph for $(\mathbb{N}, +, 0)$ and $\Gamma = \{1\}$ is an infinite chain starting in 0.

$$0 \xrightarrow{1} 1 \xrightarrow{1} 2 \xrightarrow{1} 3 \xrightarrow{1} \cdots$$

  For every $n_1, n_2 \in \mathbb{N}$ it holds that $d_\Gamma(n_1, n_2) = |n_1 - n_2|$ and in particular $|n_1|_\Gamma = n_1$.

- Let $k \in \mathbb{N}$. The Cayley graph for $(\mathbb{N}_{\leq k}, +_k, 0)$ and $\Gamma = \{1\}$ is a finite chain starting in 0 and ending in k. For $k = 3$, this can be visualised as

$$0 \xrightarrow{1} 1 \xrightarrow{1} 2 \xrightarrow{1} 3 \overset{1}{\circlearrowleft}$$

  For every $n_1, n_2 \in \mathbb{N}_{\leq k}$ it holds that $d_\Gamma(n_1, n_2) = |n_1 - n_2|$ and in particular $|n_1|_\Gamma = n_1$.

- The Cayley graph for $\mathbb{BF}$ and $\Gamma_{\mathbb{BF}}$ is a two-dimensional grid which is depicted in Figure 3.2. We note that every edge in $\text{Cay}_{\mathbb{BF}, \Gamma_{\mathbb{BF}}}$ points either to the right or upwards. Using this fact, we obtain that the set $\text{Paths}\big((0,0),(2,7)\big)$ contains the six elements $w_1, \ldots, w_6$, where

$$\begin{aligned}
w_1 &= (0,0)(0,1)(1,2)(1,3)(2,6)(2,7),\\
w_2 &= (0,0)(0,1)(1,2)(2,4)(2,5)(2,6)(2,7),\\
w_3 &= (0,0)(1,0)(1,1)(1,2)(1,3)(2,6)(2,7),\\
w_4 &= (0,0)(1,0)(1,1)(1,2)(2,4)(2,5)(2,6)(2,7),\\
w_5 &= (0,0)(1,0)(1,1)(2,2)(2,3)(2,4)(2,5)(2,6)(2,7), \text{ and}\\
w_6 &= (0,0)(1,0)(2,0)(2,1)(2,2)(2,3)(2,4)(2,5)(2,6)(2,7).
\end{aligned}$$

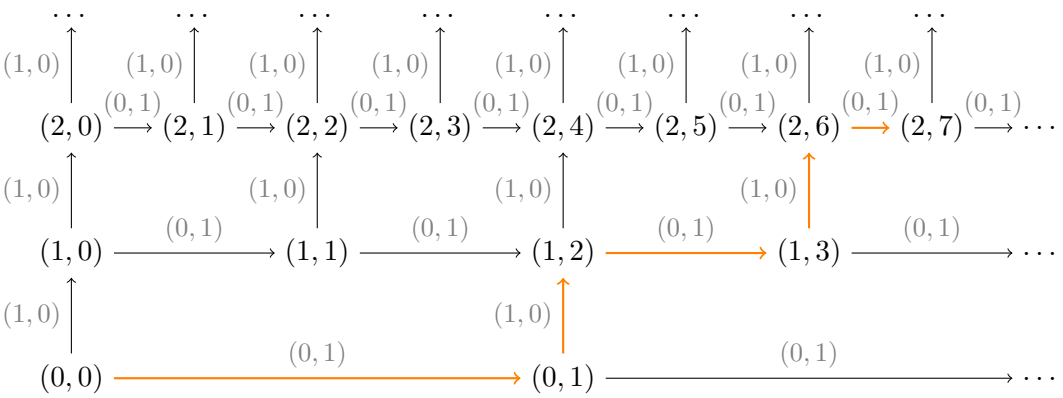


Figure 3.2: The Cayley graph for $\mathbb{BF}$ and $\Gamma_{\mathbb{BF}}$. The shortest path from $(0,0)$ to $(2,7)$ is illustrated by orange edges.

The shortest path from $(0,0)$ to $(2,7)$ is w_1 and it holds that $|w_1| = 5$. In particular, $|(2,7)|_{\Gamma_{\mathbb{BF}}} = 5$. We illustrate w_1 in Figure 3.2. ◀

Lemma 3.6. Let $m_1, m_2 \in \mathbb{M}$. It holds that

(i) $d_\Gamma(m_1, m_2) \leq |m_1|_\Gamma + |m_2|_\Gamma$ and

(ii) $|m_1 \odot m_2|_\Gamma \leq |m_1|_\Gamma + |m_2|_\Gamma$.

Proof. Let $w \in \text{Paths}(1, m_1)$ and $v \in \text{Paths}(1, m_2)$ such that $|w| = |m_1|_\Gamma$ and $|v| = |m_2|_\Gamma$. It holds that $(w, v) \in \text{FPaths}(m_1, m_2)$ and by the definition of d_Γ we obtain $d_\Gamma(m_1, m_2) \leq |(w, v)| = |m_1|_\Gamma + |m_2|_\Gamma$. This proves Inequality (i).

If $v = \varepsilon$, then Inequality (ii) holds trivially. Otherwise, assume that $v = v_1 \dots v_n$ for some $n \in \mathbb{N}$ and $v_1, \dots, v_n \in \mathbb{M}$. Surely, $v' = (m_1 \odot v_1) \dots (m_1 \odot v_n)$ is a path from m_1 to $m_1 \odot m_2$. In particular, $wv' \in \text{Paths}(1, m_1 \odot m_2)$ and $|wv'| = |w| + |v'| = |w| + |v|$. Therefore, by the definition of the Γ-length we obtain $|m_1 \odot m_2|_\Gamma \leq |wv'| = |w| + |v| = |m_1|_\Gamma + |m_2|_\Gamma$. This proves Inequality (ii). □

Lemma 3.7. Let $m_1, m_2, m \in \mathbb{M}$. It holds that $d_\Gamma(m_1, m_2) \geq d_\Gamma(m \odot m_1, m \odot m_2)$.

Proof. Let $(w, v) \in \text{FPaths}(m_1, m_2)$ such that $|(w, v)| = d_\Gamma(m_1, m_2)$. Moreover, we assume that w and v have the form

$$w = w_1 \dots w_k \quad \text{and} \quad v = v_1 \dots v_\ell,$$

where $k \in \{|w|+1, 0\}, \ell \in \{|v|+1, 0\}$, and $w_1, \ldots, w_k, v_1, \ldots, v_\ell \in \mathbb{M}$. We define the tuple (w', v'), where

$$w' = (m \odot w_1) \ldots (m \odot w_k) \quad \text{and}$$
$$v' = (m \odot v_1) \ldots (m \odot v_\ell).$$

It is obvious that $(w', v') \in \mathrm{FPaths}(m \odot m_1, m \odot m_2)$ and moreover, $|(w', v')| = |(w, v)|$. It follows from the definition of d_Γ that $|(w', v')| \geq d_\Gamma(m \odot m_1, m \odot m_2)$. Thus, we have seen that $d_\Gamma(m_1, m_2) = |(w, v)| = |(w', v')| \geq d_\Gamma(m \odot m_1, m \odot m_2)$, which proves the lemma. □

3.3.3 Divisors and Rests

We will now develop our theory of minimising divisors and minimal quotients for multisets over $\mathbb{M}$. Minimality of divisors and quotients is used in Chapter 3.5 to keep factorised states as close to the neutral element $1 \in \mathbb{M}$ as possible (with respect to d_Γ).

Definition 3.8. Let M be a finite multiset over $\mathbb{M}$. We define the set of *common (left) divisors of* M by

$$\mathrm{div}(M) = \{n \in \mathbb{M} \mid \exists N \in \mathcal{M}_{\mathrm{fin}}(\mathbb{M}) \colon n \odot N = M\}.$$

Moreover, for every $n \in \mathrm{div}(M)$ we define the set of *quotients of* M *divided by* n as the set

$$\mathrm{quot}_n(M) = \{N \in \mathcal{M}_{\mathrm{fin}}(\mathbb{M}) \mid n \odot N = M\}$$

and the set of *(Γ-)minimal quotients of* M *divided by* n as

$$\mathrm{minquot}_n(M) = \operatorname*{argmin}_{N \in \mathrm{quot}_n(M)} |N|_\Gamma.$$

Lastly, we define the set of *(Γ-)minimising divisors of* M as

$$\mathrm{mindiv}(M) = \operatorname*{argmin}_{n \in \mathrm{div}(M)} \min_{N \in \mathrm{quot}_n(M)} |N|_\Gamma.$$

If $M = \{\!\{m_1, m_2\}\!\}$ for some $m_1, m_2 \in \mathbb{M}$, then we write $\mathrm{div}(m_1, m_2)$, $\mathrm{quot}_n(m_1, m_2)$, $\mathrm{minquot}_n(m_1, m_2)$, and $\mathrm{mindiv}(m_1, m_2)$ rather than $\mathrm{div}(M)$, $\mathrm{quot}_n(M)$, $\mathrm{minquot}_n(M)$, and $\mathrm{mindiv}(M)$, respectively. ◀

Remark 3.9. We note that for every finite multiset M over $\mathbb{M}$ and $n \in \operatorname{div}(M)$, the set $\operatorname{quot}_n(M)$ is non-empty. Moreover, for every $N \in \operatorname{quot}_n(M)$ it holds that $\#N = \#M$.

Surely, there need not be a unique argmin in the definition of $\operatorname{minquot}_n(M)$. Therefore, $\operatorname{minquot}_n(M)$ is a set of all "equally minimal" quotients. Moreover, there need not be a unique argmin in the definition of $\operatorname{mindiv}(M)$. Therefore, $\operatorname{mindiv}(M)$ is the set of all "equally minimising" divisors of M.

We note that a minimising divisor n of M is "minimising" in the sense that $|N|_\Gamma$ is minimal for each $N \in \operatorname{minquot}_n(M)$. ◀

Lemma 3.10. Let M be a finite multiset over $\mathbb{M}$. It holds that

$$\operatorname{div}(M) = \bigcap_{m \in \operatorname{supp}(M)} \operatorname{div}(\{m\}).$$

Proof. Let $n \in \operatorname{div}(M)$. By definition, there exists $N \in \mathcal{M}_{\text{fin}}(\mathbb{M})$ such that $n \odot N = M$. In particular, for every $m \in \operatorname{supp}(M)$ there exists $n_m \in \operatorname{supp}(N)$ such that $n \odot n_m = m$. This shows that $n \in \bigcap_{m \in \operatorname{supp}(M)} \operatorname{div}(\{m\})$.

Now, let $n \in \bigcap_{m \in \operatorname{supp}(M)} \operatorname{div}(\{m\})$. In particular, for every $m \in \operatorname{supp}(M)$ there exists $n_m \in \mathbb{M}$ such that $n \odot n_m = m$. We define the multiset $N = \{\!\{n_m \mid m \in M\}\!\}$ and observe that $n \odot N = M$. Thus, $n \in \operatorname{div}(M)$. □

Example 3.11. We continue Example 3.5 and calculate some minimal quotients and minimising divisors.

- We consider the monoid $(\mathbb{N}, +, 0)$ and let $M \in \mathcal{M}_{\text{fin}}(\mathbb{N})$ such that $M \neq \emptyset$. Surely, a natural number $n \in \mathbb{N}$ is a common left divisor of all elements in $\operatorname{supp}(M)$ if and only if $n \leq \min(\operatorname{supp}(M))$. That is,

  $$\operatorname{div}(M) = \{0, \ldots, \min(\operatorname{supp}(M))\}.$$

  Moreover, for every $n \in \operatorname{div}(M)$ it holds that $N = \{\!\{m - n \mid m \in M\}\!\}$ is the unique element in $\operatorname{quot}_n(M)$ and hence also in $\operatorname{minquot}_n(M)$. We recall that $\mathbb{N}$ is finitely generated by $\Gamma = \{1\}$ and obtain

  $$|N|_\Gamma = \max_{m \in \operatorname{supp}(N)} |m|_\Gamma = \max_{m \in \operatorname{supp}(N)} m = \max_{m \in \operatorname{supp}(M)} (m - n) = \max(\operatorname{supp}(M)) - n$$

  which shows that $|N|_\Gamma$ is smallest if n is largest. Therefore, we have that $\operatorname{mindiv}(M) = \{\min(\operatorname{supp}(M))\}$.

- We consider the monoid $(\mathbb{N}_{\leq k}, +_k, 0)$ for some $k \in \mathbb{N}$ and let $M \in \mathcal{M}_{\text{fin}}(\mathbb{N}_{\leq k})$ such that $M \neq \emptyset$. Analogously to the monoid $(\mathbb{N}, +, 0)$, an element $n \in \mathbb{N}_{\leq k}$ is a common left divisor of all elements in $\text{supp}(M)$ if and only if $n \leq \min(\text{supp}(M))$. That is,

$$\text{div}(M) = \{0, \ldots, \min(\text{supp}(M))\}.$$

Let $n \in \text{div}(M)$ and $N \in \mathcal{M}_{\text{fin}}(\mathbb{N}_{\leq k})$. It holds that $N \in \text{quot}_n(M)$ if and only if

(a) $N(m) = M(m+n)$ for every $m \in \{0, \ldots, k-n-1\}$ and

(b) $\sum_{i=0}^{n} N(k-n+i) = M(k)$.

We recall that $\mathbb{N}_{\leq k}$ is finitely generated by $\Gamma = \{1\}$ and obtain that $|N|_\Gamma$ is minimal if $n = \min(\text{supp}(M))$ and $N(k-n) = M(k)$. Therefore, we have that $\text{mindiv}(M) = \{\min(\text{supp}(M))\}$ and $N = \{\!\{m - \min(\text{supp}(M)) \mid m \in M\}\!\}$ is the unique element in $\text{minquot}_{\min(\text{supp}(M))}(M)$.

- We consider the monoid $\mathbb{BF}$ and note that $\text{div}(\{\!\{m\}\!\}) = \{n \in \mathbb{BF} \mid \exists n \overset{w}{\rightsquigarrow\!\triangleright} m\}$ for every $m \in \mathbb{BF}$. Hence, in order to determine $\text{div}(\{\!\{m\}\!\})$, we need to determine the starting vertices of paths ending in m.

We recall that all edges in $\text{Cay}_{\mathbb{BF},\Gamma_{\mathbb{BF}}}$ point either to the right or upwards (see Figure 3.2). This illustrates the following fact. For every $m, n \in \mathbb{BF}$, there exists a path $n \overset{w}{\rightsquigarrow\!\triangleright} m$ if and only if n "lies to the bottom left" of m. Formally, it holds that

$$\text{div}(\{\!\{(m_\text{a}, m_\text{b})\}\!\}) = \left\{(n_\text{a}, n_\text{b}) \in \mathbb{BF} \;\middle|\; n_\text{a} \leq m_\text{a} \wedge n_\text{b} \leq \left\lfloor \frac{m_\text{b}}{2^{m_\text{a}-n_\text{a}}} \right\rfloor \right\} \tag{3.1}$$

for every $(m_\text{a}, m_\text{b}) \in \mathbb{BF}$. We prove Equation (3.1). Let $(m_\text{a}, m_\text{b}), (n_\text{a}, n_\text{b}) \in \mathbb{BF}$. It holds that

$$\begin{aligned}
(n_\text{a}, n_\text{b}) \in \text{div}(\{\!\{(m_\text{a}, m_\text{b})\}\!\}) &\iff \exists m' \in \mathbb{BF} \colon (n_\text{a}, n_\text{b}) \circ_{\mathbb{BF}} m' = (m_\text{a}, m_\text{b}) \\
&\overset{\star_1}{\iff} \exists m' \in \mathbb{BF} \colon m' = (m_\text{a} - n_\text{a}, m_\text{b} - 2^{m_\text{a}-n_\text{a}} \cdot n_\text{b}) \\
&\overset{\star_2}{\iff} n_\text{a} \leq m_\text{a} \wedge n_\text{b} \leq \left\lfloor \frac{m_\text{b}}{2^{m_\text{a}-n_\text{a}}} \right\rfloor.
\end{aligned}$$

Equivalence $\star_1$ follows from the fact that

$$\begin{aligned}(n_\mathrm{a}, n_\mathrm{b}) \circ_{\mathbb{BF}} (m'_\mathrm{a}, m'_\mathrm{b}) = (m_\mathrm{a}, m_\mathrm{b}) &\iff (n_\mathrm{a} + m'_\mathrm{a}, 2^{m'_\mathrm{a}} \cdot n_\mathrm{b} + m'_\mathrm{b}) = (m_\mathrm{a}, m_\mathrm{b}) \\ &\iff (m'_\mathrm{a}, m'_\mathrm{b}) = (m_\mathrm{a} - n_\mathrm{a}, m_\mathrm{b} - 2^{m_\mathrm{a} - n_\mathrm{a}} \cdot n_\mathrm{b})\end{aligned}$$

for every $(m'_\mathrm{a}, m'_\mathrm{b}) \in \mathbb{BF}$ and Equivalence $\star_2$ follows from the fact that

$$(m_\mathrm{a} - n_\mathrm{a}, m_\mathrm{b} - 2^{m_\mathrm{a} - n_\mathrm{a}} \cdot n_\mathrm{b}) \in \mathbb{N}^2 \iff m_\mathrm{a} - n_\mathrm{a} \geq 0 \wedge m_\mathrm{b} - 2^{m_\mathrm{a} - n_\mathrm{a}} \cdot n_\mathrm{b} \geq 0.$$

This concludes the proof of Equation (3.1).

We consider the multiset $M = \{\!\{(3,1),(2,8)\}\!\}$ over $\mathbb{N}^2$. From Equation (3.1) it follows that

$$\mathrm{div}(\{\!\{(3,1)\}\!\}) = \{(3,1),(3,0),(2,0),(1,0),(0,0)\}$$

and

$$\mathrm{div}(\{\!\{(2,8)\}\!\}) = \{(2,m_2),(1,m_1),(0,m_0) \mid m_2 \leq 8, m_1 \leq 4, m_0 \leq 2\}$$

which shows that $\mathrm{div}(M) = \mathrm{div}(\{\!\{(3,1)\}\!\}) \cap \mathrm{div}(\{\!\{(2,8)\}\!\}) = \{(2,0),(1,0),(0,0)\}$. We have seen in Example 2.3 that $\mathbb{BF}$ is cancellative and hence quotients in $\mathbb{BF}$ are unique. We can verify the following equations by easy calculations:

$$\begin{aligned}\mathrm{quot}_{(2,0)}(M) &= \mathrm{minquot}_{(2,0)}(M) = \{\!\{(1,1),(0,8)\}\!\}\\ \mathrm{quot}_{(1,0)}(M) &= \mathrm{minquot}_{(1,0)}(M) = \{\!\{(2,1),(1,8)\}\!\}\\ \mathrm{quot}_{(0,0)}(M) &= \mathrm{minquot}_{(0,0)}(M) = \{\!\{(3,1),(2,8)\}\!\}.\end{aligned}$$

In order to determine which divisor is a minimising divisor, we have to calculate the $\Gamma_{\mathbb{BF}}$-length of all quotients.

Similar to the calculation of $|(2,7)|_{\Gamma_{\mathbb{BF}}}$ in Example 3.5, we find that $|(1,1)|_{\Gamma_{\mathbb{BF}}} = 2$, $|(0,8)|_{\Gamma_{\mathbb{BF}}} = 8$, $|(2,1)|_{\Gamma_{\mathbb{BF}}} = 3$, $|(1,8)|_{\Gamma_{\mathbb{BF}}} = 5$, $|(3,1)|_{\Gamma_{\mathbb{BF}}} = 4$, and $|(2,8)|_{\Gamma_{\mathbb{BF}}} = 4$. Therefore, $\mathrm{mindiv}(M) = (0,0)$. ◀

Lemma 3.12. Let M be a finite multiset over $\mathbb{M}$. Moreover, let $m, n \in \mathbb{M}$ such that $m \odot n \in \mathrm{div}(M)$. It holds that

$$\{N \in \mathrm{quot}_n(K) \mid K \in \mathrm{quot}_m(M)\} = \mathrm{quot}_{m \odot n}(M).$$

Proof. Let $N \in \mathcal{M}_{\text{fin}}(\mathbb{M})$. It holds that

$$\begin{aligned} N \in \{N' \in \text{quot}_n(K) \mid K \in \text{quot}_m(M)\} \\ &\iff \exists K \in \text{quot}_m(M) \colon n \odot N = K \\ &\iff \exists K \in \mathcal{M}_{\text{fin}}(\mathbb{M}) \colon n \odot N = K \wedge m \odot K = M \\ &\iff (m \odot n) \odot N = M \iff N \in \text{quot}_{m \odot n}(M). \end{aligned}$$

This proves the claim. □

3.3.4 Factorisation Properties

In order for our factorisation approach to work, we require $\mathbb{M}$ to fulfil two properties. The first property is that $\mathbb{M}$ divides Γ-monotone, which ensures that dividing two elements of $\mathbb{M}$ by a common divisor does not increase their Cayley-distance. The second property is that $\mathbb{M}$ admits centering factorisations, which ensures that factorised states stay close to 1. We will see the specific application of these properties in Chapter 3.5 (more precisely, in Lemma 3.45).

Definition 3.13. We say that $\mathbb{M}$ *divides Γ-monotone* if for every $m_1, m_2 \in \mathbb{M}$, $n \in \text{div}(m_1, m_2)$, and $\{\!\{n_1, n_2\}\!\} \in \text{minquot}_n(m_1, m_2)$ it holds that

$$d_\Gamma(n_1, n_2) \leq d_\Gamma(m_1, m_2).$$

Let $M \in \mathcal{M}_{\text{fin}}(\mathbb{M})$, $n \in \text{mindiv}(M)$, $N \in \text{minquot}_n(M)$, and $f \colon \mathbb{N} \to \mathbb{N}$ be a strongly monotone map. We call the pair (n, N) an *f-centering factorisation of M* if for every $n_1 \in \text{supp}(N)$ there exists $n_2 \in \text{supp}(N)$ such that

$$|n_1|_\Gamma \leq f(d_\Gamma(n_1, n_2)).$$

The set of f-centering factorisations of M is denoted by $\text{CenterFact}(M, f)$. We define the strict order $\widetilde{\sqsubset}$ on $\text{CenterFact}(M, f)$ by $(n_1, N_1) \mathrel{\widetilde{\sqsubseteq}} (n_2, N_2)$ iff (1) $N_1 \sqsubset N_2$ or (2) $N_1 = N_2$ and $n_1 \leq_\Gamma n_2$. Moreover, we let $\widetilde{\sqsubset} = (\widetilde{\sqsubseteq} \setminus \text{id}_{\text{CenterFact}(M,f)})$. Furthermore, the *minimal f-centering factorisation of M* is defined as

$$\text{minCenterFact}(M, f) = \min_{\widetilde{\sqsubseteq}}\big(\text{CenterFact}(M, f)\big),$$

whenever $\mathrm{CenterFact}(M, f) \neq \emptyset$.

Let $f\colon \mathbb{N} \to \mathbb{N}$ be a strongly monotone map. We say that $\mathbb{M}$ *admits f-centering factorisations* if for every $M \in \mathcal{M}_{\mathrm{fin}}(\mathbb{M})$ there exists an f-centering factorisation (n, N) of M. We say that $\mathbb{M}$ *admits centering factorisations* if there exists a strongly monotone map $f\colon \mathbb{N} \to \mathbb{N}$ such that $\mathbb{M}$ admits f-centering factorisations. ◀

Remark 3.14. First, let $\mathbb{M}$ divide Γ-monotone and let $m_1, m_2 \in \mathbb{M}$, $n \in \mathrm{div}(m_1, m_2)$, and $\{\!\{n_1, n_2\}\!\} \in \mathrm{minquot}_n(m_1, m_2)$. By Lemma 3.7 it holds that

$$d_\Gamma(n_1, n_2) \geq d_\Gamma(n \odot n_1, n \odot n_2) = d_\Gamma(m_1, m_2)$$

and hence by the fact that $\mathbb{M}$ divides Γ-monotone, it holds that

$$d_\Gamma(n_1, n_2) = d_\Gamma(m_1, m_2).$$

Second, let $f\colon \mathbb{N} \to \mathbb{N}$ be a strongly monotone map, $M = \emptyset$, $n \in \mathrm{mindiv}(M)$, and $N \in \mathrm{minquot}_n(M)$. The fact that $n \odot N = M$ implies that $N = \emptyset$. This shows that (n, N) is an f-centering factorisation. In particular, in order to show that $\mathbb{M}$ admits f-centering factorisations, it suffices to show that for every $M \in \mathcal{M}_{\mathrm{fin}}(\mathbb{M})$ with $M \neq \emptyset$ there exists an f-centering factorisation (n, N) of M. ◀

Remark 3.15. We note that we use the adjective "centering" in centering factorisations for the following reason. Let $M \in \mathcal{M}_{\mathrm{fin}}(\mathbb{M})$ and (n, N) be an f-centering factorisation of M. For every $n_1 \in N$ and every $r > 0$ such that $N \subseteq \mathcal{B}_r(n_1)$, it holds that $1 \in \mathcal{B}_{f(r)}(n_1)$. Hence, 1 is always close to N (up to f) and therefore, N is in a sense "centered" around 1. ◀

Example 3.16. We continue Example 3.11 and determine whether our monoids divide Γ-monotone and admit centering factorisations.

- We consider $(\mathbb{N}, +, 0)$ with the generating set $\Gamma = \{1\}$ and let $m_1, m_2 \in \mathbb{N}$, $n \in \mathrm{div}(m_1, m_2)$, and $\{\!\{n_1, n_2\}\!\} \in \mathrm{quot}_n(m_1, m_2)$ such that $n + n_1 = m_1$ and $n + n_2 = m_2$. This shows

  $$d_\Gamma(n_1, n_2) \overset{\text{Ex.3.5}}{=} |n_1 - n_2| = |n_1 + n - (n_2 + n)| = |m_1 - m_2| \overset{\text{Ex.3.5}}{=} d_\Gamma(m_1, m_2)$$

  and hence, $(\mathbb{N}, +, 0)$ divides Γ-monotone.

We prove that $\mathbb{N}$ admits centering factorisations. Let $M \in \mathcal{M}_{\text{fin}}(\mathbb{M})$ such that $M \neq \emptyset$, $n \in \text{mindiv}(M)$, and $N \in \text{minquot}_n(M)$. We know from Example 3.11 that $n = \min(\text{supp}(M))$ and $N = \{\!\{m - n \mid m \in M\}\!\}$. Therefore, $0 \in \text{supp}(N)$. Let $n_1 \in \text{supp}(N)$. If $n_1 \neq 0$, then we let $n_2 = 0$, whence $|n_1|_\Gamma = d_\Gamma(n_1, n_2)$. If $n_1 = 0$, then we let $n_2 \in \text{supp}(N)$ and obtain $|n_1|_\Gamma = 0 \leq d_\Gamma(n_1, n_2)$. This shows that $(n, N) \in \text{CenterFact}(M, \text{id}_\mathbb{N})$. The case that $M = \emptyset$ is covered by Remark 3.14. This concludes the proof that $(\mathbb{N}, +, 0)$ admits centering factorisations.

- We consider $(\mathbb{N}_{\leq k}, +_k, 0)$ for some $k \in \mathbb{N}$ and recall that $\Gamma = \{1\}$ is a finite generating set of $\mathbb{N}_{\leq k}$. Let $m_1 = m_2 = n = n_1 = k$, and $n_2 = 0$. It surely holds that $n \in \text{div}(m_1, m_2)$ and $\{\!\{n_1, n_2\}\!\} \in \text{quot}_n(m_1, m_2)$. We have that

$$d_\Gamma(n_1, n_2) \overset{\text{Ex.3.5}}{=} |n_1 - n_2| = k \neq 0 = |m_1 - m_2| \overset{\text{Ex.3.5}}{=} d_\Gamma(m_1, m_2)$$

  and hence, $(\mathbb{N}_{\leq k}, +_k, 0)$ does not divide Γ-monotone.

  The fact that $\mathbb{N}_{\leq k}$ admits centering factorisations can be seen as follows. Let $M \in \mathcal{M}_{\text{fin}}(\mathbb{N}_{\leq k})$ such that $M \neq \emptyset$, $n \in \text{mindiv}(M)$, $N \in \text{minquot}_n(M)$, and $n_1 \in \text{supp}(N)$. If $\text{supp}(N) \in \mathcal{S}(\mathbb{M})$, then also $\text{supp}(M) \in \mathcal{S}(\mathbb{M})$. In this case, $\text{supp}(M) = \{n\}$ by the minimality of N, which implies that $n_1 = 0 = |n_1|_\Gamma$. Otherwise, let $n_2 \in \text{supp}(N)$ such that $n_1 \neq n_2$. Since $\mathbb{N}_{\leq k}$ is finite, it holds that $|n_1|_\Gamma \leq \#\mathbb{N}_{\leq k} \leq \#\mathbb{N}_{\leq k} \cdot d_\Gamma(n_1, n_2)$. This proves the fact that (n, N) is a $(\#\mathbb{N}_{\leq k} \cdot \text{id}_\mathbb{N})$-centering factorisation of M. The case that $M = \emptyset$ is covered by Remark 3.14. This shows that $(\mathbb{N}_{\leq k}, +_k, 0)$ admits centering factorisations. In fact, this proof can easily be generalised to show that all finite monoids admit centering factorisations for every finite generating set.

- We consider the monoid $\mathbb{BF}$ and the multiset $M = \{\!\{(3,1), (2,8)\}\!\}$ from Example 3.11. We have seen that $(2,0) \in \text{div}(M)$ and $\text{quot}_{(2,0)}(M) = \{\{\!\{(1,1), (0,8)\}\!\}\}$. We will now show that

$$d_{\Gamma_{\mathbb{BF}}}((1,1), (0,8)) > d_{\Gamma_{\mathbb{BF}}}((3,1), (2,8)),$$

  which proves that $\mathbb{BF}$ does not divide $\Gamma_{\mathbb{BF}}$-monotone.

We can easily verify that there exists only one fork-path between $(1,1)$ and $(0,8)$, namely

$$\Big(\quad (0,0)(1,0)(1,1) \quad , \quad (0,0)(0,1)\dots(0,8) \quad \Big)$$

and hence $d_{\Gamma_{\mathbb{BF}}}((1,1),(0,8)) = 10$. Moreover, the pair

$$\Big(\quad (0,0)(1,0)(2,0)(3,0)(3,1) \quad , \quad (0,0)(0,1)(0,2)(1,4)(2,8) \quad \Big)$$

is a fork-path between $(3,1)$ and $(2,8)$ with length 8. Hence,

$$d_{\Gamma_{\mathbb{BF}}}((1,1),(0,8)) = 10 > 8 \geq d_{\Gamma_{\mathbb{BF}}}((3,1),(2,8)),$$

which concludes our proof.

Next, we show that $\mathbb{BF}$ admits centering factorisations. We define the map $f\colon \mathbb{N} \to \mathbb{N}$ for every $r \in \mathbb{N}$ by $f(r) = 2 \cdot r + 2^r \cdot (r+1)$ and see that f is strongly monotone. Let $M \in \mathcal{M}_{\text{fin}}(\mathbb{BF})$ such that $M \neq \emptyset$, $n \in \text{mindiv}(M)$, $N \in \text{minquot}_n(M)$, $n_1 \in \text{supp}(N)$, and define $r = \max_{n_2 \in \text{supp}(N)} d_{\Gamma_{\mathbb{BF}}}(n_1, n_2)$. In order to show that (n, N) is an f-centering factorisation of M, we need to show $|n_1|_{\Gamma_{\mathbb{BF}}} \leq f(r)$.

We do the proof by contraposition and therefore assume that $|n_1|_{\Gamma_{\mathbb{BF}}} > f(r)$. We will now show that there exists $n' \in \text{div}(N)$ and $N' \in \text{minquot}_{n'}(N)$ such that $|N'|_{\Gamma_{\mathbb{BF}}} \leq f(r)$. This concludes the proof, as

$$|N|_{\Gamma_{\mathbb{BF}}} \geq |n_1|_{\Gamma_{\mathbb{BF}}} > f(r) \geq |N'|_{\Gamma_{\mathbb{BF}}}$$

and hence $n' \circ n$ is a "more minimising" divisor of M than n, which contradicts the fact that $n \in \text{mindiv}(M)$.

We denote the components of n_1 as $n_1 = (n_{1,\text{a}}, n_{1,\text{b}})$ and define

$$n' = \Big(n_{1,\text{a}} - r, \left\lfloor \frac{n_{1,\text{b}}}{2^r} \right\rfloor - r\Big).$$

To show that $n' \in \text{div}(N)$, we have to show that n' divides every $n_2 \in \text{supp}(N)$. Since $d_{\Gamma_{\mathbb{BF}}}(n_1, n_2) \leq r$, there exists $m \in \mathbb{BF}$ and $(w, v) \in \text{FPaths}(n_1, n_2)$ such that $m \overset{w}{\rightsquigarrow} n_1$, $m \overset{v}{\rightsquigarrow} n_2$, and $|w| + |v| \leq r$. In particular, if n' divides m, then n' divides n_2. Hence, it suffices to show that $n' \in \text{div}(K)$, where

$$K = \{m \in \mathbb{BF} \mid \exists m \overset{w}{\rightsquigarrow} n_1 \colon |w| \leq r\}.$$

Let $m \in K$ with components $m = (m_a, m_b)$. There exist $i, j_0, \dots, j_i \in \mathbb{N}$ such that $i \in \{0, \dots, r\}$, $\sum_{k=0}^{i} j_k \leq r$, and $m \circ_{\mathbb{BF}} w = n_1$ where

$$w = (0,1)^{j_0}(1,0)(0,1)^{j_1}(1,0)\dots(1,0)(0,1)^{j_i} \in \mathbb{BF}.$$

We obtain

$$m_a + i = n_{1,a} \qquad \text{and} \qquad (\dots((m_b + j_0) \cdot 2 + j_1) \cdot 2 \dots) \cdot 2 + j_i = n_{1,b},$$

which yields

$$\lfloor \frac{n_{1,\mathrm{b}}}{2^i} \rfloor = m_b + \lfloor \sum_{k=0}^{i} \frac{j_k}{2^k} \rfloor.$$

In particular, we obtain that $m = (n_{1,\mathrm{a}} - i, \lfloor \frac{n_{1,\mathrm{b}}}{2^i} \rfloor - j)$ for some $i, j \in \{0, \dots, r\}$. By Equation (3.1) it holds that n' divides m if and only if

$$n_{1,\mathrm{a}} - r \leq n_{1,\mathrm{a}} - i \qquad \text{and} \qquad \left\lfloor \frac{n_{1,\mathrm{b}}}{2^r} \right\rfloor - r \leq \left\lfloor \frac{\lfloor \frac{n_{1,\mathrm{b}}}{2^i} \rfloor - j}{2^{(n_{1,\mathrm{a}} - i) - (n_{1,\mathrm{a}} - r)}} \right\rfloor. \tag{3.2}$$

The first inequality in (3.2) holds since $i \leq r$ and the second inequality in (3.2) can be proven as follows.

$$\left\lfloor \frac{n_{1,\mathrm{b}}}{2^r} \right\rfloor - r = \left\lfloor \frac{\lfloor \frac{n_{1,\mathrm{b}}}{2^i} \rfloor}{2^{r-i}} \right\rfloor - r \leq \left\lfloor \frac{\lfloor \frac{n_{1,\mathrm{b}}}{2^i} \rfloor}{2^{r-i}} \right\rfloor - j \leq \left\lfloor \frac{\lfloor \frac{n_{1,\mathrm{b}}}{2^i} \rfloor - j}{2^{r-i}} \right\rfloor = \left\lfloor \frac{\lfloor \frac{n_{1,\mathrm{b}}}{2^i} \rfloor - j}{2^{(n_{1,\mathrm{a}} - i) - (n_{1,\mathrm{a}} - r)}} \right\rfloor$$

This concludes our proof that $n' \in \mathrm{div}(N)$.

We let $\ell = 2^{r-i} \cdot r - j$ and note that there exists $k \in \{0, \dots, 2^r\}$ such that

$$\begin{aligned} n' = \left(n_{1,\mathrm{a}} - r, \left\lfloor \frac{n_{1,\mathrm{b}}}{2^r} \right\rfloor - r\right) &\overset{(1,0)^{r-i}}{\rightsquigarrow} \left(n_{1,\mathrm{a}} - i, 2^{r-i} \cdot \left\lfloor \frac{n_{1,\mathrm{b}}}{2^r} \right\rfloor - 2^{r-i} \cdot r\right) \\ &\overset{(0,1)^k}{\rightsquigarrow} \left(n_{1,\mathrm{a}} - i, \left\lfloor 2^{r-i} \cdot \frac{n_{1,\mathrm{b}}}{2^r} \right\rfloor - 2^{r-i} \cdot r\right) \\ &= \left(n_{1,\mathrm{a}} - i, \left\lfloor \frac{n_{1,\mathrm{b}}}{2^i} \right\rfloor - 2^{r-i} \cdot r\right) \\ &\overset{(0,1)^\ell}{\rightsquigarrow} \left(n_{1,\mathrm{a}} - i, \left\lfloor \frac{n_{1,\mathrm{b}}}{2^i} \right\rfloor - j\right) = m \end{aligned}$$

is a path from n' to m. This proves that $d_{\Gamma_{\mathbb{BF}}}(n', m) \leq r - i + k + \ell \leq r + 2^r + 2^r \cdot r$ and hence $d_{\Gamma_{\mathbb{BF}}}(n', n_2) \leq d_{\Gamma_{\mathbb{BF}}}(n', m) + r \leq 2 \cdot r + 2^r \cdot (r+1) = f(r)$ for every $n_2 \in \mathrm{supp}(N)$. In particular, for the unique $N' \in \mathrm{minquot}_{n'}(N)$ it holds that $|N'|_{\Gamma_{\mathbb{BF}}} \leq f(r)$. This concludes our proof. ◀

Lemma 3.17. If $\mathbb{M}$ divides Γ-monotone, then $\mathbb{M}$ is cancellative.

Proof. We prove the statement by contradiction. Assume that $\mathbb{M}$ is not cancellative. There exist $a, b, c \in \mathbb{M}$ such that $a \odot b = a \odot c$ and $b \neq c$. We note that $a \in \operatorname{div}(a \odot b, a \odot c)$ and $\{\!\{b, c\}\!\} \in \operatorname{quot}_a(a \odot b, a \odot c)$. By the fact that $\mathbb{M}$ divides Γ-monotone, it holds that $d_\Gamma(b, c) \leq d_\Gamma(a \odot b, a \odot c)$. However, $d_\Gamma(b, c) \geq 1 > 0 = d_\Gamma(a \odot b, a \odot c)$, which is a contradiction. □

Remark 3.18. Lemma 3.17 states that cancellativity is a necessary condition for $\mathbb{M}$ to divide Γ-monotone. Moreover, in Example 3.16 we have seen that cancellativity is not a sufficient condition for $\mathbb{M}$ to divide Γ-monotone (see the monoid $\mathbb{BF}$).

Whenever we require $\mathbb{M}$ to divide Γ-monotone, we will still use the formalism of quotients and minimal quotients, even though quotients are unique in cancellative monoids. This has one major reason. At the time of writing, we strongly believe that the definition of "divides Γ-monotone" can be weakened such that all our theorems still work, but cancellativity is not implied. In order to make our proofs easily generalisable to the non-cancellative setting, we decided to write them without using unique quotients.

We also note that all monoids in Example 3.16 admit centering factorisations. At the time of writing, we are unaware of a result proving whether there exist finitely generated monoids that do not admit centering factorisations. We conjecture that every finitely generated monoid that divides Γ-monotone also admits centering factorisations. ◀

3.4 Weighted Tree Automata over $\mathcal{M}_{\text{fin}}(\mathbb{M})$ and the Twinning Property

In this chapter, we collect some useful definitions and facts about weighted tree automata over the semiring of finite multisets over $\mathbb{M}$. These will be used to implement Steps (I) and (II) from our $\mathbb{M}$-sequentialisation construction (see Chapter 3.1). Moreover, we introduce our notion of the twinning property and illustrate everything with running examples.

Throughout the rest of Chapter 3.4, we let Σ be a ranked alphabet, $(\mathbb{M}, \odot, 1)$ be a monoid, and $\mathscr{B} = (Q, T, \text{final})$ be a WTA *over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$.*

3.4.1 Weighted Tree Automata over $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$

In order to have a structured notational system for the weights occurring in calculations, we make the following convention. All values denoted by a (possibly decorated) lowercase symbol y (or z) represent elements of transition weights (or final weights, respectively) of weighted tree automata over $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$. All remaining weights have no fixed notational convention. We adhere to this convention as strictly as possible.

Definition 3.19. For every $s \in \mathbb{N}$ and $\sigma \in \Sigma^{(s)}$ we define the multiset T^{R}_σ over $Q^s \times Q \times \mathbb{M}$ by

$$T^{\mathrm{R}}_\sigma = \{\!\!\{(q_1, \ldots, q_s, q, y) \mid q_1, \ldots, q_s, q \in Q, y \in T_\sigma(q_1, \ldots, q_s, q)\}\!\!\}$$

and define the family $T^{\mathrm{R}} = (T^{\mathrm{R}}_\sigma \mid \sigma \in \Sigma)$. Moreover, we denote $T^{\mathrm{R}}_\cup = \bigcup_{\sigma \in \Sigma} T^{\mathrm{R}}_\sigma$. For every $t = (q_1, \ldots, q_s, q, y) \in \mathrm{supp}(T^{\mathrm{R}}_\sigma)$ we define $\mathrm{in}(t) = q_1 \ldots q_s$ in Q^*, $\mathrm{out}(t) = q$, and $\mathrm{wt}(t) = y$. ◀

Remark 3.20. We note that T^{R} is merely a syntactical variant of T and contains the same information as T. Analogously, one can introduce a relational variant $\mathrm{final}_{\mathrm{R}}$ of final. Then, $\mathrm{final}_{\mathrm{R}}$ is a multiset over $Q \times \mathbb{M}$. This yields a relational automaton model, where each automaton is of the form $(Q, T^{\mathrm{R}}, \mathrm{final}_{\mathrm{R}})$.

In this sense, our WTA model can be compared to the automaton model from [29, Definition 2] (cf. [31] for the word case). In fact, [29, Definition 2] is a restriction of our relational automaton model where all weights are sets (rather than multisets). In this thesis, we are interested in determinising automata over non-idempotent semirings. Therefore, we need the extension from sets to multisets. ◀

Example 3.21. Let $\Sigma = \{\alpha^{(0)}, \gamma^{(1)}, \sigma^{(2)}\}$. We consider the WTA $\mathscr{B} = (Q, T, \mathrm{final})$ over Σ and $\mathcal{M}_{\mathrm{fin}}(\mathbb{N})$, where $\mathcal{M}_{\mathrm{fin}}(\mathbb{N}) = \mathcal{M}_{\mathrm{fin}}((\mathbb{N}, +, 0))$, $Q = \{q_1, q_2, q_3\}$ and every transition weight and final weight of $\mathscr{A}$ is $\emptyset$ except

$$\begin{aligned}
&\mathrm{final}(q_1) = \mathrm{final}(q_2) = \mathrm{final}(q_3) = \{\!\!\{0\}\!\!\},\\
&T_\alpha(q_1) = \{\!\!\{1\}\!\!\},\ T_\gamma(q_2, q_1) = \{\!\!\{1, 1\}\!\!\},\\
&T_\gamma(q_1, q_1) = T_\gamma(q_2, q_3) = T_\gamma(q_3, q_3) = \{\!\!\{1\}\!\!\}, \text{ and}\\
&T_\sigma(q_i, q_j, q_2) = \{\!\!\{1\}\!\!\} \text{ for every } i, j \in \{1, 2\}.
\end{aligned}$$

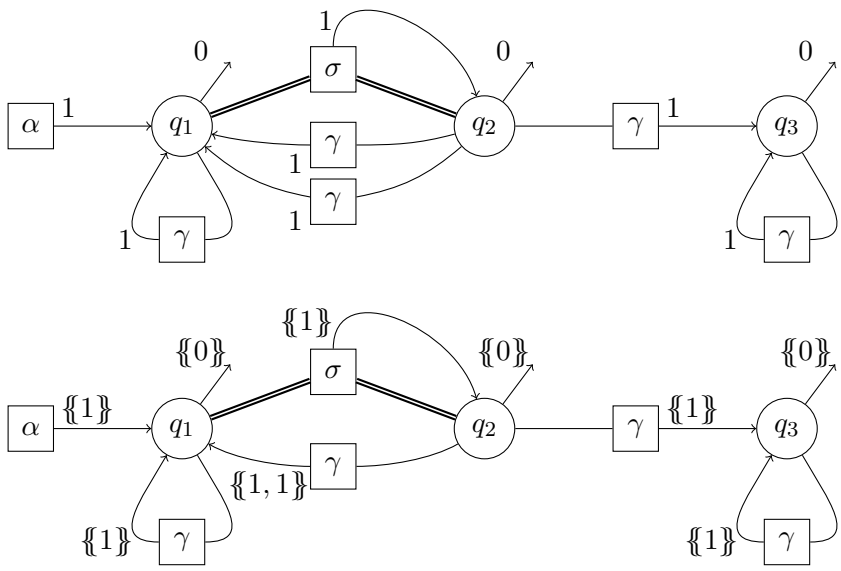


Figure 3.3: Bottom: illustration of the WTA $\mathscr{B}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{N})$ from Example 3.21. Top: illustration of the relational variant of $\mathscr{B}$. We again use the convention from page 30 to make the hypergraphs better readable.

Thus, T^{R} is given by

$$\begin{aligned}
T^{\text{R}}_{\alpha} &= \{\!\{(q_1, 1)\}\!\}, \\
T^{\text{R}}_{\gamma} &= \{\!\{(q_1, q_1, 1), (q_2, q_1, 1), (q_2, q_1, 1), (q_2, q_3, 1), (q_3, q_3, 1)\}\!\}, \text{ and} \\
T^{\text{R}}_{\sigma} &= \{\!\{(q_1, q_1, q_2, 1), (q_1, q_2, q_2, 1), (q_2, q_1, q_2, 1), (q_2, q_2, q_2, 1)\}\!\}.
\end{aligned}$$

We observe that the tuple $(q_2, q_1, 1)$ is contained twice in T^{R}_{γ}. This directly corresponds to the fact that $T_{\gamma}(q_2, q_1) = \{\!\{1, 1\}\!\}$, which is a multiset with size 2.

We give a graphical representation of $\mathscr{B}$ in Figure 3.3. ◀

Definition 3.22. We define the $\mathbb{M}$-*image of* $\mathscr{B}$, denoted by $\mathbb{M}\text{-im}(\mathscr{B})$, as the set

$$\mathbb{M}\text{-im}(\mathscr{B}) = \Big(\bigcup_{\substack{s \in \mathbb{N}, \sigma \in \Sigma^{(s)}, \\ q_1, \ldots, q_s, q \in Q}} \text{supp}(T_{\sigma}(q_1, \ldots, q_s, q)) \Big) \cup \Big(\bigcup_{q \in Q} \text{supp}(\text{final}(q)) \Big). \quad ◀$$

Besides the usual runs of WTA over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$, we also consider another notion of runs, called R-runs. In essence, an R-run of $\mathscr{B}$ is a usual run of $\mathscr{B}$ where additionally,

each local state behaviour $(q_1, \ldots, q_s, q)$ described by the run is enriched by a single element from the multiset $T_\sigma(q_1, \ldots, q_s, q)$. Intuitively, the R-runs of $\mathscr{B}$ correspond combinatorically to the paths of the hypergraph of the relational variant of $\mathscr{B}$.

Definition 3.23. Let $\xi \in \mathrm{T}_\Sigma \cup \mathrm{C}_\Sigma$ be a tree or a context. An *R-run of $\mathscr{B}$ on ξ* is a map $\rho\colon \mathrm{pos}(\xi) \to (\mathrm{supp}(T^{\mathrm{R}}_\cup) \cup Q)$ such that

- for every $w \in \mathrm{pos}_X(\xi)$ it holds that $\rho(w) \in Q$ and
- for every $w \in \mathrm{pos}_\Sigma(\xi)$ it holds that (a) $\rho(w) \in \mathrm{supp}(T^{\mathrm{R}}_\sigma)$ where $\sigma = \xi(w)$ and (b) $\mathrm{in}(\rho(w)) = \mathrm{out}(\rho(w1)) \ldots \mathrm{out}(\rho(ws))$ where $s = \mathrm{rk}(\xi(w))$.

We denote $\mathrm{out}(\rho) = \mathrm{out}(\rho(\varepsilon))$ and if $\xi \in \mathrm{C}_\Sigma$, then we denote $\mathrm{in}(\rho) = \rho(\mathrm{pos}_{\mathrm{var}}(\xi))$. The *multiset of all R-runs of $\mathscr{B}$ on ξ*, denoted by $\text{R-Runs}_{\mathscr{B}}(\xi)$, is a multiset over $(\mathrm{supp}(T^{\mathrm{R}}_\cup) \cup Q)^{\mathrm{pos}(\xi)}$ given for every $\rho\colon \mathrm{pos}(\xi) \to (\mathrm{supp}(T^{\mathrm{R}}_\cup) \cup Q)$ as follows. If ρ is not an R-run, then $\text{R-Runs}_{\mathscr{B}}(\xi)(\rho) = 0$ and if ρ is an R-run, then

$$\text{R-Runs}_{\mathscr{B}}(\xi)(\rho) = \prod_{w \in \mathrm{pos}_\Sigma(\xi)} T^{\mathrm{R}}_{\xi(w)}(\rho(w)).$$

That is, the multiplicity of ρ in $\text{R-Runs}_{\mathscr{B}}(\xi)$ equals the product of the multiplicities of all transitions occurring in ρ.

We define the map $\mathrm{wt}^{\mathrm{R}}_{\mathscr{B}}(\xi, \_, \_)\colon \mathrm{supp}(\text{R-Runs}_{\mathscr{B}}(\xi)) \times \mathrm{pos}(\xi) \to \mathbb{M}$ inductively as follows. Let $\rho \in \mathrm{supp}(\text{R-Runs}_{\mathscr{B}}(\xi))$ and $w \in \mathrm{pos}(\xi)$. If $\xi(w) = x_1$, then we define $\mathrm{wt}^{\mathrm{R}}_{\mathscr{B}}(\xi, \rho, w) = 1$. Otherwise, there exists $s \in \mathbb{N}$ such that $\xi(w) \in \Sigma^{(s)}$ and we define

$$\mathrm{wt}^{\mathrm{R}}_{\mathscr{B}}(\xi, \rho, w) = \mathrm{wt}^{\mathrm{R}}_{\mathscr{B}}(\xi, \rho, w1) \odot \cdots \odot \mathrm{wt}^{\mathrm{R}}_{\mathscr{B}}(\xi, \rho, ws) \odot \mathrm{wt}(\rho(w)).$$

Whenever the automaton $\mathscr{B}$ is clear from the context, we will omit the subscript $\mathscr{B}$ from $\mathrm{wt}^{\mathrm{R}}_{\mathscr{B}}$ and simply write wt^{R}. Moreover, we will always drop the superscript R from wt^{R} and simply write wt.

Let $\rho \in \mathrm{supp}(\text{R-Runs}_{\mathscr{B}}(\xi))$. We abbreviate $\mathrm{wt}(\xi, \rho, \varepsilon)$ by $\mathrm{wt}(\xi, \rho)$ and call $\mathrm{wt}(\xi, \rho)$ the *weight of ρ*. Moreover, we call ρ *non-vanishing* if $\mathrm{wt}(\xi, \rho) \neq 0$.

We define the multiset

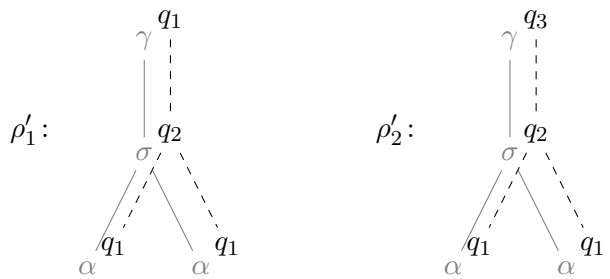
$$\text{R-Runs}_{\mathscr{B}}(\xi, q) = \{\!\{\rho \mid \rho \in \text{R-Runs}_{\mathscr{B}}(\xi), \mathrm{out}(\rho) = q\}\!\}$$

for every $q \in Q$. Moreover, if $\xi \in \mathrm{C}_\Sigma$, then we define the multiset

$$\text{R-Runs}_{\mathscr{B}}(p, \xi, q) = \{\!\!\{\rho \mid \rho \in \text{R-Runs}_{\mathscr{B}}(\xi, q), \mathrm{in}(\rho) = p\}\!\!\}$$

for every $p, q \in Q$. We say that $\rho \in \mathrm{supp}(\text{R-Runs}_{\mathscr{B}}(\xi))$ is a *loop (on ξ)* if $\xi \in \mathrm{C}_\Sigma$ and there exists $q \in Q$ such that $\rho \in \mathrm{supp}(\text{R-Runs}_{\mathscr{B}}(q, \xi, q))$. ◀

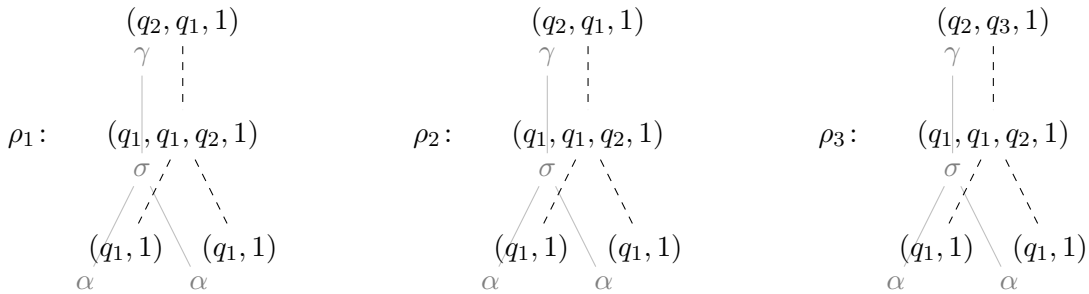
Example 3.24. We continue Example 3.21 and consider the tree $\xi = \gamma(\sigma(\alpha, \alpha))$ in T_Σ. We can easily see that there exist exactly two valid runs of $\mathscr{B}$ on w. In fact, $\text{Runs}_{\mathscr{B}}(w) = \{\rho'_1, \rho'_2\}$, where



The weights of ρ'_1 and ρ'_2 are given by

$$\begin{aligned} \mathrm{wt}(\xi, \rho'_1) &= \{\!\!\{1\}\!\!\} + \{\!\!\{1\}\!\!\} + \{\!\!\{1\}\!\!\} + \{\!\!\{1, 1\}\!\!\} = \{\!\!\{4, 4\}\!\!\} \text{ and} \\ \mathrm{wt}(\xi, \rho'_2) &= \{\!\!\{1\}\!\!\} + \{\!\!\{1\}\!\!\} + \{\!\!\{1\}\!\!\} + \{\!\!\{1\}\!\!\} \quad = \{\!\!\{4\}\!\!\}. \end{aligned}$$

We can also see that for every $i \in [2]$, ρ'_i can be associated to a multiset of R-runs that have the same state behaviour as ρ'_i. We will make this connection precise in Remark 3.25. We consider the three R-Runs of $\mathscr{B}$ on ξ given as follows.



In our case, ρ'_1 can be associated to the two (equal) R-runs ρ_1 and ρ_2. One can think of ρ_1 and ρ_2 as taking the "upper" (respectively "lower") relational γ-edge from q_2 to q_1 depicted in the top part of Figure 3.3. The run ρ'_1, on the other hand, can take both

of these relational γ-edges from q_2 to q_1 in a single accumulated step with transition weight $\{\!\{1, 1\}\!\}$. This shows how a single valid run of $\mathscr{B}$ can be associated to multiple R-runs of $\mathscr{B}$. Moreover, it holds that $\rho_1 = \rho_2$ and hence, we can see that $\text{R-Runs}_{\mathscr{B}}(w)$ is indeed a proper multiset (that is, not a set).

The only R-run that can be associated to ρ_2' is ρ_3. In fact, we have determined all R-runs of $\mathscr{B}$ on w and obtain $\text{R-Runs}_{\mathscr{B}}(w) = \{\!\{\rho_1, \rho_2, \rho_3\}\!\}$. ◀

Remark 3.25. Let $\xi \in \mathrm{T}_\Sigma \cup \mathrm{C}_\Sigma$ be a tree or a context and $\rho' \colon \mathrm{pos}(\xi) \to Q$ be a run of $\mathscr{B}$ on ξ. For every R-run $\rho \colon \mathrm{pos}(\xi) \to (\mathrm{supp}(T_\cup^{\mathrm{R}}) \cup Q)$ of $\mathscr{B}$ on ξ we say that ρ' and ρ are *associated* if

$$\rho'(w) = \mathrm{out}(\rho(w)) \text{ for every } w \in \mathrm{pos}(\xi).$$

We define the *multiset of R-runs of $\mathscr{B}$ on w associated with ρ'*, denoted by $\mathrm{R}(\rho')$, by

$$\mathrm{R}(\rho') = \{\!\{\rho \mid \rho \in \text{R-Runs}_{\mathscr{B}}(\xi), \rho' \text{ and } \rho \text{ are associated}\}\!\}.$$

If ρ' is not valid, then there is no R-run associated with ρ'. On the other hand, if ρ' is valid and there is a position $w \in \mathrm{pos}(\xi)$ such that $\#\mathrm{locwt}(\xi, \rho', w) > 1$, then there are multiple R-runs associated with ρ'.

Moreover, a straightforward proof shows that $\mathrm{wt}(\xi, \rho') = \{\!\{\mathrm{wt}(\xi, \rho) \mid \rho \in \mathrm{R}(\rho')\}\!\}$ for every $\rho' \in \mathrm{Runs}_{\mathscr{B}}(\xi)$. ◀

Remark 3.26. In the rest of this chapter, we use the following notation for R-runs.

Let $q \in Q$, $\xi \in \mathrm{T}_\Sigma$, $\rho \in \mathrm{supp}(\text{R-Runs}_{\mathscr{B}}(\xi, q))$, and denote $y = \mathrm{wt}(\xi, \rho)$. We say that ρ is *of the form* $\xrightarrow{\xi|y} q$. Analogously, let $p, q \in Q$, $\xi \in \mathrm{C}_\Sigma$, $\rho \in \mathrm{supp}(\text{R-Runs}_{\mathscr{B}}(p, \xi, q))$, and denote $y = \mathrm{wt}(\xi, \rho)$. We say that ρ is *of the form* $p \xrightarrow{\xi|y} q$. Moreover, we denote the fact that $z \in \mathrm{final}(q)$ by $q \xrightarrow{z}$. We freely compose these notations in order to express more conditions on an R-run, for example "$\xrightarrow{\xi|y_1} q \xrightarrow{\zeta|y_2} q \xrightarrow{z}$". Moreover, if such an expression contains free variables, they shall be quantified according to the quantification before the expression. For example, given $\xi \in \mathrm{T}_\Sigma$ and $\zeta \in \mathrm{C}_\Sigma$, the expression "for every R-run of the form $\xrightarrow{\xi|y_1} q \xrightarrow{\zeta|y_2} q$" universally quantifies $q \in Q$, $y_1, y_2 \in \mathbb{M}$, and (nameless) R-runs $\rho_1 \in \text{R-Runs}_{\mathscr{B}}(\xi, q)$ and $\rho_2 \in \text{R-Runs}_{\mathscr{B}}(q, \zeta, q)$ such that $\mathrm{wt}(\xi, \rho_1) = y_1$ and $\mathrm{wt}(\zeta, \rho_2) = y_2$. Whenever we deal with multiple automata simultaneously, we disambiguate such expressions by writing, e.g., $p \xrightarrow{\zeta|y} q \in \mathscr{B}$ for an R-run of $\mathscr{B}$.

If we use such expressions in multiset constructors or summation indices, then we drop the words "R-run of the form". We note that R-runs quantified in multiset constructors or summation indices like this are considered with their multiplicity. ◀

Lemma 3.27. Let $\mathscr{B} = (Q, T, \text{final})$ be a finite-run WTA over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$. For every $\xi \in \mathrm{T}_\Sigma$ it holds that

$$[\![\mathscr{B}]\!](\xi) = \{\!\!\{ y \odot z \mid \xrightarrow{\xi|y} q \xrightarrow{z} \}\!\!\}.$$

Proof. Recall that $\mathrm{R}(\rho')$ is the multiset of R-runs of $\mathscr{B}$ on ξ associated with a run $\rho' \in \mathrm{Runs}_{\mathscr{B}}(\xi)$. It is easy to see that

$$\bigcup_{\rho' \in \mathrm{Runs}_{\mathscr{B}}(\xi,q)} \mathrm{R}(\rho') = \text{R-Runs}_{\mathscr{B}}(\xi, q) \tag{3.3}$$

for every $q \in Q$.

By the definition of $[\![\mathscr{B}]\!]$ it holds that

$$\begin{aligned}
[\![\mathscr{B}]\!](\xi) &= \bigcup_{\rho' \in \mathrm{Runs}_{\mathscr{B}}(\xi)} \mathrm{wt}(\xi, \rho') \odot \mathrm{final}(\mathrm{out}(\rho')) \\
&= \bigcup_{q \in Q} \bigcup_{\rho' \in \mathrm{Runs}_{\mathscr{B}}(\xi,q)} \mathrm{wt}(\xi, \rho') \odot \mathrm{final}(q) \\
&= \bigcup_{q \in Q} \bigcup_{z \in \mathrm{final}(q)} \bigcup_{\rho' \in \mathrm{Runs}_{\mathscr{B}}(\xi,q)} \mathrm{wt}(\xi, \rho') \odot \{\!\!\{ z \}\!\!\} \\
&\overset{\star_1}{=} \bigcup_{q \in Q} \bigcup_{z \in \mathrm{final}(q)} \bigcup_{\rho' \in \mathrm{Runs}_{\mathscr{B}}(\xi,q)} \{\!\!\{ \mathrm{wt}(\xi, \rho) \mid \rho \in \mathrm{R}(\rho') \}\!\!\} \odot \{\!\!\{ z \}\!\!\} \\
&\overset{\star_2}{=} \bigcup_{q \in Q} \bigcup_{z \in \mathrm{final}(q)} \{\!\!\{ \mathrm{wt}(\xi, \rho) \mid \rho \in \text{R-Runs}_{\mathscr{B}}(\xi, q) \}\!\!\} \odot \{\!\!\{ z \}\!\!\} \\
&= \bigcup_{q \in Q} \bigcup_{z \in \mathrm{final}(q)} \{\!\!\{ y \odot z \mid \xrightarrow{\xi|y} q \}\!\!\} = \{\!\!\{ y \odot z \mid \xrightarrow{\xi|y} q \xrightarrow{z} \}\!\!\},
\end{aligned}$$

where Equation $\star_1$ follows from Remark 3.25 and Equation $\star_2$ follows from Equation (3.3). □

Example 3.28. We continue Example 3.24 and denote for every tree $\xi \in \mathrm{T}_\Sigma$ the number of times a γ occurs directly above a σ in ξ by n_ξ.

We claim that the $[\![\mathscr{B}]\!]$ is given for every $\xi \in \mathrm{T}_\Sigma$ by

$$[\![\mathscr{B}]\!](\xi) = \{\!\!\{ \overbrace{\mathrm{size}(\xi), \ldots, \mathrm{size}(\xi)}^{r(\xi) \text{ times}} \}\!\!\}, \text{ where} \tag{3.4}$$

$$r(\xi) = 2^{n_\xi} + \begin{cases} 2^{n_\xi - 1} & \text{if } \xi = \gamma^n(\sigma(\xi', \xi'')) \text{ for some } n \geq 1 \text{ and } \xi', \xi'' \in \mathrm{T}_\Sigma \\ 0 & \text{otherwise.} \end{cases}$$

It is clear from the definition of $\mathscr{B}$ that for every $\rho \in \mathrm{supp}(\text{R-Runs}_{\mathscr{B}}(\xi))$ it holds that $\mathrm{wt}(\xi, \rho) = \mathrm{size}(\xi)$. Therefore, by Lemma 3.27, if we can show that

$$\#\,\text{R-Runs}_{\mathscr{B}}(\xi) = r(\xi) \quad \text{and} \quad \#\,\text{R-Runs}_{\mathscr{B}}(\xi, q_1) + \#\,\text{R-Runs}_{\mathscr{B}}(\xi, q_2) = 2^{n_\xi}, \tag{3.5}$$

then this proves Equation (3.4). We prove Equation (3.5) by induction on ξ.

We assume that $\xi = \tau(\xi_1, \ldots, \xi_s)$ and that Equation (3.5) holds for $\xi_1, \ldots, \xi_s$. If $\tau = \alpha$, then $n_\xi = 0$ and verifiably, $\#\,\text{R-Runs}_{\mathscr{B}}(\xi) = 1$ and $\#\,\text{R-Runs}_{\mathscr{B}}(\xi, q_3) = 0$. Next, we assume that $\tau = \sigma$. It holds that

$$\begin{aligned}
\#\,\text{R-Runs}_{\mathscr{B}}(\xi) &= \sum_{p \in Q} \#\,\text{R-Runs}_{\mathscr{B}}(\xi, p) \\
&\overset{\star_1}{=} \sum_{p \in Q} \sum_{p_1, p_2 \in Q} \#T_\sigma(p_1, p_2, p) \cdot \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, p_1) \cdot \#\,\text{R-Runs}_{\mathscr{B}}(\xi_2, p_2) \\
&\overset{\star_2}{=} \sum_{p_1, p_2 \in \{q_1, q_2\}} \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, p_1) \cdot \#\,\text{R-Runs}_{\mathscr{B}}(\xi_2, p_2) \\
&\overset{\star_3}{=} \Big(\sum_{p_1 \in \{q_1, q_2\}} \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, p_1) \Big) \cdot \Big(\sum_{p_2 \in \{q_1, q_2\}} \#\,\text{R-Runs}_{\mathscr{B}}(\xi_2, p_2) \Big) \\
&\overset{\star_4}{=} 2^{n_{\xi_1}} \cdot 2^{n_{\xi_2}} = 2^{n_{\xi_1} + n_{\xi_2}} = 2^{n_\xi} = r(\xi),
\end{aligned}$$

where Equations $\star_1, \ldots, \star_4$ are justified as follows. Equation $\star_1$ can be shown with a straightforward structural induction on ξ. Equation $\star_2$ follows from the fact that $\#T_\sigma(p_1, p_2, p) = 1$ if $p_1, p_2 \in \{q_1, q_2\}$ and $p = q_2$ and $\#T_\sigma(p_1, p_2, p) = 0$ otherwise. Equations $\star_3$, and $\star_4$ follow from the distributivity of $\mathbb{N}$, and the induction assumption, respectively. It follows directly from the definition of T_σ that $\#\,\text{R-Runs}_{\mathscr{B}}(\xi, q_3) = 0$. Next, we assume that $\tau = \gamma$. We know that

$$\begin{aligned}
\#\,\text{R-Runs}_{\mathscr{B}}(\xi) &= \sum_{p_1, p \in Q} \#T_\gamma(p_1, p) \cdot \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, p_1) \\
&= \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, q_1) + 3 \cdot \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, q_2) + \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, q_3).
\end{aligned}$$

We distinguish two cases for $\xi_1(\varepsilon)$. First, let $\xi_1(\varepsilon) \neq \sigma$ and observe that $n_\xi = n_{\xi_1}$. Moreover, it follows directly from the definition of T that $\#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, q_2) = 0$ and

hence we can use the induction assumption to obtain

$$\begin{aligned}\#\,\text{R-Runs}_{\mathscr{B}}(\xi) &= \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, q_1) + \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, q_2) + \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, q_3)\\ &= \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1) = r(\xi_1).\end{aligned}$$

Since ξ is of the form $\gamma^n(\sigma(\xi', \xi''))$ if and only if ξ_1 is of that form, we obtain $r(\xi) = r(\xi_1)$, which concludes the case $\xi_1(\varepsilon) \neq \sigma$. Next, let $\xi_1(\varepsilon) = \sigma$ and observe that $n_\xi = n_{\xi_1} + 1$. Moreover, it follows directly from the definition of T_σ that $\#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, q) = 0$ for $q = q_1$ and $q = q_3$ and hence we can use the induction assumption to obtain

$$\begin{aligned}\#\,\text{R-Runs}_{\mathscr{B}}(\xi) &= 3 \cdot (\#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, q_1) + \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, q_2) + \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1, q_3))\\ &= 3 \cdot \#\,\text{R-Runs}_{\mathscr{B}}(\xi_1) = 3 \cdot 2^{n_{\xi_1}} = 2 \cdot 2^{n_{\xi_1}} + 2^{n_{\xi_1}} = 2^{n_\xi} + 2^{n_\xi - 1}.\end{aligned}$$

Considering the fact that $\text{R-Runs}_{\mathscr{B}}(\xi_1) = \text{R-Runs}_{\mathscr{B}}(\xi_1, q_2)$, we obtain

$$\#\,\text{R-Runs}_{\mathscr{B}}(\xi, q_3) = \frac{1}{3} \cdot \#\,\text{R-Runs}_{\mathscr{B}}(\xi) = 2^{n_\xi - 1}$$

from the definition of T_γ. This concludes the proof of Equations (3.5) and (3.4). ◀

Definition 3.29. Let $\mathscr{B} = (Q, T, \text{final})$ be a WTA over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$.

We call $\mathscr{B}$ *sequential* if $\mathscr{B}$ is $\mathcal{S}(\mathbb{M})$-sequential.

We call $\mathscr{B}$ *sequentialisable* if there exists a sequential WTA $\mathscr{B}'$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$ such that $\mathscr{B}'$ is equivalent to $\mathscr{B}$. We note that the class of weighted tree languages recognised by sequential finite WTA over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$ is $\text{sRec}(\Sigma, \mathcal{M}_{\text{fin}}(\mathbb{M}), \mathcal{S}(\mathbb{M}))$.

We call $\mathscr{B}$ *finitely R-ambiguous* if there exists $k \in \mathbb{N}$ such that for every $\xi \in \text{T}_\Sigma$ it holds that $\#\,\text{R-Runs}_{\mathscr{B}}(\xi) \leq k$. ◀

Example 3.30. We continue Example 3.28. It holds that $\#T_\gamma(q_2, q_1) = 2$ and hence $T_\gamma(q_2, q_1) \notin \mathcal{S}(\mathbb{M})$. This shows that $\mathscr{B}$ is not sequential.

Let $\zeta = \gamma(\sigma(\alpha, x_1)) \in \text{C}_\Sigma$. Equation (3.5) implies that $\#\,\text{R-Runs}_{\mathscr{B}}(\zeta^n[\alpha]) \geq 2^n$ for every $n \in \mathbb{N}$. Since the sequence $(2^n \mid n \in \mathbb{N})$ is unbounded, $\mathscr{B}$ is not finitely R-ambiguous.

Next, we give a finitely R-ambiguous WTA. For this, we let $\Gamma = \{2, 3\}$ and consider the submonoid $\langle \Gamma \rangle.$ of $(\mathbb{N}, \cdot, 1)$. We outline the proof that $\langle \Gamma \rangle.$ divides Γ-monotone and admits centering factorisations. First, we note that every $m \in \langle \Gamma \rangle.$ can be written

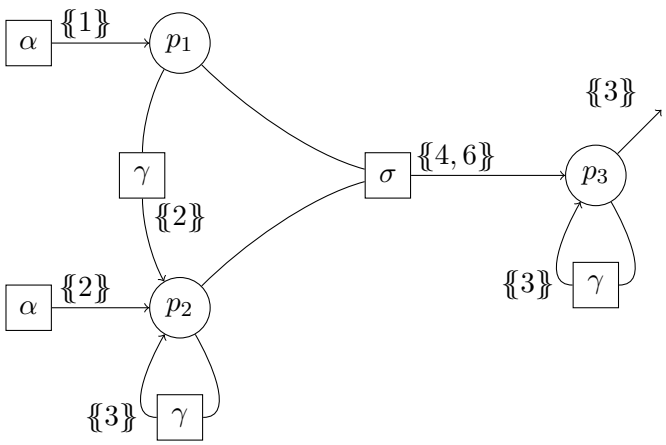


Figure 3.4: Illustration of the WTA $\mathscr{B}_1$ over $\{a, b\}$ and $\mathcal{M}_{\text{fin}}(\langle\{2,3\}\rangle.)$ from Example 3.30.

as $m = 2^k \cdot 3^\ell$ for some uniquely determined $k, \ell \in \mathbb{N}$. One can easily show that $|2^k \cdot 3^\ell|_\Gamma = k + \ell$ and $d_\Gamma(2^k \cdot 3^\ell, 2^{k'} \cdot 3^{\ell'}) = |k - k'| + |\ell - \ell'|$ for every $2^k \cdot 3^\ell, 2^{k'} \cdot 3^{\ell'} \in \mathbb{M}$. Moreover, for every $M \in \mathcal{M}_{\text{fin}}(\langle\Gamma\rangle.)$ and $2^k \cdot 3^\ell \in \text{div}(M)$ it holds that $\text{quot}_{2^k \cdot 3^\ell}(M)$ is a singleton set consisting only of $\{\!\{2^{k'-k} \cdot 3^{\ell'-\ell} \mid 2^{k'} \cdot 3^{\ell'} \in M\}\!\}$. Furthermore,

$$\text{mindiv}(M) = \{2^k \cdot 3^\ell \mid k = \min_{2^{k'} \cdot 3^{\ell'} \in M} k' \wedge \ell = \min_{2^{k'} \cdot 3^{\ell'} \in M} \ell'\}.$$

From here, one can easily show that $\langle\Gamma\rangle.$ divides Γ-monotone.

Let $f \colon \mathbb{N} \to \mathbb{N}$ be given by $f(n) = 2 \cdot n$ for every $n \in \mathbb{N}$. Next, we show that $\langle\Gamma\rangle.$ admits f-centering factorisations. Let $M \in \mathcal{M}_{\text{fin}}(\langle\Gamma\rangle.)$, $2^k \cdot 3^\ell \in \text{mindiv}(M)$, $N \in \text{minquot}_{2^k \cdot 3^\ell}(M)$, and $2^{k_1} \cdot 3^{\ell_1} \in \text{supp}(N)$. We assume that $k_1 \geq \ell_1$. We have already seen that $k = \min_{2^{k'} \cdot 3^{\ell'} \in M} k'$ and hence there exists $\ell_2 \in \mathbb{N}$ such that $3^{\ell_2} \in \text{supp}(N)$. We obtain

$$|2^{k_1} \cdot 3^{\ell_1}|_\Gamma = k_1 + \ell_1 \leq 2 \cdot k_1 \leq 2 \cdot d_\Gamma(2^{k_1} \cdot 3^{\ell_1}, 3^{\ell_2}).$$

The case $\ell_1 \geq k_1$ works analogously by replacing 3^{ℓ_2} by an element $2^{k_2} \in \text{supp}(N)$. This concludes the proof that $\langle\Gamma\rangle.$ admits f-centering factorisations.

We define the WTA $\mathscr{C} = (Q_\mathscr{C}, T_\mathscr{C}, \text{final}_\mathscr{C})$ over Σ and $\mathcal{M}_{\text{fin}}(\langle\Gamma\rangle.)$ as follows. The state set of $\mathscr{C}$ is $Q_\mathscr{C} = \{p_1, p_2, p_3\}$ and every transition weight and final weight of $\mathscr{A}$ is

$\emptyset$ except

$$\begin{aligned}
&\text{final}_{\mathscr{C}}(p_3) = \{\!\{3\}\!\},\\
&(T_{\mathscr{C}})_\alpha(p_1) = \{\!\{1\}\!\},\ (T_{\mathscr{C}})_\alpha(p_2) = \{\!\{2\}\!\},\\
&(T_{\mathscr{C}})_\gamma(p_1,p_2) = \{\!\{2\}\!\},\ (T_{\mathscr{C}})_\gamma(p_2,p_2) = (T_{\mathscr{C}})_\gamma(p_3,p_3) = \{\!\{3\}\!\},\ \text{and}\\
&(T_{\mathscr{C}})_\sigma(p_1,p_2,p_3) = \{\!\{4,6\}\!\}.
\end{aligned}$$

An illustration of $\mathscr{C}$ can be found in Figure 3.4.

Next, we prove that $\mathscr{C}$ is finitely R-ambiguous. Let $\xi \in \mathrm{T}_\Sigma$ and assume that there exists a valid run ρ of $\mathscr{C}$ on ξ. Since $\mathscr{C}$ is deterministic, ρ is the unique valid run of $\mathscr{C}$ on ξ. Hence, from Equation (3.3) we obtain $\text{R-Runs}_{\mathscr{C}}(\xi) = \mathrm{R}(\rho)$. It holds that $\#\,\mathrm{R}(\rho) = \prod_{w\in\text{pos}(\xi)} \#\text{locwt}_{\mathscr{C}}(\xi,\rho,w)$. Every transition weight of $\mathscr{C}$ is a singleton set except $(T_{\mathscr{C}})_\sigma(p_1,p_2,p_3)$ and the transition (p_1,p_2,σ,p_3) occurs at most once in a valid run of $\mathscr{C}$. This shows $\#\,\mathrm{R}(\rho) \leq 2$ and hence $\mathscr{C}$ is finitely R-ambiguous. ◀

3.4.2 The Twinning Property

Throughout the rest of Chapter 3.4, we assume $\mathbb{M}$ to be finitely generated by a finite generating set Γ.

Definition 3.31. We say that $\mathscr{B}$ *has the twinning property* (in symbols: $\mathscr{B} \vDash \text{TP}$), if for every $\xi \in \mathrm{T}_\Sigma$, $\zeta \in \mathrm{C}_\Sigma$, and R-runs of the respective form

$$\xrightarrow{\xi|y_1} q \xrightarrow{\zeta|y_2} q \qquad \text{and} \qquad \xrightarrow{\xi|y_1'} q' \xrightarrow{\zeta|y_2'} q'$$

of $\mathscr{B}$ it holds that $d_\Gamma(y_1,y_1') = d_\Gamma(y_1 \odot y_2, y_1' \odot y_2')$. ◀

Definition 3.32. We say that $\mathscr{B}$ *has the extended twinning property* (in symbols: $\mathscr{B} \vDash \text{ETP}$), if for every $\xi \in \mathrm{T}_\Sigma$, $\zeta,\eta \in \mathrm{C}_\Sigma$, and R-runs of the respective form

$$\xrightarrow{\xi|y_1} q \xrightarrow{\zeta|y_2} q \xrightarrow{\eta|y_3} p \qquad \text{and} \qquad \xrightarrow{\xi|y_1'} q' \xrightarrow{\zeta|y_2'} q' \xrightarrow{\eta|y_3'} p'$$

of $\mathscr{B}$ it holds that $d_\Gamma(y_1 \odot y_3, y_1' \odot y_3') = d_\Gamma(y_1 \odot y_2 \odot y_3, y_1' \odot y_2' \odot y_3')$. ◀

Example 3.33. We consider the WTA $\mathscr{B} = (Q,T,\text{final})$ from Example 3.21. We claim that $\mathscr{B} \vDash \text{ETP}$, which can be seen as follows. First, we note that every non-vanishing

transition weight of $\mathscr{B}$ is $\{\!\{1\}\!\}$ or $\{\!\{1,1\}\!\}$. Hence for every $\xi \in \mathrm{T}_\Sigma$, $\zeta, \eta \in \mathrm{C}_\Sigma$, and R-runs of the respective form

$$\xrightarrow{\xi|y_1} q \xrightarrow{\zeta|y_2} q \xrightarrow{\eta|y_3} p \qquad \text{and} \qquad \xrightarrow{\xi|y_1'} q' \xrightarrow{\zeta|y_2'} q' \xrightarrow{\eta|y_3'} r'$$

of $\mathscr{B}$ it holds that $y_1 = y_1' = \mathrm{size}(\xi)$, $y_2 = y_2' = \mathrm{size}(\zeta)$, and $y_3 = y_3' = \mathrm{size}(\eta)$. Thus

$$\begin{aligned} d_\Gamma(y_1 + y_3, y_1' + y_3') &= d_\Gamma(\mathrm{size}(\xi) + \mathrm{size}(\eta), \mathrm{size}(\xi) + \mathrm{size}(\eta)) = 0 \\ &= d_\Gamma(\mathrm{size}(\xi) + \mathrm{size}(\zeta) + \mathrm{size}(\eta), \mathrm{size}(\xi) + \mathrm{size}(\zeta) + \mathrm{size}(\eta)) \\ &= d_\Gamma(y_1 + y_2 + y_3, y_1' + y_2' + y_3'). \end{aligned}$$

We conclude that $\mathscr{B} \vDash \mathrm{ETP}$. ◀

Remark 3.34. We note that $\mathscr{B} \vDash \mathrm{ETP}$ implies $\mathscr{B} \vDash \mathrm{TP}$. This follows immediately from the definitions of the twinning properties. On the other hand, $\mathscr{B} \vDash \mathrm{TP}$ does not imply $\mathscr{B} \vDash \mathrm{ETP}$ in general, which we will see in the upcoming Example 3.35. ◀

Example 3.35. We recall the family of monoids $\mathbb{N}_{\leq k}$ from Example 3.5. For this example, we let $k = 5$ and consider the WTA $\mathscr{D} = (Q, T, \mathrm{final})$ over $\Sigma = \{\alpha^{(0)}, \gamma^{(1)}\}$ and $\mathcal{M}_{\mathrm{fin}}(\mathbb{N}_{\leq 5})$ which is defined follows. The set of states of $\mathscr{D}$ is $Q = \{q_1, q_2\}$ and every transition weight and final weight of $\mathscr{D}$ is $\emptyset$ except

$$\begin{aligned} &\mathrm{final}(q_2) = \{\!\{0\}\!\}, \\ &T_\alpha(q_1) = \{\!\{0\}\!\}, \\ &T_\gamma(q_1, q_1) = \{\!\{4\}\!\}, \text{ and } T_\gamma(q_1, q_2) = \{\!\{1, 2\}\!\}. \end{aligned}$$

A graphical representation of the relational variant of $\mathscr{D}$ is given in Figure 3.5.

