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On extended quasi-MV algebras

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Abstract

In this paper, we introduce a new algebraic structure called extended quasi-MV algebras, which are generalizations of quasi-MV algebras. The notions of ideals, ideal congruences and filters in Equasi-MV algebras were introduced and their mutual relationships were investigated. There is a bijection between the set of all ideals and the set of all ideal congruences on an Equasi-MV algebra.
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1 Introduction

MV-algebras were introduced by Chang Chang [1958] as an algebraic counterpart of infinite valued logic. There are many papers on MV-algebras. Also, many algebraic structures are defined, which extend the notion of MV-algebras. Quantum computation logics M. L. Dalla Chiara and Leporini [2005] received more attention in recent years, which are new forms of quantum logics G. Cattaneo and Leporini [2004]. These logics determine the meaning of a sentence with a mixture of quregisters M. L. Dalla Chiara and Greechie [2013]. Corresponding to quantum computational, Ledda, Konig, Paoli and Giuntini introduced the notion of quasi-MV algebras in A. Ledda and Giuntini [2006], which are generalizations of MV-algebras. The element 0 in a quasi-MV algebra is not necessarily a neutral element of the operation $\oplus$. Since then, many authors continued to study quasi-MV algebras. For example, Ledda etc. studied some properties of quasi-MV algebras and $\sqrt{'}$ quasi-MV algebras F. Bou and Freytes [2008], F. Paoli and Freytes [2009]; Chen introduced pseudo-quasi-MV algebras which are non-commutative generalizations of quasi-MV algebras Liu and Chen [2016].

EMV-algebras (extended MV-algebras) Dvurečenskij and Zahiri [2019] are also generalizations of MV-algebras. An EMV-algebra does not necessarily have a top element. Dvurečenskij and Zahiri gave some properties of EMV-algebras. The notions of ideals, congruences and filters in EMV-algebras were also introduced and the relationships between them were investigated. One of the main results is that every EMV-algebra can be embedded into an EMV-algebra with a top element. Liu presented EBL-algebras in Liu [2020], which extended the notion of BL-algebras. The author gave some properties of EBL-algebras. Also, the concepts of ideals, congruences and filters were introduced and the relationships between them were studied.

Inspired by Dvurečenskij and Zahiri [2019], we shall give the definition of Equasi-MV algebras. In these algebras, 0 is not necessarily the neutral element and the complement element of 0 does not necessarily exist. The structure of this paper is as follows. In Sect.2, we give some definitions and results of quasi-MV algebras. In Sect.3, we introduce Equasi-MV algebras and present some examples of Equasi-MV algebras. In Sect.4, we define ideals and ideal congruences in Equasi-MV algebras. And we study the relationships between them. In Sect.5, we introduce the notions of filters and prime ideals. Moreover, every Equasi-MV algebra has at least one maximal ideal.

2 Preliminaries

In this section, we will give some notions and results on quasi-MV algebras, which will be used in the following.

A quasi-MV algebra A. Ledda and Giuntini [2006] is an algebra $\mathbf{A} = \langle A, \oplus, ', 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ satisfying the following conditions:

QMV1) $x \oplus (y \oplus z) = (x \oplus z) \oplus y$;

QMV2) $x'' = x$;

QMV3) $x \oplus 1 = 1$;

QMV4) $(x' \oplus y)' \oplus y = (y' \oplus x)' \oplus x$;

QMV5) $(x \oplus 0)' = x' \oplus 0$;

QMV6) $(x \oplus y) \oplus 0 = x \oplus y$;

QMV7) $0' = 1$.

In any quasi-MV algebra $\mathbf{A}$, we can define the following operations:

$$x \otimes y = (x' \oplus y')'; \ x \Cup y = x \oplus (x' \otimes y); \ x \Cap y = x \otimes (x' \oplus y).$$

It is obvious that $x \Cup y = (x \Cup y) \oplus 0$ and $x \Cap y = (x \Cap y) \oplus 0$. Moreover, we can also define an binary relation $\leqslant$ on A as follows: $x \leqslant y$ iff $x \Cap y = x \oplus 0$. The relation $\leqslant$ is a preordering of A, but not a partial ordering.

Lemma 2.1. *[A. Ledda and Giuntini, 2006, Lemma 8] Let $\mathbf{A}$ be a quasi-MV algebra. For all $x, y, z \in A$, the following statements are equivalent.*

(i) $x \leqslant y$;

(ii) $x' \oplus y = 1$;

(iii) $x \Cup y = y \oplus 0$.

In the following, we give some properties of quasi-MV algebras, including a few properties of preordering $\leqslant$ and the operations $\Cap$ and $\Cup$.

Lemma 2.2. *[A. Ledda and Giuntini, 2006, Lemma 11] Let $\mathbf{A}$ be a quasi-MV algebra. For all $x, y, z, w \in A$:*

(i) $x\oplus 0 \leqslant y\oplus 0$, $y\oplus 0 \leqslant x\oplus 0$ *imply* $x\oplus 0 = y\oplus 0$;

(ii) $x \leqslant y$ *and* $z \leqslant w$ *imply* $x\oplus z \leqslant y\oplus w$;

(iii) $x \leqslant y$ *and* $z \leqslant w$ *imply* $x\otimes z \leqslant y\otimes w$;

(iv) $x \leqslant y$ *and* $z \leqslant w$ *imply* $x\Cap z \leqslant y \Cap w$;

(v) $x \leqslant y$ *and* $z \leqslant w$ *imply* $x\Cup z \leqslant y\Cup w$;

(vi) $x \leqslant x\oplus 0$ *and* $x\oplus 0 \leqslant x$;

(vii) $x\otimes y \leqslant z$ *iff* $x \leqslant y'\oplus z$;

(viii) if $x \leqslant y,$ *then* $y' \leqslant x'$;

(ix) $0 \leqslant x$, $x \leqslant 1$.

Lemma 2.3. *[A. Ledda and Giuntini, 2006, Lemma 12] Let $\mathbf{A}$ be a quasi-MV* *algebra. For all $x, y, z \in A$:*

(i) $x \Cap y = y \Cap x$;

(ii) $x \Cup y = y \Cup x$;

(iii) $x \Cap y \leqslant x$, y *and* x, $y \leqslant x \Cup y$;

(iv) if $x \leqslant y$, z, *then* $x \leqslant y \Cap z$;

(v) if x, $y \leqslant z$, *then* $x \Cup y \leqslant z$;

(vi) $x \oplus (y \Cap z) = (x \oplus y) \Cap (x \oplus z)$;

(vii) $x \otimes (y \Cup z) = (x \otimes y) \Cup (x \otimes z)$;

(viii) $x \Cap (y \Cap z) = (x \Cap y) \Cap z$;

(ix) $x \Cup (y \Cup z) = (x \Cup y) \Cup z$;

(x) $x \leqslant x \Cap x$ *and* $x \Cap x \leqslant x$;

(xi) $(x \Cap y)' = x' \Cup y'$ *and* $(x \Cup y)' = x' \Cap y'$.

The following lemma gives the distributivity between $\Cap$ and $\Cup$ on quasi-MV algebras.

Lemma 2.4. *Let* A *be a quasi-MV algebra. For all $x, y, z \in A$,*

(i) $(x \Cup y) \Cap z = (x \Cap z) \Cup (y \Cap z)$;

(ii) $(x \Cap y) \Cup z = (x \Cup z) \Cap (y \Cup z)$;

(iii) $x \Cap (y \oplus z) \leqslant (x \Cap y) \oplus (x \Cap z)$;

(iv) $(x \Cup y) \otimes (x \Cup z) \leqslant x \Cup (y \otimes z)$.

Proof. (i) For any $x, y \in A$, we have $x, y \leqslant x \Cup y$ and so $x \Cap z, y \Cap z \leqslant (x \Cup y) \Cap z$ by Lemma 2.2 (iv). It follows from Lemma 2.3 (v) that $(x \Cap z) \Cup (y \Cap z) \leqslant (x \Cup y) \Cap z$. Conversely, we have

$$\begin{aligned}(x \Cup y) \Cap z &= (x \Cup y) \otimes ((x \Cup y)' \oplus z)\\ &= (x \Cup y) \otimes ((x' \oplus z) \Cap (y' \oplus z)) \text{ (Lemma 2.3 (xi) and (vi))}\\ &\leqslant (x \otimes (x' \oplus z)) \Cup (y \otimes (y' \oplus z)) \text{ (Lemma 2.3 (vii) and (iii))}\\ &= (x \Cap z) \Cup (y \Cap z).\end{aligned}$$

Then $((x \Cap z) \Cup (y \Cap z)) \oplus 0 \leqslant ((x \Cup y) \Cap z) \oplus 0$ and $((x \Cup y) \Cap z) \oplus 0 \leqslant ((x \Cap z) \Cup (y \Cap z)) \oplus 0$. Note that $((x \Cap z) \Cup (y \Cap z)) \oplus 0 = (x \Cap z) \Cup (y \Cap z)$ and $((x \Cup y) \Cap z) \oplus 0 = (x \Cup y) \Cap z$. It follows that $(x \Cap z) \Cup (y \Cap z) = (x \Cup y) \Cap z$ by Lemma 2.2 (i).

Similarly, we can prove (ii).