We observe that $\mathscr{D} \vDash \mathrm{TP}$ for the following reason. Let $\xi \in \mathrm{T}_\Sigma$, $\zeta \in \mathrm{C}_\Sigma$, and consider two R-runs of the respective form

$$\xrightarrow{\xi|y_1} q \xrightarrow{\zeta|y_2} q \qquad \text{and} \qquad \xrightarrow{\xi|y_1'} q' \xrightarrow{\zeta|y_2'} q'$$

of $\mathscr{B}$ on $\zeta[\xi]$. We note that $\xi = \gamma^k(\alpha)$ and $\zeta = \gamma^\ell(x_1)$ for some $k, \ell \in \mathbb{N}$. We distinguish two cases.

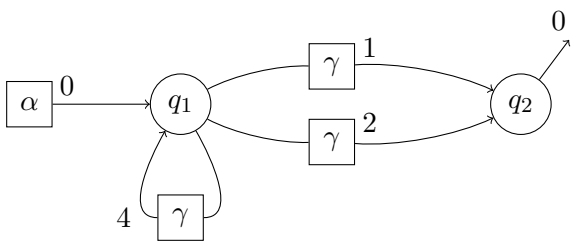


Figure 3.5: The WTA $\mathscr{D}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{N}_{\leq 5})$ from Example 3.35. $\mathscr{D}$ has the TP, but not the ETP. We have chosen the relational depiction of $\mathscr{D}$ for an easier identification of the R-runs.

1. If $\zeta = x_1$, then $y_2 = y_2' = 0$ and hence $d_\Gamma(y_1 + y_2, y_1' + y_2') = d_\Gamma(y_1, y_1')$.
2. If $\zeta \neq x_1$, then $q = q' = q_1$, $y_1 = y_1' = 4 \cdot k$, and $y_2 = y_2' = 4 \cdot \ell$. In particular, $d_\Gamma(y_1 + y_2, y_1' + y_2') = 0 = d_\Gamma(y_1, y_1')$.

However, $\mathscr{D}$ does not have the ETP, which can be seen as follows. Let $\xi = \alpha$ and $\zeta = \eta = \gamma(x_1)$. We consider the R-runs ρ_1 and ρ_2 of $\mathscr{D}$ on $\eta[\zeta[\xi]]$ given by

$$\rho_1(\varepsilon) = (q_1, q_2, 1),\ \rho_2(\varepsilon) = (q_1, q_2, 2),$$
$$\rho_1(1) = \rho_2(1) = (q_1, q_1, 4), \text{ and } \rho_1(11) = \rho_2(11) = (q_1, 0).$$

Using the notation from Definition 3.32 we obtain $y_1 = y_1' = 0$, $y_2 = y_2' = 4$, $y_3' = 1$, and $y_3' = 2$. We compute

$$y_1 + y_3 = 0 +_5 1 = 1 \qquad y_1 + y_2 + y_3 = 0 +_5 4 +_5 1 = 5$$
$$y_1' + y_3' = 0 +_5 2 = 2 \qquad y_1' + y_2' + y_3' = 0 +_5 4 +_5 2 = 5$$

and

$$d_\Gamma(0 +_5 1, 0 +_5 2) = 1 \neq 0 = d_\Gamma(0 +_5 4 +_5 1, 0 +_5 4 +_5 2).$$

This shows that $\mathscr{D}$ does not have ETP.

We note that $\text{maxrk}(\Sigma) = 1$ and $\mathbb{N}_{\leq k}$ is finite. Hence, TP and ETP are generally not equivalent, not even for weighted word automata and finite monoids. ◀

We conclude this chapter by showing that weighted tree automata having the ETP also have the property that the weights of any two R-runs on the same input are close with respect to d_Γ.

Definition 3.36. We define the two constants

$$M_{\mathscr{B}} = \max\{|m|_\Gamma \mid m \in \mathbb{M}\text{-im}(\mathscr{B})\} \quad \text{and} \quad N_{\mathscr{B}} = 2 \cdot M_{\mathscr{B}} \cdot \text{maxrk}(\Sigma)^{(\#Q^2+2)}. \quad \blacktriangleleft$$

Lemma 3.37. Let $\mathscr{B} \vDash \text{ETP}$. For every $\xi \in \mathrm{T}_\Sigma$ and every two R-runs of the respective form $\xrightarrow{\xi|y} q$ and $\xrightarrow{\xi|y'} q'$ of $\mathscr{B}$ on ξ it holds that

$$d_\Gamma(y, y') < N_{\mathscr{B}}.$$

Proof. Let $\xi \in \mathrm{T}_\Sigma$, ρ be an R-run of the form $\xrightarrow{\xi|y} q$, ρ' be an R-run of the form $\xrightarrow{\xi|y'} q'$, and denote $n = \text{size}(\xi)$. First, we assume that $n \leq \text{maxrk}(\Sigma)^{(\#Q^2+1)}$. There exist two families $(y_i \in \mathbb{M}\text{-im}(\mathscr{B}) \mid i \in [n])$ and $(y'_i \in \mathbb{M}\text{-im}(\mathscr{B}) \mid i \in [n])$ such that $y = y_1 \odot \cdots \odot y_n$ and $y' = y'_1 \odot \cdots \odot y'_n$. Since all y_i and y'_i are in $\mathbb{M}\text{-im}(\mathscr{B})$, it holds that $|y_i|_\Gamma \leq M_{\mathscr{B}}$ and $|y'_i|_\Gamma \leq M_{\mathscr{B}}$ for every $i \in [n]$. We obtain

$$d_\Gamma(y, y') \overset{\star_1}{\leq} |y|_\Gamma + |y'|_\Gamma \overset{\star_2}{\leq} \sum_{i=1}^{n} \left(|y_i|_\Gamma + |y'_i|_\Gamma\right) \overset{\star_3}{\leq} 2 \cdot M_{\mathscr{B}} \cdot n < N_{\mathscr{B}}.$$

Equations $\star_1$ and $\star_2$ follow from Inequalities (i) and (ii) from Lemma 3.6, respectively. For Equation $\star_3$, we use that $|y_i|_\Gamma \leq M_{\mathscr{B}}$ and $|y'_i|_\Gamma \leq M_{\mathscr{B}}$ for every $i \in [n]$.

We prove the lemma by induction on n. The induction base $n = 1$ follows from our above argument. Let $n \in \mathbb{N}$ and assume that the claim holds for every $n' < n$. If $n \leq \text{maxrk}(\Sigma)^{(\#Q^2+1)}$, then our above argument yields the claim for n. Now, let $n > \text{maxrk}(\Sigma)^{(\#Q^2+1)}$, which implies $\text{height}(\xi) > \#Q^2$.

We note that ρ and ρ' contain a loop on the same part of ξ. This follows from the pigeonhole principle since $\text{height}(\xi) > \#Q^2$. Formally, there exists $\xi' \in \mathrm{T}_\Sigma$, $\zeta, \zeta' \in \mathrm{C}_\Sigma$, weights $y_a, y_b, y_c, y'_a, y'_b, y'_c \in \mathbb{M}$, and states $p, p' \in Q$ such that $\xi = \zeta'[\zeta[\xi']]$, $\zeta \neq x_1$, and ρ and ρ' be of the form $\xrightarrow{\xi'|y_a} p \xrightarrow{\zeta|y_b} p \xrightarrow{\zeta'|y_c} q$ and $\xrightarrow{\xi'|y'_a} p' \xrightarrow{\zeta|y'_b} p' \xrightarrow{\zeta'|y'_c} q'$, respectively.

We obtain

$$d_\Gamma(y, y') = d_\Gamma(y_a \odot y_b \odot y_c, y'_a \odot y'_b \odot y'_c) \overset{\star_1}{=} d_\Gamma(y_a \odot y_c, y'_a \odot y'_c) \overset{\star_2}{<} N_{\mathscr{B}},$$

where Equation $\star_1$ holds since $\mathscr{B} \vDash \text{ETP}$ and Equation $\star_2$ holds by the induction hypothesis (applied to $n' = \text{size}(\zeta'[\xi']) < n$). $\square$

3.5 Sequentialisation of Weighted Tree Automata over $\mathcal{M}_{\text{fin}}(\mathbb{M})$

We have now collected the algebraic formalisms to do factorisations in monoids and deepened our understanding of multiset-weighted tree automata. This provides all the necessary tools to dive into our $\mathbb{M}$-sequentialisation construction. In this chapter, we carry out Step (II) of our $\mathbb{M}$-sequentialisation construction (see Chapter 3.1).

In particular, given a WTA $\mathscr{B}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$, we define an equivalent sequential WTA $D_{\mathscr{B}}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$ that is not necessarily finite. The automaton $D_{\mathscr{B}} = (Q', T', \text{final}')$ is given by a weighted power set construction with factorisation (cf. Definition 3.39). The states of $D_{\mathscr{B}}$ are maps of the form $X\colon Q \to \mathcal{M}_{\text{fin}}(\mathbb{M})$. If X is reached in $D_{\mathscr{B}}$ by reading an input tree $\xi \in \mathrm{T}_{\Sigma}$, then X is a "record" of the R-run weights of $\mathscr{B}$ on ξ (cf. Lemma 3.43). The transition weights of $D_{\mathscr{B}}$ are generated as follows. Given a symbol $\sigma \in \Sigma$ with rank $s = \text{rk}(\sigma)$ and states $X_1, \ldots, X_s$ of $D_{\mathscr{B}}$, we define the unfactorised successor state $\mathcal{T}_\sigma(X_1, \ldots, X_s)$ by applying the transition weight map T_σ to $X_1, \ldots, X_s$. Next, we let (y, Y) be the minimal f-centering factorisation of $\mathcal{T}_\sigma(X_1, \ldots, X_s)$ and obtain the transition weight $T'_\sigma(X_1, \ldots, X_s, Y) = y$. This step uses the fact that $\mathbb{M}$ admits centering factorisations.

> *Throughout the rest of Chapter 3.5, we assume* $(\mathbb{M}, \odot, 1)$ *to be an arbitrary finitely generated monoid with finite generating set* Γ *such that* $\mathbb{M}$ *divides* Γ*-monotone and admits* f*-centering factorisations (for some strongly monotone map* $f\colon \mathbb{N} \to \mathbb{N}$*). Moreover, we assume* $\mathscr{B} = (Q, T, \text{final})$ *to be an arbitrary finite* WTA *over* Σ *and* $\mathcal{M}_{\text{fin}}(\mathbb{M})$*.*

3.5.1 The Sequentialisation Construction

First, we lift the definitions of divisors, quotients, minimal quotients, minimising divisors, and centering factorisations from $\mathcal{M}_{\text{fin}}(\mathbb{M})$ to maps of type $Q \to \mathcal{M}_{\text{fin}}(\mathbb{M})$.

Definition 3.38. Let $X\colon Q \to \mathcal{M}_{\text{fin}}(\mathbb{M})$ be a map. We define $|X|_\Gamma = \max_{q \in Q} |X(q)|_\Gamma$ and

$$\text{div}(X) = \text{div}(\bigcup_{q \in Q} X(q)).$$

Moreover, for every $y \in \operatorname{div}(X)$ we define

$$\begin{aligned}
\operatorname{quot}_y(X) &= \{Y \colon Q \to \mathcal{M}_{\mathrm{fin}}(\mathbb{M}) \mid \forall q \in Q \colon Y(q) \in \operatorname{quot}_y(X(q))\},\\
\operatorname{minquot}_y(X) &= \{Y \colon Q \to \mathcal{M}_{\mathrm{fin}}(\mathbb{M}) \mid \forall q \in Q \colon Y(q) \in \operatorname{minquot}_y(X(q))\}, \text{ and}\\
\operatorname{mindiv}(X) &= \operatorname*{argmin}_{y \in \operatorname{div}(X)} \min_{Y \in \operatorname{minquot}_y(X)} |Y|_\Gamma .
\end{aligned}$$

Finally, we define $\operatorname{minCenterFact}(X, f)$ as the pair (y, Y) such that $y \in \operatorname{mindiv}(X)$, $Y \in \operatorname{minquot}_y(X)$, and $(y, \bigcup_{q \in Q} Y(q)) = \operatorname{minCenterFact}(\bigcup_{q \in Q} X(q), f)$. ◀

Definition 3.39. We define the WTA $D_{\mathscr{B}} = (Q', T', \mathrm{final}')$ over Σ and $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$ as follows. The set of states of $D_{\mathscr{B}}$ is $Q' = \mathcal{M}_{\mathrm{fin}}(\mathbb{M})^Q$, the final weight map is given for every $X \in Q'$ by

$$\mathrm{final}'(X) = \{\!\!\{ m \odot z \mid q \in Q, m \in X(q), z \in \mathrm{final}(q) \}\!\!\},$$

and the transition weights are constructed as follows. For every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and states $X_1, \ldots, X_s \in Q'$, we define the unfactorised successor state $\mathcal{T}_\sigma(X_1, \ldots, X_s) \colon Q \to \mathcal{M}_{\mathrm{fin}}(\mathbb{M})$ for every $q \in Q$ by

$$\begin{aligned}
\mathcal{T}_\sigma(X_1, \ldots, X_s)(q) = \{\!\!\{ & m_1 \odot \cdots \odot m_s \odot y \mid q_1, \ldots, q_s \in Q,\\
& m_1 \in X_1(q_s), \ldots, m_s \in X_s(q_s), y \in T_\sigma(q_1, \ldots, q_s, q) \}\!\!\}.
\end{aligned}$$

Moreover, we define $(y_\sigma^{X_1,\ldots,X_s}, Y_\sigma^{X_1,\ldots,X_s}) = \operatorname{minCenterFact}(\mathcal{T}_\sigma(X_1, \ldots, X_s), f)$, which exists since $\mathbb{M}$ admits f-centering factorisations.

Now, T' is given for every $s \in \mathbb{N}$, $\sigma \in \Sigma$, and $X_1, \ldots, X_s, X \in Q'$ by

$$T'_\sigma(X_1, \ldots, X_s, X) = \begin{cases} \{\!\!\{ y_\sigma^{X_1,\ldots,X_s} \}\!\!\} & \text{if } X = Y_\sigma^{X_1,\ldots,X_s} \\ \emptyset & \text{otherwise.} \end{cases}$$ ◀

Remark 3.40. We note that $D_{\mathscr{B}}$ is indeed sequential and thus finite-run. In fact, for every $\xi \in \mathrm{T}_\Sigma$ there exists a unique R-run of $D_{\mathscr{B}}$ on ξ. This follows directly from the construction.

We consider the case where $\mathcal{T}_\sigma(X_1, \ldots, X_s)(q) = \emptyset$ for every $q \in Q$ and note what happens during the factorisation of this empty unfactorized state. It clearly holds that $|\mathcal{T}_\sigma(X_1, \ldots, X_s)|_\Gamma = 0$, $\operatorname{div}(\mathcal{T}_\sigma(X_1, \ldots, X_s)) = \mathbb{M}$, and

$$\operatorname{quot}_y(\mathcal{T}_\sigma(X_1, \ldots, X_s)) = \operatorname{minquot}_y(\mathcal{T}_\sigma(X_1, \ldots, X_s)) = \{\mathcal{T}_\sigma(X_1, \ldots, X_s)\}$$

for every $y \in \mathbb{M}$. Therefore, $\text{mindiv}(\mathcal{T}_\sigma(X_1, \ldots, X_s)) = \mathbb{M}$. Since $1 = \min_{\leq_\Gamma} \mathbb{M}$, it holds that $(1, \mathcal{T}_\sigma(X_1, \ldots, X_s)) = \text{minCenterFact}(\mathcal{T}_\sigma(X_1, \ldots, X_s), f)$. That is, our factorisation keeps the empty state unchanged with the minimising divisor 1. ◀

Example 3.41. We continue Example 3.33 by constructing the sequential WTA $D_{\mathscr{B}} = (Q', T', \text{final}')$. More precisely, we construct the reachable part of $D_{\mathscr{B}}$ in a procedural manner by exploring the state space and the transitions of $D_{\mathscr{B}}$ using the definition of T'. We recall from Example 3.11 that $(\mathbb{N}, +, 0)$ admits $\text{id}_{\mathbb{N}}$-centering factorisations.

Moreover, we denote $X_0 = (q_1 \mapsto \{\!\{0\}\!\}, q_2 \mapsto \emptyset, q_3 \mapsto \emptyset)$.

We begin by exploring transitions with input symbol α. Surely, we have that $\mathcal{T}_\alpha() = (q_1 \mapsto \{\!\{1\}\!\}, q_2 \mapsto \emptyset, q_3 \mapsto \emptyset)$. In order to obtain the non-vanishing transition weight of the form $T'_\alpha(X)$, we calculate $\text{minCenterFact}(\mathcal{T}_\alpha(), \text{id}_{\mathbb{N}})$. It is clear that

$$\text{div}(\mathcal{T}_\alpha()) = \{0, 1\},$$

as $0 + \mathcal{T}_\alpha() = \mathcal{T}_\alpha()$ and $1 + X_0 = \mathcal{T}_\alpha()$. Moreover, these are the only respective quotients, which implies

$$\text{minquot}_0(\mathcal{T}_\alpha()) = \{\mathcal{T}_\alpha()\} \qquad \text{and} \qquad \text{minquot}_1(\mathcal{T}_\alpha()) = \{X_0\}.$$

In order to find minimising divisors, we calculate the Γ-length of these quotients. It holds that $|\mathcal{T}_\alpha()|_\Gamma = 1$ and $|X_0|_\Gamma = 0$ and hence

$$\text{mindiv}(\mathcal{T}_\alpha()) = \{1\}.$$

We obtain $(1, X_0) = \text{minCenterFact}(\mathcal{T}_\alpha(), \text{id}_{\mathbb{N}})$ since $\mathbb{M}$ admits $\text{id}_{\mathbb{N}}$-centering factorisations. Therefore, we have determined the transition weight

$$T'_\alpha(X_0) = \{\!\{1\}\!\}.$$

Next, we explore all transitions starting in X_0 and note that $\text{final}'(X_0) = \{\!\{0\}\!\}$.

We begin with the tranisition starting in X_0 with input symbol γ. We have $\mathcal{T}_\gamma(X_0) = (q_1 \mapsto \{\!\{1\}\!\}, q_2 \mapsto \emptyset, q_3 \mapsto \emptyset)$. From our previous calculations for $\mathcal{T}_\alpha()$ we obtain $T'_\gamma(X_0, X_0) = \{\!\{1\}\!\}$.

We continue with the transition starting in (X_0, X_0) with input symbol σ. We have $\mathcal{T}_\sigma(X_0, X_0) = (q_1 \mapsto \emptyset, q_2 \mapsto \{\!\{1\}\!\}, q_3 \mapsto \emptyset)$. In order to obtain the non-vanishing transition weight of the form $T'_\sigma(X_0, X_0, X)$, we calculate $\mathrm{minCenterFact}(\mathcal{T}_\sigma(X_0, X_0), \mathrm{id}_\mathbb{N})$. It is clear that

$$\mathrm{div}(\mathcal{T}_\sigma(X_0, X_0)) = \{0, 1\},$$

as $0 + \mathcal{T}_\sigma(X_0, X_0) = \mathcal{T}_\sigma(X_0, X_0)$ and $1 + X'_0 = \mathcal{T}_\sigma(X_0, X_0)$, where

$$X'_0 = (q_1 \mapsto \emptyset, q_2 \mapsto \{\!\{0\}\!\}, q_3 \mapsto \emptyset).$$

Moreover, these are the only respective quotients, that is,

$$\mathrm{minquot}_0(\mathcal{T}_\sigma(X_0, X_0)) = \{\mathcal{T}_\sigma(X_0, X_0)\} \quad \text{and} \quad \mathrm{minquot}_1(\mathcal{T}_\sigma(X_0, X_0)) = \{X'_0\}.$$

In order to find minimising divisors, we calculate the Γ-length of these quotients. It holds that $|\mathcal{T}_\sigma(X_0, X_0)|_\Gamma = 1$ and $|X'_0|_\Gamma = 0$ and hence

$$\mathrm{mindiv}(\mathcal{T}_\sigma(X_0, X_0)) = \{1\}$$

Hence $\mathrm{minCenterFact}(\mathcal{T}_\sigma(X_0, X_0), \mathrm{id}_\mathbb{N}) = (1, X'_0)$. Therefore, we have determined the transition weight

$$T'(X_0, X_0, \sigma, X'_0) = \{\!\{1\}\!\}.$$

Next, we explore all unexplored transitions starting in X_0 and X'_0 and note that $\mathrm{final}'(X'_0) = \{\!\{0\}\!\}$. We easily calculate

$$\mathcal{T}_\gamma(X'_0) = (q_1 \mapsto \{\!\{1, 1\}\!\}, q_2 \mapsto \emptyset, q_3 \mapsto \{\!\{1\}\!\}), \text{ and}$$
$$\mathcal{T}_\sigma(X_0, X'_0) = \mathcal{T}_\sigma(X'_0, X_0) = \mathcal{T}_\sigma(X'_0, X'_0) = (q_1 \mapsto \emptyset, q_2 \mapsto \{\!\{1\}\!\}, q_3 \mapsto \emptyset).$$

Using similar calculations as in our explorations starting in X_0, we obtain

$$T'_\gamma(X'_0, X_1) = T'_\sigma(X_0, X'_0, X'_0) = T'_\sigma(X'_0, X_0, X'_0) = T'_\sigma(X'_0, X'_0, X'_0) = \{\!\{1\}\!\},$$

where

$$X_1 = (q_1 \mapsto \{\!\{0, 0\}\!\}, q_2 \mapsto \emptyset, q_3 \mapsto \{\!\{0\}\!\}).$$

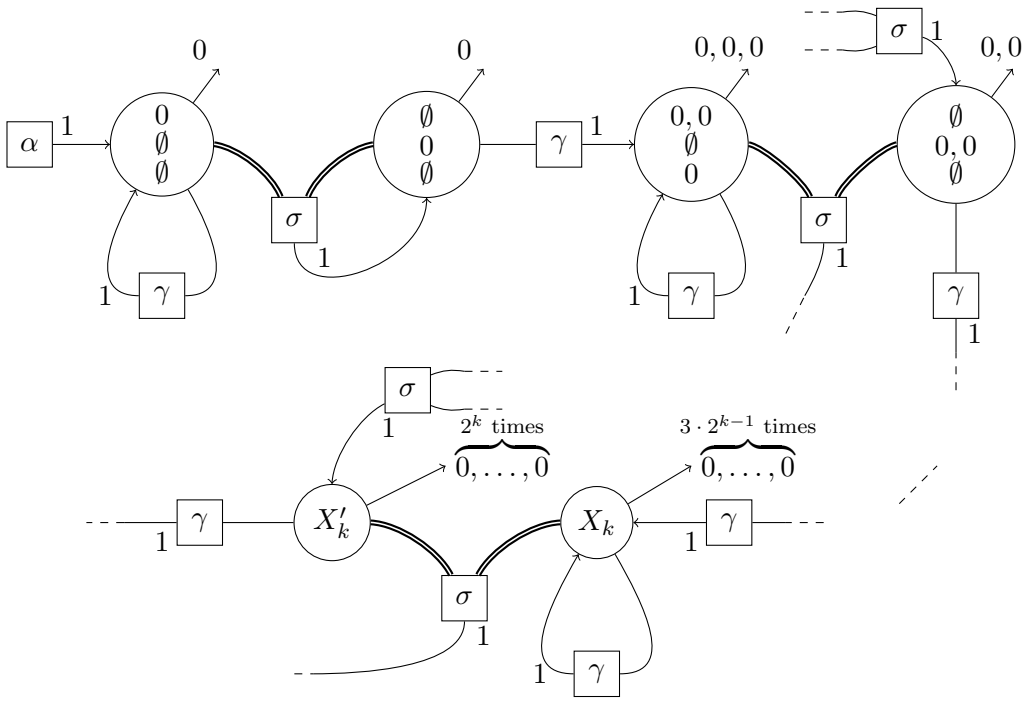


Figure 3.6: Illustration of the sequential WTA $D_{\mathscr{B}}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$.

By continuing in this fashion, we explore the reachable state space and arrive at the following states and transition weights. For every $k \in \mathbb{N}_+$ we define

$$X_k = (q_1 \mapsto \{\!\{\overbrace{0, \ldots, 0}^{2^k \text{ times}}\}\!\}, q_2 \mapsto \emptyset, q_3 \mapsto \{\!\{\overbrace{0, \ldots, 0}^{2^{k-1} \text{ times}}\}\!\}) \text{ and}$$

$$X'_k = (q_1 \mapsto \emptyset, q_2 \mapsto \{\!\{\overbrace{0, \ldots, 0}^{2^k \text{ times}}\}\!\}, q_3 \mapsto \emptyset)$$

and have $\text{final}'(X_k) = \{\!\{\overbrace{0, \ldots, 0}^{3 \cdot 2^{k-1} \text{ times}}\}\!\}$ and $\text{final}'(X'_k) = \{\!\{\overbrace{0, \ldots, 0}^{2^k \text{ times}}\}\!\}$. Moreover, we obtain the following transition weights.

$$T'_\gamma(X_k, X_k) = T'_\gamma(X'_k, X_{k+1}) = \{\!\{1\}\!\} \text{ for every } k \in \mathbb{N}$$

$$T'_\sigma(Y_i, Y_j, X'_{i+j}) = \{\!\{1\}\!\} \text{ for every } i, j \in \mathbb{N}, Y_i \in \{X_i, X'_i\}, Y_j \in \{X_j, X'_j\}$$

A graphical illustration of the reachable part of $D_{\mathscr{B}}$ can be found in Figure 3.6. Since $D_{\mathscr{B}}$ has many non-vanishing transition weights even within the first few states, we have depicted only some σ-transitions and many of those only partially using dotted lines.

We have also removed the curly braces from all multisets to aid readability. Surely, $D_{\mathscr{B}}$ is sequential. ◀

Definition 3.42. Let $\xi \in \mathrm{T}_\Sigma$ and $q \in Q$. We define the multiset

$$\mathrm{W}_\xi^{\mathscr{B}}(q) = \{\!\{ y \mid \xrightarrow{\xi|y} q \in \mathscr{B} \}\!\}. \qquad ◀$$

Lemma 3.43. Let $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ be commutative. Moreover, let $\xi \in \mathrm{T}_\Sigma$ and let the form of the unique R-run of $D_{\mathscr{B}}$ on ξ be $\xrightarrow{\xi|y} X$. For every $q \in Q$ it holds that

$$X(q) \in \mathrm{quot}_y(\mathrm{W}_\xi^{\mathscr{B}}(q)).$$

Proof. We prove the claim by structural induction on ξ. Assume that $\xi = \sigma(\xi_1, \ldots, \xi_s)$ such that the claim is proven for ξ_i for every $i \in [s]$. Let the form of the unique R-run of $D_{\mathscr{B}}$ on ξ_i be $\xrightarrow{\xi_i|y_i} X_i$ for every $i \in [s]$. By the induction assumption it holds that

$$X_i(q) \in \mathrm{quot}_{y_i}(\mathrm{W}_{\xi_i}^{\mathscr{B}}(q))$$

for every $q \in Q$ and $i \in [s]$. Moreover, let $\{\!\{y'\}\!\} = T'_\sigma(X_1, \ldots, X_s, X)$ and note that $y = y_1 \odot \cdots \odot y_s \odot y'$. Furthermore, by the definition of T' it holds that

$$(y', X) = \mathrm{minCenterFact}(\mathcal{T}_\sigma(X_1, \ldots, X_s), f).$$

We obtain $X(q) \in \mathrm{quot}_{y'}(\mathcal{T}_\sigma(X_1, \ldots, X_s)(q))$ for every $q \in Q$. Thus, if we show that

$$\mathcal{T}_\sigma(X_1, \ldots, X_s)(q) \in \mathrm{quot}_{y_1 \odot \cdots \odot y_s}(\mathrm{W}_\xi^{\mathscr{B}}(q)), \tag{3.6}$$

for every $q \in Q$, then we have $X(q) \in \{Y \in \mathrm{quot}_{y'}(Y') \mid Y' \in \mathrm{quot}_{y_1 \odot \cdots \odot y_s}(\mathrm{W}_\xi^{\mathscr{B}}(q))\}$. This lets us conclude $X(q) \in \mathrm{quot}_y(\mathrm{W}_\xi^{\mathscr{B}}(q))$ by Lemma 3.12. An illustration of this proof idea can be found in Figure 3.7.

In order to prove Equation (3.6), we note that, by definition of $\mathcal{T}_\sigma(X')$,

$$\mathcal{T}_\sigma(X_1, \ldots, X_s)(q) = \bigcup_{q_1, \ldots, q_s \in Q} \Big(X_1(q_1) \odot \cdots \odot X_s(q_s) \odot T_\sigma(q_1, \ldots, q_s, q) \Big). \tag{3.7}$$

Moreover, it holds that

$$\begin{aligned}
\mathrm{W}_\xi^{\mathscr{B}}(q) &= \{\!\{ y'' \mid \xrightarrow{\xi|y''} q \in \mathscr{B} \}\!\} \\
&= \bigcup_{q_1, \ldots, q_s \in Q} \{\!\{ y'_1 \odot \cdots \odot y'_s \odot \hat{y} \mid \xrightarrow{\xi_1|y'_1} q_1, \ldots, \xrightarrow{\xi_s|y'_s} q_s \in \mathscr{B}, \hat{y} \in T_\sigma(q_1, \ldots, q_s, q) \}\!\} \\
&= \bigcup_{q_1, \ldots, q_s \in Q} \Big(\mathrm{W}_{\xi_1}^{\mathscr{B}}(q_1) \odot \cdots \odot \mathrm{W}_{\xi_s}^{\mathscr{B}}(q_s) \odot T_\sigma(q_1, \ldots, q_s, q) \Big).
\end{aligned}$$

Therefore, Equation (3.6) holds if and only if

$$\Big(\bigcup_{q_1,\ldots,q_s\in Q}\Big(X_1(q_1)\odot\cdots\odot X_s(q_s)\odot T_\sigma(q_1,\ldots,q_s,q)\Big)\Big)$$
$$\in \operatorname{quot}_{y_1\odot\cdots\odot y_s}\Big(\bigcup_{q_1,\ldots,q_s\in Q}\Big(\mathrm{W}^{\mathscr{B}}_{\xi_1}(q_1)\odot\cdots\odot \mathrm{W}^{\mathscr{B}}_{\xi_s}(q_s)\odot T_\sigma(q_1,\ldots,q_s,q)\Big)\Big),$$

which holds if for every $q_1,\ldots,q_s\in Q$ we have that

$$\Big(X_1(q_1)\odot\cdots\odot X_s(q_s)\odot T_\sigma(q_1,\ldots,q_s,q)\Big)$$
$$\in \operatorname{quot}_{y_1\odot\cdots\odot y_s}\Big(\mathrm{W}^{\mathscr{B}}_{\xi_1}(q_1)\odot\cdots\odot \mathrm{W}^{\mathscr{B}}_{\xi_s}(q_s)\odot T_\sigma(q_1,\ldots,q_s,q)\Big). \tag{3.8}$$

Equation (3.8) holds if $\big(X_1(q_1)\odot\cdots\odot X_s(q_s)\big)\in\operatorname{quot}_{y_1\odot\cdots\odot y_s}\big(\mathrm{W}^{\mathscr{B}}_{\xi_1}(q_1)\odot\cdots\odot \mathrm{W}^{\mathscr{B}}_{\xi_s}(q_s)\big)$ and since $y_i\odot X_i(q_i)=\mathrm{W}^{\mathscr{B}}_{\xi_i}(q_i)$ for every $i\in[s]$ by the induction assumption, we have ultimately determined that Equation (3.6) holds if

$$\big(X_1(q_1)\odot\cdots\odot X_s(q_s)\big)\in\ \operatorname{quot}_{y_1\odot\cdots\odot y_s}\big(y_1\odot X_1(q_1)\odot\cdots\odot y_s\odot X_s(q_s)\big). \tag{3.9}$$

If $\Sigma=\Sigma^{(0)}\cup\Sigma^{(1)}$, then $s\leq 1$ and hence Equation (3.9) is true. If $\mathbb{M}$ is commutative, then $y_1\odot X_1(q_1)\odot\cdots\odot y_s\odot X_s(q_s)=y_1\odot\cdots\odot y_s\odot X_1(q_1)\odot\cdots\odot X_s(q_s)$ and hence Equation (3.9) is true. □

Lemma 3.44. If $\Sigma=\Sigma^{(0)}\cup\Sigma^{(1)}$ or $\mathbb{M}$ is commutative, then $D_{\mathscr{B}}$ is equivalent to $\mathscr{B}$.

Proof. Let $\xi\in\mathrm{T}_\Sigma$ and let the form of the unique R-run of $D_{\mathscr{B}}$ on ξ be $\xrightarrow{\xi|y}X$. It holds that

$$\begin{aligned}
[\![D_{\mathscr{B}}]\!](\xi) &\overset{\star_1}{=} \{\!\{y\odot z\mid X\xrightarrow{z}\in D_{\mathscr{B}}\}\!\}\\
&\overset{\star_2}{=} \{\!\{y\odot m\odot z'\mid q\in Q, m\in X(q), z'\in\mathrm{final}(q)\}\!\}\\
&= \{\!\{m\odot z'\mid q\in Q, m\in(y\odot X(q)), z'\in\mathrm{final}(q)\}\!\}\\
&\overset{\star_3}{=} \{\!\{m\odot z'\mid q\in Q, m\in \mathrm{W}^{\mathscr{B}}_{\xi}(q), z'\in\mathrm{final}(q)\}\!\}\\
&\overset{\star_4}{=} \{\!\{y'\odot z'\mid q\in Q, \xrightarrow{\xi|y'}q\in\mathscr{B}, z'\in\mathrm{final}(q)\}\!\}\\
&\overset{\star_5}{=} \{\!\{y'\odot z'\mid \xrightarrow{\xi|y'}q\xrightarrow{z'}\in\mathscr{B}\}\!\}\overset{\star_6}{=}[\![\mathscr{B}]\!](\xi),
\end{aligned}$$

where Equations $\star_1$ – $\star_5$ can be seen as follows. Equation $\star_1$ follows from Lemma 3.27 and the uniqueness of the R-run of $D_{\mathscr{B}}$ on ξ. Equation $\star_2$ follows from the

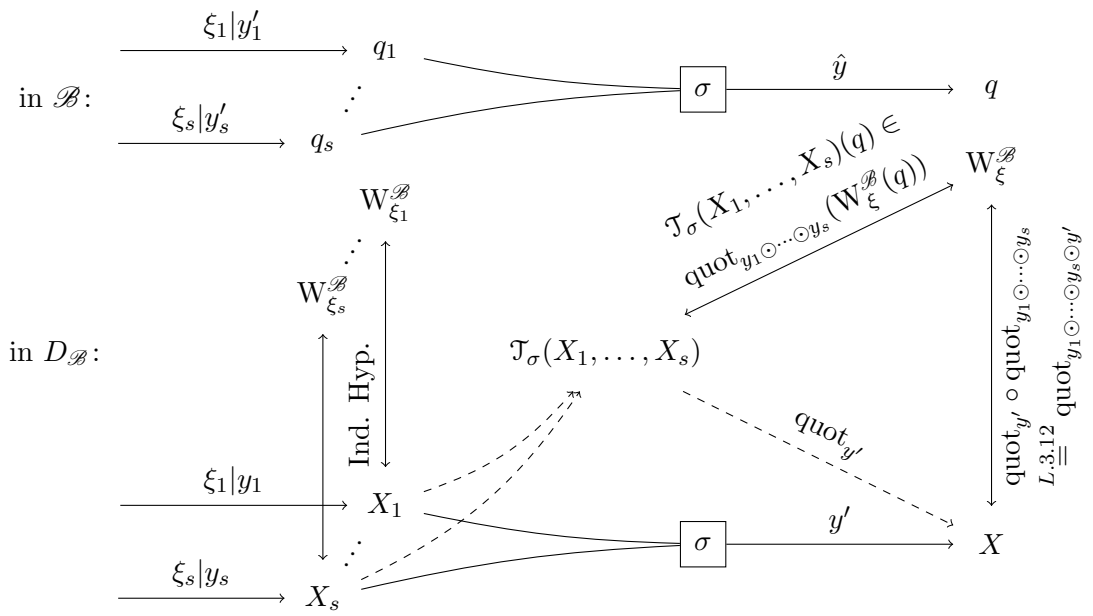


Figure 3.7: Illustration of the proof idea from Lemma 3.43.

definition of final′. Equation $\star_3$ follows from Lemma 3.43. More precisely, the fact that $X(q) \in \mathrm{quot}_y(\mathrm{W}_\xi^{\mathscr{B}}(q))$ implies that $y \odot X(q) = \mathrm{W}_\xi^{\mathscr{B}}(q)$. We note that this uses the assumption that $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ is commutative. Equation $\star_4$ follows from the definition of $\mathrm{W}_\xi^{\mathscr{B}}(q)$. Equation $\star_5$ is an application of our notation for R-runs and final weights (cf. Remark 3.26). Equation $\star_6$ follows from Lemma 3.27. □

Lemma 3.45. Let $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ be commutative. Moreover, let $\mathscr{B} \vDash \mathrm{ETP}$. For every reachable state $X \in Q'$ of $D_{\mathscr{B}}$ and every $m_1 \in X(q)$ for some $q \in Q$ it holds that

$$|m_1|_\Gamma < f(N_{\mathscr{B}}).$$

Proof. Let $X \in Q'$ be a reachable state of $D_{\mathscr{B}}$ and $m_1 \in X(q)$ for some $q \in Q$. The fact that X is reachable implies the existence of $\xi \in \mathrm{T}_\Sigma$ and $y \in \mathbb{M}$ such that the form of the unique R-run of $D_{\mathscr{B}}$ on ξ is $\xrightarrow{\xi|y} X$.

Assume that $\xi = \sigma(\xi_1, \ldots, \xi_s)$ and let $X_1, \ldots, X_s \in Q'$ and $y_1, \ldots, y_s \in \mathbb{M}$ such that the unique R-run of $D_{\mathscr{B}}$ on ξ_i is of the form $\xrightarrow{\xi_i|y_i} X_i$ for every $i \in [s]$. Furthermore, let

$\{\!\{y'\}\!\} = T'_\sigma(X_1, \ldots, X_s, X)$ and recall from Definition 3.39 that

$$(y', X) = \text{minCenterFact}(\mathcal{T}_\sigma(X_1, \ldots, X_s), f).$$

Thus, by the definition of an f-centering factorisation, there exists $q' \in Q$ and $m_2 \in \text{supp}(X(q'))$ such that $|m_1|_\Gamma \leq f(d_\Gamma(m_1, m_2))$.

By Lemma 3.43 it holds that $X(q) \in \text{quot}_y\big(\text{W}_\xi^{\mathscr{B}}(q)\big)$ and $X(q') \in \text{quot}_y\big(\text{W}_\xi^{\mathscr{B}}(q')\big)$. Thus there exists $y'_1 \in \text{supp}(\text{W}_\xi^{\mathscr{B}}(q))$ and $y'_2 \in \text{supp}(\text{W}_\xi^{\mathscr{B}}(q'))$ such that $y'_1 = y \odot m_1$ and $y'_2 = y \odot m_2$. We note that this uses the assumption that $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ is commutative. We obtain

$$|m_1|_\Gamma \leq f(d_\Gamma(m_1, m_2)) \overset{\star_1}{\leq} f(d_\Gamma(y \odot m_1, y \odot m_2)) = f(d_\Gamma(y'_1, y'_2)) \overset{\star_2}{<} f(N_{\mathscr{B}}),$$

where Equation $\star_1$ holds since $\mathbb{M}$ divides Γ-monotone and f is strongly monotone and Equation $\star_2$ follows from Lemma 3.37 and the fact that f is strongly monotone. This concludes the proof. □

Corollary 3.46. Let $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ be commutative. Moreover, let $\mathscr{B} \vDash \text{ETP}$. For every reachable state $X \in Q'$ of $D_{\mathscr{B}}$ and every $q \in Q$ it holds that

$$\text{supp}(X(q)) \subseteq \mathcal{B}_{f(N_{\mathscr{B}})}(1).$$

In particular, it holds that $\#\text{supp}(X(q)) < \infty$.

Proof. This follows directly from Lemma 3.45. □

3.5.2 The Finitely R-Ambiguous Case

We have seen that all values occurring in states of $D_{\mathscr{B}}$ are close to 1 if $\mathscr{B}$ satisfies the ETP. However, each occurring value can have arbitrarily large multiplicity, which allows the set of reachable states of $D_{\mathscr{B}}$ to be infinite nonetheless. Fortunately, if $\mathscr{B}$ is finitely R-ambiguous, then the size of all reachable states of $D_{\mathscr{B}}$ is bounded (recall from Lemma 3.43 that the size of a reachable state of $D_{\mathscr{B}}$ equals the number of R-runs of $D_{\mathscr{B}}$ on some tree $\xi \in \text{T}_\Sigma$) and hence, the reachable part of $D_{\mathscr{B}}$ is finite.

Nonetheless, even if the reachable part of $D_{\mathscr{B}}$ is infinite, then there are still cases for which our $\mathbb{M}$-sequentialisation construction works, as we will see in Chapter 3.7.

Corollary 3.47. Let $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ be commutative. Moreover, let $\mathscr{B} \vDash \mathrm{ETP}$. If $\mathscr{B}$ is finitely R-ambiguous, then the set of reachable states of $D_{\mathscr{B}}$ is finite.

In particular, if $\mathscr{B}$ is finitely R-ambiguous, then $\mathscr{B}$ is sequentialisable.

Proof. We recall that since $\mathscr{B}$ is finitely R-ambiguous, there exists $k \in \mathbb{N}$ such that

$$\#\,\mathrm{R\text{-}Runs}_{\mathscr{B}}(\xi) \leq k$$

for every $\xi \in \mathrm{T}_{\Sigma}$. Moreover, we define the set

$$K = \{M \in \mathcal{M}(\mathcal{B}_{f(N_{\mathscr{B}})}(1)) \mid \#M \leq k\}.$$

It is clear that K, and thus also K^Q, is finite. Let $X \in Q'$ be reachable state of $D_{\mathscr{B}}$. We will show that $X \in K^Q$, which proves the claim.

By the reachability of X, there exists $\xi \in \mathrm{T}_{\Sigma}$, $y \in \mathbb{M}$, and an R-run of the form $\xrightarrow{\xi|y} X$ of $D_{\mathscr{B}}$. Moreover, let $q \in Q$. By the finite R-ambiguity of $\mathscr{B}$ it holds that $\#\mathrm{W}^{\mathscr{B}}_{\xi}(q) \leq k$. By Lemma 3.43, we have $X(q) \in \mathrm{quot}_y(\mathrm{W}^{\mathscr{B}}_{\xi}(q))$ and thus also $\#X(q) \leq k$. Moreover, by Corollary 3.46, we obtain that $X(q) \in \mathcal{M}(\mathcal{B}_{f(N_{\mathscr{B}})}(1))$. This proves that $X \in K^Q$. $\square$

Example 3.48. We illustrate Corollary 3.47 by sequentialising the finitely R-ambiguous WTA $\mathscr{C}$ from Example 3.30. More precisely, we construct the reachable part of $D_{\mathscr{C}} = (Q', T', \mathrm{final}')$ in a procedural manner by exploring the state space and the transitions of $D_{\mathscr{C}}$ using the definition of T'. We recall that $\mathscr{C}$ is a WTA over $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$, where $\mathbb{M} = (\langle \Gamma \rangle., \cdot, 1)$ and $\Gamma = \{2, 3\}$. Moreover, we have seen in Example 3.30 that $\mathbb{M}$ admits f-centering factorisations, where $f(n) = 2 \cdot n$ for every $n \in \mathbb{N}$.

We begin by exploring transitions with input symbol α. Surely, we have that $\mathcal{T}_{\alpha}() = (q_1 \mapsto \{\!\{1\}\!\}, q_2 \mapsto \{\!\{2\}\!\}, q_3 \mapsto \emptyset)$. Using similar calculations as in Example 3.41, we obtain the transition weight

$$T'_{\alpha}(X_0) = \{\!\{1\}\!\},$$

where $X_0 = (q_1 \mapsto \{\!\{1\}\!\}, q_2 \mapsto \{\!\{2\}\!\}, q_3 \mapsto \emptyset)$.

Next, we explore all transitions starting in X_0 and note that $\text{final}'(X_0) = \emptyset$. We have

$$\mathcal{T}_\gamma(X_0) = (q_1 \mapsto \emptyset, q_2 \mapsto \{\!\{2, 6\}\!\}, q_3 \mapsto \emptyset) \text{ and}$$
$$\mathcal{T}_\sigma(X_0, X_0) = (q_1 \mapsto \emptyset, q_2 \mapsto \emptyset, q_3 \mapsto \{\!\{8, 12\}\!\}).$$

Again, using similar calculations as in Example 3.41, we obtain the transition weights $T'_\gamma(X_0, X_1) = \{\!\{2\}\!\}$ and $\mathrm{T}'_\sigma(X_0, X_0, X_2) = \{\!\{4\}\!\}$, where

$$X_1 = (q_1 \mapsto \emptyset, q_2 \mapsto \{\!\{1, 3\}\!\}, q_3 \mapsto \emptyset) \text{ and}$$
$$X_2 = (q_1 \mapsto \emptyset, q_2 \mapsto \emptyset, q_3 \mapsto \{\!\{2, 3\}\!\}).$$

By continuing in this fashion, we explore only two more states, namely

$$X_3 = (q_1 \mapsto \emptyset, q_2 \mapsto \emptyset, q_3 \mapsto \{\!\{2, 3, 6, 9\}\!\}) \text{ and}$$
$$X_4 = (q_1 \mapsto \emptyset, q_2 \mapsto \emptyset, q_3 \mapsto \emptyset),$$

the final weights $\text{final}'(X_1) = \emptyset$, $\text{final}'(X_2) = \{\!\{6, 9\}\!\}$, $\text{final}'(X_3) = \{\!\{6, 9, 18, 27\}\!\}$, $\text{final}'(X_4) = \emptyset$, and the following further transition weights.

$$T'_\gamma(X_1, X_1) = T'_\gamma(X_2, X_2) = T'_\gamma(X_3, X_3) = \{\!\{3\}\!\},\ T'_\gamma(X_4, X_4) = \{\!\{1\}\!\},$$
$$T'_\sigma(X_0, X_1, X_3) = \{\!\{2\}\!\},\ T'_\sigma(X_1, X_0, X_4) = \{\!\{1\}\!\}, \text{ and}$$
$$T'_\sigma(X_i, X_j, X_4) = \{\!\{1\}\!\}\ \forall i, j \in \{0, 1, 2, 3\} \text{ s.th. } i + j \geq 2.$$

In particular, the reachable state space of $D_{\mathscr{C}}$ equals $\{X_0, X_1, X_2, X_3, X_4\}$, which is a finite set, as predicted by Corollary 3.47. We note that X_4 acts as a "sink" state with an empty final weight and no outgoing non-vanishing transition. An illustration of the reachable part of $D_{\mathscr{C}}$ can be found in Figure 3.8. To aid readability, we omitted the sink state X_4 and the curly braces of all multiset weights. ◀

3.6 Relating WTA over $\mathcal{M}_{\text{fin}}(\mathbb{M})$ and WTA over S

We will now execute Step (I) of our $\mathbb{M}$-sequentialisation construction (see Chapter 3.1) using a concept of relatedness of weighted tree automata. This allows us to translate

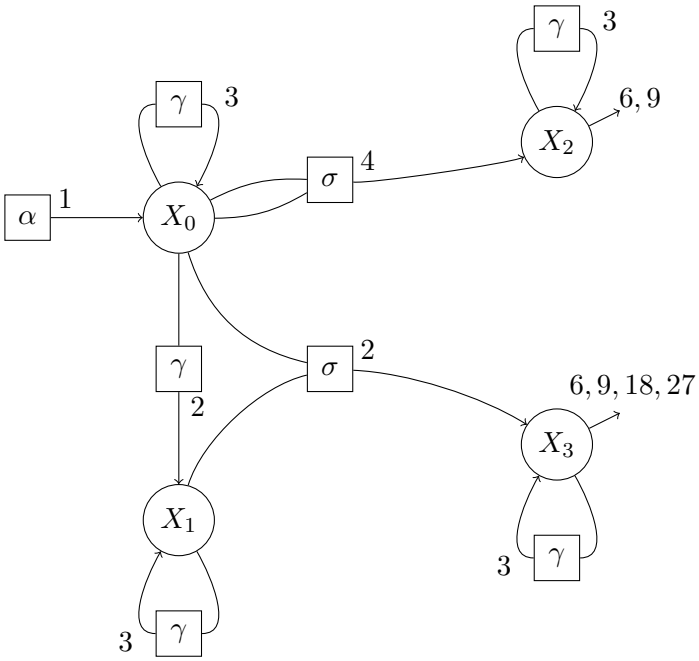


Figure 3.8: Illustration of the sequential finite WTA $D_{\mathscr{B}_1}$ over $\{a, b\}$ and $\mathcal{M}_{\text{fin}}(\langle\{2, 3\}\rangle.)$ from Example 3.48. We have omitted the sink state X_4 to aid readability.

our $\mathcal{S}(\mathbb{M})$-sequentialisation results for WTA over $\mathcal{M}_{\text{fin}}(\mathbb{M})$ (see Chapter 3.5) to WTA over an arbitrary $\mathbb{M}$-semiring S.

More precisely, given a WTA $\mathscr{A}$ over Σ and S and a WTA $\mathscr{B}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$, we say that $\mathscr{A}$ and $\mathscr{B}$ are related if $\mathscr{A}$ and $\mathscr{B}$ have the same sets of states and every transition weight and final weight of $\mathscr{A}$ equals the evaluation of the respective multiset weight of $\mathscr{B}$ via $\oplus$ in S (cf. Definition 3.51). Moreover, we will introduce the notion of strong relatedness which expresses that, in addition to relatedness, the sets of runs of $\mathscr{A}$ and $\mathscr{B}$ have certain combinatorial similarities. We will use these combinatorial similarities in Chapter 3.7.

Throughout the rest of Chapter 3.6, we assume $(S, \oplus, \odot, 0, 1)$ to be an arbitrary $\mathbb{M}$-semiring for some finitely generated monoid $\mathbb{M}$ with finite generating set Γ.

Definition 3.49. We define the operator $[\![\_]\!]_S \colon \mathcal{M}_{\text{fin}}(\mathbb{M}) \to S$ where for every multiset

$M \in \mathcal{M}_{\text{fin}}(\mathbb{M})$ we let

$$[\![M]\!]_S = \bigoplus_{m \in M} m.$$

For every set A, we can naturally extend $[\![\_]\!]_S$ to a map $[\![\_]\!]_S \colon (\mathcal{M}_{\text{fin}}(\mathbb{M}))^A \to S^A$ where for every $L \colon A \to \mathcal{M}_{\text{fin}}(\mathbb{M})$ and $a \in A$ we let

$$[\![L]\!]_S(a) = [\![L(a)]\!]_S$$

Furthermore, $[\![\_]\!]_S$ naturally extends to subsets of $(\mathcal{M}_{\text{fin}}(\mathbb{M}))^A$ by elementwise application. ◀

Lemma 3.50. Let $\mathscr{B} = (Q, T, \text{final})$ be a finite-run WTA over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$. For every $\xi \in \mathrm{T}_\Sigma$ it holds that

$$[\![[\![\mathscr{B}]\!]]\!]_S(\xi) = \bigoplus_{\xrightarrow{\xi|y} q \xrightarrow{z}} y \odot z.$$

Proof. This follows immediately from Lemma 3.27 and the definition of $[\![\cdot]\!]_S$. □

Definition 3.51. Let $\mathscr{A} = (Q, T, \text{final})$ and $\mathscr{B} = (Q', T', \text{final}')$ be a WTA over Σ and S and a WTA over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$, respectively. We say that $\mathscr{A}$ *and* $\mathscr{B}$ *are related* if $Q = Q'$ and for every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q$ it holds that

$$T_\sigma(q_1, \ldots, q_s, q) = [\![T'_\sigma(q_1, \ldots, q_s, q)]\!]_S \qquad \text{and} \qquad \text{final}(q) = [\![\text{final}'(q)]\!]_S.$$

Moreover, we say that $\mathscr{A}$ *and* $\mathscr{B}$ *are strongly related* if $\mathscr{A}$ and $\mathscr{B}$ are related and for every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q$ it holds that

(a) if $T_\sigma(q_1, \ldots, q_s, q) = 0$, then $T'_\sigma(q_1, \ldots, q_s, q) = \emptyset$ and

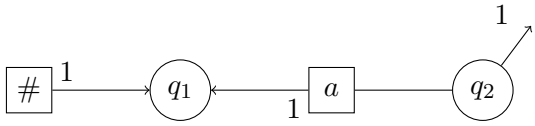
(b) if $T_\sigma(q_1, \ldots, q_s, q) \in \mathbb{M} \setminus \{0\}$, then $T'_\sigma(q_1, \ldots, q_s, q) = \{\!\{T_\sigma(q_1, \ldots, q_s, q)\}\!\}$.

Given a WTA $\mathscr{A}$ over Σ and S, we denote the set of WTAs over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$ that are related (or strongly related) to $\mathscr{A}$ by $\text{Rel}(\mathscr{A})$ (or $\text{StrongRel}(\mathscr{A})$, respectively). Analogously, given a WTA $\mathscr{B}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$, we denote the set of WTAs over Σ and S that are related (or strongly related) to $\mathscr{B}$ by $\text{Rel}(\mathscr{B})$ (or $\text{StrongRel}(\mathscr{B})$, respectively). ◀

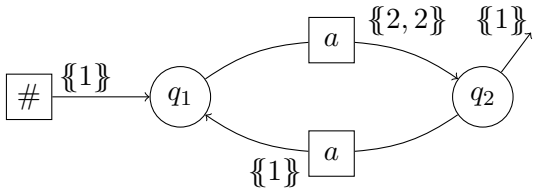
Remark 3.52. Let $\mathscr{A}$ and $\mathscr{B}$ be as in Definition 3.51 and moreover, let $\mathscr{A}$ and $\mathscr{B}$ be strongly related. Let $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q$. We note that $(q_1, \ldots, q_s, \sigma, q)$ is a transition in a loop of $\mathscr{A}$ if and only if $(q_1, \ldots, q_s, \sigma, q)$ is a transition in a loop of $\mathscr{B}$. This follows from the fact that $\mathscr{A}$ and $\mathscr{B}$ have the same graph structure.

We note that this is not true if $\mathscr{A}$ and $\mathscr{B}$ are only related. More precisely, if $\mathscr{A}$ and $\mathscr{B}$ are only related, then $\mathscr{B}$ can have non-vanishing transition weights that correspond to vanishing transition weights in $\mathscr{A}$. In particular, $\mathscr{B}$ can have valid runs that are non-valid runs of $\mathscr{A}$. We illustrate this fact using the WTA $\hat{\mathscr{A}} = (Q, T, \mathrm{final})$ over $\{\#^{(0)}, a^{(1)}\}$ and $(\mathbb{Z}_4, \oplus_4, \odot_4, 0, 1)$ and the WTA $\hat{\mathscr{B}} = (Q, T', \mathrm{final}')$ over $\{\#^{(0)}, a^{(1)}\}$ and $\mathcal{M}_{\mathrm{fin}}((\mathbb{Z}_4, \odot_4, 1))$. We define the automata by their graphical representation as follows.

$\hat{\mathscr{A}}$:



$\hat{\mathscr{B}}$:



Clearly, $\hat{\mathscr{A}}$ and $\hat{\mathscr{B}}$ are related, since $T_a(q_1, q_2) = 0 = 2 \oplus_4 2 = [\![T'_a(q_1, q_2)]\!]_{\mathbb{Z}_4}$ (and similarly for all other occurring weights). However, they are not strongly related, since condition (a) of strong relatedness is violated (see Definition 3.51). In fact, $\hat{\mathscr{B}}$ has a valid run on $\xi = a(\#)$ while $\hat{\mathscr{A}}$ has none. Moreover, (q_1, σ, q_2) is a transition in a loop of $\hat{\mathscr{B}}$ while it is not a transition in a loop of $\hat{\mathscr{A}}$. ◀

Lemma 3.53. Let $\mathscr{A}$ be a WTA over Σ and S and let $\mathscr{B}$ be a WTA over Σ and $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$. If $\mathscr{A}$ and $\mathscr{B}$ are related, then $[\![\mathscr{A}]\!] = [\![[\![\mathscr{B}]\!]]\!]_S$.

Proof. We denote $\mathscr{A} = (Q, T, \mathrm{final})$ and $\mathscr{B} = (Q, T', \mathrm{final}')$. Let $\xi \in \mathrm{T}_\Sigma$. It holds that

$$\forall q \in Q \colon \mathrm{Runs}^v_{\mathscr{A}}(\xi, q) \subseteq \mathrm{Runs}^v_{\mathscr{B}}(\xi, q), \tag{3.10}$$

which can be proven as follows. Let $q \in Q$ and $\rho \in (Q^{\text{pos}(\xi)} \setminus \text{Runs}^v_{\mathscr{B}}(\xi, q))$. That is, ρ is a run of $\mathscr{B}$ on ξ that is not valid. In particular, there exists $w \in \text{pos}(\xi)$ such that $T'_{\xi(w)}(\rho(w1), \ldots, \rho(ws), \rho(w)) = \emptyset$, where $s = \text{rk}(\xi(w))$. Since $\mathscr{A}$ and $\mathscr{B}$ are related we obtain $T_{\xi(w)}(\rho(w1), \ldots, \rho(ws), \rho(w)) = 0$ and hence, ρ is also not a valid run of $\mathscr{A}$ on ξ. Therefore, $\rho \in (Q^{\text{pos}(\xi)} \setminus \text{Runs}^v_{\mathscr{A}}(\xi, q))$, which concludes the proof of Equation (3.10).

Let $\rho \in \big(\text{Runs}^v_{\mathscr{B}}(\xi, q) \setminus \text{Runs}^v_{\mathscr{A}}(\xi, q)\big)$ for some $q \in Q$. Since ρ is not a valid run of $\mathscr{A}$, there exists $w \in \text{pos}(\xi)$ such that $T_{\xi(w)}(\rho(w1), \ldots, \rho(ws), \rho(w)) = 0$ where $s = \text{rk}(\xi(w))$. Therefore, $[\![T'_{\xi(w)}(\rho(w1), \ldots, \rho(ws), \rho(w))]\!]_S = 0$ by the definition of relatedness and hence $[\![\text{wt}_{\mathscr{B}}(\xi, \rho)]\!]_S = 0$. This proves the following Equation (3.11).

$$\forall q \in Q \colon \forall \rho \in \big(\text{Runs}^v_{\mathscr{B}}(\xi, q) \setminus \text{Runs}^v_{\mathscr{A}}(\xi, q)\big) \colon [\![\text{wt}_{\mathscr{B}}(\xi, \rho)]\!]_S = 0 \tag{3.11}$$

Next we show the following Equation (3.12) by structural induction on ξ.

$$\forall q \in Q \colon \forall \rho \in \text{Runs}_{\mathscr{A}}(\xi, q) \colon \text{wt}_{\mathscr{A}}(\xi, \rho) = [\![\text{wt}_{\mathscr{B}}(\xi, \rho)]\!]_S. \tag{3.12}$$

Assume that $\xi = \sigma(\xi_1, \ldots, \xi_s)$ and that Equation (3.12) holds for $\xi_1, \ldots, \xi_s$. Let $q \in Q$ and $\rho \in \text{Runs}^v_{\mathscr{A}}(\xi, q)$ and denote $q_i = \rho(i)$ for every $i \in [s]$. Moreover, for every $i \in [s]$ let ρ_i be the restriction of ρ to ξ_i, that is, $\rho_i \in \text{Runs}^v_{\mathscr{A}}(\xi_i, q_i)$ such that $\rho_i(w) = \rho(iw)$ for every $w \in \text{pos}(\xi_i)$. We obtain

$$\begin{aligned}
\text{wt}_{\mathscr{A}}(\xi, \rho) &\overset{\star_1}{=} \text{wt}_{\mathscr{A}}(\xi_1, \rho_1) \odot \cdots \odot \text{wt}_{\mathscr{A}}(\xi_s, \rho_s) \odot T_\sigma(q_1, \ldots, q_s, q) \\
&\overset{\star_2}{=} [\![\text{wt}_{\mathscr{A}}(\xi_1, \rho_1)]\!]_S \odot \cdots \odot [\![\text{wt}_{\mathscr{A}}(\xi_s, \rho_s)]\!]_S \odot T_\sigma(q_1, \ldots, q_s, q) \\
&\overset{\star_3}{=} [\![\text{wt}_{\mathscr{A}}(\xi_1, \rho_1)]\!]_S \odot \cdots \odot [\![\text{wt}_{\mathscr{A}}(\xi_s, \rho_s)]\!]_S \odot [\![T'_\sigma(q_1, \ldots, q_s, q)]\!]_S \\
&\overset{\star_4}{=} [\![\text{wt}_{\mathscr{A}}(\xi_1, \rho_1) \odot \cdots \odot \text{wt}_{\mathscr{A}}(\xi_s, \rho_s) \odot T'_\sigma(q_1, \ldots, q_s, q)]\!]_S \overset{\star_5}{=} [\![\text{wt}_{\mathscr{B}}(\xi, \rho)]\!]_S,
\end{aligned}$$

where Equations $\star_1$ and $\star_5$ follow from the definition of the weight of a run, Equation $\star_2$ follows from the induction assumption, Equation $\star_3$ follows from the definition of relatedness, and Equation $\star_4$ follows from the distributivity law. This proves the induction step and therefore concludes the proof of Equation (3.12).

We obtain that for every $\xi \in \mathrm{T}_\Sigma$ it holds that

$$\begin{aligned}
[\![\mathscr{A}]\!](\xi) &= \bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{Runs}^v_{\mathscr{A}}(\xi,q)} \mathrm{wt}_{\mathscr{A}}(\xi,\rho) \odot \mathrm{final}(q) \\
&\overset{\star_1}{=} \bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{Runs}^v_{\mathscr{A}}(\xi,q)} \mathrm{wt}_{\mathscr{A}}(\xi,\rho) \odot [\![\mathrm{final}'(q)]\!]_S \\
&\overset{\star_2}{=} \bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{Runs}^v_{\mathscr{B}}(\xi,q)} [\![\mathrm{wt}_{\mathscr{B}}(\xi,\rho)]\!]_S \odot [\![\mathrm{final}'(q)]\!]_S \\
&\overset{\star_3}{=} \bigoplus_{\rho \in \mathrm{Runs}^v_{\mathscr{B}}(\xi)} [\![\mathrm{wt}_{\mathscr{B}}(\xi,\rho) \odot \mathrm{final}'(\rho(\varepsilon))]\!]_S \overset{\star_4}{=} [\![[\![\mathscr{B}]\!]]\!]_S(w),
\end{aligned}$$

where Equation $\star_1$ follows from relatedness of $\mathscr{A}$ and $\mathscr{B}$, Equation $\star_2$ follows from Equations (3.11) and (3.12), Equation $\star_3$ follows from the distributivity of S, and Equation $\star_4$ follows from the definition of $[\![\_]\!]_S$ and $[\![\mathscr{B}]\!]$. This concludes the proof of the lemma. □

Example 3.54. The concept of relatedness links the WTA $\mathscr{A}$ from Example 3.1 (also cf. Figure 3.1) and the WTA $\mathscr{B}$ from Example 3.21 (also cf. Figure 3.3). In fact, $\mathscr{A}$ and $\mathscr{B}$ are strongly related. This can easily be seen as follows.

For every $q \in Q$ it holds that $\mathrm{final}(q) = \{\!\{\mathrm{final}_{\mathscr{A}}(q)\}\!\}$. Furthermore, for every $s \geq 0$, $\tau \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q$ such that $(T_{\mathscr{A}})_\tau(q_1, \ldots, q_s, q) \in \mathbb{N}$ it holds that $(T_{\mathscr{A}})_\tau(q_1, \ldots, q_s, q) = 1$ and $T_\tau(q_1, \ldots, q_s, q) = \{\!\{1\}\!\}$. The only other transition with a non-vanishing transition weight in $\mathscr{B}$ or $\mathscr{A}$ is (q_2, γ, q_1), where $(T_{\mathscr{A}})_\gamma(q_2, q_1) = \top$ and $T_\gamma(q_2, q_1) = \{\!\{1, 1\}\!\}$. The fact that $[\![\{\!\{1, 1\}\!\}]\!]_{\mathbb{X}} = 1 \vee 1 = \top$ concludes the proof that $\mathscr{A}$ and $\mathscr{B}$ are strongly related.

It folllows from Lemma 3.53 that $[\![\mathscr{A}]\!] = [\![[\![\mathscr{B}]\!]]\!]_S$. We recall from Example 3.28 that for every $\xi \in \mathrm{T}_\Sigma$ it holds that

$$[\![\mathscr{B}]\!](\xi) = \{\!\{\overbrace{\mathrm{size}(\xi), \ldots, \mathrm{size}(\xi)}^{r(\xi)\text{ times}}\}\!\}, \text{ where}$$

$$r(\xi) = 2^{n_\xi} + \begin{cases} 2^{n_\xi - 1} & \text{if } \xi(\varepsilon) = \gamma \text{ and } \xi(1) = \sigma \\ 0 & \text{otherwise.} \end{cases}$$

and n_ξ is the number of times a γ occurs directly above a σ in ξ. Thus, by the definition

of $[\![\_]\!]_S$,

$$[\![[\![\mathscr{B}]\!]]\!]_S(\xi) = \begin{cases} \text{size}(\xi) & \text{if } n_\xi = 0 \\ \top & \text{otherwise.} \end{cases}$$

By Lemma 3.53, $[\![\mathscr{A}]\!] = [\![[\![\mathscr{B}]\!]]\!]_S$. This proves our conjecture about the weighted tree language $[\![\mathscr{A}]\!]$ from Example 3.1. ◀

We will now show that for every WTA $\mathscr{A}$ over Σ and $\langle\mathbb{M}\rangle_\oplus$ there exists a strongly related WTA $\mathscr{B}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$ and for every WTA $\mathscr{B}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$ there exists a related WTA $\mathscr{A}$ over Σ and S.

Lemma 3.55. Let $\mathscr{A} = (Q, T, \text{final})$ be a WTA over Σ and $\langle\mathbb{M}\rangle_\oplus$. It holds that

$$\text{StrongRel}(\mathscr{A}) \neq \emptyset.$$

Moreover, if $\mathscr{A}$ is $\mathbb{M}$-sequential, then there exists a $\mathscr{B} \in \text{StrongRel}(\mathscr{A})$ which is sequential.

Proof. Let $\mathscr{B} = (Q', T', \text{final}')$ be the WTA over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$ where $Q' = Q$ and T' and final$'$ are defined as follows.

For every $q \in Q$, the fact that $\text{final}(q) \in \langle\mathbb{M}\rangle_\oplus$ implies the existence of $n_q \in \mathbb{N}$ and $z_1^q, \ldots, z_{n_q}^q \in \mathbb{M}$ such that $\text{final}(q) = \bigoplus_{i=1}^{n_q} z_i^q$. We fix an arbitrary instance of $n_q, z_1^q, \ldots, z_{n_q}^q$ and define the final weight map of $\mathscr{B}$ by $\text{final}'(q) = \{\!\{z_i^q \mid i \in [n_q]\}\!\}$ for every $q \in Q$.

Analogously, for every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q$, the fact that $T_\sigma(q_1, \ldots, q_s, q) \in \langle\mathbb{M}\rangle_\oplus$ implies the existence of $n_t \in \mathbb{N}$ and $y_1^t, \ldots, y_{n_t}^t \in \mathbb{M}$ where $t = (q_1, \ldots, q_s, \sigma, q)$ such that $T_\sigma(q_1, \ldots, q_s, q) = \bigoplus_{i=1}^{n_t} y_i^t$. If $T_\sigma(q_1, \ldots, q_s, q) = 0$, then we let $n_t = 0$. If $T_\sigma(q_1, \ldots, q_s, q) \in \mathbb{M}\backslash\{0\}$, then we let $n_t = 1$ and $y_1^t = T_\sigma(q_1, \ldots, q_s, q)$. Otherwise, we fix an arbitrary instance of $n_t, y_1^t, \ldots, y_{n_t}^t$. We define the transition weight map of $\mathscr{B}$ by $T'_\sigma(q_1, \ldots, q_s, q) = \{\!\{y_i^t \mid i \in [n_t]\}\!\}$ for every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q$.

By the construction it is clear that $\mathscr{A}$ and $\mathscr{B}$ are strongly related. This proves that $\text{StrongRel}(\mathscr{A}) \neq \emptyset$.

The second claim can be seen as follows. Let every non-vanishing weight occurring in T be in $\mathbb{M}$. It holds that for every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q$ either

$T_\sigma(q_1, \ldots, q_s, q) = 0$ (in this case $T'_\sigma(q_1, \ldots, q_s, q) = \emptyset$) or $T_\sigma(q_1, \ldots, q_s, q) \in \mathbb{M}$ (in this case $T'_\sigma(q_1, \ldots, q_s, q) = \{\!\{T_\sigma(q_1, \ldots, q_s, q)\}\!\}$). Hence, every transition weight of $\mathscr{B}$ is either $\emptyset$ or in $\mathcal{S}(\mathbb{M})$. Moreover, if $\mathscr{A}$ is deterministic, then $\mathscr{B}$ is also deterministic[1]. In total, if $\mathscr{A}$ is $\mathbb{M}$-sequential, then $\mathscr{B}$ is deterministic and every non-vanishing transition weight of $\mathscr{B}$ is in $\mathcal{S}(\mathbb{M})$. This proves the claim. □

Lemma 3.56. Let $\mathscr{B} = (Q, T, \mathrm{final})$ be a WTA over Σ and $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$. It holds that

$$\mathrm{Rel}(\mathscr{B}) \neq \emptyset.$$

Moreover, if $\mathscr{B}$ is sequential, then every $\mathscr{A} \in \mathrm{Rel}(\mathscr{B})$ is $\mathbb{M}$-sequential.

Proof. Let $\mathscr{A} = (Q', T', \mathrm{final}')$ be the WTA over Σ and S defined by $Q' = Q$ and for every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q$ by

$$T'_\sigma(q_1, \ldots, q_s, q) = [\![T_\sigma(q_1, \ldots, q_s, q)]\!]_S \qquad \text{and} \qquad \mathrm{final}'(q) = [\![\mathrm{final}(q)]\!]_S.$$

By definition, $\mathscr{A}$ and $\mathscr{B}$ are related, which proves the first claim. We note that $\mathscr{A}$ is indeed the only WTA over Σ and S that is related to $\mathscr{B}$.

The second claim can be seen as follows. Let $\mathscr{B}$ be sequential. Since $\mathscr{B}$ is deterministic, also $\mathscr{A}$ is deterministic. Let $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q$. Since $\mathscr{B}$ is sequential, it holds that either $T_\sigma(q_1, \ldots, q_s, q) = \emptyset$ (in which case $T'_\sigma(q_1, \ldots, q_s, q) = 0$) or $T_\sigma(q_1, \ldots, q_s, q) = \{\!\{x\}\!\}$ for some $x \in \mathbb{M}$ (in which case $T'_\sigma(q_1, \ldots, q_s, q) = x$) by construction. Hence, all non-vanishing weights occurring in T' are in $\mathbb{M}$, which yields the claim. □

Corollary 3.57. It holds that

$$[\![\mathrm{Rec}(\Sigma, \mathcal{M}_{\mathrm{fin}}(\mathbb{M}))]\!]_S = \mathrm{Rec}(\Sigma, \langle \mathbb{M} \rangle_\oplus).$$

In particular, it holds that $[\![\mathrm{Rec}(\Sigma, \mathcal{M}_{\mathrm{fin}}((S, \odot, 1)))]\!]_S = \mathrm{Rec}(\Sigma, S)$.

Proof. We show the equality $[\![\mathrm{Rec}(\Sigma, \mathcal{M}_{\mathrm{fin}}(\mathbb{M}))]\!]_S = \mathrm{Rec}(\Sigma, \langle \mathbb{M} \rangle_\oplus)$ by proving the two set inclusions "$\subseteq$" and "$\supseteq$" using relatedness of automata.

For "$\subseteq$": Let $\mathscr{B} = (Q, T, \mathrm{final})$ be a finite WTA over Σ and $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$. By Lemma 3.56 there exists $\mathscr{A} \in \mathrm{Rel}(\mathscr{B})$ and by Lemma 3.53 it holds that $[\![[\![\mathscr{B}]\!]]\!]_S = [\![\mathscr{A}]\!]$.

[1] We note that this is not true if $\mathscr{A}$ and $\mathscr{B}$ are only related, see Remark 3.52

Moreover, $\mathscr{A}$ is finite and every weight occurring in $\mathscr{A}$ is in $\langle\mathbb{M}\rangle_\oplus$ by definition of relatedness. Thus, $[\![\mathscr{A}]\!] \in \text{Rec}(\Sigma, \langle\mathbb{M}\rangle_\oplus)$ and hence also $[\![[\![\mathscr{B}]\!]]\!]_S \in \text{Rec}(\Sigma, \langle\mathbb{M}\rangle_\oplus)$. This proves the inclusion $[\![\text{Rec}(\Sigma, \mathcal{M}_{\text{fin}}(\mathbb{M}))]\!]_S \subseteq \text{Rec}(\Sigma, \langle\mathbb{M}\rangle_\oplus)$.

For "$\supseteq$": Let $\mathscr{A} = (Q, T, \text{final})$ be a finite WTA over Σ and $\langle\mathbb{M}\rangle_\oplus$. By Lemma 3.55 there exists $\mathscr{B} \in \text{StrongRel}(\mathscr{A})$ and by Lemma 3.53 it holds that $[\![\mathscr{A}]\!] = [\![[\![\mathscr{B}]\!]]\!]_S$. Together with the fact that $\mathscr{B}$ is finite, we obtain $[\![\mathscr{A}]\!] \in [\![\text{Rec}(\Sigma, \mathcal{M}_{\text{fin}}(\mathbb{M}))]\!]_S$. This proves the inclusion $[\![\text{Rec}(\Sigma, \mathcal{M}_{\text{fin}}(\mathbb{M}))]\!]_S \supseteq \text{Rec}(\Sigma, \langle\mathbb{M}\rangle_\oplus)$. □

Corollary 3.58. It holds that

$$[\![\text{sRec}(\Sigma, \mathcal{M}_{\text{fin}}(\mathbb{M}), \mathcal{S}(\mathbb{M}))]\!]_S \subseteq \text{dRec}(\Sigma, \langle\mathbb{M}\rangle_\oplus).$$

Moreover, it holds that $[\![\text{sRec}(\Sigma, \mathcal{M}_{\text{fin}}(S), \mathcal{S}(S))]\!]_S = \text{dRec}(\Sigma, S)$.

Proof. Let $\mathscr{B}$ be a sequential finite WTA over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$. Since $\mathscr{B}$ is sequential, Lemma 3.56 implies the existence of a deterministic $\mathscr{A} \in \text{Rel}(\mathscr{B})$. Similarly to Corollary 3.57 one can prove the first claim and the inclusion "$\subseteq$" of the second claim.

It remains to show the inclusion $[\![\text{sRec}(\Sigma, \mathcal{M}_{\text{fin}}(S), \mathcal{S}(S))]\!]_S \supseteq \text{dRec}(\Sigma, S)$. Let $\mathscr{A}$ be a deterministic WTA over Σ and S. We note that in this case, the weights of $\mathscr{A}$ are already in the monoid $(S, \odot, 1)$. Hence, by Lemma 3.55 there exists a $\mathcal{S}(S)$-sequential $\mathscr{B} \in \text{StrongRel}(\mathscr{A})$. This concludes the proof. □

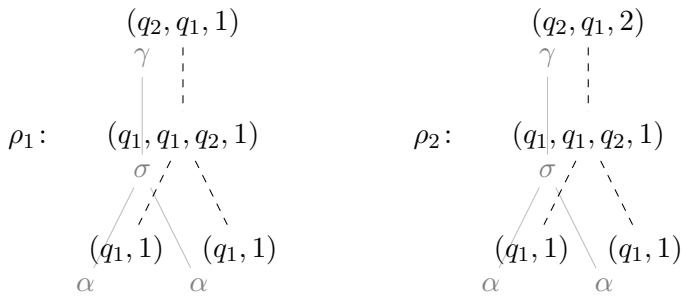
Definition 3.59. Let $\mathscr{A} = (Q, T, \text{final})$ be a finite WTA over Σ and S.

We say that $\mathscr{A}$ *has the extended twinning property* (in symbols: $\mathscr{A} \vDash \text{ETP}$) if there exists $\mathscr{B} \in \text{StrongRel}(\mathscr{A})$ such that $\mathscr{B} \vDash \text{ETP}$. ◀

Example 3.60. We recall the WTA $\mathscr{B}$ from Example 3.21 and the WTA $\mathscr{A}$ from Example 3.1. In Example 3.33 we have seen that $\mathscr{B} \vDash \text{ETP}$ and from Example 3.54 we know that $\mathscr{B} \in \text{StrongRel}(\mathscr{A})$. Thus, by Definition 3.59, $\mathscr{A} \vDash \text{ETP}$.

We note that the fact "$\mathscr{A} \vDash \text{ETP}$" does not imply that every WTA which is strongly related to $\mathscr{A}$ has the ETP. For illustration, we consider the WTA $\mathscr{B}'$, which is equal to $\mathscr{B}$ except that we replace the transition weight $T_\gamma(q_2, q_1)$ (which equals $\{\!\{1, 1\}\!\}$ in $\mathscr{B}$) by $\{\!\{1, 2\}\!\}$. It is easy to see that $\mathscr{B}'$ is still strongly related to $\mathscr{A}$ (since $1 \vee 2 = \top = 1 \vee 1$). However, $\mathscr{B}'$ does not have the ETP.

In order to see this, we let $\xi = \gamma(\sigma(\alpha, \alpha))$. Similar to Example 3.24 we fix the following two R-runs of $\mathscr{B}'$ on ξ.



We let $\xi' = \alpha$, $\zeta = \gamma(\sigma(x_1, \alpha))$ and note that ρ_1 and ρ_2 are of the respective form

$$\xrightarrow{\xi|y_1} q_1 \xrightarrow{\zeta|y_2} q_1 \qquad \text{and} \qquad \xrightarrow{\xi|y_1'} q_1 \xrightarrow{\zeta|y_2'} q_1,$$

where $y_1 = y_1' = 1$, $y_2 = 1 + 1 + 1 = 3$, and $y_2' = 1 + 1 + 2 = 4$. It holds that

$$d_\Gamma(y_1, y_1') = 0 \neq 1 = d_\Gamma(y_1 + y_2, y_1' + y_2'),$$

which directly proves that $\mathscr{B}'$ does not have the TP and hence, $\mathscr{B}'$ does not have the ETP either. ◀

Definition 3.61. Let $\mathscr{A} = (Q, T, \text{final})$ be a finite WTA over Σ and S.