(iii) For any $x, y, z \in A$, since $x \leqslant x \oplus 0 \leqslant x \oplus y$, we have

$$\begin{aligned}(x \Cap y) \oplus (x \Cap z) &= ((x \Cap y) \oplus x) \Cap ((x \Cap y) \oplus z) \text{ (Lemma 2.3 (vi))}\\ &= (x \oplus x) \Cap (y \oplus x) \Cap (x \oplus z) \Cap (y \oplus z)\\ &\geqslant x \Cap x \Cap x \Cap (y \oplus z)\\ &= (x \oplus 0) \Cap x \Cap (y \oplus z) \text{ (Lemma 2.3 (x))}\\ &= (x \oplus 0) \Cap (y \oplus z).\end{aligned}$$

Note that $(x \oplus 0) \Cap (y \oplus z) = x \Cap (y \oplus z)$. It follows that $x \Cap (y \oplus z) \leqslant (x \Cap y) \oplus (x \Cap z)$.

(iv) For any $x, y, z \in A$, it follows from $(x \otimes y)' \oplus y = x' \oplus y' \oplus y = 1$ that

$x \otimes y \leqslant y$. Then we have

$$\begin{aligned}(x \Cup y) \otimes (x \Cup z) &= ((x \Cup y) \otimes x) \Cup ((x \Cup y) \otimes z) \text{ (Lemma 2.3 (vii))}\\ &= (x \otimes x) \Cup (y \otimes x) \Cup (x \otimes z) \Cup (y \otimes z)\\ &\leqslant x \Cup x \Cup x \Cup (y \otimes z)\\ &= (x \oplus 0) \Cup x \Cup (y \otimes z) \text{ (Lemma 2.3 (x))}\\ &= (x \oplus 0) \Cup (y \otimes z).\end{aligned}$$

Note that $(x \oplus 0) \Cup (y \otimes z) = x \Cup (y \otimes z)$. It follows that $(x \Cup y) \otimes (x \Cup z) \leqslant x \Cup (y \otimes z)$.□

Let $\mathbf{A}$ be a quasi-MV algebra and $a \in A$. If $a \oplus a = a$, we call a to be idempotent. We use $\mathcal{I}(\mathbf{A})$ to denote the set of all idempotent elements of A. For $a \in A$, we call a regular if $a \oplus 0 = a$. We denote the set of all regular elements of A by $\mathcal{R}(\mathbf{A})$.

Lemma 2.5. *Let $\mathbf{A}$ be a quasi-MV algebra. For any $x \in A$, $a \in \mathcal{I}(\mathbf{A})$, we have*

(i) $x \oplus a = x \Cup a$;

(ii) $x \otimes a = x \Cap a$.

Proof. (i) For any $x \in A$ and $a \in \mathcal{I}(\mathbf{A})$, we have $x, a \leqslant x \oplus a$. Then $x \Cup a \leqslant x \oplus a$ by Lemma 2.3 (v). Conversely,

$$\begin{aligned}(x \oplus a) \otimes (x \Cup a)' &= (x \oplus a) \otimes (x' \Cap a') \text{ (Lemma 2.3 (xi))}\\ &\leqslant ((x \oplus a) \otimes x') \Cap ((x \oplus a) \otimes a') \text{ (Lemma 2.2(iii) and 2.3(iv))}\\ &= (a \Cap x') \Cap (x \Cap a')\\ &= (a \Cap a') \Cap (x \Cap x')\\ &= 0 \Cap (x \Cap x') = 0.\end{aligned}$$

This means that $(x \oplus a)' \oplus (x \Cup a) = 1$. It follows that $x \oplus a \leqslant x \Cup a$.

(ii) By (i), we have $x' \oplus a'$=$x' \Cup a'$, that is $(x' \oplus a')' = (x' \Cup a')' = x \Cap a$. It follows that $x \Cap a = x \otimes a$.□

The application of the above lemma will be reflected in the following proof process.

Example 2.1. *[A. Ledda and Giuntini, 2006, Example 3] The Diamond is the 4-element quasi-MV algebra, where the operations $\oplus$ and $'$ are defined as following tables:*

$\oplus$	0	a	b	1
0	0	b	b	1
a	b	1	1	1
b	b	1	1	1
1	1	1	1	1

$'$	
0	1
a	a
b	b
1	0

Remark that $a \oplus a = 1$, but $a \Cap a = (a' \oplus (a' \oplus a)')' = (a \oplus (a \oplus a)')' = b \neq 1$.

3 Equasi-MV algebras

In the section, we shall define the notion of extended quasi-MV algebras, which are generalizations of quasi-MV algebras. Some basic properties of these algebras are presented.

Definition 3.1. *A extended quasi-MV algebra (abbreviated as Equasi-MV algebra) is an algebra* $\mathbf{A}$=$\langle A, \oplus, 0\rangle$*, if the following conditions are satisfied:*

EQMV1) $\langle A, \oplus, 0\rangle$ *is a commutative preordered semigroup and* $(x \oplus y) \oplus 0 = x \oplus y$ *for all* $x, y \in A$*;*

EQMV2) for each $x \in A$*, there is* $b \in \mathcal{I}(\mathbf{A})$ *such that* $x \leqslant b$*, and the element*

$$\lambda_b(x) = \mathit{min}\{z \in [0, b] : z \oplus x = b\}$$

exists in A *for all* $x \in [0, b]$ *such that* $\langle [0, b], \oplus, \lambda_b, 0, b\rangle$ *is a quasi-MV algebra.*

Note that for any $x, y \in A$, there exist $a, b \in \mathcal{I}(\mathbf{A})$ such that $x \leqslant a$ and $y \leqslant b$. Then there exists $c \in \mathcal{I}(\mathbf{A})$ such that $a, b \leqslant c$. In fact, take $c = a \oplus b$. It is obvious that $a, b \leqslant a \oplus b$ and $a \oplus b \in \mathcal{I}(\mathbf{A})$. Therefore, an Equasi-MV algebra has enough idempotent elements. That is, for all $x \in A$, there is $a \in \mathcal{I}(\mathbf{A})$ such that $x \leqslant a$.

Let $\mathbf{A}$ be an Equasi-MV algebra. For all $n \in \mathbb{N}$ and $x \in A$, we define

$$0.x = 0,\ 1.x = x,\ \cdots,\ (n+1).x = n.x \oplus x.$$

An Equasi-MV algebra $\langle A, \oplus, 0\rangle$ is called a proper Equasi-MV algebra if 0 has no complement element.

Example 3.1. *If* $\langle A, \oplus, ', 0, 1\rangle$ *is a quasi-MV algebra, then* $\langle A, \oplus, 0\rangle$ *is an Equasi-MV algebra. Also, if* $\langle A, \vee, \wedge, \oplus, 0\rangle$ *is an EMV-algebra, it is obvious that* $\langle A, \oplus, 0\rangle$ *is an Equasi-MV algebra.*

Example 3.2. *Let* $\langle A, \oplus, ', 0, 1\rangle$ *be a quasi-MV algebra and* $\langle B, \vee, \wedge, \oplus, 0\rangle$ *be an EMV-algebra. We define that the operation on the algebra* $A \times B$ *is point by point. That is, for any* $\langle x_1, x_2\rangle, \langle y_1, y_2\rangle \in A \times B$*,*

$$\langle x_1, x_2\rangle \oplus \langle y_1, y_2\rangle = \langle x_1 \oplus y_1, x_2 \oplus y_2\rangle.$$

And the least element of $A \times B$ *is* $0 = \langle 0, 0\rangle$*. For any* $x \in B$*, there exists* $b \in \mathcal{I}(\mathbf{B})$ *such that* $x \leqslant b$*. Then for any* $\langle x_1, x_2\rangle \in A \times B$*, there exists* $\langle 1, b\rangle \in \mathcal{I}(\mathbf{A}) \times \mathcal{I}(\mathbf{B})$*. It suffices to show that* $\langle [\langle 0, 0\rangle, \langle 1, b\rangle], \oplus, \lambda_{\langle 1,b\rangle}, \langle 0, 0\rangle, \langle 1, b\rangle\rangle$ *is a quasi-MV algebra. We define* $\lambda_{\langle 1,b\rangle}(\langle x_1, x_2\rangle) = \langle (x_1)', \lambda_b(x_2)\rangle$*, for all* $\langle x_1, x_2\rangle \in A \times B$*. As a result,* $A \times B$ *is an Equasi-MV algebra.*

Example 3.3. *Let* e *be a smallest idempotent of an Equasi-MV algebra* $\mathbf{A}$*. Then an Equasi-MV algebra is the algebra* $\mathbf{S}$=$\langle A \times A, \oplus^S, 0^S\rangle$*, where:*

(i) $0^S = \langle 0, \frac{e}{2}\rangle$*;*

(ii) $x^S \oplus^S y^S = \langle x_1 \oplus y_1, \frac{e}{2}\rangle$*, for all* $x^S = \langle x_1, x_2\rangle$ *and* $y^S = \langle y_1, y_2\rangle$*.*

For any $a \in \mathcal{I}(\mathbf{A})$*, we define* $a^S = \langle a, \frac{e}{2}\rangle$*. Then* $a^S = a^S \oplus a^S \in \mathcal{I}(\mathbf{S})$*. Now we show that* $\langle [0^S, a^S], \oplus^S, \lambda_{a^S}, 0^S, a^S\rangle$ *is a quasi-MV algebra, where* $\lambda_{a^S}(x^S) =$

$\langle \lambda_a(x_1), x_2\rangle$ and $a \in \mathcal{I}(\mathbf{A})$. It is easy to show that $\lambda_{a^S}(x^S)$ is the least element such that $x^S \oplus z^S = a^S$ for all $x^S \in [0^S, a^S]$.