We call $\mathscr{A}$ *finitely* $\mathbb{M}$*-ambiguous* if (a) there exists $k \in \mathbb{N}$ such that for every $\xi \in \mathrm{T}_\Sigma$ it holds that $\#\mathrm{Runs}^v_{\mathscr{A}}(\xi) \leq k$ and (b) for every transition $(q_1, \ldots, q_s, \sigma, q)$ in a loop of $\mathscr{A}$ such that $q_1, \ldots, q_s$ are reachable, it holds that $T_\sigma(q_1, \ldots, q_s, q) \in \mathbb{M}$. ◀

Lemma 3.62. Let $\mathscr{A} = (Q, T, \text{final})$ be a finite WTA over Σ and S. It holds that

$$\mathscr{A} \text{ is finitely } \mathbb{M}\text{-ambiguous} \iff \forall \mathscr{B} \in \mathrm{StrongRel}(\mathscr{A}) \colon \mathscr{B} \text{ is finitely R-ambiguous.}$$

Proof. For " $\Longleftarrow$ ": We note that $\mathrm{StrongRel}(\mathscr{A}) \neq \emptyset$ by Lemma 3.55. We let $\mathscr{B} \in \mathrm{StrongRel}(\mathscr{A})$ and denote $\mathscr{B} = (Q, T', \text{final}')$. By assumption, $\mathscr{B}$ is finitely R-ambiguous and hence, there exists $k \in \mathbb{N}$ such that for every $\xi \in \mathrm{T}_\Sigma$ it holds that $\#\,\mathrm{R\text{-}Runs}_{\mathscr{B}}(\xi) \leq k$. The fact that

$$\#\mathrm{Runs}^v_{\mathscr{A}}(\xi) \leq \#\mathrm{Runs}^v_{\mathscr{B}}(\xi) \leq \#\,\mathrm{R\text{-}Runs}_{\mathscr{B}}(\xi)$$

yields condition (a) from the definition of finite $\mathbb{M}$-ambiguity for $\mathscr{A}$.

It remains to show that condition (b) holds. First, let $s \in \mathbb{N}, \sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q$ be reachable such that $(q_1, \ldots, q_s, \sigma, q)$ is a transition in a loop of $\mathscr{B}$. The finite R-ambiguity of $\mathscr{B}$ implies that $\#T'_\sigma(q_1, \ldots, q_s, q) = 1$. This can be shown by a pumping argument as follows. There exists $\xi \in \mathrm{T}_\Sigma$, $\zeta \in \mathrm{C}_\Sigma$, $\rho \in \mathrm{Runs}_{\mathscr{A}}(\xi)$, $\rho' \in \mathrm{Runs}_{\mathscr{A}}(\zeta)$, and $w \in \mathrm{pos}(\zeta)$ such that ρ' is a loop, $\rho'(wi) = q_i$ for every $i \in [s]$, $\xi(w) = \sigma$, and $\rho'(w) = q$. It holds that $(\rho')^\ell[\rho] \in \mathrm{Runs}_{\mathscr{B}}(\zeta^\ell[\xi])$ for every $\ell \in \mathbb{N}$. If $\#T'_\sigma(q_1, \ldots, q_s, q) \geq 2$, then $(\rho')^\ell[\rho]$ is associated to at least 2^ℓ pairwise different R-runs of $\mathscr{B}$ on $\zeta^\ell[\xi]$. In particular, $\#\,\mathrm{R\text{-}Runs}_{\mathscr{B}}(\zeta^\ell[\xi]) \geq 2^\ell$ and hence, $\mathscr{B}$ is not finitely R-ambiguous.

Now, let $s \in \mathbb{N}, \sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q$ be reachable such that $(q_1, \ldots, q_s, \sigma, q)$ is a transition in a loop of $\mathscr{A}$. By Remark 3.52, $(q_1, \ldots, q_s, \sigma, q)$ is a transition in a loop of $\mathscr{B}$ and from our above argument we obtain $\#T'_\sigma(q_1, \ldots, q_s, q) = 1$. Therefore, by the relatedness of $\mathscr{A}$ and $\mathscr{B}$, it holds that $T'_\sigma(q_1, \ldots, q_s, q) = \{\!\{T_\sigma(q_1, \ldots, q_s, q)\}\!\}$ and hence $T_\sigma(q_1, \ldots, q_s, q) \in \mathbb{M}$. This yields condition (b) from the definition of finite $\mathbb{M}$-ambiguity for $\mathscr{A}$.

For "$\Longrightarrow$": Let $\mathscr{B} \in \mathrm{StrongRel}(\mathscr{A})$ and denote $\mathscr{B} = (Q, T', \mathrm{final}')$. Moreover, we define the constant

$$c_{\mathscr{B}} = \max_{\substack{s \in \mathbb{N}, \sigma \in \Sigma^{(s)}, \\ q_1, \ldots, q_s, q \in Q}} \#T'_\sigma(q_1, \ldots, q_s, q).$$

Now let $\xi \in \mathrm{T}_\Sigma$. It holds that $\#\mathrm{Runs}^v_{\mathscr{B}}(\xi) = \#\mathrm{Runs}^v_{\mathscr{A}}(\xi)$ by the definition of strong relatedness and hence $\#\mathrm{Runs}^v_{\mathscr{B}}(\xi) \leq k$ by the finite $\mathbb{M}$-ambiguity of $\mathscr{A}$. Furthermore, from Equation (3.3) we obtain that $\bigcup_{\rho \in \mathrm{Runs}_{\mathscr{B}}(\xi)} \mathrm{R}(\rho) = \mathrm{R\text{-}Runs}_{\mathscr{B}}(\xi)$. Hence, if we show

$$\forall \rho \in \mathrm{Runs}^v_{\mathscr{B}}(\xi) \colon \#\,\mathrm{R}(\rho) \leq (c_{\mathscr{B}})^{\mathrm{maxrk}(\Sigma)^{\#Q}}, \tag{3.13}$$

then we obtain $\#\,\mathrm{R\text{-}Runs}_{\mathscr{B}}(\xi) \leq k \cdot (c_{\mathscr{B}})^{\mathrm{maxrk}(\Sigma)^{\#Q}}$, which implies the claim.

Let $\rho \in \mathrm{Runs}^v_{\mathscr{B}}(\xi)$. Since the R-runs in $\mathrm{R}(\rho)$ must have the same "state behavior" as ρ, it surely holds that

$$\#\,\mathrm{R}(\rho) = \prod_{w \in \mathrm{pos}(\xi)} \#T'_{\xi(w)}(\rho(w1) \ldots \rho(ws), \rho(w)) \leq (c_{\mathscr{B}})^{\mathrm{size}(\xi)}.$$

In particular, if $\mathrm{size}(\xi) \leq \mathrm{maxrk}(\Sigma)^{\#Q}$, then Equation (3.13) holds.

Now let $\mathrm{size}(\xi) > \mathrm{maxrk}(\Sigma)^{\#Q}$ and note that this implies $\mathrm{height}(\xi) > \#Q$. Thus, by the pigeonhole principle, ρ contains a loop. Formally, there exists $\xi' \in \mathrm{T}_\Sigma$, $\zeta, \zeta' \in \mathrm{C}_\Sigma$,

$\rho_1 \in \mathrm{Runs}_{\mathscr{B}}(\xi')$, $\rho_2 \in \mathrm{Runs}_{\mathscr{B}}(\zeta)$, and $\rho_3 \in \mathrm{Runs}_{\mathscr{B}}(\zeta')$ such that $\zeta \neq x_1$, ρ_2 is a loop, $\zeta'[\zeta[\xi']] = \xi$, and $\rho_3[\rho_2[\rho_1]] = \rho$. Since every transition occurring in ρ_2 occurs in a loop (namely ρ_2), every transition weight occurring in ρ_2 (considered over $\mathscr{A}$) is in $\mathbb{M}$ and hence, every transition weight occurring in ρ_2 (considered over $\mathscr{B}$) is a singleton set. In particular,

$$\begin{aligned}\#\,\mathrm{R}(\rho) &= \prod_{w \in \mathrm{pos}(\xi)} \#T'_{\xi(w)}(\rho(w1)\dots\rho(ws), \rho(w)) \\ &= \prod_{w \in \mathrm{pos}(\zeta'[\xi'])} \#T'_{\xi(w)}(\rho(w1)\dots\rho(ws), \rho(w)) = \#\,\mathrm{R}(\rho_3[\rho_1]).\end{aligned}$$

Moreover, $\mathrm{size}(\zeta'[\xi']) < \mathrm{size}(\xi)$. By iteratively removing loops from ξ in this way, we arrive at a tree $\hat{\xi} \in \mathrm{T}_{\Sigma}$ and a run $\hat{\rho}$ of $\mathscr{B}$ on $\hat{\xi}$ such that $\mathrm{size}(\hat{\xi}) \leq \mathrm{maxrk}(\Sigma)^{\#Q}$ and $\#\,\mathrm{R}(\hat{\rho}) = \#\,\mathrm{R}(\rho)$. It holds that $\#\,\mathrm{R}(\hat{\rho}) \leq (c_{\mathscr{B}})^{\mathrm{maxrk}(\Sigma)^{\#Q}}$ by our above argument and hence, we obtain that $\#\,\mathrm{R}(\rho) \leq (c_{\mathscr{B}})^{\mathrm{maxrk}(\Sigma)^{\#Q}}$, which concludes the proof of Equation (3.13) and therefore the proof of the lemma. □

3.7 $\mathbb{M}$-Sequentialisation of Weighted Tree Automata

In this chapter, we execute Step (III) from our $\mathbb{M}$-sequentialisation construction (see Chapter 3.1). First, given the sequentialisation $D_{\mathscr{B}}$ of some WTA $\mathscr{B}$ over Σ and $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$ (cf. Definition 3.39), we define the concept of an accumulator $\sim$ of $D_{\mathscr{B}}$ and the accumulation $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ of $D_{\mathscr{B}}$ via $\sim$ (cf. Definitions 3.63 and 3.64). This accumulation process combines states with the same local state behaviour and S-evaluates all multiset weights. In particular, $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ is a WTA over Σ and S. Second, we show that $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ is indeed $\mathbb{M}$-sequential cf. Lemma 3.66) and equivalent to $\mathscr{B}$ up to S-evaluation (cf. Lemma 3.67). Next, we provide classes of weighted tree automata $\mathscr{B}$ over $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$ and accumulators $\sim$ such that $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ is finite (cf. Definition 3.69 and Lemmas 3.70 and 3.72).

We ultimately combine the concept of accumulators with the concept of strong relatedness from Chapter 3.6 to obtain $\mathbb{M}$-sequentialisation results for weighted tree automata over Σ and $\langle\mathbb{M}\rangle_{\oplus}$. This chapter also concludes the main contribution of Chapter 3. We restate our $\mathbb{M}$-sequentialisation results in a closed form in Theorems 3.77 and 3.78.

Throughout the rest of Chapter 3.7, we assume $(S, \oplus, \odot, 0, 1)$ to be an arbitrary $\mathbb{M}$-semiring for some finitely generated monoid $\mathbb{M}$ with finite generating set Γ. Moreover, we assume that $\mathbb{M}$ divides Γ-monotone and admits f-centering factorisations (for some strongly monotone map $f\colon \mathbb{N} \to \mathbb{N}$).

3.7.1 Accumulation of $D_{\mathscr{B}}$

Throughout the rest of Chapter 3.7.1, we assume $\mathscr{B} = (Q, T, \mathrm{final})$ to be an arbitrary finite WTA *over Σ and $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$. Moreover, we consider the sequential* WTA *$D_{\mathscr{B}} = (Q', T', \mathrm{final}')$ as defined in Definition 3.39 and let $Q'' \subseteq Q'$ be the set of reachable states of $D_{\mathscr{B}}$.*

Definition 3.63. An equivalence relation $\sim \subseteq Q'' \times Q''$ is called *accumulator of $D_{\mathscr{B}}$* if

- for every $X, X' \in Q''$ such that $X \sim X'$ it holds that $[\![\mathrm{final}'(X)]\!]_S = [\![\mathrm{final}'(X')]\!]_S$ and
- for every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $X_1, \ldots, X_s, X'_1, \ldots, X'_s, X, X' \in Q''$ such that $X_1 \sim X'_1, \ldots, X_s \sim X'_s$, $T'_\sigma(X_1, \ldots, X_s, X) \neq \emptyset$, and $T'_\sigma(X'_1, \ldots, X'_s, X') \neq \emptyset$ it holds that $X \sim X'$ and $T'_\sigma(X_1, \ldots, X_s, X) = T'_\sigma(X'_1, \ldots, X'_s, X')$.

We visualise the second condition in Figure 3.9.

We call an accumulator $\sim$ of $D_{\mathscr{B}}$ *finite* if $Q''/_\sim$ is finite. ◀

Definition 3.64. Let $\sim \subseteq Q'' \times Q''$ be an accumulator of $D_{\mathscr{B}}$. The *accumulation of $D_{\mathscr{B}}$ via $\sim$* is the tuple

$$\mathrm{acc}_\sim(D_{\mathscr{B}}) = (\widetilde{Q}, \widetilde{T}, \widetilde{\mathrm{final}})$$

defined as follows.

- $\widetilde{Q} = Q''/_\sim$
- $\widetilde{\mathrm{final}}\colon \widetilde{Q} \to S$ is given for every $\widetilde{X} \in \widetilde{Q}$ by

$$\widetilde{\mathrm{final}}(\widetilde{X}) = [\![\mathrm{final}'(X)]\!]_S,$$

  where $X \in Q''$ such that $\widetilde{X} = [X]_\sim$.

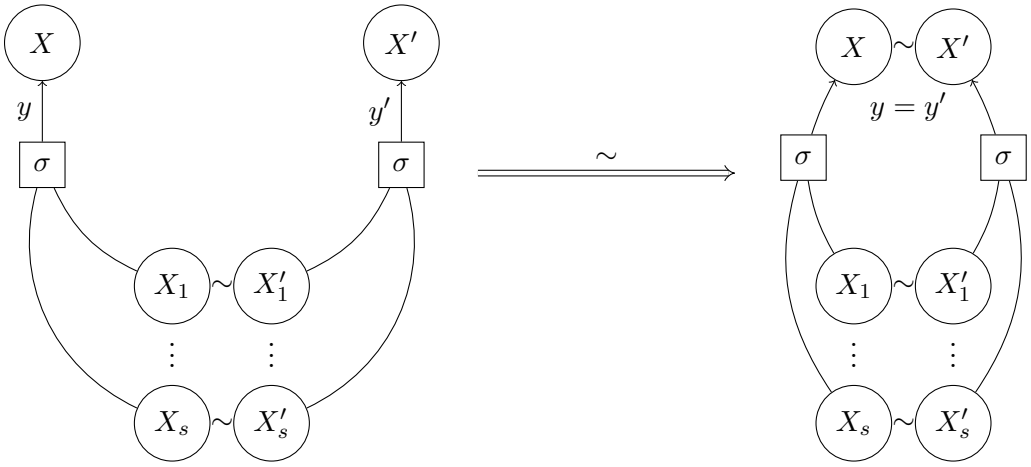


Figure 3.9: An illustration of the second property of Definition 3.63. Two transitions in $D_{\mathscr{B}}$ where $X_i \sim X'_i$ for every $i \in [s]$ also satisfy $X \sim X'$ and $y = y'$.

- For every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $\widetilde{X}_1, \ldots, \widetilde{X}_s, \widetilde{X} \in \widetilde{Q}$, the transition weight $\widetilde{T}_\sigma(\widetilde{X}_1, \ldots, \widetilde{X}_s, \widetilde{X})$ is given by

$$\widetilde{T}_\sigma(\widetilde{X}_1, \ldots, \widetilde{X}_s, \widetilde{X}) = \begin{cases} y & \text{if } y \in \mathbb{M} \wedge \exists X_1, \ldots, X_s, X \in Q'' \colon \Big(\widetilde{X} = [X]_\sim \wedge \\ & \qquad (\forall i \in [s] \colon \widetilde{X}_i = [X_i]_\sim) \wedge T'_\sigma(X_1, \ldots, X_s, X) = \{\!\{y\}\!\}\Big) \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sim$ is an accumulator of $D_{\mathscr{B}}$, it follows from Definition 3.63 that $\mathrm{acc}_\sim(D_{\mathscr{B}})$ is well-defined. ◀

Example 3.65. We continue Example 3.54 by defining an accumulator $\sim_{\mathbb{X}}$ of $D_{\mathscr{B}}$ and the accumulation of $D_{\mathscr{B}}$ with $\sim_{\mathbb{X}}$. In Example 3.41, we have constructed the set of reachable states Q'' of $D_{\mathscr{B}}$, which is $Q'' = \{X_i, X'_i \mid i \in \mathbb{N}\}$. Let $\sim_{\mathbb{X}} \subseteq Q'' \times Q''$ be the equivalence relation defined by

$$[X_0]_{\sim_{\mathbb{X}}} = \{X_0\}, \qquad [X'_0]_{\sim_{\mathbb{X}}} = \{X'_0\}, \qquad \text{and} \qquad [X_1]_{\sim_{\mathbb{X}}} = \{X_i, X'_i \mid i \in \mathbb{N}_+\}.$$

An illustration of the relation $\sim_{\mathbb{X}}$ can be found in Figure 3.10. Next we prove that $\sim_{\mathbb{X}}$ is an accumulator of $D_{\mathscr{B}}$.

We prove the first condition from Definition 3.63. Let $Y, Y' \in Q''$ such that $Y \sim_{\mathbb{X}} Y'$. Hence, $Y = Y'$ or $Y, Y' \in [X_1]_{\sim_{\mathbb{X}}}$. If $Y = Y'$, then trivially $[\![\mathrm{final}'(Y)]\!]_S = [\![\mathrm{final}'(Y')]\!]_S$.

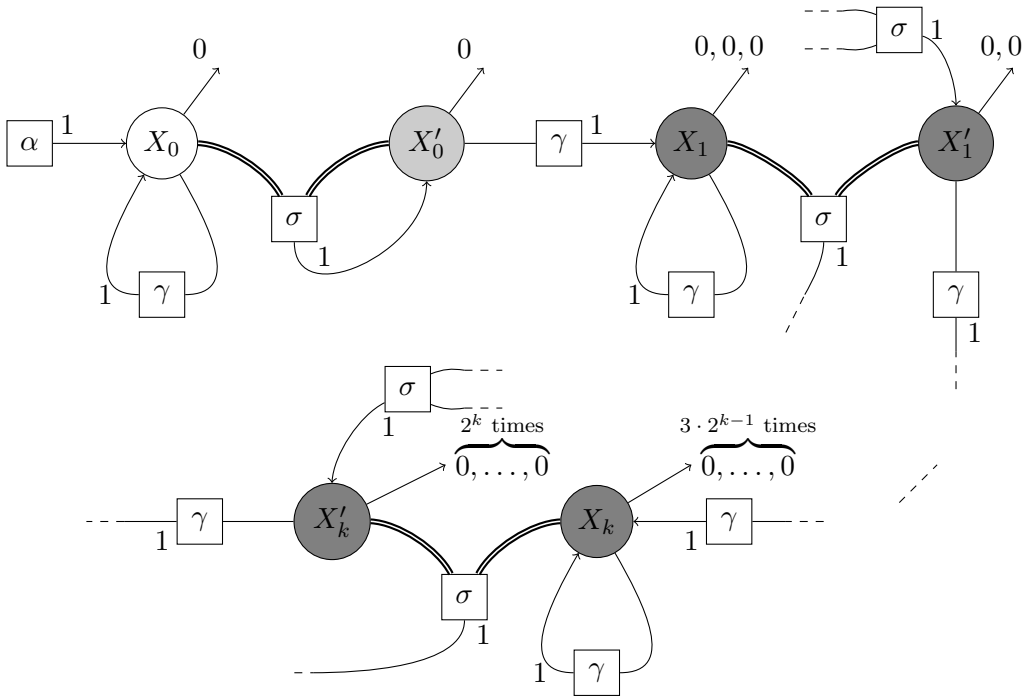


Figure 3.10: An illustration of $D_{\mathscr{B}}$ (cf. Example 3.41), where the state coloring illustrates the equivalence relation $\sim_{\mathbb{X}}$. The states X_0, X'_0, and X_1 are in pairwise different equivalence classes under $\sim_{\mathbb{X}}$, whereas all states of the form X_i and X'_i with $i \geq 1$ are in the same equivalence class.

If $Y, Y' \in [X_1]_{\sim_{\mathbb{X}}}$, then from the definition of final$'$ we obtain $\#\text{final}'(Y) \geq 2$ and $\#\text{final}'(Y') \geq 2$, which proves $[\![\text{final}'(Y)]\!]_S = \top = [\![\text{final}'(Y')]\!]_S$. This concludes the proof of the first condition from Definition 3.63.

Next, we prove the second condition from Definition 3.63. Let $s \in \mathbb{N}, \tau \in \Sigma^{(s)}$, and $Y_1, \ldots, Y_s, Y'_1, \ldots, Y'_s, Y, Y' \in Q''$ such that $T'_\tau(Y_1, \ldots, Y_s, Y)$ and $T'_\tau(Y'_1, \ldots, Y'_s, Y')$ are non-empty and $Y_i \sim_{\mathbb{X}} Y'_i$ for every $i \in [s]$. The definition of T' shows that $T'_\tau(Y_1, \ldots, Y_s, Y) = \{\!\{1\}\!\} = T'_\tau(Y'_1, \ldots, Y'_s, Y')$. It remains to show that $Y \sim_{\mathbb{X}} Y'$. If $\tau = \alpha$, then $Y \sim_{\mathbb{X}} Y'$ holds trivially. Let $\tau = \gamma$. We know that $Y_1, Y'_1 \in \{X_i \mid i \in \mathbb{N}\}$. Since $Y_1 \sim_{\mathbb{X}} Y'_1$, we obtain that either $Y_1 = Y'_1 = X_0$ (in which case $Y = Y' = X_0$) or $Y_1, Y'_1 \in [X_1]_{\sim_{\mathbb{X}}}$ (in which case $Y, Y' \in [X_1]_{\sim_{\mathbb{X}}}$). In any case, $Y \sim_{\mathbb{X}} Y'$. Now let $\tau = \sigma$. We note that $Y \neq X_0$ and $Y' \neq X_0$ by the definition of T'. In fact, the only two cases where $Y \not\sim_{\mathbb{X}} Y'$ are (i) $Y = X'_0$ and $Y' \in [X_1]_{\sim_{\mathbb{X}}}$ and (ii) $Y \in [X_1]_{\sim_{\mathbb{X}}}$ and $Y' = X'_0$. We prove $Y \sim_{\mathbb{X}} Y'$ by contradiction and assume without loss of generality that $Y = X'_0$ and $Y' \in [X_1]_{\sim_{\mathbb{X}}}$. From the definition of T' we obtain that $Y_1, Y_2 \in \{X_0, X'_0\}$. Therefore, since $Y_1 \sim_{\mathbb{X}} Y'_1$ and $Y_2 \sim_{\mathbb{X}} Y'_2$, we have that $Y'_1, Y'_2 \in \{X_0, X'_0\}$. For every such Y'_1, Y'_2, and Y' it holds that $T'_\sigma(Y'_1, Y'_2, Y') = \emptyset$, which is a contradiction to the quantification of Y'_1, Y'_2, and Y'. This concludes the proof of the second condition from Definition 3.63.

To keep the example readable, we abbreviate the equivalence classes of $\sim_{\mathbb{X}}$ by

$$Q_1 = [X_0]_{\sim_{\mathbb{X}}}, \qquad Q_2 = [X'_0]_{\sim_{\mathbb{X}}}, \quad \text{and} \quad Q_3 = [X_1]_{\sim_{\mathbb{X}}}.$$

From Definition 3.64 we obtain that $\text{acc}_{\sim_{\mathbb{X}}}(\text{D}_{\mathscr{B}})$ is well-defined and denote its components by $\text{acc}_{\sim_{\mathbb{X}}}(\text{D}_{\mathscr{B}}) = (\widetilde{Q}, \widetilde{\text{init}}, \widetilde{T}, \widetilde{\text{final}})$. We know that $\widetilde{Q} = \{Q_1, Q_2, Q_3\}$. Moreover, we can easily calculate the values of $[\![\text{final}'(X)]\!]_S$ for $X \in Q''$ and obtain

$$\widetilde{\text{final}} = (Q_1 \mapsto 0, Q_2 \mapsto 0, Q_3 \mapsto \top).$$

In order to obtain the values of $\widetilde{T}$, we simply need to find the corresponding transition weights from $D_{\mathscr{B}}$. Every value of $\widetilde{T}$ is $\perp$ except

$$\begin{aligned}
&\widetilde{T}_\alpha(Q_1) = 1,\\
&\widetilde{T}_\gamma(Q_1, Q_1) = \widetilde{T}_\gamma(Q_2, Q_3) = \widetilde{T}_\gamma(Q_3, Q_3) = 1,\\
&\widetilde{T}_\sigma(Q_i, Q_j, Q_2) = 1 \text{ for every } i, j \in \{1, 2\}, \text{ and}\\
&\widetilde{T}_\sigma(Q_i, Q_j, Q_3) = 1 \text{ for every } i, j \in \{1, 2, 3\} \text{ such that } i = 3 \text{ or } j = 3.
\end{aligned}$$

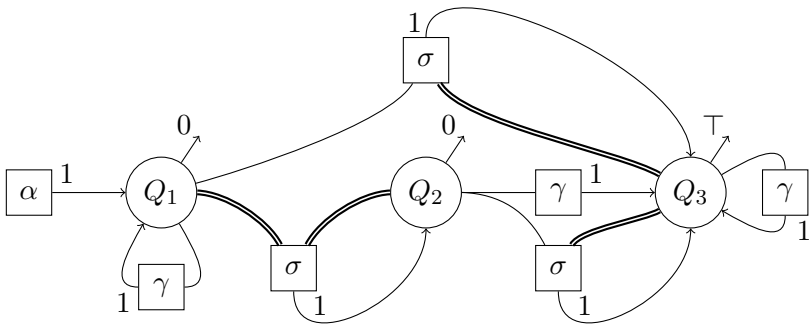


Figure 3.11: The deterministic WTA $\mathrm{acc}_{\sim_{\mathbb{X}}}(D_{\mathscr{B}})$ from Example 3.65.

An illustration of $\mathrm{acc}_{\sim_{\mathbb{X}}}(D_{\mathscr{B}})$ can be found in Figure 3.11. Clearly, $\mathrm{acc}_{\sim_{\mathbb{X}}}(D_{\mathscr{B}})$ is a deterministic WTA over Σ and $\mathbb{X}$ and every non-vanishing weight occuring in $\widetilde{T}$ is in $\mathbb{N}$. Moreover, one can show that $[\![[\![\mathscr{B}]\!]]\!]_S = [\![\mathrm{acc}_{\sim_{\mathbb{X}}}(D_{\mathscr{B}})]\!]$. ◀

We have seen in Example 3.65 how the accumulation of $D_{\mathscr{B}}$ via an accumulator results in an $\mathbb{M}$-sequential WTA that is equivalent to $\mathscr{B}$ up to S-evaluation. In the upcoming Lemmas 3.66 and 3.67, we show that this holds in general, that is, for every WTA $\mathscr{B}$ over Σ and $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$ and every accumulator $\sim$ of $D_{\mathscr{B}}$.

Lemma 3.66. Let $\sim \subseteq Q'' \times Q''$ be an accumulator of $D_{\mathscr{B}}$. It holds that $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ is an $\mathbb{M}$-sequential WTA over Σ and $\langle \mathbb{M} \rangle_{\oplus}$.

Proof. Surely, $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ is a WTA over Σ and S. Using Definition 3.64 and the fact that $D_{\mathscr{B}}$ is sequential, one easily sees that $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ is deterministic. Moreover, the non-vanishing transition weights of $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ are by construction in $\mathbb{M}$ and thus, $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ is $\mathbb{M}$-sequential. Additionally, the final weights of $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ are by construction in $\langle \mathbb{M} \rangle_{\oplus}$, which proves that $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ is a WTA over Σ and $\langle \mathbb{M} \rangle_{\oplus}$. □

Lemma 3.67. Let $\sim \subseteq Q'' \times Q''$ be an accumulator of $D_{\mathscr{B}}$. It holds that

$$[\![[\![D_{\mathscr{B}}]\!]]\!]_S = [\![\mathrm{acc}_{\sim}(D_{\mathscr{B}})]\!] \,.$$

Proof. Let $\xi \in \mathrm{T}_{\Sigma}$ and let the form of the unique R-run ρ of $D_{\mathscr{B}}$ on ξ be $\xrightarrow{\xi|y} X$. One can easily show (using structural induction on ξ) that there exists a run ρ' of $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$

on ξ such that $\mathrm{out}(\rho') = [X]_\sim$ and $\mathrm{wt}_{\mathrm{acc}_\sim(D_{\mathscr{B}})}(\xi, \rho') = y$. By Lemma 3.66, $\mathrm{acc}_\sim(D_{\mathscr{B}})$ is deterministic and hence, ρ' is the unique valid run of $\mathrm{acc}_\sim(D_{\mathscr{B}})$ on ξ. It holds that

$$\begin{aligned}
[\![[\![D_{\mathscr{B}}]\!]]\!]_S(\xi) &\overset{\star_1}{=} \bigoplus_{\xi|y' \to X' \overset{z'}{\to} \in D_{\mathscr{B}}} y' \odot z' \overset{\star_2}{=} \bigoplus_{X \overset{z'}{\to} \in D_{\mathscr{B}}} y \odot z' \\
&\overset{\star_3}{=} y \odot \Big(\bigoplus_{X \overset{z'}{\to} \in D_{\mathscr{B}}} z' \Big) = y \odot [\![\mathrm{final}'(X)]\!]_S \overset{\star_4}{=} y \odot \widetilde{\mathrm{final}}([X]_\sim) \\
&\overset{\star_5}{=} \mathrm{wt}_{\mathrm{acc}_\sim(D_{\mathscr{B}})}(\xi, \rho') \odot \widetilde{\mathrm{final}}(\mathrm{out}(\rho')) \overset{\star_6}{=} [\![\mathrm{acc}_\sim(D_{\mathscr{B}})]\!](\xi)
\end{aligned}$$

where $\star_1$ follows from Lemma 3.50, $\star_2$ follows from the fact that ρ is the unique R-run of $D_{\mathscr{B}}$ on ξ, $\star_3$ follows from the distributivity law, $\star_4$ follows from the definition of $\widetilde{\mathrm{final}}$, $\star_5$ follows from the definition of ρ', and $\star_6$ follows from the fact that ρ' is the unique run of $\mathrm{acc}_\sim(D_{\mathscr{B}})$ on ξ. □

Lemma 3.68. Let $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ be commutative and let $\sim \subseteq Q'' \times Q''$ be an accumulator of $D_{\mathscr{B}}$. It holds that $[\![[\![\mathscr{B}]\!]]\!]_S = [\![\mathrm{acc}_\sim(D_{\mathscr{B}})]\!]$.

Proof. By Lemma 3.44 it holds that $[\![\mathscr{B}]\!] = [\![D_{\mathscr{B}}]\!]$ and by Lemma 3.67 it holds that $[\![[\![D_{\mathscr{B}}]\!]]\!]_S = [\![\mathrm{acc}_\sim(D_{\mathscr{B}})]\!]$, which proves the claim. □

We will now define accumulators for entire classes of weighted tree automata $\mathscr{B}$. First, the identity relation on Q' is always an accumulator of $D_{\mathscr{B}}$. This will be useful in the case that $\mathscr{B}$ is finitely R-ambiguous. Second, if S is additively idempotent, then "equality under taking support" is an accumulator of $D_{\mathscr{B}}$.

Definition 3.69. We define the equivalence relations $\sim_{\mathrm{fa}}, \sim_{\mathrm{idp}} \subseteq Q' \times Q'$ where for every $X_1, X_2 \in Q'$ it holds that

- $X_1 \sim_{\mathrm{fa}} X_2$ iff $X_1 = X_2$,
- $X_1 \sim_{\mathrm{idp}} X_2$ iff $\mathrm{supp}(X_1) = \mathrm{supp}(X_2)$,

where for every $X \in Q'$ we define $\mathrm{supp}(X) \colon Q \to \mathcal{P}_{\mathrm{fin}}(\mathbb{M})$ by $\mathrm{supp}(X)(q) = \mathrm{supp}(X(q))$ for every $q \in Q$. Moreover, we also denote the restrictions of $\sim_{\mathrm{fa}}$ and $\sim_{\mathrm{idp}}$ to Q'' by $\sim_{\mathrm{fa}}$ and $\sim_{\mathrm{idp}}$, respectively. ◀

Lemma 3.70. $\sim_{\mathrm{fa}}$ is an accumulator of $D_{\mathscr{B}}$.

Proof. The claim holds trivially. □

Lemma 3.71. Let $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $X_1, \ldots, X_s, X'_1, \ldots, X'_s \in Q''$ such that $X_i \sim_{\mathrm{idp}} X'_i$ for every $i \in [s]$. The following holds.

1. $\mathcal{T}_\sigma(X_1, \ldots, X_s) \sim_{\mathrm{idp}} \mathcal{T}_\sigma(X'_1, \ldots, X'_s)$

2. $\mathrm{mindiv}(\mathcal{T}_\sigma(X_1, \ldots, X_s)) = \mathrm{mindiv}(\mathcal{T}_\sigma(X'_1, \ldots, X'_s))$

3. For every $y \in \mathrm{mindiv}(\mathcal{T}_\sigma(X_1, \ldots, X_s))$ it holds that

$$\{\mathrm{supp}(X) \mid X \in \mathrm{minquot}_y(\mathcal{T}_\sigma(X_1, \ldots, X_s))\} \\ = \{\mathrm{supp}(X') \mid X' \in \mathrm{minquot}_y(\mathcal{T}_\sigma(X'_1, \ldots, X'_s))\}.$$

4. Let (y, X) and (y', X') be the minimal f-centering factorisations of $\mathcal{T}_\sigma(X_1, \ldots, X_s)$ and $\mathcal{T}_\sigma(X'_1, \ldots, X'_s)$, respectively. It holds that $y = y'$ and $X \sim_{\mathrm{idp}} X'$.

Proof. We abbreviate $\mathcal{T}_\sigma(X_{[s]}) = \mathcal{T}_\sigma(X_1, \ldots, X_s)$ and $\mathcal{T}_\sigma(X'_{[s]}) = \mathcal{T}_\sigma(X'_1, \ldots, X'_s)$. For every $q \in Q$ it holds that

$$\begin{aligned}
&\mathrm{supp}(\mathcal{T}_\sigma(X_{[s]}))(q) \\
&\quad \overset{\star_1}{=} \mathrm{supp}\Big(\bigcup_{q_1,\ldots,q_s \in Q} \big(X_1(q_1) \odot \cdots \odot X_s(q_s) \odot T_\sigma(q_1, \ldots, q_s, q)\big)\Big) \\
&\quad = \mathrm{supp}\Big(\bigcup_{q_1,\ldots,q_s \in Q} \Big(\mathrm{supp}(X_1(q_1)) \odot \cdots \odot \mathrm{supp}(X_s(q_s)) \odot T_\sigma(q_1, \ldots, q_s, q)\Big)\Big) \\
&\quad \overset{\star_2}{=} \mathrm{supp}\Big(\bigcup_{q_1,\ldots,q_s \in Q} \Big(\mathrm{supp}(X'_1(q_1)) \odot \cdots \odot \mathrm{supp}(X'_s(q_s)) \odot T_\sigma(q_1, \ldots, q_s, q)\Big)\Big) \\
&\quad = \mathrm{supp}\Big(\bigcup_{q_1,\ldots,q_s \in Q} \Big(X'_1(q_1) \odot \cdots \odot X'_s(q_s) \odot T_\sigma(q_1, \ldots, q_s, q)\Big)\Big) \\
&\quad \overset{\star_3}{=} \mathrm{supp}(\mathcal{T}_\sigma(X'_{[s]}))(q),
\end{aligned}$$

where Equations $\star_1$ and $\star_3$ follow from Equation (3.7) (proof of Lemma 3.43) and Equation $\star_2$ follows from the fact that $X_1 \sim_{\mathrm{idp}} X_2$. This proves Claim 1.

Claim 2 follows from the definition of mindiv. In fact, for every multiset M over $\mathbb{M}$ it holds that $\mathrm{mindiv}(M) = \mathrm{mindiv}(\mathrm{supp}(M))$. Together with Claim 1 we obtain

$$\begin{aligned}\mathrm{mindiv}(\mathcal{T}_\sigma(X_{[s]})) &= \mathrm{mindiv}(\mathrm{supp}(\mathcal{T}_\sigma(X_{[s]}))) \\ &= \mathrm{mindiv}(\mathrm{supp}(\mathcal{T}_\sigma(X'_{[s]}))) = \mathrm{mindiv}(\mathcal{T}_\sigma(X_{[s]})).\end{aligned}$$

Claim 3 follows from the definition of minquot. First, note that for every multiset M over $\mathbb{M}$ and $y \in \mathrm{mindiv}(M)$ it holds that

$$\{\mathrm{supp}(N) \mid N \in \mathrm{minquot}_y(M)\} = \{\mathrm{supp}(N) \mid N \in \mathrm{minquot}_y(\mathrm{supp}(M))\}. \tag{3.14}$$

Now let $y \in \mathrm{mindiv}(\mathcal{T}_\sigma(X_{[s]}))$. We recall that $\mathcal{T}_\sigma(X_{[s]})$ is a map of type $Q \to \mathcal{M}_{\mathrm{fin}}(\mathbb{M})$. By applying Equation (3.14) pointwise we obtain

$$\begin{aligned}&\{\mathrm{supp}(X) \mid X \in \mathrm{minquot}_y(\mathcal{T}_\sigma(X_{[s]}))\} \\ &\qquad = \{\mathrm{supp}(X) \mid X \in \mathrm{minquot}_y(\mathrm{supp}(\mathcal{T}_\sigma(X_{[s]})))\} \\ &\qquad \overset{\star}{=} \{\mathrm{supp}(X') \mid X' \in \mathrm{minquot}_y(\mathrm{supp}(\mathcal{T}_\sigma(X'_{[s]})))\} \\ &\qquad = \{\mathrm{supp}(X') \mid X' \in \mathrm{minquot}_y(\mathcal{T}_\sigma(X'_{[s]}))\},\end{aligned}$$

where Equation $\star$ follows from Claim 1. This proves Claim 3.

Claim 4 can be seen as follows. First, we note that

$$\begin{aligned}&\{(y, \mathrm{supp}(X)) \mid (y, X) \in \mathrm{CenterFact}(\mathcal{T}_\sigma(X_{[s]}), f)\} \\ &\quad = \{(y, \mathrm{supp}(X)) \mid y \in \mathrm{mindiv}(\mathcal{T}_\sigma(X_{[s]}), X \in \mathrm{minquot}_y(\mathcal{T}_\sigma(X_{[s]})) \\ &\qquad\qquad , (y, X) \text{ is an } f\text{-centering factorisation of } \mathcal{T}_\sigma(X_{[s]})\} \\ &\quad \overset{\star_1}{=} \{(y', \mathrm{supp}(X)) \mid y' \in \mathrm{mindiv}(\mathcal{T}_\sigma(X'_{[s]})), X \in \mathrm{minquot}_{y'}(\mathcal{T}_\sigma(X_{[s]})) \\ &\qquad\qquad , (y', X) \text{ is an } f\text{-centering factorisation of } \mathcal{T}_\sigma(X_{[s]})\} \\ &\quad \overset{\star_2}{=} \{(y', \mathrm{supp}(X)) \mid y' \in \mathrm{mindiv}(\mathcal{T}_\sigma(X'_{[s]})), X \in \mathrm{minquot}_{y'}(\mathrm{supp}(\mathcal{T}_\sigma(X_{[s]}))) \\ &\qquad\qquad , (y', X) \text{ is an } f\text{-centering factorisation of } \mathrm{supp}(\mathcal{T}_\sigma(X_{[s]}))\} \\ &\quad \overset{\star_3}{=} \{(y', \mathrm{supp}(X)) \mid y' \in \mathrm{mindiv}(\mathcal{T}_\sigma(X'_{[s]})), X' \in \mathrm{minquot}_{y'}(\mathrm{supp}(\mathcal{T}_\sigma(X'_{[s]}))) \\ &\qquad\qquad , (y', X') \text{ is an } f\text{-centering factorisation of } \mathrm{supp}(\mathcal{T}_\sigma(X'_{[s]}))\} \\ &\quad \overset{\star_4}{=} \{(y', \mathrm{supp}(X')) \mid y' \in \mathrm{mindiv}(\mathcal{T}_\sigma(X'_{[s]})), X' \in \mathrm{minquot}_{y'}(\mathcal{T}_\sigma(X'_{[s]})) \\ &\qquad\qquad , (y', X') \text{ is an } f\text{-centering factorisation of } \mathcal{T}_\sigma(X'_{[s]})\} \\ &\quad = \{(y', \mathrm{supp}(X')) \mid (y', X') \in \mathrm{CenterFact}(\mathcal{T}_\sigma(X'_{[s]}), f)\},\end{aligned} \tag{3.15}$$

where Equation $\star_1$ follows from Claim 2, Equations $\star_2$ and $\star_4$ follow from pointwise application of Equation (3.14), and Equation $\star_3$ follows from Claim 3.

Let $(y, X) = \mathrm{minCenterFact}(\mathcal{T}_\sigma(X_{[s]}), f)$ and $(y', X') = \mathrm{minCenterFact}(\mathcal{T}_\sigma(X'_{[s]}), f)$. It holds that

$$\begin{aligned}(y, \mathrm{supp}(X)) &\overset{\star_1}{=} \min_{\widetilde{\sqsubseteq}}\big\{(\hat{y}, \mathrm{supp}(\hat{X})) \;\big|\; (\hat{y}, \hat{X}) \in \mathrm{CenterFact}(\mathcal{T}_\sigma(X_{[s]}), f)\big\} \\ &\overset{\star_2}{=} \min_{\widetilde{\sqsubseteq}}\big\{(\hat{y}', \mathrm{supp}(\hat{X}')) \;\big|\; (\hat{y}', \hat{X}') \in \mathrm{CenterFact}(\mathcal{T}_\sigma(X'_{[s]}), f)\big\} \\ &\overset{\star_3}{=} (y', \mathrm{supp}(X')),\end{aligned}$$

where Equation $\star_2$ follows from Equation (3.15) and Equations $\star_1$ and $\star_3$ can be seen as follows. We only prove Equation $\star_1$ by showing both "$\widetilde{\sqsubseteq}$" and "$\widetilde{\sqsupseteq}$". Clearly, "$\widetilde{\sqsupseteq}$" holds, as $(y, \mathrm{supp}(X))$ is an element of the set on the right hand side of Equation $\star_1$. Let $(\widetilde{y}, \widetilde{X}) = \min_{\widetilde{\sqsubseteq}}\big\{(\hat{y}, \mathrm{supp}(\hat{X})) \;\big|\; (\hat{y}, \hat{X}) \in \mathrm{CenterFact}(\mathcal{T}_\sigma(X_{[s]}), f)\big\}$ and assume that $(y, \mathrm{supp}(X)) \;\widetilde{\sqsubseteq}\; (\widetilde{y}, \widetilde{X})$ does not hold. Therefore, it must hold that (1) $\widetilde{X} \sqsubset \mathrm{supp}(X)$ or (2) $\widetilde{X} = \mathrm{supp}(X)$ and $\widetilde{y} \leq_\Gamma y$ and $\widetilde{y} \neq y$. In both cases, there exists $(\hat{y}, \hat{X}) \in \mathrm{CenterFact}(\mathcal{T}_\sigma(X_{[s]}), f)$ such that $(\hat{y}, \hat{X}) \;\widetilde{\sqsubset}\; (y, X)$, which is not possible since $(y, X) = \mathrm{minCenterFact}(\mathcal{T}_\sigma(X_{[s]}), f)$. This concludes the proof of Claim 4. □

Lemma 3.72. If S is additively idempotent, then $\sim_{\mathrm{idp}}$ is an accumulator of $D_{\mathscr{B}}$.

Proof. Let $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $X_1, \ldots, X_s, X'_1, \ldots, X'_s, X, X' \in Q''$ such that $X_i \sim X'_i$ for every $i \in [s]$, $T'_\sigma(X_1, \ldots, X_s, X) \neq \emptyset$, and $T'_\sigma(X'_1, \ldots, X'_s, X') \neq \emptyset$. Since $D_{\mathscr{B}}$ is sequential, all non-empty transition weights of $D_{\mathscr{B}}$ are singleton sets and hence, there exist $y, y' \in \mathbb{M}$ such that $T'_\sigma(X_1, \ldots, X_s, X) = \{\!\{y\}\!\}$ and $T'_\sigma(X'_1, \ldots, X'_s, X') = \{\!\{y'\}\!\}$. By the Definition of T', it holds that $(y, X') = \mathrm{minCenterFact}(\mathcal{T}_\sigma(X_1, \ldots, X_s), f)$ and $(y', X') = \mathrm{minCenterFact}(\mathcal{T}_\sigma(X'_1, \ldots, X'_s), f)$.

By Lemma 3.71, it holds that $y = y'$ and $X \sim_{\mathrm{idp}} X'$. Thus, we have shown the second property of Definition 3.63. It remains to show the first property of Definition 3.63. However, one can easily verify (using the idempotency of S) that

$$[\![\mathrm{final}'(X)]\!]_S = [\![\mathrm{final}'(\mathrm{supp}(X))]\!]_S = [\![\mathrm{final}'(\mathrm{supp}(X'))]\!]_S = [\![\mathrm{final}'(X')]\!]_S,$$

which concludes the proof. □

3.7.2 $\mathbb{M}$-Sequentialisation Results

In this chapter, we turn to the original problem of $\mathbb{M}$-sequentialisabilty. We combine all of our previous results to obtain a general $\mathbb{M}$-sequentialisation result. Given a WTA $\mathscr{A}$ over Σ and $\langle \mathbb{M} \rangle_{\oplus}$, we first use Lemma 3.55 to obtain a strongly related WTA $\mathscr{B}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$. Next, we use Definition 3.39 to construct the $\mathcal{S}(\mathbb{M})$-sequential but not necessarily finite WTA $D_{\mathscr{B}}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{M})$. Finally, we apply Definition 3.64 to accumulate $D_{\mathscr{B}}$ if an appropriate accumulator $\sim$ exists. This yields a finite $\mathbb{M}$-sequential WTA $\text{acc}_{\sim}(D_{\mathscr{B}})$ over Σ and S that is equivalent to $\mathscr{A}$ and concludes the $\mathbb{M}$-sequentialisation. This entire process is illustrated by our running example in the upcoming Example 3.73.

Example 3.73. We continue Example 3.65 by recalling the entire $\mathbb{N}$-sequentialisation process of $\mathscr{A}$ and depicting it in Figure 3.12.

In Example 2.5 we were given a weighted tree automaton $\mathscr{A}$ over the ranked alphabet $\Sigma = \{\alpha^{(0)}, \gamma^{(1)}, \sigma^{(2)}\}$ and the semiring $\mathbb{X} = (\mathbb{N} \cup \{\bot, \top\}, \vee, +, \bot, 0)$ from Example 2.4. Since the monoid $(\mathbb{N}, +, 0)$ is well-studied and computational methods can easily be applied to it, we wanted to $\mathbb{N}$-sequentialise the automaton $\mathscr{A}$, rather than simply determinise it. That is, we wanted to find a deterministic automaton $\mathscr{A}'$ such that $\mathscr{A}$ and $\mathscr{A}'$ are equivalent and all non-vanishing transition weights of $\mathscr{A}'$ are in $\mathbb{N}$.

First, via Examples 3.21 and 3.54, we fixed a WTA $\mathscr{B}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{N})$ such that $\mathscr{B}$ and $\mathscr{A}$ are strongly related. This step "reduces" the weight space from the semiring $\mathbb{X}$ to the monoid $\mathbb{N}$.

Next, in Example 3.41, we sequentialised $\mathscr{B}$ by constructing the WTA $D_{\mathscr{B}}$ over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{N})$. $D_{\mathscr{B}}$ is sequential (which is a shorthand for $\mathcal{S}(\mathbb{N})$-sequential) and equivalent to $\mathscr{B}$. However, the reachable part of $D_{\mathscr{B}}$ is not finite, which we tackled in the following step.

In Example 3.65, we defined the accumulator $\sim_{\mathbb{X}}$ of $D_{\mathscr{B}}$ and constructed the finite WTA $\text{acc}_{\sim_{\mathbb{X}}}(D_{\mathscr{B}})$ over Σ and $\mathbb{X}$. This step removes the infinity of $D_{\mathscr{B}}$ while keeping all transition weights in $\mathbb{N}$. Hence, the automaton $\text{acc}_{\sim_{\mathbb{X}}}(D_{\mathscr{B}})$ is the desired $\mathbb{N}$-sequential WTA over Σ and $\mathbb{X}$ that is equivalent to $\mathscr{A}$. This successfully concludes our $\mathbb{N}$-sequentialisation process. ◀

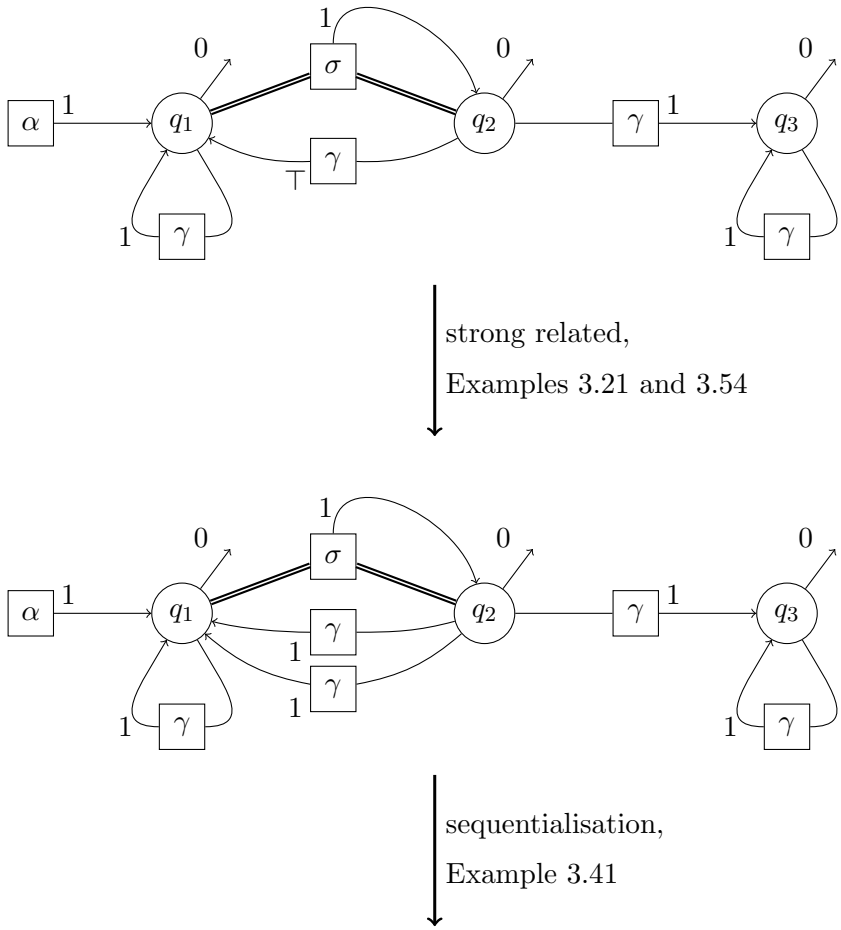
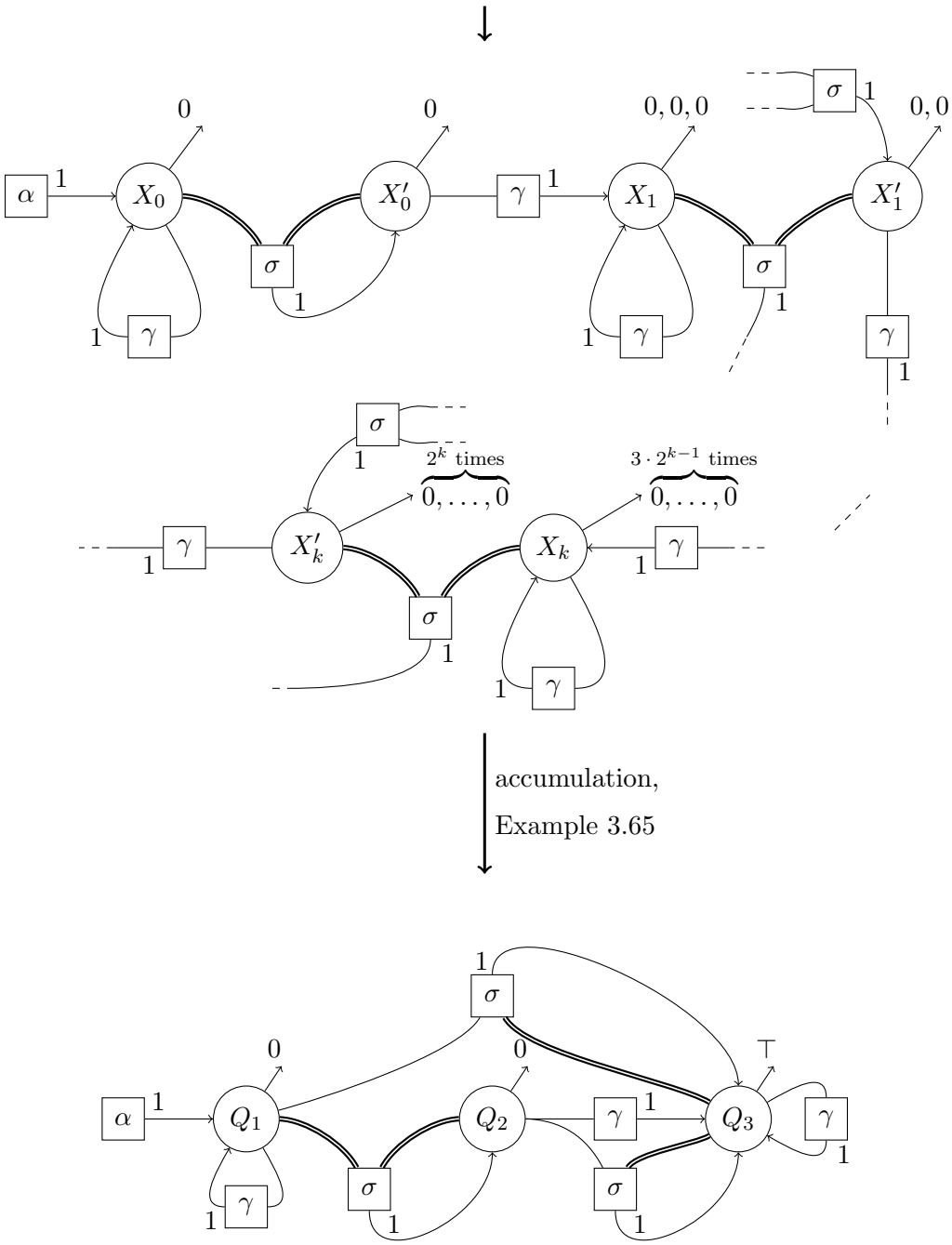


Figure 3.12: (Continues on next page) Illustration of the $\mathbb{N}$-sequentialisation process for $\mathscr{A}$ from Example 2.5. We omit the double curly braces from all multiset weights and use our notation for doubly stroked hyperedge lines from page 30.


accumulation,
Example 3.65

Lemma 3.74. Let $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ be commutative and let $\mathscr{A} = (Q, T, \mathrm{final})$ be a finite WTA over Σ and $\langle \mathbb{M} \rangle_{\oplus}$ such that $\mathscr{A} \vDash \mathrm{ETP}$. Moreover, let $\mathscr{B} \in \mathrm{StrongRel}(\mathscr{A})$ such that $\mathscr{B} \vDash \mathrm{ETP}$.

If there exists a finite accumulator $\sim$ of $D_{\mathscr{B}}$, then $\mathscr{A}$ is $\mathbb{M}$-sequentialisable.

Proof. Let $\sim$ be an accumulator of $D_{\mathscr{B}}$ such that $Q''/_{\sim}$ is finite. Let Q'' and $\widetilde{Q}$ be the set of reachable states of $D_{\mathscr{B}}$ and the set of states of $\mathrm{acc}_{\sim_{\mathrm{fa}}}(D_{\mathscr{B}})$, respectively.

By assumption, $\widetilde{Q}$ is finite. Hence, using Lemma 3.66, we obtain that $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$ is an $\mathbb{M}$-sequential finite WTA over Σ and $\langle \mathbb{M} \rangle_{\oplus}$.

By Lemma 3.53 it holds that $[\![\mathscr{A}]\!] = [\![[\![\mathscr{B}]\!]]\!]_S$ and by Lemma 3.68 it holds that $[\![[\![\mathscr{B}]\!]]\!]_S = [\![\mathrm{acc}_{\sim}(D_{\mathscr{B}})]\!]$ (note that this uses the assumption that $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ is commutative). This yields the fact that $\mathscr{A}$ is equivalent to $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$. Hence, $\mathscr{A}$ is $\mathbb{M}$-sequentialisable. An equivalent finite $\mathbb{M}$-sequential WTA is $\mathrm{acc}_{\sim}(D_{\mathscr{B}})$. $\square$

Lemma 3.75. Let $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ be commutative and let $\mathscr{A} = (Q, T, \mathrm{final})$ be a finitely $\mathbb{M}$-ambiguous finite WTA over Σ and $\langle \mathbb{M} \rangle_{\oplus}$.

If $\mathscr{A} \vDash \mathrm{ETP}$, then $\mathscr{A}$ is $\mathbb{M}$-sequentialisable.

Proof. Let $\mathscr{A} \vDash \mathrm{ETP}$. By Definition 3.59 there exists a WTA $\mathscr{B}$ over Σ and $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$ such that $\mathscr{B}$ is strongly related to $\mathscr{A}$ and $\mathscr{B} \vDash \mathrm{ETP}$. By Lemma 3.62, $\mathscr{B}$ is finitely R-ambiguous. Let Q'' and $\widetilde{Q}$ be the set of reachable states of $D_{\mathscr{B}}$ and the set of states of $\mathrm{acc}_{\sim_{\mathrm{fa}}}(D_{\mathscr{B}})$, respectively.

By Lemma 3.70, $\sim_{\mathrm{fa}}$ is an accumulator of $D_{\mathscr{B}}$. Thus, by Lemma 3.74, it suffices to show that $\widetilde{Q}$ is finite.

By Lemma 3.47, Q'' is finite (note that this uses the assumption that $\mathscr{B} \vDash \mathrm{ETP}$). By the definition of $\sim_{\mathrm{fa}}$ and $\widetilde{Q}$, it holds that $\widetilde{Q}$ and Q'' are bijective. Hence, $\widetilde{Q}$ is finite. $\square$

Lemma 3.76. Let $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ be commutative and let $\langle \mathbb{M} \rangle_{\oplus}$ be additively idempotent. Moreover, let $\mathscr{A} = (Q, T, \mathrm{final})$ be a finite WTA over Σ and $\langle \mathbb{M} \rangle_{\oplus}$.

If $\mathscr{A} \vDash \mathrm{ETP}$, then $\mathscr{A}$ is $\mathbb{M}$-sequentialisable.

Proof. Let $\mathscr{A} \vDash \mathrm{ETP}$. By Definition 3.59 there exists a WTA $\mathscr{B}$ over Σ and $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$ such that $\mathscr{B}$ is strongly related to $\mathscr{A}$ and $\mathscr{B} \vDash \mathrm{ETP}$. Let Q'' and $\widetilde{Q}$ be the set of reachable states of $D_{\mathscr{B}}$ and the set of states of $\mathrm{acc}_{\sim_{\mathrm{idp}}}(D_{\mathscr{B}})$, respectively.

By Lemma 3.72, $\sim_{\mathrm{idp}}$ is an accumulator of $D_{\mathscr{B}}$. Thus, by Lemma 3.74, it suffices to show that $\widetilde{Q}$ is finite. We define the set

$$K = \mathcal{P}_{\mathrm{fin}}(\mathcal{B}_{f(N_{\mathscr{B}})}(1))$$

and note that K is finite.

Let $X \in Q''$ and $q \in Q$. By Corollary 3.46, we have $\mathrm{supp}(X(q)) \in K$ and hence $\mathrm{supp}(X) \in K^Q$. We have obtained that the set $P = \{\mathrm{supp}(X) \mid X \in Q''\}$ is a subset of K^Q and thus P is finite. We define the map $h\colon \widetilde{Q} \to P$ by $h([X]_{\sim_{\mathrm{idp}}}) = \mathrm{supp}(X)$. By the definition of $\sim_{\mathrm{idp}}$, h is well-defined and bijective. This shows that $\#\widetilde{Q} = \#P$ and since P is finite, also $\widetilde{Q}$ is finite. □

We will now state our main results in a closed form. In particular, we collect all previous assumptions and repeat them in Theorems 3.77 and 3.78. An important difference to the word case [31, Theorems 77 and 78] is that we require $\mathbb{M}$ to be commutative if $\Sigma \neq \Sigma^{(0)} \cup \Sigma^{(1)}$. This is due to Lemma 3.43, where we move around factors in weights.

Theorem 3.77. Let S be an $\mathbb{M}$-semiring such that $\mathbb{M}$ is finitely generated by Γ, divides Γ-monotone, and admits centering factorisations. Let $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ be commutative and let $\mathscr{A}$ be a finite WTA over Σ and $\langle \mathbb{M} \rangle_{\oplus}$ such that $\mathscr{A} \vDash \mathrm{ETP}$. Let $\mathscr{B} \in \mathrm{StrongRel}(\mathscr{A})$ such that $\mathscr{B} \vDash \mathrm{ETP}$.

If there exists a finite accumulator $\sim$ of $D_{\mathscr{B}}$, then $\mathscr{A}$ is $\mathbb{M}$-sequentialisable.

Proof. This is an alternative formulation of Lemma 3.74 in a closed form. □

Theorem 3.78. Let S be an $\mathbb{M}$-semiring such that $\mathbb{M}$ is finitely generated by Γ, divides Γ-monotone, and admits centering factorisations. Let $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or $\mathbb{M}$ be commutative and let $\mathscr{A}$ be a finite WTA over Σ and $\langle \mathbb{M} \rangle_{\oplus}$. Moreover, let one of the following conditions hold.

1. $\mathscr{A}$ is finitely $\mathbb{M}$-ambiguous
2. $\langle \mathbb{M} \rangle_{\oplus}$ is additively idempotent

If $\mathscr{A} \vDash \mathrm{ETP}$, then $\mathscr{A}$ is $\mathbb{M}$-sequentialisable.

Proof. Lemmas 3.75 and 3.76 yield the result for Cases 1 and 2, respectively. □

Corollary 3.79. Let S be a semiring such that $(S, \odot, 1)$ is finitely generated by Γ, divides Γ-monotone, and admits centering factorisations. Let $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$ or S be commutative and let $\mathscr{A}$ be a finite WTA over Σ and S. Moreover, let one of the following conditions hold.

1. $\mathscr{A}$ is finitely S-ambiguous
2. S is additively idempotent

If $\mathscr{A} \models$ ETP, then $\mathscr{A}$ is determinisable.

3.8 Comparison of our Results to the Literature

This chapter is dedicated to an in-depth comparison of some determinisation results from the literature to our main $\mathbb{M}$-sequentialisation results (Theorems 3.77 and 3.78). We cover the publications [5, 14, 29], which in turn cover [21, 80].

Each of these publications is considered in a separate subchapter and all of these subchapters follow the same overall structure. First, we compare the notations and terminology from the respective literature to ours, followed by a restated version of the respective determinisation result in our terminology. Finally, we either prove that our result covers the result from the literature or we provide examples showing the incomparability of the results.

3.8.1 Determinisation of Unweighted Tree Automata

Before we compare our results to other weighted determinisation results, we briefly show that we cover the unweighted case.

Remark 3.80. We show that every WTA over Σ and $\mathbb{B}$ is determinisable in our framework. For this, we note that $\mathbb{B}$ is a $(\{\top\}, \wedge, \top)$-semiring.

In the following, we will show that (i) $(\{\top\}, \wedge, \top)$ has all properties necessary for our sequentialisation procedure and (ii) we can apply Corollary 3.79 to every unweighted tree automaton.

First, we note that $(\{\top\}, \wedge, \top)$ is finitely generated by $\Gamma = \{\top\}$. We proceed to show that $\mathbb{B}$ divides Γ-monotone and admits centering factorisations.

Let $m_1, m_2 \in \{\top\}$, $n \in \operatorname{div}(m_1, m_2)$, and $\{\!\{n_1, n_2\}\!\} \in \operatorname{quot}_n(m_1, m_2)$. The fact that $m_1 = m_2 = n = n_1 = n_2 = \top$ proves $d_\Gamma(m_1, m_2) = 0 = d_\Gamma(n_1, n_2)$ and hence, $\mathbb{B}$ divides Γ-monotone.

Now let $M \in \mathcal{M}_{\text{fin}}(\{\top\})$. It holds that $|n_1|_\Gamma = 0 = d_\Gamma(n_1, n_2)$ for every $n_1, n_2 \in \mathbb{B}$. Thus, every factorisation of M is an $\mathrm{id}_{\mathbb{N}}$-centering factorisation of M. We conclude that $\mathbb{B}$ admits centering factorisations.

Next we prove our claim (ii). Let $\mathscr{A}$ be a WTA over Σ and $\mathbb{B}$. It is obvious that every WTA over Σ and $\mathcal{M}_{\text{fin}}(\{\top\})$ has the ETP, since the only transition weight that may occur in an R-run is $\top$. Hence also $\mathscr{A} \vDash \text{ETP}$. Moreover, $\mathbb{B}$ is commutative and additively idempotent. Thus, by Theorem 3.78, $\mathscr{A}$ is determinisable. ◀

3.8.2 The Free Monoid Case

We now proceed to show how our $\mathbb{M}$-sequentialisation result (Theorem 3.78) can be applied to obtain the sequentialisation result from [5]. First, we dedicate some remarks to a comparison of the notations and terminology from [5] with the notations and terminology from Chapter 3. After that, we state the implication "third bullet $\Longrightarrow$ first bullet" from [5, Proposition 7] as a corollary of Theorem 3.78 (cf. Corollary 3.84) and prove it using our terminology.

We note that [5] considers only weighted word automata. Therefore we only deal with the case that $\Sigma = \Sigma_A$ for some alphabet A in this chapter. We recall that the components of a WA $\mathscr{A}$ over Σ and A are $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$.

Throughout Chapter 3.8.2 we assume Γ to be a finite set and consider the monoid $(\Gamma^*, \circ, \varepsilon)$, which is finitely generated by Γ. Moreover, we assume A to be an alphabet.

Remark 3.81. We have seen in Remark 3.20 that a WA $\mathscr{A}$ over A and $\mathcal{P}_{\text{fin}}(\Gamma^*)$ can be written equivalently in a relational way. In this sense, our model "WA over A and $\mathcal{P}_{\text{fin}}(\Gamma^*)$" can be compared to the model "transducer over $A^* \times \Gamma^*$" from [5]. In fact, besides the fact that [5] only allows for initial and final states (rather than weights), the only difference between the two automaton models is that the transducer model from [5] allows transitions to be in $Q \times A^* \times Q \times \Gamma^*$, whereas our WA model requires

transitions to be in $Q \times A \times Q \times \Gamma^*$. That is, the transducers introduced in [5] read entire words as inputs of transitions (*word-transitions*), where a WA reads only single symbols from A (*symbol-transitions*).

In the sequentialisation result [5, Proposition 7] it is required that every language accepted by a transducer has the type $A^* \to (\mathcal{S}(\Gamma^*) \cup \{\emptyset\})$ (there: "partial function"). It has been shown in [113, Proposition 1.1] that transducers recognizing such partial functions can be transformed into equivalent transducers using only symbol-transitions.

In total, we have seen that the transducer model from [5] is covered by our model of WA over A and $\mathcal{P}_{\text{fin}}(\Gamma^*)$ for the purpose of stating [5, Proposition 7].

Since $\mathcal{P}_{\text{fin}}(\Gamma^*) \subseteq \mathcal{M}_{\text{fin}}(\Gamma^*)$, every WA $\mathscr{B}$ over A and $\mathcal{P}_{\text{fin}}(\Gamma^*)$ can be considered as a WA over A and $\mathcal{M}_{\text{fin}}(\Gamma^*)$. Hence, with the term "$\mathscr{B} \models$ ETP" we refer to Definition 3.32 (rather than Definition 3.59). ◀

Remark 3.82. We now define a property of WA over A and $\mathcal{P}_{\text{fin}}(\Gamma^*)$ which resembles the twinning property from [5]. Let $\mathscr{B}$ be a WA over A and $\mathcal{P}_{\text{fin}}(\Gamma^*)$. We say that $\mathscr{B}$ has the BC–*twinning property* (short: $\mathscr{B} \models$ BCTP) if for every $u, v \in A^*$ and R-runs of the respective form

$$\xrightarrow{\#|x} p \xrightarrow{u|y_1} q \xrightarrow{v|y_2} q \qquad \text{and} \qquad \xrightarrow{\#|x'} p' \xrightarrow{u|y_1'} q' \xrightarrow{v|y_2'} q'$$

of $\mathscr{B}$ it holds that either $y_2 = y_2' = \varepsilon$ or there exists $w \in A^*$ such that either (a) $x \circ y_1 = x' \circ y_1' \circ w$ and $w \circ y_2 = y_2' \circ w$ or (b) $x' \circ y_1' = x \circ y_1 \circ w$ and $w \circ y_2' = y_2 \circ w$.

An easy comparison of the "twinning property" from [5] with the BCTP shows that the BCTP is a natural extension of the twinning property from [5] to the case of initial weights (rather than initial states). ◀

Remark 3.83. The definition of "sequential" from [5] can be adapted to allow initial weights, which reads in our terminology as follows. Let $\mathscr{B}$ be a WA over A and $\mathcal{P}_{\text{fin}}(\Gamma^*)$. We say that $\mathscr{B}$ is BC-*sequential* if (a) there exist at most one $q \in Q$ and $x \in \Gamma^*$ such that $x \in \text{init}(q)$ and (b) for every $p \in Q$ and $a \in A$ there exists at most one $q \in Q$ such that $T(p, a, q) \neq \emptyset$.

This yields that $\mathscr{B}$ is BC-sequential if and only if $\mathscr{B}$ is deterministic and all weights occurring in init are singleton sets over Γ^*. Hence, our $\mathcal{S}(\Gamma^*)$-sequentiality (see Chap-

ter 3.2) implies BC-sequentiality. This motivates why we state Corollary 3.84 using $\mathcal{S}(\Gamma^*)$-sequentialisability. ◀

We can now state the sequentialisation result from [5] in our notation.

Corollary 3.84 (Proposition 7 from [5], "third bullet $\implies$ first bullet")**.** Let $\mathscr{B}$ be a WA over A and $\mathcal{P}_{\text{fin}}(\Gamma^*)$. If $\mathscr{B} \vDash$ BCTP, then $\mathscr{B}$ is $\mathcal{S}(\Gamma^*)$-sequentialisable.

The rest of this chapter is dedicated to a formal proof of Corollary 3.84 using our sequentialisation results.

Lemma 3.85. $(\Gamma^*, \circ, \varepsilon)$ divides Γ-monotone and admits centering factorisations.

Proof. For every $m \in \Gamma^*$ we denote the set of *prefixes of* m by $\text{Prefix}(m)$. For every $n \in \text{Prefix}(m)$ we denote by ${}_n\backslash^m$ the unique element $n' \in \Gamma^*$ such that $n \circ n' = m$. For every $M \subseteq \Gamma^*$ we denote the *longest common prefix of all elements in* M by $\text{lcp}(M)$ and abbreviate $\text{lcp}(m, m') = \text{lcp}(M)$ if $M = \{m, m'\}$ for some $m, m' \in \Gamma^*$. We recall that $|m|$ is the length of the word $m \in \Gamma^*$. Since m is the unique path from ε to m in $\text{Cay}_{\Gamma^*,\Gamma}$ for every $m \in \Gamma^*$, one can easily see that

$$d_\Gamma(m, m') = |m| + |m'| - 2 \cdot |\text{lcp}(m, m')| \qquad \text{and} \qquad |m|_\Gamma = |m|.$$

Let $m, m' \in \Gamma^*$. It is clear that $\text{div}(m, m') = \text{Prefix}(m) \cap \text{Prefix}(m')$ and moreover, $\text{quot}_n(m, m') = \text{minquot}_n(m, m') = \{\{\!\{{}_n\backslash^m, {}_n\backslash^{m'}\}\!\}\}$ for every $n \in \text{div}(m, m')$. Now, let $n \in \text{div}(m, m')$. In order to show that Γ^* divides Γ-monotone, we need to show that

$$d_\Gamma({}_n\backslash^m, {}_n\backslash^{m'}) \leq d_\Gamma(m, m').$$

One can easily see that $\text{lcp}(m, m') = n \circ \text{lcp}({}_n\backslash^m, {}_n\backslash^{m'})$ and hence

$$\begin{aligned} d_\Gamma({}_n\backslash^m, {}_n\backslash^{m'}) &= |{}_n\backslash^m| + |{}_n\backslash^{m'}| - 2 \cdot |\text{lcp}({}_n\backslash^m, {}_n\backslash^{m'})| \\ &= |m| - |n| + |m'| - |n| - 2 \cdot (|\text{lcp}(m, m')| - |n|) \\ &= |m| + |m'| - 2 \cdot |\text{lcp}(m, m')| = d_\Gamma(m, m'). \end{aligned}$$

Next, we show that Γ^* admits centering factorisations. Let $M \in \mathcal{M}_{\text{fin}}(\Gamma^*)$. It is clear that $\text{div}(M) = \bigcap_{m \in \text{supp}(M)} \text{Prefix}(m)$ and for every $n \in \text{div}(M)$ we have

$$\text{quot}_n(M) = \text{minquot}_n(M) = \{\{\!\{{}_n\backslash^m \mid m \in M\}\!\}\}.$$

Therefore, the unique minimising divisor of M is $n = \mathrm{lcp}(\mathrm{supp}(M))$. Next, we let $N \in \mathrm{minquot}_n(M)$ and show that (n, N) is an $\mathrm{id}_{\mathbb{N}}$-centering factorization of M. Let $n_1 \in \mathrm{supp}(N)$. We note that for every $n_2 \in \mathrm{supp}(N)$ it holds that if $\mathrm{lcp}(n_1, n_2) = \varepsilon$, then the only fork-path connecting n_1 and n_2 is of the form (w, v) where $1 \overset{w}{\rightsquigarrow} n_1$ and $1 \overset{v}{\rightsquigarrow} n_2$ and hence, $|n_1|_\Gamma \leq |w| \leq |(w, v)| = d_\Gamma(n_1, n_2)$.

We assume that (n, N) is not an $\mathrm{id}_{\mathbb{N}}$-centering factorization of M. In particular, for every $n_2 \in \mathrm{supp}(N)$ it holds that $|n_1|_\Gamma > d_\Gamma(n_1, n_2)$. Thus, by our previous argument, it holds that $\mathrm{lcp}(n_1, n_2) \neq \varepsilon$ for every $n_2 \in \mathrm{supp}(N)$. Hence, $n_1[1]$ is a common prefix of all elements in N and in particular, $n \circ n_1[1]$ is a common prefix of M. This contradicts the fact that n is the longest common prefix of M. Thus, the assumption that (n, N) is not an $\mathrm{id}_{\mathbb{N}}$-centering factorisation of M must be dropped, which yields that Γ^* admits centering factorisations. □

Lemma 3.86. For every WA $\mathscr{B}$ over A and $\mathcal{P}_{\mathrm{fin}}(\Gamma^*)$ the following holds: if $\mathscr{B} \vDash$ BCTP, then $\mathscr{B} \vDash$ ETP.

Proof. Let $\mathscr{B} = (Q, T, \mathrm{final})$ be a WA over A and $\mathcal{P}_{\mathrm{fin}}(\Gamma^*)$ such that $\mathscr{B} \vDash$ BCTP. Moreover, let $u, v, w \in \Sigma^*$ and

$$\xrightarrow{\#|x} p \xrightarrow{u|y_1} q \xrightarrow{v|y_2} q \xrightarrow{w|y_3} r \qquad \text{and} \qquad \xrightarrow{\#|x'} p' \xrightarrow{u|y_1'} q' \xrightarrow{v|y_2'} q' \xrightarrow{w|y_3'} r'$$

be the respective forms of two R-runs of $\mathscr{B}$. Since $\mathscr{B} \vDash$ BCTP, it holds that either $y_2 = y_2' = \varepsilon$ or there exists $w \in \Sigma^*$ such that either (a) $x \circ y_1 = x' \circ y_1' \circ w$ and $w \circ y_2 = y_2' \circ w$ or (b) $x' \circ y_1' = x \circ y_1 \circ w$ and $w \circ y_2' = y_2 \circ w$.

If $y_2 = y_2' = \varepsilon$, then $d_\Gamma(x \circ y_1 \circ y_2 \circ y_3, x' \circ y_1' \circ y_2' \circ y_3') = d_\Gamma(x \circ y_1 \circ y_3, x' \circ y_1' \circ y_3')$.

Otherwise, since the conditions (a) and (b) from the BCTP are symmetrical, we assume without loss of generality that (a) holds and hence $x \circ y_1 = x' \circ y_1' \circ w$ and $w \circ y_2 = y_2' \circ w$. Moreover, we obtain that

$$x \circ y_1 \circ y_2 = x' \circ y_1' \circ w \circ y_2 = x' \circ y_1' \circ y_2' \circ w.$$

This shows that

$$\begin{aligned} d_\Gamma(x \circ y_1 \circ y_3, x' \circ y_1' \circ y_3') &= d_\Gamma(x' \circ y_1' \circ w \circ y_3, x' \circ y_1' \circ y_3') \\ &\overset{\star}{=} d_\Gamma(w \circ y_3, y_3') \\ &\overset{\star}{=} d_\Gamma(x' \circ y_1' \circ y_2' \circ w \circ y_3, x' \circ y_1' \circ y_2' \circ y_3') \\ &= d_\Gamma(x \circ y_1 \circ y_2 \circ y_3, x' \circ y_1' \circ y_2' \circ y_3'), \end{aligned}$$

where the equations marked with $\star$ follow from the fact that $d_\Gamma(y, y') = d_\Gamma(y'' \circ y, y'' \circ y')$ for every $y, y', y'' \in \Gamma^*$. This shows that $\mathscr{B} \vDash \text{ETP}$. □

Proof of Corollary 3.84. Let $S = \mathcal{P}_{\text{fin}}(\Gamma^*)$ and $\mathbb{M} = (\mathcal{S}(\Gamma^*), \odot, \{\varepsilon\})$. It surely holds that S is an $\mathbb{M}$-semiring and $S = \langle \mathbb{M} \rangle_\cup$. It is clear that $\mathbb{M}$ is finitely generated by $\mathcal{S}(\Gamma)$. By Lemma 3.85, it holds that Γ^* divides Γ-monotone and admits centering factorisations. This clearly implies that $\mathbb{M}$ divides $\mathcal{S}(\Gamma)$-monotone and admits centering factorisations. Moreover, $\Sigma_A = \Sigma_A^{(0)} \cup \Sigma_A^{(1)}$ and S is additively idempotent. Finally, by Lemma 3.86, it holds that $\mathscr{B} \vDash \text{ETP}$. Hence, we can apply Theorem 3.78 and obtain that $\mathscr{B}$ is $\mathbb{M}$-sequentialisable, which concludes the proof. □

3.8.3 The Group Case

In this chapter, we show how our $\mathbb{M}$-sequentialisation result (Theorem 3.78) can be applied to obtain the sequentialisation result from [29]. First, we dedicate some remarks to a comparison of the notations and terminology from [29] with the notations and terminology from Chapter 3. After that, we state the implication [29, Theorem 18] as a corollary of Theorem 3.78 (cf. Corollary 3.89) and prove it using our terminology.

We note that [29] generalises parts of [21] from the word case to the tree case. Therefore, Chapter 3.8.3 also provides a proof that our sequentialisation results cover the implication "*ii*) $\implies$ *iii*)" from [21, Theorem 2, case $k = 1$].

Throughout Chapter 3.8.3 we assume $(\mathbb{G}, \odot, 1)$ *to be a group.*

Remark 3.87. We briefly explain the difference between finitely generated groups and finitely generated monoids.

A set $\mathbb{U} \subseteq \mathbb{G}$ is called a *subgroup* of $\mathbb{G}$, in symbols $\mathbb{U} \leq \mathbb{G}$, if $(\mathbb{U}, \odot|_{\mathbb{U}\times\mathbb{U}}, 1)$ is a group. Given a set $\Gamma \subseteq \mathbb{G}$, we define the *subgroup of* $\mathbb{G}$ *generated by* Γ, denoted by $\langle \Gamma \rangle_{\odot}^{\mathrm{G}}$, as the smallest subgroup $\mathbb{U}$ of $\mathbb{G}$ such that $\Gamma \subseteq \mathbb{U}$. If $\langle \Gamma \rangle_{\odot}^{\mathrm{G}} = \mathbb{G}$, we say that $\mathbb{G}$ is *finitely generated as a group by* Γ. Using some elementary algebra, one can show that $\langle \Gamma \rangle_{\odot}^{\mathrm{G}} = \langle \Gamma \cup \bar{\Gamma} \rangle_{\odot}$, where $\bar{\Gamma} = \{\gamma^{-1} \mid \gamma \in \Gamma\}$. Hence, if $\mathbb{G}$ is finitely generated as a group by Γ, then $\mathbb{G}$ is finitely generated (as a monoid) by $\Gamma \cup \bar{\Gamma}$. ◀

Remark 3.88. We compare the terminology from [29] with our terminology.