It is clear that $\lambda_{a^S}\lambda_{a^S}(x^S) = \lambda_{a^S}\langle \lambda_a(x_1), x_2\rangle = \langle x_1, x_2\rangle = x^S$. And $\lambda_{a^S}(x^S \oplus^S 0^S) = \lambda_{a^S}\langle x_1 \oplus 0, \frac{e}{2}\rangle = \langle \lambda_a(x_1) \oplus 0, \frac{e}{2}\rangle$, $\lambda_{a^S}(x^S) \oplus 0^S = \langle \lambda_a(x_1), x_2\rangle \oplus 0^S = \langle \lambda_a(x_1) \oplus 0, \frac{e}{2}\rangle$. What's more, $\lambda_{a^S}(0^S) = \langle \lambda_a(0), \frac{e}{2}\rangle = \langle a, \frac{e}{2}\rangle = a^S$.

Example 3.4. *Let $\langle A, \vee, \wedge, 0\rangle$ be a generalized Boolean algebra Conrad and Darnel [1997]. For any $x, y \in [0, b]$, where $\oplus = \vee$ and $\lambda_b(x)$ is the unique relative complement of x in $[0, b]$. Then $\langle A, \oplus, 0\rangle$ is an EMV-algebra by Example 3.2 (2) in Dvurečenskij and Zahiri [2019]. Hence, $\langle A, \oplus, 0\rangle$ is an Equasi-MV algebra.*

Example 3.5. *Let $\langle A, \oplus, ', 0, 1\rangle$ be a quasi-MV algebra and $\langle B, \vee, \wedge, 0\rangle$ be a generalized Boolean algebra. It is easy to show that $A \times B$ is an Equasi-MV algebra.*

Proof. The operation $\oplus$ on $A \times B$ is defined pointwise. For all $\langle x, y\rangle \in A \times B$, there exist $a \in \mathcal{I}(\mathbf{A})$ and $b \in \mathcal{I}(\mathbf{B})$ such that $\langle x, y\rangle \leqslant \langle a, b\rangle$ and $\langle [\langle 0, 0\rangle, \langle a, b\rangle], \oplus, \lambda_{\langle a,b\rangle}, \langle 0, 0\rangle, \langle a, b\rangle\rangle$ is a quasi-MV algebra.

Let's give a specific description of the above example. Let the Diamond (Example 2.6) be the 4-element quasi-MV algebra $\mathbf{A}$ and $\mathbf{M} = \langle M, \vee, \wedge, 0\rangle$ be the generalized Boolean algebra Conrad and Darnel [1997], where M is the set of components of any positive element $\mathbb{N}^+$ and the least element $0 := \emptyset$. That is, $M = \{N : N \subseteq \mathbb{N}^+\}$. Then every element N in M is idempotent. It is easily shown that $A \times M$ with the pointwise operation is an Equasi-MV algebra. □

Example 3.6. *Let $\mathbf{S} = \langle [0, 1] \times [0, 1], \oplus, ', 0, 1\rangle$ be a standard quasi-MV algebra A. Ledda and Giuntini [2006, Example 5]. Let $\mathbf{A} = \mathbf{S} \oplus \mathbf{S} \oplus \mathbf{S} \oplus \cdots$. Then $\mathbf{A}$ is an Equasi-MV algebra.*

Proof. Obviously, $\langle A, \oplus, 0\rangle$ is a commutative preordered semigroup and $(x \oplus y) \oplus 0 = x \oplus y$ for all $x, y \in A$. For any $x, y \in \mathbf{A}$. Suppose $x = (x_i)$, $y = (y_i)$. If $x_i \neq 0$ or $y_i \neq 0$, there exists $u_i \in \mathcal{I}(\mathbf{A})$ such that $x_i, y_i \leqslant u_i$ for all $i \geqslant 1$. If $x_i = y_i = 0$, take $u_i = 0$. We have an idempotent $u = (u_i) \in A$ such that $x, y \leqslant u$ and $\langle [0, u], \oplus, \lambda_u, 0, u\rangle$ is a quasi-MV algebra. □

Remark 3.1. *Let $\mathbf{A}$ be an Equasi-MV algebra. For all $x, y \in A$, there exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \in [0, b]$. In the quasi-MV algebra $\langle [0, b], \oplus, \lambda_b, 0, b\rangle$, we denote*

$$x \Cup_b y = \lambda_b(\lambda_b(x) \oplus y) \oplus y,\ x \Cap_b y = \lambda_b(\lambda_b(x) \oplus \lambda_b(\lambda_b(x) \oplus y)).$$

Proposition 3.1. *Let $\mathbf{A}$ be an Equasi-MV algebra and $a, b \in \mathcal{I}(\mathbf{A})$ such that $a \leqslant b$. For each $x \in [0, a]$, we have*

(i) $\lambda_b(a)$ is an idempotent, and $\lambda_a(a)$=0;
(ii) $\lambda_a(x) \oplus 0 = \lambda_b(x) \Cap a$;
(iii) $\lambda_b(x) \oplus 0 = \lambda_a(x) \oplus \lambda_b(a)$;
(iv) $\lambda_a(x) \leqslant \lambda_b(x)$.

Proof. Since $\langle [0,b], \oplus, \lambda_b, 0, b\rangle$ is a quasi-MV algebra and $a \in \mathcal{I}(\mathbf{A})$, by Lemma 2.5 (i) we get that $x \oplus a = x \Cup a$ for all $x \in [0,b]$.

(i) Since $\langle [0,b], \oplus, \lambda_b, 0, b\rangle$ is a quasi-MV algebra, $\lambda_b(a)$ is also an idempotent element by Lemma 26 in A. Ledda and Giuntini [2006]. It is obvious $\lambda_a(a) = 0$ in the quasi-MV algebra $\langle [0,a], \oplus, \lambda_a, 0, a\rangle$.

(ii) For all $x \in [0,a]$, we have

$$(\lambda_b(x) \Cap a) \oplus (x \oplus 0) = (\lambda_b(x) \oplus (x \oplus 0)) \Cap (a \oplus (x \oplus 0)) \text{ (Lemma 2.3 (vi))}$$
$$= b \Cap a = a.$$

It follows that $\lambda_a(x) \oplus 0 = \lambda_a(x \oplus 0) \leqslant \lambda_b(x) \Cap a$ in the quasi-MV algebra $\langle [0,b], \oplus, \lambda_b, 0, b\rangle$. Conversely, since $b = a \oplus \lambda_b(a) = x \oplus (\lambda_a(x) \oplus \lambda_b(a))$, we get $\lambda_b(x) \leqslant \lambda_a(x) \oplus \lambda_b(a)$. Since $\lambda_b(a)$ is an idempotent, by Lemma 2.5 (i) we have $\lambda_a(x) \oplus \lambda_b(a) = \lambda_a(x) \Cup \lambda_b(a)$. Hence, $\lambda_b(x) \leqslant \lambda_a(x) \Cup \lambda_b(a)$. Thus

$$\lambda_b(x) \Cap a \leqslant (\lambda_a(x) \Cup \lambda_b(a)) \Cap a \text{ (Lemma 2.2 (iv))}$$
$$= \lambda_a(x) \oplus 0 \text{ (Lemma 2.4 (i))}.$$

Summary of the above results, we get that $\lambda_a(x) \oplus 0 = \lambda_b(x) \Cap a$.

(iii) By (ii) we have

$$\lambda_a(x) \oplus \lambda_b(a) = (\lambda_a(x) \oplus 0) \oplus \lambda_b(a)$$
$$= (\lambda_b(x) \Cap a) \oplus \lambda_b(a)$$
$$= \lambda_b(x) \Cup \lambda_b(a) \text{ (Lemma 2.3 (vi) and Lemma 2.5 (i))}.$$

It follows from $x \leqslant a$ that $\lambda_b(a) \leqslant \lambda_b(x)$. Then $\lambda_b(x) \Cup \lambda_b(a) = \lambda_b(x) \oplus 0$. Therefore, $\lambda_b(x) \oplus 0 = \lambda_a(x) \oplus \lambda_b(a)$.

(iv) It follows from (ii) or (iii).□

The following statement shows that $\Cup_a$ and $\Cap_a$ on $[0,a]$ are coincide with $\Cup$ and $\Cap$ on A, respectively.

Proposition 3.2. *Let $\mathbf{A}$ be an Equasi-MV algebra. For all $x, y \in A$, there exist $a, b \in \mathcal{I}(\mathbf{A})$ such that $x, y \in [0,a]$ and $x, y \in [0,b]$. Then we have*

(i) $x \Cap_a y = x \Cap_b y$;

(ii) $x \Cup_a y = x \Cup_b y$.