- We have seen in Remark 3.20 that the automaton model from [29] is equivalent to the model of WTA over the semiring $\mathcal{P}_{\mathrm{fin}}(\mathbb{G})$.
- In [29, p. 270] it is required that the group $\mathbb{G}$ is "infinitary". However, this property of $\mathbb{G}$ is not used during the proof of [29, Theorem 2] and hence, we omit it in Corollary 3.89.
- We now define a property of WTA over Σ and $\mathcal{P}_{\mathrm{fin}}(\mathbb{G})$ which resembles the "twinning property" from [29, Definition 14]. Let $\mathscr{B}$ be a WTA over Σ and $\mathcal{P}_{\mathrm{fin}}(\mathbb{G})$. We say that $\mathscr{B}$ *has the* DFS-*twinning property* (in symbols: $\mathscr{B} \vDash$ DFSTP), if for every $\xi \in \mathrm{T}_{\Sigma}$, $\zeta \in \mathrm{C}_{\Sigma}$, and R-runs of the respective form

  $$\xrightarrow{\xi|y_1} q \xrightarrow{\zeta|y_2} q \qquad \text{and} \qquad \xrightarrow{\xi|y_1'} q' \xrightarrow{\zeta|y_2'} q'$$

  of $\mathscr{B}$ it holds that $y_2 = y_2'$. An easy comparison of the twinning property from [29] with DFSTP shows that the two properties are equivalent.

  We note that in [29] it is assumed that $\mathbb{G}$ is commutative, which has the side effect that the DFSTP has this seemingly very restrictive form. However, there exists an equivalent definition of the DFSTP which represents our idea behind it better. More precisely, one can replace the condition "$y_2 = y_2'$" by

  $$(y_1)^{-1} \odot y_1' = (y_1 \odot y_2)^{-1} \odot y_1' \odot y_2'.$$

  These two conditions are equivalent if $\mathbb{G}$ is commutative and the latter one is less restrictive if $\mathbb{G}$ is non-commutative, without affecting the implied sequentialisability (cf. also [21]).

- Since $\mathcal{P}_{\text{fin}}(\mathbb{G}) \subseteq \mathcal{M}_{\text{fin}}(\mathbb{G})$, every WTA $\mathscr{B}$ over Σ and $\mathcal{P}_{\text{fin}}(\mathbb{G})$ can be considered as a WTA over Σ and $\mathcal{M}_{\text{fin}}(\mathbb{G})$. Hence, with the term "$\mathscr{B} \vDash$ ETP" we refer to Definition 3.32.

- The definition of "sequential" from [29, Definition 9] reads in our terminology as follows. Let $\mathscr{B} = (Q, T, \text{final})$ be a WTA over Σ and $\mathcal{P}_{\text{fin}}(\mathbb{G})$. We call $\mathscr{B}$ DFS-*sequential* if for every $s \geq 0$, $\sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s \in Q$ there exist at most one $(q, g) \in Q \times \mathbb{G}$ such that $g \in T_\sigma(q_1, \ldots, q_s, q)$.

  This yields that $\mathscr{B}$ is DFS-sequential if and only if $\mathscr{B}$ is deterministic and all weights occurring in T are singleton sets over $\mathbb{G}$. We obtain that DFS-sequentiality is equivalent to $\mathcal{S}(\mathbb{G})$-sequentiality. ◀

We can now state the sequentialisation result from [21] in our notation.

Corollary 3.89 (Theorem 18 from [29])**.** Let $\mathbb{G}$ be commutative and finitely generated as a group by some set Γ and let $\mathscr{B}$ be a WTA over Σ and $\mathcal{P}_{\text{fin}}(\mathbb{G})$. If $\mathscr{B} \vDash$ DFSTP, then $\mathscr{B}$ is $\mathcal{S}(\mathbb{G})$-sequentialisable.

The rest of this chapter is dedicated to a formal proof of Corollary 3.89 in our terminology.

Throughout the rest of Chapter 3.8.3 we assume $\mathbb{G}$ to be commutative and finitely generated as a group by some set Γ.

Lemma 3.90. Let $M \in \mathcal{M}_{\text{fin}}(\mathbb{G})$. It holds that $\text{div}(M) = \mathbb{G}$. Moreover, for every $g \in \mathbb{G}$ it holds that $\text{quot}_g(M) = \text{minquot}_g(M) = \{g^{-1} \odot M\}$.

Proof. In order to show that the first claim holds, we let $g' \in \mathbb{G}$ and $N = g'^{-1} \odot M$ and note that $g' \odot N = M$. The second claim holds since $\mathbb{G}$ being a group implies that the equation $g \odot N = M$ has the unique solution $N = g^{-1} \odot M$. □

Lemma 3.91. It holds that $(\mathbb{G}, \odot, 1)$ divides $(\Gamma \cup \bar{\Gamma})$-monotone and admits centering factorisations.

Proof. We obtain from Remark 3.87 that $\mathbb{G}$ is finitely generated (as a monoid) by $\Gamma \cup \bar{\Gamma}$.

First we show that $\mathbb{G}$ divides $(\Gamma \cup \bar{\Gamma})$-monotone. Let $g_1, g_2, g \in \mathbb{G}$. It follows from Lemma 3.7 that

$$\begin{aligned} d_{(\Gamma\cup\bar{\Gamma})}(g_1, g_2) &\geq d_{(\Gamma\cup\bar{\Gamma})}(g \odot g_1, g \odot g_2) \\ &\geq d_{(\Gamma\cup\bar{\Gamma})}(g^{-1} \odot g \odot g_1, g^{-1} \odot g \odot g_2) = d_{(\Gamma\cup\bar{\Gamma})}(g_1, g_2). \end{aligned}$$

Therefore, we obtain

$$\forall g_1, g_2, g \in \mathbb{G} \colon d_{(\Gamma\cup\bar{\Gamma})}(g_1, g_2) = d_{(\Gamma\cup\bar{\Gamma})}(g \odot g_1, g \odot g_2). \tag{3.16}$$

By Lemma 3.90, $\operatorname{div}(g_1, g_2) = \mathbb{G}$ and $\operatorname{quot}_g(g_1, g_2) = \{\!\{g^{-1} \odot g_1, g^{-1} \odot g_2\}\!\}$ for every $g \in \mathbb{G}$. Therefore, we can apply Equation (3.16) where g is replaced by g^{-1} and obtain that $\mathbb{G}$ divides $(\Gamma \cup \bar{\Gamma})$-monotone.

Next we show that $\mathbb{G}$ admits $\mathrm{id}_{\mathbb{N}}$-centering factorisations. Let $M \in \mathcal{M}_{\mathrm{fin}}(\mathbb{G})$, $g \in \operatorname{mindiv}(M)$, and $N \in \operatorname{minquot}_g(M)$. By Lemma 3.90, $N = g^{-1} \odot M$. We will show that (g, N) is an $\mathrm{id}_{\mathbb{N}}$-centering factorisation of M by contraposition. Assume that (g, N) is not an $\mathrm{id}_{\mathbb{N}}$-centering factorisation of M. Then, there exists an element $n_1 \in \operatorname{supp}(N)$ such that $|n_1|_{(\Gamma\cup\bar{\Gamma})} > d_{(\Gamma\cup\bar{\Gamma})}(n_1, n_2)$ for every $n_2 \in \operatorname{supp}(N)$. Let $r = \max_{n_2 \in \operatorname{supp}(N)} d_{(\Gamma\cup\bar{\Gamma})}(n_1, n_2)$ and observe that $|N|_{(\Gamma\cup\bar{\Gamma})} \geq |n_1|_{(\Gamma\cup\bar{\Gamma})} > r$.

It holds that $g \odot n_1 \in \operatorname{supp}(M)$ and we consider $N' = (g \odot n_1)^{-1} \odot M$. Surely, $N' = n_1^{-1} \odot N$ and hence it holds that $\operatorname{supp}(N') = \operatorname{supp}(n_1^{-1} \odot N) = n_1^{-1} \odot \operatorname{supp}(N)$. Moreover, since $\operatorname{supp}(N) \subseteq \mathcal{B}_r(n_1)$, we obtain that $\operatorname{supp}(N') \subseteq n_1^{-1} \odot \mathcal{B}_r(n_1)$.

For every $n_2 \in \mathbb{G}$ it holds that

$$\begin{aligned} n_2 \in \mathcal{B}_r(n_1) &\iff \exists w \in \operatorname{Paths}(n_1, n_2) \colon |w| \leq r \\ &\iff \exists w \in \operatorname{Paths}(n_1^{-1} \odot n_1, n_1^{-1} \odot n_2) \colon |w| \leq r \\ &\iff \exists w \in \operatorname{Paths}(1, n_1^{-1} \odot n_2) \colon |w| \leq r \iff n_1^{-1} \odot n_2 \in \mathcal{B}_r(1). \end{aligned}$$

This shows that $n_1^{-1} \odot \mathcal{B}_r(n_1) = \mathcal{B}_r(1)$ and hence we obtain $\operatorname{supp}(N') \subseteq \mathcal{B}_r(1)$. This proves that $|N'|_{(\Gamma\cup\bar{\Gamma})} \leq r$ and hence $|N'|_{(\Gamma\cup\bar{\Gamma})} < |N|_{(\Gamma\cup\bar{\Gamma})}$, which contradicts the fact that $g \in \operatorname{mindiv}(M)$. Hence, the assumption that (g, N) is not an $\mathrm{id}_{\mathbb{N}}$-centering factorisation is false. □

Lemma 3.92. For every WTA $\mathscr{B}$ over Σ and $\mathcal{P}_{\mathrm{fin}}(\mathbb{G})$ it holds that if $\mathscr{B} \vDash$ DFSTP, then $\mathscr{B} \vDash$ ETP.

Proof. We note that for every $g_1, g_2 \in \mathcal{S}(\mathbb{G})$ it holds that

$$d_{(\Gamma\cup\bar{\Gamma})}(g_1, g_2) = |g_1^{-1} \odot g_2|_{(\Gamma\cup\bar{\Gamma})}, \tag{3.17}$$

which can be seen as follows. We obtain $d_{(\Gamma\cup\bar{\Gamma})}(g_1, g_2) = d_{(\Gamma\cup\bar{\Gamma})}(1, g_1^{-1} \odot g_2)$ from Equation (3.16) by letting $g = g_1^{-1}$. Moreover, $d_{(\Gamma\cup\bar{\Gamma})}(1, g_1^{-1} \odot g_2) = |g_1^{-1} \odot g_2|_{(\Gamma\cup\bar{\Gamma})}$ since (1) every $(w, v) \in \mathrm{FPaths}(1, g_1^{-1} \odot g_2)$ yields the path $u \in \mathrm{Paths}(1, g_1^{-1} \odot g_2)$ given by $u = w[|w|]^{-1} \dots w[1]^{-1} v$, which satisfies $|(w, v)| = |u|$ and (2) every path $u \in \mathrm{Paths}(1, g_1^{-1} \odot g_2)$ satisfies $(1, u) \in \mathrm{FPaths}(1, g_1^{-1} \odot g_2)$.

Let $\mathscr{B} = (Q, T, \mathrm{final})$ be a WTA over Σ and $\mathcal{P}_{\mathrm{fin}}(\mathbb{G})$ such that $\mathscr{B} \vDash \mathrm{DFSTP}$. Let $\xi \in \mathrm{T}_\Sigma$, $\zeta, \eta \in \mathrm{C}_\Sigma$, and

$$\xrightarrow{\xi|y_1} q \xrightarrow{\zeta|y_2} q \xrightarrow{\eta|y_3} p \qquad \text{and} \qquad \xrightarrow{\xi|y_1'} q' \xrightarrow{\zeta|y_2'} q' \xrightarrow{\eta|y_3'} p'$$

be the respectife forms of two R-runs of $\mathscr{B}$. We obtain that

$$\begin{aligned}
d_{(\Gamma\cup\bar{\Gamma})}(x \odot y_1 \odot y_3, x' \odot y_1' \odot y_3') &\overset{\star_1}{=} |(x \odot y_1 \odot y_3)^{-1} \odot x' \odot y_1' \odot y_3'|_{(\Gamma\cup\bar{\Gamma})} \\
&\overset{\star_2}{=} |y_3^{-1} \odot y_1^{-1} \odot x^{-1} \odot x' \odot y_1' \odot y_3'|_{(\Gamma\cup\bar{\Gamma})} \\
&\overset{\star_3}{=} |y_3^{-1} \odot y_2^{-1} \odot y_2 \odot y_1^{-1} \odot x^{-1} \odot x' \odot y_1' \odot y_3'|_{(\Gamma\cup\bar{\Gamma})} \\
&\overset{\star_4}{=} |y_3^{-1} \odot y_2^{-1} \odot y_1^{-1} \odot x^{-1} \odot x' \odot y_1' \odot y_2' \odot y_3'|_{(\Gamma\cup\bar{\Gamma})} \\
&\overset{\star_5}{=} |(x \odot y_1 \odot y_2 \odot y_3)^{-1} \odot x' \odot y_1' \odot y_2' \odot y_3'|_{(\Gamma\cup\bar{\Gamma})} \\
&\overset{\star_6}{=} d_{(\Gamma\cup\bar{\Gamma})}(x \odot y_1 \odot y_2 \odot y_3, x' \odot y_1' \odot y_2' \odot y_3'),
\end{aligned}$$

where Equations $\star_1$ and $\star_6$ follow from Equation (3.17), Equations $\star_2$ and $\star_5$ follow from standard group arithmetics, Equation $\star_3$ follows from the fact that $y_2^{-1} \odot y_2 = 1$, and Equation $\star_4$ follows from the fact that $\mathscr{B} \vDash \mathrm{DFSTP}$ (whence $y_2 = y_2'$) and the commutativity of $\mathbb{G}$. Therefore, we have shown that $\mathscr{B} \vDash \mathrm{ETP}$. □

Proof of Corollary 3.89. Let $S = \mathcal{P}_{\mathrm{fin}}(\mathbb{G})$ and $\mathbb{M} = (\mathcal{S}(\mathbb{G}), \odot, \{1\})$. It surely holds that S is an $\mathbb{M}$-semiring and $S = \langle \mathbb{M} \rangle_\cup$. By Remark 3.87, $\mathbb{M}$ is finitely generated by $\mathcal{S}(\Gamma \cup \bar{\Gamma})$ and by Lemma 3.91, $\mathbb{G}$ divides $(\Gamma \cup \bar{\Gamma})$-monotone and admits centering factorisations. This clearly yields that $\mathbb{M}$ divides $\mathcal{S}(\Gamma \cup \bar{\Gamma})$-monotone and admits centering factorisations. Moreover, S is additively idempotent and commutative. Finally, by Lemma 3.92, it holds that $\mathscr{B} \vDash \mathrm{ETP}$. Hence, we can apply Theorem 3.78 and obtain that $\mathscr{B}$ is $\mathbb{M}$-sequentialisable, which concludes the proof. □

3.8.4 The Extremal Case

In this chapter, we compare our sequentialisation result (Theorems 3.77 and 3.78) and the determinisation result from [14]. First, we recall some of the notations and terminology from [14] and then state the determinisation result [14, Theorem 5.2] using our terminology in Theorem 3.94. Next, we compare our ETP to the "twins property" from [14] and show in Example 3.95 that these properties are incomparable. This shows that Theorem 3.77 and Theorem 3.94 are incomparable.

To fully understand the applicability of the different approaches, we believe that it is also necessary to analyse the algebraic assumptions of Theorem 3.77 and Theorem 3.94. In Example 3.96 we provide semirings where Theorem 3.77 can be applied but not Theorem 3.94. In Lemma 3.98 we collect algebraic properties of S that are implied by the existence of "maximal factorisations" from [14]. It remains an open problem whether there exist semirings where Theorem 3.94 can be applied but not Theorem 3.77. However, we believe that such semirings do not exist and formulate this in Conjecture 3.99.

Throughout Chapter 3.8.4 we assume S to be an extremal, commutative semiring and $\mathscr{A} = (Q, T, \mathrm{final})$ to be a WTA *over Σ and S.*

Remark 3.93. We introduce some terminology from [14] and compare it to our terminology. We note that scalar-vector multiplications are evaluated componentwise.

- An important assumption in [14] is that S admits the following factorisation approach. Let $n \in \mathbb{N}$. A pair of maps (f, g) where $f\colon S^n \to S^n$ and $g\colon S^n \to S$ is called *maximal factorisation of dimension* n if for every $u \in S^n \setminus \{0^n\}$ we have that $g(u) \odot f(u) = u$ and $f(c \odot u) = f(u)$ for every $c \in S$. We omit writing "of dimension n" whenever n is clear from the context.

- We now define a property of WTA over Σ and S which resembles the "twins property" from [14]. Let $\mathscr{A}$ be a WTA over Σ and S. We say that $\mathscr{A}$ *has the* BMV*-twinning property* (in symbols: $\mathscr{A} \vDash$ BMVTP), if for every $\xi \in \mathrm{T}_\Sigma$, $\zeta \in \mathrm{C}_\Sigma$ and $q, q' \in Q$ such that $\mathrm{Runs}^v_{\mathscr{A}}(\xi, q) \neq \emptyset$, $\mathrm{Runs}^v_{\mathscr{A}}(\xi, q') \neq \emptyset$, $\mathrm{Runs}^v_{\mathscr{A}}(q, \zeta, q) \neq \emptyset$,

and $\mathrm{Runs}^v_{\mathscr{A}}(q', \zeta, q') \neq \emptyset$ it holds that

$$\bigoplus_{\rho \in \mathrm{Runs}_{\mathscr{A}}(q,\zeta,q)} \mathrm{wt}(\zeta, \rho) = \bigoplus_{\rho \in \mathrm{Runs}_{\mathscr{A}}(q',\zeta,q')} \mathrm{wt}(\zeta, \rho).$$

An easy comparison of the "twins property" from [14] with the BMVTP shows that these properties are equivalent.

- In [14, Theorem 5.2], the claim is that, under certain conditions, the determinisation construction given in [14] is successful. We weaken this formulation and rather claim the determinisability of $\mathscr{A}$.

  Moreover, [14, Theorem 5.2] gives four cases (separated by bullet points) in which their determinisation construction is successful. We only consider the last case in Theorem 3.94 and briefly recall the other three cases now. The first case is that $\mathscr{A}$ is *non-recursive*, which implies that $\mathscr{A}$ does not contain any valid loops. The second case is that S is locally finite. The third case is that $\mathscr{A}$ is already deterministic. In any of these three cases, it is known (and can straightforwardly be proven) that $\mathscr{A}$ is determinisable by a weighted power set construction without factorisation. ◀

We can now state the determinisation result from [14] in our notation.

Theorem 3.94 (Theorem 5.2 from [14], last case)**.** Assume that there exists a maximal factorisation of dimension $\#Q$. If $\mathscr{A} \vDash$ BMVTP, then $\mathscr{A}$ is determinisable.

The following example shows that the twinning properties of Theorem 3.77 and 3.94 are incomparable in general. In fact, we provide weighted (word) automata that already witness this incomparability.

Example 3.95. We consider the ranked alphabet Σ_A for the alphabet $A = \{a\}$ and the semiring

$$S = (\mathbb{N}_{\leq k} \cup \{-\infty\}, \max, +_k, -\infty, 0)$$

for $k = 5$. S is an $\mathbb{N}_{\leq k}$-semiring and we consider the finite generating set $\Gamma = \{1\}$ of $\mathbb{N}_{\leq k}$. We recall from Example 3.5 that $d_\Gamma(n_1, n_2) = |n_1 - n_2|$ for every $n_1, n_2 \in \mathbb{N}_{\leq k}$.

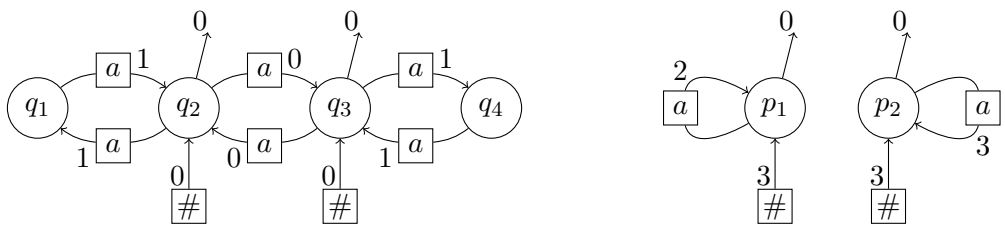


Figure 3.13: Two WTA over $\Sigma = \{a\}$ and S from Example 3.95. The one on the left has the BMVTP but not the ETP. The one on the right has the ETP but not the BMVTP.

- We start by giving a WTA $\mathscr{A}_1$ over Σ_A and S such that $\mathscr{A}_1 \vDash$ BMVTP and $\mathscr{A}_1 \nvDash$ ETP. Formally, let $\mathscr{A}_1 = (Q_1, \mathrm{init}_1, T_1, \mathrm{final}_1)$, where $Q_1 = \{q_1, q_2, q_3, q_4\}$ and every transition weight and final weight of $\mathscr{A}_1$ is $-\infty$ except

$$\begin{aligned}
&\mathrm{final}_1(q_2) = \mathrm{final}_1(q_3) = 0,\\
&(T_1)_\#(q_2) = (T_1)_\#(q_3) = 0,\\
&(T_1)_a(q_3, q_2) = (T_1)_a(q_2, q_3) = 0, \text{ and}\\
&(T_1)_a(q_1, q_2) = (T_1)_a(q_2, q_1) = (T_1)_a(q_4, q_3) = (T_1)_a(q_3, q_4) = 1.
\end{aligned}$$

  A graphical representation of $\mathscr{A}_1$ can be found in the left of Figure 3.13.

  Since $\mathrm{im}(\mathscr{A}_1) \subseteq (\mathbb{N}_{\leq k} \cup \{-\infty\})$, that is, each weight occuring in $\mathscr{A}_1$ is either in $\mathbb{N}_{\leq k}$ or the zero element of S, there exists a unique WTA $\mathscr{B}_1$ over Σ_A and $\mathcal{M}_{\mathrm{fin}}(\mathbb{N}_{\geq k})$ that is strongly related to $\mathscr{A}_1$. Intuitively, $\mathscr{B}_1$ can be obtained from $\mathscr{A}_1$ by replacing every weight n occuring in $\mathscr{A}_1$ with $\{\!\{n\}\!\}$. The fact that $\mathscr{B}_1$ does not have the ETP is witnessed by the unique R-runs of the respective form $\xrightarrow{\#|0} q_3 \xrightarrow{aa|0} q_3$ and $\xrightarrow{\#|0} q_3 \xrightarrow{aa|2} q_3$ of $\mathscr{B}_1$, as $d_\Gamma(0+0, 0+2) = 2 \neq 0 = d_\Gamma(0,0)$. The fact that $\mathscr{A}_1$ has the BMVTP can be seen as follows. Let $\zeta \in \mathrm{C}_{\Sigma_A}$ and $q \in Q_1$. If $\zeta = a^n(x_1)$ for an odd $n \in \mathbb{N}$, then $\mathrm{Runs}_{\mathscr{A}}(q, \zeta, q) = \emptyset$. If $\zeta = a^n(x_1)$ for an even $n \in \mathbb{N}$, then an easy calculation shows that

$$\bigoplus_{\rho \in \mathrm{Runs}_{\mathscr{A}}(q,\zeta,q)} \mathrm{wt}(\zeta, \rho) = \min\{n, 5\},$$

  where the right-hand side of the equation does not depend on q. Hence, $\mathscr{A}_1$ has the BMVTP.

- We continue by giving a WTA $\mathscr{A}_2$ over Σ_A and S such that $\mathscr{A}_2 \not\models$ BMVTP and $\mathscr{A}_2 \models$ ETP. Formally, $\mathscr{A}_2 = (Q_2, \mathrm{init}_2, T_2, \mathrm{final}_2)$, where $Q_2 = \{p_1, p_2\}$ and every transition weight and final weight of $\mathscr{A}_2$ is $-\infty$ except

$$\begin{aligned}
&\mathrm{final}_2(p_1) = \mathrm{final}_2(p_2) = 0,\\
&(T_2)_{\#}(p_1) = (T_2)_{\#}(p_2) = 3,\\
&(T_2)_a(p_1, p_1) = 2, \text{ and } = (T_2)_a(p_2, p_2) = 3.
\end{aligned}$$

A graphical representation of $\mathscr{A}_2$ can be found in the right of Figure 3.13.

Surely, it holds that $\mathscr{A}_2 \not\models$ BMVTP. Since $\mathrm{im}(\mathscr{A}_1) \subseteq (\mathbb{N}_{\leq k} \cup \{-\infty\})$, there exists a unique WTA $\mathscr{B}_2$ over Σ_A and $\mathcal{M}_{\mathrm{fin}}(\mathbb{M})$ that is strongly related to $\mathscr{A}_2$. Let $\xi \in \mathrm{T}_{\Sigma_A}$, $\zeta, \eta \in \mathrm{C}_{\Sigma_A}$ and

$$\xrightarrow{\xi|y_1} q \xrightarrow{\zeta|y_2} q \xrightarrow{\eta|y_3} p \qquad \text{and} \qquad \xrightarrow{\xi|y'_1} q' \xrightarrow{\zeta|y'_2} q' \xrightarrow{\eta|y'_3} p'$$

be the respective form of two R-runs of $\mathscr{B}_2$. We have to show

$$d_\Gamma(y_1 +_k y_2 +_k y_3, y'_1 +_k y'_2 +_k y'_3) = d_\Gamma(y_1 +_k y_3, y'_1 +_k y'_3). \tag{3.18}$$

First, we note that if $\xi = a^n(\#)$ for some $n \geq 1$, then $y_1 = y'_1 = 5$ and hence Equation (3.18) holds. Moreover, if $\zeta = x_1$, then we have $y_2 = y'_2 = 0$, which immediately yields Equation (3.18). Now assume that $\xi = \#$ and $\zeta \neq x_1$. By the definition of $\mathscr{A}_2$ it holds that $y_1 = y'_1 = 3$, $y_2 \geq 2$, and $y'_2 \geq 2$. Therefore

$$y_1 +_k y_2 +_k y_3 = y'_1 +_k y'_2 +_k y'_3 = 5$$

and hence the left hand side of Equation (3.18) is equal to 0. Now, if $\eta = x_1$, then $y_3 = y'_3 = 0$ and if $\eta \neq x_1$, then $y_3 \geq 2$ and $y'_3 \geq 2$. In both cases, the right hand side of Equation (3.18) is equal to 0 as well. Therefore, $\mathscr{A} \models$ ETP. ◀

We have seen that our sequentialisation result (Theorem 3.77) and the deteremini-sation result from [14] (Theorem 3.94) are incomparable because of the different notions of the twinning property. We will now turn towards the algebraic assumptions of the respective theorems.

Example 3.96. We show that Theorem 3.77 covers semirings that are not covered by Theorem 3.94. More precisely, we give both a non-extremal and an extremal semiring where Theorem 3.77 can be applied but not Theorem 3.94.

- The semiring $\mathbb{X}$ is not idempotent and hence also not extremal. Thus, Theorem 3.94 is not applicable.

  However, $\mathbb{X}$ is an $\mathbb{N}$-semiring and $\mathbb{N}$ divides Γ-monotone and admits centering factorisations for $\Gamma = \{1\}$. Therefore, we can apply Theorem 3.77 to every finite WTA $\mathscr{A}$ over Σ and $\mathbb{X}$ such that (a) $\mathscr{A} \vDash$ ETP and (b) there exists a finite accumulator $\sim$ of $D_{\mathscr{B}}$ for some $\mathscr{B} \in \mathrm{StrongRel}(\mathscr{A})$ satisfying $\mathscr{B} \vDash$ ETP. One such automaton $\mathscr{A}$ is given in Example 3.73.

- We consider the monoid $(\mathbb{N}^2, +, (0,0))$ and the subset $\Gamma = \big\{(2,0),(1,1),(0,2)\big\}$ of $\mathbb{N}^2$. It holds that $\langle\Gamma\rangle_+ = \{(i,j) \in \mathbb{N}^2 \mid i+j \text{ is even}\}$. Moreover, we consider the order $\leq$ on $\mathbb{N}^2$ where $(i,j) \leq (i',j')$ if and only if $2^i \cdot 3^j \leq 2^{i'} \cdot 3^{j'}$. The semiring

  $$S = \Big(\langle\Gamma\rangle_+ \cup \{-\infty\}, \max_{\leq}, +, -\infty, (0,0)\Big)$$

  is an extremal and commutative $\langle\Gamma\rangle_+$-semiring. Assume that there exists a maximal factorisation (f,g) of dimension 2. It holds that $(2,0) + \big((1,1),(0,2)\big) = \big((3,1),(2,2)\big) = (1,1) + \big((2,0),(1,1)\big)$ and hence

  $$f\big((1,1),(0,2)\big) = f\big((3,1),(2,2)\big) = f\big((2,0),(1,1)\big). \tag{3.19}$$

  The only common divisor of $(1,1)$ and $(0,2)$ is $g\big((1,1),(0,2)\big) = (0,0)$ and hence $f\big((1,1),(0,2)\big) = \big((1,1),(0,2)\big)$. Analogously, $f\big((2,0),(1,1)\big) = \big((2,0),(1,1)\big)$. This is a contradiction to Equation (3.19). In fact, this example is a slightly simplified version of the counterexample given in [80, pages 9–10].

  Next, we outline the proof that $\langle\Gamma\rangle_+$ divides Γ-monotone and admits centering factorisations. This shows that we can apply Theorem 3.78 to every finite WTA $\mathscr{A}$ over Σ and S such that $\mathscr{A} \vDash$ ETP. We denote $\mathbb{M} = \langle\Gamma\rangle_+$.

  Let $M \in \mathcal{M}_{\mathrm{fin}}(\mathbb{M})$. Surely, for $(i,j),(i',j') \in \mathbb{M}$ we have $(i,j) \in \mathrm{div}((i',j'))$ if and only if $i \leq i'$ and $j \leq j'$. Thus,

  $$\mathrm{div}(M) = \big\{(i,j) \in \mathbb{M} \;\big|\; i \leq \min_{(i',j')\in M} i' \;\text{ and }\; j \leq \min_{(i',j')\in M} j'\big\}$$

and since $\mathbb{M}$ is cancellative (as a submonoid of $(\mathbb{N}^2, +, (0,0))$), we obtain

$$\mathrm{quot}_{(i,j)}(M) = \mathrm{minquot}_{(i,j)}(M) = \left\{ \{\!\{ (i'-i, j'-j) \mid (i',j') \in M \}\!\} \right\}$$

for every $(i,j) \in \mathrm{div}(M)$. We continue by determining d_Γ. Let $(i,j) \in \mathbb{M}$ and $(i',j') \in \mathrm{div}((i,j))$. Every $(i_\gamma, j_\gamma) \in \Gamma$ satisfies $i_\gamma + j_\gamma = 2$ and hence for every path $w \in \mathrm{Paths}((i',j'),(i,j))$ it holds that $|w| = \frac{i-i'+j-j'}{2}$. Therefore,

$$\min_{w \in \mathrm{Paths}((i',j'),(i,j))} |w| = \frac{i-i'+j-j'}{2}.$$

This also shows that

$$|(i,j)|_\Gamma = \min_{w \in \mathrm{Paths}((0,0),(i,j))} |w| = \frac{i+j}{2}. \tag{3.20}$$

Let $(i_1,j_1),(i_2,j_2) \in \mathbb{M}$. From the definition of d_Γ we obtain that

$$d_\Gamma((i_1,j_1),(i_2,j_2)) = \frac{i_1-i'+j_1-j'}{2} + \frac{i_2-i'+j_2-j'}{2} \tag{3.21}$$

for some $(i',j') \in \mathbb{M}$. It is an easy observation that the right hand side of Equation (3.21) is minimal if i' and j' are as close to $\min(i_1,i_2)$ and $\min(j_1,j_2)$ as possible. Let $i'' = \min(i_1,i_2)$ and $j'' = \min(j_1,j_2)$. If $(i'',j'') \in \mathbb{M}$, then $(i',j') = (i'',j'')$ is the solution to Equation (3.21). Otherwise, $(i',j') = (i''-1,j'')$ and $(i',j') = (i'',j''-1)$ are the two possible solutions to Equation (3.21). Therefore, a straightforward calculation shows

$$d_\Gamma((i_1,j_1),(i_2,j_2) = d_\Gamma((i,j)+(i_1,j_1),(i,j)+(i_2,j_2)) \tag{3.22}$$

for every $(i,j) \in \mathbb{M}$. Since $\mathbb{M}$ is cancellative, Equation (3.22) implies that $\mathbb{M}$ divides Γ-monotone. We continue to prove that $\mathbb{M}$ admits f-centering factorisations, where f is given by $f(n) = 2 \cdot (n+1)$ for every $n \in \mathbb{N}$. Let $M \in \mathcal{M}_{\mathrm{fin}}(\mathbb{M})$, $(i',j') \in \mathrm{mindiv}(M)$, and $N \in \mathrm{minquot}_{(i',j')}(M)$. Moreover, we denote $i_M = \min_{(i,j)\in\mathrm{supp}(M)} i$ and $j_M = \min_{(i,j)\in\mathrm{supp}(M)} j$. Since (i',j') minimises $|N|_\Gamma$, Equation (3.20) implies that (i',j') is as close to (i_M,j_M) as possible. If $(i_M,j_M) \in \mathbb{M}$, then $(i',j') = (i_M,j_M)$. Otherwise, both $(i',j') = (i_M-1,j_M)$ and $(i',j') = (i_M,j_M-1)$ are minimising divisors of M. Since we only need to find one f-centering factorisation of M, we assume that $j' = j_M$. In any case,

there exists $j_N \in \mathbb{N}$ and $\delta \in \{0,1\}$ such that $(\delta, j_N) \in \operatorname{supp}(N)$. Moreover, there exists $i_N \in \mathbb{N}$ such that $(i_N, 0) \in \operatorname{supp}(N)$. Let $(i_1, j_1) \in \operatorname{supp}(N)$ and note that $d_\Gamma((i_1,j_1),(\delta,j_N)) \geq \frac{i_1}{2} - 1$ and $d_\Gamma((i_1,j_1),(i_N,0)) \geq \frac{j_1}{2}$. If $i_1 \leq j_1$, then

$$|(i_1,j_1)|_\Gamma = \frac{i_1+j_1}{2} \leq j_1 \leq 2 \cdot d_\Gamma((i_1,j_1),(i_N,0)).$$

If $i_1 \geq j_1$, then

$$|(i_1,j_1)|_\Gamma = \frac{i_1+j_1}{2} \leq i_1 \leq 2 \cdot \big(d_\Gamma((i_1,j_1),(\delta,j_N)) + 1\big).$$

This concludes the proof that $\big((i',j'),N\big)$ is an f-centering factorisation of M and hence shows that $\mathbb{M}$ admits centering factorisations. ◀

The question whether there exist semirings such that Theorem 3.94 can be applied but not Theorem 3.77 is very involved and we do not know the answer at the time of writing. In the rest of this chapter, we present an implication of the existence of maximal factorisations and conjecture that if the algebraic assumptions of Theorem 3.94 are satisfied, then the algebraic assumptions of Theorem 3.77 are satisfied.

Definition 3.97. Let $(\mathbb{M}, \odot, 1)$ be a commutative monoid and let $x, y, c \in \mathbb{M}$. We call c a *greatest common divisor of x and y* (short: *gcd of x and y*) if $c \in \operatorname{div}(x,y)$ and for every $d \in \operatorname{div}(x,y)$ it holds that $d \in \operatorname{div}(\{\!\{c\}\!\})$. The set of gcds of x and y is denoted $\gcd(x,y)$. We say that $\mathbb{M}$ is a *gcd-monoid* if for every $x, y \in \mathbb{M}$ it holds that $\gcd(x,y) \neq \emptyset$. ◀

Lemma 3.98. Let (f,g) be a maximal factorization of dimension $n \geq 2$. It holds that $(S, \odot, 1)$ is a cancellative gcd-monoid.

Proof. Let $\operatorname{proj}_{1,2}\colon S^n \to S^2$ be the projection map onto the first two components. We define the maps $f'\colon S^2 \to S^2$ and $g'\colon S^2 \to S$ where for every $x, y \in S$ we let $f'(x,y) = \operatorname{proj}_{1,2} f(x,y,\ldots,y)$ and $g'(x,y) = g(x,y,\ldots,y)$. It surely holds that (f',g') is a maximal factorisation of dimension 2. To aid readability, we denote (f',g') by (f,g) as well.

Let $x, y, c \in S$ and assume that $c \odot x = c \odot y = 1$. We obtain

$$g(1,1) \odot f(1,1) = (1,1) = c \odot (x,y) = c \odot g(x,y) \odot f(x,y) \tag{3.23}$$

and moreover

$$\begin{aligned}(x,x) &= x \odot (1,1) = x \odot g(1,1) \odot f(1,1) \overset{(3.23)}{=} x \odot c \odot g(x,y) \odot f(x,y) \\ &= y \odot c \odot g(x,y) \odot f(x,y) \overset{(3.23)}{=} y \odot g(1,1) \odot f(1,1) = y \odot (1,1) = (y,y),\end{aligned}$$

which shows that $x = y$. In total, we have thus seen

$$\forall x, y, c \in S\colon (c \odot x = c \odot y = 1) \implies x = y. \tag{3.24}$$

Now, let $x, y, c \in S$ such that $c \odot x = c \odot y$. It holds that

$$f(1,1) = f(c \odot x, c \odot y) = f(x,y)$$

since (f,g) is a maximal factorisation and $c \odot x = c \odot y$. Therefore, we have that $(x,y) = g(x,y) \odot f(x,y) = g(x,y) \odot f(1,1)$. Let $z, z', d \in S$ such that $(z,z') = f(1,1)$ and $d = g(1,1)$. We obtain from $(1,1) = g(1,1) \odot f(1,1)$ that $d \odot z = d \odot z' = 1$ and apply Equation (3.24) to obtain $z = z'$. In total, we obtain that $(x,y) = g(x,y) \odot (z,z)$ and hence, $x = y$. Thus we have proven that $(S, \odot, 1)$ is cancellative.

Next, let $x, y \in S$, $c \in \operatorname{div}(x,y)$, and $\frac{x}{c}$ and $\frac{y}{c}$ be the unique quotients of x and y divided by c, respectively. We note that these quotients are unique because $(S, \odot, 1)$ is cancellative. It surely holds that $g(x,y) \in \operatorname{div}(x,y)$. Moreover, it holds that

$$\begin{aligned}g(x,y) \odot f\Big(\frac{x}{c}, \frac{y}{c}\Big) &= g(x,y) \odot f(x,y) = (x,y) \\ &= c \odot \Big(\frac{x}{c}, \frac{y}{c}\Big) = c \odot g\Big(\frac{x}{c}, \frac{y}{c}\Big) \odot f\Big(\frac{x}{c}, \frac{y}{c}\Big).\end{aligned}$$

Since S is cancellative, this yields $g(x,y) = c \odot g(\frac{x}{c}, \frac{y}{c})$. In particular, c divides $g(x,y)$. Therefore, $g(x,y)$ is a gcd of x and y and hence, $(S, \odot, 1)$ is a gcd-monoid. □

Conjecture 3.99. *Let $(S, \odot, 1)$ be a commutative, cancellative gcd-monoid and let I be a finite subset of S. There exists a finitely generated monoid $\mathbb{M} \leq (S, \odot, 1)$ with finite generating set Γ such that $I \subseteq \mathbb{M}$ and $\mathbb{M}$ divides Γ-monotone and admits centering factorisations.*

In particular, whenever Theorem 3.94 is applicable, then Theorem 3.77 is applicable.

3.9 Conclusion

In this chapter, we have introduced a framework for the determinisation of weighted tree automata, called $\mathbb{M}$-sequentialisation, which captures different approaches from the literature. This framework emerged from our observation that the run-semantics of a WTA $\mathscr{A}$ over S only needs the multiplicative monoid $(S, \odot, 1)$ to describe every possible "behaviour" (that is, run weight) of $\mathscr{A}$. The additive monoid $(S, \oplus, 0)$ is then only a way of accumulating these behaviours. Therefore, our approach separates the two operations of S and deals with each of them individually.

We have given an $\mathbb{M}$-sequentialisation construction that involves multiple steps and requires many non-trivial mathematical tools (cf. also Chapter 3.1). First, we presented a theory of factorisations in monoids, which we subsequently used to provide an $\mathcal{S}(\mathbb{M})$-sequentialisation construction for WTA $\mathscr{B}$ over $\mathcal{M}_{\text{fin}}(\mathbb{M})$, resulting in possibly infinite $\mathcal{S}(\mathbb{M})$-sequential WTA $D_{\mathscr{B}}$ over $\mathcal{M}_{\text{fin}}(\mathbb{M})$. This acted as our core determinisation result and our remaining focus was to translate this result from WTA over $\mathcal{M}_{\text{fin}}(\mathbb{M})$ to WTA over S. For this, we first defined a notion of relatedness of WTA as a way to transform a WTA $\mathscr{A}$ over $\langle \mathbb{M} \rangle_{\oplus}$ into a WTA $\mathscr{B}$ over $\mathcal{M}_{\text{fin}}(\mathbb{M})$. Next, we introduced a way of accumulating $D_{\mathscr{B}}$ via an appropriate equivalence relation $\sim$, which returns an $\mathbb{M}$-sequential WTA $\text{acc}_{\sim}(D_{\mathscr{B}})$ over S. Then, we provided some cases in which $\text{acc}_{\sim}(D_{\mathscr{B}})$ is finite and connected all our steps to obtain an $\mathbb{M}$-sequentialisation result (cf. Theorems 3.77 and 3.78). Ultimately, we compared our approach to different determinisation approaches from the literature.

Even though Chapter 3 is very involved and rather lengthy, we were not able to answer all related questions and leave some opportunities for further research. We list three open problems. Firstly, we did not characterise what it means for a monoid $\mathbb{M}$ to divide Γ-monotone and admit centering factorisations. In future research, these properties should be investigated more thoroughly. This also includes Conjecture 3.99. Secondly, we believe that the case "additively idempotent" from Theorem 3.78 can be weakened to "additively locally finite". This generalisation should be attempted in future research. Thirdly, one should study the decidability of our ETP and the complexity of our constructions.

4

Approximated Determinisation of Weighted Tree Automata

This chapter is an alternative presentation of Dörband, Feller, and Stier [28]. While the original paper gave an approximated determinisation construction via an algorithm in pseudo-code, this chapter replaces the pseudo-code with formal mathematical constructions. Besides this change, the present chapter closely follows [28], which is a canonical generalisation of [4] from weighted (word) automata to weighted tree automata.

Throughout Chapter 4, we assume Σ to be a ranked alphabet.

4.1 Introduction

One endeavour to simplify weighted automata that cannot be determinised is to aim for *approximated determinisation*. Different approaches to this paradigm have been proposed, see e.g. [4, 8, 9]. The main idea of these papers is to take a weighted automaton $\mathscr{A}$ and then construct a deterministic weighted automaton that recognizes a "similar" language to the one of $\mathscr{A}$. The notions of similarity differ in the literature. As this chapter aims to generalise Aminof et al. [4] from the word case to the tree case, we subsequently focus on [4].

The weight structure considered in [4] is the tropical semiring $(\mathbb{R}_\infty, \min, +, \infty, 0)$, where $\mathbb{R}_\infty = \{x \in \mathbb{R} \mid x \geq 0\} \cup \{\infty\}$ and the notion of approximation is given as follows.

Let $t \geq 1$ be a real number, called the *approximation factor*. A weighted automaton $\mathscr{A}'$ *t-approximates* $\mathscr{A}$, if for every input word $w \in \Sigma^*$ it holds that

$$[\![\mathscr{A}]\!](w) \leq [\![\mathscr{A}']\!](w) \leq t \cdot [\![\mathscr{A}]\!](w).$$

In [4], Aminof et al. provide an algorithm, called tDet, that takes as input a weighted (word) automaton $\mathscr{A}$ and an approximation factor $t \geq 1$ and (if the algorithm terminates) outputs a deterministic weighted automaton $\mathscr{A}'$ such that $\mathscr{A}'$ t-approximates $\mathscr{A}$. The algorithm tDet executes a weighted power set construction (with a fixed factorisation) similar to the one given by Kirsten and Mäurer [80]. That is, the states of $\mathscr{A}'$ are maps from the set of states of $\mathscr{A}$ to weights from the semiring, which are considered as residual weights. These residual weights keep track of the difference between the weights of runs of $\mathscr{A}'$ and runs of $\mathscr{A}$. For approximated determinisation, however, tDet keeps track of two *bounds* for every state of $\mathscr{A}$ rather than a single residual weight. Namely, a *lower bound* and an *upper bound*. These bounds describe intervals of residual weights, which need to be taken into account during the choice of final weights in order to ensure t-approximation.

Next, Aminof et al. prove that tDet terminates if $\mathscr{A}$ satisfies the so-called *t-twinning property*. The t-twinning property is a generalisation of the classical twinning property from [80, 90] to the approximated setting. Ultimately, it is proven in [4], that the t-twinning property is decidable.

We follow the approach by Aminof et al. [4], although we provide a formal construction for approximated determinisation instead of generalising the algorithm from [4]. In Chapter 4.2, we introduce some elementary technical machinery and our automaton model. Next, in Chapter 4.3, we define t-approximation for weighted tree automata, give a construction for t-approximate determinisation, and prove its correctness in the cases where it returns a finite WTA (see Theorem 4.21). In Chapter 4.4, we introduce the t-twinning property for weighted tree automata, show that it is a sufficient condition for the finiteness of our construction (see Theorem 4.27), and prove that our t-twinning property is decidable (see Theorem 4.31). We conclude Chapter 4 by posing some open research questions in Chapter 4.5.

4.2 Preliminaries

We define the sets $\mathbb{R}_\infty = \{x \in \mathbb{R} \mid x \geq 0\} \cup \{\infty\}$ and $\mathbb{Q}_\infty = \{x \in \mathbb{Q} \mid x \geq 0\} \cup \{\infty\}$. For every $x, y \in \mathbb{R}$ we define the *interval* $[x, y] = \{z \in \mathbb{R} \mid x \leq z \leq y\}$ and denote the set $[\infty, \infty] = \{\infty\}$.

Similarly to the arctic semiring (see Arct in Example 2.4), we consider the semiring $\mathrm{Trop} = (\mathbb{R}_\infty, \min, +, \infty, 0)$, where for every $x_1, x_2 \in \mathbb{R}_\infty$ we define

$$\min(x_1, x_2) = \begin{cases} x_{3-i} & \text{if } x_i = \infty \text{ for some } i \in \{1, 2\} \\ \min(x_1, x_2) & \text{otherwise} \end{cases}$$

$$x_1 + x_2 = \begin{cases} \infty & \text{if } x_1 = \infty \text{ or } x_2 = \infty \\ x_1 + x_2 & \text{otherwise.} \end{cases}$$

Trop is called the *tropical semiring (over $\mathbb{R}$)*. We note that Trop is commutative and extremal. Analogously, one can introduce the tropical semiring over $\mathbb{Q}$.

> *Throughout Chapter 4, if not stated differently, the term "weighted tree automaton" stands for "weighted tree automaton over Σ and* Trop*".*

We use the following notation for a run ρ of $\mathscr{A}$ on a tree or context $\xi \in \mathrm{T}_\Sigma \cup \mathrm{C}_\Sigma$. Let $q = \rho(\varepsilon)$ and $x = \mathrm{wt}(\xi, \rho)$. If $\xi \in \mathrm{T}_\Sigma$, then we write $\xrightarrow{\xi|\rho|x} q$. If $\xi \in \mathrm{C}_\Sigma$, then we write $p \xrightarrow{\xi|\rho|x} q$, where $p = \rho(\mathrm{pos}_{\mathrm{var}}(\xi))$. Whenever we do not care about the name of the run, we simply write $\xrightarrow{\xi|x} q$ and $p \xrightarrow{\xi|x} q$, respectively.

For every $\xi \in \mathrm{T}_\Sigma$, $\zeta \in \mathrm{C}_\Sigma$, and $p, q \in Q$ we define the values

$$\theta_{\mathscr{A}}(\xi, q) = \min\big\{\mathrm{wt}(\xi, \rho) \mid \rho \in \mathrm{Runs}_{\mathscr{A}}(\xi, q)\big\} \text{ and}$$
$$\theta_{\mathscr{A}}(p, \zeta, q) = \min\big\{\mathrm{wt}(\zeta, \rho) \mid \rho \in \mathrm{Runs}_{\mathscr{A}}(p, \zeta, q)\big\}.$$

Moreover, we define $\theta_{\mathscr{A}}(\xi) = \min\{\theta_{\mathscr{A}}(\xi, q') \mid q' \in Q\}$. If $\mathscr{A}$ is deterministic and $\mathrm{Runs}_{\mathscr{A}}(\xi) \neq \emptyset$, then $\theta_{\mathscr{A}}(\xi) = \mathrm{wt}(\xi, \rho)$, where ρ is the unique run in $\mathrm{Runs}_{\mathscr{A}}(\xi)$.

Remark 4.1. Let $\mathscr{A} = (Q, T, \mathrm{final})$ be a finite WTA. For every $\xi \in \mathrm{T}_\Sigma$ it holds that

$$[\![\mathscr{A}]\!](\xi) = \min_{q \in Q} \big(\theta_{\mathscr{A}}(\xi, q) + \mathrm{final}(q)\big).$$

This can easily be shown using the distributivity law in Trop. ◀

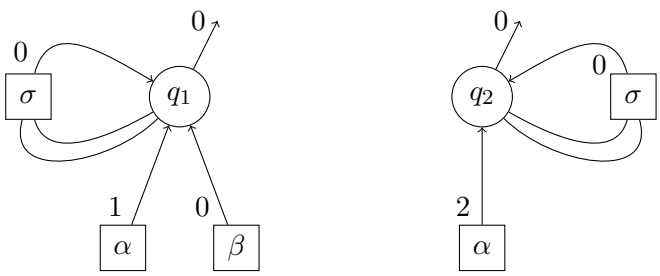


Figure 4.1: The non-deterministic finite WTA $\mathscr{A}$ from Example 4.2.

Example 4.2. Let $\Sigma = \{\alpha^{(0)}, \beta^{(0)}, \sigma^{(2)}\}$ and consider the finite WTA $\mathscr{A} = (Q, T, \text{final})$, where $Q = \{q_1, q_2\}$, final $= 0$, and T is ∞ except in the cases

$$T_\alpha(q_1) = 1, \qquad T_\alpha(q_2) = 2, \qquad T_\sigma(q_1, q_1, q_1) = 0,$$
$$T_\beta(q_1) = 0, \qquad T_\sigma(q_2, q_2, q_2) = 0.$$

The hypergraph of $\mathscr{A}$ is depicted in Figure 4.1.

Let $\xi \in \mathrm{T}_\Sigma$. One easily verifies the following statements using the definition of $\mathscr{A}$. If ξ contains at least one β, then there exists a unique non-vanishing run ρ of $\mathscr{A}$ on ξ (namely the constant map to q_1) and it holds that $\mathrm{wt}(\xi, \rho) = \#\mathrm{pos}_\alpha(\xi)$. If ξ contains no β, then there exist exactly two non-vanishing runs ρ_1 and ρ_2 of $\mathscr{A}$ on ξ (namely the constant maps to q_1 and q_2, respectively) and it holds that $\mathrm{wt}(\xi, \rho_1) = \#\mathrm{pos}_\alpha(\xi)$ and $\mathrm{wt}(\xi, \rho_2) = 2 \cdot \#\mathrm{pos}_\alpha(\xi)$. In total, we obtain that $[\![\mathscr{A}]\!](\xi) = \#\mathrm{pos}_\alpha(\xi)$.

Clearly, $\mathscr{A}$ is not deterministic, as the two transition weights $T_\alpha(q_1)$ and $T_\alpha(q_2)$ are both non-vanishing. ◀

4.3 Approximated Determinisation

In this chapter, we define t-approximation of weighted tree automata. Moreover, we present an approximated determinisation construction that takes a finite WTA $\mathscr{A}$ and an approximation factor $t \geq 1$ as input and yields a potentially infinite WTA $\mathscr{A}'$ over Σ and Trop. After applying the construction to the WTA from Example 4.2, we then proceed to prove that if the tuple $\mathscr{A}'$ has finite components, then $\mathscr{A}'$ is a deterministic finite WTA that t-approximates $\mathscr{A}$.

Throughout the rest of Chapter 4.3 we assume $\mathscr{A} = (Q, T, \mathrm{final})$ *to be an arbitrary finite* WTA.

4.3.1 The Approximated Determinisation Construction

Definition 4.3. Let $t \in \mathbb{R}$ be a real number such that $t \geq 1$ and let $\mathscr{B} = (Q', T', \mathrm{final}')$ be a finite-run[1] WTA.

We say that $\mathscr{B}$ *t-approximates* $\mathscr{A}$ if for every $\xi \in \mathrm{T}_\Sigma$ it holds that

$$[\![\mathscr{A}]\!](\xi) \leq [\![\mathscr{B}]\!](\xi) \leq t \cdot [\![\mathscr{A}]\!](\xi). \tag{4.1}$$

Moreover, we call $\mathscr{A}$ *t-approximate deterministic* (or *t-determinisable*) if there exists a deterministic WTA $\mathscr{B}$ such that $\mathscr{B}$ t-approximates $\mathscr{A}$. ◀

Remark 4.4. Note that if $\mathscr{B}$ t-approximates $\mathscr{A}$, then $\mathrm{supp}([\![\mathscr{A}]\!]) = \mathrm{supp}([\![\mathscr{B}]\!])$. Moreover, $\mathscr{B}$ 1-approximates $\mathscr{A}$ if and only if $[\![\mathscr{A}]\!] = [\![\mathscr{B}]\!]$. ◀

Throughout the rest of Chapter 4.3, we assume that $t \in \mathbb{R}$ *with* $t \geq 1$.

Remark 4.5. Note that, in general, $\mathscr{A}$ is not t-determinisable. In fact, for every Σ containing two distinct symbols $\sigma^{(r)}$ and $\tau^{(s)}$ (where $r, s > 0$), there exists a finite WTA $\mathscr{B}$ such that $\mathscr{B}$ is not t'-determinisable for any $t' \geq 1$.

This was already proven for words in [4, Theorem 1] and the constructions can easily be adapted to the tree case by considering so-called comb trees over σ and τ, which behave similarly to words. ◀

Next we introduce our approximate determinisation construction. For a summary of the conceptional details of our approach and how it fits into the existing literature, we refer to Section 4.1. Recall that our construction is a weighted power set construction with factorisation ([80], cf. also Chapter 3). We now present the intuitive idea behind the steps of our construction.

Given the finite WTA $\mathscr{A}$ and an approximation factor $t \geq 1$, we define an ascending sequence $(\mathscr{A}_n \mid n \in \mathbb{N})$ of weighted tree automata ("ascending" with respect to component-wise set inclusion) and let $\mathscr{A}'$ be the limit (that is, component-wise union)

[1] We recall that this property is required in this thesis in order for $[\![\mathscr{B}]\!]$ to be defined.

of this sequence. More precisely, we let $\mathscr{A}_0$ be a WTAlike tuple with an empty set of states and then iteratively explore more states of the goal WTA $\mathscr{A}'$ using a weighted power set construction. This exploration is done by a map called $\mathrm{step}_{\mathscr{A},t}$ (see Definition 4.8), which takes as input any $\mathscr{A}_n$ and returns its (slightly larger) successor $\mathscr{A}_{n+1}$. To do this in a mathematically comfortable way, we introduce *partial weighted tree automata*, which are our standard WTA with the added liberty that all transition weight maps are partial maps. This allows us to explore new states without having to define all possible new transition weights at the same time.

We denote the components of $\mathscr{A}_n$ by $(Q_n, T_n, \mathrm{final}_n)$ and the components of $\mathscr{A}'$ by $(Q', T', \mathrm{final}')$. The state set Q' is a subset of $(\mathbb{R}_\infty \times \mathbb{R}_\infty)^Q$, which we think about as follows. A state $P \in Q'$ maps every state $q \in Q$ to a *lower bound* l_q^P and an *upper bound* u_q^P. Thus, we denote $(l_q^P, u_q^P) = P(q)$. These bounds represent an interval in $\mathbb{R}_\infty$ and will be determined by our construction such that the following holds.

Let ρ be the (unique) non-vanishing run of $\mathscr{A}'$ on a tree ξ and let $\rho(\varepsilon) = P$. For every $q \in Q$ it holds that

$$[l_q^P + \mathrm{wt}(\xi, \rho), u_q^P + \mathrm{wt}(\xi, \rho)] \subseteq [\theta_{\mathscr{A}}(\xi, q), t \cdot \theta_{\mathscr{A}}(\xi, q)]$$

(see Lemma 4.20). We note that $[\theta_{\mathscr{A}}(\xi, q), t \cdot \theta_{\mathscr{A}}(\xi, q)]$ is the relevant interval we need to consider in order to achieve t-approximation of $\mathscr{A}$. Therefore, $\mathscr{A}'$ t-approximates $\mathscr{A}$ as long as the final weight map of $\mathscr{A}'$ respects the lower and upper bounds stored in the states of $\mathscr{A}'$.

Moreover, we use of the following concept. Given two maps $P, P' \colon Q \to \mathbb{R}_\infty \times \mathbb{R}_\infty$, we say that P' *refines* P if for every $q \in Q$ it holds that $[l_q^{P'}, u_q^{P'}] \subseteq [l_q^P, u_q^P]$. That is, P' describes tighter bounds than P. Refinement plays a major role in ensuring the finiteness of $\mathscr{A}'$.

The overall structure of the definition of $\mathrm{step}_{\mathscr{A},t}$ (Definition 4.8) is the following. Given $s \in \mathbb{N}, \sigma \in \Sigma^{(s)}$ and already explored states $P_1, \ldots, P_s$, we explore the unique successor state P' and transition weight c resulting in a new transition $(P_1, \ldots, P_s, \sigma, c, P')$ as follows. First we accumulate the lower bounds and the upper bounds of $P_1, \ldots, P_s$ respectively with the transition weights given by T. This results in the accumulated lower bounds $(l_q \mid q \in Q)$ and upper bounds $(u_q \mid q \in Q)$. Next we determine the

new transition weight c as $\min_{q \in Q} u_q$. Then we define the intermediate successor state P pointwise by $P(q) = (l_q - c, u_q - c)$ (Equation (4.2)). If P is refined by some already existing state P'', then we let $P' = P''$. Otherwise, we let $P' = P$ and call $(P_1, \ldots, P_s, \sigma, c, P')$ *discovering*.

A transition $(P_1, \ldots, P_s, \sigma, c, P')$ is discovering if and only if it was the first non-vanishing transition with successor state P' that was generated by our approximate determinisation construction. Runs that correspond at each position to a discovering transition have certain combinatorial properties that we will use later on in our proofs.

We will now give formal definitions for these ideas.

Definition 4.6. A *partial weighted tree automaton* (short: partial WTA) is a tuple $\hat{\mathscr{A}} = (\widehat{Q}, \widehat{T}, \widehat{\text{final}})$, where

- $\widehat{Q}$ is a set,
- $\widehat{T} = (\widehat{T}_\sigma \colon \widehat{Q}^s \times \widehat{Q} \dashrightarrow \text{Trop} \mid s \in \mathbb{N}, \sigma \in \Sigma^{(s)})$ is a family of partial maps, and
- $\widehat{\text{final}} \colon \widehat{Q} \to \text{Trop}$ is a map.

If $\widehat{Q}$ is finite, then we call $\hat{\mathscr{A}}$ *finite*. ◀

Definition 4.7. Let $P \colon Q \to \mathbb{R}_\infty \times \mathbb{R}_\infty$ and $q \in Q$. We denote the components of $P(q)$ by $(l_q^P, u_q^P) = P(q)$.

Let $P, P' \colon Q \to \mathbb{R}_\infty \times \mathbb{R}_\infty$. We say that P' *refines* P if for every $q \in Q$ it holds that $[l_q^{P'}, u_q^{P'}] \subseteq [l_q^P, u_q^P]$. ◀

Definition 4.8. Let $\hat{\mathscr{A}} = (\widehat{Q}, \widehat{T}, \widehat{\text{final}})$ be a finite partial WTA over Σ and $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$ with $\widehat{Q} \subseteq (\mathbb{R}_\infty \times \mathbb{R}_\infty)^Q$.

First, let $s \in \mathbb{N}, \sigma \in \Sigma^{(s)}$, and $P_1, \ldots, P_s \in \widehat{Q}$. For every $q \in Q$ we define

$$l_q = \min\{l_{q_1}^{P_1} + \cdots + l_{q_s}^{P_s} + T_\sigma(q_1, \ldots, q_s, q) \mid q_1, \ldots, q_s \in Q\} \text{ and}$$
$$u_q = \min\{u_{q_1}^{P_1} + \cdots + u_{q_s}^{P_s} + t \cdot T_\sigma(q_1, \ldots, q_s, q) \mid q_1, \ldots, q_s \in Q\}.$$

Moreover, we define the constant $c = \min_{q \in Q} u_q$ and the map $P \colon Q \to (\mathbb{R}_\infty \times \mathbb{R}_\infty)$ by

$$P(q) = (l_q - c, u_q - c) \text{ for every } q \in Q, \tag{4.2}$$

where $\infty - \infty = \infty$. If P is refined by a state in $\widehat{Q}$, then we fix an arbitrary such refining state P'. Otherwise we let $P' = P$. We define

$$\mathcal{T}_\sigma(P_1, \ldots, P_s) = (c, P').$$

We note that this defines a map $\mathcal{T}_{\_}(\_)\colon (\bigcup_{s\in\mathbb{N}} \Sigma^{(s)} \times \hat{Q}^s) \to \mathrm{Trop} \times (\mathbb{R}_\infty \times \mathbb{R}_\infty)^Q$.

Next, we define the finite partial WTA

$$\mathrm{step}_{\mathscr{A},t}(\hat{\mathscr{A}}) = (\widehat{Q}', \widehat{T}', \widehat{\mathrm{final}}')$$

over Σ and $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$ as follows. The set of states is

$$\widehat{Q}' = \hat{Q} \cup \big\{P \mid (c, P) \in \mathrm{im}\big(\mathcal{T}_{\_}(\_)\big)\big\},$$

the transition weight maps are $\widehat{T}' = (\widehat{T}'_\sigma\colon (\widehat{Q}')^s \times \widehat{Q}' \dashrightarrow \mathrm{Trop} \mid s \in \mathbb{N}, \sigma \in \Sigma^{(s)})$, where for every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}, P_1, \ldots, P_s \in \widehat{Q}$, and $P \in \widehat{Q}'$, we define

$$\widehat{T}'_\sigma(P_1, \ldots, P_s, P) = \begin{cases} \widehat{T}_\sigma(P_1, \ldots, P_s, P) & \text{if } \widehat{T}_\sigma(P_1, \ldots, P_s, P) \text{ exists} \\ c & \text{if } \mathcal{T}_\sigma(P_1, \ldots, P_s) = (c, P) \\ \infty & \text{otherwise,} \end{cases}$$

and the final weight map is given by $(\widehat{\mathrm{final}}')(P) = \min_{q\in Q}(u_q^P + t \cdot \mathrm{final}(q))$ for every $P \in \widehat{Q}'$. Clearly, $\mathrm{step}_{\mathscr{A},t}(\hat{\mathscr{A}})$ is a finite partial WTA with weights in $\mathbb{R} \cup \{\infty\}$. ◀

Definition 4.9. We define the sequence $\big(\mathscr{A}_n = (Q_n, T_n, \mathrm{final}_n) \mid n \in \mathbb{N}\big)$ of finite partial WTA inductively by

$$\mathscr{A}_0 = (\emptyset, (\emptyset \mid \sigma \in \Sigma), \emptyset)$$

and $\mathscr{A}_{n+1} = \mathrm{step}_{\mathscr{A},t}(\mathscr{A}_n)$ for every $n \in \mathbb{N}$. ◀

Remark 4.10. Let $k \in \mathbb{N}$. It is an easy fact that $Q_k \subseteq Q_{k+1}$, $(T_k)_\sigma \subseteq (T_{k+1})_\sigma$ for every $\sigma \in \Sigma$, and $\mathrm{final}_k \subseteq \mathrm{final}_{k+1}$. That is, $(\mathscr{A}_n \mid n \in \mathbb{N})$ is an ascending sequence of finite partial WTA. In particular, we can define the tuple

$$\mathscr{A}' = (Q', T', \mathrm{final}'),$$

where $Q' = \bigcup_{n\in\mathbb{N}} Q_n$, $T'_\sigma = \bigcup_{n\in\mathbb{N}} (T_n)_\sigma$ for every $\sigma \in \Sigma$, and $\mathrm{final}' = \bigcup_{n\in\mathbb{N}} \mathrm{final}_n$. It is clear that $\mathscr{A}'$ is a (possibly infinite) partial WTA. Moreover, for every $s \in \mathbb{N}$ and

$\sigma \in \Sigma^{(s)}$ we have that $(T_{k+1})_\sigma$ is a map of type $Q_k^s \times Q_{k+1} \to S$, whence T'_σ is a map of type $(Q')^s \times Q' \to S$. Therefore, $\mathscr{A}'$ is in fact a (possibly infinite) WTA over Σ and $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$.

Clearly, if $\mathscr{A}_n = \mathscr{A}_{n+1}$ for some $n \in \mathbb{N}$, then it also holds that $\mathscr{A}_n = \mathscr{A}_k$ for every $k \geq n$. In this case, $\mathscr{A}' = \mathscr{A}_n$ and therefore $\mathscr{A}'$ is a finite WTA over Σ and $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$. ◀

Lemma 4.11. For every $s \geq 0$, $\sigma \in \Sigma^{(s)}$, and $P_1, \ldots, P_s, P \in Q'$ it holds that $T'_\sigma(P_1, \ldots, P_s, P) = \infty$ or $P = \mathrm{proj}_2(\mathcal{T}_\sigma(P_1, \ldots, P_s))$.

In particular, $\mathscr{A}'$ is a deterministic WTA over Σ and $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$.

Proof. Let $s \geq 0$, $\sigma \in \Sigma^{(s)}$, and $P_1, \ldots, P_s, P \in Q'$ such that $T'_\sigma(P_1, \ldots, P_s, P) \neq \infty$. There exists $n \in \mathbb{N}$ where $(T_n)_\sigma(P_1, \ldots, P_s, P) \neq \infty$ and $(T_{n-1})_\sigma(P_1, \ldots, P_s, P) = \infty$. From the definition of T_n it now follows that $\mathcal{T}_\sigma(P_1, \ldots, P_s) = (c, P)$ for some $c \in \mathbb{R}$. This proves the first claim.

The second claim can be seen as follows. By Remark 4.10, $\mathscr{A}'$ is a WTA over Σ and $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$ and the fact that $\mathscr{A}'$ is deterministic follows directly from the first claim. □

Example 4.12. We continue Example 4.2 by constructing the sequence $(\mathscr{A}_n \mid n \in \mathbb{N})$ for $t \geq 2$. We denote $\mathscr{A}_n = (Q_n, T_n, \mathrm{final}_n)$ for every $n \in \mathbb{N}$.

By definition we have $\mathscr{A}_0 = (\emptyset, (\emptyset \mid \sigma \in \Sigma), \emptyset)$.

Next, we determine $\mathscr{A}_1 = \mathrm{step}_{\mathscr{A},t}(\mathscr{A}_0)$ and denote $\mathrm{step}_{\mathscr{A},t}(\mathscr{A}_0) = (Q'_0, T'_0, \mathrm{final}'_0)$. We need to calculate $\mathcal{T}_\tau(P_1, \ldots, P_s)$ for every s, τ, and $P_1, \ldots, P_s$. Since $Q_0 = \emptyset$, we only need to consider the case $s = 0$.

First, we consider $\alpha \in \Sigma^{(0)}$. We calculate

$$l_{q_1} = \min\{T_\alpha(q_1)\} = 1 \qquad u_{q_1} = \min\{t \cdot T_\alpha(q_1)\} = t$$
$$l_{q_2} = \min\{T_\alpha(q_2)\} = 2 \qquad u_{q_2} = \min\{t \cdot T_\alpha(q_1)\} = 2 \cdot t$$

and obtain $c = \min\{t, 2 \cdot t\} = t$. Next, we determine

$$P(q_1) = (1 - t, 0) \qquad \text{and} \qquad P(q_2) = (2 - t, t).$$

As Q_0 is empty, P is not refined by any state in Q_0 and hence, we have

$$\mathcal{T}_\alpha() = (t, P_1'),$$

where $P_1' = P = \big(q_1 \mapsto (1-t, 0), q_2 \mapsto (2-t, t)\big)$.

Next, we consider $\beta \in \Sigma^{(0)}$. We calculate

$$l_{q_1} = \min\{T_\beta(q_1)\} = 0 \qquad u_{q_1} = \min\{t \cdot T_\beta(q_1)\} = 0$$
$$l_{q_2} = \min\{T_\beta(q_2)\} = \infty \qquad u_{q_2} = \min\{t \cdot T_\beta(q_1)\} = \infty$$

and obtain $c = \min\{0, \infty\} = 0$. Next, we determine

$$P(q_1) = (0, 0) \qquad \text{and} \qquad P(q_2) = (\infty, \infty).$$

As Q_0 is empty, P is not refined by any state in Q_0 and hence, we have

$$\mathcal{T}_\beta() = (0, P_2'),$$

where $P_2' = P = \big(q_1 \mapsto (0, 0), q_2 \mapsto (\infty, \infty)\big)$.

We have now determined all new values for $\mathcal{T}_{\_}(\_)$ and obtain $Q_0' = \{P_1', P_2'\}$,

$$(T_0')_\alpha(P_1') = t, \qquad (T_0')_\alpha(P_2') = \infty,$$
$$(T_0')_\beta(P_2') = \infty, \qquad (T_0')_\beta(P_2') = 0,$$

and $\text{final}_0'(P_1') = \text{final}_0'(P_2') = 0$. This defines the partial WTA $\mathscr{A}_1$, which we depict in Figure 4.2.

Next, we determine $\mathscr{A}_2 = \text{step}_{\mathscr{A},t}(\mathscr{A}_1)$ and denote $\text{step}_{\mathscr{A},t}(\mathscr{A}_1) = (Q_1', T_1', \text{final}_1')$. We again need to calculate $\mathcal{T}_\tau(P_1, \ldots, P_s)$ for every s, τ, and $P_1, \ldots, P_s$. However, since the values of $\mathcal{T}_\tau(P_1, \ldots, P_s)$ do not depend on T_1 or final_1, we obtain that $\mathcal{T}_\alpha()$ and $\mathcal{T}_\beta()$ are the same as in the calculation of $\text{step}_{\mathscr{A},t}(\mathscr{A}_0)$.

We consider $\sigma \in \Sigma^{(2)}$ and $P_1 = P_2 = P_1'$ from Q_1. Similarly to the construction of

$\mathrm{step}_{\mathscr{A},t}(\mathscr{A}_0)$, we calculate

$$\begin{aligned}
l_{q_1} &= \min\{l_{q_1}^{P_1} + l_{q_1}^{P_2} + T_\sigma(q_1,q_1,q_1), l_{q_1}^{P_1} + l_{q_2}^{P_2} + T_\sigma(q_1,q_2,q_1),\\
&\qquad l_{q_2}^{P_1} + l_{q_1}^{P_2} + T_\sigma(q_2,q_1,q_1), l_{q_2}^{P_1} + l_{q_2}^{P_2} + T_\sigma(q_2,q_2,q_1)\}\\
&= \min\{1-t+1-t+0,\infty,\infty,\infty\} = 2-2\cdot t,\\
u_{q_1} &= \min\{0+0+0,\infty,\infty,\infty\} = 0,\\
l_{q_2} &= \min\{\infty,\infty,\infty,2-t+2-t+0\} = 4-2\cdot t, \text{ and}\\
u_{q_2} &= \min\{\infty,\infty,\infty,t+t+0\} = 2\cdot t.
\end{aligned}$$

Moreover we obtain $c = 0$ and

$$P(q_1) = (2-2\cdot t, 0) \qquad \text{and} \qquad P(q_2) = (4-2\cdot t, 2\cdot t).$$

Note that P_2' does not refine P and that P_1' refines P if and only if

$$2-2t \leq 1-t, \qquad 0 \leq 0, \qquad 4-2t \leq 2-t, \qquad \text{and} \qquad t \leq 2t.$$

That is, P is refined by P_1' if and only if $t \geq 2$, which is true by assumption. In total we obtain

$$\mathcal{T}_\sigma(P_1', P_1') = (0, P_1').$$

By continuing in the same fashion for every other combination of P_1' and P_2', we obtain the values

$$\mathcal{T}_\sigma(P_1', P_2') = (0, P_2'), \qquad \mathcal{T}_\sigma(P_2', P_1') = (0, P_2'), \qquad \text{and} \qquad \mathcal{T}_\sigma(P_2', P_2') = (0, P_2').$$

We have now determined all new values for $\mathcal{T}_{\_}(\_)$ and obtain $Q_1' = \{P_1', P_2'\}$, $\mathrm{final}_0'(P_1') = \mathrm{final}_0'(P_2') = 0$, and T_1' is ∞ except in the cases

$$\begin{aligned}
&(T_1')_\alpha(P_1') = t, \qquad (T_1')_\sigma(P_1', P_1', P_1') = 0, \qquad (T_1')_\sigma(P_1', P_2', P_2') = 0,\\
&(T_1')_\beta(P_2') = 0, \qquad (T_1')_\sigma(P_2', P_1', P_2') = 0, \qquad (T_1')_\sigma(P_2', P_2', P_2') = 0.
\end{aligned}$$

This defines the partial WTA $\mathscr{A}_2$. Moreover, since $Q_2 = Q_1$, we obtain that $\mathscr{A}_n = \mathscr{A}_2$ for every $n \geq 2$ and hence $\mathscr{A}' = \mathscr{A}_2$. Clearly, $\mathscr{A}'$ is a finite WTA and one can easily see that $[\![\mathscr{A}']\!](\xi) = t \cdot \#\mathrm{pos}_\alpha(\xi)$ for every $\xi \in \mathrm{T}_\Sigma$. In particular, $\mathscr{A}'$ t-approximates $\mathscr{A}$. A depiction of $\mathscr{A}'$ can be found in Figure 4.2. ◀

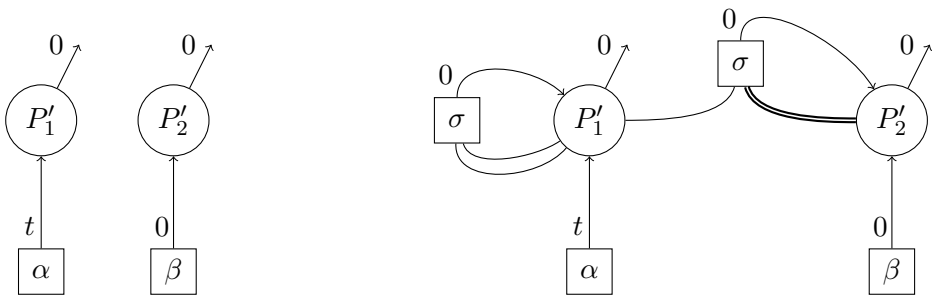


Figure 4.2: Partial WTA $\mathscr{A}_1$ (left) and $\mathscr{A}_2$ (right) calculated in Example 4.12, where $\mathscr{A}_2 = \mathscr{A}'$ is a deterministic WTA t-approximating the WTA $\mathscr{A}$ from Example 4.2.

Remark 4.13. Note that the construction of $\mathscr{A}'$ does not preserve the weighted language of $\mathscr{A}$ in general. This is due to the fact that the state normalisation in Definition 4.8 is done with respect to the *upper* bounds u_q^P. This yields that if $\mathscr{A}$ is deterministic, then $\mathscr{A}'$ recognises $t \cdot [\![\mathscr{A}]\!]$ rather than $[\![\mathscr{A}]\!]$. This can be proved in a straightforward way. ◀

4.3.2 Correctness of the Construction

We now set up the proof that if $\mathscr{A}' = \mathscr{A}_n$ for some $n \in \mathbb{N}$, then $\mathscr{A}'$ is a deterministic WTA which t-approximates $\mathscr{A}$ (see Theorem 4.21).

The following auxiliary lemma is a simple result that can be proven in a straightforward manner by induction on ξ using distributivity (cf. Equation 7 of the proof of [94, Theorem 4.1.] and [10, Lemma 4.1.13]).

Lemma 4.14. Let $\xi \in \mathrm{T}_\Sigma$ and $q \in Q$ and assume that $\xi = \sigma(\xi_1, \ldots, \xi_s)$. It holds that

$$\theta_{\mathscr{A}}(\xi, q) = \min\{\Big(\sum_{i=1}^{s} \theta_{\mathscr{A}}(\xi_i, q_i)\Big) + T_\sigma(q_1, \ldots, q_s, q) \mid q_1, \ldots, q_s \in Q\}.$$

Lemma 4.15. Let $P \in Q'$. For every $q \in Q$ it holds that $u_q^P \geq 0$. Moreover, if $u_q^P < \infty$ for some $q \in Q$, then $u_{q'}^P = 0$ for some $q' \in Q$.

Proof. Let $n \in \mathbb{N}$ such that $P \in Q_{n+1} \setminus Q_n$. By the definition of $\mathrm{step}_{\mathscr{A},t}$ it holds that $(c, P) = \mathcal{T}_\sigma(P_1, \ldots, P_s)$ for some $c \in \mathrm{Trop}, s \geq 0, \sigma \in \Sigma^{(s)}$, and $P_1, \ldots, P_s \in Q_n$. Since $P \notin Q_n$, it must hold that P is not refined by a state in Q_n and therefore

$P(q) = (l_q - c, u_q - c)$, where $l_q, u_q \in \mathrm{Trop}$ for every $q \in Q$ and $c = \min_{q \in Q} u_q$ (see Definition 4.8). In particular, $u_q^P = u_q - \min_{q \in Q} u_q \geq 0$ for every $q \in Q$. Moreover, if $u_q^P < \infty$ for some $q \in Q$, then $u_q < \infty$ and hence for $q' = \mathrm{argmin}_{q'' \in Q} u_{q''}$ we have $u_{q'} < \infty$ and $u_{q'}^P = u_{q'} - u_{q'} = 0$. This concludes the proof. □

Lemma 4.16. For all $\sigma \in \Sigma$ it holds that $\mathrm{im}(T'_\sigma) \subseteq \mathbb{R}_\infty$ and $\mathrm{im}(\mathrm{final}') \subseteq \mathbb{R}_\infty$.

Proof. One easily sees that all occurring weights are in $\mathbb{R} \cup \{\infty\}$. Therefore, we only show their nonnegativity.

Let $s \geq 0, \sigma \in \Sigma^{(s)}$, and $P_1, \ldots, P_s, P \in Q'$. If $T'_\sigma(P_1, \ldots, P_s, P) = \infty$, then we are done. Otherwise, $T'_\sigma(P_1, \ldots, P_s, P) = c$, where $(c, P) = \mathcal{T}_\sigma(P_1, \ldots, P_s)$ for $c = \min_{q \in Q} u_q$ and $u_q = \min\{u_{q_1}^{P_1} + \cdots + u_{q_s}^{P_s} + t \cdot T_\sigma(q_1, \ldots, q_s, q) \mid q_1, \ldots, q_s \in Q\}$ for every $q \in Q$ (see Definition 4.8). Thus, it suffices to prove that $u_q \geq 0$ for every $q \in Q$. This easily follows from Lemma 4.15, the fact that $t \geq 1$, and the fact that $\mathrm{im}(T_\sigma) \subseteq \mathbb{R}_\infty$ for every $\sigma \in \Sigma$.

Let $P \in Q'$. It holds that $\mathrm{final}'(P) = \min_{q \in Q}(u_q^P + t \cdot \mathrm{final}(q))$, which is non-negative by Lemma 4.15, the fact that $t \geq 1$, and the fact that $\mathrm{im}(\mathrm{final}) \subseteq \mathbb{R}_\infty$. □

Corollary 4.17. If $\mathscr{A}' = \mathscr{A}_n$ for some $n \in \mathbb{N}$, then $\mathscr{A}'$ is a deterministic finite WTA.

Proof. This follows from Remark 4.10, Lemma 4.16, and Lemma 4.11. □

We now turn towards the proof that $\mathscr{A}'$ indeed t-approximates $\mathscr{A}$.

Definition 4.18. For every $P \in Q'$ we define the *depth of* P (*in* $\mathscr{A}'$), denoted $\mathrm{depth}(P)$, as the unique $n \in \mathbb{N}$ such that $P \in Q_n \setminus Q_{n-1}$, where we let $Q_{-1} = \emptyset$.

We call a transition $(P_1, \ldots, P_s, \sigma, c, P)$ of $\mathscr{A}'$ *discovering* if $c < \infty$ and for every $i \in [s]$ we have $\mathrm{depth}(P) > \mathrm{depth}(P_i)$. In this case, P is introduced in $Q_{\mathrm{depth}(P)}$ as the second component of $\mathcal{T}_\sigma(P_1, \ldots, P_s)$ (and is not refined by some state $P' \in Q_{\mathrm{depth}(P)-1}$).

Let $\xi \in \mathrm{T}_\Sigma$ and $\rho \in \mathrm{Runs}_{\mathscr{A}'}(\xi)$. We call ρ *discovering* if for every $w \in \mathrm{pos}(\xi)$ it holds that $(P_1, \ldots, P_s, \sigma, c, P)$ is discovering, where $\sigma = \xi(w)$, $s = \mathrm{rk}(\sigma)$, $P_i = \rho(wi)$ for every $i \in [s]$, $P = \rho(w)$, and $c = \mathrm{locwt}_{\mathscr{A}'}(\xi, \rho, w)$. ◀

Lemma 4.19. Let $\xi \in \mathrm{T}_\Sigma$. If there exists a non-vanishing run $\rho \in \mathrm{Runs}_{\mathscr{A}}(\xi)$, then there exists a non-vanishing run $\rho' \in \mathrm{Runs}_{\mathscr{A}'}(\xi)$ such that $u_{\rho(\varepsilon)}^{\rho'(\varepsilon)} < \infty$.

In particular, if $[\![\mathscr{A}]\!](\xi) \neq \infty$, then $[\![\mathscr{A}']\!](\xi) \neq \infty$.

Proof. We prove the claim by structural induction on ξ. We assume $\xi = \sigma(\xi_1, \dots, \xi_s)$ and that the claim holds for $\xi_1, \dots, \xi_s$. Let $\rho \in \mathrm{Runs}_{\mathscr{A}}(\xi)$ be non-vanishing and denote for every $i \in [s]$ the restriction[1] of ρ to ξ_i by ρ_i. Since ρ is non-vanishing, it holds that ρ_i is non-vanishing for every $i \in [s]$. By the induction assumption, there exists a non-vanishing run $\rho'_i \in \mathrm{Runs}_{\mathscr{A}'}(\xi_i)$ for every $i \in [s]$. We denote $(c, P) = \mathcal{T}_\sigma(\rho'_1(\varepsilon), \dots, \rho'_s(\varepsilon))$ and define the run $\rho' \colon \mathrm{pos}(\xi) \to Q'$ of $\mathscr{A}'$ on ξ by

$$\rho'(w) = \begin{cases} P & \text{if } w = \varepsilon \\ \rho'_i(v) & \text{if } w = iv \text{ for some } i \in [s] \text{ and } v \in \mathrm{pos}(\xi_i), \end{cases}$$

for every $w \in \mathrm{pos}(\xi)$. We need to prove that ρ' is non-vanishing and $u^P_{\rho(\varepsilon)} < \infty$. Since $\mathrm{wt}(\xi, \rho') = \mathrm{wt}(\xi, \rho'_1) + \cdots + \mathrm{wt}(\xi, \rho'_s) + T'_\sigma(\rho'_1(\varepsilon), \dots, \rho'_s(\varepsilon), P)$ and $\mathrm{wt}(\xi, \rho'_i) < \infty$ for every $i \in [s]$, we know that ρ' is non-vanishing if $T'_\sigma(\rho'_1(\varepsilon), \dots, \rho'_s(\varepsilon), P) < \infty$. It is an easy fact that $u^P_{\rho(\varepsilon)} < \infty$ implies $T'_\sigma(\rho'_1(\varepsilon), \dots, \rho'_s(\varepsilon), P) < \infty$ and hence, it suffices to show $u^P_{\rho(\varepsilon)} < \infty$. We know that $u^P_{\rho(\varepsilon)} \leq u_{\rho(\varepsilon)} - c$, where $c = \min_{q \in Q} u_q$ and

$$u_q = \min\{u_{q_1}^{\rho'_1(\varepsilon)} + \cdots + u_{q_s}^{\rho'_s(\varepsilon)} + t \cdot T_\sigma(q_1, \dots, q_s, q) \mid q_1, \dots, q_s \in Q\}$$

for every $q \in Q$. This follows from Definition 4.8.

By the induction assumption, it holds that $u_{\rho_i(\varepsilon)}^{\rho'_i(\varepsilon)} < \infty$ for every $i \in [s]$ and since ρ is non-vanishing we also have $T_\sigma(\rho_1(\varepsilon), \dots, \rho_s(\varepsilon), \rho(\varepsilon)) < \infty$. This shows that $u_{\rho(\varepsilon)} < \infty$, $c < \infty$, and hence $u^P_{\rho(\varepsilon)} \leq u_{\rho(\varepsilon)} - c < \infty$. □

Lemma 4.20. Let $\xi \in \mathrm{T}_\Sigma$ such that $\theta_{\mathscr{A}'}(\xi) \neq \infty$ and let $P \in Q'$ such that there exists a run $\xrightarrow{\xi|\theta_{\mathscr{A}'}(\xi)} P$. For every $q \in Q$, it holds that

$$\theta_{\mathscr{A}}(\xi, q) - \theta_{\mathscr{A}'}(\xi) \overset{\star}{\leq} l^P_q \leq u^P_q \overset{\star}{\leq} t \cdot \theta_{\mathscr{A}}(\xi, q) - \theta_{\mathscr{A}'}(\xi). \tag{4.3}$$

Moreover, if there exists a discovering run $\xrightarrow{\xi|\theta_{\mathscr{A}'}(\xi)} P$, the $\star$-inequalities hold as equalities.

Proof. Let $n = \mathrm{depth}(P)$. We first prove the inequality "$l^P_q \leq u^P_q$" by induction on n. The induction base $n = 0$ is clear since $Q_0 = \emptyset$. Now assume that $n \geq 1$ and the claimed inequality holds for every $P' \in Q_{n-1}$. By the definition of $\mathrm{step}_{\mathscr{A},t}$ it holds that $(c, P) = \mathcal{T}_\sigma(P_1, \dots, P_s)$ for some $c \in \mathrm{Trop}, s \geq 0, \sigma \in \Sigma^{(s)}$, and $P_1, \dots, P_s \in Q_{n-1}$.

[1] That is, $\rho_i(w) = \rho(iw)$ for every $i \in [s]$ and $w \in \mathrm{pos}(\xi_i)$.

Since $P \notin Q_{n-1}$, P is not refined by some state in Q_{n-1} and in particular, P is defined by Equation (4.2) as $P(q) = (l_q - c, u_q - c)$, where

$$l_q = \min\{l_{q_1}^{P_1} + \cdots + l_{q_s}^{P_s} + T_\sigma(q_1, \ldots, q_s, q) \mid q_1, \ldots, q_s \in Q\} \text{ and}$$
$$u_q = \min\{u_{q_1}^{P_1} + \cdots + u_{q_s}^{P_s} + t \cdot T_\sigma(q_1, \ldots, q_s, q) \mid q_1, \ldots, q_s \in Q\}$$

for every $q \in Q$ and $c = \min_{q \in Q} u_q$. Therefore we can show that $l_q^P \leq u_q^P$ by proving $l_q \leq u_q$. However, for every $q_1, \ldots, q_s \in Q$ it holds that

$$l_{q_1}^{P_1} + \cdots + l_{q_s}^{P_s} + T_\sigma(q_1, \ldots, q_s, q) \leq u_{q_1}^{P_1} + \cdots + u_{q_r}^{P_s} + t \cdot T_\sigma(q_1, \ldots, q_s, q)$$

by the induction assumption and the fact that $t \geq 1$, which concludes the proof of this inequality.

Next we prove the $\star$-inequalities. The proof is done by structural induction on ξ. We assume that $\xi = \sigma(\xi_1, \ldots, \xi_s)$. By the induction assumption, the claimed inequalities hold for ξ_i for every $i \in [s]$. Moreover, let $P_i \in Q'$ be the (unique) state such that there exists a run $\xrightarrow{\xi_i | \theta_{\mathscr{A}'}(\xi_i)} P_i$ for every $i \in [s]$. Since $\mathscr{A}'$ is deterministic, it holds that

$$\theta_{\mathscr{A}'}(\xi) = \Big(\sum_{i=1}^{s} \theta_{\mathscr{A}'_i}(\xi_i)\Big) + T'_\sigma(P_1, \ldots, P_s, P). \tag{4.4}$$

We define the weight

$$c = \min\{u_{q_1}^{P_1} + \cdots + u_{q_s}^{P_s} + t \cdot T_\sigma(q_1, \ldots, q_s, q) \mid q, q_1, \ldots, q_s \in Q\}.$$

Since $T'_\sigma(P_1, \ldots, P_s, P) < \infty$, it must hold that $\mathcal{T}_\sigma(P_1, \ldots, P_s) = (T'_\sigma(P_1, \ldots, P_s, P), P)$. Moreover, from Definition 4.8 we obtain that $c = \mathrm{proj}_1(\mathcal{T}_\sigma(P_1, \ldots, P_s))$, which yields $c = T'_\sigma(P_1, \ldots, P_s, P)$. In total, we obtain the following inequality chain for every $q \in Q$.

$$\begin{aligned}
&\theta_{\mathscr{A}}(\xi, q) - \theta_{\mathscr{A}'}(\xi) \\
&\overset{\star_1}{=} \min\{\Big(\sum_{i=1}^{s} \theta_{\mathscr{A}}(\xi_i, q_i)\Big) + T_\sigma(q_1, \ldots, q_s, q) \mid q_1, \ldots, q_s \in Q\} - \theta_{\mathscr{A}'}(\xi) \\
&\overset{\star_2}{=} \min\{\Big(\sum_{i=1}^{s} \theta_{\mathscr{A}}(\xi_i, q_i)\Big) + T_\sigma(q_1, \ldots, q_s, q) - \sum_{i=1}^{s} \theta_{\mathscr{A}'}(\xi_i) \mid q_1, \ldots, q_s \in Q\} - c \\
&\overset{\star_3}{=} \min\{\Big(\sum_{i=1}^{s} \big(\theta_{\mathscr{A}}(\xi_i, q_i) - \theta_{\mathscr{A}'}(\xi_i)\big)\Big) + T_\sigma(q_1, \ldots, q_s, q) \mid q_1, \ldots, q_s \in Q\} - c \\
&\overset{\star_4}{\leq} \min\{l_{q_1}^{P_1} + \cdots + l_{q_s}^{P_s} + T_\sigma(q_1, \ldots, q_s, q) \mid q_1, \ldots, q_s \in Q\} - c \overset{\star_5}{\leq} l_q^P
\end{aligned}$$

Equality $\star_1$ uses Lemma 4.14. Equality $\star_2$ first uses Equation (4.4) and then pulls the term $\sum_{i=1}^{s} \theta_{\mathscr{A}'}(\xi_i)$ inside the minimum. Equality $\star_3$ simply rearranges the weights. Inequality $\star_4$ applies the induction hypothesis. Inequality $\star_5$ follows from Definition 4.8.

Similarly to $\star_1, \ldots, \star_5$, we can prove the following inequality chain for every $q \in Q$.

$$\begin{aligned}
u_q^P &\leq \min\{u_{q_1}^{P_1} + \cdots + u_{q_s}^{P_s} + t \cdot T_\sigma(q_1, \ldots, q_s, q) \mid q_1, \ldots, q_s \in Q\} - c \\
&\leq \min\{\Big(\sum_{i=1}^{s} \big(t \cdot \theta_{\mathscr{A}}(\xi_i, q_i) - \theta_{\mathscr{A}'}(\xi_i)\big)\Big) + t \cdot T_\sigma(q_1, \ldots, q_s, q) \mid q_1, \ldots, q_s \in Q\} - c \\
&= \min\{t \cdot \Big(\Big(\sum_{i=1}^{s} \theta_{\mathscr{A}}(\xi_i, q_i)\Big) + T_\sigma(q_1, \ldots, q_s, q)\Big) - \sum_{i=1}^{s} \theta_{\mathscr{A}'}(\xi_i) \mid q_1, \ldots, q_s \in Q\} - c \\
&= t \cdot \min\{\Big(\sum_{i=1}^{s} \theta_{\mathscr{A}}(\xi_i, q_i)\Big) + T_\sigma(q_1, \ldots, q_s, q) \mid q_1, \ldots, q_s \in Q\} - \theta_{\mathscr{A}'}(\xi) \\
&= t \cdot \theta_{\mathscr{A}}(\xi, q) - \theta_{\mathscr{A}'}(\xi)
\end{aligned}$$

This concludes the proof of Equation (4.3).

If there exists a discovering run $\xrightarrow{\xi|\theta_{\mathscr{A}'}(\xi)} P$, then inequality $\star_5$ holds as an equality, since P is not refined by a state $P' \in Q_{\mathrm{depth}(P)-1}$ and is hence defined by Equation (4.2). In this case, inequality $\star_4$ also holds as an equality by the induction assumption, as the unique run $\xrightarrow{\xi_i|x_i} P_i$ such that $x_i < \infty$ is a discovering run for every $i \in [s]$. These arguments analogously apply to the inequalities regarding u_q^P. This concludes the proof of the lemma. □

The following theorem proves the partial correctness of our approximate determinisation construction (Definitions 4.8 and 4.9) and follows from Lemma 4.20 and the definition of final′.

Theorem 4.21. If $\mathscr{A}' = \mathscr{A}_n$ for some $n \in \mathbb{N}$, then $\mathscr{A}'$ is a deterministic finite WTA that t-approximates $\mathscr{A}$. In this case, $\mathscr{A}$ is in particular t-determinisable.

Proof. From Corollary 4.17 we obtain that $\mathscr{A}'$ is a deterministic finite WTA. It remains to show that $\mathscr{A}'$ t-approximates $\mathscr{A}$, that is, we need to prove

$$[\![\mathscr{A}]\!](\xi) \leq [\![\mathscr{A}']\!](\xi) \leq t \cdot [\![\mathscr{A}]\!](\xi) \tag{4.5}$$

for every $\xi \in \mathrm{T}_\Sigma$.

Let $\xi \in \mathrm{T}_\Sigma$. If $\theta_{\mathscr{A}'}(\xi) = \infty$, then $[\![\mathscr{A}]\!](\xi) = \infty$ by Lemma 4.19, which proves Equation (4.5). Now assume that $\theta_{\mathscr{A}'}(\xi) \neq \infty$. Since $\mathscr{A}'$ is deterministic, there exists a unique run $\xrightarrow{\xi|\theta_{\mathscr{A}'}(\xi)} P$ of $\mathscr{A}'$ on ξ, where $P \in Q'$. Moreover, $[\![\mathscr{A}']\!](\xi) = \theta_{\mathscr{A}'}(\xi) + \mathrm{final}'(P)$. Therefore, after subtracting $\theta_{\mathscr{A}'}(\xi)$ from all sides of Equation (4.5) and using the fact that $[\![\mathscr{A}]\!](\xi) = \min_{q\in Q}\big(\theta_{\mathscr{A}}(\xi, q) + \mathrm{final}(q)\big)$, we only need to show that

$$\begin{aligned}\min_{q\in Q}\Big(\theta_{\mathscr{A}}(\xi, q) + \mathrm{final}(q)\Big) - \theta_{\mathscr{A}'}(\xi) &\overset{\star_1}{\leq} \mathrm{final}'(P)\\ &\overset{\star_2}{\leq} t \cdot \min_{q\in Q}\Big(\theta_{\mathscr{A}}(\xi, q) + \mathrm{final}(q)\Big) - \theta_{\mathscr{A}'}(\xi).\end{aligned}$$

It holds that $\mathrm{final}'(P) \overset{\diamond}{=} \min_{q\in Q}(u_q^P + t \cdot \mathrm{final}(q))$ by definition. We can use Lemma 4.20 to estimate every u_q^P in Equality $\diamond$ from below and above, which yields the inequalities

$$\begin{aligned}\min_{q\in Q}\Big(\theta_{\mathscr{A}}(\xi, q) + t \cdot \mathrm{final}(q)\Big) - \theta_{\mathscr{A}'}(\xi) &\overset{\star_3}{\leq} \mathrm{final}'(P)\\ &\overset{\star_4}{\leq} \min_{q\in Q}\Big(t \cdot \theta_{\mathscr{A}}(\xi, q) + t \cdot \mathrm{final}(q)\Big) - \theta_{\mathscr{A}'}(\xi).\end{aligned}$$

Inequality $\star_4$ is equivalent to $\star_2$ and since $t \geq 1$, Inequality $\star_3$ implies Inequality $\star_1$. This concludes the proof. □

4.4 The Approximated Twinning Property

We start this chapter by defining the so-called t-twinning property for weighted tree automata, which is a natural extension of both, the word case [4] and the tree case without approximation (that is, $t = 1$) [14]. We then prove that the t-twinning property is a sufficient condition for the finiteness of our approximated determinisation construction.

Throughout the rest of Chapter 4.4, if not stated differently, we assume $\mathscr{A} = (Q, T, \mathrm{final})$ to be a finite WTA *and $t \in \mathbb{R}$ to be a real number such that $t \geq 1$.*

Definition 4.22. For every $p, q \in Q$ we say that *p and q are siblings* if there exists a tree $\xi \in \mathrm{T}_\Sigma$ and non-vanishing runs $\rho_1 \in \mathrm{Runs}_{\mathscr{A}}(\xi, p)$ and $\rho_2 \in \mathrm{Runs}_{\mathscr{A}}(\xi, q)$. Let p and q be siblings. We say that *p and q are t-twins* if for every $\zeta \in \mathrm{C}_\Sigma$ it holds that $\theta(p, \zeta, p) = \infty$, $\theta(q, \zeta, q) = \infty$, or

$$\frac{1}{t} \cdot \theta(q, \zeta, q) \leq \theta(p, \zeta, p) \leq t \cdot \theta(q, \zeta, q). \tag{4.6}$$

We say that $\mathscr{A}$ has the *t-twinning property* if for all siblings $p, q \in Q$ it holds that p and q are t-twins. ◀

Example 4.23. We continue Example 4.2 by showing that $\mathscr{A}$ satisfies the 2-twinning property but not the 1-twinning property. First, note that q_1 and q_1 are trivially siblings and t-twins (analogously for q_2 and q_2). Moreover, q_1 and q_2 are siblings as there are two runs ρ_1 and ρ_2 on $\xi = \alpha$, where $\rho_1(\varepsilon) = q_1$ and $\rho_2(\varepsilon) = q_2$.

Let $\zeta \in \mathrm{C}_\Sigma$. If ζ contains a β, then we have that $\theta(q_2, \zeta, q_2) = \infty$. We now assume that ζ does not contain a β. Let ρ be a non-vanishing run of $\mathscr{A}$ on ζ. One easily sees that ρ either maps each position to q_1 (in this case $\mathrm{wt}(\zeta, \rho) = \#\mathrm{pos}_\alpha(\zeta)$) or ρ maps each position to q_2 (in this case $\mathrm{wt}(\zeta, \rho) = 2 \cdot \#\mathrm{pos}_\alpha(\zeta)$). Since these are the only valid runs of $\mathscr{A}$ on ζ, we obtain $\theta(q_2, \zeta, q_2) = 2 \cdot \theta(q_1, \zeta, q_1)$. This proves that $\mathscr{A}$ satisfies the 2-twinning property.

Moreover, $\mathscr{A}$ does not satisfy the 1-twinning property, as q_1 and q_2 are siblings but for $\zeta = \sigma(\alpha, x_1)$ it holds that $\theta(q_1, \zeta, q_1) = 1 \neq 2 = \theta(q_2, \zeta, q_2)$. ◀

4.4.1 Implications for Approximated Determinisability

In order to show that the t-twinning property is a sufficient condition for the finiteness of our approximate determinisation construction, we provide three supporting technical remarks. We begin by defining a multiplication between factors $d \in \mathbb{R}_\infty$ and partial weighted tree automata whose states are in $(\mathbb{R}_\infty \times \mathbb{R}_\infty)^Q$.

Remark 4.24. Let $d \in \mathbb{R}_\infty$ such that $d > 0$. For every $P \colon Q \to \mathbb{R}_\infty \times \mathbb{R}_\infty$ we define $d \cdot P \colon Q \to \mathbb{R}_\infty \times \mathbb{R}_\infty$ by $(d \cdot P)(q) = d \cdot P(q)$ for every $q \in Q$. Let $\mathscr{B} = (\widehat{Q}, \widehat{T}, \widehat{\mathrm{final}})$ be a partial WTA such that $\widehat{Q} \subseteq (\mathbb{R}_\infty \times \mathbb{R}_\infty)^Q$. We denote by $d \cdot \mathscr{B}$ the partial WTA

$$d \cdot \mathscr{B} = (\widehat{Q}_d, \widehat{T}_d, \widehat{\mathrm{final}}_d),$$

where $\widehat{Q}_d = \{d \cdot P \mid P \in \widehat{Q}\}$,

$$(\widehat{T}_d)_\sigma(P_1, \ldots, P_s) = d \cdot \widehat{T}_\sigma(d^{-1} \cdot P_1, \ldots, d^{-1} \cdot P_s), \text{ and}$$
$$\widehat{\mathrm{final}}_d(P) = d \cdot \widehat{\mathrm{final}}(d^{-1} \cdot P)$$

for every $s \geq 0$, $\sigma \in \Sigma$, and $P_1, \ldots, P_s, P \in \widehat{Q}_d$. That is, $d \cdot \mathscr{B}$ is the partial WTA constructed from $\mathscr{B}$ by multiplying all transition weights and final weights by d and renaming all states by multiplying all lower and upper residues by d.

A straightforward inductive proof shows that

$$(d \cdot \mathscr{A})_n = d \cdot \mathscr{A}_n \text{ for every } n \in \mathbb{N}, \tag{4.7}$$

and hence also $(d \cdot \mathscr{A})' = d \cdot \mathscr{A}'$. We briefly outline the proof of Equation (4.7). The case $n = 0$ is trivial. Assume that Equation (4.7) holds for some $n \in \mathbb{N}$. We need to show that $\mathrm{step}_{d \cdot \mathscr{A}, t}((d \cdot \mathscr{A})_n) = d \cdot \mathrm{step}_{\mathscr{A}, t}(\mathscr{A}_n)$. Let $s \geq 0$, $\sigma \in \Sigma$, and $P_1, \ldots, P_s \in Q_n$ and let $(c, P) = \mathcal{T}_\sigma(P_1, \ldots, P_s)$ in the construction of $\mathrm{step}_{\mathscr{A}, t}(\mathscr{A}_n)$ and $(c', P') = \mathcal{T}_\sigma(d \cdot P_1, \ldots, d \cdot P_s)$ in the construction of $\mathrm{step}_{d \cdot \mathscr{A}, t}((d \cdot \mathscr{A})_n)$. One can easily verify that $c' = d \cdot c$ and $P' = d \cdot P$ using Definition 4.8 and the induction assumption $(d \cdot \mathscr{A})_n = d \cdot \mathscr{A}_n$. This also uses the fact that for two maps $P_1, P_2 \in (\mathbb{R}_\infty \times \mathbb{R}_\infty)^Q$ it holds that P_1 refines P_2 if and only if $d \cdot P_1$ refines $d \cdot P_2$. Similarly, one sees that the final weights of $(d \cdot \mathscr{A})_{n+1}$ correspond to the final weights of $d \cdot \mathscr{A}_{n+1}$. ◀

Next we show that every state of $\mathscr{A}'$ occurs at the root of a discovering run of $\mathscr{A}'$. We will use this later to obtain tighter inequalities (than the ones holding for arbitrary runs).

Remark 4.25. Let $P \in Q'$ be a state of $\mathscr{A}'$. There exists a tree $\xi \in \mathrm{T}_\Sigma$ and a discovering run $\rho \in \mathrm{Runs}_{\mathscr{A}'}(\xi, P)$. This can be seen as follows. Let $n = \mathrm{depth}(P)$. We proceed by induction on n. The induction base $n = 0$ trivially holds, since $Q_0 = \emptyset$. Now assume that $n \geq 1$ such that the claim holds for every $P' \in Q_{n-1}$. Since $P \in Q_n$, there exists $s \geq 0$, $\sigma \in \Sigma^{(s)}$, $P_1, \ldots, P_s \in Q_{n-1}$, and $c \in \mathbb{R}_\infty$ such that $(c, P) = \mathcal{T}_\sigma(P_1, \ldots, P_s)$ in the definition of $\mathrm{step}_{\mathscr{A}, t}(\mathscr{A}_n)$. We know that P is not refined by some state in Q_{n-1}, because $P \notin Q_{n-1}$. In particular, the transition $(P_1, \ldots, P_s, \sigma, c, P)$ is a discovering transition. By assumption, there exist trees $\xi_1, \ldots, \xi_s \in \mathrm{T}_\Sigma$ and for every $i \in [s]$ there exists a discovering run $\rho_i \in \mathrm{Runs}_{\mathscr{A}}(\xi_i, P_i)$. Therefore, the run ρ of $\mathscr{A}'$ on the tree $\xi = \sigma(\xi_1, \ldots, \xi_s)$, defined for every $w \in \mathrm{pos}(\xi)$ by

$$\rho(w) = \begin{cases} P & \text{if } w = \varepsilon, \\ \rho_i(v) & \text{if } w = iv \text{ for some } i \in [s] \text{ and } v \in \mathbb{N}^* \end{cases}$$

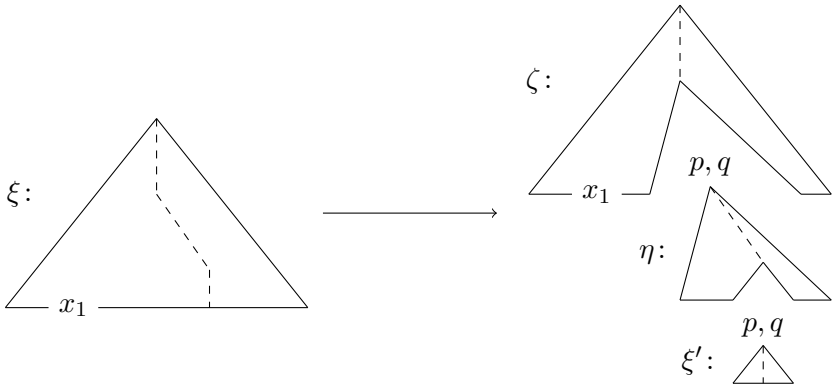


Figure 4.3: The identification of synchronised loops of two runs ρ_1 and ρ_2 on a context tree ξ. The dashed line in ξ marks a path of height $> \#Q^2 + 1$. Somewhere along this path, ξ can be decomposed into the three parts ζ, η, and ξ' on the right, such that $\rho_1|_\eta$ and $\rho_2|_\eta$ are loops (in the respective states p and q). Depending on ξ, the original variable x_1 might occur in ζ, η, or ξ'.

is a discovering run. ◀

In the following remark, we discuss some properties of loops in runs. This will come in handy multiple times throughout the rest of Chapter 4.

Remark 4.26. Let $\xi \in \mathrm{T}_\Sigma \cup \mathrm{C}_\Sigma$. In this remark, we consider ξ as an element of $\mathrm{T}_{\Sigma \cup \{x_1\}}$, that is, we do not consider x_1 as a variable, but rather as a symbol of the underlying ranked alphabet. Then, for a context $\zeta \in \mathrm{C}_{\Sigma \cup \{x_1\}}$, the variable occurrence in ζ is distinguishable from any (other) occurrence of x_1 in ζ.

For every run $\rho \in \mathrm{Runs}_{\mathscr{A}}(\xi)$ and every $\zeta, \eta \in \mathrm{C}_{\Sigma \cup \{x_1\}}$, and $\xi' \in \mathrm{T}_{\Sigma \cup \{x_1\}}$ such that $\zeta[\eta[\xi']] = \xi$, we define the runs $\rho|_\eta \in \mathrm{Runs}_{\mathscr{A}}(\eta)$ and $\rho|_{\zeta[\xi']} \in \mathrm{Runs}_{\mathscr{A}}(\zeta[\xi'])$ where

$$\rho|_\eta(w) = \rho(\mathrm{pos}_{\mathrm{var}}(\zeta)w)$$

for every $w \in \mathrm{pos}(\eta)$ and

$$\rho|_{\zeta[\xi']}(w) = \begin{cases} \rho(w) & \text{if } w \in (\mathrm{pos}(\zeta) \setminus \{\mathrm{pos}_{\mathrm{var}}(\zeta)\}), \\ \rho(\mathrm{pos}_{\mathrm{var}}(\zeta)\mathrm{pos}_{\mathrm{var}}(\eta)v) & \text{if } w = \mathrm{pos}_{\mathrm{var}}(\zeta)v \text{ for some } v \in \mathbb{N}^* \end{cases}$$

for every $w \in \mathrm{pos}(\zeta[\xi'])$. We note that $\rho|_{\zeta[\xi']}$ is well-defined as the two given cases cover $\mathrm{pos}(\zeta[\xi'])$. Moreover, it holds that if $\mathrm{wt}(\xi, \rho) = \theta(\xi, \rho(\varepsilon))$, then we also have

$\mathrm{wt}(\eta, \rho|_\eta) = \theta(\eta, \rho|_\eta(\varepsilon))$ and $\mathrm{wt}(\zeta[\xi'], \rho|_{\zeta[\xi']}) = \theta(\zeta[\xi'], \rho|_{\zeta[\xi']}(\varepsilon))$. That is, if ρ is the run with minimal weight on ξ, then $\rho|_{\zeta[\xi']}$ and $\rho|_\eta$ are the runs with minimal weights on $\zeta[\xi']$ and η, respectively. This fact follows from the commutativity of Trop, as $\mathrm{wt}(\xi, \rho) = \mathrm{wt}(\zeta[\xi'], \rho|_{\zeta[\xi']}) + \mathrm{wt}(\eta, \rho|_\eta)$.

Now let $\mathrm{size}(\xi) > \mathrm{maxrk}(\Sigma)^{\#Q^2+1}$ and let $\rho_1, \rho_2 \in \mathrm{Runs}_{\mathscr{A}}(\xi)$ be two runs of $\mathscr{A}$ on ξ. There exist $\zeta, \eta \in \mathrm{C}_{\Sigma\cup\{x_1\}}$ and $\xi' \in \mathrm{T}_{\Sigma\cup\{x_1\}}$ such that $\mathrm{size}(\xi) > \mathrm{size}(\eta) > 1$, $\zeta[\eta[\xi']] = \xi$, and $\rho_1|_\eta$ and $\rho_2|_\eta$ are loops. This follows from the pigeonhole principle, as $\mathrm{height}(\xi) \geq \#Q^2+1$ and the maximal number of different pairs of states $(\rho_1(w), \rho_2(w))$ is $\#Q^2$. If $\mathrm{size}(\xi) > \mathrm{maxrk}(\Sigma)^{2\cdot\#Q^2+1}$, then we can moreover ensure that $\eta \in \mathrm{C}_\Sigma$, that is, any variable possibly occurring in ξ is not part of η. We depict the decomposition of a context into the parts ζ, η, and ξ' in Figure 4.3. ◀

Throughout the rest of Chapter 4.4.1, we assume that $\mathrm{im}(\mathscr{A}) \subseteq \mathbb{Q}_\infty$.

We are now ready to prove that the t-twinning property is a sufficient condition for the finiteness of our approximate determinisation construction (Theorem 4.27). The proof is very similar to the proof of [4, Theorem 8]. Note that in [4, Theorem 8], t is a rational number, whereas we allow for t to be a real number. We resolve this by multiplying t and all weights occurring in $\mathscr{A}$ by $\frac{1}{t}$.

Theorem 4.27. If $\mathscr{A}$ satisfies the t-twinning property, then $\mathscr{A}' = \mathscr{A}_n$ for some $n \in \mathbb{N}$.

Proof. We define $d \in \mathbb{R}$ as the least common multiple of the elements in the set

$$\left\{ b \;\middle|\; a, b \in \mathbb{N}, \frac{a}{b} \in \mathrm{im}(\mathscr{A}), \text{ and } a \text{ and } b \text{ are coprime} \right\},$$

where two elements $a, b \in \mathbb{N}$ are *coprime* if 1 is the only common divisor of a and b (cf. [78] for more details on least common multiples and coprimality). Furthermore we define $c = \frac{1}{t} \cdot d$. We note that all weights of $c \cdot \mathscr{A}$ are in $\frac{1}{t} \cdot \mathbb{N} \cup \{\infty\}$ by the definition of d. By Remark 4.24, the claim holds for $c \cdot \mathscr{A}$ (that is, $(c \cdot \mathscr{A})' = (c \cdot \mathscr{A})_n$ for some $n \in \mathbb{N}$) if and only if it holds for $\mathscr{A}$. Therefore, we henceforth assume that all weights of $\mathscr{A}$ are in $\frac{1}{t} \cdot \mathbb{N} \cup \{\infty\}$.

Assume that $\mathscr{A}$ satisfies the t-twinning property and $\mathscr{A}' \neq \mathscr{A}_n$ for every $n \in \mathbb{N}$. Hence, $\mathscr{A}'$ has infinitely many states. In particular, there exists an infinite sequence

$\pi = (P_k \mid k \in \mathbb{N})$ of pairwise different states of $\mathscr{A}'$. Next, observe that there exist states $\hat{q}, \bar{q} \in Q$ and a subsequence $\pi' = (P_{i_k} \mid k \in \mathbb{N})$ of π such that (i) $(l_{\hat{q}}^{P_{i_k}} \mid k \in \mathbb{N})$ is a sequence of elements in $\mathbb{R}$ that monotonically increases towards infinity for $k \to \infty$ and (ii) $u_{\bar{q}}^{P_{i_k}} = 0$ for every $k \in \mathbb{N}$. In fact, the very same property is proven in three steps during the proof of [4, Theorem 8] (for rational t). Our argumentation differs merely in the fact that all weights are multiplied by the factor $\frac{1}{t}$ from the argumentation presented in [4] and hence we omit the proof of this property.

We define the value

$$x = \max\{\mathrm{wt}(\xi, \hat{\rho}) - t \cdot \mathrm{wt}(\xi, \bar{\rho}) \mid \xi \in \mathrm{T}_\Sigma, \mathrm{height}(\xi) \leq \#Q^2, \\ \hat{\rho} \in \mathrm{Runs}_{\mathscr{A}}(\xi, \hat{q}), \bar{\rho} \in \mathrm{Runs}_{\mathscr{A}}(\xi, \bar{q})\}.$$

By Remark 4.25 it holds that for every $k \in \mathbb{N}$, the state P_{i_k} is reachable by a discovering run on some tree $\xi_k \in \mathrm{T}_\Sigma$. Therefore, by Lemma 4.20, it holds that

$$\theta_{\mathscr{A}}(\xi_k, \hat{q}) - \theta_{\mathscr{A}'}(\xi_k) = l_{\hat{q}}^{P_{i_k}} \qquad \text{and} \qquad t \cdot \theta_{\mathscr{A}}(\xi_k, \bar{q}) - \theta_{\mathscr{A}'}(\xi_k) = u_{\bar{q}}^{P_{i_k}} (= 0)$$

for every $k \in \mathbb{N}$. Subtracting the right equation from the left equation, we obtain

$$\theta_{\mathscr{A}}(\xi_k, \hat{q}) - t \cdot \theta_{\mathscr{A}}(\xi_k, \bar{q}) = l_{\hat{q}}^{P_{i_k}}$$

for every $k \in \mathbb{N}$. Since $(l_{\hat{q}}^{P_{i_k}} \mid k \in \mathbb{N})$ monotonically increases towards infinity, there exists $k \in \mathbb{N}$ such that $\theta_{\mathscr{A}}(\xi_k, \hat{q}) - t \cdot \theta_{\mathscr{A}}(\xi_k, \bar{q}) > x$. Therefore, by the definition of x we know that $\mathrm{height}(\xi_k) > \#Q^2$. The fact that $l_{\hat{q}}^{P_{i_k}} \in \mathbb{R}$ implies that $\theta_{\mathscr{A}}(\xi_k, \hat{q}) < \infty$ and $\theta_{\mathscr{A}}(\xi_k, \bar{q}) < \infty$ and therefore, there exist two runs $\hat{\rho} \in \mathrm{Runs}_{\mathscr{A}}(\xi_k, \hat{q})$ and $\bar{\rho} \in \mathrm{Runs}_{\mathscr{A}}(\xi_k, \bar{q})$ such that $\mathrm{wt}(\xi_k, \hat{\rho}) = \theta_{\mathscr{A}}(\xi_k, \hat{q})$ and $\mathrm{wt}(\xi_k, \bar{\rho}) = \theta_{\mathscr{A}}(\xi_k, \bar{q})$. By Remark 4.26 there exist contexts $\zeta', \eta \in \mathrm{C}_\Sigma$ and a tree $\xi' \in \mathrm{T}_\Sigma$ such that $\xi_k = \zeta'[\eta[\xi']]$, $\mathrm{size}(\eta) > 1$, and both $\hat{\rho}|_\eta$ and $\bar{\rho}|_\eta$ are loops on η. Moreover, since $\mathrm{wt}(\xi_k, \hat{\rho}) = \theta_{\mathscr{A}}(\xi_k, \hat{q})$, we know that $\mathrm{wt}(\xi_k, \hat{\rho}) - \mathrm{wt}(\zeta'[\xi'], \hat{\rho}|_{\zeta'[\xi']}) = \theta(\hat{q}, \zeta, \hat{q})$ by Remark 4.26. Analogously, we have $\mathrm{wt}(\xi_k, \bar{\rho}) - \mathrm{wt}(\zeta'[\xi'], \bar{\rho}_{\zeta'[\xi']}) = \theta(\bar{q}, \zeta, \bar{q})$. Therefore, the fact that $\mathscr{A}$ satisfies the t-twinning property implies that

$$\mathrm{wt}(\xi_k, \hat{\rho}) - \mathrm{wt}(\zeta'[\xi'], \hat{\rho}_{\zeta'[\xi']}) = \theta(\hat{q}, \zeta, \hat{q}) \\ \leq t \cdot \theta(\bar{q}, \zeta, \bar{q}) = t \cdot (\mathrm{wt}(\xi_k, \bar{\rho}) - \mathrm{wt}(\zeta'[\xi'], \bar{\rho}_{\zeta'[\xi']})). \tag{4.8}$$

In total we obtain

$$\begin{aligned}\mathrm{wt}(\zeta'[\xi'], \hat{\rho}_{\zeta'[\xi']}) - t \cdot \mathrm{wt}(\zeta'[\xi'], \bar{\rho}_{\zeta'[\xi']}) &\overset{\star}{\geq} \mathrm{wt}(\xi, \hat{\rho}) - t \cdot \mathrm{wt}(\xi, \bar{\rho}) \\ &= \theta_{\mathscr{A}}(\xi_k, \hat{q}) - t \cdot \theta_{\mathscr{A}}(\xi_k, \bar{q}) > x.\end{aligned}$$

where inequality $\star$ is a rearranged version of Equation (4.8). We have thus found the tree $\zeta[\xi']$ with $\mathrm{size}(\zeta[\xi']) < \mathrm{size}(\xi_k)$, yet still $\mathrm{height}(\zeta[\xi']) > \#Q^2$. By repeatedly removing loops from $\hat{\rho}$ and $\bar{\rho}$ in this manner, we can find a contradiction. This concludes the proof of this theorem. □

Corollary 4.28. If $\mathscr{A}$ satisfies the t-twinning property, then $\mathscr{A}$ is t-determinisable.

Proof. This follows immediately from Theorems 4.21 and 4.27. □

Example 4.29. We continue Example 4.23. We have seen that for $t = 2$, it holds that $\mathscr{A}_2 = \mathscr{A}_1$. For $t = 1$, one can show that $\mathscr{A}_{n+1} \neq \mathscr{A}_n$ for every $n \in \mathbb{N}$. That is, our approximate determinisation construction yields a finite automaton for $t = 2$, but not for $t = 1$. This was expected by Theorem 4.27, as $\mathscr{A}$ satisfies the 2-twinning property, but not the 1-twinning property.

We now provide a short argumentation as to why $\mathscr{A}_{n+1} \neq \mathscr{A}_n$ for every $n \in \mathbb{N}$ in the case that $t = 1$. In Example 4.12, we calculated $\mathscr{A}_2 = \mathrm{step}_{\mathscr{A},t}(\mathscr{A}_1)$ and generated the successor state $P' = \{(q_1, (2-2t, 0)), (q_2, (4-2t, 2t))\}$ by considering the input symbol σ and the predecessor states P'_1 and P'_1. If $t = 2$, then P' is refined by P'_1 and therefore not added as a new state in Q_2. For $t = 1$, however, P' is equal to $\{(q_1, (0, 0)), (q_2, (2, 2))\}$ and hence P' is not refined by any previously existing state. Therefore, P' is added to the state space. Next, considering the input symbol σ and the predecessor states P' and P', we obtain another unrefineable state, namely $P'' = \{(q_1, (0, 0)), (q_2, (4, 4))\}$. One easily sees that the construction continues to generate every state of the form $\{(q_1, (0, 0)), (q_2, (2^k, 2^k))\}$ as a new state in Q_{k+1} and hence the sequence $(\mathscr{A}_n \mid n \in \mathbb{N})$ does not stagnate after some $n \in \mathbb{N}$. ◀

4.4.2 Decidability of the Twinning Property

In the following theorem, we prove the decidability of the t-twinning property. This is due to the fact that if a WTA $\mathscr{A}$ does not satisfy the t-twinning property, then this

non-satisfaction is already witnessed by a small context tree.

Lemma 4.30. Let $\mathscr{A} = (Q, T, \text{final})$ be a WTA and $t \in \mathbb{R}$ such that $t \geq 1$.

If $\mathscr{A}$ does not satisfy the t-twinning property, then there exist siblings $p, q \in Q$ and a context $\zeta \in \mathrm{C}_\Sigma$ such that $\text{size}(\zeta) \leq \text{maxrk}(\Sigma)^{2 \cdot \#Q^2+1}$ and

$$t \cdot \theta(q, \zeta, q) < \theta(p, \zeta, p) < \infty.$$

Proof. Assume that $\mathscr{A}$ does not satisfy the t-twinning property. Thus, there exist siblings $p, q \in Q$ and a context $\zeta \in \mathrm{C}_\Sigma$ such that

$$t \cdot \theta(q, \zeta, q) < \theta(p, \zeta, p) < \infty$$

(after swapping p and q, if necessary). Among all possible such ζ, we fix a context such that $\text{size}(\zeta)$ is minimal. We assume that $\text{size}(\zeta) > \text{maxrk}(\Sigma)^{2 \cdot \#Q^2+1}$. In particular, $\text{height}(\zeta) \geq 2 \cdot \#Q^2 + 1$.

Let $\rho_1 \in \text{Runs}_{\mathscr{A}}(p, \zeta, p)$ and $\rho_2 \in \text{Runs}_{\mathscr{A}}(q, \zeta, q)$ such that $\text{wt}(\zeta, \rho_1) = \theta(p, \zeta, p)$ and $\text{wt}(\zeta, \rho_2) = \theta(q, \zeta, q)$. By Remark 4.26, there exist $\zeta', \zeta'' \in \mathrm{C}_{\Sigma \cup \{x_1\}}$ and $\eta \in \mathrm{C}_\Sigma$ such that $\zeta = \zeta'[\eta[\zeta'']]$, $\text{size}(\zeta) > \text{size}(\eta) > 1$, and both ρ_1 and ρ_2 loop on η (in states q_1 and q_2, respectively). It holds that $\theta(q_1, \eta, q_1) \leq t \cdot \theta(q_2, \eta, q_2)$ since ζ is by assumption a minimal witness of the non-satisfaction of the t-twinning property and $\text{size}(\eta) < \text{size}(\zeta)$. We ultimately obtain

$$\theta(p, \zeta'[\zeta''], p) \stackrel{\star}{=} \theta(p, \zeta, p) - \theta(q_1, \eta, q_1) > t \cdot (\theta(q, \zeta, q) - \theta(q_2, \eta, q_2)) \stackrel{\star}{=} t \cdot \theta(q, \zeta'[\zeta''], q),$$

where the $\star$-equations follow from Remark 4.26, as $\text{wt}(\zeta'[\zeta''], \rho_1|_{\zeta'[\zeta'']}) = \theta(p, \zeta'[\zeta''], p)$ and $\text{wt}(\eta, \rho_1|_\eta) = \theta(q_1, \eta, q_1)$, and analogously for ρ_2. In particular, we have found a smaller witness of the non-satisfaction of the t-twinning property than ζ, namely $\zeta'[\zeta'']$, which is a contradiction. Hence, the assumption that $\text{size}(\zeta) > \text{maxrk}(\Sigma)^{2 \cdot \#Q^2+1}$ must be dropped. □

Theorem 4.31. The t-twinning property is decidable for every WTA $\mathscr{A}$ and $t \geq 1$.

Proof. First, note that we can determine the set of siblings in Q by only considering trees $\xi \in \mathrm{T}_\Sigma$ such that $\text{size}(\xi) \leq \text{maxrk}(\Sigma)^{\#Q^2+1}$. This fact is proven analogously to Lemma 4.30 by removing synchronised loops from runs on bigger input trees.

By Lemma 4.30, $\mathscr{A}$ does not satisfy the t-twinning property if and only if there is a small witness to the non-satisfaction of the t-twinning property.

Hence, we can decide the t-twinning property by (1) determining the set of siblings of Q, (2) calculating $\theta(p, \zeta, p)$ for every state $p \in Q$ and every context $\zeta \in \mathrm{C}_\Sigma$ such that $\mathrm{size}(\zeta) \leq \mathrm{maxrk}(\Sigma)^{2 \cdot \#Q^2+1}$, and (3) checking the t-twinning property only on the values calculated in (2). $\square$

4.5 Conclusion

In this chapter, we generalised [4] from the word case to the tree case. First, we gave a t-approximated determinisation construction by defining a sequence $(\mathscr{A}_n \mid n \in \mathbb{N})$ of partial WTA for a given input automaton $\mathscr{A}$ and considering the limit $\mathscr{A}'$ of this sequence. Next, we proved that $\mathscr{A}'$ is a deterministic WTA that t-approximates $\mathscr{A}$, whenever $\mathscr{A}' = \mathscr{A}_n$ for some $n \in \mathbb{N}$. Then, we introduced the t-twinning property for weighted tree automata and showed that the t-twinning property implies that $\mathscr{A}' = \mathscr{A}_n$ for some $n \in \mathbb{N}$ (under the assumption that all weights of $\mathscr{A}$ are in $\mathbb{Q}_\infty$). We ultimately showed that our t-twinning property is decidable.

This chapter is a rather compact excursion to approximated determinisation and some research directions remain untouched. For example, recent work has shown that the twinning property is equivalent to determinisability in some cases (e.g. [21]). It would be worthwhile to determine whether in our case, the t-twinning property is necessary for t-determinisability. Another interesting research direction is to introduce approximated determinisation for general classes of semirings rather than only considering the tropical semiring. Moreover, it seems rather arbitrary to say $x \in \mathbb{R}$ is approximated exactly by the values in the interval $[x, t \cdot x]$. It would be interesting to introduce more general notions of "approximation" and find sufficient conditions for this general approximated determinisability.

5

Kleene and Büchi Theorems for Weighted Forest Languages over M-Monoids

This chapter is a presentation of Dörband [27] with minor changes.

Throughout Chapter 5, we assume Σ to be a ranked alphabet.

5.1 Introduction

We have studied formal languages throughout the previous chapters via the general model of weighted tree automata over semirings. From a more algebraic perspective, one can study classes of formal languages through their closure properties under certain operations. For example, the class of *rational word languages* is defined as the smallest class containing the finite languages and that is closed under union, concatenation, and Kleene star (cf. [81]). A third way of looking at formal languages is from the logician's point of view. One can study classes of formal languages through the logical formulas that they satisfy. A prominent example of such a logic formalism is called monadic second-order logic (or short: MSO-logic) and was introduced in [13]. The formal languages represented by MSO-logic formulas are called *MSO-definable*.

Since a long time, it is established that these different perspectives on formal languages are strongly related in the word case. The so-called *Kleene Theorem* [81] states that the class of recognisable word languages equals the class of rational word languages. On the other hand, the so-called *Büchi Theorem* [13] states that the class of recognisable word languages equals the class of MSO-definable word languages.

Over the years, many different generalisations of the Kleene and Büchi theorems have been published, both, in the unweighted and the weighted setting. In this chapter, we consider so-called *weighted forest automata*. For us, forests are tuples of trees and weighted forest languages only consider forests containing a fixed number of trees. A different approach to weighted forest languages is to consider forests with an arbitrary amount of trees. This has been done by Mathissen in [87], where forests are called *hedges*. Various other syntactic objects have been considered and Kleene and Büchi theorems have been established. However, for our purposes, we will focus on the tree and forest cases. We collect the relevant literature in the following table.

	Trees	**Forests**	**Hedges**
Kleene	[107] (unweighted) [38] (semirings) [51] (M-monoids) [70] (tv-monoids)	[104] (unweighted) [26] (semirings) **this chapter** (M-monoids)	
Büchi	[107] (unweighted) [40] (semirings) [52] (M-monoids) [33, 34] (tv-monoids)	**this chapter** (M-monoids)	[87] (semirings)

As weight structures, we consider so-called *M-monoids*. Up to some technicalities, an M-monoid (short for "multioperator"-monoid) is a monoid $(\mathbb{M}, \oplus, 0)$ together with a family of operations $\Omega \subseteq \mathrm{Ops}(\mathbb{M})$. A *weighted tree automaton* (short: WTA) *over* $\mathbb{M}$ is a tuple $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$, which is defined similarly to weighted tree automata over semirings, except that all weights are operations from Ω, rather than elements of the underlying algebra. Moreover, init provides designated leaf weights for variable positions, which weighted tree automata do not provide. The weight of a run of $\mathscr{A}$ on a tree ξ is the composition of the assigned operations in the natural way and $[\![\mathscr{A}]\!](\xi)$ is an operation on $\mathbb{M}$ whose arity depends on the number of variables occurring in ξ. We

have chosen this very general weighted automaton model in order to extend as many of the existing results as possible.

There is an even more general class of weight structures, the so-called *tree valuation monoids* (tv-monoids), over which weighted tree automata (WTA) have been considered. Also, Kleene-like and Büchi-like theorems have been proven for WTA over tree valuation monoids [33, 34, 70]. We now briefly justify the fact that we only consider M-monoids, rather than tv-monoids. In [106, Lemma 20], the authors have shown that for each tv-monoid $\mathbb{T}$, there is an M-monoid $\mathbb{M}$, such that for each WTA $\mathscr{A}$ over $\mathbb{T}$, there is a WTA $\mathscr{B}$ over $\mathbb{M}$, such that $[\![\mathscr{A}]\!] = \mathrm{proj}_1([\![\mathscr{B}]\!])$. Hence, the only theoretical difference between recognisable weighted tree languages over $\mathbb{T}$ and recognisable weighted tree languages over $\mathbb{M}$ is the application of a projection map. We believe that this close connection between tv-monoids and M-monoids supports our choice to only consider M-monoids.

Mathissen's work [87] on weighted hedge automata is related to our weighted forest automata model, but the two approaches cannot be compared directly. Two reasons are the following. First of all, Mathissen considers unranked hedges, whereas we consider forests over ranked trees. Moreover, the hedges in [87] can consist of an arbitrary (finite) number of trees, whereas we consider forests with a fixed number of trees. However, restricting [87] to ranked hedges with a fixed number of trees and restricting our weighted forest automata to the case of semirings (see Remark 5.14) results in the same class of recognisable languages.

We note that the restriction from "unranked" to "ranked" is only a technical difference in the semiring-weighted setting. In fact, it has been shown in [23] that weighted (ranked) tree automata over semirings together with a so-called binarisation are equivalent to weighted unranked tree automata over semirings.

We now briefly outline the rest of Chapter 5 and our main contributions. In Chapter 5.2, we establish the necessary mathematical foundations. We define M-monoids and some M-monoid properties, including absorptivity, distributivity, and $(1,\star)$-composition closure. Moreover, we define weighted tree automata over M-monoids. In Chapter 5.3, we introduce (b,n)-forests for every $b,n \in \mathbb{N}$ as b-tuples of trees in $\mathrm{T}_\Sigma(X_n)$. Moreover, we generalise the weighted tree automaton model from Chapter 5.2 to the case

of (b, n)-forests by simply allowing for b root distributions $\mathrm{final}_1, \ldots, \mathrm{final}_b$. In order to define the language of such a weighted (b, n)-forest automaton, we assume that there is a second operation on $\mathbb{M}$, which distributes over the monoid operation and which is contained in Ω. Naturally extending the tree case, the weight of a run on a (b, n)-forest is a tuple of b operations on $\mathbb{M}$, which we then multiply using the second operation on $\mathbb{M}$. This yields the class $\mathrm{Rec}(\Sigma, \mathbb{M}, b, n)$ (and $\mathrm{Rec}_{\mathrm{f}}(\Sigma, \mathbb{M}, b, n)$) of languages recognised by weighted (b, n)-forest automata over $\mathbb{M}$ (with final states). As a central result, we obtain that every $\varphi \in \mathrm{Rec}(\Sigma, \mathbb{M}, b, n)$ can be decomposed into a product of b elements $\varphi_1, \ldots, \varphi_b \in \mathrm{Rec}(\Sigma, \mathbb{M}, 1, n)$. Here, "product" refers to the second operation on $\mathbb{M}$ applied pointwise. In Chapters 5.4 and 5.5, we lift the Kleene-like result from [51] and the Büchi-like result from [52] from the tree case to the forest case, respectively. First, we define the class $\mathrm{Rat}(\Sigma, \mathbb{M}, b, n)$ of *rational weighted* (b, n)*-forest languages over* $\mathbb{M}$. Then, we prove our Kleene-like theorem, Theorem 5.37, which generalises the Kleene-like theorems from [26, 38, 51, 104, 107].

Theorem 5.37 (Kleene result for forests)**.** If $\mathbb{M}$ is distributive, then

$$\mathrm{Rec}(\Sigma, \mathbb{M}, b, n) \subseteq \bigcup_{k \in \mathbb{N}} \mathrm{Rat}(\Sigma, \mathbb{M}, b, k)|_{\mathrm{T}_\Sigma(X_n)}.$$

If $\mathbb{M}$ is distributive, closed under sum, and $(1,\star)$-composition closed, then

$$\mathrm{Rat}(\Sigma, \mathbb{M}, b, n) \subseteq \mathrm{Rec}(\Sigma, \mathbb{M}, b, n).$$

We note that the direction "Rec $\subseteq$ Rat" does not preserve the number of variables n, which we discuss towards the end of Chapter 5.1. Next, we define the class $\mathrm{MDef}(\Sigma, \mathbb{M}, b)$ of *languages defined by* $(b, 0)$*-forest M-expressions.* Then, we prove our Büchi-like theorem, Theorem 5.45, which generalises the Büchi-like theorems from [40, 52, 107].

Theorem 5.45 (Büchi result for forests)**.** Let $\mathbb{M}$ be absorptive. It holds that

$$\mathrm{Rec}_{\mathrm{f}}(\Sigma, \mathbb{M}, b, 0) = \mathrm{MDef}(\Sigma, \mathbb{M}, b).$$

Our definitions of rational forest expressions and forest M-expressions are not inductive, which is why we dedicate Chapter 5.4.3 to a discussion on why straightforward inductive definitions fail. Lastly, we give some concluding remarks in Chapter 5.6.

We close this introductory chapter by briefly elaborating the use of variables in Chapter 5. We recall that our unified automaton model will be used to generalise both a Kleene-like result [51] and a Büchi-like result [52]. Hence, our automaton model needs to take into account the way that [51, 52] consider variables.

For the Kleene-like result variables are used during the analysis of an automaton. Let $\mathscr{A}$ be a weighted tree automaton with state space Q and denote $k = \#Q$. We consider X_k, where each state of Q corresponds to a variable in X_k. In order to construct a rational expression generating the language $[\![\mathscr{A}]\!]$, one decomposes $[\![\mathscr{A}]\!]$ into "easier" intermediate languages, using the variables from X_k. More specifically, a state $q \in Q$ is chosen and one only considers runs of $\mathscr{A}$ that can use q only at the root of a tree and at leaves labelled by the variable corresponding to q. This restriction of runs is repeated until all states in Q are processed. Thus, our automaton model needs to be capable of handling variables in trees. As a side product, one can consider trees that already contain variables from X_n for some n and simply shift the index of the variables in X_k by n.

For the Büchi-like result, no variables are used [52, 87]. In order to have a unified automaton model, we disallow variables by requiring $n = 0$ for our Büchi-like result.

We note that, for our purposes, variables are merely a technical tool and do not count towards qualitative properties of trees and forests (like the size, height, etc.).

5.2 Preliminaries

M-Monoids A *unit-less semiring* is an algebra $(S, \oplus, \odot, 0)$ satisfying the same properties as a semiring, except there need not be a unit element with respect to $\odot$.

Let $(\mathbb{M}, \oplus, 0)$ be a commutative monoid and $\omega \in \mathrm{Ops}^k(\mathbb{M})$ for some $k \geq 0$. We say that ω is *distributive* (*with respect to* $(\mathbb{M}, \oplus, 0)$) if for every $i \in [k]$, $m, m' \in \mathbb{M}$, $\mathbf{m} \in \mathbb{M}^{i-1}$, and $\mathbf{m}' \in \mathbb{M}^{k-i}$ it holds that

$$\omega(\mathbf{m}, m \oplus m', \mathbf{m}') = \omega(\mathbf{m}, m, \mathbf{m}') \oplus \omega(\mathbf{m}, m', \mathbf{m}') \text{ and} \tag{5.1}$$

$$\omega(\mathbf{m}, 0, \mathbf{m}') = 0. \tag{5.2}$$

We call (5.1) the *distributivity law* and (5.2) the *annihilation law*. If Equation (5.2) is satisfied (but not necessarily Equation (5.1)), then we say that ω is *absorptive*.

A *multioperator monoid* (or short: *M-monoid*) is a tuple $(\mathbb{M}, \oplus, 0, \Omega_{\mathbb{M}})$, where $(\mathbb{M}, \oplus, 0)$ is a commutative monoid (also called *underlying monoid*) and $\Omega_{\mathbb{M}} \subseteq \mathrm{Ops}(\mathbb{M})$ such that $\mathrm{id}_{\mathbb{M}} \in \Omega_{\mathbb{M}}$ and $0^{(k)} \in \Omega_{\mathbb{M}}^{(k)}$ for every $k \geq 0$, where $0^{(k)} \in \mathrm{Ops}^k(\mathbb{M})$ is the constant map to 0.

An M-monoid is called

- *distributive* if every $\omega \in \Omega_{\mathbb{M}}$ is distributive with respect to $(\mathbb{M}, \oplus, 0)$ and
- *absorptive* if every $\omega \in \Omega_{\mathbb{M}}$ is absorptive with respect to $(\mathbb{M}, \oplus, 0)$.

Whenever the context is clear, we refer to an M-monoid $(\mathbb{M}, \oplus, 0, \Omega_{\mathbb{M}})$ by the set $\mathbb{M}$.

It is an easy fact that to every semiring S one can naturally associate an M-monoid, denoted $\mathbb{M}(S)$. The operations of $\mathbb{M}(S)$ are simply finite multiplications in S. Formally, for every $k \geq 0$ we define the k-ary operation $\Pi_k \colon S^k \to S$ on S by $\Pi_k(s_1, \ldots, s_k) = s_1 \odot \cdots \odot s_k$ for every $s_1, \ldots, s_k \in S$. We note that $\Pi_1 = \mathrm{id}_{\mathbb{M}}$ and moreover, since S is a semiring, Π_k is distributive with respect to $(S, \oplus, 0)$. Now, the *M-monoid induced by* S is $\mathbb{M}(S) = (S, \oplus, 0, \Omega)$, where $\Omega = \{\Pi_k, 0^{(k)} \mid k \geq 0\}$. For an alternative M-monoid which can be naturally associated to S, confer [51, Lemma 8.6.].

Given an M-monoid $(\mathbb{M}, \oplus, 0, \Omega_{\mathbb{M}})$, we say that $\mathbb{M}$ *contains a semiring* if there exists a unit-less semiring with ground set $\mathbb{M}$ such that $\Omega_{\mathbb{M}} \supseteq (\Pi_k \mid k \geq 0)$.

Throughout the rest of Chapter 5, if $\mathbb{M}$ is left unspecified, then it stands for an arbitrary M-monoid.

A $(\Sigma, \mathbb{M})$-*weighted tree language* is a map $\varphi \colon \mathrm{T}_\Sigma \to \mathbb{M}$. We make the same conventions as in the semiring-weighted case (see Chapter 2.2).

Let $n \in \mathbb{N}$. A *tree valuation over Σ, $\mathbb{M}$, and n* is a map $\varphi \colon \mathrm{T}_\Sigma(X_n) \to \mathrm{Ops}(\mathbb{M})$. If for every $\xi \in \mathrm{T}_\Sigma(X_n)$ the arity of $\varphi(\xi)$ equals the $|\xi|_{X_n}$, then we call φ *uniform*. We denote the set of all uniform tree valuations over $\Sigma, \mathbb{M}$, and n by $\mathrm{Uvals}(\Sigma, \mathbb{M}, n)$. We note that $\mathrm{Uvals}(\Sigma, \mathbb{M}, 0) \cong \mathbb{M}^{\mathrm{T}_\Sigma}$ (as sets).

For every $\xi \in \mathrm{T}_\Sigma(X_n)$ and $\omega \in \mathrm{Ops}^{|\xi|_{X_n}}(\mathbb{M})$ we define the uniform tree valuation $\omega.\xi \colon \mathrm{T}_\Sigma \to \mathrm{Ops}(\mathbb{M})$ by

$$\omega.\xi(\zeta) = \begin{cases} \omega & \text{if } \zeta = \xi \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

for every $\zeta \in \mathrm{T}_\Sigma$.

Let $k \in \mathbb{N}$ and $\omega_1, \omega_2 \in \mathrm{Ops}^k(\mathbb{M})$. We define the *sum of* ω_1 *and* ω_2 as the k-ary operation $\omega_1 \oplus \omega_2$ on $\mathbb{M}$ given for every $\mathbf{m} \in \mathbb{M}^k$ by

$$(\omega_1 \oplus \omega_2)(\mathbf{m}) = \omega_1(\mathbf{m}) \oplus \omega_2(\mathbf{m}).$$

Now let $\omega \in \mathrm{Ops}^k(\mathbb{M})$, $l_1, \dots, l_k \in \mathbb{N}$, and $\omega_1 \in \mathrm{Ops}^{l_1}(\mathbb{M}), \dots, \omega_k \in \mathrm{Ops}^{l_k}(\mathbb{M})$. We define the *composition of* ω *with* $(\omega_1, \dots, \omega_k)$ as the $\big(\sum_{j=1}^k l_j\big)$-ary operation $\omega(\omega_1, \dots, \omega_k)$ given for every $\mathbf{m}_1 \in \mathbb{M}^{l_1}, \dots, \mathbf{m}_k \in \mathbb{M}^{l_k}$ by

$$\omega(\omega_1, \dots, \omega_k)(\mathbf{m}_1, \dots, \mathbf{m}_k) = \omega(\omega_1(\mathbf{m}_1), \dots, \omega_k(\mathbf{m}_k)).$$

This definition naturally extends to the case where $\omega \in \big(\mathrm{Ops}^k(\mathbb{M})\big)^\ell$ for some $\ell \geq 1$, that is, ω is a tuple of operations. More precisely, given $\omega = (\omega^1, \dots, \omega^\ell)$, we define

$$\omega(\omega_1, \dots, \omega_k) = (\omega^1(\omega_1, \dots, \omega_k), \dots, \omega^\ell(\omega_1, \dots, \omega_k)).$$

Example 5.1. Consider the set $\mathbb{M} = \mathcal{P}(\mathbb{N}^*)$ and the set

$$\Omega = \{\mathrm{id}_\mathbb{M}\} \cup \{\omega_s, \emptyset^{(s)} \mid s \geq 0\}$$

of operations on $\mathbb{M}$, where for every $s \geq 0$ and $P_1, \dots, P_s \in \mathbb{M}$ we define

$$\omega_s(P_1, \dots, P_s) = \begin{cases} \emptyset & \text{if } P_i = \emptyset \text{ for some } i \in [s] \\ \{\varepsilon\} \cup \bigcup_{i \in [s]} i \cdot P_i & \text{otherwise.} \end{cases}$$

The algebra $(\mathbb{M}, \cup, \emptyset, \Omega)$ is certainly an M-monoid. Note that we have forced the annihilation law in the definition of the ω_s. Moreover, one can check that the distributivity law holds for the ω_s and hence $\mathbb{M}$ is a distributive M-monoid. ◀

Logics In preparation for our Büchi Theorem, we briefly recall unweighted MSO-logic for trees in this chapter. Most of the definitions are taken from [52, pages 246–247].

In our MSO-logic, we use first-order variables (denoted by lowercase symbols, like $x, y, z, \dots$) and second-order variables (denoted by uppercase symbols, like $X, Y, Z, \dots$). We will also use the concept of *extended Backus-Naur forms* [1] (short: *EBNF*) in order to compactly define logic formalisms.

We define the set $\mathrm{MSO}(\Sigma)$ of *MSO-logic formulas* over Σ by the following EBNF with nonterminal ϕ.

$$\phi = \mathrm{label}_\sigma(x) \mid \mathrm{edge}_i(x,y) \mid x \in X \mid \neg\phi \mid \phi \vee \phi \mid \exists x.\phi \mid \exists X.\phi,$$

where $i \in [\mathrm{maxrk}(\Sigma)], \sigma \in \Sigma$, x and y are first-order variables, and X is a second-order variable. The *set of free variables* of a formula $\phi \in \mathrm{MSO}(\Sigma)$ is defined as usual and denoted by $\mathrm{Free}(\phi)$ (cf. [105, Chapter II.2]).

Let $\mathcal{V}$ be a finite set of first-order and second-order variables. We define the ranked alphabet $\Sigma_\mathcal{V}$ by $\Sigma_\mathcal{V}^{(s)} = \Sigma^{(s)} \times \mathcal{P}(\mathcal{V})$ for every $s \geq 0$. We note that $\Sigma_\emptyset \cong \Sigma$ as sets. A tree ξ in $\mathrm{T}_{\Sigma_\mathcal{V}}$ is called *valid* if every first-order variable $x \in \mathcal{V}$ occurs at exactly one position in ξ. The set of valid trees in $\mathrm{T}_{\Sigma_\mathcal{V}}$ is denoted $\mathrm{T}^{\mathrm{v}}_{\Sigma_\mathcal{V}}$. For every $\xi \in \mathrm{T}^{\mathrm{v}}_{\Sigma_\mathcal{V}}$ and $\phi \in \mathrm{MSO}(\Sigma)$, we define the relation "ξ *satisfies* ϕ" as usual (cf. [105, Chapter II.2]) and denote it by $\xi \models \phi$. Moreover, we define the set $\mathcal{L}_\mathcal{V}(\phi) = \{\xi \in \mathrm{T}^{\mathrm{v}}_{\Sigma_\mathcal{V}} \mid \xi \models \phi\}$ and abbreviate $\mathcal{L}(\phi) = \mathcal{L}_{\mathrm{Free}(\phi)}(\phi)$.

In order to be able to manipulate valid trees $\xi \in \mathrm{T}^{\mathrm{v}}_{\Sigma_\mathcal{V}}$, we introduce the following notations. For every first-order variable x and position $w \in \mathrm{pos}(\xi)$, we denote by $\xi[x \mapsto w]$ the valid tree obtained from ξ by moving the unique occurrence of x to the position w. Analogously for every second-order variable X and set of positions $W \subseteq \mathrm{pos}(\xi)$, we denote by $\xi[X \mapsto W]$ the valid tree obtained from ξ by assigning X exactly to the positions in W and removing it from the remaining positions.

For every ranked alphabet Δ, we define a *Δ-family of operations in* $\mathbb{M}$ as a family $\omega = (\omega_\sigma \mid \sigma \in \Delta)$, where $\omega_\sigma \in \Omega_\mathbb{M}^{(\mathrm{rk}(\sigma))}$ for every $\sigma \in \Delta$. Given a Δ-family ω of operations in $\mathbb{M}$, we define the map $\mathrm{h}_\omega \colon \mathrm{T}_\Delta \to \mathbb{M}$ inductively for every $\xi \in \mathrm{T}_\Delta$ as follows. We assume that $\xi = \sigma(\xi_1, \ldots, \xi_s)$ and let

$$\mathrm{h}_\omega(\xi) = \omega_\sigma(\mathrm{h}_\omega(\xi_1), \ldots, \mathrm{h}_\omega(\xi_s)).$$

The experienced reader might observe that h_ω is the unique homomorphism from the initial Δ-algebra T_Δ to the Δ-algebra $(\mathbb{M}, \omega)$.

Another technical tool we will use is the extension of index sets of families of operations. Let $\mathcal{U}$ and $\mathcal{V}$ be finite sets of variables satisfying $\mathcal{U} \subseteq \mathcal{V}$ and let ω be a $\Sigma_\mathcal{U}$-family of operations in $\mathbb{M}$. We define $\omega[\mathcal{U} \rightsquigarrow \mathcal{V}]$ as the $\Sigma_\mathcal{V}$-family of operations in $\mathbb{M}$ defined for every $\sigma \in \Sigma$ and $V \subseteq \mathcal{V}$ by $\omega[\mathcal{U} \rightsquigarrow \mathcal{V}]_{(\sigma,V)} = \omega_{(\sigma,\mathcal{U}\cap V)}$.

Weighted Tree Automata We now recall the concept of weighted tree automata over M-monoids [51, Definition 3.5.], [52, Section 2.6]. Note that the cited definitions differ in two regards and our definition will unify these differences. This allows us to provide both, a Kleene-like result and a Büchi-like result for the same automaton model. A general survey on weighted tree automata has been done in [36, Chapter 9].

On the one hand, [51] makes use of a finite set Z, which represents variables. The semantics of an automaton is then a uniform tree valuation over $\Sigma, \#Z$, and $\mathbb{M}$. We replace this use of a finite set Z by just the number $\#Z$. This adaptation is purely syntactical and does not affect any results. However, it is consistent with the notation in [26]. In [52], variables do not occur in the automaton model, which amounts to the special case where $\#Z = 0$.

On the other hand, the semantics of weighted tree automata differ in [51] and [52]. Given an automaton $\mathscr{A}$ with state set Q, [51] defines the semantics of $\mathscr{A}$ pointwise via

$$[\![\mathscr{A}]\!](\xi) = \bigoplus_{q \in Q} \mathrm{final}_q\Big(\bigoplus_{\substack{\text{run } \rho \text{ s.th.} \\ \rho(\varepsilon)=q}} \mathrm{wt}_{\mathscr{A}}(\xi, \rho)\Big),$$

whereas [52] defines the semantics of $\mathscr{A}$ pointwise via

$$[\![\mathscr{A}]\!](\xi) = \bigoplus_{\rho \text{ run}} \mathrm{final}_{\rho(\varepsilon)}(\mathrm{wt}_{\mathscr{A}}(\xi, \rho)).$$

These semantics of $\mathscr{A}$ are not equal in general but they coincide if the M-monoid $\mathbb{M}$ is distributive [51, Observation 3.9.]. Fortunately, the distributivity of $\mathbb{M}$ is assumed in [51] for the proof of the Kleene result. Therefore, we can safely use the semantics of weighted tree automata introduced in [52] without diverging from [51].

We arrive at the following unified weighted tree automata model over M-monoids.

Let $n \in \mathbb{N}$. An *M-weighted tree automaton over Σ, $\mathbb{M}$, and X_n* (short: $(\Sigma, \mathbb{M}, n)$-MWTA) is a tuple $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$, where

- Q is a finite and non-empty set of *states*,
- $\mathrm{init}\colon X_n \times Q \to \Omega_{\mathbb{M}}^{(1)}$ is the *variable assignment*,
- $T = (T_\sigma\colon Q^s \times Q \to \Omega_{\mathbb{M}}^{(s)} \mid s \geq 0, \sigma \in \Sigma^{(s)})$ is a family of *transition weight* maps,
- $\mathrm{final}\colon Q \to \Omega_{\mathbb{M}}^{(1)}$ is the *final weight* map.

For every $i \in [n]$, we abbreviate the map $\mathrm{init}(x_i, \_)\colon Q \to \Omega_{\mathbb{M}}^{(1)}$ by init_i. Moreover, we abbreviate $\mathrm{final}(q)$ by final_q for every $q \in Q$. If Σ and $\mathbb{M}$ are clear from the context, then we write n-MWTA rather than $(\Sigma, \mathbb{M}, n)$-MWTA.

Let $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$ be an n-MWTA. Moreover, let $\xi \in \mathrm{T}_\Sigma(X_n)$ and $q \in Q$. The set of all *runs of* $\mathscr{A}$ *on* ξ *ending in* q, denoted $\mathrm{Runs}_{\mathscr{A}}(\xi, q)$, is the set

$$\{\rho \in \mathrm{T}_{\Sigma\times Q}(X_n \times Q) \mid \mathrm{proj}_2(\rho(\varepsilon)) = q, \mathrm{pos}(\rho) = \mathrm{pos}(\xi), \text{ and } \mathrm{proj}_1(\rho(w)) = \xi(w) \text{ for every } w \in \mathrm{pos}(\xi)\}.$$

We denote the sets $\mathrm{Runs}_{\mathscr{A}}(\xi) = \bigcup_{q\in Q} \mathrm{Runs}_{\mathscr{A}}(\xi, q)$ and $\mathrm{Runs}_{\mathscr{A}} = \bigcup_{\xi\in \mathrm{T}_\Sigma} \mathrm{Runs}_{\mathscr{A}}(\xi)$. We note that $\mathrm{Runs}_{\mathscr{A}} = \mathrm{T}_{\Sigma\times Q}(X_n \times Q)$.

Furthermore, we define the map $\mathrm{wt}_{\mathscr{A}}\colon \mathrm{Runs}_{\mathscr{A}} \to \mathrm{Ops}(\mathbb{M})$ by induction as follows. Let $\rho \in \mathrm{Runs}_{\mathscr{A}}$. If $\rho = (x_i, q)$ for some $i \in [n]$ and $q \in Q$, then we define

$$\mathrm{wt}_{\mathscr{A}}(\rho) = \mathrm{init}_i(q).$$

Otherwise, we assume that $\rho = (\sigma, q)(\rho_1, \ldots, \rho_s)$ and define

$$\mathrm{wt}_{\mathscr{A}}(\rho) = T_\sigma(\mathrm{proj}_2(\rho_1(\varepsilon)), \ldots, \mathrm{proj}_2(\rho_k(\varepsilon)), q)\big(\mathrm{wt}_{\mathscr{A}}(\rho_1), \ldots, \mathrm{wt}_{\mathscr{A}}(\rho_s)\big),$$

which is a composition of operations on $\mathbb{M}$ as introduced in Chapter 5.2.

The *uniform tree valuation recognised by* $\mathscr{A}$ is the map $[\![\mathscr{A}]\!] \in \mathrm{Uvals}(\Sigma, \mathbb{M}, n)$ given for every $\xi \in \mathrm{T}_\Sigma(X_n)$ by

$$[\![\mathscr{A}]\!](\xi) = \bigoplus_{\rho\in\mathrm{Runs}_{\mathscr{A}}(\xi)} \mathrm{final}_{\mathrm{proj}_2(\rho(\varepsilon))}(\mathrm{wt}_{\mathscr{A}}(\rho)).$$

We say that $\mathscr{A}$ is an MWTA *with final states* if $\mathrm{im}(\mathrm{final}) \subseteq \{\mathrm{id}_{\mathbb{M}}, 0\}$.

Let $\varphi \in \mathrm{Uvals}(\Sigma, \mathbb{M}, n)$. We call φ *M-recognisable* if there exists an n-MWTA $\mathscr{A}$, such that $[\![\mathscr{A}]\!] = \varphi$. The class of all recognisable uniform tree valuations over Σ, $\mathbb{M}$, and n is denoted $\mathrm{Rec}(\Sigma, \mathbb{M}, n)$. We call φ *M-recognisable with final states* if there exists an n-MWTA with final states, denoted $\mathscr{A}$, such that $[\![\mathscr{A}]\!] = \varphi$. The class of all uniform tree valuations over Σ, $\mathbb{M}$, and n that are recognisable with final states is denoted $\mathrm{Rec}_{\mathrm{f}}(\Sigma, \mathbb{M}, n)$. We abbreviate $\mathrm{Rec}_{\mathrm{f}}(\Sigma, \mathbb{M}) = \mathrm{Rec}_{\mathrm{f}}(\Sigma, \mathbb{M}, 0)$.

Example 5.2. Recall the M-monoid $\mathbb{M} = (\mathcal{P}(\mathbb{N}^*), \cup, \emptyset, \Omega)$ from Example 5.1 and consider the ranked alphabet $\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \beta^{(0)}, \alpha^{(0)}\}$. We define an 0-MWTA over Σ and $\mathbb{M}$ that calculates the set of positions of an input tree.

We define $\mathscr{A} = (Q, \emptyset, T, \mathrm{final})$, where $Q = \{q\}$, $\mathrm{final} = (q \mapsto \mathrm{id}_{\mathbb{M}})$, and for every $s \geq 0$ and $\tau \in \Sigma^{(s)}$ we have $T_\tau(q, \ldots, q, q) = \omega_s$. We note that $\mathscr{A}$ is a 0-MWTA with final states. For every tree $\xi \in \mathrm{T}_\Sigma$ there exists a unique run ρ of $\mathscr{A}$ on ξ, namely the one labeling every position with the state q. Therefore $[\![\mathscr{A}]\!](\xi) = \mathrm{final}_q(\mathrm{wt}_{\mathscr{A}}(\rho)) = \mathrm{wt}_{\mathscr{A}}(\rho)$. By induction on ξ, we easily obtain $\mathrm{wt}_{\mathscr{A}}(\rho) = \mathrm{pos}(\xi)$ and hence $[\![\mathscr{A}]\!](\xi) = \mathrm{pos}(\xi)$. We note that in this case, $\mathrm{pos}(\xi)$ is considered as a 0-ary operation on $\mathbb{M}$.

We can also construct a finite WTA $\mathscr{B}$ over Σ and the semiring $(\mathcal{P}(\mathbb{N}^*), \cup, \cdot, \emptyset, \{\varepsilon\})$ such that $[\![\mathscr{B}]\!] = [\![\mathscr{A}]\!]$. Namely, $\mathscr{B}$ has two states, an "active" state x and a "passive" state o. Moreover, the transition maps are chosen such that the non-vanishing runs of $\mathscr{B}$ correspond bijectively to the positions of ξ and if a run ρ corresponds to $w \in \mathrm{pos}(\xi)$, then $\mathrm{wt}_{\mathscr{B}}(\xi, \rho) = \{w\}$. However, the automaton $\mathscr{B}$ has less desirable properties than $\mathscr{A}$. For example, $\mathscr{A}$ is deterministic, whereas the number of valid runs of $\mathscr{B}$ on a tree ξ equals $\mathrm{size}(\xi)$. This demonstrates the advantage of introducing the more technical M-monoids instead of only considering semirings. ◀

We note that, for every $n \in \mathbb{N}$, the M-weighted tree automaton model presented in this chapter is equivalent to the weighted tree automaton model from [51, Definition 3.5.], given that $\mathbb{M}$ is distributive. In this case, for every set Z such that $\#Z = n$, our class $\mathrm{Rec}(\Sigma, \mathbb{M}, n)$ equals the class $\mathrm{Rec}(\Sigma, Z, \mathbb{M})$ defined in [51, Definition 3.8.].

In the case that $n = 0$, our automaton model degenerates as follows. The variable assignment becomes irrelevant, uniform tree valuations are simply weighted languages ($\mathrm{Uvals}(\Sigma, \mathbb{M}, 0) = \mathbb{M}^{\mathrm{T}_\Sigma}$), and the weight of a run ρ, $\mathrm{wt}_{\mathscr{A}}(\rho)$, is an element of $\mathbb{M}$. Therefore, our 0-MWTA with final states coincide with the weighted tree automaton model from [52, Section 2.6]. In particular, our class $\mathrm{Rec}_{\mathrm{f}}(\Sigma, \mathbb{M})$ equals the class $\mathrm{Rec}(\Sigma, \mathbb{M})$ defined in [52, Section 2.6].

Moreover, the semiring-weighted tree automaton model from Chapter 2.3 is a special case of our M-weighted tree automaton model. More precisely, given a semiring S, the two classes $\mathrm{Rec}(\Sigma, S)$ and $\mathrm{Rec}(\Sigma, \mathbb{M}(S), 0)$ coincide up to identification of S^{T_Σ} and $\mathrm{Uvals}(\Sigma, \mathbb{M}(S), 0)$.

Using our unified M-weighted tree automaton model, we can state both Kleene's Theorem [51] and Büchi's Theorem [52] in a single setting.

5.3 Weighted Forest Automata

In this chapter, we formally introduce forests and weighted forest automata. As we will see in Theorems 5.21 and 5.22, this formalism recognises a class of languages which is very close to the class of languages recognised by weighted tree automata. This connection is called "rectangularity" and will be a dominant force in proving the Kleene and Büchi results for weighted forest automata.