Proof. (i) By Definition 3.1, for all $a, b \in \mathcal{I}(\mathbf{A})$, there exists $c \in \mathcal{I}(\mathbf{A})$ such that $a, b \leqslant c$. Then we have

$$x \Cup_c y = x \oplus \lambda_c(x \oplus \lambda_c(y) \oplus 0)$$
$$= x \oplus \lambda_c(x \oplus \lambda_a(y) \oplus \lambda_c(a)) \text{ (Proposition 3.1 (iii))}$$
$$= x \oplus (\lambda_c(x \oplus \lambda_a(y)) \otimes_c a) \text{ (the definition of } \otimes_c)$$
$$= x \oplus (\lambda_c(x \oplus \lambda_a(y)) \Cap a) \text{ (Lemma 2.5 (ii))}$$
$$= x \oplus ((\lambda_a(x \oplus \lambda_a(y)) \Cup \lambda_c(a)) \Cap a) \text{ (Proposition 3.1(iii), Lemma 2.5(i))}$$
$$= x \oplus (\lambda_a(x \oplus \lambda_a(y)) \Cap a) \text{ (Lemma 2.4 (i))}$$
$$= x \oplus (\lambda_a(x \oplus \lambda_a(y)) \oplus 0) = x \Cup_a y.$$

Similarly, we can show that $x \Cup_c y = x \Cup_b y$. Hence, $x \Cup_a y = x \Cup_b y$.

(ii) We also have

$$\begin{aligned} x \Cap_c y &= \lambda_c(\lambda_c(x) \oplus \lambda_c(\lambda_c(x) \oplus y)) \\ &= \lambda_c(\lambda_c(x) \oplus \lambda_c(\lambda_a(x) \oplus y \oplus \lambda_c(a))) \text{ (Proposition 3.1 (iii))} \\ &= \lambda_c(\lambda_c(x) \oplus (\lambda_c(\lambda_a(x) \oplus y) \otimes_c a)) \text{ (definition of } \otimes_c) \\ &= \lambda_c(\lambda_c(x) \oplus ((\lambda_a(\lambda_a(x) \oplus y) \oplus \lambda_c(a)) \otimes_c a)) \text{ (Proposition 3.1 (iii))} \\ &= \lambda_c(\lambda_c(x) \oplus ((\lambda_a(\lambda_a(x) \oplus y) \oplus \lambda_c(a)) \Cap a)) \text{ (Lemma 2.5 (i))} \\ &= \lambda_c(\lambda_c(x) \oplus (\lambda_a(\lambda_a(x) \oplus y) \Cap a)) \text{ (Lemma 2.4 (i))} \\ &= \lambda_c(\lambda_c(x) \oplus \lambda_a(\lambda_a(x) \oplus y)) \\ &= \lambda_c(\lambda_a(x) \oplus \lambda_a(\lambda_a(x) \oplus y) \oplus \lambda_c(a)) \text{ (Proposition 3.1 (iii))} \\ &= \lambda_c(\lambda_a(x) \oplus \lambda_a(\lambda_a(x) \oplus y)) \otimes_c a \text{ (definition of } \otimes_c) \\ &= (\lambda_a(\lambda_a(x) \oplus \lambda_a(\lambda_a(x) \oplus y)) \oplus \lambda_c(a)) \otimes_c a \text{ (Proposition 3.1 (iii))} \\ &= \lambda_a(\lambda_a(x) \oplus \lambda_a(\lambda_a(x) \oplus y)) \Cap a \\ &= x \Cap_a y. \end{aligned}$$

Similarly, we can show that $x \Cap_c y = x \Cap_b y$ and so $x \Cap_a y = x \Cap_b y$.□

Definition 3.2. *Let* $\mathbf{A}$ *be an Equasi-MV algebra and* $x, y \in [0, a]$ *where* $a \in \mathcal{I}(\mathbf{A})$. *A preordering* $\leqslant_a$ *on the quasi-MV algebra* $\langle [0, a], \oplus, \lambda_a, 0, a\rangle$ *defined as follows:*

$$x \leqslant_a y \iff x \Cap_a y = x \oplus 0.$$

By Proposition 3.2, for any $x, y \leqslant a, b$, where $a, b \in \mathcal{I}(\mathbf{A})$, we have $x \leqslant_a y \iff x \leqslant y \iff x \leqslant_b y$. Then we can also define a preordering $\leqslant$ on A by $x \leqslant y \iff x \Cap y = x \oplus 0$, where $x \Cap y = x \Cap_a y$.

Lemma 3.1. *Let* $\mathbf{A}$ *be an Equasi-MV algebra. For all* $x, y \in A$*, the operation* $\otimes$*:* $A \times A \to A$ *defined by* $x \otimes y = \lambda_a(\lambda_a(x) \oplus \lambda_a(y))$*, where* $a \in \mathcal{I}(\mathbf{A})$ *and* $x, y \leqslant a$*. Then*

(i) the well-defined binary operation $\otimes$ *on* A *is not determined by the choice of* a *and is also order preserving and associative.*

(ii) if $x, y \in A$*,* $x \leqslant y$*, then* $y \otimes \lambda_a(x) = y \otimes \lambda_b(x)$ *and* $y \oplus 0 = x \oplus (y \otimes \lambda_a(x))$ *for all* $a, b \in \mathcal{I}(\mathbf{A})$ *and* $x, y \leqslant a, b$*.*

(iii) if $x, y \in [0, a]$ *and* $a \in \mathcal{I}(\mathbf{A})$*, then* $x \otimes \lambda_a(y) = x \otimes \lambda_a(x \Cap y)$ *and* $x \oplus 0 = (x \Cap y) \oplus (x \otimes \lambda_a(y))$*.*

(iv) an element $a \in A$ *is idempotent iff* $a \otimes a = a$*.*

Proof. (i) Let $x, y \in A$ and $a, b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leqslant a, b$. We claim that $\lambda_a(\lambda_a(x) \oplus \lambda_a(y)) = \lambda_b(\lambda_b(x) \oplus \lambda_b(y))$. Indeed, there exists an element $c \in \mathcal{I}(\mathbf{A})$

such that $a, b \leqslant c$. Then

$$\begin{aligned}\lambda_c(\lambda_c(x) \oplus \lambda_c(y)) &= \lambda_c(\lambda_a(x) \oplus \lambda_c(a) \oplus \lambda_a(y) \oplus \lambda_c(a)) \text{ (Proposition 3.1 (iii))}\\ &= \lambda_c(\lambda_a(x) \oplus \lambda_a(y)) \otimes_c \lambda_c(\lambda_c(a)) \text{ (Propsition 3.1 (i))}\\ &= \lambda_c(\lambda_a(x) \oplus \lambda_a(y)) \Cap a \text{ (Lemma 2.5 (ii))}\\ &= (\lambda_a(\lambda_a(x) \oplus \lambda_a(y)) \oplus \lambda_c(a)) \Cap a \text{ (Lemma 3.1 (iii))}\\ &= (\lambda_a(\lambda_a(x) \oplus \lambda_a(y)) \Cup \lambda_c(a)) \Cap a \text{ (Lemma 2.5 (i))}\\ &= \lambda_a(\lambda_a(x) \oplus \lambda_a(y)) \Cap a\\ &= \lambda_a(\lambda_a(x) \oplus \lambda_a(y)).\end{aligned}$$

Similarly, we have $\lambda_c(\lambda_c(x) \oplus \lambda_c(y)) = \lambda_b(\lambda_b(x) \oplus \lambda_b(y))$.

Let $x, y, z \in A$. There exists $c \in \mathcal{I}(\mathbf{A})$ such that $x, y, z \leqslant c$. It follows from the definition of $\otimes$ that $x \otimes y$, $y \otimes z \in [0, c]$. Then

$$\begin{aligned}(x \otimes y) \otimes z &= \lambda_c(\lambda_c(x \otimes y) \oplus \lambda_c(z))\\ &= \lambda_c((\lambda_c(x) \oplus \lambda_c(y)) \oplus \lambda_c(z))\\ &= \lambda_c(\lambda_c(x) \oplus (\lambda_c(y) \oplus \lambda_c(z)))\\ &= \lambda_c(\lambda_c(x) \oplus \lambda_c(y \otimes z)) = x \otimes (y \otimes z).\end{aligned}$$

This proves that $\otimes$ is associative. It is easy to prove that $\otimes$ is order preserving.

(ii) Let $x \leqslant y$ and $x, y \leqslant a, b$, where $a, b \in \mathcal{I}(\mathbf{A})$. There exists $c \in \mathcal{I}(\mathbf{A})$ such that $a, b \leqslant c$. By Proposition 3.1, we have

$$\begin{aligned}y \otimes \lambda_a(x) &= \lambda_c(\lambda_c(y) \oplus \lambda_c(\lambda_a(x)))\\ &= \lambda_c(\lambda_c(y) \oplus \lambda_c(\lambda_a(x)) \oplus 0)\\ &= \lambda_c(\lambda_c(y) \oplus \lambda_c(\lambda_a(x) \oplus 0))\\ &= y \otimes (\lambda_a(x) \oplus 0).\end{aligned}$$

Then

$$\begin{aligned}y \otimes \lambda_c(x) &= y \otimes (\lambda_c(x) \oplus 0)\\ &= y \otimes (\lambda_a(x) \oplus \lambda_c(a))\\ &= y \otimes (\lambda_a(x) \Cup \lambda_c(a)) \text{ (Lemma 2.5 (i))}\\ &= (y \otimes \lambda_a(x)) \Cup (y \otimes \lambda_c(a)) \text{ (Lemma 2.3 (vii))}.\end{aligned}$$

Since $\lambda_c(a) \leqslant \lambda_c(y)$, we have $y \otimes \lambda_c(a) \leqslant y \otimes \lambda_c(y) = 0$, where $y \leqslant a \leqslant c$. This implies $y \otimes \lambda_c(x) = y \otimes \lambda_a(x)$. Similarly, we have $y \otimes \lambda_c(x) = y \otimes \lambda_b(x)$. It follows that $y \otimes \lambda_a(x) = y \otimes \lambda_b(x)$.