5.3.1 Forests

Definition 5.3. Let $b, n \in \mathbb{N}$. We define the set

$$\mathrm{F}(\Sigma)_n^b = \{n\} \times \mathrm{T}_\Sigma(X_n)^b$$

of (b, n)-*forests over* Σ. The set of all *forests over* Σ is then defined as

$$\mathrm{F}(\Sigma) = \bigcup_{b', n' \in \mathbb{N}} \mathrm{F}(\Sigma)_{n'}^{b'}.$$

We also abbreviate the set $\mathrm{F}(\Sigma)^b = \bigcup_{n' \in \mathbb{N}} \mathrm{F}(\Sigma)_{n'}^b$.

Moreover, we will denote forests using angle brackets, to aid readability of examples. In our notation, the lowercase greek letter ξ will be used to denote trees and the uppercase greek letter Ξ will be used to denote forests[1]. ◀

Throughout the rest of Chapter 5, given parameters $b, n \in \mathbb{N}$, we write "Let $\Xi = \langle n, \xi_1, \ldots, \xi_b \rangle \in \mathrm{F}(\Sigma)_n^b$" as an abbreviation for "Let $\Xi \in \mathrm{F}(\Sigma)_n^b$ and $\xi_1, \ldots, \xi_b \in \mathrm{T}_\Sigma(X_n)$ such that $\Xi = \langle n, \xi_1, \ldots, \xi_b \rangle$".

Remark 5.4. We note that the sets in $(\mathrm{F}(\Sigma)_n^b \mid b, n \in \mathbb{N})$ are pairwise disjoint. This is an immediate consequence of the definition, as forests with different numbers of variables have different first components and forests with different numbers of trees have different tuple sizes.

[1] Because, convincingly enough, forests are the "big version" of trees.

Moreover, we note that $\mathrm{F}(\Sigma)^1_n \cong \mathrm{T}_\Sigma(X_n)$ as sets. This fact is clearly witnessed by the bijection $f\colon \mathrm{F}(\Sigma)^1_n \to \mathrm{T}_\Sigma(X_n)$ given by $f(\langle n, \xi\rangle) = \xi$. ◀

Definition 5.5. Let $b, n \in \mathbb{N}$ and $\Xi = \langle n, \xi_1, \ldots, \xi_b\rangle \in \mathrm{F}(\Sigma)^b_n$. For every $i \in [b]$ we define $\pi^b_i(\Xi) = \langle n, \xi_i\rangle$. Note that this induces maps $\pi^b_1, \ldots, \pi^b_b\colon \mathrm{F}(\Sigma)^b_n \to \mathrm{F}(\Sigma)^1_n$.

Furthermore, we define the maps

$$\begin{aligned}
\mathrm{pos}\colon \mathrm{F}(\Sigma)^b_n &\longrightarrow \mathcal{P}(\mathbb{N} \times \mathbb{N}^*),\\
\mathrm{size}\colon \mathrm{F}(\Sigma)^b_n &\longrightarrow \mathbb{N},\\
\mathrm{height}\colon \mathrm{F}(\Sigma)^b_n &\longrightarrow \mathbb{N},
\end{aligned}$$

given for every $\Xi = \langle n, \xi_1, \ldots, \xi_b\rangle \in \mathrm{F}(\Sigma)^b_n$ by

$$\begin{aligned}
\mathrm{pos}(\Xi) &= \bigcup_{i=1}^{b} \{i\} \times \mathrm{pos}(\xi_i),\\
\mathrm{size}(\Xi) &= \sum_{i\in[b]} \mathrm{size}(\xi_i),\\
\mathrm{height}(\Xi) &= \max\{\mathrm{height}(\xi_i) \mid i \in [b]\}.
\end{aligned}$$

We use the same abbreviations as in the tree case, like $\mathrm{pos}_\Delta(\Xi)$ and $|\Xi|_\Delta$ for a subset $\Delta \subseteq \Sigma \cup X_n$ and $\mathrm{leaf}(\Xi)$ for the set $\mathrm{pos}_{\Sigma^{(0)}}(\Xi)$. Moreover, for every $(i, w) \in \mathrm{pos}(\Xi)$ we define the *label of* Ξ *at* (i, w), denoted by $\Xi((i, w))$, as $\xi_i(w)$. ◀

Definition 5.6. Let $b, n \in \mathbb{N}$. A $(\Sigma, \mathbb{M})$-*weighted* (b, n)-*forest language* is a map $\varphi\colon \mathrm{F}(\Sigma)^b_n \to \mathbb{M}$. We drop the parameter $(\Sigma, \mathbb{M})$ whenever it is clear from the context.

A (b, n)-*forest valuation* over Σ and $\mathbb{M}$ is a map $\varphi\colon \mathrm{F}(\Sigma)^b_n \to \mathrm{Ops}(\mathbb{M})$. If for every $\Xi \in \mathrm{F}(\Sigma)^b_n$ the arity of $\varphi(\Xi)$ equals $|\Xi|_{X_n}$, we call φ *uniform*. The set of all uniform (b, n)-forest valuations over Σ and $\mathbb{M}$ is denoted $\mathrm{Uvals}(\Sigma, \mathbb{M}, b, n)$. We note that $\mathrm{Uvals}(\Sigma, \mathbb{M}, b, 0) \cong \mathbb{M}^{\mathrm{F}(\Sigma)^b_0}$ and $\mathrm{Uvals}(\Sigma, \mathbb{M}, 1, n) \cong \mathrm{Uvals}(\Sigma, \mathbb{M}, n)$. ◀

5.3.2 Weighted Forest Automata

Subsequently, we will consider weighted forest languages and forest valuations generated by weighted forest automata. In order to do this, we fix the values b and n for the forests we are interested in. We note, however, that our automaton model will be capable of

processing forests with arbitrary ranks. In fact, the choice of b and n only restricts the language of an automaton to be a uniform (b, n)-forest valuation.

Throughout the rest of Chapter 5, we let b and n be arbitrary elements of $\mathbb{N}$. Moreover, we assume that $\mathbb{M}$ contains a semiring[1].

Definition 5.7. A *weighted (b, n)-forest automaton* over Σ and $\mathbb{M}$ (short: (b, n)-WFA) is a tuple $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$, where

- Q is a finite and non-empty set of *states*,
- $\mathrm{init}\colon X_n \times Q \to \Omega_{\mathbb{M}}^{(1)}$ is the *variable assignment*,
- $T = (T_\sigma\colon Q^s \times Q \to \Omega_{\mathbb{M}}^{(s)} \mid s \geq 0, \sigma \in \Sigma^{(s)})$ is a family of *transition weight maps*,
- $\mathrm{final}\colon [b] \times Q \to \Omega_{\mathbb{M}}^{(1)}$ is the *final weight map*.

For every $i \in [n]$ and $j \in [b]$, we abbreviate the map $\mathrm{init}(x_i, \_)\colon Q \to \Omega_{\mathbb{M}}^{(1)}$ by init_i and the map $\mathrm{final}(j, \_)\colon Q \to \Omega_{\mathbb{M}}^{(1)}$ by final_j. Furthermore, we fix the map $\mathbf{F}\colon Q^b \to (\Omega_{\mathbb{M}}^{(b)})^b$ such that for every $(q_1, \ldots, q_b) \in Q^b$ we have

$$\mathbf{F}(q_1, \ldots, q_b) = (\mathrm{final}_1(q_1) \circ \mathrm{proj}_1, \ldots, \mathrm{final}_b(q_b) \circ \mathrm{proj}_b). \qquad \blacktriangleleft$$

In the upcoming Definition 5.8, we define a family $(E_{k,k'}^{\mathscr{A}} \mid k, k' \in \mathbb{N})$ of maps of the form $E_{k,k'}^{\mathscr{A}}\colon \mathrm{F}(\Sigma)_k^{k'} \times Q^{k'} \to \mathrm{Ops}(M)^{k'}$. Such a map $E_{k,k'}^{\mathscr{A}}$ takes as input a (k', k)-forest Ξ and k' states $q_1, \ldots, q_{k'} \in Q$ (one for each root position of Ξ) and evaluates $\mathscr{A}$ on Ξ, given the states $q_1, \ldots, q_{k'}$. This evaluation is done inductively on the structure of Ξ. Variables are evaluated as 1-ary operations given by init, tuples are evaluated component-wise, and Σ-positions w in trees are evaluated in an initial algebra fashion, similar to [94]. In Definition 5.9, we use these evaluation maps in order to define the *uniform (b, n)-forest valuation i-recognised by $\mathscr{A}$* – where "i" stands for "initial".

[1]Note that every M-monoid contains a semiring by default: multiplication is simply the constant map to 0. However, our statement here is to encourage the use of arbitrary multiplication operations. We compare our approach to an approach that does not require $\mathbb{M}$ to contain a semiring in Remark 5.27.

Definition 5.8. Let $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$ be a (b,n)-WFA over Σ and $\mathbb{M}$. We lift the transition weight maps to forests as follows. Let $k \in \mathbb{N}$. We define

$$E^{\mathscr{A}}_{k,0} \colon \mathrm{F}(\Sigma)^0_k \times Q^0 \to \mathrm{Ops}(\mathbb{M})^0$$

as the constant map to the empty tuple $()$. Moreover we define the family of maps

$$\Big(E^{\mathscr{A}}_{k,k'} \colon \mathrm{F}(\Sigma)^{k'}_k \times Q^{k'} \to \mathrm{Ops}(\mathbb{M})^{k'} \mid k, k' \in \mathbb{N}\Big)$$

by simultaneous induction.

The case $k' = 1$ is given for any $q \in Q$ and $\langle k, \xi\rangle \in \mathrm{F}(\Sigma)^1_k$ inductively on the structure of ξ. If $\xi = x_i$ for some $i \in [k]$, then we define

$$E^{\mathscr{A}}_{k,1}(\langle k, \xi\rangle, q) = \mathrm{init}_i(q).$$

Otherwise, we assume that $\xi = \sigma(\xi_1, \ldots, \xi_s)$ and define

$$E^{\mathscr{A}}_{k,1}(\langle k, \xi\rangle, q) = \bigoplus_{p_1,\ldots,p_s \in Q} T_\sigma(p_1, \ldots, p_s, q)\Big(E^{\mathscr{A}}_{k,s}(\langle k, \xi_1, \ldots, \xi_s\rangle, (p_1, \ldots, p_s))\Big).$$

The case $k' > 1$ is given for every $q_1, \ldots, q_{k'} \in Q$ and $\Xi = \langle k, \xi_1, \ldots, \xi_{k'}\rangle \in T(\Sigma)^{k'}_k$ by

$$E^{\mathscr{A}}_{k,k'}(\Xi, (q_1, \ldots, q_{k'})) = \Big(E^{\mathscr{A}}_{k,1}(\langle k, \xi_1\rangle, q_1), \ldots, E^{\mathscr{A}}_{k,1}(\langle k, \xi_{k'}\rangle, q_{k'})\Big).$$

We will subsequently omit the brackets from the second parameter of $E^{\mathscr{A}}_{k,k'}$. ◀

Definition 5.9. Let $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$ be a (b,n)-WFA over Σ and $\mathbb{M}$. The *uniform (b,n)-forest valuation i-recognised by $\mathscr{A}$* is the map $[\![\mathscr{A}]\!]_{\mathrm{i}} \in \mathrm{Uvals}(\Sigma, \mathbb{M}, b, n)$ given for every $\Xi \in \mathrm{F}(\Sigma)^b_n$ by

$$[\![\mathscr{A}]\!]_{\mathrm{i}}(\Xi) = \bigoplus_{q \in Q^b} \Pi_b\Big(\mathbf{F}_q\big(E^{\mathscr{A}}_{n,b}(\Xi, q)\big)\Big).$$

Let $\varphi \in \mathrm{Uvals}(\Sigma, \mathbb{M}, b, n)$. We call φ *i-recognisable* if there exists a (b,n)-WFA $\mathscr{A}$ over Σ and $\mathbb{M}$, such that $[\![\mathscr{A}]\!]_{\mathrm{i}} = \varphi$. The class of all i-recognisable uniform (b,n)-forest valuations over Σ and $\mathbb{M}$ is denoted $\mathrm{Rec}_{\mathrm{i}}(\Sigma, \mathbb{M}, b, n)$. ◀

In the upcoming Definition 5.10, we define runs of weighted forest automata $\mathscr{A}$ and weights of such runs. Given a forest Ξ, a run ρ of $\mathscr{A}$ on Ξ associates to every position

of Ξ a state of $\mathscr{A}$. The transition weight maps in T and the variable assignment init induce for every $w \in \mathrm{pos}(\Xi)$ a local weight of ρ at w. The overall weight of ρ is the composition of the local weights of ρ in the way that the structure of Ξ induces. This leads to the definition of the *uniform (b,n)-forest valuation r-recognised by $\mathscr{A}$* – where "r" stands for "run". This semantic of $\mathscr{A}$ is similar to the run semantics from [51, 94]

Definition 5.10. Let $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$ be a (b,n)-WFA over Σ and $\mathbb{M}$, $k, k' \in \mathbb{N}$, $\Xi \in \mathrm{F}(\Sigma)_k^{k'}$, and $q_1, \ldots, q_{k'} \in Q$. A *run of $\mathscr{A}$ on Ξ ending in $q = (q_1, \ldots, q_{k'})$* is a map $\rho\colon \mathrm{pos}(\Xi) \longrightarrow Q$ such that

$$\rho(j, \varepsilon) = q_j$$

for every $j \in [k']$.

The set of runs of $\mathscr{A}$ on Ξ ending in q is denoted by $\mathrm{Runs}_{\mathscr{A}}(\Xi, q)$. Moreover we denote $\mathrm{Runs}_{\mathscr{A}}(\Xi) = \bigcup_{q' \in Q^{k'}} \mathrm{Runs}_{\mathscr{A}}(\Xi, q')$.

Let ρ be a run of $\mathscr{A}$ on Ξ ending in q. For every $w = (i, u) \in \mathrm{pos}(\Xi)$ we define the *weight of w in Ξ under ρ* as the operation $\mathrm{wt}_{\mathscr{A}}(\Xi, \rho, w) \in \mathrm{Ops}(\mathbb{M})$ given as follows. If $\Xi(w) = x_i$ for some $i \in [k']$, then we define

$$\mathrm{wt}_{\mathscr{A}}(\Xi, \rho, w) = \mathrm{init}_i(\rho(w)),$$

and if $\Xi(w) = \sigma$ for some $s \geq 0$ and $\sigma \in \Sigma^{(s)}$, then we define

$$\mathrm{wt}_{\mathscr{A}}(\Xi, \rho, w) = T_\sigma\big(\rho(i, u1), \ldots, \rho(i, us), \rho(i, u)\big)\Big(\omega_1, \ldots, \omega_s\Big),$$

where $\omega_\ell = \mathrm{wt}_{\mathscr{A}}(\Xi, \rho, (i, u\ell))$ for every $\ell \in [s]$. Moreover we define the *weight of Ξ under ρ* as the tuple of operations $\mathrm{wt}_{\mathscr{A}}(\Xi, \rho) \in (\mathrm{Ops}(\mathbb{M}))^{k'}$ given by

$$\mathrm{wt}_{\mathscr{A}}(\Xi, \rho) = \Big(\mathrm{wt}_{\mathscr{A}}(\Xi, \rho, (1, \varepsilon)), \ldots, \mathrm{wt}_{\mathscr{A}}(\Xi, \rho, (k', \varepsilon))\Big).$$

We note that $\mathrm{wt}_{\mathscr{A}}(\Xi, \rho)$ is the empty tuple $()$ for $k' = 0$. We say that ρ is *vanishing* if $\mathrm{wt}_{\mathscr{A}}(\Xi, \rho) = (0^{(\ell_1)}, \ldots, 0^{(\ell_{k'})})$ for some $\ell_1, \ldots, \ell_{k'} \geq 0$.

We define the *uniform (b,n)-forest valuation r-recognised by $\mathscr{A}$*, as the map $[\![\mathscr{A}]\!]_{\mathrm{r}} \in \mathrm{Uvals}(\Sigma, \mathbb{M}, b, n)$, defined for every $\Xi \in \mathrm{F}(\Sigma)_n^b$ by

$$[\![\mathscr{A}]\!]_{\mathrm{r}}(\Xi) = \bigoplus_{\rho \in \mathrm{Runs}_{\mathscr{A}}(\Xi)} \Pi_b\Big(\mathbf{F}_{(\rho(1,\varepsilon), \ldots, \rho(b,\varepsilon))}\big(\mathrm{wt}_{\mathscr{A}}(\Xi, \rho)\big)\Big).$$

Let $\varphi \in \mathrm{Uvals}(\Sigma, \mathbb{M}, b, n)$. We call φ *r-recognisable* if there exists a (b,n)-WFA $\mathscr{A}$ over Σ and $\mathbb{M}$ such that $[\![\mathscr{A}]\!]_{\mathrm{r}} = \varphi$. The class of all r-recognisable uniform (b,n)-forest valuations over Σ and $\mathbb{M}$ is denoted $\mathrm{Rec}_{\mathrm{r}}(\Sigma, \mathbb{M}, b, n)$. Moreover, we call φ *recognisable with final states* if there exists a (b,n)-WFA $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$ over Σ and $\mathbb{M}$ such that $[\![\mathscr{A}]\!]_{\mathrm{r}} = \varphi$ and $\mathrm{im}(\mathrm{final}) \subseteq \{\mathrm{id}_{\mathbb{M}}, 0\}$. The class of all uniform (b,n)-forest valuations over Σ and $\mathbb{M}$ that are recognisable with final states is denoted $\mathrm{Rec}_{\mathrm{f}}(\Sigma, \mathbb{M}, b, n)$. ◀

Example 5.11. Recall the M-monoid $\mathbb{M} = (\mathcal{P}(\mathbb{N}^*), \cup, \emptyset, \Omega)$ and the ranked alphabet Σ from Example 5.2. We extend $\mathbb{M}$ as follows. For every $s \geq 0$ we define the operation $\upsilon_s \in \mathrm{Ops}^s(\mathbb{M})$ for every $P_1, \ldots, P_s \in \mathbb{M}$ by

$$\upsilon_s(P_1, \ldots, P_s) = \begin{cases} \emptyset & \text{if } P_i = \emptyset \text{ for some } i \in [s] \\ \bigcup_{i \in [s]} \{i\} \circ P_i & \text{otherwise.} \end{cases}$$

Furthermore, consider the intersection operation $\cap$ as another operation on $\mathbb{M}$ and denote the corresponding finite intersection operators by $(\Pi_s \mid s \geq 0)$. We define $\mathbb{M}' = (\mathcal{P}(\mathbb{N}^*), \cup, \emptyset, \Omega')$, where

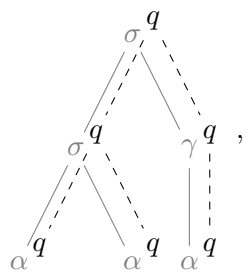
$$\Omega' = \{\upsilon_s, \omega_s, \Pi_s, \emptyset^{(s)} \mid s \geq 0\} \cup \{\mathrm{id}_{\mathbb{M}}, \{\varepsilon\}\}.$$

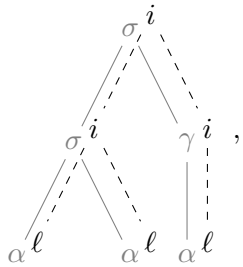
The element $\{\varepsilon\}$ of Ω' denotes a constant map. One can prove in a straightforward way that $\mathbb{M}'$ is a distributive M-monoid containing a semiring.

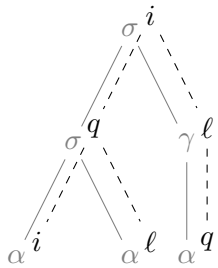
We define the $(2,0)$-WFA $\mathscr{C} = (Q, \emptyset, T, \mathrm{final})$, where $Q = \{q, \ell, i\}$, final maps every input to 0 except for $\mathrm{final}_1(q) = \mathrm{final}_2(\ell) = \mathrm{final}_2(i) = \mathrm{id}_{\mathbb{M}}$, and T is given for every $s \geq 0, \tau \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q' \in Q$ by

$$T_\tau(q_1, \ldots, q_s, q') = \begin{cases} \omega_s & \text{if } q_1 = \cdots = q_s = q' = q \\ \{\varepsilon\} & \text{if } s = 0 \text{ and } q' = \ell \\ \upsilon_s & \text{if } s > 0, q_1, \ldots, q_s \in \{\ell, i\}, \text{ and } q' = i \\ \emptyset^{(s)} & \text{otherwise.} \end{cases}$$

Consider the $(1,0)$-forest $\Xi = \langle 0, \sigma(\sigma(\alpha, \alpha), \gamma(\beta))\rangle$ and the following three runs ρ_1, ρ_2, ρ_3 of $\mathscr{C}$ on Ξ (depicted slightly above Ξ).

ρ_1: 

, ρ_2: 

, ρ_3: 

We can now determine

$$\begin{aligned}\mathrm{wt}_{\mathscr{C}}(\Xi, \rho_1) &= T_\sigma(q,q,q)(T_\sigma(q,q,q)(T_\alpha(q), T_\alpha(q)), T_\gamma(q,q)(T_\alpha(q))) \\ &= \omega_2(\omega_2(\omega_0, \omega_0), \omega_1(\omega_0)) = \{\varepsilon, 1, 11, 12, 2, 21\}.\end{aligned}$$

In a similar fashion, we obtain

$$\mathrm{wt}_{\mathscr{C}}(\Xi, \rho_2) = \{11, 12, 21\} \qquad \text{and} \qquad \mathrm{wt}_{\mathscr{C}}(\Xi, \rho_3) = \emptyset,$$

which shows that $\mathrm{wt}_{\mathscr{C}}(\Xi, \rho_1) = \mathrm{pos}(\Xi)$ and $\mathrm{wt}_{\mathscr{C}}(\Xi, \rho_2) = \mathrm{leaf}(\Xi)$, up to identification of $\mathrm{F}(\Sigma)_0^1$ and T_Σ. It follows from the definition of T that ρ_1 and ρ_2 are in fact the only non-vanishing runs of $\mathscr{C}$ on Ξ. More generally, for every tree $\Xi' \in \mathrm{F}(\Sigma)_0^1$, there exist only two non-vanishing runs of $\mathscr{C}$ on Ξ', namely the run labeling every position with q and the run labeling every leaf position with ℓ and all other positions with i. Using these observations, we will calculate the forest language (r-)recognised by $\mathscr{C}$ in Example 5.26. ◀

Theorem 5.12. Let $b = 1$ and $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$ be a $(1, n)$-WFA over Σ and $\mathbb{M}$. It holds that

$$[\![\mathscr{A}]\!]_{\mathrm{r}} = [\![\mathscr{A}]\!],$$

where the right hand side denotes the uniform tree valuation[1] recognised by $\mathscr{A}$ as an n-MWTA.

Proof. Given $\Xi = \langle n, \xi \rangle \in \mathrm{F}(\Sigma)_n^1$, it holds that $\mathrm{Runs}_{\mathscr{A}}(\Xi) \cong \mathrm{Runs}_{\mathscr{A}}(\xi)$ as sets. That is, the set of runs of $\mathscr{A}$ on Ξ as a WFA is bijective to the set of runs of $\mathscr{A}$ on ξ as an

[1] Here, we use the identification of $\mathrm{F}(\Sigma)_n^1$ and $\mathrm{T}_\Sigma(X_n)$ to make the types of the maps compatible. Moreover we use the fact that $(1, n)$-WFAs have the same components as n-MWTAs up to isomorphism and hence, $\mathscr{A}$ can be seen as a WTA.

n-MWTA. The bijection $f\colon \mathrm{Runs}_{\mathscr{A}}(\Xi) \to \mathrm{Runs}_{\mathscr{A}}(\xi)$ is given for every $\rho \in \mathrm{Runs}_{\mathscr{A}}(\Xi)$ by

$$f(\rho)(w) = (\xi(w), \rho(1, w))$$

for every $w \in \mathrm{pos}(\xi)$. A straightforward proof by structural induction on ξ shows that $\mathrm{wt}_{\mathscr{A}}(\Xi, \rho) = \mathrm{wt}_{\mathscr{A}}(f(\rho))$. Therefore,

$$\begin{aligned}
[\![\mathscr{A}]\!]_{\mathrm{r}}(\Xi) &= \bigoplus_{\rho \in \mathrm{Runs}_{\mathscr{A}}(\Xi)} \Pi_1\Big(\mathbf{F}_{(\rho(1,\varepsilon))}\big(\mathrm{wt}_{\mathscr{A}}(\Xi, \rho)\big)\Big) \\
&= \bigoplus_{\rho \in \mathrm{Runs}_{\mathscr{A}}(\Xi)} \mathrm{final}_{\rho(1,\varepsilon)}\big(\mathrm{wt}_{\mathscr{A}}(\Xi, \rho)\big)\Big) \\
&= \bigoplus_{\rho \in \mathrm{Runs}_{\mathscr{A}}(\Xi)} \mathrm{final}_{\mathrm{proj}_2(f(\rho)(\varepsilon))}\big(\mathrm{wt}_{\mathscr{A}}(f(\rho))\big)\Big) \\
&= \bigoplus_{\rho \in \mathrm{Runs}_{\mathscr{A}}(\xi)} \mathrm{final}_{\mathrm{proj}_2(\rho(\varepsilon))}(\mathrm{wt}_{\mathscr{A}}(\rho)) = [\![\mathscr{A}]\!](\xi).
\end{aligned}$$

This concludes the proof. □

Corollary 5.13. Up to identification of $\mathrm{F}(\Sigma)_n^1$ and $\mathrm{T}_{\Sigma}(X_n)$, it holds that

$$\mathrm{Rec}_{\mathrm{r}}(\Sigma, \mathbb{M}, 1, n) = \mathrm{Rec}(\Sigma, \mathbb{M}, n).$$

Remark 5.14. Let S be a semiring and let $\mathbb{M}$ be the M-monoid $(S, \oplus, 0, \Omega)$, where $\Omega = (0^{(k)}, \Pi_k^a \mid a \in S, k \geq 0)$ and $\Pi_k^a = a \cdot \Pi_k$ is the multiplication of k operands and a.

In this case, our automaton model degenerates into the automaton model introduced in [26, Definition 3.2.1.]. This can easily be verified and validates our present approach to weighted forest automata. ◀

5.3.3 Rectangularity

We now introduce rectangularity and prove that weighted forest automata only generate rectangular weighted forest languages. This is the key result of Chapter 5.3, acting as the driving force in the rest of Chapter 5.

Definition 5.15. Let $b_1, b_2 \in \mathbb{N}$ and let τ_1 and τ_2 be a (b_1, n)-forest valuation and a (b_2, n)-forest valuation over Σ and $\mathbb{M}$, respectively. We define the *horizontal concatenation* of τ_1 and τ_2 as the $(b_1 + b_2, n)$-forest valuation $\tau_1 \times \tau_2$ given for every

$\langle n, \xi_1, \ldots, \xi_{b_1}, \zeta_1, \ldots, \zeta_{b_2} \rangle \in \mathrm{F}(\Sigma)_n^{b_1+b_2}$ by

$$\tau_1 \times \tau_2(\langle n, \xi_1, \ldots, \xi_{b_1}, \zeta_1, \ldots, \zeta_{b_2} \rangle) = \Pi_2(\tau_1(\langle n, \xi_1, \ldots, \xi_{b_1} \rangle), \tau_2(\langle n, \zeta_1, \ldots, \zeta_{b_2} \rangle)).$$

The definition of horizontal concatenation naturally extends to finitely many operands by the associativity of the operation that induces $(\Pi_k \mid k \geq 0)$. ◀

Definition 5.16. A (b, n)-forest valuation τ over Σ and $\mathbb{M}$ is called *rectangular* if there exist $(1, n)$-forest valuations $\tau_1, \ldots, \tau_b$ over Σ and $\mathbb{M}$ such that

$$\tau = \tau_1 \times \cdots \times \tau_b.$$

In this case, we call the forest valuations $\tau_1, \ldots, \tau_b$ the *rectangular components of* τ. ◀

Remark 5.17. In general, the decomposition of rectangular forest valuations into rectangular components is not unique. A very simple example is the case $\mathbb{M} = \mathbb{M}(S)$ for a commutative semiring S. The rectangular forest valuations $(c\tau_1) \times \tau_2$ and $\tau_1 \times (c\tau_2)$ are equal for every $c \in S$ and forest valuations τ_1 and τ_2. In Chapter 5.4.3, we will discuss the implications of this ambiguity and why this even invalidates the results presented in [26]. ◀

Throughout the rest of Chapter 5.3.3, we let $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$ *be an arbitrary* (b, n)-WFA.

Remark 5.18. Let $\Xi \in \mathrm{F}(\Sigma)_n^b$ and $q = (q_1, \ldots, q_b) \in Q^b$. It holds that

$$\mathrm{Runs}_{\mathscr{A}}(\Xi, q) \cong \mathrm{Runs}_{\mathscr{A}}(\pi_1^b(\Xi), q_1) \times \cdots \times \mathrm{Runs}_{\mathscr{A}}(\pi_b^b(\Xi), q_b).$$

This can easily be seen by the fact that each run $\rho \in \mathrm{Runs}_{\mathscr{A}}(\Xi, q)$ can be restricted to the $(1, n)$-forest $\pi_i^b(\Xi)$ for every $i \in [b]$. Formally, given $i \in [b]$, we define the run $\rho_i \in \mathrm{Runs}_{\mathscr{A}}(\pi_i^b(\Xi), q_i)$ by $\rho_i((1, u)) = \rho((i, u))$ for every $(1, u) \in \mathrm{pos}(\pi_i^b(\Xi))$. We obtain the desired bijection by letting $\rho \mapsto (\rho_1, \ldots, \rho_b)$. ◀

Throughout the rest of Chapter 5, whenever a run ρ *on a* (b, n)*-forest is quantified, we let* $\rho_1, \ldots, \rho_b$ *be defined as in Remark 5.18.*

Lemma 5.19. Let $\Xi \in \mathrm{F}(\Sigma)_n^b$, $q \in Q^b$, and $\rho \in \mathrm{Runs}_{\mathscr{A}}(\Xi, q)$ and denote $\Xi_i = \pi_i^b(\Xi)$ for every $i \in [b]$. It holds that

$$\mathrm{wt}_{\mathscr{A}}(\Xi, \rho) = \Big(\mathrm{wt}_{\mathscr{A}}(\Xi_1, \rho_1), \ldots, \mathrm{wt}_{\mathscr{A}}(\Xi_b, \rho_b)\Big)$$

Proof. By the definition of $\mathrm{wt}_{\mathscr{A}}(\Xi, \rho)$, we have

$$\mathrm{wt}_{\mathscr{A}}(\Xi, \rho) = \Big(\mathrm{wt}_{\mathscr{A}}(\Xi, \rho, (1, \varepsilon)), \ldots, \mathrm{wt}_{\mathscr{A}}(\Xi, \rho, (b, \varepsilon))\Big).$$

Therefore, we only need to show equation $\star$ in

$$\mathrm{wt}_{\mathscr{A}}(\Xi, \rho, (i, \varepsilon)) \stackrel{\star}{=} \mathrm{wt}_{\mathscr{A}}(\Xi_i, \rho_i, (1, \varepsilon)) = \mathrm{wt}_{\mathscr{A}}(\Xi_i, \rho_i),$$

for every $i \in [b]$ in order to prove the lemma.

In fact, we will prove the slightly more general statement that

$$\mathrm{wt}_{\mathscr{A}}(\Xi, \rho, (i, u)) = \mathrm{wt}_{\mathscr{A}}(\Xi_i, \rho_i, (1, u)), \tag{5.3}$$

for every $(i, u) \in \mathrm{pos}(\Xi)$. Our proof uses a variant of structural induction on Ξ.

Let $(i, u) \in \mathrm{pos}(\Xi)$. Firstly, if $\Xi((i, u)) = x_j$ for some $j \in [n]$, then

$$\mathrm{wt}_{\mathscr{A}}(\Xi, \rho, (i, u)) = \mathrm{init}_j(\rho(i, u)) = \mathrm{init}_j(\rho_i(1, u)) = \mathrm{wt}_{\mathscr{A}}(\Xi_i, \rho_i, (1, u)).$$

This case proves Equation (5.3) for the set $\mathrm{pos}_{X_n}(\Xi)$.

Now assume that $\Xi(i, u) = \sigma$ for some $\sigma \in \Sigma^{(s)}$ and $s \geq 0$, such that Equation (5.3) holds for $(i, u\ell) \in \mathrm{pos}(\Xi)$ for every $\ell \in [s]$. We obtain

$$\mathrm{wt}_{\mathscr{A}}(\Xi, \rho, (i, u)) = T_\sigma(\rho(i, u1), \ldots, \rho(i, us), \rho(i, u))\Big(c_1, \ldots, c_s\Big),$$

where $c_\ell = \mathrm{wt}_{\mathscr{A}}(\Xi, \rho, (i, u\ell))$ for every $\ell \in [s]$. Moreover it holds that

$$\mathrm{wt}_{\mathscr{A}}(\Xi_i, \rho_i, (1, u)) = T_\sigma(\rho_i(1, u1), \ldots, \rho_i(1, us), \rho_i(1, u))\Big(c'_1, \ldots, c'_s\Big),$$

where $c'_\ell = \mathrm{wt}_{\mathscr{A}}(\Xi_i, \rho_i, (1, u\ell))$ for every $\ell \in [s]$. The induction hypothesis implies that $c_\ell = c'_\ell$ for every $\ell \in [s]$ and since

$$T_\sigma(\rho(i, u1), \ldots, \rho(i, us), \rho(i, u)) = T_\sigma(\rho_i(1, u1), \ldots, \rho_i(1, us), \rho_i(1, u))$$

by the definition of ρ_i, the proof of Equation (5.3) is complete. $\square$

Remark 5.20. For every $i \in [b]$ we define the $(1,n)$-WFA

$$\mathscr{A}_i = (Q, \mathrm{init}, T, \mathrm{final}_i).$$

Clearly, for every $\Xi \in \mathrm{F}(\Sigma)_n^1$ and every $q \in Q$ we have $E_{n,1}^{\mathscr{A}}(\Xi, q) = E_{n,1}^{\mathscr{A}_i}(\Xi, q)$ and $\mathrm{Runs}_{\mathscr{A}}(\Xi, q) = \mathrm{Runs}_{\mathscr{A}_i}(\Xi, q)$. Moreover, for every $\rho \in \mathrm{Runs}_{\mathscr{A}}(\Xi, q)$, we have $\mathrm{wt}_{\mathscr{A}}(\Xi, \rho) = \mathrm{wt}_{\mathscr{A}_i}(\Xi, \rho)$. ◀

Theorem 5.21. It holds that

$$[\![\mathscr{A}]\!]_{\mathrm{r}} = [\![\mathscr{A}_1]\!]_{\mathrm{r}} \times \cdots \times [\![\mathscr{A}_b]\!]_{\mathrm{r}}.$$

In particular, $[\![\mathscr{A}]\!]_{\mathrm{r}}$ is rectangular.

Proof. Let $\Xi = \langle n, \xi_1, \ldots, \xi_b \rangle \in \mathrm{F}(\Sigma)_n^b$ and denote $\Xi_i = \pi_i^b(\Xi)$ for every $i \in [b]$.

Using Lemma 5.19, we obtain that for every $\rho \in \mathrm{Runs}_{\mathscr{A}}(\Xi)$ we have

$$\mathbf{F}_{(\rho(1,\varepsilon),\ldots,\rho(b,\varepsilon))}\big(\mathrm{wt}_{\mathscr{A}}(\Xi, \rho)\big) = \Big(\mathrm{final}_1(q_1)(\mathrm{wt}_{\mathscr{A}}(\Xi_1, \rho_1)), \ldots, \mathrm{final}_b(q_b)(\mathrm{wt}_{\mathscr{A}}(\Xi_b, \rho_b))\Big).$$

Using this fact, we can now prove the claimed equation.

$$\begin{aligned}
[\![\mathscr{A}]\!]_{\mathrm{r}}(\Xi) &= \bigoplus_{\rho \in \mathrm{Runs}_{\mathscr{A}}(\Xi)} \Pi_b\Big(\mathbf{F}_{(\rho(1,\varepsilon),\ldots,\rho(b,\varepsilon))}\big(\mathrm{wt}_{\mathscr{A}}(\Xi, \rho)\big)\Big) \\
&= \bigoplus_{\rho \in \mathrm{Runs}_{\mathscr{A}}(\Xi)} \Pi_b\Big(\mathrm{final}_1(q_1)\big(\mathrm{wt}_{\mathscr{A}}(\Xi_1, \rho_1)\big), \ldots, \mathrm{final}_b(q_b)\big(\mathrm{wt}_{\mathscr{A}}(\Xi_b, \rho_b)\big)\Big) \\
&\overset{\star_1}{=} \bigoplus_{\substack{\rho_1 \in \mathrm{Runs}_{\mathscr{A}}(\Xi_1) \\ \cdots \\ \rho_b \in \mathrm{Runs}_{\mathscr{A}}(\Xi_b)}} \Pi_b\Big(\mathrm{final}_1(q_1)\big(\mathrm{wt}_{\mathscr{A}}(\Xi_1, \rho_1)\big), \ldots, \mathrm{final}_b(q_b)\big(\mathrm{wt}_{\mathscr{A}}(\Xi_b, \rho_b)\big)\Big) \\
&\overset{\star_2}{=} \Pi_b\Big(\bigoplus_{\rho_1 \in \mathrm{Runs}_{\mathscr{A}}(\Xi_1)} \mathrm{final}_1(q_1)\big(\mathrm{wt}_{\mathscr{A}}(\Xi_1, \rho_1)\big), \ldots, \bigoplus_{\rho_b \in \mathrm{Runs}_{\mathscr{A}}(\Xi_b)} \mathrm{final}_b(q_b)\big(\mathrm{wt}_{\mathscr{A}}(\Xi_b, \rho_b)\big)\Big),
\end{aligned}$$

where in Equation $\star_1$ we use Remark 5.18 and in Equation $\star_2$ we use distributivity of Π_b over $\oplus$ ($\mathbb{M}$ contains a semiring, see also the definition of $\mathbb{M}(S)$ on page 156). It holds that

$$\bigoplus_{\rho_i \in \mathrm{Runs}_{\mathscr{A}}(\Xi_i)} \mathrm{final}_i(q_i)(\mathrm{wt}_{\mathscr{A}}(\Xi_i, \rho_i)) \overset{\star_3}{=} \bigoplus_{\rho_i \in \mathrm{Runs}_{\mathscr{A}_i}(\Xi_i)} \mathrm{final}_i(q_i)(\mathrm{wt}_{\mathscr{A}_i}(\Xi_i, \rho_i)) = [\![\mathscr{A}_i]\!]_{\mathrm{r}}(\Xi_i)$$

for every $i \in [b]$, where equation $\star_3$ follows from Remark 5.20. In total we obtain

$$[\![\mathscr{A}]\!]_{\mathrm{r}}(\Xi) = \Pi_b\big([\![\mathscr{A}_1]\!]_{\mathrm{r}}(\Xi_1), \ldots, [\![\mathscr{A}_b]\!]_{\mathrm{r}}(\Xi_b)\big),$$

which proves the claim. □

Theorem 5.22. It holds that

$$[\![\mathscr{A}]\!]_{\mathrm{i}} = [\![\mathscr{A}_1]\!]_{\mathrm{i}} \times \cdots \times [\![\mathscr{A}_b]\!]_{\mathrm{i}}$$

In particular, $[\![\mathscr{A}]\!]_{\mathrm{i}}$ is rectangular.

Proof. Let $\Xi = \langle n, \xi_1, \ldots, \xi_b \rangle \in \mathrm{F}(\Sigma)_n^b$ and denote $\Xi_i = \pi_i^b(\Xi)$ for every $i \in [b]$. It holds that

$$\begin{aligned}
[\![\mathscr{A}]\!]_{\mathrm{i}}(\Xi) &= \bigoplus_{(q_1,\ldots,q_b)\in Q^b} \Pi_b(\mathbf{F}_{(q_1,\ldots,q_b)}(E_{n,b}^{\mathscr{A}}(\Xi, q_1, \ldots, q_b))) \\
&= \bigoplus_{(q_1,\ldots,q_b)\in Q^b} \Pi_b\Big(\mathbf{F}_{(q_1,\ldots,q_b)}\big(E_{n,1}^{\mathscr{A}}(\Xi_1, q_1), \ldots, E_{n,1}^{\mathscr{A}}(\Xi_b, q_b)\big)\Big) \\
&\overset{\star}{=} \bigoplus_{(q_1,\ldots,q_b)\in Q^b} \Pi_b\Big(\mathbf{F}_{(q_1,\ldots,q_b)}\big(E_{n,1}^{\mathscr{A}_1}(\Xi_1, q_1), \ldots, E_{n,1}^{\mathscr{A}_b}(\Xi_b, q_b)\big)\Big) \\
&= \bigoplus_{(q_1,\ldots,q_b)\in Q^b} \Pi_b\Big(\mathrm{final}_1(q_1)\big(E_{n,1}^{\mathscr{A}_1}(\Xi_1, q_1)\big), \ldots, \mathrm{final}_b(q_b)\big(E_{n,1}^{\mathscr{A}_b}(\Xi_b, q_b)\big)\Big) \\
&= \Pi_b\Big(\bigoplus_{q_1\in Q} \mathrm{final}_1(q_1)\big(E_{n,1}^{\mathscr{A}_1}(\Xi_1, q_1)\big), \ldots, \bigoplus_{q_b\in Q} \mathrm{final}_b(q_b)\big(E_{n,1}^{\mathscr{A}_b}(\Xi_b, q_b)\big)\Big) \\
&= \Pi_b\Big([\![\mathscr{A}_1]\!]_{\mathrm{i}}(\Xi_1), \ldots, [\![\mathscr{A}_b]\!]_{\mathrm{i}}(\Xi_b)\Big),
\end{aligned}$$

where we use Remark 5.20 in Equation $\star$. □

5.3.4 I-recognisable is R-recognisable

Theorem 5.23. Let $\mathscr{A}$ be a (b, n)-WFA. Whenever

$$[\![\mathscr{A}_i]\!]_{\mathrm{r}} = [\![\mathscr{A}_i]\!]_{\mathrm{i}}$$

holds for every $i \in [b]$, then also

$$[\![\mathscr{A}]\!]_{\mathrm{r}} = [\![\mathscr{A}]\!]_{\mathrm{i}}$$

holds.

Proof. By Theorems 5.21 and 5.22, we obtain

$$[\![\mathscr{A}]\!]_{\mathrm{r}} = [\![\mathscr{A}_1]\!]_{\mathrm{r}} \times \cdots \times [\![\mathscr{A}_b]\!]_{\mathrm{r}} \qquad \text{and} \qquad [\![\mathscr{A}]\!]_{\mathrm{i}} = [\![\mathscr{A}_1]\!]_{\mathrm{i}} \times \cdots \times [\![\mathscr{A}_b]\!]_{\mathrm{i}}.$$

By assumption, the two right hand sides are equal, which proves the claim. □

Corollary 5.24. Let $\mathscr{A}$ be a (b,n)-WFA. If $\mathbb{M}$ is distributive, then $[\![\mathscr{A}]\!]_{\mathrm{r}} = [\![\mathscr{A}]\!]_{\mathrm{i}}$.

Proof. By Theorem 5.23 it suffices to prove the claim for the case $b = 1$. The case $b = 1$ has been proven for the case of strong bimonoids (rather than M-monoids) in [94]. We can apply a similar proof idea, which we sketch subsequently.

Let $\mathscr{A}$ be a $(1,n)$-WFA and $\xi \in \mathrm{F}(\Sigma)^1_n$. First, one applies the distributivity of $\mathbb{M}$ to show that $[\![\mathscr{A}]\!]_{\mathrm{r}}(\xi) = [\![\mathscr{A}]\!]_{\mathrm{i}}(\xi)$ holds if the following equation holds.

$$\bigoplus_{\rho \in \mathrm{Runs}_{\mathscr{A}}(\xi)} \mathrm{wt}_{\mathscr{A}}(\xi,\rho) = \bigoplus_{q \in Q} E^{\mathscr{A}}_{n,1}(\xi,q) \tag{5.4}$$

The proof of Equation (5.4) is done by a straightforward structural induction on ξ. The case $\xi = x_i$ follows immediately from the definition and the case $\xi = \sigma(\xi_1,\ldots,\xi_s)$ follows from the fact that $\mathrm{Runs}_{\mathscr{A}}(\xi) \cong Q \times \mathrm{Runs}_{\mathscr{A}}(\xi_1) \times \cdots \times \mathrm{Runs}_{\mathscr{A}}(\xi_s)$ and the distributivity of $\mathbb{M}$. □

We aim at generalising [51, 52], both of which use the run semantic as the main semantic of weighted tree automata. Hence, we also choose the run semantic as our main semantic in the following Definition 5.25.

Definition 5.25. We define the *uniform (b,n)-forest valuation recognised by $\mathscr{A}$*, as the map $[\![\mathscr{A}]\!] = [\![\mathscr{A}]\!]_{\mathrm{r}}$. We naturally define *recognisability* as r-recognisability and denote $\mathrm{Rec}(\Sigma,\mathbb{M},b,n) = \mathrm{Rec}_{\mathrm{r}}(\Sigma,\mathbb{M},b,n)$. ◀

Example 5.26. We continue Example 5.11 by calculating the uniform $(2,0)$-forest valuation (that is, the weighted $(2,0)$-forest language) recognised by $\mathscr{C}$. Recall that $\mathbb{M}'$ is distributive and hence $[\![\mathscr{C}]\!]_{\mathrm{i}} = [\![\mathscr{C}]\!]_{\mathrm{r}}$.

For every $\Xi \in \mathrm{F}(\Sigma)^1_0$ we have already observed (see Example 5.11) that there are exactly two non-vanishing runs of $\mathscr{C}$ on Ξ. The first case is that q is assigned to all positions of Ξ. In this case, a straightforward induction over Ξ proves that the cost of the run is $\mathrm{pos}(\Xi)$. We denote this run by $\rho_{\Xi,1}$. The second case is that ℓ is assigned to all leaf positions of Ξ and i is assigned to the remaining positions. In this case, a similar induction over Ξ proves that the cost of the run is $\mathrm{leaf}(\Xi)$. We denote this run by $\rho_{\Xi,2}$.

By Theorem 5.21 we know that $[\![\mathscr{C}]\!]_{\mathrm{r}} = [\![\mathscr{C}_1]\!]_{\mathrm{r}} \times [\![\mathscr{C}_2]\!]_{\mathrm{r}}$, where $\mathscr{C}_i = (Q, \emptyset, T, \mathrm{final}_i)$ for $i \in [2]$. Surely, given a $(1,0)$-forest $\Xi \in \mathrm{F}(\Sigma)_0^1$, the runs of $\mathscr{C}_1$ (respectively, $\mathscr{C}_2$) on Ξ are the same as the runs of $\mathscr{C}$ on Ξ. We obtain

$$\begin{aligned}
[\![\mathscr{C}_1]\!]_{\mathrm{r}}(\Xi) &= \mathrm{final}_1(\rho_{\Xi,1}(\varepsilon))(\mathrm{wt}_{\mathscr{C}_1}(\Xi, \rho_{\Xi,1})) + \mathrm{final}_1(\rho_{\Xi,2}(\varepsilon))(\mathrm{wt}_{\mathscr{C}_1}(\Xi, \rho_{\Xi,2})) \\
&= \mathrm{wt}_{\mathscr{C}_1}(\Xi, \rho_{\Xi,1}) = \mathrm{pos}(\Xi) \text{ and} \\
[\![\mathscr{C}_2]\!]_{\mathrm{r}}(\Xi) &= \mathrm{final}_2(\rho_{\Xi,1}(\varepsilon))(\mathrm{wt}_{\mathscr{C}_2}(\Xi, \rho_{\Xi,1})) + \mathrm{final}_2(\rho_{\Xi,2}(\varepsilon))(\mathrm{wt}_{\mathscr{C}_2}(\Xi, \rho_{\Xi,2})) \\
&= \mathrm{wt}_{\mathscr{C}_2}(\Xi, \rho_{\Xi,2}) = \mathrm{leaf}(\Xi).
\end{aligned}$$

Now, let $\Xi \in \mathrm{F}(\Sigma)_0^2$ be a $(2,0)$-forest and $\Xi_i = \pi_i^2(\Xi)$ for $i \in [2]$. It then holds

$$[\![\mathscr{C}]\!]_{\mathrm{r}}(\Xi) = [\![\mathscr{C}_1]\!]_{\mathrm{r}}(\Xi_1) \cap [\![\mathscr{C}_2]\!]_{\mathrm{r}}(\Xi_2) = \mathrm{pos}(\Xi_1) \cap \mathrm{leaf}(\Xi_2).$$

This proves that $[\![\mathscr{C}]\!]_{\mathrm{r}} = \mathrm{pos} \times \mathrm{leaf}$.

We note that our automaton model is capable of recognising other similar languages as well. For example, we consider the weighted forest languages $\varphi_1, \varphi_2, \varphi_3 \colon \mathrm{F}(\Sigma)_0^2 \to \mathbb{N}$, defined by

$$\begin{aligned}
\varphi_1(\Xi) &= \#\mathrm{pos}(\Xi_1) \cdot \#\mathrm{leaf}(\Xi_2), \\
\varphi_2(\Xi) &= \max(\#\mathrm{pos}(\Xi_1), \#\mathrm{pos}(\Xi_2)), \text{ and} \\
\varphi_3(\Xi) &= \mathrm{height}(\Xi_1) + \mathrm{height}(\Xi_2)
\end{aligned}$$

for every $\Xi \in \mathrm{F}(\Sigma)_0^2$, where $\Xi_i = \pi_i^2(\Xi)$ for every $i \in [2]$. We can easily find an M-monoid $\mathbb{M}_j$ and a $(2,0)$-WFA $\mathscr{C}_j$ such that $[\![\mathscr{C}_j]\!] = \varphi_j$, for every $j \in [3]$. To support this claim, we give $\mathbb{M}_1$ and $\mathscr{C}_1$.

Consider $\mathbb{M}_1 = \mathbb{N}$ and the set of operations $\Omega_{\mathbb{M}_1} = \{\mathrm{id}_{\mathbb{N}}\} \cup \{\widehat{\omega}_s, \widehat{\upsilon}_s, \Pi_s, 0^{(s)} \mid s \geq 0\}$ on $\mathbb{M}_1$, where for every $s \geq 0$ and $n_1, \ldots, n_s \in \mathbb{M}_1$ we define

$$\widehat{\omega}_s(n_1, \ldots, n_s) = 1 + \sum_{i=1}^{s} n_i, \qquad \widehat{\upsilon}_s(n_1, \ldots, n_s) = \begin{cases} 1 & \text{if } s = 0, \\ \sum_{i=1}^{s} n_i & \text{otherwise,} \end{cases}$$

and $\Pi_s(n_1, \ldots, n_s) = \prod_{i=1}^{s} n_i$. It certainly holds that $(\mathbb{M}_1, +, 0, \Omega_{\mathbb{M}_1})$ is an M-monoid which contains a semiring (namely $(\mathbb{N}, +, \cdot, 0, 1)$).

We define the $(2,0)$-WFA $\mathscr{C}_1 = (\widehat{Q}, \emptyset, \widehat{T}, \widehat{\mathrm{final}})$ over Σ and $\mathbb{M}_1$, where $\widehat{Q} = \{q, \ell, i\}$, $\widehat{\mathrm{final}}$ maps every input to 0 except for $\widehat{\mathrm{final}}_1(q) = \widehat{\mathrm{final}}_2(\ell) = \widehat{\mathrm{final}}_2(i) = \mathrm{id}_{\mathbb{N}}$, and for every $s \geq 0, \tau \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q' \in \widehat{Q}$, we define

$$\widehat{T}_\tau(q_1, \ldots, q_s, q') = \begin{cases} \widehat{\omega}_s & \text{if } q_1 = \cdots = q_s = q' = q \\ \widehat{\upsilon}_0 & \text{if } s = 0, q' = \ell \\ \widehat{\upsilon}_s & \text{if } s > 0, q_1, \ldots, q_s \in \{\ell, i\}, q' = i \\ \emptyset^{(s)} & \text{otherwise.} \end{cases}$$

Indeed, $\mathscr{C}_1$ is very similar to $\mathscr{C}$ and one can show analogously to the case of $\mathscr{C}$ that $[\![\mathscr{C}_1]\!] = \varphi_1$. ◀

Remark 5.27. We conclude this chapter by discussing our choice to require that $\mathbb{M}$ contains a semiring. In particular, we illustrate an alternative, more general automaton model without this requirement on $\mathbb{M}$ and show why precisely the case that $\mathbb{M}$ contains a semiring results in a robust theory.

Let $(\mathbb{M}, \oplus, 0, \Omega_{\mathbb{M}})$ be an arbitrary M-monoid. A *generalised* (b,n)*-wfa* over Σ and $\mathbb{M}$ is a tuple $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final}, \pi)$, where Q, init, T, and final are as in the definition of (b,n)-WFA and $\pi \in \Omega_{\mathbb{M}}^{(b)}$ is called the *horizontal multiplication operation*. Both, the i-semantic and r-semantic of $\mathscr{A}$ can be defined similar to the WFA case by replacing Π_b by the new horizontal multiplication operation π.

$$[\![\mathscr{A}]\!]_{\mathrm{i}}(\Xi) = \bigoplus_{q \in Q^b} \pi(\mathbf{F}_q(E^{\mathscr{A}}_{n,b}(\Xi, q)))$$

$$[\![\mathscr{A}]\!]_{\mathrm{r}}(\Xi) = \bigoplus_{\rho \in \mathrm{Runs}_{\mathscr{A}}(\Xi)} \pi\Big(\mathbf{F}_{(\rho(1,\varepsilon),\ldots,\rho(b,\varepsilon))}\big(\mathrm{wt}_{\mathscr{A}}(\Xi, \rho)\big)\Big)$$

We observe that, similar to Corollary 5.24, one obtains $[\![\mathscr{A}]\!]_{\mathrm{i}} = [\![\mathscr{A}]\!]_{\mathrm{r}}$ only if $\mathbb{M}$ (and hence also π) is distributive.

In order to define a horizontal concatenation operation on the set of uniform forest valuations, similar to $\times$ (see Definition 5.15), one can do the following. Let $\omega \in \Omega_{\mathbb{M}}^{(k)}$ for some $k \in \mathbb{N}$ and $\tau_i \in \mathrm{Uvals}(\Sigma, \mathbb{M}, b_i, n)$ for every $i \in [k]$. We define the horizontal concatenation of $\tau_1, \ldots, \tau_k$ via ω, denoted $\omega(\tau_1, \ldots, \tau_k)$, for every forest $\Xi = \langle n, \xi^1_1, \ldots, \xi^1_{b_1}, \ldots, \xi^k_1, \ldots, \xi^k_{b_k} \rangle \in \mathrm{F}(\Sigma)^{b_1+\ldots+b_k}_n$ by

$$\omega(\tau_1, \ldots, \tau_k)(\Xi) = \omega(\tau_1(\langle n, \xi^1_1, \ldots, \xi^1_{b_1} \rangle), \ldots, \tau_k(\langle n, \xi^k_1, \ldots, \xi^k_{b_k} \rangle)).$$

Analogously to our upcoming Theorem 5.36, one would desire that the class of languages recognised by generalised WFA is closed under this horizontal concatenation with any $\omega \in \Omega_{\mathbb{M}}$. However, for such a proof, one needs the distributivity and associativity of ω and the closedness of $\Omega_{\mathbb{M}}$ under composition.

This shows that in exactly the case where $\mathbb{M}$ contains a semiring, we obtain an automaton model that satisfies the requirements to yield both (a) that the i-semantic and r-semantic coincide and (b) that the class of languages recognised by WFA is closed under horizontal concatenation. ◀

5.4 Kleene's Theorem

In [51], a Kleene-like result for weighted tree automata over M-monoids has been established, which in essence states that rational tree expressions and WTA generate the same class of uniform tree valuations. We recall the theorem and the main definitions and then utilise the rectangularity of recognisable uniform forest valuations in order to lift the results to forests. In Chapter 5.4.3, we discuss the question, why our definition of rational forest expressions is not inductive.

5.4.1 Kleene's Theorem for Trees

We begin by introducing rational operations on $\mathrm{Uvals}(\Sigma, \mathbb{M}, n)$.

Definition 5.28 (Definition 4.3 from [51])**.** Let $\Omega \subseteq \mathrm{Ops}(\mathbb{M})$.

We say Ω is *sum closed* if $\omega_1 \oplus \omega_2 \in \Omega$ for every $k \geq 0$ and $\omega_1, \omega_2 \in \Omega^{(k)}$.

We say Ω is $(1, \star)$-*composition closed* if $\omega(\omega') \in \Omega$ for every $\omega \in \Omega^{(1)}$ and $\omega' \in \Omega$.

We say that $\mathbb{M}$ is sum closed (resp. $(1, \star)$-composition closed) if $\Omega_{\mathbb{M}}$ is so. ◀

Definition 5.29 (Definition 4.9 from [51])**.** Let $i \in [n]$. A uniform tree valuation $\varphi \in \mathrm{Uvals}(\Sigma, \mathbb{M}, n)$ is *i-proper* if $\varphi(x_i) = 0$. ◀

We now define the operator $\circ_{\xi,i}$ for every $i \in [n]$ and $\xi \in \mathrm{T}_\Sigma(X_n)$. The intuition for this operator is the following. Let $\ell = \#\mathrm{pos}_{X_n}(\xi)$ and $k = \#\mathrm{pos}_{x_i}(\xi)$. The operator $\circ_{\xi,i}$ takes as input an ℓ-ary operator ω and k many operators $\omega_1, \ldots, \omega_k$. We assign to the j-th parameter of ω the j-th occurrence of a variable in the tree ξ (counted from left to

right). The operation $\circ_{\xi,i}$ then composes ω with $\omega_1, \ldots, \omega_k$ at exactly the parameters that have assigned the variable x_i.

Definition 5.30 (Definition 5.1 from [51])**.** Let $i \in [n]$ and $\xi \in \mathrm{T}_\Sigma(X_n)$. Denote $\{w_1, \ldots, w_\ell\} = \mathrm{pos}_{X_n}(\xi)$ and $\{v_1, \ldots, v_k\} = \mathrm{pos}_{x_i}(\xi)$, where $w_1 <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} w_\ell$ and $v_1 <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} v_k$.

We define the map

$$\circ_{\xi,i} \colon \mathrm{Ops}^{(\ell)}(M) \times \mathrm{Ops}(M)^k \to \mathrm{Ops}(M)$$

for every $\omega \in \mathrm{Ops}^{(\ell)}(M)$ and $\omega_1, \ldots, \omega_k \in \mathrm{Ops}(M)$ by

$$\omega \circ_{\xi,i} (\omega_1, \ldots, \omega_k) = \omega(\omega'_1, \ldots, \omega'_\ell),$$

where we define

$$\omega'_j = \begin{cases} \omega_m & \text{if } w_j = v_m \text{ for some } m \in [k] \\ \mathrm{id}_{\mathbb{M}} & \text{otherwise.} \end{cases}$$

for every $j \in [\ell]$. ◀

Definition 5.31 (Definition 5.3 from [51])**.** Let $\varphi, \psi \in \mathrm{Uvals}(\Sigma, \mathbb{M}, n)$. The (pointwise) *sum* of φ and ψ is denoted by $\varphi \oplus \psi$.

Let $k \geq 0, \sigma \in \Sigma^{(k)}$, and $\omega \in \Omega_{\mathbb{M}}^{(k)}$. The *top-concatenation with σ via ω* is the k-ary operation $\mathrm{top}_{\sigma,\omega}$ on $\mathrm{Uvals}(\Sigma, \mathbb{M}, n)$ given for every $\varphi_1, \ldots, \varphi_k \in \mathrm{Uvals}(\Sigma, \mathbb{M}, n)$ by

$$\mathrm{top}_{\sigma,\omega}(\varphi_1, \ldots, \varphi_k) = \bigoplus_{\xi_1,\ldots,\xi_k \in \mathrm{T}_\Sigma(X_n)} \omega(\varphi_1(\xi_1), \ldots, \varphi_k(\xi_k)).\sigma(\xi_1, \ldots, \xi_k).$$

Let $i \in [n]$. The *i-concatenation* is the binary operation $\cdot_i$ on $\mathrm{Uvals}(\Sigma, \mathbb{M}, n)$ given for every $\varphi, \psi \in \mathrm{Uvals}(\Sigma, \mathbb{M}, n)$ by[1]

$$\varphi \cdot_i \psi = \bigoplus_{\substack{\xi \in \mathrm{T}_\Sigma(X_n), |\xi|_{x_i} = \ell, \\ \xi_1,\ldots,\xi_\ell \in \mathrm{T}_\Sigma(X_n)}} \Big(\varphi(\xi) \circ_{\xi,i} \big(\psi(\xi_1), \ldots, \psi(\xi_\ell)\big)\Big).\xi[x_i \leftarrow (\xi_1, \ldots, \xi_\ell)].$$

Let $i \in [n]$ and $\varphi \in \mathrm{Uvals}(\Sigma, \mathbb{M}, n)$. For every $k \geq 0$ we define the *k-th i-power* of φ, denoted φ_i^k, inductively via $\varphi_i^0 = 0$ and $\varphi_i^{k+1} = (\varphi \cdot_i \varphi_i^k) \oplus \mathrm{id}_{\mathbb{M}}.x_i$. Moreover, if φ

[1]Note that both, $\mathrm{top}_{\sigma,\omega}$ and $\cdot_i$, are well-defined for the following reason. For every $\xi \in \mathrm{T}_\Sigma(X_n)$ there is only a finite number of summands in both sums which assign a non-vanishing value to ξ.

is i-proper, we define the *i-Kleene star* of φ, denoted φ_i^*, by $\varphi_i^*(\xi) = \varphi_i^{\mathrm{height}(\xi)+1}(\xi)$ for every $\xi \in \mathrm{T}_\Sigma(X_n)$. We extend this definition by 0, that is, if φ is not i-proper, then we define the i-Kleene star of φ as the constant uniform tree valuation 0. ◀

Definition 5.32 (Definition 5.6 from [51])**.** We define the set of all *rational expressions over Σ, $\mathbb{M}$, and n*, denoted $\mathrm{RatExp}(\Sigma, \mathbb{M}, n)$, inductively as the smallest set R satisfying conditions (i)–(v). For every $\eta \in \mathrm{RatExp}(\Sigma, \mathbb{M}, n)$ we define its *semantics* $[\![\eta]\!] \in \mathrm{Uvals}(\Sigma, \mathbb{M}, n)$ simultaneously.

(i) For every $i \in [n]$ and $\omega \in \Omega_{\mathbb{M}}^{(1)}$, it holds that $\omega.x_i \in R$ and $[\![\omega.x_i]\!] = \omega.x_i$.

(ii) For every $s \geq 0$, $\sigma \in \Sigma^{(s)}$, $\omega \in \Omega_{\mathbb{M}}^{(s)}$, and $\eta_1, \ldots, \eta_s \in R$, it holds that $\mathrm{top}_{\sigma,\omega}(\eta_1, \ldots, \eta_s) \in R$ and $[\![\mathrm{top}_{\sigma,\omega}(\eta_1, \ldots, \eta_s)]\!] = \mathrm{top}_{\sigma,\omega}([\![\eta_1]\!], \ldots, [\![\eta_s]\!])$.

(iii) For every $\eta_1, \eta_2 \in R$, it holds that $\eta_1 \oplus \eta_2 \in R$ and $[\![\eta_1 \oplus \eta_2]\!] = [\![\eta_1]\!] \oplus [\![\eta_2]\!]$.

(iv) For every $\eta_1, \eta_2 \in R$ and $i \in [n]$, it holds that $\eta_1 \cdot_i \eta_2 \in R$ and $[\![\eta_1 \cdot_i \eta_2]\!] = [\![\eta_1]\!] \cdot_i [\![\eta_2]\!]$.

(v) For every $\eta \in R$ and $i \in [n]$, it holds that $\eta_i^* \in R$ and $[\![\eta_i^*]\!] = [\![\eta]\!]_i^*$.

We call $\varphi \in \mathrm{Uvals}(\Sigma, \mathbb{M}, n)$ *rational* if there exists $\eta \in \mathrm{RatExp}(\Sigma, \mathbb{M}, n)$ such that $\varphi = [\![\eta]\!]$. The class of all rational uniform tree valuations over Σ, $\mathbb{M}$, and n is denoted by $\mathrm{Rat}(\Sigma, \mathbb{M}, n)$. We denote the union $\bigcup_{k \in \mathbb{N}} \mathrm{Rat}(\Sigma, \mathbb{M}, k)$ by $\mathrm{Rat}(\Sigma, \mathbb{M}, \mathrm{fin})$. ◀

Example 5.33. We continue Example 5.2 and consider the rational expression η over Σ, $\mathbb{M}$, and 1 where

$$\eta = \mathrm{top}_{\alpha,\omega_0}() \oplus \mathrm{top}_{\beta,\omega_0}() \oplus \mathrm{top}_{\gamma,\omega_1}(\mathrm{id}_{\mathbb{M}}.x_1) \oplus \mathrm{top}_{\sigma,\omega_2}(\mathrm{id}_{\mathbb{M}}.x_1, \mathrm{id}_{\mathbb{M}}.x_1).$$

From Definition 5.32 we obtain that $[\![\eta]\!](\alpha) = [\![\eta]\!](\beta) = \omega_0$, $[\![\eta]\!](\gamma(x_1)) = \omega_1$, and $[\![\eta]\!](\sigma(x_1, x_1)) = \omega_2$. Moreover, $[\![\eta]\!]$ maps every other tree to an element of $\{0^{(k)} \mid k \geq 0\}$.

We note that η is 1-proper. One can determine that $[\![(\eta)_1^*]\!](\xi) = \mathrm{pos}(\xi)$ for every $\xi \in \mathrm{T}_\Sigma$ and hence it holds that $[\![(\eta)_1^*]\!]|_{\mathrm{T}_\Sigma} = [\![\mathscr{A}]\!]$. In fact, we have calculated the rational expression $\eta' = (\eta)_1^*$ by following the construction from [51, Theorem 6.8] applied to $\mathscr{A}$ and simplifying some terms. This also illustrates how the analysis of an automaton introduces new variables that are used in the rational expression. In our case, $\mathscr{A}$ did not consider any variables, whereas η' considers one variable. This is expected, as $\mathscr{A}$ has exactly one state. ◀

Theorem 5.34 (Theorems 6.8 and 7.10 from [51])**.** If $\mathbb{M}$ is distributive, then

$$\mathrm{Rec}(\Sigma, \mathbb{M}, n) \subseteq \mathrm{Rat}(\Sigma, \mathbb{M}, \mathrm{fin})|_{\mathrm{T}_\Sigma(X_n)}.$$

More specifically, for every n-WTA $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$, there exists a rational expression $\eta \in \mathrm{RatExp}(\Sigma, \mathbb{M}, n + \#Q)$ such that $[\![\mathscr{A}]\!] = [\![\eta]\!]|_{\mathrm{T}_\Sigma(X_n)}$.

Moreover, if $\mathbb{M}$ is distributive, closed under sum, and $(1,\star)$-composition closed, then

$$\mathrm{Rat}(\Sigma, \mathbb{M}, n) \subseteq \mathrm{Rec}(\Sigma, \mathbb{M}, n).$$

In [51], the authors additionally introduce a lift map, which 0-extends the occurring uniform tree valuations to a countably infinite set of variables. This allows the authors to combine the two directions of Theorem 5.34 into the equation

$$\mathrm{lift}(\mathrm{Rec}(\Sigma, \mathbb{M}, \mathrm{fin})) = \mathrm{lift}(\mathrm{Rat}(\Sigma, \mathbb{M}, \mathrm{fin})),$$

where $\mathrm{Rec}(\Sigma, \mathbb{M}, \mathrm{fin}) = \bigcup_{k \in \mathbb{N}} \mathrm{Rec}(\Sigma, \mathbb{M}, k)$.

5.4.2 Kleene's Theorem for Forests

We introduce rational forest expressions as the "horizontal concatenation" of rational tree expressions and prove that the class of uniform forest valuations generated by rational forest expressions equals the class of recognisable forest valuations. Since the rectangularity of recognisable forest valuations gives us a powerful tool to perform automata analysis, we are primarily concerned with closure properties of $\mathrm{Rec}(\Sigma, \mathbb{M}, b, n)$.

Definition 5.35. We define the set of all *rational* (b, n)*-forest expressions* over Σ and $\mathbb{M}$, denoted $\mathrm{RatExp}(\Sigma, \mathbb{M}, b, n)$, by

$$\mathrm{RatExp}(\Sigma, \mathbb{M}, b, n) = \mathrm{RatExp}(\Sigma, \mathbb{M}, n)^b.$$

We denote $(\eta_1, \ldots, \eta_b) \in \mathrm{RatExp}(\Sigma, \mathbb{M}, b, n)$ by $\eta_1 \times \cdots \times \eta_b$ and define its *semantics* $[\![\eta_1 \times \cdots \times \eta_b]\!] \in \mathrm{Uvals}(\Sigma, \mathbb{M}, b, n)$ by

$$[\![\eta_1 \times \cdots \times \eta_b]\!] = [\![\eta_1]\!] \times \cdots \times [\![\eta_b]\!].$$

That is, rational (b, n)-forest expressions over Σ and $\mathbb{M}$ are "rectangles" over rational tree valuations over Σ, $\mathbb{M}$, and n.

A uniform (b,n)-forest valuation $\varphi \in \mathrm{Uvals}(\Sigma, \mathbb{M}, b, n)$ is called *rational* if there exists $\eta \in \mathrm{RatExp}(\Sigma, \mathbb{M}, b, n)$ such that $\varphi = [\![\eta]\!]$. The class of all rational uniform (b,n)-forest valuations is denoted $\mathrm{Rat}(\Sigma, \mathbb{M}, b, n)$. We denote the union $\bigcup_{k\in\mathbb{N}} \mathrm{Rat}(\Sigma, \mathbb{M}, b, k)$ by $\mathrm{Rat}(\Sigma, \mathbb{M}, b, \mathrm{fin})$. ◀

Theorem 5.36. Let $b_1, b_2, n \in \mathbb{N}$, $\varphi \in \mathrm{Rec}(\Sigma, \mathbb{M}, b_1, n)$, and $\psi \in \mathrm{Rec}(\Sigma, \mathbb{M}, b_2, n)$. It holds that

$$\varphi \times \psi \in \mathrm{Rec}(\Sigma, \mathbb{M}, b_1 + b_2, n).$$

Proof. We prove the statement by an explicit construction. Let $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$ be a (b_1, n)-WFA such that $[\![\mathscr{A}]\!] = \varphi$ and let $\mathscr{B} = (Q', \mathrm{init}', T', \mathrm{final}')$ be a (b_2, n)- WFA such that $[\![\mathscr{B}]\!] = \psi$. We moreover assume that $Q \cap Q' = \emptyset$.

Consider the $(b_1 + b_2, n)$-WFA $\mathscr{C} = (Q'', \mathrm{init}'', T'', \mathrm{final}'')$ where $Q'' = Q \cup Q'$, init'' and final'' are given for every $k \in [b_1 + b_2]$, $i \in [n]$, and $q \in Q''$ by

$$\mathrm{init}''_i(q) = \begin{cases} \mathrm{init}(q) & \text{if } q \in Q \\ \mathrm{init}'_i(q) & \text{if } q \in Q' \end{cases} \quad \text{and} \quad \mathrm{final}''_k(q) = \begin{cases} \mathrm{final}_k(q) & \text{if } k \le b_1, q \in Q \\ \mathrm{final}'_{k-b_1}(q) & \text{if } k > b_1, q \in Q' \\ 0^{(1)} & \text{otherwise,} \end{cases}$$

and T'' is given for every $s \ge 0$, $\sigma \in \Sigma^{(s)}$, and $q_1, \ldots, q_s, q \in Q''$ by

$$T''_\sigma(q_1, \ldots, q_s, q) = \begin{cases} T_\sigma(q_1, \ldots, q_s, q) & \text{if } q_1, \ldots, q_s, q \in Q \\ T'_\sigma(q_1, \ldots, q_s, q) & \text{if } q_1, \ldots, q_s, q \in Q' \\ 0^{(s)} & \text{otherwise.} \end{cases}$$

We claim that $[\![\mathscr{C}]\!] = [\![\mathscr{A}]\!] \times [\![\mathscr{B}]\!]$. The proof of this claim is straightforward and hence we just sketch it. First one shows that a run ρ of $\mathscr{C}$ on a *tree* vanishes if states from both Q and Q' occur in ρ. On the other hand, if ρ only uses states from Q (or Q'), then the cost of ρ in $\mathscr{C}$ equals the cost of ρ in $\mathscr{A}$ (or $\mathscr{B}$, respectively). One ultimately uses the fact that $\Pi_{b_1+b_2} = \Pi_2(\Pi_{b_1}, \Pi_{b_2})$ to match the definition of $[\![\mathscr{C}]\!]$ with the definition of $[\![\mathscr{A}]\!] \times [\![\mathscr{B}]\!]$. □

Theorem 5.37 (Kleene result for forests)**.** If $\mathbb{M}$ is distributive, then

$$\mathrm{Rec}(\Sigma, \mathbb{M}, b, n) \subseteq \mathrm{Rat}(\Sigma, \mathbb{M}, b, \mathrm{fin})|_{\mathrm{F}(\Sigma)^b_n}.$$

More specifically, for every (b,n)-WFA $\mathscr{A} = (Q, \mathrm{init}, T, \mathrm{final})$, there exists a rational forest expression $\eta \in \mathrm{RatExp}(\Sigma, \mathbb{M}, b, n + \#Q)$ such that $[\![\mathscr{A}]\!] = [\![\eta]\!]|_{\mathrm{F}(\Sigma)^b_n}$.

If $\mathbb{M}$ is distributive, closed under sum, and $(1,\star)$-composition closed, then

$$\mathrm{Rat}(\Sigma, \mathbb{M}, b, n) \subseteq \mathrm{Rec}(\Sigma, \mathbb{M}, b, n).$$

Proof. Let $\mathbb{M}$ be distributive, closed under sum, and $(1,\star)$-composition closed and let $\varphi \in \mathrm{Rat}(\Sigma, \mathbb{M}, b, n)$. By definition, there exist $\varphi_1, \ldots, \varphi_b \in \mathrm{Rat}(\Sigma, \mathbb{M}, n)$ such that $\varphi = \varphi_1 \times \cdots \times \varphi_b$. By Theorem 5.34, we obtain that $\varphi_i \in \mathrm{Rec}(\Sigma, \mathbb{M}, 1, n)$ for every $i \in [b]$. Theorem 5.36 now proves that $\varphi_1 \times \cdots \times \varphi_b \in \mathrm{Rec}(\Sigma, \mathbb{M}, b, n)$.

Now let $\mathbb{M}$ be distributive and $\varphi \in \mathrm{Rec}(\Sigma, \mathbb{M}, b, n)$. From Theorem 5.21 we obtain that there exist $\varphi_1, \ldots, \varphi_b \in \mathrm{Rec}(\Sigma, \mathbb{M}, 1, n)$ such that $\varphi = \varphi_1 \times \cdots \times \varphi_b$. Corollary 5.13 shows that $\varphi_1, \ldots, \varphi_b \in \mathrm{Rec}(\Sigma, \mathbb{M}, n)$ and thus we can apply Theorem 5.34 to obtain that $\varphi_i \in \mathrm{Rat}(\Sigma, \mathbb{M}, \mathrm{fin})|_{\mathrm{T}_\Sigma(X_n)}$ for every $i \in [b]$. For every $n', n'' \in \mathbb{N}$ such that $n' < n''$, it clearly holds that $\mathrm{Rat}(\Sigma, M, n') \subseteq \mathrm{Rat}(\Sigma, \mathbb{M}, n'')$ and therefore, there exists $n' \in \mathbb{N}$ such that $\varphi_i \in \mathrm{Rat}(\Sigma, \mathbb{M}, n')|_{\mathrm{T}_\Sigma(X_n)}$ for every $i \in [b]$. Since $\mathrm{T}_\Sigma(X_n) \cong \mathrm{F}(\Sigma)^1_n$, we obtain $\mathrm{Rat}(\Sigma, \mathbb{M}, 1, n')|_{\mathrm{F}(\Sigma)^1_n} = \mathrm{Rat}(\Sigma, \mathbb{M}, n')|_{\mathrm{T}_\Sigma(X_n)}$, whence $\varphi_i \in \mathrm{Rat}(\Sigma, \mathbb{M}, 1, n')|_{\mathrm{F}(\Sigma)^1_n}$ for every $i \in [b]$. This ultimately proves that $\varphi_1 \times \cdots \times \varphi_b \in \mathrm{Rat}(\Sigma, \mathbb{M}, b, n')|_{F(\Sigma)^b_n}$ and hence $\varphi \in \mathrm{Rat}(\Sigma, \mathbb{M}, b, \mathrm{fin})|_{F(\Sigma)^b_n}$.

If an automaton $\mathscr{A}$ is given such that $[\![\mathscr{A}]\!] = \varphi$, then we know that $[\![\mathscr{A}_i]\!] = \varphi_i$ and also the number of states in $\mathscr{A}_i$ equals the number of states in $\mathscr{A}$. In particular, Theorem 5.34 yields that $n' = n + \#Q$ is a possible choice for n', which concludes our proof. □

Example 5.38. We continue Example 5.26 and recall the rational expression η over Σ, $\mathbb{M}$, and 1 from Example 5.33. Moreover, we define the rational expression τ over Σ, $\mathbb{M}'$, and 1, where

$$\eta = \mathrm{top}_{\alpha,\omega_0}() \oplus \mathrm{top}_{\beta,\omega_0}() \oplus \mathrm{top}_{\gamma,\omega_1}(\mathrm{id}_{\mathbb{M}}.x_1) \oplus \mathrm{top}_{\sigma,\omega_2}(\mathrm{id}_{\mathbb{M}}.x_1, \mathrm{id}_{\mathbb{M}}.x_1)$$
$$\tau = \mathrm{top}_{\alpha,\upsilon_0}() \oplus \mathrm{top}_{\beta,\upsilon_0}() \oplus \mathrm{top}_{\gamma,\upsilon_1}(\mathrm{id}_{\mathbb{M}'}.x_1) \oplus \mathrm{top}_{\sigma,\upsilon_2}(\mathrm{id}_{\mathbb{M}'}.x_1, \mathrm{id}_{\mathbb{M}'}.x_1).$$

The definition of η and τ shows that η and τ are 1-proper. Moreover, since the M-monoid $\mathbb{M}$ is contained in the M-monoid $\mathbb{M}'$, η can be viewed as a rational expression

over $\mathbb{M}'$. Therefore, it holds that $\eta_1^* \times \tau_1^*$ is a rational $(2,1)$-forest expression over Σ and $\mathbb{M}'$.

In Example 5.33 we have seen that $[\![\eta_1^*]\!]|_{\mathrm{T}_\Sigma} = \mathrm{pos}$. Moreover, one can show that $[\![\tau_1^*]\!]|_{\mathrm{T}_\Sigma} = \mathrm{leaf}$. This proves

$$[\![\eta_1^* \times \tau_1^*]\!]|_{\mathrm{F}(\Sigma)_0^2} = \mathrm{pos} \times \mathrm{leaf}.$$

Recall from Example 5.26 that $[\![\mathscr{C}]\!]_{\mathrm{r}} = \mathrm{pos} \times \mathrm{leaf}$, whence $[\![\eta_1^* \times \tau_1^*]\!]|_{\mathrm{F}(\Sigma)_0^2} = [\![\mathscr{C}]\!]_{\mathrm{r}}$. ◀

5.4.3 An Inductive Approach

In Chapter 5.4.2 we proved a straightforward Kleene Theorem for our weighted (b,n)-forest automaton model. However, our rational forest expressions are very simple and not defined inductively (which is rather contradictory to the idea behind the word "rational"). In this chapter, we discuss an approach to an inductive definition of rational forest expressions and why it fails.

One reasonable way to introduce rational forest expressions would be to define appropriate operations that lift summation, top-concatenation, i-concatenation, and the i-Kleene star (for every i) from the case of trees to the case of forests. However, defining a vertical concatenation operation on uniform forest valuations (or even weighted forest languages over semirings, for that matter) turns out to be challenging.

First, we define a vertical concatenation of forests. For every $a, m, o \in \mathbb{N}$ and forests $\Xi_1 \in \mathrm{F}(\Sigma)_m^a$ and $\Xi_2 \in \mathrm{F}(\Sigma)_o^m$, the *vertical concatenation* of Ξ_1 and Ξ_2, denoted $\Xi_1 \cdot \Xi_2$, is defined as the (a,o)-forest $\langle o, \xi_1, \ldots, \xi_a \rangle$, where for every $i \in [a]$ we define $\xi_i = \pi_i^a(\Xi_1)[\pi_1^m(\Xi_2), \ldots, \pi_m^m(\Xi_2)]$.