In the quasi-MV algebra $\langle [0, a], \oplus, \lambda_a, 0, a \rangle$, we have

$$x \oplus (y \otimes \lambda_a(x)) = x \oplus \lambda_a(\lambda_a(y) \oplus x) = x \Cup y = y \oplus 0.$$

(iii) Let $x, y \leqslant a$ and $a \in \mathcal{I}(\mathbf{A})$. We have

$$\begin{aligned}x \otimes \lambda_a(x \Cap y) &= x \otimes (\lambda_a(x) \Cup \lambda_a(y))\\ &= (x \otimes \lambda_a(x)) \Cup (x \otimes \lambda_a(y)) \text{ (Lemma 2.3 (vii))}\\ &= x \otimes \lambda_a(y).\end{aligned}$$

$$\begin{aligned}(x \Cap y) \oplus (x \otimes \lambda_a(y)) &= (x \Cap y) \oplus (x \otimes \lambda_a(x \Cap y)) \\ &= (x \Cap y) \oplus \lambda_a(\lambda_a(x) \oplus (x \Cap y)) \\ &= x \oplus \lambda_a(x \oplus \lambda_a(x \Cap y)) \text{ (QMV 4)} \\ &= x \oplus \lambda_a(x \oplus \lambda_a(x) \oplus \lambda_a(\lambda_a(x) \oplus y)) \\ &= x \oplus 0.\end{aligned}$$

(iv) $\Longrightarrow$: Suppose $a, b \in \mathcal{I}(\mathbf{A})$ with $a \leqslant b$. We have $\lambda_b(a) \oplus \lambda_b(a) = \lambda_b(a)$ by Proposition 3.1 (i). In the quasi-MV algebra $\langle [0, b], \oplus, \lambda_b, 0, b\rangle$, we have $a \otimes a = \lambda_b(\lambda_b(a) \oplus \lambda_b(a)) = \lambda_b(\lambda_b(a)) = a$.

$\Longleftarrow$: For each $a \in A$, there exists $b \in \mathcal{I}(\mathbf{A})$ such that $a \leqslant b$. Suppose $a \otimes a = a$. We have $\lambda_b(\lambda_b(a) \oplus \lambda_b(a)) = a$. Then $\lambda_b(\lambda_b(\lambda_b(a) \oplus \lambda_b(a))) = \lambda_b(a)$. It follows from $\lambda_b(a) \oplus \lambda_b(a) = \lambda_b(a)$ that $\lambda_b(a) \in \mathcal{I}(\mathbf{A})$. By Proposition 3.1 (i), we have $\lambda_b(\lambda_b(a)) \oplus \lambda_b(\lambda_b(a)) = \lambda_b(\lambda_b(a))$. That is $a \oplus a = a$. It implies $a \in \mathcal{I}(\mathbf{A})$. □

Theorem 3.1. *Let* $\mathbf{A}$ *be an Equasi-MV algebra. Then* $\langle \mathcal{R}(\mathbf{A}), \Cup_R, \Cap_R, \oplus_R, 0_R\rangle$ *is an EMV-subalgebra of* $\mathbf{A}$.

Proof. It is obvious that $\mathcal{R}(\mathbf{A})$ is closed under the operations $\Cup_R, \Cap_R, \oplus_R, 0_R$. For all $x, y \in \mathcal{R}(\mathbf{A})$, there exists $a \in \mathcal{I}(\mathbf{A})$ such that $x, y \leqslant a$. Then $[0, a] \cap \mathcal{R}(\mathbf{A})$ is an MV-algebra of $[0, a]$ by Lemma 15 in A. Ledda and Giuntini [2006]. This means that $\mathcal{R}(\mathbf{A})$ is an EMV-subalgebra of $\mathbf{A}$. □

4 Ideals and congruences

In this section, we give the notions of ideals and ideal congruences of Equasi-MV algebras. We also give an equivalent definition of ideals. Moreover, there is a one-to-one correspondence between the set of all ideals and the set of all ideal congruences.

Definition 4.1. *Let* $\mathbf{A}$ *be an Equasi-MV algebra. An equivalence relation* θ *on* A *is called a congruence, if the following conditions hold:*

(i) θ *is compatible with* $\oplus$*;*

(ii) for all $b \in \mathcal{I}(\mathbf{A}), \theta \cap ([0, b] \times [0, b])$ *is a congruence on the quasi-MV algebra* $\langle [0, b], \oplus, \lambda_b, 0, b\rangle$.

The set of all congruences on A represented by Con(A).

Definition 4.2. *Let* $\mathbf{A_1}, \mathbf{A_2}$ *be two Equasi-MV algebras. We call a map* $f : A_1 \longrightarrow A_2$ *to be an Equasi-MV homomorphism, if it satisfies the following statements:*

(i) $f(x \oplus y) = f(x) \oplus f(y)$ *and* $f(0) = 0$*, for all* $x, y \in A_1$*;*

(ii) for all $x, y \in [0, a]$ *and* $a \in \mathcal{I}(\mathbf{A_1})$*,* $f(\lambda_a(x)) = \lambda_{f(a)}(f(x))$.

Example 4.1. *Let $f : A_1 \to A_2$ be an Equasi-MV homomorphism. We can define $\theta = \{(x,y) \in A_1 \times A_1 : f(x) = f(y)\}$, then θ is a congruence.*

Let $\mathbf{A}$ be an Equasi-MV algebra and θ be a congruence on $\mathbf{A}$. We denote

$$A/\theta = \{x/\theta : x \in A\}, \text{ where } x/\theta = \{y \in A : \langle x, y\rangle \in \theta\}.$$

We define operations $\Cap$, $\Cup$, $\oplus$ on A/θ as follows: for any $x, y \in A$,

$x/\theta \Cap y/\theta = (x \Cap y)/\theta$, $x/\theta \Cup y/\theta = (x \Cup y)/\theta$, $x/\theta \oplus y/\theta = (x \oplus y)/\theta$.

Suppose $x/\theta \leqslant y/\theta$. Then $(x \Cap y)/\theta \geqslant x/\theta$. For all $z \in A$, we have

$$\begin{aligned} x/\theta \oplus z/\theta &= (x \oplus z)/\theta \\ &\leqslant ((x \Cap y) \oplus z)/\theta \\ &\leqslant (y \oplus z)/\theta \\ &= y/\theta \oplus z/\theta. \end{aligned}$$

This proves that $\langle A/\theta, \oplus, 0/\theta\rangle$ is a commutative preordered semigroup and $(x/\theta \oplus y/\theta) \oplus 0/\theta = x/\theta \oplus y/\theta$.

For all $x \in A$, there exists $a \in \mathcal{I}(\mathbf{A})$ such that $x \leqslant a$. It is easily shown that a/θ is an idempotent element and $x/\theta \leqslant a/\theta$. Since $\mathbf{A}$ is an Equasi-MV algebra, we have that $\langle [0,a], \oplus, \lambda_a, 0, a\rangle$ is a quasi-MV algebra. And let $\theta_a = \theta \cap ([0,a] \times [0,a])$ be an ideal congruence on $\langle [0,a], \oplus, \lambda_a, 0, a\rangle$. For any $x/\theta_a \in [0/\theta_a, a/\theta_a]$, we define $\lambda_{a/\theta_a}(x/\theta_a) = \lambda_a(x)/\theta_a$. Then $[0/\theta_a, a/\theta_a]$ is a quasi-MV algebra.

Now we show that $\langle [0/\theta, a/\theta], \oplus, \lambda_{a/\theta}, 0/\theta, a/\theta\rangle$ is a quasi-MV algebra. For all $x/\theta \in [0/\theta, a/\theta]$, there exists $y/\theta \in [0/\theta, a/\theta]$ such that $x/\theta \oplus y/\theta = a/\theta$. It follows that $\langle x \oplus y, a\rangle \in \theta$. And since $x, y \leqslant a$, we have $\langle x \oplus y, a\rangle \in \theta_a$. That is, $x/\theta_a \oplus y/\theta_a = a/\theta_a$. Thus $y/\theta_a \geqslant \lambda_a(x)/\theta_a$ and so $y/\theta \geqslant \lambda_a(x)/\theta$. This implies that $\lambda_{a/\theta}(x/\theta)$ exists and equals to $\lambda_a(x)/\theta$. It can be easily shown that $\langle [0/\theta, a/\theta], \oplus, \lambda_{a/\theta}, 0/\theta, a/\theta\rangle$ is a quasi-MV algebra. Thus, $\langle A/\theta, \oplus, 0/\theta\rangle$ is an Equasi-MV algebra.

And the map $\pi : \langle A, \oplus, 0\rangle \longrightarrow \langle A/\theta, \oplus, 0/\theta\rangle$ defined by $x \longmapsto x/\theta$ is an Equasi-MV homomorphism from A onto A/θ.

Definition 4.3. *Let $\mathbf{A}$ be an Equasi-MV algebra and I be a nonempty subset of A. We call I to be an ideal of A if the following conditions hold:*

(I1) $0 \in I$;

(I2) for all $x, y \in I$, then $x \oplus y \in I$;

(I3) $x \in I$ and $y \leqslant x$ imply $y \in I$.

If I is an ideal of A and $x \in A$, we have $x \in I$ iff $x \oplus 0 \in I$ by (I3).

Definition 4.4. *Let $\mathbf{A}$ be an Equasi-MV algebra and I be a nonempty subset of A. If the following statements hold, I is a weak ideal of A:*

(W1) $0 \in I$;

(W2) for all $x, y \in I$, then $x \oplus y \in I$;

(W3) $x \in I$ and $y \in A$ imply $x \otimes y \in I$.

Lemma 4.1. *Let I be an ideal of an Equasi-MV algebra $\mathbf{A}$. Then I is a weak ideal.*

Proof. Let I be an ideal of A and $x \in I$. If $y \in A$ with $y \leqslant x$, there exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leqslant b$. Then we have

$$\begin{aligned}(x \otimes y) \Cap x &= \lambda_b(\lambda_b(x) \oplus \lambda_b(y)) \Cap x \\ &= \lambda_b(\lambda_b(x) \oplus \lambda_b(y) \oplus \lambda_b(\lambda_b(x) \oplus \lambda_b(y) \oplus x)) \\ &= \lambda_b(\lambda_b(x) \oplus \lambda_b(y) \oplus \lambda_b(b)) = x \otimes y.\end{aligned}$$

It follows that $x \otimes y \leqslant x$. Thus $x \otimes y \in I$ and so I is a weak ideal of A.□

The converse of Lemma 4.1 is not true. For example, $\{0\}$ is a weak ideal, but not an ideal.

Proposition 4.1. *Let I be a nonempty subset of an Equasi-MV algebra $\mathbf{A}$ and $0 \in I$. Then I is an ideal iff for all $x, y \in A$, $a \in \mathcal{I}(\mathbf{A})$ with $x, y \leqslant a$, $\lambda_a(x) \otimes y \in I$ and $x \in I$ implies $y \in I$.*

Proof. $\Longrightarrow$: Let I be an ideal of A. For all $x, y \in A$ and $a \in \mathcal{I}(\mathbf{A})$ with $x, y \leqslant a$, if $\lambda_a(x) \otimes y \in I$ and $x \in I$, we have $(\lambda_a(x) \otimes y) \oplus x \in I$. Since

$$\begin{aligned}\lambda_a(y) \oplus ((\lambda_a(x) \otimes y) \oplus x) &= \lambda_a(y) \oplus (\lambda_a(x \oplus \lambda_a(y)) \oplus x) \\ &= \lambda_a(y) \oplus (\lambda_a(\lambda_a(x) \oplus y) \oplus y) \text{ (QMV4)} \\ &= \lambda_a(y) \oplus y \oplus \lambda_a(\lambda_a(x) \oplus y) \\ &= a,\end{aligned}$$

we have $y \leqslant (\lambda_a(x) \otimes y) \oplus x \in I$ and $y \in I$.

$\Longleftarrow$: For any $x, y \in I$ and $a \in \mathcal{I}(\mathbf{A})$ with $x \leqslant y$ and $x, y \leqslant a$, we have $\lambda_a(x) \otimes y = 0 \in I$. Hence, $y \in I$ is obtained from propositional conditions. And then

$$\begin{aligned}\lambda_a(x) \otimes (x \oplus y) &= \lambda_a(x \oplus \lambda_a(x \oplus y)) \\ &= \lambda_a(x) \Cap y \\ &\leqslant y \in I.\end{aligned}$$

Then $\lambda_a(x) \otimes (x \oplus y) \in I$. It follows from $x \in I$ that $x \oplus y \in I$.□

Definition 4.5. *Let $\mathbf{A}$ be an Equasi-MV algebra. We define a binary relation $\preccurlyeq$ as follows: for all $x, y \in A$,*

$$x \preccurlyeq y \text{ iff } x \Cap y = x.$$

The binary relation $\preccurlyeq$ satisfies antisymmetry and transitivity, but when x is a regular element, it satisfies reflexivity.

Lemma 4.2. *Let $\mathbf{A}$ be an Equasi-MV algebra and $x, y \in A$. Then*

$$x \preccurlyeq y \text{ iff } x \leqslant y \text{ and } x \in \mathcal{R}(\mathbf{A}).$$

Proof. If $x \preccurlyeq y$, we have $x \Cap y = x$ and $x \Cap y = (x \Cap y) \oplus 0 = x \oplus 0$. It follows that $x \leqslant y$ and $x \oplus 0 = x$. Thus $x \in \mathcal{R}(\mathbf{A})$. Conversely, if $x \leqslant y$ and $x \in \mathcal{R}(\mathbf{A})$, we have $x \Cap y = x \oplus 0 = x$ and so $x \preccurlyeq y$.□

Lemma 4.3. *Let* $\mathbf{A}$ *be an Equasi-MV algebra and* $J \subseteq A$*. Then the following statements are equivalent:*

(i) J *is a weak ideal of* A*;*

(ii) (1) if $x, y \in J$*, then* $x \oplus y \in J$*; (2) if* $x \in J$*,* $y \preccurlyeq x$*, then* $y \in J$*.*

Proof. (i)$\Longrightarrow$(ii): Suppose $x \in J$ and $y \preccurlyeq x$. There exists $b \in \mathcal{I}(\mathbf{A})$ such that $x \leqslant b$. Then $x \otimes (\lambda_b(x) \oplus y) = x \Cap y \in J$. Since $y \preccurlyeq x$, we have $x \Cap y = y \in J$.

(ii)$\Longleftarrow$(i): For any $x \in J$, $y \in A$, there exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leqslant b$. Since $x \otimes y \leqslant x$ and $x \otimes y \in \mathcal{R}(\mathbf{A})$ by Lemma 4.2, we have $x \otimes y \preccurlyeq x$. Therefore, $x \otimes y \in J$.□

Let $\mathbf{A}$ be an Equasi-MV algebra and H be a subset of A. The ideal generated by H is the smallest ideal of $\mathbf{A}$ containing H, denoted by $\langle H \rangle$.

Lemma 4.4. *Let* $\mathbf{A}$ *be an Equasi-MV algebra and* $H \subseteq A$*, then*

(i) $\langle H \rangle = \{x \in A$*: there exist* $h_1, \cdots, h_n \in H, n \in \mathbb{N}$ *such that* $x \leqslant h_1 \oplus \cdots \oplus h_n\}$*;*

(ii) $\langle 0 \rangle$ *is the smallest ideal of* $\mathbf{A}$*;*

(iii) If I *is an ideal of* A *and* $x \in A$*, we have*

$$\langle I \cup \{x\} \rangle = \{z \in A : z \leqslant a \oplus n.x \text{ for some } a \in I \text{ and } n \in \mathbb{N}\}.$$

Proof. (i) We write $M = \{x \in A$: there exist $h_1, \cdots, h_n \in H, n \in \mathbb{N}$ such that $x \leqslant h_1 \oplus \cdots \oplus h_n\}$. Then M is an ideal of A. Now we show that M is the smallest ideal of $\mathbf{A}$ containing H. Suppose M' is an ideal of $\mathbf{A}$ containing H. For any $x \in M$, there exist $h_1, \cdots, h_n \in H$ such that $x \leqslant h_1 \oplus \cdots \oplus h_n$. As $H \subseteq M'$, we get $x \in M'$ and so $M \subseteq M'$.

(ii) By (i) we obvious get the result.□

Definition 4.6. *An ideal* I *of an Equasi-MV algebra* $\mathbf{A}$ *is maximal if for all* $x \in A \setminus I$*,* $\langle I \cup \{x\} \rangle = A$*.*

Definition 4.7. *Let* $\mathbf{A}$ *be an Equasi-MV algebra and* θ *be a congruence on* $\mathbf{A}$*.* θ *is an ideal congruence if for all* $x, y \in A$*,* $(x \oplus 0)\theta(y \oplus 0) \Rightarrow x\theta y$*.*

Example 4.2. *Let* $\mathbf{A}$ *be an Equasi-MV algebra and* $x, y \in A$*. A binary relation* χ *defined as follows:* $x\chi y$ *iff* $x \leqslant y$ *and* $y \leqslant x$*.*

It is easy to show that χ *is compatible with* $\oplus$*. We now show that for all* $b \in \mathcal{I}(\mathbf{A})$*,* $\chi \cap ([0, b] \times [0, b])$ *is congruence on the quasi-MV algebra* $\langle [0, b], \oplus, \lambda_b, 0, b \rangle$*. Suppose* $\langle x, y \rangle \in \chi \cap ([0, b] \times [0, b])$*. It follows from* $\langle x, y \rangle \in \chi$ *that* $x \leqslant y$ *and* $y \leqslant x$*. Hence,* $\lambda_b(y) \leqslant \lambda_b(x)$ *and* $\lambda_b(x) \leqslant \lambda_b(y)$*. Therefore,* $\langle \lambda_b(x), \lambda_b(y) \rangle \in \chi \cap ([0, b] \times [0, b])$*. That is,* χ *is a congruence on* A*. As a result,* χ *is an ideal congruence.*

Definition 4.8. *Let* $\mathbf{A}$ *be an Equasi-MV algebra,* I *be an ideal of* $\mathbf{A}$ *and* θ *be an ideal congruence on* $\mathbf{A}$*. We define two relations* $f(J)$ *on* $A \times A$ *and* $g(\theta)$ *on* A *as follows:*

$\langle x, y\rangle \in f(J)$ *iff there exists* $b \in \mathcal{I}(\mathbf{A})$ *such that* $x \otimes \lambda_b(y), y \otimes \lambda_b(x) \in J$;

$g(\theta) = 0/\theta = \{x \in A : x\theta 0\}$.