Using an analogous approach to [26] and [104], we can now try to define a vertical concatenation of uniform forest valuations. Let $a, m, o \in \mathbb{N}$, $\varphi_1 \in \mathrm{Uvals}(\Sigma, \mathbb{M}, a, m)$, and $\varphi_2 \in \mathrm{Uvals}(\Sigma, \mathbb{M}, m, o)$. Then, the vertical concatenation of φ_1 and φ_2 is the (a,o)-forest valuation $\varphi_1 \cdot \varphi_2$, given for every $\Xi \in \mathrm{F}(\Sigma)_o^a$ as follows. $(\varphi_1 \cdot \varphi_2)(\Xi)$ is "$\varphi_1(\Xi_1)$ applied to $\varphi_2(\Xi_2)$" summed up over all $\Xi_1 \in \mathrm{F}(\Sigma)_m^a$ and $\Xi_2 \in \mathrm{F}(\Sigma)_o^m$ such that $\Xi_1 \cdot \Xi_2 = \Xi$. Because Ξ_1 can have multiple or zero occurrences of a single variable x_i and moreover, variables do not have to be ordered from left to right in Ξ_1, we need to access rectangular components of φ_2, so we can account for the variables in

Ξ_1 properly. If φ_2 is rectangular with rectangular components $\varphi_{2,1}, \ldots, \varphi_{2,m}$, then our desired definition of $(\varphi_1 \cdot \varphi_2)(\Xi)$ is

$$(\varphi_1 \cdot \varphi_2)(\Xi) = \sum_{\Xi_1 \cdot \Xi_2 = \Xi} \varphi_1(\Xi_1)\Big(\varphi_{2,i_1}(\pi^m_{i_1}(\Xi_2)), \ldots, \varphi_{2,i_\ell}(\pi^m_{i_\ell}(\Xi_2))\Big), \tag{5.5}$$

where ℓ is the number of variables in Ξ_1 and for every $j \in [\ell]$ the j-th occurrence of a variable in Ξ_1 (from left to right) is x_{i_j}. However, the rectangular components of φ_2 need not be unique, as we have seen in Remark 5.17. Indeed, different choices of the rectangular components of φ_2 result in different maps $\varphi_1 \cdot \varphi_2$. Hence, Equation (5.5) is not well-defined.

Indeed, one can easily show that Equation (5.5) is not even well-defined in the case of commutative semirings. For example, choose the field $(\mathbb{R}, +, \cdot, 0, 1)$ of real numbers. Because every element has a multiplicative inverse, we can move factors between rectangular components as described in Remark 5.17. This proves that our results from [26] are based on an ill-defined vertical concatenation and are hence invalid.

To end this chapter on a positive note, we want to point out that if the M-monoid $\mathbb{M}$ admits unique rectangular decompositions, then Equation (5.5) is indeed a good definition of the vertical concatenation. This is the case if, for example, $\mathbb{M}$ is the Boolean semiring or the multiplication in $\mathbb{M}$ is a "free" operation. Therefore, the correctness of [104] is not affected by the present thesis.

5.5 Büchi's Theorem

In [52], a Büchi-like result for weighted tree automata over M-monoids has been established, which in essence states that tree M-expressions and WTA (with final states) generate the same class of weighted tree languages. We recall this theorem and the relevant definitions and then utilise the rectangularity of the weighted forest languages generated by WFA (without variables) in order to lift the results to forests. An analogous argument as in Chapter 5.4.3 explains why our definition of forest M-expressions is not inductive. However, in this chapter we do not designate a seperate chapter for this discussion.

5.5.1 Büchi's Theorem for Trees

Definition 5.39 (Definition 3.1 from [52])**.** The set $\mathrm{MExp}(\Sigma, \mathbb{M})$ of *multioperator expressions* (short: *M-expressions*) over Σ and $\mathbb{M}$ is defined by the following EBNF with nonterminal e.

$$e = \mathrm{H}(\omega) \mid e \oplus e \mid \sum_x e \mid \sum_X e \mid \phi \triangleright e,$$

where ω is a $\Sigma_{\mathcal{V}}$-family of operations in $\mathbb{M}$ (for some finite set $\mathcal{V}$ of first-order and second-order variables), x is a first-order variable, X is a second-order variable, and $\phi \in \mathrm{MSO}(\Sigma)$. ◀

Definition 5.40 (Definition 3.2 from [52])**.** For every $e \in \mathrm{MExp}(\Sigma, \mathbb{M})$, the set of *free variables* of e, denoted $\mathrm{Free}(e)$, is defined inductively on the structure of M-expressions.

$$\begin{aligned}
\mathrm{Free}(\mathrm{H}(\omega)) &= \mathcal{V}, \\
\mathrm{Free}(e_1 \oplus e_2) &= \mathrm{Free}(e_1) \cup \mathrm{Free}(e_2), \\
\mathrm{Free}(\sum_x e) &= \mathrm{Free}(e) \setminus \{x\}, \\
\mathrm{Free}(\sum_X e) &= \mathrm{Free}(e) \setminus \{X\}, \\
\mathrm{Free}(\phi \triangleright e) &= \mathrm{Free}(\phi) \cup \mathrm{Free}(e),
\end{aligned}$$

for every finite set $\mathcal{V}$ of first-order and second-order variables, $\Sigma_{\mathcal{V}}$-family ω of operations in $\mathbb{M}$, first-order variable x, second-order variable X, and $\phi \in \mathrm{MSO}(\Sigma)$. If $\mathrm{Free}(e) = \emptyset$, then we call e a *sentence*. ◀

Definition 5.41 (Definition 3.3 from [52])**.** For every $e \in \mathrm{MExp}(\Sigma, \mathbb{M})$ and finite set $\mathcal{V} \supseteq \mathrm{Free}(e)$ of first-order and second-order variables, the *semantics of e with respect to $\mathcal{V}$* is defined as the $(\Sigma_{\mathcal{V}}, \mathbb{M})$-weighted tree language $[\![e]\!]_{\mathcal{V}} \colon \mathrm{T}_{\Sigma_{\mathcal{V}}} \to \mathbb{M}$ as follows. For every $\xi \in (\mathrm{T}_{\Sigma_{\mathcal{V}}} \setminus \mathrm{T}^{\mathrm{v}}_{\Sigma_{\mathcal{V}}})$, we let $[\![e]\!]_{\mathcal{V}}(\xi) = 0$. Moreover, for every $\xi \in \mathrm{T}^{\mathrm{v}}_{\Sigma_{\mathcal{V}}}$, we define

$[\![e]\!]_{\mathcal{V}}(\xi)$ inductively using the structure of M-expressions.

$$\begin{aligned}
[\![\mathrm{H}(\omega)]\!]_{\mathcal{V}}(\xi) &= \mathrm{h}_{\omega[\mathcal{U}\rightsquigarrow\triangleright\mathcal{V}]}(\xi),\\
[\![e_1 \oplus e_2]\!]_{\mathcal{V}}(\xi) &= [\![e_1]\!]_{\mathcal{V}}(\xi) \oplus [\![e_2]\!]_{\mathcal{V}}(\xi),\\
[\![\sum_x e]\!]_{\mathcal{V}}(\xi) &= \sum_{w\in\mathrm{pos}(\xi)} [\![e]\!]_{\mathcal{V}\cup\{x\}}(\xi[x\mapsto w]),\\
[\![\sum_X e]\!]_{\mathcal{V}}(\xi) &= \sum_{W\subseteq\mathrm{pos}(\xi)} [\![e]\!]_{\mathcal{V}\cup\{X\}}(\xi[X\mapsto W]),\\
[\![\phi \triangleright e]\!]_{\mathcal{V}}(\xi) &= \begin{cases} [\![e]\!]_{\mathcal{V}}(\xi) & \text{if } \xi\in\mathcal{L}_{\mathcal{V}}(\phi)\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

for every finite set $\mathcal{U}$ of first-order and second-order variables, $\Sigma_{\mathcal{U}}$-family ω of operations in $\mathbb{M}$, first-order variable x, second-order variable X, and $\phi\in\mathrm{MSO}(\Sigma)$.

We abbreviate $[\![e]\!] = [\![e]\!]_{\mathrm{Free}(e)}$. A $(\Sigma,\mathbb{M})$-weighted tree language φ is called *definable by M-expressions over Σ and $\mathbb{M}$*, if there exists a sentence $e\in\mathrm{MExp}(\Sigma,\mathbb{M})$ such that $[\![e]\!]=\varphi$. The class of $(\Sigma,\mathbb{M})$-weighted tree languages that are definable by M-expressions over Σ and $\mathbb{M}$ is denoted by $\mathrm{MDef}(\Sigma,\mathbb{M})$. ◀

Example 5.42. We continue Example 5.33 and consider the Σ-family of operations $\omega = (\omega_\sigma \mid \sigma\in\Sigma)$, where $\omega_\sigma = \omega_{\mathrm{rk}(\sigma)}$ for every $\sigma\in\Sigma$. Moreover, we define the M-expression e_1 over Σ and $\mathbb{M}$, where

$$e_1 = \mathrm{H}(\omega).$$

A straightforward proof shows that the homomorphism $h_\omega\colon \mathrm{T}_\Sigma\to\mathbb{M}$ is given by $h_\omega(\xi)=\mathrm{pos}(\xi)$ for every $\xi\in\mathrm{T}_\Sigma$. This shows that $[\![e_1]\!]=\mathrm{pos}$ and hence also $[\![e_1]\!]=[\![\mathscr{A}]\!]$.

In Example 5.33, we calculated η following the construction from [51]. We could have done the same for the case of M-expressions by following the construction from [52, Lemma 4.2]. However, the M-expression resulting from the analysis of the automaton $\mathscr{A}$ is very long and not very easy to grasp. Hence, we choose to refrain from depicting such a constructed M-expression and rather consider e_1. ◀

Theorem 5.43 (Theorem 4.1 from [52])**.** Let $\mathbb{M}$ be absorptive. It holds that

$$\mathrm{Rec_f}(\Sigma,\mathbb{M}) = \mathrm{MDef}(\Sigma,\mathbb{M}).$$

5.5.2 Büchi's Theorem for Forests

Definition 5.44. We define the set of all *b-forest multioperator expressions* (short: *b-forest M-expressions*) over Σ and $\mathbb{M}$, denoted $\mathrm{MExp}(\Sigma, \mathbb{M}, b)$, by

$$\mathrm{MExp}(\Sigma, \mathbb{M}, b) = \mathrm{MExp}(\Sigma, \mathbb{M})^b.$$

We denote $(e_1, \ldots, e_b) \in \mathrm{MExp}(\Sigma, \mathbb{M}, b)$ by $e_1 \times \cdots \times e_b$ and define its *semantics* $[\![e_1 \times \cdots \times e_b]\!] \colon \mathrm{F}(\Sigma)_0^b \to \mathbb{M}$ by

$$[\![e_1 \times \cdots \times e_b]\!] = [\![e_1]\!] \times \cdots \times [\![e_b]\!].$$

A weighted $(b, 0)$-forest language $\varphi \colon \mathrm{F}(\Sigma)_0^b \to \mathbb{M}$ is called *definable by M-expressions* if there exists $e \in \mathrm{MExp}(\Sigma, \mathbb{M}, b)$ such that $\varphi = [\![e]\!]$. The class of all weighted $(b, 0)$-forest languages that are definable by M-expressions is denoted $\mathrm{MDef}(\Sigma, \mathbb{M}, b)$. ◀

Theorem 5.45 (Büchi result for forests)**.** Let $\mathbb{M}$ be absorptive. It holds that

$$\mathrm{Rec_f}(\Sigma, \mathbb{M}, b, 0) = \mathrm{MDef}(\Sigma, \mathbb{M}, b)$$

Proof. First, let $\varphi \in \mathrm{Rec_f}(\Sigma, \mathbb{M}, b, 0)$. From Theorem 5.21 we obtain the existence of $\varphi_1, \ldots, \varphi_b \in \mathrm{Rec}(\Sigma, \mathbb{M}, 0)$ such that $\varphi = \varphi_1 \times \cdots \times \varphi_b$. Moreover, the proof of Theorem 5.21 shows that φ_i is recognisable with final states for every $i \in [b]$, whence $\varphi_1, \ldots, \varphi_b \in \mathrm{Rec_f}(\Sigma, \mathbb{M})$. By Theorem 5.43, we have that $\varphi_1, \ldots, \varphi_b \in \mathrm{MDef}(\Sigma, \mathbb{M})$, which yields $\varphi \in \mathrm{MDef}(\Sigma, \mathbb{M}, b)$. This concludes the proof of the inclusion "$\subseteq$".

Now let $\varphi \in \mathrm{MDef}(\Sigma, \mathbb{M}, b)$. By the definition of $\mathrm{MExp}(\Sigma, \mathbb{M}, b)$, there exist $e_1, \ldots, e_b \in \mathrm{MExp}(\Sigma, \mathbb{M})$ such that $\varphi = [\![e_1]\!] \times \cdots \times [\![e_b]\!]$. By Theorem 5.43, we have that $[\![e_i]\!] \in \mathrm{Rec_f}(\Sigma, \mathbb{M})$ for every $i \in [b]$ and hence by Theorems 5.12 and 5.36, we obtain that $\varphi \in \mathrm{Rec}(\Sigma, \mathbb{M}, b, 0)$. The proof of Theorem 5.36 moreover shows that $\varphi \in \mathrm{Rec_f}(\Sigma, \mathbb{M}, b, 0)$. This concludes the proof of the inclusion "$\supseteq$". □

Example 5.46. We continue Example 5.42 and consider the Σ-family of operations $\upsilon = (\upsilon_\sigma \mid \sigma \in \Sigma)$, where $\upsilon_\sigma = \upsilon_{\mathrm{rk}(\sigma)}$ for every $\sigma \in \Sigma$. We recall the M-expression e_1 over Σ and $\mathbb{M}$ from Example 5.42 and define the M-expression e_2 over Σ and $\mathbb{M}'$, where

$$e_1 = \mathrm{H}(\omega) \qquad \text{and} \qquad e_2 = \mathrm{H}(\upsilon).$$

Since the M-monoid $\mathbb{M}$ is contained in the M-monoid $\mathbb{M}'$, e_1 can be considered as an M-expression over $\mathbb{M}'$. Therefore, it holds that $e_1 \times e_2$ is a 2-forest M-expression over Σ and $\mathbb{M}'$.

In Example 5.42 we have seen that $[\![e_1]\!] = \mathrm{pos}$. Moreover, one can show that the homomorphism $h_\upsilon \colon \mathrm{T}_\Sigma \to M'$ is given by $h_\upsilon(\xi) = \mathrm{leaf}(\xi)$ for every $\xi \in \mathrm{T}_\Sigma$. This shows that $[\![e_2]\!] = \mathrm{leaf}$ and hence we obtain

$$[\![e_1 \times e_2]\!] = \mathrm{pos} \times \mathrm{leaf}.$$

We have seen in Example 5.26 that $[\![\mathscr{C}]\!]_{\mathrm{r}} = \mathrm{pos} \times \mathrm{leaf}$ and hence, $[\![e_1 \times e_2]\!] = [\![\mathscr{C}]\!]_{\mathrm{r}}$. ◀

5.6 Conclusion

In this chapter, we have introduced a weighted forest automata model over M-monoids. We then showed that this model recognises exactly the finite products of recognisable weighted tree languages over M-monoids. Next, we introduced rational weighted forest expressions and forest M-expressions over M-monoids and showed that the classes of languages generated by these formalisms coincide with our recognisable weighted forest languages under certain conditions on the underlying M-monoid.

In upcoming research, it would be worthwhile to study other ways to define weighted forest automata, especially models where the tree components in forests are not handled independently. This could be done while simultaneously considering unranked hedges (rather than ranked forests) and tree valuation monoids (rather than M-monoids).

6

Rational Weighted Tree Languages with Storage

This chapter is a presentation of Dörband, Fülöp, and Vogler [30] with minor changes.

Throughout Chapter 6, we assume Σ to be a ranked alphabet and S to be a complete semiring.

6.1 Introduction

Nondeterministic one-way finite-state string automata have been used very successfully for the specification of aspects of programming languages and of natural languages. Of course, there is also a need for more powerful automata which can model more sophisticated aspects of such languages, like block-structuredness or cross-serial dependencies. As a consequence, a number of models of automata with auxiliary data store were introduced; examples of such storages are pushdown [18], stack [63], checking-stack [45], checking-stack pushdown [111], nested stack [2], iterated pushdown [20, 46, 71, 86], tree-stack [22], and monoid [75, Example 3] (cf. [79]). Each of these automata models are constructed according to the recipe

"finite-state automaton + data store" [101]

where (i) the finite-state automaton uses predicates and instructions in its transitions, (ii) the data store [46, 67, 101] is a set of configurations on which the predicates and

instructions are interpreted, and there is a distinguished initial configuration. A string $u \in \Sigma^*$ is accepted if there is a sequence d of transitions such that (a) d leads the automaton from some initial to some final state, (b) the projection of d to Σ^* is u, and (c) the projection of d to the sequence of involved instructions is *executable* on the storage, i.e., it is applicable to the initial configuration (e.g., pop pop is not executable on the pushdown, because initially it contains only one symbol and pop is not defined for the empty pushdown).

In the 60's and 70's of the previous century, a theory of classes of formal languages was established [60, 61, 62] where the classes were defined

(A) by closure properties (like closure under union, concatenation, Kleene-star, and a-transductions) or

(B) by "finite state automata + data store".

The first type of definition led to the concept of *abstract family of languages* (AFL) and the second one to the concept of *abstract family of acceptors* (AFA). A fundamental theorem is that, roughly speaking, each class of formal languages is a full principle AFL if and only if it is the class of formal languages accepted by some finitely-encoded AFA [60].

In [67, 68, 69], Goldstine criticised the complexity of the definition of AFA. As alternative, he advocated another approach to automata with storage by applying the implication "recognisable $\Rightarrow$ rational" of Kleene's theorem [81] to the automaton part of the recipe, thereby obtaining

"rational expressions + data store" [67, 68, 69]

(see also the introduction of [55]). More precisely, he defined an *automaton with data store* [68, p. 276] as a rational subset A of the monoid $(\Sigma^* \times I^*)^*$ where Σ is an alphabet and I is the set of instructions of the data store. The *language* defined by A is the set

$$L(A) = \{u \in \Sigma^* \mid \exists (v \in \hat{A}) \text{ such that } u = \mathrm{proj}_1(v) \text{ and } \mathrm{proj}_2(v) \text{ is executable}\}$$

where $\hat{(.)}$ is the natural morphism from the free monoid $(\Sigma^* \times I^*)^*$ to the product monoid $\Sigma^* \times I^*$ (with string concatenation in both components), i.e., $\hat{A} \subseteq \Sigma^* \times I^*$. It

is very important to notice that, although each sequence $v \in \hat{A}$ is built up according to the rational set A, the executability of the sequence $\mathrm{proj}_2(v) \in I^*$ is checked *outside* of $\hat{A}$ (by the requirement "$\mathrm{proj}_2(v)$ is executable"). By means of an example using the pushdown storage, Goldstine demonstrates [67] that the description of the context-free language $\{a^n b^n \mid n \in \mathbb{N}\}$ is much easier in terms of a rational expression + data store than in terms of a classical state-based pushdown automaton (also cf. [55]).

The recipe "finite-state automaton + data store" has also been applied in the weighted case, for example, to the combination of weighted word automata and data store [54, 74, 112]. Moreover, in [55], Goldstine's approach was applied to the recipe

"weighted regular tree grammar + storage" [50][1],

where the considered weight structures are commutative, complete semirings. More precisely, the implication "regular $\Rightarrow$ rational" of the Kleene-theorem [38, Theorem 7.1] was applied to the weighted regular tree grammar part and executability of instruction sequences was checked outside of the rational weighted tree languages. Actually, sequences of instructions now turn into trees, which we call *behaviours* (cf. [48] where they are called *approximations*). This led to the new concept of *rational weighted tree language with storage*. In the present chapter, we will repeat this definition and add some more discussion. Then the *Kleene-Goldstine theorem* [55, Theorem 3] (see also Theorem 6.8) says the following:

$$\mathrm{Reg}(\mathfrak{S}, \Sigma, S) = \mathrm{Rat}(\mathfrak{S}, \Sigma, S)$$

where S is commutative, $\mathfrak{S}$ is a storage type, and $\mathrm{Reg}(\mathfrak{S}, \Sigma, S)$ and $\mathrm{Rat}(\mathfrak{S}, \Sigma, S)$ are the classes of weighted tree languages of type $\mathrm{T}_\Sigma \to S$ that are generated by weighted tree grammars with storage type $\mathfrak{S}$ and that are rational weighted tree languages with storage type $\mathfrak{S}$, respectively.

In the following, we discuss the definition of rational weighted tree languages with storage in more detail. For this, we recall the class of rational weighted tree languages [38, Definition 3.17] (without any reference to storage). Let Θ be a ranked alphabet.

[1] Cf. [46] for the unweighted case of such grammars. We also note that regular tree grammars and finite-state tree automata are equivalent, cf., e.g., [58, Theorem 2.3.6]. The same holds for the weighted case, cf. [49, Corollary 3.6] and [53, Theorem 3.40].

The class $\mathrm{Rat}(\Theta, S)$ of *rational weighted tree languages over Θ and S* is the smallest class of weighted tree languages of type $\mathrm{T}_\Theta \to S$ which is closed under the rational operations, that is, under top-concatenation, scalar multiplication, sum, tree concatenation, and Kleene-star (see Chapter 2.2).

Now assume additionally that $\mathfrak{S}$ is a storage type (in the sense of [46, 101]) and Σ is a ranked alphabet. Intuitively, we obtain $(\mathfrak{S}, \Sigma, S)$-rational weighted tree languages by imposing on the set of behaviours of $\mathfrak{S}$ a rational structure (as in Goldstine's approach). To make this more precise, we first discuss the concept of behaviour. For this, let P' and F' be some finite subsets of the sets of predicates and instructions of $\mathfrak{S}$, respectively. We pair Boolean combinations over P' with finite sequences over F' into a ranked alphabet Δ (where the length of the sequence in $(F')^*$ determines the rank). Then a tree $b \in \mathrm{T}_\Delta$ is a Δ-behaviour if each path of b is executable on the storage type $\mathfrak{S}$.

For instance, let $\mathfrak{S}$ be the storage type COUNT which has $\mathbb{N}$ as set of configurations, 0 as initial configuration, the predicate zero?, and the instructions inc, dec, and id (standing for increment, decrement, and identity, respectively). Figure 6.1 shows a behaviour $b \in \mathrm{T}_\Delta$ (right, solid part). Indeed, each path of b is executable on the initial configuration 0. For instance, the execution of the path

$$(\mathrm{true}, \mathrm{inc})\ (\mathrm{true}, \mathrm{inc})\ (\mathrm{true}, \mathrm{id})\ (\neg\mathrm{zero?}, \mathrm{dec})\ (\neg\mathrm{zero?}, \mathrm{dec})\ (\mathrm{zero?}, \varepsilon)$$

which leads from the root of b to its rightmost leaf, may be illustrated as follows (where tt denotes the truth value true):

$$\begin{aligned} 0 &\xrightarrow{\mathrm{true}(0)=tt} \mathrm{inc}(0) = 1 \xrightarrow{\mathrm{true}(1)=tt} \mathrm{inc}(1) = 2 \xrightarrow{\mathrm{true}(2)=tt} \mathrm{id}(2) = 2 \\ &\xrightarrow{\neg\mathrm{zero?}(2)=tt} \mathrm{dec}(2) = 1 \xrightarrow{\neg\mathrm{zero?}(1)=tt} \mathrm{dec}(1) = 0 \xrightarrow{\mathrm{zero?}(0)=tt} \mathrm{stop}\ . \end{aligned}$$

Figure 6.1 also shows the two sequences of configurations (right, grey part).

Next, we combine symbols from Δ and symbols of $\Sigma \cup \{*\}$, where the symbol $*$ allows to change configurations independent from Σ-symbols. This results in the ranked alphabet $\langle \Delta, \Sigma \rangle$. We denote by $\mathrm{proj}_1 \colon \mathrm{T}_{\langle \Delta, \Sigma \rangle} \to \mathrm{T}_\Delta$ and $\mathrm{proj}_2 \colon \mathrm{T}_{\langle \Delta, \Sigma \rangle} \to \mathrm{T}_\Sigma$ the projections to the first component (without the symbol $*$) and second component of the labels, respectively. For instance, Figure 6.1 shows a tree $\zeta \in \mathrm{T}_{\langle \Delta, \Sigma \rangle}$ (left), where $\Sigma = \{\sigma^{(2)}, \delta^{(1)}, \alpha^{(0)}\}$ is a ranked alphabet, the behaviour $b = \mathrm{proj}_1(\zeta)$ in T_Δ (right),

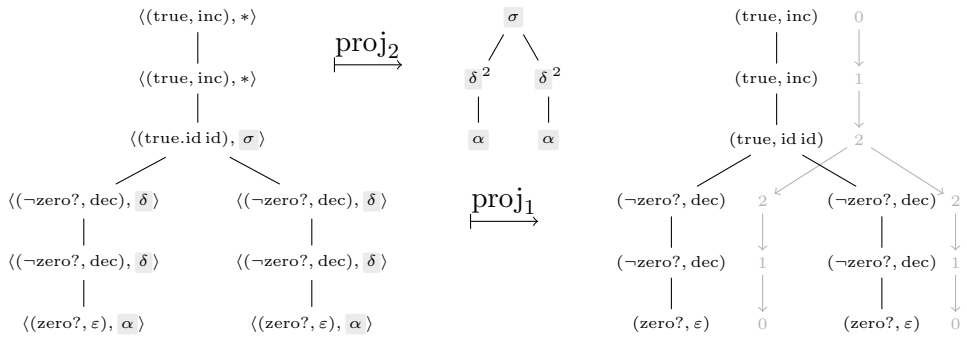


Figure 6.1: A tree $\zeta \in \mathrm{T}_{\langle \Delta, \Sigma \rangle}$ over the extended ranked alphabet $\langle \Delta, \Sigma \rangle$ (left), a tree $\xi \in \mathrm{T}_{\Sigma}$ (middle), and a behaviour $b \in \mathrm{T}_{\Delta}$ (right). In the behaviour b, the intermediate configurations along each of its paths are indicated in grey.

and the tree $\mathrm{proj}_2(\zeta) = \xi$ in T_{Σ} (middle). We note that the set $\mathrm{T}_{\langle \Delta, \Sigma \rangle}$ corresponds to the product monoid $\Sigma^* \times I^*$ of Goldstine[1][67, 68, 69].

Finally, we define the map $\mathcal{B}_{\Delta}$ which maps each tree $\xi \in \mathrm{T}_{\Sigma}$ to the set of all $\zeta \in \mathrm{T}_{\langle \Delta, \Sigma \rangle}$ such that $\mathrm{proj}_1(\zeta)$ is a Δ-behaviour and $\mathrm{proj}_2(\zeta) = \xi$. Then, a weighted tree language $\varphi \colon \mathrm{T}_{\Sigma} \to S$ is $(\mathfrak{S}, \Sigma, S)$-*rational* if there is a weighted tree language $\psi \colon \mathrm{T}_{\langle \Delta, \Sigma \rangle} \to S$ in $\mathrm{Rat}^{\infty}(\langle \Delta, \Sigma \rangle, S)$ (where ∞ indicates that ψ may use for tree concatenation an arbitrary but finite number of nullary symbols additionally to $\langle \Delta, \Sigma \rangle$) such that

$$\varphi = \mathcal{B}_{\Delta}; \psi \ , \tag{6.1}$$

where the map $(\mathcal{B}_{\Delta}; \psi) \colon \mathrm{T}_{\Sigma} \to S$ is defined for every $\xi \in \mathrm{T}_{\Sigma}$ by

$$(\mathcal{B}_{\Delta}; \psi)(\xi) = \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)} \psi(\zeta) \ ,$$

and where $\bigoplus$ is an infinitary summation in S. We note that the weighted tree languages in $\mathrm{Rat}^{\infty}(\langle \Delta, \Sigma \rangle, S)$ do not account for executability of sequences of instructions (as in Goldstine's approach).

Usually, a class of rational objects is defined as the smallest class which contains some finite set of objects and which is closed under some appropriate sum operation, concatenation operation, and iteration operation. For instance, the class $\mathrm{Rat}(\Theta, S)$

[1]However, our notation differs from Goldstine's one as follows: in $\Sigma^* \times I^*$ instructions are in the second components, while in our $\mathrm{T}_{\langle \Delta, \Sigma \rangle}$ instructions are in the first components.

(for every ranked alphabet Θ) is defined in this way. However, this is not true for our class $\mathrm{Rat}(\mathfrak{S}, \Sigma, S)$ of rational weighted tree languages with storage, because each such weighted tree language has the form $\mathcal{B}_\Delta; \psi$ (cf. Equation (6.1)) and we do not know how to *decompose* behaviours under tree concatenation and Kleene-star. That means we could not prove the following: for every $\psi_1, \psi_2, \psi \in \mathrm{Rat}^\infty(\langle \Delta, \Sigma \rangle, S)$ and nullary symbol $\bar{\alpha} = \langle (p, \varepsilon), \alpha \rangle$ in $\langle \Delta, \Sigma \rangle$ there are $\psi_1', \psi_2', \psi' \in \mathrm{Rat}^\infty(\langle \Delta, \Sigma \rangle, S)$ such that ψ_i' only depends on ψ_i for $i \in \{1, 2\}$, ψ' only depends on ψ, and

$$\mathcal{B}_\Delta; (\psi_1 \circ_{\bar{\alpha}} \psi_2) = (\mathcal{B}_\Delta; \psi_1') \circ_\alpha (\mathcal{B}_\Delta; \psi_2') \quad \text{and} \quad \mathcal{B}_\Delta; \psi_{\bar{\alpha}}^* = (\mathcal{B}_\Delta; \psi')_\alpha^* \ .$$

We even claim that these statements are wrong. Thus, the definition of $\mathrm{Rat}(\mathfrak{S}, \Sigma, S)$ does not imply that $\mathrm{Rat}(\mathfrak{S}, \Sigma, S)$ is closed under the rational operations, in particular, tree concatenation and Kleene-star.

However, in [56, Lemma 6.6 and 6.7] it was proved that, if S is commutative, then the class $\mathrm{Reg}(\mathfrak{S}, \Sigma, S)$ is closed under tree concatenation and Kleene-star if $\mathfrak{S}$ contains a reset instruction (i.e., an instruction ¢ which transforms each configuration into the initial configuration). The proofs consist of direct constructions and their correctness proofs. Then, it follows from our Kleene-Goldstine theorem that also $\mathrm{Rat}(\mathfrak{S}, \Sigma, S)$ is closed under tree concatenation and Kleene-star (again assuming that $\mathfrak{S}$ contains a reset instruction).

In this chapter, we show alternative proofs of the closure of $\mathrm{Rat}(\mathfrak{S}, \Sigma, S)$ under the rational operations, where closure under tree concatenation and Kleene-star assumes that $\mathfrak{S}$ contains a reset instruction (cf. Theorem 6.21); here we do not assume that S is commutative. For this, we show how to *compose* behaviours. In particular, we prove the following for every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, $\psi, \psi_1, \psi_2, \ldots, \psi_s \in \mathrm{Rat}^\infty(\langle \Delta, \Sigma \rangle, S)$, and $a \in S$ it holds that

- $\mathrm{top}_\sigma((\mathcal{B}_\Delta; \psi_1), \ldots, (\mathcal{B}_\Delta; \psi_s)) = \mathcal{B}_\Delta; (\mathrm{top}_{\langle (\mathrm{true}, \mathrm{id} \cdots \mathrm{id}), \sigma \rangle}(\psi_1, \ldots, \psi_s))$ (cf. Lemma 6.9)
- $a \odot (\mathcal{B}_\Delta; \psi) = \mathcal{B}_\Delta; (a \odot \psi)$ (cf. Lemma 6.10),
- $(\mathcal{B}_\Delta; \psi_1) \oplus (\mathcal{B}_\Delta; \psi_2) = \mathcal{B}_\Delta; (\psi_1 \oplus \psi_2)$ (cf. Lemma 6.11),
- $(\mathcal{B}_\Delta; \psi_1) \circ_\alpha (\mathcal{B}_\Delta; \psi_2) = \mathcal{B}_\Delta; (\psi_1^\downarrow \circ_{\bar{\alpha}} \psi_2)$ (cf. Lemma 6.15), and

- $(\mathcal{B}_\Delta;\psi)^*_\alpha = \mathcal{B}_\Delta;(\psi^\downarrow)^*_{\bar{\alpha}}$ if $\mathcal{B}_\Delta;\psi$ is α-proper (cf. Lemma 6.20).

Here, $\psi_1^\downarrow$ and $\psi^\downarrow$ are extensions of ψ_1 and ψ, respectively, which use the reset instruction. Both $\psi_1^\downarrow$ and $\psi^\downarrow$ are elements of $\mathrm{Rat}^\infty(\langle\Delta,\Sigma\rangle,S)$ (cf. Lemma 6.13).

We now point out the benefit of our new proofs compared to the existing ones. In [56], proofs were given by constructions and their correctness proofs. This is a quite unrewarding approach, as very similar proofs have already been done in [38] for the case of the trivial storage type. The fact that $\mathrm{Rat}(\mathfrak{S},\Sigma,S)$ essentially equals $\mathrm{Rat}^\infty(\langle\Delta,\Sigma\rangle,S)$ precomposed by $\mathcal{B}_\Delta$ is enough knowledge to prove the closure properties. Hence, we do not need to know the formalism generating $\mathrm{Reg}(\mathfrak{S},\Sigma,S)$ in order to prove the closure properties. We much rather show that the way, in which storage is introduced, *preserves* these properties.

This chapter supplements [55]. However, we will also repeat the main definitions and the main theorem of [55], i.e., the Kleene-Goldstine theorem.

6.2 Preliminaries

0-Extensions A 0-*extension of* Σ is a ranked alphabet Θ such that $\Sigma\subseteq\Theta$, $\mathrm{rk}_\Theta(\sigma)=\mathrm{rk}_\Sigma(\sigma)$ for each $\sigma\in\Sigma$, and $\mathrm{rk}_\Theta(\sigma)=0$ for each $\sigma\in\Theta\setminus\Sigma$.

Let $\varphi\colon T_\Sigma\to S$ be a weighted tree language. A 0-*extension of* φ is a weighted tree language $\varphi'\colon \mathrm{T}_\Theta\to S$ such that Θ is a 0-extension of Σ, $\varphi'|_{\mathrm{T}_\Sigma}=\varphi$ and $\varphi'(\xi)=0$ for every $\xi\in\mathrm{T}_\Theta\setminus\mathrm{T}_\Sigma$.

Let Θ be a ranked alphabet, $\tau\colon \mathrm{T}_\Sigma\to\mathcal{P}(\mathrm{T}_\Theta)$ a map, and $\varphi\colon\mathrm{T}_\Theta\to S$. We define the (Σ,S)-weighted tree language $(\tau;\varphi)\colon\mathrm{T}_\Sigma\to S$ for every $\xi\in\mathrm{T}_\Sigma$ by

$$(\tau;\varphi)(\xi)=\bigoplus_{\zeta\in\tau(\xi)}s(\zeta),$$

where $\bigoplus$ is an infinitary sum operation on S (recall that S is complete).

Storage Types and Behaviours We recall the (slightly modified) concept of storage type from [46]. Storage types are a reformulation of the concept of machines [101] and data stores [68, 69].

A *storage type* is a tuple $\mathfrak{S} = (C, P, F, c_0)$, where C is a set of *configurations*, $c_0 \in C$ is the *initial configuration*, P is a set of maps each having the type $p\colon C \to \{0,1\}$, called *predicates*, and F is a nonempty set of partial maps $f\colon C \dashrightarrow C$, called *instructions*. Moreover, we assume that F contains id_C, which we abbreviate by id and call *identity instruction (on C)*.

The *reset instruction (for $\mathfrak{S}$)* is the map $¢\colon C \to C$ which maps each configuration to c_0. We denote by $\mathfrak{S}¢$ the storage type $(C, P, F¢, c_0)$ where $F¢ = F \cup \{¢\}$. Clearly, if $¢ \in F$, then $\mathfrak{S} = \mathfrak{S}¢$.

The *trivial storage type* is the storage type $\mathrm{TRIV} = (\{c\}, \emptyset, \{\mathrm{id}_{\{c\}}\}, c)$, where c is some arbitrary but fixed symbol. Clearly, $\mathrm{TRIV}¢ = \mathrm{TRIV}$. Another example of a storage type is the *counter* $\mathrm{COUNT} = (\mathbb{N}, \{\mathrm{zero?}\}, \{\mathrm{id}_{\mathbb{N}}, \mathrm{inc}, \mathrm{dec}\}, 0)$, where for each $n \in \mathbb{N}$, we let $\mathrm{zero?}(n) = 1$ iff $n = 0$, $\mathrm{inc}(n) = n + 1$, and $\mathrm{dec}(n) = n - 1$ if $n \geq 1$ and undefined otherwise. Surely, $\mathrm{COUNT}¢ \neq \mathrm{COUNT}$.

We define the maps $\mathrm{true}\colon C \to \{0,1\}$ and $\mathrm{false}\colon C \to \{0,1\}$ by $\mathrm{true}(c) = 1$ and $\mathrm{false}(c) = 0$ for every $c \in C$. We denote by $\mathrm{BC}(P)$ the *Boolean closure of P*, i.e., the smallest set of maps of type $C \to \{0,1\}$ which contains true, false, and all predicates in P and which is closed under negation $\neg$, disjunction $\vee$, and conjunction $\wedge$. If P is finite, then $\mathrm{BC}(P)$ is also finite because it is a finitely generated subalgebra of the Boolean algebra of all predicates over C (cf. [64, Corollary 2]).

> *Throughout the rest of Chapter 6, if $\mathfrak{S}$ is left unspecified, then it stands for an arbitrary storage type $\mathfrak{S} = (C, P, F, c_0)$. Also, if P and F are left unspecified, then they stand for the sets of predicates and instructions, respectively, of some storage type $\mathfrak{S}$.*

Let $P' \subseteq P$ be a finite set and $F' \subseteq F$ be a non-empty, finite set. Moreover, let $n \in \mathbb{N}$. We define the ranked alphabet

$$\Delta = \bigcup_{0 \leq k \leq n} \Delta^{(k)} \quad \text{with} \quad \Delta^{(k)} = \mathrm{BC}(P') \times (F')^k \ .$$

We call Δ the *ranked alphabet n-corresponding to P' and F'*. We write elements $(p, f_1, \ldots, f_k)$ of Δ in the slightly shorter form $(p, f_1 \cdots f_k)$. Note that the parameter n is used to put an upper bound on the rank of symbols in Δ, whence $\mathrm{maxrk}(\Delta) = n$. The

ranked alphabet corresponding to Σ, P', and F' is the ranked alphabet n-corresponding to P' and F' where $n = \max\{\mathrm{maxrk}(\Sigma), 1\}$.

We now turn towards the concept of behaviour, which is inspired by the set $L_{\mathcal{D}}$ of all executable sequences of instructions (defined on [60, p. 148]; also cf. the notion of storage tracks in [68]). In [48, Definition 3.23] a family of behaviours is put together into a tree by sharing common prefixes; such a tree is called approximation. Here we recall the concept of approximation from [50], but we keep the original name "behaviour".

Formally, let $c \in C$, $n \in \mathbb{N}$, and Δ be the ranked alphabet n-corresponding to $P' \subseteq P$ and $F' \subseteq F$. Then a tree $b \in \mathrm{T}_\Delta$ is a (Δ, c)-*behaviour* if there exists a family $(c_w \in C \mid w \in \mathrm{pos}(b))$ of configurations such that $c_\varepsilon = c$ and for every $w \in \mathrm{pos}(b)$ it holds that if $b(w) = (p, f_1 \cdots f_k)$, then (i) $p(c_w) = 1$ and (ii) for every $i \in [k]$ the configuration $f_i(c_w)$ is defined and $c_{wi} = f_i(c_w)$. If b is a (Δ, c)-behaviour, then we call $(c_w \in C \mid w \in \mathrm{pos}(b))$ the *family of configurations determined by b and c*. A Δ-*behaviour* is a (Δ, c_0)-behaviour. The right part of Figure 6.1 shows an example b of a Δ-behaviour; the grey-shaded tree is the family of configurations determined by b and 0. We denote the set of all Δ-behaviours by $\mathcal{B}(\Delta)$. We refer the reader to [50, Figure 2] for an example of a behaviour of the pushdown storage.

In Figure 6.2, we provide a table which compares our concepts of storage types and of behaviours with the corresponding concepts of data stores and storage tracks from [68, Definition 3.1].

6.3 Rational Weighted Tree Languages with Storage

In this chapter, we generalise the approach of [68, 69] from the unweighted case to the weighted case and from strings to trees. For the definition of rational weighted tree languages (disregarding storage for the time being), we first recall what it means for weighted tree languages to be closed under rational operations. We refer the reader to Chapter 2.2 for the definition of the rational operations.

Let $\mathcal{L} \subseteq S^{\mathrm{T}_\Sigma}$ be a class of weighted tree languages. We say that $\mathcal{L}$ is

- *closed under scalar multiplication* if for every $\varphi \in \mathcal{L}$ and $a \in S$, the (Σ, S)-weighted tree language $a \odot \varphi$ is in $\mathcal{L}$,

data store [68, Definition 3.1]	storage type [present chapter]
$\mathcal{D} = (D_0, \iota, D_1)$ (data store)	$S = (C, P, F, c_0)$ (storage type)
D (set of configurations)	C (set of configurations)
$D_0 \subseteq D$ (initial configurations)	$c_0 \in C$ (initial configuration)
$D_1 \subseteq D$ (terminal configurations)	–
–	P (set of predicates)
I (set of instructions)	F (set of instructions)
$\iota(i) \subseteq D \times D$ (interpretation of instruction $i \in I$)	$f : C \to C$ (instruction)
$v \in I^*$ such that $\iota(v) : D_0 \mapsto D_1$ (storage track)	Δ-behaviour $b \in \mathcal{B}(\Delta)$

Figure 6.2: Comparison of data storage and storage types.

- *closed under sum* if for every $\varphi_1, \varphi_2 \in \mathcal{L}$, the (Σ, S)-weighted tree language $\varphi_1 \oplus \varphi_2$ is in $\mathcal{L}$,

- *closed under top-concatenation* if for every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $\varphi_1, \ldots, \varphi_s \in \mathcal{L}$, the (Σ, S)-weighted tree language $\mathrm{top}_\sigma(\varphi_1, \ldots, \varphi_s)$ is in $\mathcal{L}$,

- *closed under tree concatenation* if for every $\varphi_1, \varphi_2 \in \mathcal{L}$ and for every $\alpha \in \Sigma^{(0)}$, the (Σ, S)-weighted tree language $\varphi_1 \circ_\alpha \varphi_2$ is in $\mathcal{L}$, and

- *closed under Kleene-star* if for every $\alpha \in \Sigma^{(0)}$ and $\varphi \in \mathcal{L}$ such that φ is α-proper, the (Σ, S)-weighted tree language φ_α^* is in $\mathcal{L}$.

Definition 6.1. The set of *rational weighted tree languages over Σ and S*, denoted by $\mathrm{Rat}(\Sigma, S)$, is the smallest class of (Σ, S)-weighted tree languages which is closed under the *rational operations*, that is, scalar multiplication, sum, top-concatenation, tree concatenation, and Kleene star. ◀

By iterating top-concatenations and using scalar multiplication and sum, we can build up each weighted tree language with finite support. Hence each weighted tree language with finite support is in $\mathrm{Rat}(\Sigma, S)$.

We note that the class $\mathrm{Rat}(\Sigma, S)$ is the same as the class $A^{\mathrm{rat}}\langle\!\langle \mathrm{T}_\Sigma \rangle\!\rangle$ from [38].

However, the Kleene-theorem [38, Theorem 7.1] only applies to the class

$$A^{\mathrm{rat}}\langle\!\langle \mathrm{T}_\Sigma(Q_\infty)\rangle\!\rangle = \bigcup_{\substack{Q \text{ finite set of} \\ \text{nullary symbols}}} A^{\mathrm{rat}}\langle\!\langle \mathrm{T}_{\Sigma\cup Q}\rangle\!\rangle \ .$$

These nullary symbols are needed for the analysis of chain-free (Σ, S)-regular tree grammars (cf. Section 6.4), specifically to do a certain q-concatenation with $q \in Q_\infty$ (see [38, Theorem 5.2], cf. also [107, Theorem 9]). We therefore extend our definition to capture this phenomenon by adding 0-extensions.

Definition 6.2. The *set of extended* (Σ, S)*-rational weighted tree languages*, denoted by $\mathrm{Rat}^\infty(\Sigma, S)$, contains each (Σ, S)-weighted tree language φ such that there exists a 0-extension $\varphi' \in \mathrm{Rat}(\Sigma', S)$ of φ for some 0-extension Σ' of Σ. ◀

In particular, $\mathrm{Rat}(\Sigma, S) \subseteq \mathrm{Rat}^\infty(\Sigma, S)$.

Remark 6.3. Let $A^{\mathrm{rat}}\langle\!\langle \mathrm{T}_\Sigma\rangle\!\rangle$ and $A^{\mathrm{rat}}\langle\!\langle \mathrm{T}_\Sigma(Q_\infty)\rangle\!\rangle$ be defined as in [38, Definitions 3.18 and 4.2]. It holds that

$$\begin{aligned} \mathrm{Rat}(\Sigma, S) &= A^{\mathrm{rat}}\langle\!\langle \mathrm{T}_\Sigma\rangle\!\rangle \\ \mathrm{Rat}^\infty(\Sigma, S) &= A^{\mathrm{rat}}\langle\!\langle \mathrm{T}_\Sigma(Q_\infty)\rangle\!\rangle \ . \end{aligned}$$

We note that in [55], the sets $\mathrm{Rat}(\Sigma, S)$ and $\mathrm{Rat}^\infty(\Sigma, S)$ are denoted by $\mathcal{L}(\Sigma, S)$ and $\mathrm{Rat}(\Sigma, S)$, respectively. We have changed these notations here in order to be more compatible with the notations used in [38, 83]. ◀

Remark 6.4. Let Σ' be a ranked alphabet such that $\Sigma \subseteq \Sigma'$. One can easily verify that $\mathrm{Rat}^\infty(\Sigma, S) \subseteq \mathrm{Rat}^\infty(\Sigma', S)$. ◀

In our definition of rational weighted tree languages over Σ and S with storage $\mathfrak{S}$, we will use the rational operations to build up trees and each such tree ζ combines a tree $\xi \in \mathrm{T}_\Sigma$ and a tree b over the ranked alphabet Δ corresponding to Σ and some finite sets $P' \subseteq P$ and $F' \subseteq F$. Then, according to Goldstine's idea, we check outside of the building process whether b is a behaviour. In order to allow manipulation of the storage via P' and F' also independently from the generation of a Σ-symbol, we use the symbol $*$ as a padding symbol of rank 1 such that $* \notin \Sigma$. (We refer the reader to

pages 205 and 213 for a short discussion on the usage and necessity of $*$.) Formally, we define the *Σ-extension of Δ*, denoted by $\langle \Delta, \Sigma \rangle$, to be the ranked alphabet where

$$\langle \Delta, \Sigma \rangle^{(1)} = \Delta^{(1)} \times (\Sigma^{(1)} \cup \{*\}) \quad \text{and} \quad \langle \Delta, \Sigma \rangle^{(k)} = \Delta^{(k)} \times \Sigma^{(k)} \quad \text{for } k \neq 1.$$

Additionally, we define the maps $\mathrm{proj}_1 \colon \mathrm{T}_{\langle \Delta, \Sigma \rangle} \to \mathrm{T}_\Delta$ and $\mathrm{proj}_2 \colon \mathrm{T}_{\langle \Delta, \Sigma \rangle} \to \mathrm{T}_\Sigma$ such that for every $k \in \mathbb{N}$, $\langle \delta, \sigma \rangle \in \langle \Delta, \Sigma \rangle^{(k)}$, and $\zeta_1, \ldots, \zeta_k \in \mathrm{T}_{\langle \Delta, \Sigma \rangle}$, we have

$$\mathrm{proj}_1(\langle \delta, \sigma \rangle(\zeta_1, \ldots, \zeta_k)) = \delta(\mathrm{proj}_1(\zeta_1), \ldots, \mathrm{proj}_1(\zeta_k)) \ ,$$

and

$$\mathrm{proj}_2(\langle \delta, \sigma \rangle(\zeta_1, \ldots, \zeta_k)) = \begin{cases} \sigma(\mathrm{proj}_2(\zeta_1), \ldots, \mathrm{proj}_2(\zeta_k)) & \text{if } \sigma \neq * \\ \mathrm{proj}_2(\zeta_1) & \text{otherwise.} \end{cases}$$

We note that proj_1 is a tree relabeling in the sense of [43]. Then, we define the map $\mathcal{B}_\Delta \colon \mathrm{T}_\Sigma \to \mathcal{P}(\mathrm{T}_{\langle \Delta, \Sigma \rangle})$ for each $\xi \in \mathrm{T}_\Sigma$ by

$$\mathcal{B}_\Delta(\xi) = \{\zeta \in \mathrm{T}_{\langle \Delta, \Sigma \rangle} \mid \mathrm{proj}_1(\zeta) \in \mathcal{B}(\Delta) \text{ and } \mathrm{proj}_2(\zeta) = \xi\} \ .$$

We call $\mathcal{B}_\Delta(\xi)$ the set of *ξ-extended behaviours*. We note that this definition of $\mathcal{B}_\Delta(\xi)$ is equivalent to the one on [55, p. 143], and that $\mathcal{B}_\Delta(\xi)$ is infinite due to the presence of the padding symbol $*$ and the fact that $\mathrm{id} \in F$. We refer to Figure 6.1 for an example of $\xi \in \mathrm{T}_\Sigma$ and $\zeta \in \mathcal{B}_\Delta(\xi)$.

Let $\zeta \in \mathcal{B}_\Delta(\xi)$. We define $\simeq_\zeta$ as the smallest equivalence relation on $\mathrm{pos}(\zeta)$ such that for every $w' \in \mathrm{pos}(\zeta)$ satisfying $\zeta(w') = \langle \delta, * \rangle$ for some $\delta \in \Delta$, we have $w' \simeq_\zeta w'1$. In Figure 6.3, we visualise the equivalence classes of $\simeq_\zeta$ by the light gray boxes. Moreover, Figure 6.3 illustrates that each equivalence class of $\simeq_\zeta$ corresponds to exactly one position of ξ. We use this intuition of correspondence of equivalence classes of $\simeq_\zeta$ to positions of ξ in order to formally define the following relation. Given $w' \in \mathrm{pos}(\zeta)$ and $w \in \mathrm{pos}(\xi)$ we say that w' *corresponds to* w if $[w']_{\simeq_\zeta}$ corresponds to w. The following table shows the complete list of correspondences of positions for ζ and ξ from Figure 6.3.

position w' of ζ	position of ξ corresponding to w'
ε, 1	ε
$11, 111, 1111$	1
11111	11
$12, 121$	2

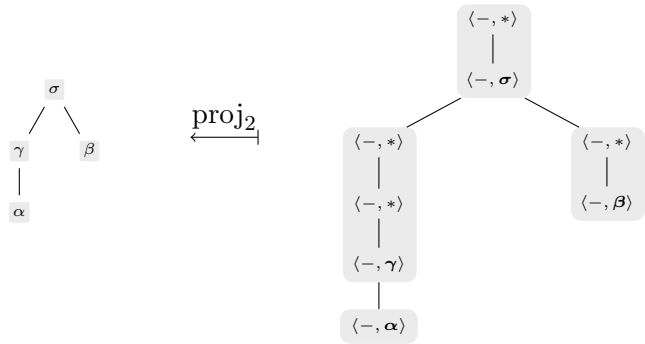


Figure 6.3: A tree $\xi \in \mathrm{T}_\Sigma$ (left) and a tree $\zeta \in \mathcal{B}_\Delta(\xi)$ (right) for which $\xi = \mathrm{proj}_2(\zeta)$ holds. Each occurrence of the hyphen $(-)$ in ζ represents an element of Δ; different occurrences of the hyphen may represent different elements of Δ.

Definition 6.5. Let $\varphi\colon \mathrm{T}_\Sigma \to K$ be a weighted tree language. We say that φ is $(\mathfrak{S}, \Sigma, S)$*-rational* if there are finite sets $P' \subseteq P$ and $F' \subseteq F$ and a weighted tree language $\psi \in \mathrm{Rat}^\infty(\langle \Delta, \Sigma\rangle, S)$ such that

$$\varphi = \mathcal{B}_\Delta; \psi, \qquad \text{that is,} \qquad \varphi(\xi) = \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} \psi(\zeta) \text{ for every } \xi \in \mathrm{T}_\Sigma.$$

We denote by $\mathrm{Rat}(\mathfrak{S}, \Sigma, S)$ the class of all $(\mathfrak{S}, \Sigma, S)$*-rational weighted tree languages*, which we also call the class of *rational weighted tree languages with storage.* ◀

Now we compare rational weighted tree languages with storage and automata with data store [68]. As a technical tool, we use the concept of characteristic functions. For every set B and every $B' \subseteq B$, the *characteristic function of* B' *in* B is the map $\chi_{(B,B')}\colon B \to \mathbb{B}$ defined for every $a \in B$ by $\chi_{(B,B')}(a) = 1$ if $a \in B'$ and $\chi_{(B,B')}(a) = 0$ otherwise. If we transcribe the definition of the language $L(A) \subseteq \Sigma^*$ defined by a Goldstine-automaton A (as given in [68, p. 276] for some alphabet Σ) by replacing the membership test $u \in L(A)$ (for some string $u \in \Sigma^*$) by the equation $\chi_{(\Sigma^*, L(A))}(u) = 1$, then the definition of $L(A)$ reads:

$$\chi_{(\Sigma^*, L(A))}(u) = \bigvee_{\substack{v \in I^* \text{ s.th.} \\ \iota(v)\colon D_0 \mapsto D_1}} \chi_{(\Sigma^* \times I^*, \hat{A})}(u, v) \tag{6.2}$$

where $\iota(v)\colon D_0 \mapsto D_1$ says that there is an initial configuration $c_0 \in D_0$ and a final configuration $c_1 \in D_1$ such that the sequence v of instructions can transform c_0 into

language defined by an automaton over data store [68]	(S, Σ, K)-rational weighted tree language [present chapter]
string $u \in \Sigma^*$	tree $\xi \in \mathrm{T}_\Sigma$
sequence $v \in I^*$ of instructions	tree $b \in \mathrm{T}_\Delta$
combination of $v \in I^*$ and $u \in \Sigma^*$	tree $\zeta \in \mathrm{T}_{\langle\Delta,\Sigma\rangle}$
$v \in I^*$ is storage track if $\iota(v) : D_0 \mapsto D_1$	$b \in \mathrm{T}_\Delta$ is a behaviour if $b \in \mathcal{B}(\Delta)$
automaton A	rational expression (not contained here)
trace $\hat{A}$	$(\langle\Delta, \Sigma\rangle, S)$-rational weighted tree language ψ
$\chi_{(\Sigma^* \times I^*, \hat{A})}(u, v)$	$\psi(\zeta)$
addition $\vee$ in the Boolean semiring	addition $+$ in an arbitrary complete semiring S
truth value $\chi_{(\Sigma^*, L(A))}(u)$	value $\varphi(\xi)$ in S

Figure 6.4: Comparison of concepts in [68] and in the present chapter.

c_1; and $\hat{A}$ is the trace of A. Moreover, the operator $\bigvee$ is the generalisation of binary disjunction of truth values to a finite number of arguments. Then, Equation (6.2) can easily be compared to the definition of rational weighted tree languages with storage:

$$\varphi(\xi) = \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} \psi(\zeta) \ . \tag{6.3}$$

Thus our concept of $(\mathfrak{S}, \Sigma, S)$-rational weighted tree languages generalises the concept of automata over data stores [68] from the unweighted to the weighted case (i.e., from the Boolean semiring $\mathbb{B}$ to any complete semiring S) and from strings to trees. The correspondences between the quantities involved in (6.2) and (6.3) are shown in Figure 6.4. For an example of a rational weighted tree language with storage we refer the reader to [55, Example 1].

6.4 The Kleene-Goldstine Theorem

In this chapter, we briefly recall from [55] the definition of weighted regular tree grammars with storage and the Kleene-Goldstine theorem. Our grammar model is the weighted version of regular tree grammar with storage [48], where we take the weights from a complete semiring S. Our concept slightly extends the form of rules of

$(\mathfrak{S}, \Sigma, S)$-regular tree grammars as defined in [50, Section 3.1] by allowing any number of terminal symbols in the right hand side of rules. Also, in [50], the weight algebras are complete M-monoids, which are more general than complete semirings.

A *weighted regular tree grammar over Σ with storage $\mathfrak{S}$ and weights in S* (for short: $(\mathfrak{S}, \Sigma, S)$-rtg) is a tuple $\mathcal{G} = (N, Z, R, \mathrm{wt})$, where N is a finite set of *nonterminals* such that $N \cap \Sigma = \emptyset$, $Z \subseteq N$ is the set of *initial nonterminals*, R is a finite and nonempty set of *rules*, such that each rule has the form

$$A(p) \to \xi \ ,$$

where $A \in N$, $p \in \mathrm{BC}(P)$, and $\xi \in \mathrm{T}_\Sigma(N(F))$ with $N(F) = \{A(f) \mid A \in N, f \in F\}$, and $\mathrm{wt}\colon R \to S$ is the *weight function*. We recall that P and F are the sets of predicates and of instructions, respectively, of the storage type $\mathfrak{S}$.

If r is a rule of the form $A(p) \to B(f)$, then it is called a *chain rule*. If $\mathcal{G}$ does not have chain rules, then we call it *chain-free*. We say that $\mathcal{G}$ is *in normal form* if each rule contains at most one symbol from Σ. We let $P_\mathcal{G}$ and $F_\mathcal{G}$ denote the finite sets of predicates and instructions, respectively, which occur in $\mathcal{G}$. Moreover, if $F_\mathcal{G} \neq \emptyset$, then we denote the ranked alphabet corresponding to R, $P_\mathcal{G}$, and $F_\mathcal{G}$ by $\Delta_\mathcal{G}$, where we view R as a ranked alphabet such that the rank of a rule $r \in R$ is the number of nonterminals in its right hand side.

For the formal definition of the *weighted tree language generated by* $\mathcal{G}$, denoted by $[\![\mathcal{G}]\!] : \mathrm{T}_\Sigma \to S$, we refer the reader to [55]; here we only give an intuition by showing a slight modification of [55, Example 2]. We consider the semiring $(\mathbb{N}_\infty, +, \cdot, 0, 1)$ where $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ and the $(\mathrm{COUNT}, \Sigma, \mathbb{N}_\infty)$-rtg $\mathcal{G} = (\{Z, A\}, Z, R, \mathrm{wt})$ where all rules in R and their corresponding weights are shown in Figure 6.5. Clearly, $\mathcal{G}$ is in normal form (in contrast to [55, Example 2]) and the topmost rule in Figure 6.5 is a chain rule.

At the left of Figure 6.6, a *derivation tree* $d \in \mathrm{D}_\mathcal{G}(\xi)$ for the input tree ξ (middle) is shown, where $\mathrm{D}_\mathcal{G}(\xi)$ is the set of all derivation trees for ξ [55, Section 5]. We note that $\mathrm{D}_\mathcal{G}(\xi) \subseteq \mathrm{T}_R$. The involved maps have the types $\pi\colon \mathrm{T}_R \to \mathrm{T}_\Sigma$ and $\beta\colon \mathrm{T}_R \to \mathrm{T}_{\Delta_\mathcal{G}}$ and they extract from a given derivation tree d the tree ξ that is generated by d and the behaviour b that was used to generate ξ, respectively. Additionally, the map $\mathrm{wt}_\mathcal{G}\colon \mathrm{T}_R \to S$ multiplies the weights of the rules which occur in its argument in a fixed

rule r:			wt(r):
$Z(\text{true})$	$\to$	$Z(\text{inc})$	2
$Z(\text{true})$	$\to$	$\sigma(A(\text{id}), A(\text{id}))$	1
$A(\neg\text{zero?})$	$\to$	$\delta(A(\text{dec}))$	2
$A(\text{zero?})$	$\to$	α	1

Figure 6.5: Rules of a $(\text{COUNT}, \Sigma, \mathbb{N}_\infty)$-rtg.

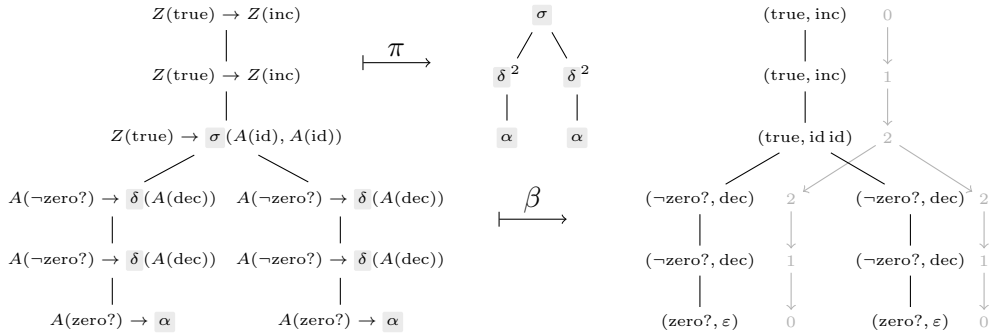


Figure 6.6: A derivation tree $d \in \mathrm{D}_{\mathcal{G}}(\xi)$ (left), the input tree $\xi = \pi(d) = \sigma(\delta^2(\alpha), \delta^2(\alpha))$ (up middle), the $\Delta_{\mathcal{G}}$-behaviour $b = \beta(d)$ (right), the family $(c_w \mid w \in \mathrm{pos}(b))$ of configurations (in grey) determined by b and 0.

order. The *weighted tree language generated by* $\mathcal{G}$ is the map $[\![\mathcal{G}]\!] \colon \mathrm{T}_\Sigma \to S$ defined for each $\xi \in \mathrm{T}_\Sigma$ by

$$[\![\mathcal{G}]\!](\xi) = \bigoplus_{d \in \mathrm{D}_{\mathcal{G}}(\xi)} \mathrm{wt}_{\mathcal{G}}(d) \ .$$

Due to chain rules, the index set $\mathrm{D}_{\mathcal{G}}(\xi)$ can be infinite. Also in this case, the sum is well defined because S is a complete semiring.

For our example grammar $\mathcal{G}$, it is rather easy to see that $[\![\mathcal{G}]\!] : \mathrm{T}_\Sigma \to \mathbb{N}_\infty$ and

$$[\![\mathcal{G}]\!](\xi) = \begin{cases} (2^n)^3 & \text{if } \xi = \sigma(\delta^n(\alpha), \delta^n(\alpha)) \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

In general, let φ be a (Σ, S)-weighted tree language. We say that φ is $(\mathfrak{S}, \Sigma, S)$-*regular* if there exists an $(\mathfrak{S}, \Sigma, S)$-rtg $\mathcal{G}$ such that $\varphi = [\![\mathcal{G}]\!]$. The class of all $(\mathfrak{S}, \Sigma, S)$-regular tree languages is denoted by $\mathrm{Reg}(\mathfrak{S}, \Sigma, S)$. We note that a (TRIV, Σ, S)-rtg in which

rule r:			wt(r):
Z	$\to$	$\langle(\text{true}, \text{inc}), *\rangle\big(Z\big)$	2
Z	$\to$	$\langle(\text{true}, \text{id id}), \sigma\rangle\big(A, A\big)$	1
A	$\to$	$\langle(\neg\text{zero?}, \text{dec}), \delta\rangle\big(A\big)$	2
A	$\to$	$\langle(\text{zero?}, \varepsilon), \alpha\rangle$	1

Figure 6.7: Rules of a $(\text{TRIV}, \langle\Delta, \Sigma\rangle, \mathbb{N}_\infty)$-rtg.

every rule contains exactly one terminal symbol, is just another syntactic form of a weighted tree automaton over Σ and S [53].

The Kleene-Goldstine theorem is based on two theorems: (a) a decomposition theorem and (b) a Kleene theorem. The decomposition theorem (a) was proved in [50, Theorem 5.3] for $(\mathfrak{S}, \Sigma, S)$-rtg in normal form. We recall the slightly extended version from [55] for the case of arbitrary $(\mathfrak{S}, \Sigma, S)$-rtg.

Theorem 6.6 (Theorem 1 from [55])**.** Let $\varphi\colon \mathrm{T}_\Sigma \to S$. The following are equivalent:

(i) $\varphi = [\![\mathcal{G}]\!]$ for some $(\mathfrak{S}, \Sigma, S)$-rtg $\mathcal{G}$.

(ii) There are finite sets $P' \subseteq P$ and $F' \subseteq F$, and a chain-free $(\text{TRIV}, \langle\Delta, \Sigma\rangle, S)$-rtg $\mathcal{G}$ such that Δ is the ranked alphabet corresponding to Σ, P', and F', and $\varphi = \mathcal{B}_\Delta; [\![\mathcal{G}]\!]$.

We illustrate the easy construction involved in $(i) \Rightarrow (ii)$ of Theorem 6.6 by considering the $(\text{COUNT}, \Sigma, \mathbb{N}_\infty)$-rtg $\mathcal{G}$ with rules given in Figure 6.5. The rules of the chain-free $(\text{TRIV}, \langle\Delta, \Sigma\rangle, \mathbb{N}_\infty)$-rtg resulting from that construction are shown in Figure 6.7 where we have dropped the Boolean combination true from the left hand side of each rule and the instruction id from each nonterminal of the right hand side of each rule. This example indicates how the symbol $*$ is used as a padding symbol. Indeed, weighted regular tree grammars over TRIV are equivalent to weighted regular tree grammars without storage type (cf. special case (ii) of [50, page 13] and [56, page 9]).

The used Kleene theorem (b) was proved in [38] for weighted tree automata over a ranked alphabet Θ and commutative semiring S, which are equivalent to chain-free (TRIV, Θ, S)-rtg in normal form (cf. special cases (ii) on [50, p. 13]).

Theorem 6.7 (Theorems 5.2 and 6.8(2) from [38])**.** For every ranked alphabet Θ and commutative semiring S, we have

$$\mathrm{Reg}_{\mathrm{nc}}(\mathrm{TRIV}, \Theta, S) = \mathrm{Rat}^{\infty}(\Theta, S),$$

where $\mathrm{Reg}_{\mathrm{nc}}(\mathrm{TRIV}, \Theta, S)$ is the class of (Θ, S)-weighted tree languages generated by chain-free $(\mathrm{TRIV}, \Theta, S)$-rtg.

Now we recall the Kleene-Goldstine theorem for weighted regular tree grammars with storage, which follows easily from Theorems 6.6 and 6.7 (by choosing $\Theta = \langle \Delta, \Sigma \rangle$) and from the definition of rational weighted tree languages with storage. This theorem generalises [38, Theorem 7.1] from the trivial storage type TRIV to an arbitrary storage type.

Theorem 6.8 (Theorem 3 from [55])**.** Let S be commutative. It holds that

$$\mathrm{Reg}(\mathfrak{S}, \Sigma, S) = \mathrm{Rat}(\mathfrak{S}, \Sigma, S).$$

We refer the reader to [55] for a discussion of some special cases.

6.5 Closure of $\mathrm{Rat}(\mathfrak{S}¢, \Sigma, S)$ under Rational Operations

In this chapter, we prove that $\mathrm{Rat}(\mathfrak{S}¢, \Sigma, S)$ is closed under the rational operations.

Throughout the rest of Chapter 6.5, we assume that P' is a finite subset of P, F' is a finite and nonempty subset of F, and Δ is the ranked alphabet corresponding to Σ, P', and F'.

6.5.1 Top-Concatenation, Scalar Multiplication, and Sum

The next three lemmas are preparations for the proof that $\mathrm{Rat}(\mathfrak{S}, \Sigma, S)$ is closed under top-concatenation, scalar multiplication, and sum, respectively.

Lemma 6.9. For every $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $\varphi_1, \ldots, \varphi_s \in \mathrm{Rat}^{\infty}(\langle \Delta, \Sigma \rangle, S)$ it holds that

$$\mathrm{top}_{\sigma}((\mathcal{B}_{\Delta}; \varphi_1), \ldots, (\mathcal{B}_{\Delta}; \varphi_s)) = \mathcal{B}_{\Delta}; \mathrm{top}_{\langle (\mathrm{true}, \mathrm{id} \cdots \mathrm{id}), \sigma \rangle}(\varphi_1, \ldots, \varphi_s)$$

with s occurrences of id.

Proof. Let $\xi \in \mathrm{T}_\Sigma$. First, we assume that there do not exist $\xi_1, \ldots, \xi_s \in T_\Sigma$ such that $\xi = \sigma(\xi_1, \ldots, \xi_s)$. Then, $\mathrm{top}_\sigma((\mathcal{B}_\Delta; \varphi_1), \ldots, (\mathcal{B}_\Delta; \varphi_s))(\xi) = 0$. Moreover, using $\bar{\varphi}$ as abbreviation of $\mathrm{top}_{\langle(\mathrm{true},\mathrm{id}\cdots\mathrm{id}),\sigma\rangle}(\varphi_1, \ldots, \varphi_s)$, it holds

$$(\mathcal{B}_\Delta; \bar{\varphi})(\xi) = \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} \bar{\varphi}(\zeta) = 0$$

by definition of $\mathcal{B}_\Delta(\xi)$ and the forms of $\bar{\varphi}$ and ξ.

Next, we assume that there exist $\xi_1, \ldots, \xi_s \in T_\Sigma$ such that $\xi = \sigma(\xi_1, \ldots, \xi_s)$. Using the same abbreviation as above, it holds that

$$\begin{aligned}
&\mathrm{top}_\sigma((\mathcal{B}_\Delta; \varphi_1), \ldots, (\mathcal{B}_\Delta; \varphi_s))(\xi) \\
&= (\mathcal{B}_\Delta; \varphi_1)(\xi_1) \odot \ldots \odot (\mathcal{B}_\Delta; \varphi_s)(\xi_s) \\
&= \left(\bigoplus_{\zeta_1 \in \mathcal{B}_\Delta(\xi_1)} \varphi_1(\zeta_1) \right) \odot \ldots \odot \left(\bigoplus_{\zeta_s \in \mathcal{B}_\Delta(\xi_s)} \varphi_s(\zeta_s) \right) \\
&\overset{\star_1}{=} \bigoplus_{\zeta_1 \in \mathcal{B}_\Delta(\xi_1)} \cdots \bigoplus_{\zeta_s \in \mathcal{B}_\Delta(\xi_s)} \varphi_1(\zeta_1) \odot \ldots \odot \varphi_s(\zeta_s) \\
&= \bigoplus_{\zeta_1 \in \mathcal{B}_\Delta(\xi_1)} \cdots \bigoplus_{\zeta_s \in \mathcal{B}_\Delta(\xi_s)} \bar{\varphi}(\langle(\mathrm{true}, \mathrm{id} \ldots \mathrm{id}), \sigma\rangle(\zeta_1, \ldots, \zeta_s)) \\
&\overset{\star_2}{=} \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} \bar{\varphi}(\zeta) = (\mathcal{B}_\Delta; \bar{\varphi})(\xi)
\end{aligned}$$

where in Equation $\star_1$ we have used the generalised distributivity law (see Equation (2.4) on page 21) and in Equation $\star_2$ we use the definition of $\mathcal{B}_\Delta$ and the fact that $\bar{\varphi}(\zeta) = 0$ for every $\zeta \in \mathcal{B}_\Delta(\xi)$ such that $\zeta(\varepsilon) \neq \langle(\mathrm{true}, \mathrm{id} \ldots \mathrm{id}), \sigma\rangle$. □

Lemma 6.10. For every $a \in S$ and $\varphi \in \mathrm{Rat}^\infty(\langle\Delta, \Sigma\rangle, S)$ it holds that

$$a \odot (\mathcal{B}_\Delta; \varphi) = \mathcal{B}_\Delta; (a \odot \varphi).$$

Proof. Let $\xi \in \mathrm{T}_\Sigma$. Then

$$\begin{aligned}
(a \odot (\mathcal{B}_\Delta; \varphi))(\xi) = a \odot (\mathcal{B}_\Delta; \varphi)(\xi) &= a \odot \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} \varphi(\zeta) \\
&\overset{\star}{=} \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} a \odot \varphi(\zeta) = \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} (a \odot \varphi)(\zeta) = (\mathcal{B}_\Delta; a \odot \varphi)(\xi)
\end{aligned}$$

where in Equality $\star$ we have used the generalised distributivity law, Equation (2.4). □

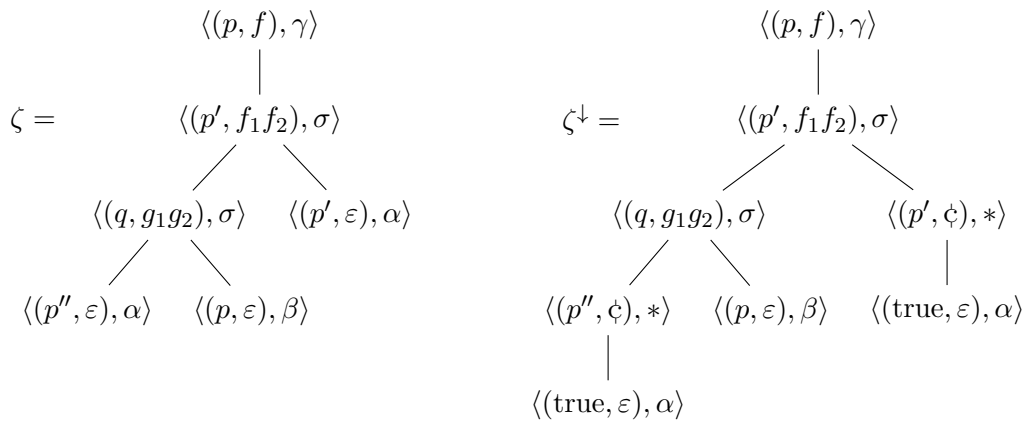


Figure 6.8: A tree ζ and its α-extension $\zeta^{\downarrow}$.

Lemma 6.11. For every $\varphi_1, \varphi_2 \in \mathrm{Rat}^{\infty}(\langle \Delta, \Sigma \rangle, S)$ it holds that

$$(\mathcal{B}_{\Delta}; \varphi_1) \oplus (\mathcal{B}_{\Delta}; \varphi_2) = \mathcal{B}_{\Delta}; (\varphi_1 \oplus \varphi_2).$$

Proof. Let $\xi \in \mathrm{T}_{\Sigma}$. Then

$$\begin{aligned}((\mathcal{B}_{\Delta}; \varphi_1) \oplus (\mathcal{B}_{\Delta}; \varphi_2))(\xi) &= \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)} \varphi_1(\zeta) \oplus \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)} \varphi_2(\zeta) \\ &= \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)} (\varphi_1 \oplus \varphi_2)(\zeta) = (\mathcal{B}_{\Delta}; (\varphi_1 \oplus \varphi_2))(\xi). \qquad \square\end{aligned}$$

6.5.2 α-Concatenation

In [56, Lemma 6.6] it was proved that $\mathrm{Reg}(\mathfrak{S}¢, \Sigma, S)$ is closed under α-concatenation. Thus, by the Kleene-Goldstine theorem (cf. Theorem 6.8), also $\mathrm{Rat}(\mathfrak{S}¢, \Sigma, S)$ is closed under α-concatenation. Next we prepare an alternative proof of this fact.

Throughout the rest of Chapter 6.5, we assume that $¢ \in F'$ and $\alpha \in \Sigma^{(0)}$.

For our closure proof, we would like to concatenate a tree ζ over $\langle \Delta, \Sigma \rangle$ at each leaf which contains α with another tree ζ'. But then the following problem occurs, cf. Figure 6.8. The tree ζ has two leaves which are labeled by $\langle (p'', \varepsilon), \alpha \rangle$ and $\langle (p', \varepsilon), \alpha \rangle$, respectively. Now if p' and p'' are different elements of $\mathrm{BC}(P)$, then there is no nullary symbol $\langle x, \alpha \rangle \in \langle \Delta, \Sigma \rangle$ such that the $\langle x, \alpha \rangle$-concatenation replaces both leaves by ζ'.

We solve this problem by transforming ζ into its α-extension $\zeta^{\downarrow}$ (cf. Figure 6.8) which involves the padding symbol $*$. Thereby we have homogenised the labels containing α and we can use $\zeta^{\downarrow} \circ_{\langle(\mathrm{true},\varepsilon),\alpha\rangle} \zeta$ for the desired concatenation.

Definition 6.12. Let $\zeta \in \mathrm{T}_{\langle\Delta,\Sigma\rangle}$. We define the *$\alpha$-extension of* ζ, denoted $\zeta^{\downarrow}$, as the tree obtained from ζ by replacing each occurrence of the leaf $\langle(p,\varepsilon),\alpha\rangle$, where $p \in \mathrm{BC}(P)$, by the tree $\langle(p,\dot{\mathrm{c}}),*\rangle(\langle(\mathrm{true},\varepsilon),\alpha\rangle)$. This defines an injective map $-^{\downarrow}\colon \mathrm{T}_{\langle\Delta,\Sigma\rangle} \to \mathrm{T}_{\langle\Delta,\Sigma\rangle}$. We extend the definition of $-^{\downarrow}$ to sets of trees in the natural way.

Given a weighted tree language $\varphi\colon \mathrm{T}_{\langle\Delta,\Sigma\rangle} \to S$, we define the *$\alpha$-extension of* φ as the weighted tree language $\varphi^{\downarrow}\colon \mathrm{T}_{\langle\Delta,\Sigma\rangle} \to S$, given for every $\zeta \in \mathrm{T}_{\langle\Delta,\Sigma\rangle}$ by

$$\varphi^{\downarrow}(\zeta) = \begin{cases} \varphi(\eta) & \text{if } \zeta = \eta^{\downarrow} \text{ for some } \eta \in \mathrm{T}_{\langle\Delta,\Sigma\rangle} \\ 0 & \text{otherwise.} \end{cases}$$ ◀

Lemma 6.13. Let $\varphi \in \mathrm{Rat}^{\infty}(\langle\Delta,\Sigma\rangle, S)$. It holds that $\varphi^{\downarrow} \in \mathrm{Rat}^{\infty}(\langle\Delta,\Sigma\rangle, S)$.

Proof. Let $\varphi \in \mathrm{Rat}^{\infty}(\langle\Delta,\Sigma\rangle, S)$ and let us consider a rational expression R which represents φ (cf. [38, Definition 3.17]). Let $\{p_1,\ldots,p_n\}$ be the set of all elements in $\mathrm{BC}(P) \setminus \{\mathrm{true}\}$ which occur in R. For each $p \in \{p_1,\ldots,p_n\} \cup \{\mathrm{true}\}$ we define

$$R_p = \langle(p,\dot{\mathrm{c}}),*\rangle(\langle(\mathrm{true},\varepsilon),\alpha\rangle) \ .$$

Now we define the rational expression R' as follows:

$$R' = (\cdots((R \circ_{\langle(\mathrm{true},\varepsilon),\alpha\rangle} R_{\mathrm{true}}) \circ_{\langle(p_1,\varepsilon),\alpha\rangle} R_{p_1})\cdots) \circ_{\langle(p_n,\varepsilon),\alpha\rangle} R_{p_n} \ .$$

We note that it is possible that true does not occur in R. In this case, the $\langle(\mathrm{true},\varepsilon),\alpha\rangle$-concatention has no effect in R'.

By standard arguments, we can prove that R' represents $\varphi^{\downarrow}$. Hence we have shown that $\varphi^{\downarrow} \in \mathrm{Rat}^{\infty}(\langle\Delta,\Sigma\rangle, S)$. □

Let Θ be a ranked alphabet (later, we will instantiate $\Theta = \Sigma$ or $\Theta = \langle\Delta,\Sigma\rangle$), $\xi \in \mathrm{T}_{\Theta}$, $\beta \in \Theta^{(0)}$, $n \in \mathbb{N}$, and $\widetilde{w} = (w_1,\ldots,w_n) \in \mathrm{pos}(\xi)^n$ such that $w_1 <_l \cdots <_l w_n$. We denote

$$\xi_{\beta}^{\widetilde{w},0} = \xi[\beta]_{\widetilde{w}} \quad \text{and} \quad \xi_{\beta}^{\widetilde{w},i} = \xi|_{w_i}$$

for every $i \in [n]$. Whenever β is clear from the context, for example if $\widetilde{w} \in \mathrm{cut}_\beta(\xi)$, we drop the annotation of β and simply write $\xi^{\widetilde{w},i}$. In particular, $\xi^{(),0} = \xi$.