Theorem 4.1. *Let* $\mathbf{A}$ *be an Equasi-MV algebra,* J *be an ideal of* $\mathbf{A}$ *and* θ *be an ideal congruence on* $\mathbf{A}$*.*

(i) $f(J)$ *is an ideal congruence on* $\mathbf{A}$*;*

(ii) $g(\theta)$ *is an ideal of* $\mathbf{A}$*;*

(iii) $J = g(f(J))$*;*

(iv) $\theta = f(g(\theta))$*.*

Proof. (i) Obviously, $f(J)$ is a congruence on A. Now we show that $f(J)$ is an ideal congruence. Let $\langle x \oplus 0, y \oplus 0\rangle \in f(J)$. There exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leqslant b$. Then $\lambda_b(x \oplus 0) \otimes (y \oplus 0)$, $\lambda_b(y \oplus 0) \otimes (x \oplus 0) \in J$. It follows that $\lambda_b(x) \otimes y$=$\lambda_b(x \oplus 0) \otimes (y \oplus 0) \in J$. Similarly, $\lambda_b(y) \otimes x \in J$. Thus, $\langle x, y\rangle \in f(J)$. Therefore, $f(J)$ is an ideal congruence on A.

(ii) Suppose $\langle x, 0\rangle \in \theta$ and $y \leqslant x$. We have $\langle \lambda_b(x), b\rangle \in \theta$. That implies $\langle \lambda_b(x) \oplus y, b\rangle \in \theta$ and so $\langle x \otimes (\lambda_b(x) \oplus y), x \otimes b\rangle \in \theta$. That is, $\langle x \Cap y, x \oplus 0\rangle \in \theta$. It follows from $y \leqslant x$ that $x \Cap y = y \oplus 0$. Thus, $\langle y \oplus 0, x \oplus 0\rangle \in \theta$. Since θ is an ideal congruence on $\mathbf{A}$, we have $\langle y, x\rangle \in \theta$. This together with $\langle 0, x\rangle \in \theta$ implies that $\langle y, 0\rangle \in \theta$ and so $y \in g(\theta)$. Therefore, $g(\theta)$ is an ideal of $\mathbf{A}$.

(iii) It is easily seen that $g(f(J)) = \{x \in A : x \oplus 0 \in J\}$. For all $x \in A$, we have $x \in J$ iff $x \oplus 0 \in J$. Thus $g(f(J)) = \{x \in A : x \in J\}$.

(iv) For any $x, y \in A$, if $\langle x, y\rangle \in f(g(\theta))$, there exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leqslant b$, $\langle \lambda_b(x) \otimes y, 0\rangle \in \theta$ and $\langle \lambda_b(y) \otimes x, 0\rangle \in \theta$. Then $\langle (\lambda_b(x) \otimes y) \oplus x, 0 \oplus x\rangle \in \theta$. By $(\lambda_b(x) \otimes y) \oplus x = x \Cup y$, we get $\langle x \Cup y, 0 \oplus x\rangle \in \theta$. Similarly, we have $\langle x \Cup y, 0 \oplus y\rangle \in \theta$. Thus, $\langle 0 \oplus x, 0 \oplus y\rangle \in \theta$. Since θ is an ideal congruence on $\mathbf{A}$, we have $\langle x, y\rangle \in \theta$. Therefore, $f(g(\theta)) \subseteq \theta$.

Conversely, if $\langle x, y\rangle \in \theta$, there exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leqslant b$ and so $\langle y \otimes \lambda_b(x), x \otimes \lambda_b(x)\rangle \in \theta$. This together with $x \otimes \lambda_b(x) = 0$ implies $\langle y \otimes \lambda_b(x), 0\rangle \in \theta$. Similarly, $\langle x \otimes \lambda_b(y), 0\rangle \in \theta$. Thus, $\langle x, y\rangle \in f(g(\theta))$. Therefore, $\theta \subseteq f(g(\theta))$.□

Let I be an ideal of an Equasi-MV algebra $\mathbf{A}$. The relation θ_I is defined as follows: for all $x, y \in A$,

$(x, y) \in \theta_I \iff \exists b \in \mathcal{I}(\mathbf{A})$ with $x, y \leqslant b$ such that $\lambda_b(\lambda_b(x) \oplus y), \lambda_b(\lambda_b(y) \oplus x) \in I$.

Proposition 4.2. *Let* $\mathbf{A}$ *be an Equasi-MV algebra. If* I *is an ideal of* A*, the relation* θ_I *is an ideal congruence on* A*.*

Proof. Let I be an ideal of A. Suppose $\langle x, y\rangle, \langle y, z\rangle \in \theta_I$. We have $\lambda_b(\lambda_b(x) \oplus y), \lambda_b(\lambda_b(y) \oplus x) \in I$ and $\lambda_b(\lambda_b(z) \oplus y), \lambda_b(\lambda_b(y) \oplus z) \in I$ where $b \in \mathcal{I}(\mathbf{A})$ such that $x, y, z \leqslant b$. Since I is an ideal of A, we have $\lambda_b(\lambda_b(x) \oplus y) \oplus \lambda_b(\lambda_b(y) \oplus z) \in I$ and $\lambda_b(\lambda_b(y) \oplus x) \oplus \lambda_b(\lambda_b(z) \oplus y) \in I$. And $(\lambda_b(x) \oplus z) \oplus (\lambda_b(\lambda_b(x) \oplus y) \oplus \lambda_b(\lambda_b(y) \oplus z)) = b$. It follows that $\lambda_b(\lambda_b(x) \oplus z) \in I$. Similarly, $\lambda_b(\lambda_b(z) \oplus x) \in I$. Then $\langle x, z \rangle \in \theta_I$. The reflexivity and symmetry is clear.

It is easy to prove that θ_I is compatible with $\oplus$. For all $u \in \mathcal{I}(\mathbf{A})$ such that $x, y, z \leqslant u$. Now, we show that $\theta_{I_u} = \theta_I \cap ([0, u] \times [0, u])$ is a congruence on the quasi-MV algebra $\langle [0, u], \oplus, \lambda_u, 0, u \rangle$. Suppose $\langle x, y \rangle \in \theta_{I_u}$, we have $\lambda_u(\lambda_u(x) \oplus y), \lambda_u(\lambda_u(y) \oplus x) \in I \cap ([0, u] \times [0, u])$. Then

$$\begin{aligned}
&(\lambda_u(x \oplus z) \oplus (y \oplus z)) \oplus \lambda_u(\lambda_u(x) \oplus y) \\
=&\lambda_u(x \oplus z) \oplus x \oplus z \oplus \lambda_u(\lambda_u(y) \oplus x) \\
=&\lambda_u(\lambda_u(x) \oplus \lambda_u(z)) \oplus \lambda_u(z) \oplus z \oplus \lambda_u(\lambda_u(y) \oplus x) \\
=&u.
\end{aligned}$$

It follows that $\lambda_u(\lambda_u(x \oplus z) \oplus (y \oplus z)) \leqslant \lambda_u(\lambda_u(x) \oplus y) \in \theta_I$. Then $\lambda_u(\lambda_u(x \oplus z) \oplus (y \oplus z)) \in \theta_I$. Similarly, $\lambda_u(\lambda_u(y \oplus z) \oplus (x \oplus z)) \in \theta_I$. Thus, $\langle x \oplus z, y \oplus z \rangle \in \theta_{I_u}$. And $\langle \lambda_u(x), \lambda_u(z) \rangle \in \theta_{I_u}$ is obvious. Therefore, θ_I is a congruence on A.