Let $\xi \in \mathrm{T}_\Sigma$ and $\zeta \in \mathcal{B}_\Delta(\xi)$. We recall that each position $w' \in \mathrm{pos}(\zeta)$ corresponds to a unique $w \in \mathrm{pos}(\xi)$ (cf. Figure 6.3). Therefore, for every $\widetilde{w}' = (w'_1, \ldots, w'_n)$ in $\mathrm{cut}_{\langle(\mathrm{true},\varepsilon),\alpha\rangle}(\zeta)$, there is a unique tuple $\widetilde{w} = (w_1, \ldots, w_n)$ in $\mathrm{pos}(\xi)^n$ such that w'_i corresponds to w_i. We say that $\widetilde{w}$ *corresponds to* $\widetilde{w}'$. We note that $w_1 <_l \cdots <_l w_n$ and $\mathrm{proj}_2(\zeta^{\widetilde{w}',i}) = \xi_\alpha^{\widetilde{w},i}$ for every $0 \le i \le n$. However, $\widetilde{w}$ does not necessarily cover $\mathrm{pos}_\alpha(\xi)$. In fact, $\widetilde{w} \in \mathrm{cut}_\alpha(\xi)$ iff $\widetilde{w}'$ covers all positions in ζ which contain α (as opposed to just $\mathrm{pos}_{\langle(\mathrm{true},\varepsilon),\alpha\rangle}(\zeta)$).

Lemma 6.14. Let $\xi \in \mathrm{T}_\Sigma$. Consider the sets

$$\mathcal{D} = \{(\widetilde{w}, \zeta_0, \ldots, \zeta_{|\widetilde{w}|}) \mid \widetilde{w} \in \mathrm{cut}_\alpha(\xi), \forall (0 \le i \le |\widetilde{w}|) \colon \zeta_i \in \mathcal{B}_\Delta(\xi^{\widetilde{w},i})\},$$
$$\mathcal{I} = \{(\zeta, \widetilde{w}') \mid \zeta \in \mathcal{B}_\Delta(\xi), \widetilde{w}' \in \mathrm{cut}_{\langle(\mathrm{true},\varepsilon),\alpha\rangle}(\zeta)\},$$

and the map $\kappa \colon \mathcal{D} \to \mathcal{I}$ defined by

$$(\widetilde{w}, \zeta_0, \ldots, \zeta_{|\widetilde{w}|}) \mapsto (\zeta_0^{\downarrow}[\zeta_1, \ldots, \zeta_{|\widetilde{w}|}]_{\widetilde{w}'}, \widetilde{w}'),$$

where $\widetilde{w}'$ consists of the elements of $\mathrm{pos}_{\langle(\mathrm{true},\varepsilon),\alpha\rangle}(\zeta_0^{\downarrow})$ in lexicographic order. We visualise the ingredients of the map κ in Figure 6.9.

It holds that

1. κ is well defined,
2. κ is injective,
3. $\mathrm{im}(\kappa) = \{(\zeta, \widetilde{w}') \in \mathcal{I} \mid \widetilde{w} \in \mathrm{pos}(\xi)^{|\widetilde{w}'|}$, $\widetilde{w}$ corresponds to $\widetilde{w}'$,
$$\zeta^{\widetilde{w}',0} \in \mathrm{im}(-^{\downarrow}), \text{ and } \forall (0 \le i \le |\widetilde{w}'|) \colon \zeta^{\widetilde{w}',i} \in \mathcal{B}_\Delta(\xi_\alpha^{\widetilde{w},i})\}.$$

Proof. First, we prove 1. Let $(\widetilde{w}, \zeta_0, \ldots, \zeta_{|\widetilde{w}|}) \in \mathcal{D}$. We note that there are $|\widetilde{w}|$ occurrences of $\langle(\mathrm{true},\varepsilon),\alpha\rangle$ in $\zeta_0^{\downarrow}$, hence $|\widetilde{w}'| = |\widetilde{w}|$ and thus the substitution $\zeta_0^{\downarrow}[\zeta_1, \ldots, \zeta_{|\widetilde{w}|}]_{\widetilde{w}'}$ is legal. Let $\zeta = \zeta_0^{\downarrow}[\zeta_1, \ldots, \zeta_{|\widetilde{w}|}]_{\widetilde{w}'}$. In order to prove well-definedness, we still have to show that $(\zeta, \widetilde{w}') \in \mathcal{I}$. For every $0 \le i \le |\widetilde{w}|$, ζ_i is a $\xi^{\widetilde{w},i}$-extended behaviour by definition. The forced reset instructions in the α-extension of ζ_0 now ensure that

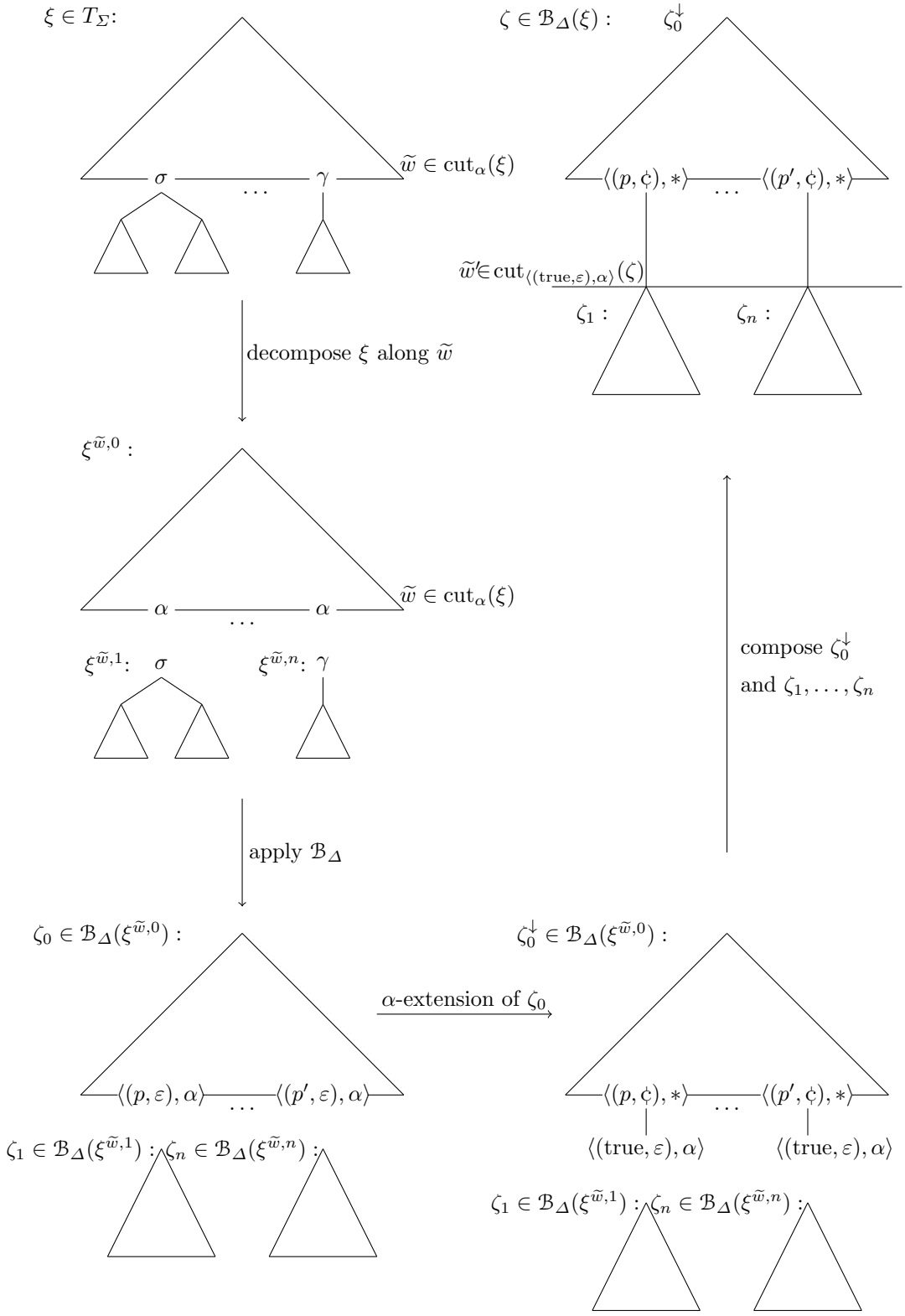


Figure 6.9: Illustration of $(\widetilde{w}, \zeta_0, \ldots, \zeta_n) \in \mathcal{D} \overset{\kappa}{\rightsquigarrow} (\zeta, \widetilde{w}') \in \mathcal{I}$, realising the map $\kappa\colon \mathcal{D} \to \mathcal{I}$.

$\mathrm{proj}_1(\zeta)$ is indeed a Δ-behaviour and hence, $\zeta \in \mathcal{B}_\Delta(\xi)$. Moreover, it surely holds that $\widetilde{w}' \in \mathrm{cut}_{\langle(\mathrm{true},\varepsilon),\alpha\rangle}(\zeta)$ because $\widetilde{w}'$ covers all positions of ζ which contain α. In fact, $\widetilde{w}$ corresponds to $\widetilde{w}'$.

Next, we prove 2. Let $d_1, d_2 \in \mathcal{D}$ with the components $d_1 = (\widetilde{w}, \zeta_0, \ldots, \zeta_{|\widetilde{w}|})$ and $d_2 = (\widetilde{v}, \eta_0, \ldots, \eta_{|\widetilde{v}|})$. Assume that $\kappa(d_1) = \kappa(d_2)$. The equality of the second components of $\kappa(d_1)$ and $\kappa(d_2)$ means that the first components of $\kappa(d_1)$ and $\kappa(d_2)$ are obtained from ζ_0 and η_0 respectively by replacing the same set of positions $\widetilde{w}'$ and hence also $|\widetilde{w}| = |\widetilde{v}|$. Together with the fact that

$$\zeta_0^{\downarrow}[\zeta_1, \ldots, \zeta_{|\widetilde{w}|}]_{\widetilde{w}'} = \mathrm{proj}_1(\kappa(d_1)) = \mathrm{proj}_1(\kappa(d_2)) = \eta_0^{\downarrow}[\eta_1, \ldots, \eta_{|\widetilde{v}|}]_{\widetilde{w}'},$$

we can deduce that $\zeta_0^{\downarrow} = \eta_0^{\downarrow}$ (and hence $\zeta_0 = \eta_0$) and $\zeta_i = \eta_i$ for every $i \in [|\widetilde{w}|]$. Moreover, $\widetilde{w}$ and $\widetilde{v}$ both correspond to the vector which consists of the positions in $\mathrm{pos}_{\langle(\mathrm{true},\varepsilon),\alpha\rangle}(\zeta_0^{\downarrow})$ in lexicographic order. This implies $\widetilde{w} = \widetilde{v}$. In total, we have seen that $d_1 = d_2$, which proves claim 2.

Next, we prove 3. Surely, every $(\zeta, \widetilde{w}') \in \mathrm{im}(\kappa)$ is an element of the set on the right hand side of claim 3. Conversely, let $(\zeta, \widetilde{w}')$ be an element of the set on the right hand side of claim 3. Moreover, let $\eta_0 \in \mathrm{T}_{\langle\Delta,\Sigma\rangle}$ such that $\eta_0^{\downarrow} = \zeta^{\widetilde{w}',0}$ and let $\widetilde{w} \in \mathrm{pos}(\xi)^{|\widetilde{w}'|}$ be the tuple corresponding to $\widetilde{w}'$. It surely holds that

$$\kappa(\widetilde{w}, \eta_0, \zeta^{\widetilde{w}',1}, \ldots, \zeta^{\widetilde{w}',|\widetilde{w}'|}) = (\zeta, \widetilde{w}').$$

In particular, $(\zeta, \widetilde{w}') \in \mathrm{im}(\kappa)$, which concludes our proof of claim 3. □

Lemma 6.15. For every $\varphi_1, \varphi_2 \in \mathrm{Rat}^\infty(\langle\Delta, \Sigma\rangle, S)$ it holds that

$$(\mathcal{B}_\Delta; \varphi_1) \circ_\alpha (\mathcal{B}_\Delta; \varphi_2) = \mathcal{B}_\Delta; (\varphi_1^{\downarrow} \circ_{\langle(\mathrm{true},\varepsilon),\alpha\rangle} \varphi_2).$$

Proof. Denote $\mathcal{I}$ and κ as in Lemma 6.14. Let $\xi \in \mathrm{T}_\Sigma$. It holds that

$$
\begin{aligned}
&((\mathcal{B}_\Delta; \varphi_1) \circ_\alpha (\mathcal{B}_\Delta; \varphi_2))(\xi) \\
&= \bigoplus_{\widetilde{w} \in \mathrm{cut}_\alpha(\xi)} (\mathcal{B}_\Delta; \varphi_1)(\xi^{\widetilde{w},0}) \odot \bigodot_{i=1}^{|\widetilde{w}|} (\mathcal{B}_\Delta; \varphi_2)(\xi^{\widetilde{w},i}) \\
&= \bigoplus_{\widetilde{w} \in \mathrm{cut}_\alpha(\xi)} \Big(\bigoplus_{\zeta_0 \in \mathcal{B}_\Delta(\xi^{\widetilde{w},0})} \varphi_1(\zeta_0) \Big) \odot \bigodot_{i=1}^{|\widetilde{w}|} \Big(\bigoplus_{\zeta_i \in \mathcal{B}_\Delta(\xi^{\widetilde{w},i})} \varphi_2(\zeta_i) \Big) \\
&\overset{\star_1}{=} \bigoplus_{\widetilde{w} \in \mathrm{cut}_\alpha(\xi)} \bigoplus_{\substack{\zeta_0 \in \mathcal{B}_\Delta(\xi^{\widetilde{w},0}), \\ \dots, \\ \zeta_{|\widetilde{w}|} \in \mathcal{B}_\Delta(\xi^{\widetilde{w},|\widetilde{w}|})}} \varphi_1^{\downarrow}(\zeta_0^{\downarrow}) \odot \bigodot_{i=1}^{|\widetilde{w}|} \varphi_2(\zeta_i) \\
&\overset{\star_2}{=} \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} \bigoplus_{\substack{\widetilde{w}' \in \mathrm{cut}_{\langle(\mathrm{true},\varepsilon),\alpha\rangle}(\zeta) \\ \text{s.th. } (\zeta,\widetilde{w}') \in \mathrm{im}(\kappa)}} \varphi_1^{\downarrow}(\zeta^{\widetilde{w}',0}) \odot \bigodot_{i=1}^{|\widetilde{w}'|} \varphi_2(\zeta^{\widetilde{w}',i}) \\
&\overset{\star_3}{=} \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} \bigoplus_{\widetilde{w}' \in \mathrm{cut}_{\langle(\mathrm{true},\varepsilon),\alpha\rangle}(\zeta)} \varphi_1^{\downarrow}(\zeta^{\widetilde{w}',0}) \odot \bigodot_{i=1}^{|\widetilde{w}'|} \varphi_2(\zeta^{\widetilde{w}',i}) \\
&= \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} (\varphi_1^{\downarrow} \circ_{\langle(\mathrm{true},\varepsilon),\alpha\rangle} \varphi_2)(\zeta) \;=\; (\mathcal{B}_\Delta; (\varphi_1^{\downarrow} \circ_{\langle(\mathrm{true},\varepsilon),\alpha\rangle} \varphi_2))(\xi) \;.
\end{aligned}
$$

In Equation $\star_1$ we apply the generalised distributivity law (Equation (2.4)) in S and the definition of α-extensions of weighted tree languages. In Equation $\star_2$ we use Lemma 6.14. Equation $\star_3$ is justified by the fact that for every $(\zeta, \widetilde{w}') \in \mathcal{I} \setminus \mathrm{im}(\kappa)$ it holds that

$$\varphi_1^{\downarrow}(\zeta^{\widetilde{w}',0}) \odot \bigodot_{i=1}^{|\widetilde{w}'|} \varphi_2(\zeta^{\widetilde{w}',i}) = 0.$$

This can be proven as follows. If $\zeta^{\widetilde{w}',0} \notin \mathrm{im}(-^{\downarrow})$, then $\varphi_1^{\downarrow}(\zeta^{\widetilde{w}',0}) = 0$. If $\zeta^{\widetilde{w}',0} \in \mathrm{im}(-^{\downarrow})$, then by Lemma 6.14(3), there exists $0 \leq i \leq |\widetilde{w}'|$ such that $\zeta^{\widetilde{w}',i} \notin \mathcal{B}_\Delta(\xi^{\widetilde{w},i})$ where $\widetilde{w} \in \mathrm{pos}(\xi)^{|\widetilde{w}'|}$ is the tuple corresponding to $\widetilde{w}'$. This cannot occur, as $\zeta \in \mathcal{B}_\Delta(\xi)$ is an extended behaviour. $\square$

Via the use of α-extension, we have homogenised the labels containing α. This solution has the price that the padding symbol $*$ was inserted into the tree. The question may arise whether the homogenisation problem can be solved without the use

of $*$. Let us briefly discuss a possible solution. Instead of pushing the α down into a new position, we can push the $p \in \mathrm{BC}(P)$ up to the parent of the current position. There we can replace the corresponding instruction f by a new instruction f' where $f'(c) = f(c)$ if $p(f(c)) = \mathrm{true}$, and undefined otherwise. Of course, this is only possible if F is closed under such a filtering with Boolean closures of predicates. For this reason we refrained from elaborating this solution.

6.5.3 α-Kleene Star

In [56, Lemma 6.7] it was proved that $\mathrm{Reg}(\mathfrak{S}, \Sigma, S)$ is closed under α-Kleene star. Thus, by the Kleene-Goldstine theorem (cf. Theorem 6.8), also $\mathrm{Rat}(\mathfrak{S}, \Sigma, S)$ is closed under Kleene-star. Next, we prepare an alternative proof of this fact.

We recall that $\alpha \in \Sigma^{(0)}$ is fixed.

Throughout the rest of Chapter 6.5, we assume that $\varphi \in \mathrm{Rat}^\infty(\langle \Delta, \Sigma \rangle, S)$ *and we abbreviate* $\langle (\mathrm{true}, \varepsilon), \alpha \rangle \in \langle \Delta, \Sigma \rangle^{(0)}$ *by* $\bar{\alpha}$.

Lemma 6.16. For every $n \in \mathbb{N}$, it holds that

$$(\mathcal{B}_\Delta; \varphi)^n_\alpha = \mathcal{B}_\Delta; (\varphi^\downarrow)^n_{\bar{\alpha}}. \tag{6.4}$$

Proof. The proof is done by induction on n. The case $n = 0$ is trivial as both sides in (6.4) are equal to the constant map 0.

Assume that Equation (6.4) holds for some $n \in \mathbb{N}$. It holds that

$$\begin{aligned}(\mathcal{B}_\Delta; \varphi)^{n+1}_\alpha &= (\mathcal{B}_\Delta; \varphi) \circ_\alpha (\mathcal{B}_\Delta; \varphi)^n_\alpha \oplus 1.\alpha \overset{\star_1}{=} (\mathcal{B}_\Delta; \varphi) \circ_\alpha (\mathcal{B}_\Delta; (\varphi^\downarrow)^n_{\bar{\alpha}}) \oplus \mathcal{B}_\Delta; (1.\bar{\alpha}) \\ &\overset{\star_2}{=} \mathcal{B}_\Delta; (\varphi^\downarrow \circ_{\bar{\alpha}} (\varphi^\downarrow)^n_{\bar{\alpha}} \oplus 1.\bar{\alpha}) = \mathcal{B}_\Delta; (\varphi^\downarrow)^{n+1}_{\bar{\alpha}}.\end{aligned}$$

In Equation $\star_1$, we use the induction hypothesis and the fact that $1.\alpha = \mathcal{B}_\Delta; (1.\bar{\alpha})$. In Equation $\star_2$ we use Lemmas 6.15 and 6.11. This concludes the induction step and the proof of the lemma. □

Let us assume that $\mathcal{B}_\Delta; \varphi$ is α-proper. We can hope for the equation

$$(\mathcal{B}_\Delta; \varphi)^*_\alpha = \mathcal{B}_\Delta; (\varphi^\downarrow)^*_{\bar{\alpha}}$$

to hold, yet (for every $\xi \in \mathrm{T}_\Sigma$) the left hand side satisfies

$$\begin{aligned}(\mathcal{B}_\Delta; \varphi)^*_\alpha(\xi) &= (\mathcal{B}_\Delta; \varphi)^{\mathrm{height}(\xi)+1}_\alpha(\xi) \\ &= (\mathcal{B}_\Delta; (\varphi^\downarrow)^{\mathrm{height}(\xi)+1}_{\bar\alpha})(\xi) = \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} (\varphi^\downarrow)^{\mathrm{height}(\xi)+1}_{\bar\alpha}(\zeta),\end{aligned} \tag{6.5}$$

whereas the right hand side satisfies

$$(\mathcal{B}_\Delta; (\varphi^\downarrow)^*_{\bar\alpha})(\xi) = \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} (\varphi^\downarrow)^*_{\bar\alpha}(\zeta) = \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} (\varphi^\downarrow)^{\mathrm{height}(\zeta)+1}_{\bar\alpha}(\zeta) \ . \tag{6.6}$$

Note that in Equation (6.5) we use the same $\bar\alpha$-power of $\varphi^\downarrow$ for every ζ, whereas in Equation (6.6) the $\bar\alpha$-power of $\varphi^\downarrow$ depends on ζ. As $\mathcal{B}_\Delta(\xi)$ is an infinite set, there is no upper bound for $\mathrm{height}(\zeta) + 1$ that only depends on ξ. Hence, we can not simply apply Lemma 6.16 to prove closure under α-Kleene star.

The weighted tree language $(\varphi^\downarrow)^{\mathrm{height}(\xi)+1}_{\bar\alpha}$ cuts every input ζ into $\mathrm{height}(\xi) + 1$ "stripes" and measures each stripe separately with $\varphi^\downarrow$, whereas $(\varphi^\downarrow)^{\mathrm{height}(\zeta)+1}_{\bar\alpha}$ cuts ζ into $\mathrm{height}(\zeta) + 1$ stripes. Unfortunately, cutting off only $*$-positions from ζ does not yield the weight 0 in $\varphi^\downarrow$. That is, we cannot prove $(\varphi^\downarrow)^{\mathrm{height}(\xi)+1}_{\bar\alpha}(\zeta) = (\varphi^\downarrow)^{\mathrm{height}(\zeta)+1}_{\bar\alpha}(\zeta)$. However, we can use the fact that

$$0 = (\mathcal{B}_\Delta; \varphi)(\alpha) = \sum_{\zeta \in \mathcal{B}_\Delta(\alpha)} \varphi(\zeta) = \sum_{\zeta \in \mathcal{B}_\Delta(\alpha)^\downarrow} \varphi^\downarrow(\zeta) \tag{6.7}$$

(by α-properness of $(\mathcal{B}_\Delta; \varphi)$) to show that cutting off only $*$-positions from ζ does indeed give us weight 0 after summing over *all* α-extended behaviours.

To make these thoughts precise, we need some more notation.

Definition 6.17. Let $\xi \in \mathrm{T}_\Sigma$. We define the map $\mathrm{shift}\colon \mathcal{B}_\Delta(\xi) \to \mathbb{N}$ which counts the number of consecutive $*$-positions at the root of ξ-extended behaviours inductively by

$$\begin{aligned}\mathrm{shift}(\langle \delta, * \rangle(\zeta_1)) &= 1 + \mathrm{shift}(\zeta_1) \quad \text{and} \\ \mathrm{shift}(\langle \delta, \sigma \rangle(\zeta_1, \ldots, \zeta_{\mathrm{rk}(\sigma)})) &= 0\end{aligned}$$

for every $\delta \in \Delta$ and $\sigma \in \Sigma$.

Moreover, let $k \in \mathbb{N}$ and $\widetilde{w}' \in \bigcup_{n\in\mathbb{N}}(\mathbb{N}^*)^n$. We define the sets

$$\begin{aligned}
\mathcal{B}_\Delta(\xi)^{=k} &= \{\zeta \in \mathcal{B}_\Delta(\xi) \mid \mathrm{shift}(\zeta) = k\},\\
\mathcal{B}_\Delta^{\widetilde{w}'}(\xi)^{=k} &= \{\zeta \in \mathcal{B}_\Delta(\xi)^{=k} \mid \widetilde{w}' \in \mathrm{cut}_{\bar{\alpha}}(\zeta)\},\\
\mathcal{B}_\Delta^{\widetilde{w}',0}(\xi)^{=k} &= \{\zeta^{\widetilde{w}',0} \mid \zeta \in \mathcal{B}_\Delta^{\widetilde{w}'}(\xi)^{=k}\}.
\end{aligned}$$

Furthermore we call $\widetilde{w}'$ *k-extended* if for every component w_i' of $\widetilde{w}'$ there exists an element $v_i \in \mathbb{N}^* \setminus \{\varepsilon\}$ such that $w_i' = 1\cdots 1 v_i$ with k occurrences of 1. The set of k-extended tuples $\widetilde{w}'$ is denoted by k-Ext. For every $0 \le \ell \le k$, we define $\overline{\boldsymbol{k-\ell}} = (1\cdots 1)$ with $k-\ell$ occurrences of 1. In other words, $\overline{\boldsymbol{k-\ell}}$ is a vector with one component, and its only component is the position $1\cdots 1$ with $k-\ell$ occurrences of 1. ◀

Lemma 6.18. Let ξ, k, and $\widetilde{w}'$ as in Definition 6.17.

1. For every $0 \le \ell \le k$, it holds that

$$\{\zeta \in \mathcal{B}_\Delta^{\overline{\boldsymbol{k-\ell}}}(\xi)^{=k} \mid \zeta^{\overline{\boldsymbol{k-\ell}},0} \in \mathrm{im}(-^{\downarrow})\} \cong \big(\mathcal{B}_\Delta(\alpha)^{=k-\ell} \cap \mathrm{im}(-^{\downarrow})\big) \times \mathcal{B}_\Delta(\xi)^{=\ell} \tag{6.8}$$

2. It holds that

$$\{\zeta \in \mathcal{B}_\Delta^{\widetilde{w}'}(\xi)^{=k} \mid \zeta^{\widetilde{w}',0} \in \mathrm{im}(-^{\downarrow})\} \cong \big(\mathcal{B}_\Delta^{\widetilde{w}',0}(\xi)^{=k} \cap \mathrm{im}(-^{\downarrow})\big) \times \mathcal{B}_\Delta(\xi_\alpha^{\widetilde{w},1}) \times \ldots \times \mathcal{B}_\Delta(\xi_\alpha^{\widetilde{w},|\widetilde{w}|}), \tag{6.9}$$

   where $\widetilde{w}$ corresponds to[1] $\widetilde{w}'$.

Proof. First we prove claim 2. In fact, we show that the map f defined by

$$\zeta \overset{f}{\mapsto} (\zeta^{\widetilde{w}',0}, \ldots, \zeta^{\widetilde{w}',|\widetilde{w}'|})$$

is a bijection from the left hand side (lhs) to the right hand side (rhs) of Equation (6.9). First, note that f is indeed a well-defined map of type "lhs $\to$ im(f)". Moreover, f is clearly injective. It remains to show that im(f) = rhs.

[1] We recall that $\widetilde{w}$ need not be an element of $\mathrm{cut}_\alpha(\xi)$, because $\widetilde{w}'$ does not necessarily cut off all positions of ζ which contain α.

Let $(\zeta_0, \ldots, \zeta_{\widetilde{w}})$ be an element of rhs. Since $\widetilde{w}$ corresponds to $\widetilde{w}'$, we know that $|\widetilde{w}| = |\widetilde{w}'|$. Therefore, the substitution $\zeta_0[\zeta_1, \ldots, \zeta_{\widetilde{w}}]_{\widetilde{w}'}$ exists and is in lhs since $\zeta_0 \in \mathrm{im}(-^{\downarrow})$. Moreover, $f(\zeta_0[\zeta_1, \ldots, \zeta_{\widetilde{w}}]_{\widetilde{w}'}) = (\zeta_0, \ldots, \zeta_{\widetilde{w}})$, whence rhs $\subseteq \mathrm{im}(f)$.

Next, let ζ be in lhs. By assumption we have that $\zeta^{\widetilde{w}',0} \in \big(\mathcal{B}_{\Delta}^{\widetilde{w}',0}(\xi)^{=k} \cap \mathrm{im}(-^{\downarrow})\big)$. We also have $\zeta^{\widetilde{w}',i} \in \mathcal{B}_{\Delta}(\xi_{\alpha}^{\widetilde{w},i})$ for every $1 \leq i \leq |\widetilde{w}'|$. This follows from the fact that $\zeta \in \mathcal{B}_{\Delta}(\xi)$ and $\widetilde{w}$ corresponds to $\widetilde{w}'$. This concludes the proof of claim 2.

The proof of claim 1 is very similar. For this, we note that (ε) corresponds to $\overline{\boldsymbol{k} - \boldsymbol{\ell}}$ and it decomposes ξ into α and ξ. $\square$

Lemma 6.19. Let $\mathcal{B}_{\Delta}; \varphi$ be α-proper. For every $\xi \in \mathrm{T}_{\Sigma}$ it holds that

$$\bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)} (\varphi^{\downarrow})_{\bar{\alpha}}^{\mathrm{height}(\xi)+1}(\zeta) = \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)} (\varphi^{\downarrow})_{\bar{\alpha}}^{\mathrm{height}(\zeta)+1}(\zeta).$$

Proof. The proof is by structural induction on ξ. Let $n \colon \mathcal{B}_{\Delta}(\xi) \to \mathbb{N}$ be an arbitrary map (later, it will be instantiated by $\zeta \mapsto \mathrm{height}(\zeta)$ and $\zeta \mapsto \mathrm{height}(\xi)$).

Case $\xi = \alpha$: We have that

$$\begin{aligned}
&\bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)} (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)+1}(\zeta) \\
&= \bigoplus_{k \in \mathbb{N}} \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)^{=k}} (\varphi^{\downarrow} \circ_{\bar{\alpha}} (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)} \oplus 1.\bar{\alpha})(\zeta) \\
&= \bigoplus_{k \in \mathbb{N}} \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)^{=k}} (\varphi^{\downarrow} \circ_{\bar{\alpha}} (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)})(\zeta) \oplus \bigoplus_{k \in \mathbb{N}} \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)^{=k}} (1.\bar{\alpha})(\zeta) \\
&\overset{\star_1}{=} 1 \oplus \bigoplus_{k \in \mathbb{N}} \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)^{=k}} \bigoplus_{\widetilde{w}' \in \mathrm{cut}_{\bar{\alpha}}(\zeta)} \varphi^{\downarrow}(\zeta^{\widetilde{w}',0}) \odot (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)}(\zeta^{\widetilde{w}',1}) \\
&\overset{\star_2}{=} 1 \oplus \bigoplus_{k \in \mathbb{N}} \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)^{=k}} \bigoplus_{\ell=0}^{k} \varphi^{\downarrow}(\zeta^{\overline{\boldsymbol{k}-\boldsymbol{\ell}},0}) \odot (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)}(\zeta^{\overline{\boldsymbol{k}-\boldsymbol{\ell}},1}) \\
&\overset{\star_3}{=} 1 \oplus \bigoplus_{\ell \in \mathbb{N}} \bigoplus_{k \geq \ell} \bigoplus_{\zeta \in \mathcal{B}_{\Delta}^{\overline{\boldsymbol{k}-\boldsymbol{\ell}}}(\xi)^{=k}} \varphi^{\downarrow}(\zeta^{\overline{\boldsymbol{k}-\boldsymbol{\ell}},0}) \odot (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)}(\zeta^{\overline{\boldsymbol{k}-\boldsymbol{\ell}},1}) \\
&\overset{\star_4}{=} 1 \oplus \bigoplus_{\ell \in \mathbb{N}} \bigoplus_{k \geq \ell} \bigoplus_{\substack{\zeta \in \mathcal{B}_{\Delta}^{\overline{\boldsymbol{k}-\boldsymbol{\ell}}}(\xi)^{=k}, \\ \zeta^{\overline{\boldsymbol{k}-\boldsymbol{\ell}},0} \in \mathrm{im}(-^{\downarrow})}} \varphi^{\downarrow}(\zeta^{\overline{\boldsymbol{k}-\boldsymbol{\ell}},0}) \odot (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)}(\zeta^{\overline{\boldsymbol{k}-\boldsymbol{\ell}},1}) \\
&\overset{\star_5}{=} 1 \oplus \bigoplus_{\ell \in \mathbb{N}} \bigoplus_{k \geq \ell} \Big(\bigoplus_{\zeta_0 \in \mathcal{B}_{\Delta}(\alpha)^{=k-\ell} \cap \mathrm{im}(-^{\downarrow})} \varphi^{\downarrow}(\zeta_0) \Big) \odot \Big(\bigoplus_{\zeta_1 \in \mathcal{B}_{\Delta}(\xi)^{=\ell}} (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)}(\zeta_1) \Big).
\end{aligned}$$

In Equation $\star_1$ we use the fact that $(1.\bar{\alpha})(\zeta) = 0$ whenever $\zeta \neq \bar{\alpha}$. Equation $\star_2$ is justified by the fact that the set of $\bar{\alpha}$-cuts of ζ are exactly the $\overline{\boldsymbol{k-\ell}}$ for $k \in \mathbb{N}$ and $0 \leq \ell \leq k$. In Equation $\star_3$ we first swap the sum for ζ with the sum for ℓ and then swap the sum for k with the sum for ℓ (which is possible due to Equation (2.3)). Equation $\star_4$ holds because $\varphi^{\downarrow}$ vanishes if $\zeta^{\overline{\boldsymbol{k-\ell}},0}$ is not in $\mathrm{im}(-^{\downarrow})$. Finally, in Equation $\star_5$ we use Lemma 6.18(1) and the distributivity in S. Moreover, for every $\ell \in \mathbb{N}$ we have that

$$\bigoplus_{k \geq \ell} \Big(\bigoplus_{\zeta_0 \in \mathcal{B}_\Delta(\alpha)^{=k-\ell} \cap \mathrm{im}(-^{\downarrow})} \varphi^{\downarrow}(\zeta_0) \Big) = \bigoplus_{\zeta_0 \in \mathcal{B}_\Delta(\alpha)^{\downarrow}} \varphi^{\downarrow}(\zeta_0),$$

which vanishes by Equation (6.7). In total, we obtain that $\bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)+1}(\zeta) = 1$. As n was arbitrary, this proves the claim.

Case $\xi = \beta$ for $\beta \in \Sigma^{(0)} \setminus \{\alpha\}$: The proof of this case is analogous to the case $\xi = \alpha$ except for the following parts. In Equation $\star_1$ we use that $(1.\bar{\alpha})(\zeta) = 0$ for every ζ and hence we lose the "$1\oplus$" in all subsequent equations. On the right hand side of Equation $\star_2$, we need to consider the remaining case $\widetilde{w}' = ()$, that is, we do not cut ζ at all. Therefore, we get the additional term $\bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} \varphi^{\downarrow}(\zeta) = (\mathcal{B}_\Delta; \varphi^{\downarrow})(\beta)$ on the right hand side of $\star_2$ and all the subsequent equations.

Case $\xi = \sigma(\xi_1, \ldots, \xi_s)$ for $s \geq 1, \sigma \in \Sigma^{(s)}$: Then

$$\begin{aligned}
&\bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)+1}(\zeta) \\
&\overset{\blacklozenge_1}{=} \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)} \bigoplus_{\widetilde{w}' \in \mathrm{cut}_{\bar{\alpha}}(\zeta)} \varphi^{\downarrow}(\zeta^{\widetilde{w}',0}) \odot \bigodot_{i=1}^{|\widetilde{w}'|} (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)}(\zeta^{\widetilde{w}',i}) \\
&\overset{\blacklozenge_2}{=} \bigoplus_{k \in \mathbb{N}} \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)^{=k}} \bigoplus_{\ell=0}^{k} \varphi^{\downarrow}(\zeta^{\overline{\boldsymbol{k-\ell}},0}) \odot (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)}(\zeta^{\overline{\boldsymbol{k-\ell}},1}) \\
&\quad \oplus \bigoplus_{k \in \mathbb{N}} \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)^{=k}} \bigoplus_{\widetilde{w}' \in \mathrm{cut}_{\bar{\alpha}}(\zeta) \cap k\text{-Ext}} \varphi^{\downarrow}(\zeta^{\widetilde{w}',0}) \odot \bigodot_{i=1}^{|\widetilde{w}'|} (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)}(\zeta^{\widetilde{w}',i}) \\
&\overset{\blacklozenge_3}{=} 0 \oplus \bigoplus_{k \in \mathbb{N}} \bigoplus_{\zeta \in \mathcal{B}_\Delta(\xi)^{=k}} \bigoplus_{\widetilde{w}' \in \mathrm{cut}_{\bar{\alpha}}(\zeta) \cap k\text{-Ext}} \varphi^{\downarrow}(\zeta^{\widetilde{w}',0}) \odot \bigodot_{i=1}^{|\widetilde{w}'|} (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)}(\zeta^{\widetilde{w}',i}) \\
&\overset{\blacklozenge_4}{=} \bigoplus_{k \in \mathbb{N}} \bigoplus_{\widetilde{w}' \in k\text{-Ext}} \bigoplus_{\substack{\zeta \in \mathcal{B}_\Delta^{\widetilde{w}'}(\xi)^{=k}, \\ \zeta^{\widetilde{w}',0} \in \mathrm{im}(-^{\downarrow})}} \varphi^{\downarrow}(\zeta^{\widetilde{w}',0}) \odot \bigodot_{i=1}^{|\widetilde{w}'|} (\varphi^{\downarrow})_{\bar{\alpha}}^{n(\zeta)}(\zeta^{\widetilde{w}',i}).
\end{aligned}$$

In Equation $\blacklozenge_1$ we use the definition of $(\varphi^{\downarrow})^{n(\zeta)+1}_{\bar{\alpha}}$ and the definition of $\circ_{\bar{\alpha}}$. We observe that $(1.\bar{\alpha})(\zeta)$ vanishes as ζ can never be $\bar{\alpha}$ in this case. Equation $\blacklozenge_2$ is justified by the fact that the $\bar{\alpha}$-cuts of $\zeta \in \mathcal{B}_{\Delta}(\xi)^{=k}$ are either of the form $\overline{\boldsymbol{k-\ell}}$ for some $0 \leq \ell \leq k$ or in k-Ext. We note that the empty $\bar{\alpha}$-cut $\widetilde{w} = ()$ of ζ is in k-Ext. In Equation $\blacklozenge_3$ we argue analogously to the cases $\xi = \alpha$ and $\xi = \beta$ to see that the sum for the case $\widetilde{w}' = \overline{\boldsymbol{k-\ell}}$ vanishes. More precisely, we first swap the sums for ζ, k, and ℓ, and then use Lemma 6.18(1) and Equation (6.7):

$$
\begin{aligned}
&\bigoplus_{k\in\mathbb{N}} \bigoplus_{\zeta\in\mathcal{B}_{\Delta}(\xi)^{=k}} \bigoplus_{\ell=0}^{k} \varphi^{\downarrow}(\zeta^{\overline{\boldsymbol{k-\ell}},0}) \odot (\varphi^{\downarrow})^{n(\zeta)}_{\bar{\alpha}}(\zeta^{\overline{\boldsymbol{k-\ell}},1}) \\
&\qquad = \bigoplus_{\ell\in\mathbb{N}} \bigoplus_{k\geq\ell} \bigoplus_{\zeta\in\mathcal{B}_{\Delta}(\xi)^{=k}} \varphi^{\downarrow}(\zeta^{\overline{\boldsymbol{k-\ell}},0}) \odot (\varphi^{\downarrow})^{n(\zeta)}_{\bar{\alpha}}(\zeta^{\overline{\boldsymbol{k-\ell}},1}) \\
&\qquad \overset{(6.8)}{=} \bigoplus_{\ell\in\mathbb{N}} \bigoplus_{k\geq\ell} \bigoplus_{\substack{\zeta_0\in\mathcal{B}_{\Delta}(\alpha)^{=k-\ell}\cap\mathrm{im}(-^{\downarrow}),\\ \zeta_1\in\mathcal{B}_{\Delta}(\xi)^{=\ell}}} \varphi^{\downarrow}(\zeta_0) \odot (\varphi^{\downarrow})^{n(\zeta)}_{\bar{\alpha}}(\zeta_1) \\
&\qquad = \bigoplus_{\ell\in\mathbb{N}} \Big(\bigoplus_{\zeta_0\in\mathcal{B}_{\Delta}(\alpha)^{\downarrow}} \varphi^{\downarrow}(\zeta_0) \Big) \odot \Big(\bigoplus_{\zeta_1\in\mathcal{B}_{\Delta}(\xi)^{=\ell}} (\varphi^{\downarrow})^{n(\zeta)}_{\bar{\alpha}}(\zeta_1) \Big) \overset{(6.7)}{=} 0
\end{aligned}
$$

In Equation $\blacklozenge_4$ we first swap the sums for $\widetilde{w}'$ and ζ and then restrict $\zeta^{\widetilde{w}',0}$ to be in the image of $-^{\downarrow}$, as otherwise $\varphi^{\downarrow}(\zeta^{\widetilde{w}',0}) = 0$.

Now we instantiate n with $\zeta \mapsto \mathrm{height}(\zeta)$ and reduce a subterm of the right hand side of $\blacklozenge_4$ to show the claim.

$$
\begin{aligned}
&\bigoplus_{\substack{\zeta\in\mathcal{B}^{\widetilde{w}'}_{\Delta}(\xi)^{=k},\\ \zeta^{\widetilde{w}',0}\in\mathrm{im}(-^{\downarrow})}} \varphi^{\downarrow}(\zeta^{\widetilde{w}',0}) \odot \bigodot_{i=1}^{|\widetilde{w}'|} (\varphi^{\downarrow})^{\mathrm{height}(\zeta)}_{\bar{\alpha}}(\zeta^{\widetilde{w}',i}) \\
&\overset{\blacklozenge_5}{=} \bigoplus_{\zeta_0\in\mathcal{B}^{\widetilde{w}',0}_{\Delta}(\xi)^{=k}\cap\mathrm{im}(-^{\downarrow})} \bigoplus_{\substack{\zeta_1\in\mathcal{B}_{\Delta}(\xi^{\widetilde{w},1}),\\ \dots,\\ \zeta_{|\widetilde{w}|}\in\mathcal{B}_{\Delta}(\xi^{\widetilde{w},|\widetilde{w}|})}} \varphi^{\downarrow}(\zeta_0) \odot \bigodot_{i=1}^{|\widetilde{w}|} (\varphi^{\downarrow})^{\mathrm{height}(\zeta)}_{\bar{\alpha}}(\zeta_i) \\
&\overset{\blacklozenge_6}{=} \bigoplus_{\zeta_0\in\mathcal{B}^{\widetilde{w}',0}_{\Delta}(\xi)^{=k}\cap\mathrm{im}(-^{\downarrow})} \varphi^{\downarrow}(\zeta_0) \odot \bigodot_{i=1}^{|\widetilde{w}|} \Big(\bigoplus_{\zeta_i\in\mathcal{B}_{\Delta}(\xi^{\widetilde{w},i})} (\varphi^{\downarrow})^{\mathrm{height}(\zeta_i)+1}_{\bar{\alpha}}(\zeta_i) \Big),
\end{aligned}
$$

where on the right hand side of Equations $\blacklozenge_5$ and $\blacklozenge_6$, $\widetilde{w}$ corresponds to $\widetilde{w}'$. In Equation $\blacklozenge_5$ we apply Lemma 6.18(2). In Equation $\blacklozenge_6$ we use the generalised distributivity law

in S, Equation (2.4), as the $\zeta_1, \ldots, \zeta_{|\widetilde{w}|}$ are independent of each other (once ζ_0 is fixed). Moreover, we use the fact that $\mathrm{height}(\zeta) \geq \mathrm{height}(\zeta_i) + 1$, whence by [38, Lemma 3.10] we have that $(\varphi^{\downarrow})_{\bar{\alpha}}^{\mathrm{height}(\zeta)}(\zeta_i) = (\varphi^{\downarrow})_{\bar{\alpha}}^{\mathrm{height}(\zeta_i)+1}(\zeta_i)$.

We now apply the induction hypothesis to obtain

$$\begin{aligned}
\bigoplus_{\zeta_i \in \mathcal{B}_{\Delta}(\xi^{\widetilde{w},i})} (\varphi^{\downarrow})_{\bar{\alpha}}^{\mathrm{height}(\zeta_i)+1}(\zeta_i) &\overset{\mathrm{IH}}{=} \bigoplus_{\zeta_i \in \mathcal{B}_{\Delta}(\xi^{\widetilde{w},i})} (\varphi^{\downarrow})_{\bar{\alpha}}^{\mathrm{height}(\xi^{\widetilde{w},i})+1}(\zeta_i) \\
&= (\mathcal{B}_{\Delta}; (\varphi^{\downarrow})_{\bar{\alpha}}^{\mathrm{height}(\xi^{\widetilde{w},i})+1})(\xi^{\widetilde{w},i}) \overset{\blacklozenge_7}{=} (\mathcal{B}_{\Delta}; \varphi^{\downarrow})_{\alpha}^{\mathrm{height}(\xi^{i})+1}(\xi^{\widetilde{w},i}) \\
&\overset{\blacklozenge_8}{=} (\mathcal{B}_{\Delta}; \varphi^{\downarrow})_{\alpha}^{\mathrm{height}(\xi)}(\xi^{\widetilde{w},i}) = \bigoplus_{\zeta_i \in \mathcal{B}_{\Delta}(\xi^{i})} (\varphi^{\downarrow})_{\bar{\alpha}}^{\mathrm{height}(\xi)}(\zeta_i),
\end{aligned}$$

where in Equation $\blacklozenge_7$ we use Lemma 6.16 and Equation $\blacklozenge_8$ we apply [38, Lemma 3.10] to $\mathrm{height}(\xi) \geq \mathrm{height}(\xi^{\widetilde{w},i}) + 1$.

Now, starting on the right hand side of the very last equation, we can trace back all the steps up to the right hand side of Equation $\blacklozenge_4$ by always choosing $\mathrm{height}(\xi)$ as power; then we end up at the right hand side of Equation $\blacklozenge_4$ with n instantiated by $\zeta \mapsto \mathrm{height}(\xi)$. This completes the proof of both, this case and the lemma. □

Lemma 6.20. Let $(\mathcal{B}_{\Delta}; \varphi)$ be α-proper. It holds that

$$(\mathcal{B}_{\Delta}; \varphi)_{\alpha}^{*} = \mathcal{B}_{\Delta}; (\varphi^{\downarrow})_{\bar{\alpha}}^{*}. \tag{6.10}$$

Proof. Let $\xi \in \mathrm{T}_{\Sigma}$. We have

$$\begin{aligned}
(\mathcal{B}_{\Delta}; \varphi)_{\alpha}^{*}(\xi) &= (\mathcal{B}_{\Delta}; \varphi)_{\alpha}^{\mathrm{height}(\xi)+1}(\xi) \overset{\star_1}{=} (\mathcal{B}_{\Delta}; (\varphi^{\downarrow})_{\bar{\alpha}}^{\mathrm{height}(\xi)+1})(\xi) \\
&= \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)} (\varphi^{\downarrow})_{\bar{\alpha}}^{\mathrm{height}(\xi)+1}(\zeta) \overset{\star_2}{=} \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)} (\varphi^{\downarrow})_{\bar{\alpha}}^{\mathrm{height}(\zeta)+1}(\zeta) \\
&= \bigoplus_{\zeta \in \mathcal{B}_{\Delta}(\xi)} (\varphi^{\downarrow})_{\bar{\alpha}}^{*}(\zeta) = (\mathcal{B}_{\Delta}; (\varphi^{\downarrow})_{\bar{\alpha}}^{*})(\xi).
\end{aligned}$$

In Equations $\star_1$ and $\star_2$ we use Lemma 6.16 and 6.19, respectively. □

We now collect our previous results in order to prove the closure of rational weighted tree languages with storage under the rational operations.

Theorem 6.21. Let S be an arbitrary complete semiring. Then $\mathrm{Rat}(\mathfrak{S}, \Sigma, S)$ is closed under top-concatenation, multiplication with a scalar, and sum. Moreover, $\mathrm{Rat}(\mathfrak{S}¢, \Sigma, S)$ is closed under α-concatenation and α-Kleene star.

Proof. *Top-concatenation:* Let $s \in \mathbb{N}$, $\sigma \in \Sigma^{(s)}$, and $\varphi_1, \ldots, \varphi_s \in \mathrm{Rat}(\mathfrak{S}, \Sigma, S)$. Thus, for every $i \in [s]$, there are a finite subsets $P_i \subseteq P$ and $F_i \subseteq F$ and a weighted tree language $\psi_i \in \mathrm{Rat}^\infty(\langle \Delta_i, \Sigma\rangle, S)$ such that $\varphi_i = \mathcal{B}_\Delta; \psi_i$, where Δ_i is the ranked alphabet corresponding to Σ, P_i, and F_i.

Now let $\hat{P} = \bigcup_{i=1}^s P_i$ and $\hat{F} = \bigcup_{i=1}^s F_i \cup \{\mathrm{id}\}$ and $\hat{\Delta}$ be the ranked alphabet corresponding to Σ, $\hat{P}$, and $\hat{F}$. By Remark 6.4, $\psi_i \in \mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ for every $i \in [s]$. Moreover, $\langle(\mathrm{true}, \mathrm{id} \cdots \mathrm{id}), \sigma\rangle \in \langle \hat{\Delta}, \Sigma\rangle$. Since $\mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ is closed under top-concatenation, we have that $\mathrm{top}_{\langle(\mathrm{true},\mathrm{id}\cdots\mathrm{id}),\sigma\rangle}(\psi_1, \ldots, \psi_s) \in \mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$. Hence, we obtain that $\mathrm{top}_\sigma(\varphi_1, \ldots, \varphi_k) \in \mathrm{Rat}(\mathfrak{S}, \Sigma, S)$ by Lemma 6.9.

Multiplication with a scalar: The proof easily follows from Lemma 6.10 and the fact that $\mathrm{Rat}^\infty(\langle \Delta, \Sigma\rangle, S)$ is closed under multiplication with scalar.

Sum: Let $\varphi_1, \varphi_2 \in \mathrm{Rat}(\mathfrak{S}, \Sigma, S)$. For every $i \in \{1, 2\}$, there exist finite subsets $P_i \subseteq P$ and $F_i \subseteq F$ and a weighted tree language $\psi_i \in \mathrm{Rat}^\infty(\langle \Delta_i, \Sigma\rangle, S)$ such that $\varphi_i = \mathcal{B}_{\Delta_i}; \psi_i$, where Δ_i is the ranked alphabet corresponding to Σ, P_i, and F_i.

Now let $\hat{P} = P_1 \cup P_2$, $\hat{F} = F_1 \cup F_2$, and $\hat{\Delta}$ be the ranked alphabet corresponding to Σ, $\hat{P}$, and $\hat{F}$. By Remark 6.4 we have $\psi_i \in \mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ for $i \in \{1, 2\}$ and thus $\psi_1 \oplus \psi_2 \in \mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ because $\mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ is closed under sum. Then by Lemma 6.11 we obtain that $\varphi_1 \oplus \varphi_2 \in \mathrm{Rat}(\mathfrak{S}, \Sigma, S)$.

α-concatenation: Let $\varphi_1, \varphi_2 \in \mathrm{Rat}(\mathfrak{S}¢, \Sigma, S)$. For $i \in \{1, 2\}$, let $P_i \subseteq P$, $F_i \subseteq F¢$, Δ_i, and ψ_i be the same as in the case of sum. Let $\hat{P} = P_1 \cup P_2$, $\hat{F} = F_1 \cup F_2 \cup \{¢\}$, and $\hat{\Delta}$ be the ranked alphabet corresponding to Σ, $\hat{P}$, and $\hat{F}$. By Remark 6.4 we have $\psi_i \in \mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ for every $i \in \{1, 2\}$. Moreover, $\psi_1^\downarrow \in \mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ by Lemma 6.13 and thus $\psi_1^\downarrow \circ_{\bar{\alpha}} \psi_2 \in \mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ because $\bar{\alpha} \in \langle \hat{\Delta}, \Sigma\rangle$ and $\mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ is closed under $\bar{\alpha}$-concatenation. By Lemma 6.15 we obtain that $\varphi_1 \circ_\alpha \varphi_2 = \mathcal{B}_{\hat{\Delta}}; (\psi_1^\downarrow \circ_{\bar{\alpha}} \psi_2)$ and hence $\varphi_1 \circ_\alpha \varphi_2 \in \mathrm{Rat}(\mathfrak{S}¢, \Sigma, S)$.

α-Kleene star: Let $\varphi \in \mathrm{Rat}(\mathfrak{S}¢, \Sigma, S)$ be α-proper. There exist finite subsets $\hat{P} \subseteq P$ and $\hat{F} \subseteq F¢$ and a weighted language $\psi \in \mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ such that $\varphi = \mathcal{B}_\Delta; \psi$, where $\hat{\Delta}$ is the ranked alphabet corresponding to Σ, $\hat{P}$, and $\hat{F}$. We may assume without loss of generality that $¢ \in \hat{F}$. Due to Lemma 6.13, we have that $\psi^\downarrow \in \mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ and thus $(\psi^\downarrow)^*_{\bar{\alpha}} \in \mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ because $\bar{\alpha} \in \langle \hat{\Delta}, \Sigma\rangle$ and $\mathrm{Rat}^\infty(\langle \hat{\Delta}, \Sigma\rangle, S)$ is closed under $\bar{\alpha}$-Kleene star. Hence by Lemma 6.20 we obtain that $\varphi^*_\alpha \in \mathrm{Rat}(\mathfrak{S}¢, \Sigma, S)$. $\square$

6.6 Conclusion

We have recalled the Kleene-Goldstine theorem which states the equivalence of rational and regular weighted tree languages with storage for commutative complete semirings. Moreover, we have shown that $\mathrm{Rat}(\mathfrak{S}, \Sigma, S)$ is closed under the rational operations for arbitrary complete semirings. More precisely, we proved that our way of introducing storage to weighted regular tree grammars, *preserves* closure properties from the case without storage.

We believe that this preservation property can be proven also for other formalisms generating classes of formal languages, such as regular string languages, recognisable forest languages, and their weighted analoga.

7

Outlook

In this thesis, we investigated different theoretical questions concerning weighted automata models over tree-like input structures. First, we studied exact and approximated determinisation and then, we turned to Kleene-like and Büchi-like characterisations. We considered multiple weighted automata models, including weighted tree automata over semirings (Chapters 3 and 4), weighted forest automata over M-monoids (Chapter 5), and rational weighted tree languages with storage (Chapter 6). For an explanation as to why the last class can be considered as a weighted automaton model, we refer to page 190. We will now summarise the main contributions of the thesis.

In Chapter 3, we focused on the determinisation of weighted tree automata and presented our determinisation framework, called $\mathbb{M}$-sequentialisation, which can model different notions of determinisation from the existing literature. Then, we provided a positive $\mathbb{M}$-sequentialisation result for the case of additively idempotent semirings or finitely $\mathbb{M}$-ambiguous weighted tree automata. Another important contribution of Chapter 3 is Theorem 3.77, where we provide a blueprint theorem that can be used to find determinisation results for more classes of semirings and weighted tree automata easily. In fact, instead of repeating an entire determinisation construction, Theorem 3.77 allows us to prove a determinisation result by finding certain finite equivalence relations. This is a very potent tool for future research in the area of determinisation.

In Chapter 4, we moved from exact determinisation towards approximate determinisation. We lifted the formalisms and the main results from [4] from the word

case to the tree case. This successfully resulted in an approximated determinisation construction for weighted tree automata over the tropical semiring. We also provided a formal mathematical description of the approximated determinisation construction, rather than an algorithmic description as in [4, 28].

In Chapter 5, we turned away from determinisation and instead considered Kleene-like and Büchi-like characterisations of weighted recognisability. We introduced weighted forest automata over M-monoids, which are a generalisation of weighted tree automata over M-monoids and weighted forest automata over semirings. Then, we proved that our recognisable weighted forest languages can be decomposed into a finite product of recognisable weighted tree languages. We also proved that the initial algebra semantic and the run semantic for weighted forest automata are equivalent under certain conditions. Lastly, we defined rational forest expressions and forest M-expressions and and proved that the classes of languages generated by these formalisms coincide with recognisable weighted forest languages under certain conditions.

In Chapter 6, we considered rational weighted tree languages with storage, where the storage is introduced by composing rational weighted tree languages without storage with a storage map. In [56], it was proven that rational weighted tree languages with storage are closed under the rational operations. In Chapter 6, we provided alternative proofs of these closure propertiess. In fact, we proved that our way of introducing storage to rational weighted tree languages preserves the closure properties from rational weighted tree languages without storage.

Our results raise many new research questions, especially surrounding the mathematical machinery introduced for our $\mathbb{M}$-sequentialisation approach in Chapter 3. We have designated separate subchapters at the end of Chapters 3 to 6, which capture some of the most important questions and possible future research directions.

References

[1] ISO/IEC 14977:1996(E). **Information technology — Syntactic metalanguage — Extended BNF**. Standard, International Organization for Standardization, Geneva, CH, December 1996. 157

[2] Alfred Vaino Aho. **Nested stack automata**. *Journal of the Association for Computing Machinery*, **16**:383–406, 1969. 189

[3] Athanasios Alexandrakis and Symeon Bozapalidis. **Weighted grammars and Kleene's theorem**. *Information Processing Letters*, **24**(1):1–4, 1987. 30

[4] Benjamin Aminof, Orna Kupferman, and Robby Lampert. **Rigorous approximated determinization of weighted automata**. *Theoretical Computer Science*, **480**:104–117, 2013. 5, 7, 125, 126, 129, 141, 145, 146, 149, 223, 224

[5] Marie-Pierre Béal and Olivier Carton. **Determinization of transducers over finite and infinite words**. *Theoretical Computer Science*, **289**(1):225–251, 2002. 3, 4, 6, 34, 35, 36, 38, 43, 105, 106, 107, 108

[6] Jean Berstel and Christophe Reutenauer. **Recognizable formal power series on trees**. *Theoretical Computer Science*, **18**(2):115–148, 1982. 30

[7] Jean Berstel and Christophe Reutenauer. **Rational Series and Their Languages**, 12 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, 1988. 3

[8] Udi Boker and Thomas Henzinger. **Approximate determinization of quantitative automata**. In *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2012)*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2012. 5, 125

[9] Udi Boker and Thomas Henzinger. **Exact and Approximate Determinization of Discounted-Sum Automata**. *Logical Methods in Computer Science*, **10**(1), 2014. 5, 125

[10] Björn Borchardt. **The Theory of Recognizable Tree Series**. *Verlag für Wissenschaft und Forschung, Berlin.* PhD thesis, Technische Universität Dresden, Dresden, 2004. 136

[11] Björn Borchardt. **The Myhill-Nerode Theorem for Recognizable Tree Series**. In Z. Ésik and Z. Fülöp, editors, *Developments in Language Theory*, pages 146–158. Springer Berlin Heidelberg, 2003. 4

[12] Björn Borchardt and Heiko Vogler. **Determinization of finite state weighted tree automata**. *Journal of Automata, Languages and Combinatorics*, **8**(3):417–463, 2003. 3, 33

[13] Julius Büchi. **Weak second-order arithmetic and finite automata**. *Mathematical Logic Quarterly*, **6**(1-6):66–92, 1960. 2, 151, 152

[14] Matthias Büchse, Jonathan May, and Heiko Vogler. **Determinization of Weighted Tree Automata Using Factorizations.** *Journal of Automata, Languages and Combinatorics*, **15**(3/4):229–254, 2010. 4, 6, 34, 39, 105, 115, 116, 118, 141

[15] Arindam Chakrabarti, Krishnendu Chatterjee, Thomas Henzinger, Orna Kupferman, and Rupak Majumdar. **Verifying Quantitative Properties Using Bound Functions**. In *Proceedings of the 13th International Conference on Correct Hardware Design and Verification Methods (CHARME 2005)*, pages 50–64, Berlin, Heidelberg, 2005. Springer-Verlag. 5

[16] Krishnendu Chatterjee, Laurent Doyen, and Thomas Henzinger. **Quantitative Languages**. *ACM Transactions on Computational Logic - TOCL*, **11**:385–400, 2008. 5

[17] Christian Choffrut. **Une Caracterisation des Fonctions Sequentielles et des Fonctions Sous-Sequentielles en tant que Relations Rationnelles**. *Theoretical Computer Science*, **5**(3):325–337, 1977. 3, 34

[18] Avram Noam Chomsky. **Context-free grammars and pushdown storage**. Technical report, MIT Research Lab of Electronics, Quaterly Progress Report 65, 1962. 2, 189

[19] Hubert Comon, Max Dauchet, Rémi Gilleron, Christof Löding, Florent Jacquemard, Denis Lugiez, Sophie Tison, and Marc Tommasi. **Tree Automata Techniques and Applications**, 2007. see also: `https://jacquema.gitlabpages.inria.fr/files/tata.pdf`. 4

[20] Werner Damm and Andreas Goerdt. **An automata-theoretical characterization of the OI-hierarchy**. *Information and Control*, **71**:1–32, 1986. 189

[21] Laure Daviaud, Ismaël Jecker, Pierre-Alain Reynier, and Didier Villevalois. **Degree of Sequentiality of Weighted Automata**. In J. Esparza and A. Murawski, editors, *Proceedings of the 20th International Conference on Foundations of Software Science and Computation Structures (FOSSACS 2017)*, **10203** of LNCS, pages 215–230, 2017. 3, 4, 34, 35, 36, 38, 43, 105, 110, 111, 112, 149

[22] Tobias Denkinger. **An automata characterisation for multiple context-free languages**. In S. Brlek and C. Reutenauer, editor, *Proceedings of the 20th International Conference on Developments in Language Theory (DLT 2016)*, pages 138–150. Springer, 2016. 189

[23] Toni Dietze. **Equivalences between Ranked and Unranked Weighted Tree Automata via Binarization**. In B. Jurish, A. Maletti, K.-M.

Würzner, and U. Springmann, editors, *Proceedings of the ACL Workshop on Statistical Natural Language Processing and Weighted Automata (StatFSM 2016)*, pages 1–10, 2016. 153

[24] John Doner. **Decidability of the weak second-order theory of two successors**. *Notices of the American Mathematical Society*, **12**, 1965. Abstract 65T-468, 819. 4

[25] John Doner. **Tree acceptors and some of their applications**. *Journal of Computer and System Sciences*, **4**:406–451, 1970. 4

[26] Frederic Dörband. **A Kleene Theorem for Weighted Forest Automata**. Master's thesis, Technische Universität Dresden, Dresden, 2019. 6, 152, 154, 159, 169, 170, 183, 184

[27] Frederic Dörband. **Kleene and Büchi Theorems for Weighted Forest Languages over M-Monoids**. *Special issue of Information and Computation devoted to WATA 2020*, **281**:104765, 2021. 7, 151

[28] Frederic Dörband, Thomas Feller, and Kevin Stier. **Approximated Determinisation of Weighted Tree Automata**. In A. Leporati, C. Martín-Vide, D. Shapira, and C. Zandron, editors, *Proceedings of the 15th International Conference on Language and Automata Theory and Applications (LATA 2021)*, **12638** of LNCS, pages 255–266. Springer International Publishing, 2021. 7, 125, 224

[29] Frederic Dörband, Thomas Feller, and Kevin Stier. **Sequentiality of Group-Weighted Tree Automata**. In A. Leporati, C. Martín-Vide, D. Shapira, and C. Zandron, editors, *Proceedings of the 15th International Conference on Language and Automata Theory and Applications (LATA 2021)*, **12638** of LNCS, pages 267–278. Springer International Publishing, 2021. 4, 6, 7, 33, 34, 35, 36, 38, 43, 56, 105, 110, 111, 112

[30] Frederic Dörband, Zoltán Fülöp, and Heiko Vogler. **Rational Weighted Tree Languages with Storage**. *Special issue of Information and Computation devoted to CAI 2019*, 2019. Accepted. 7, 189

[31] Frederic Dörband and Richard Mörbitz. **A General Approach to Determinisation of Weighted Automata**. Manuscript, 2021. 7, 31, 33, 56, 104

[32] Manfred Droste and Paul Gastin. **Weighted automata and weighted logics**. *Theoretical Computer Science*, **380**(1):69–86, 2007. Automata, Languages and Programming. 3

[33] Manfred Droste, Doreen Götze, Steffen Märcker, and Ingmar Meinecke. **Weighted Tree Automata over Valuation Monoids and Their Characterization by Weighted Logics**. In W. Kuich and G. Rahonis, editors, *Algebraic Foundations in Computer Science: Essays Dedicated to Symeon Bozapalidis on the Occasion of His Retirement*, pages 30–55. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011. 6, 152, 153

[34] Manfred Droste, Doreen Heusel, and Heiko Vogler. **Weighted Unranked Tree Automata over Tree Valuation Monoids and Their Characterization by Weighted Logics**. In *Proceedings of the 6th International Conference on Algebraic Informatics (CAI 2015)*, **9270** of *LNCS*, pages 90–102, 2016. 6, 152, 153

[35] Manfred Droste and Werner Kuich. **Semirings and formal power series**. In M. Droste, W. Kuich, and H. Vogler, editors, *Handbook of Weighted Automata*, pages 3–28. Springer-Verlag, 2009. 21

[36] Manfred Droste, Werner Kuich, and Heiko Vogler, editors. **Handbook of Weighted Automata**. EATCS Monographs in Theoretical Computer Science. Springer-Verlag, 2009. 3, 20, 31, 159

[37] Manfred Droste and Dietrich Kuske. **Weighted Automata**. In J.-É. Pin, editor, *Handbook of Automata Theory*, pages 113–150. EMS Press, Berlin, 2021. 23

[38] Manfred Droste, Christian Pech, and Heiko Vogler. **A Kleene theorem for weighted tree automata**. *Theory of Computing Systems*, **38**:1–38, 2005. 4, 6, 24, 25, 26, 152, 154, 191, 195, 198, 199, 205, 206, 209, 220

[39] Manfred Droste, Torsten Stüber, and Heiko Vogler. **Weighted finite automata over strong bimonoids**. *Information Sciences*, **180**:156–166, 2010. 34, 35

[40] Manfred Droste and Heiko Vogler. **Weighted Tree Automata and Weighted Logics**. *Theoretical Computer Science*, **366**:228–247, 2006. 4, 6, 30, 152, 154

[41] Samuel Eilenberg. **Automata, Languages, and Machines – Volume A**, 59 of *Pure and Applied Mathematics*. Academic Press, 1974. 21

[42] Calvin Creston Elgot. **Decision Problems of Finite Automata Design and Related Arithmetics**. *Transactions of the American Mathematical Society*, **98**(1):21–51, 1961. 2, 6

[43] Joost Engelfriet. **Bottom-up and top-down tree transformations - a comparison**. *Mathematical Systems Theory*, **9**(3):198–231, 1975. 200

[44] Joost Engelfriet. **Tree automata and tree grammars**. Technical Report DAIMI FN-10, Institute of Mathematics, University of Aarhus, Aarhus, 1975. see also: arXiv:1510.02036v1 [cs.FL] 7 Oct 2015. 4

[45] Joost Engelfriet. **Two-way automata and checking automata**. *Mathematical Centre Tracts*, **108**:1–69, 1979. 189

[46] Joost Engelfriet. **Context–free grammars with storage**. Technical Report 86-11, University of Leiden, Leiden, 1986. see also: arXiv:1408.0683v1 [cs.FL] 04 Aug 2014. 189, 191, 192, 195

[47] Joost Engelfriet. **Alternative Kleene theorem for weighted automata**. personal communication, 2003. 25, 26

[48] Joost Engelfriet and Heiko Vogler. **Pushdown machines for the macro tree transducer**. *Theoretical Computer Science*, **42**(3):251–368, 1986. 191, 197, 202

[49] Zoltán Ésik and Werner Kuich. **Formal Tree Series**. *Journal of Automata, Languages, and Combinatorics*, **8**(2):219–285, 2003. 191

[50] Zoltán Fülöp, Luisa Herrmann, and Heiko Vogler. **Weighted Regular Tree Grammars with Storage**. *Discrete Mathematics and Theoretical Computer Science*, **20(1)**:#26, 2018. see also: arXiv:1705.06681v6 [cs.FL] 02 Jul 2020. 191, 197, 203, 205

[51] Zoltán Fülöp, Andreas Maletti, and Heiko Vogler. **A Kleene theorem for weighted tree automata over distributive multioperator monoids**. *Theory of Computing Systems*, **44**:455–499, 2009. 4, 6, 152, 154, 155, 156, 159, 161, 162, 166, 174, 177, 178, 179, 180, 186

[52] Zoltán Fülöp, Thorsten Stüber, and Heiko Vogler. **A Büchi-like theorem for weighted tree automata over multioperator monoids**. *Theory of Computing Systems*, **50**(2):241–278, 2012. 4, 6, 152, 154, 155, 157, 159, 161, 162, 174, 184, 185, 186

[53] Zoltán Fülöp and Heiko Vogler. **Weighted tree automata and tree transducers**. In M. Droste, W. Kuich, and H. Vogler, editors, *Handbook of Weighted Automata*, pages 313–403. Springer-Verlag, 2009. 24, 30, 191, 205

[54] Zoltán Fülöp and Heiko Vogler. **Weighted iterated linear control**. *Acta Informatica*, **56(5)**:447–469, 2018. 191

[55] Zoltán Fülöp and Heiko Vogler. **Rational Weighted Tree Languages with Storage and the Kleene-Goldstine Theorem**. In M. Ćirić, M. Droste, and J.-É. Pin, editors, *Algebraic Informatics (CAI 2019)*, **11545** of LNCS, pages 138–150, 2019. 6, 7, 190, 191, 195, 199, 200, 202, 203, 205, 206

[56] Zoltán Fülöp and Heiko Vogler. **Principal Abstract Families of Weighted Tree Languages**. *Special issue of Information and Computation devoted to WATA 2018*, **282**:104657, 2022. 194, 195, 205, 208, 214, 224

[57] Zoltán Fülöp, Dávid Kószó, and Heiko Vogler. **Crisp-determinization of weighted tree automata over strong bimonoids**. *Discrete Mathematics & Theoretical Computer Science*, **23(1)**, 2021. 4, 6, 34, 35, 40

[58] Ferenc Gécseg and Magnus Steinby. **Tree Automata**. Akadémiai Kiadó, Budapest, 1984. see also: arXiv:1509.06233v1 [cs.FL] 21 Sep 2015. 30, 31, 191

[59] Ferenc Gécseg and Magnus Steinby. **Tree Languages**. In G. Rozenberg and A. Salomaa, editors, *Handbook of Formal Languages*, **3(1)**, pages 1–68. Springer, 1997. 30, 31

[60] Seymour Ginsburg. **Algebraic and automata-theoretic properties of formal languages**. North-Holland, 1975. 190, 197

[61] Seymour Ginsburg and Sheila Adele Greibach. **Abstract families of languages**. *Memoirs of the American Mathematical Society*, **87**:1–32, 1969. 190

[62] Seymour Ginsburg and Sheila Adele Greibach. **Principal AFL**. *Journal of Computer and System Sciences*, **4**:308–338, 1970. 190

[63] Seymour Ginsburg, Sheila Adele Greibach, and Michael Alexander Harrison. **One-Way Stack Automata**. *Journal of the Association for Computing Machinery*, **14**(2):389–418, 1967. 189

[64] Steven Givant and Paul Halmos. **Introduction to Boolean Algebras**. Springer, 2009. 196

[65] Jonathan Samuel Golan. **Semirings and their Applications**. Kluwer Academic Publishers, Dordrecht, 1999. 20, 21

[66] Jonathan Samuel Golan. **Complete semirings**. In *Semirings and Affine Equations over Them: Theory and Applications*, pages 39–47. Springer Netherlands, Dordrecht, 2003. 21

[67] Jonathan Goldstine. **Automata with data storage**. In *Proceedings of the Conference on Theoretical Computer Science, University of Waterloo, Ontario*, pages 239–246, 1977. 189, 190, 191, 193

[68] Jonathan Goldstine. **A rational theory of AFLs**. In H.A. Maurer, editor, *Proceedings of the 6th International Colloquium on Automata, Languages and Programming (ICALP 1979)*, **71** of *LNCS*, pages 271–281. Springer, 1979. 190, 193, 195, 197, 198, 201, 202

[69] Jonathan Goldstine. **Formal languages and their relation to automata: What Hopcroft & Ullman didn't tell us**. In R. V. Book, editor, *Formal Language Theory: Perspectives and Open Problems*, pages 109–140. Academic Press, New York, 1980. 190, 193, 195, 197

[70] Doreen Götze, Zoltán Fülöp, and Manfred Droste. **A Kleene theorem for weighted tree automata over tree valuation monoids**. *Information and Computation*, **269**:104445, 2019. 6, 152, 153

[71] Seymour Greibach. **Full AFLs and nested iterated substitution**. *Information and Control*, **16**:7–35, 1970. 189

[72] Michael Alexander Harrison. **Introduction to formal language theory**. Addison-Wesley series in computer science. Addison-Wesley Longman Publishing Co., Inc, 1978. 1

[73] Udo Hebisch and Hanns Joachim Weinert. **Semirings - Algebraic Theory and Applications in Computer Science**. World Scientific, Singapore, 1998. 21

[74] Luisa Herrmann and Heiko Vogler. **A Chomsky-Schützenberger Theorem for Weighted Automata with Storage**. In *Proceedings of the 6th International Conference on Algebraic Informatics (CAI 2015)*, **9270** of LNCS, pages 115–127, 2015. 191

[75] Luisa Herrmann, Heiko Vogler, and Manfred Droste. **Weighted automata with storage**. *Special Issue of Information and Computation devoted to LATA 2016*, **269**:104447, 2019. 189

[76] John Edward Hopcroft and Jeffrey David Ullman. **Introduction to Automata Theory, Languages, and Computation**. Addison-Wesley, USA, 1979. 1, 2, 4, 31, 33

[77] Yasuyoshi Inagaki and Teruo Fukumura. **On the description of fuzzy meaning of context-free languages**. In *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, pages 301–328. Elsevier, 1975. 3

[78] Gareth Aneurin Jones and Josephine Mary Jones. **Elementary Number Theory**. Springer Science & Business Media, 1998. 145

[79] Mark Kambites. **Formal languages and groups as memory**. *Communications in Algebra*, **37(1)**:193–208, 2009. 189

[80] Daniel Kirsten and Ina Mäurer. **On the Determinization of Weighted Automata.** *Journal of Automata, Languages and Combinatorics*, **10**:287–312, 2005. 4, 34, 105, 119, 126, 129

[81] Stephen Kleene. **Representation of events in nerve nets and finite automata**. In J.McCarthy C.E. Shannon, editor, *Automata Studies*, pages 3–42, 1956. 1, 2, 6, 151, 152, 190

[82] Kevin Knight and Jonathan Graehl. **An Overview of Probabilistic Tree Transducers for Natural Language Processing**. In Alexander Gelbukh, editor, *Computational Linguistics and Intelligent Text Processing*, pages 1–24, Berlin, Heidelberg, 2005. Springer Berlin Heidelberg. 4

[83] Werner Kuich. **Formal power series over trees**. In S. Bozapalidis, editor, *Proceedings of the 3rd International Conference on Developments in Language Theory (DLT 1997)*, pages 61–101. Aristotle University of Thessaloniki, 1998. 199

[84] Werner Kuich and Arto Salomaa. **Semirings, Automata, Languages**, 5 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, 1986. 3

[85] Sylvain Lombardy and Jacques Sakarovitch. **Sequential?** *Theoretical Computer Science*, **356**(1):224–244, 2006. In honour of Professor Christian Choffrut on the occasion of his 60th birthday. 3, 33, 34

[86] Aleksandr Nikolaevich Maslov. **Multilevel stack automata**. *Problems of Information Transmission*, **12**:38–43, 1976. 189

[87] Christian Mathissen. **Weighted Automata and Weighted Logics over Tree-like Structures**. PhD thesis, Universität Leipzig, Leipzig, 2009. 3, 6, 152, 153, 155

[88] Warren Sturgis McCulloch and Walter Pitts. **A logical calculus of the ideas immanent in nervous activity**. *The Bulletin of Mathematical Biophysics*, **5**(4):115–133, 1943. 1

[89] Robert McNaughton and Hisao Yamada. **Regular Expressions and State Graphs for Automata**. *IRE Transactions on Electronic Computers*, **EC-9**(1):39–47, 1960. 1

[90] Mehryar Mohri. **Finite-State Transducers in Language and Speech Processing**. *Computational Linguistics*, **23**(2):269–311, 1997. 4, 34, 126

[91] Anil Nerode. **Linear Automaton Transformations**. *Proceedings of the American Mathematical Society*, **9**(4):541–544, 1958. 2

[92] Erik Paul. **Finite sequentiality of unambiguous max-plus tree automata**. In R. Niedermeier and C. Paul, editors, *Proceedings of the 36th International Symposium on Theoretical Aspects of Computer Science (STACS 2019)*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019. 34

[93] Michael Rabin and Dana Scott. **Finite Automata and Their Decision Problems**. *IBM Journal of Research and Development*, **3**:114–125, 1959. 1, 2, 33

[94] Dragica Radovanovic. **Weighted tree automata over strong bimonoids**. *Novi Sad Journal of Mathematics*, **40**(3):89–108, 2010. 136, 164, 166, 174

[95] George Rahonis. **Fuzzy Languages**. In Manfred Droste, Werner Kuich, and Heiko Vogler, editors, *Handbook of Weighted Automata*, pages 481–517. Springer Berlin Heidelberg, 2009. 3

[96] Grzegorz Rozenberg and Arto Salomaa, editors. **Handbook of Formal Languages, Vol. 1: Word, Language, Grammar**. Springer-Verlag, Berlin, Heidelberg, 1997. 3

[97] Grzegorz Rozenberg and Arto Salomaa, editors. **Handbook of Formal Languages, Vol. 3: Beyond Words**. Springer-Verlag, Berlin, Heidelberg, 1997. 3, 4

[98] Arto Salomaa and Matti Soittola. **Automata-Theoretic Aspects of Formal Power Series**. Texts and Monographs in Computer Science, Springer-Verlag, 1978. 3

[99] Marcel-Paul Schützenberger. **On the definition of a family of automata**. *Information and Control*, **4**:245–270, 1961. 3, 23, 24

[100] Thomas Schwentick. **Foundations of XML Based on Logic and Automata: A Snapshot**. In *Proceedings of the 7th International Symposium on Foundations of Information and Knowledge Systems (FoIKS 2012)*, pages 23–33, 2012. 4

[101] Dana Scott. **Some definitional suggestions for automata theory**. *Journal of Computer and System Sciences*, **1**:187–212, 1967. 189, 192, 195

[102] Imre Simon. **Limited Subsets of a Free Monoid**. In *19th Annual Symposium on Foundations of Computer Science*, pages 143–150. IEEE Computer Society, 1978. 3

[103] Imre Simon. **Recognizable Sets with Multiplicities in the Tropical Semiring**. In *Proceedings of the 13th International Symposium on Mathematical*

Foundations of Computer Science (MFCS 1988), pages 107–120. Springer-Verlag, 1988. 3

[104] Lutz Strassburger. **A Kleene theorem for forest languages**. In *Language and automata theory and applications*, **5457** of LNCS, pages 715–727. Springer, Berlin, 2009. 6, 152, 154, 183, 184

[105] Howard Straubing. **Finite Automata, Formal Logic, and Circuit Complexity**. Birkhauser Verlag, CHE, 1994. 158

[106] Markus Teichmann and Johannes Osterholzer. **A link between multioperator and tree valuation automata and logics**. *Theoretical Computer Science*, **594**:106 – 119, 2015. 153

[107] James Thatcher and Jesse Wright. **Generalized finite automata theory with an application to a decision problem of second-order logic**. *Mathematical Systems Theory*, **2**(1):57–81, 1968. 6, 152, 154, 199

[108] Ken Thompson. **Programming Techniques: Regular Expression Search Algorithm**. **11**(6):419–422, 1968. 1

[109] Boris Avraamovich Trakhtenbrot. **Finite automata and the logic of one-place predicates**. *American Mathematical Society Translations : Series 2*, **59**:23–55, 1966. 2

[110] Alan Mathison Turing. **On Computable Numbers, with an Application to the Entscheidungsproblem**. *Proceedings of the London Mathematical Society*, **s2-42**(1):230–265, 1937. 1

[111] Jan van Leeuwen. **Variations of a new machine model**. In *Proceedings of the 17th Annual Symposium on Foundations of Computer Science (FOCS 1976)*, pages 228–235, 1976. 189

[112] Heiko Vogler, Manfred Droste, and Luisa Herrmann. **A Weighted MSO Logic with Storage Behaviour and Its Büchi-Elgot-Trakhtenbrot Theorem**. In A.-H. Dediu, J. Janoušek, C. Martín-Vide, and B. Truthe,

editors, *Proceedings of the 10th International Conference on Language and Automata Theory and Applications (LATA 2016)*, pages 127–139. Springer International Publishing, 2016. 191

[113] Andreas Weber and Reinhard Klemm. **Economy of Description for Single-Valued Transducers**. *Information and Computation*, **118**(2):327–340, 1995. 107

[114] Wolfgang Wechler. **The Concept of Fuzziness in Automata and Language Theory**. In *Studien zur Algebra und ihre Anwendungen*. Akademie-Verlag, Berlin, 1978. 3