For each $\langle x \oplus 0, y \oplus 0 \rangle \in \theta_I$, we have $\lambda_b(\lambda_b(x \oplus 0) \oplus (y \oplus 0)), \lambda_b(\lambda_b(y \oplus 0) \oplus (x \oplus 0)) \in I$. That is, $\lambda_b(\lambda_b(x) \oplus y), \lambda_b(\lambda_b(y) \oplus x) \in I$. Thus $\langle x, y \rangle \in \theta_I$. Therefore, θ_I is an ideal congruence.□

Theorem 4.2. *Let* $\mathbf{A}$ *be an Equasi-MV algebra. There is a one-to-one correspondence between the set of all ideals and the set of all ideal congruences.*

Proof. Let I be an ideal of A and θ_I be an ideal congruence induced by I. Now we show that $I = 0/\theta_I$. Since $0 \in I$, we have $\langle x, 0 \rangle \in \theta_I$, for all $x \in I$. It follows that $x \in 0/\theta_I$. Conversely, suppose $x \in 0/\theta_I$. There exists $a \in \mathcal{I}(\mathbf{A})$ such that $x \leqslant a$. By Proposition 4.1, since $\lambda_a(x) \otimes 0 \in I$ and $0 \in I$, we have $x \in I$. Hence, $I = 0/\theta_I$.

Let θ be an ideal congruence on A. Let $I = 0/\theta$. Suppose $\langle x, y \rangle \in \theta_I$. There exists $a \in \mathcal{I}(\mathbf{A})$ such that $x, y \leqslant a$ and $\lambda_b(\lambda_b(x) \oplus y), \lambda_b(\lambda_b(y) \oplus x) \in I = 0/\theta$. That is, $\langle \lambda_b(\lambda_b(x) \oplus y), 0 \rangle \in \theta$ and $\langle \lambda_b(\lambda_b(y) \oplus x), 0 \rangle \in \theta$. Hence, $\langle \lambda_b(\lambda_b(x) \oplus y) \oplus y, 0 \oplus y \rangle \in \theta$ and $\langle \lambda_b(\lambda_b(y) \oplus x) \oplus x, 0 \oplus x \rangle \in \theta$. Since $\lambda_b(\lambda_b(x) \oplus y) \oplus y = \lambda_b(\lambda_b(y) \oplus x) \oplus x$, we have $\langle x \oplus 0, y \oplus 0 \rangle \in \theta$. And since θ is an ideal congruence on A, we have $\langle x, y \rangle \in \theta$.

Conversely, let $\langle x, y \rangle \in \theta$. There exists $a \in \mathcal{I}(\mathbf{A})$ such that $x, y \leqslant a$. Then $\langle \lambda_a(x), \lambda_a(y) \rangle \in \theta$ and $\langle \lambda_a(x) \otimes y, \lambda_a(y) \otimes y \rangle \in \theta$. Since $\lambda_a(y) \otimes y = 0$, we have $\lambda_a(x) \otimes y \in 0/\theta$. Similarly, $\lambda_a(y) \otimes x \in 0/\theta$. That is, $\langle x, y \rangle \in \theta_I$. Therefore, $\theta = \theta_I$.□

Theorem 4.3. *Let* $\mathbf{A}$ *be an Equasi-MV algebra. Then* $f(I) \circ f(J) = f(J) \circ f(I)$ *is vaild, where* I *and* J *are ideals of* $\mathbf{A}$.

Proof. Suppose $f(I), f(J) \in \mathrm{Con}I(\mathbf{A})$ and $\langle x, y\rangle \in f(I) \circ f(J)$ for $x, y \in A$. So there exists $z \in A$ such that $\langle x, z\rangle \in f(I)$ and $\langle z, y\rangle \in f(J)$. There exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y, z \leqslant b$. Let p be a ternary term defined as follows:

$$p_b(x, y, z) = (x \otimes (\lambda_b(y) \oplus (y \Cap z))) \Cup (z \otimes (\lambda_b(y) \oplus (y \Cap x))).$$

Then

$$(x \otimes (\lambda_b(z) \oplus (z \Cap y))) \Cup (y \otimes (\lambda_b(z) \oplus (z \Cap x)))\ f(I)\ p_b(z, z, y) = y \oplus 0$$

and

$$(x \otimes (\lambda_b(z) \oplus (z \Cap y))) \Cup (y \otimes (\lambda_b(z) \oplus (z \Cap x)))\ f(J)\ p_b(x, y, y) = x \oplus 0.$$

Let

$$(x \otimes (\lambda_b(z) \oplus (z \Cap y))) \Cup (y \otimes (\lambda_b(z) \oplus (y \Cap x))) = t,$$

where $t \leqslant b \in \mathcal{I}(\mathbf{A})$. It follows from $\langle t, y \oplus 0\rangle \in f(I)$ and $\langle t, x \oplus 0\rangle \in f(J)$ that

$$(y \oplus 0) \otimes \lambda_b(t),\ \lambda_b(y \oplus 0) \otimes t \in I;$$

$$(x \oplus 0) \otimes \lambda_b(t),\ \lambda_b(x \oplus 0) \otimes t \in J.$$

Now, $y \otimes \lambda_b(t) \leqslant (y \oplus 0) \otimes \lambda_b(t) \in I$, $x \otimes \lambda_b(t) \leqslant (x \oplus 0) \otimes \lambda_b(t) \in J$. Similarly, $\lambda_b(y) \otimes t \leqslant \lambda_b(y \oplus 0) \otimes t \in I$, $\lambda_b(x) \otimes t \leqslant \lambda_b(x \oplus 0) \otimes t \in J$. Thus, $\langle t, y\rangle \in f(I)$ and $\langle t, x\rangle \in f(J)$. That is, $\langle x, y\rangle \in f(J) \circ f(I)$.□

Lemma 4.5. *If* $\mathbf{A}$ *is an Equasi-MV algebra, the lattice Con*$I(\mathbf{A})$ *of ideal congruences on* $\mathbf{A}$ *is a sublattice of Con*$(\mathbf{A})$.

Proof. Let I, J be two ideals of $\mathbf{A}$. It is easy to prove that $f(I \cap J) = f(I) \cap f(J)$. Now we show that $f(I \vee J) = f(I) \vee f(J)$.

Since $g(f(I \vee J)) = I \vee J$ and $g(f(I)) \vee g(f(J)) = I \vee J$, we claim that $g(f(I) \vee f(J)) = g(f(I)) \vee g(f(J))$. Let $x \in g(f(I)) \vee g(f(J))$ such that $x \leqslant y \oplus z$ where $y \in g(f(I))$ and $z \in g(f(J))$. Then we get $\langle y, 0\rangle \in f(I)$, $\langle z, 0\rangle \in f(J)$ and $\langle y, z\rangle \in f(I) \circ f(J) = f(I) \vee f(J)$. It follows that $\langle z \oplus 0, 0\rangle \in f(J)$, $\langle y \oplus z, z \oplus 0\rangle \in f(I)$ and $\langle y \oplus z, 0\rangle \in f(I) \circ f(J) = f(I) \vee f(J)$. And then $x \leqslant y \oplus z \in g(f(I) \vee f(J))$. Therefore, $g(f(I)) \vee g(f(J) \subseteq g(f(I) \vee f(J))$.

Conversely, for any $x \in g(f(I) \vee f(J))$, we have $\langle x, 0\rangle \in f(I) \vee f(J) = f(I) \circ f(J)$. Then there exist $z \in A$ and $b \in \mathcal{I}(\mathbf{A})$ such that $\langle x, z\rangle \in f(I)$ and $\langle z, 0\rangle \in f(J)$. And $\langle x \otimes \lambda_b(z), 0\rangle \in f(I)$, $\langle z, 0\rangle \in f(J)$. Then $x \leqslant (x \otimes \lambda_b(z)) \oplus z$. Since $x \otimes \lambda_b(z) \in g(f(I))$ and $z \in g(f(J))$, we have $x \in g(f(I)) \vee g(f(J))$. Thus, $g(f(I) \vee f(J)) \subseteq g(f(I)) \vee g(f(J))$.□

Theorem 4.4. *Con*$I(\mathbf{A})$ *is distributive.*

Proof. By Theorem 4.2, we only need to prove that the lattice of ideals on A is distributive. Suppose I, J, K are ideals on A and $x \in I \cap (J \vee K)$. Then $x \in I$ and $x \leqslant y \oplus z$, for some $y \in J$, $z \in K$. Hence, $x \leqslant (x \Cap y) \oplus (x \Cap z)$. It follows from $x \Cap y \in I \cap J$, $x \Cap z \in I \cap K$ that $x \in (I \cap J) \vee (I \cap K)$.□

5 Filters and prime ideals

In this section, we introduce the notions of filters and prime ideals of Equasi-MV algebras. Moreover, we study some properties of them. We prove that every Equasi-MV algebra has at least one maximal ideal. Also, we get prime theorem on Equasi-MV algebras.

Definition 5.1. *Let $\langle A, \oplus, 0\rangle$ be an Equasi-MV algebra and F be a nonempty subset of A. F is called a filter if the following conditions are satisfied:*

(i) for all $x, y \in A$, if $x \leqslant y$ and $x \in F$, then $y \in F$;

(ii) for all $x, y \in F$, then $x \otimes y \in F$.

Definition 5.2. *We call a filter F is proper if $F \neq A$. A proper filter F is maximal, if for all $x \in A \setminus F$, $\langle F \cup \{x\}\rangle = A$.*

Let $\mathbf{A}$ be an Equasi-MV algebra. For $x \in A$ and $n \in \mathbb{N}$, we define

$$x^1 = x, \ \cdots, x^n = x^{n-1} \otimes x, n \geqslant 2.$$

Proposition 5.1. *Let $\mathbf{A}$ be an Equasi-MV algebra and F be a filter of $\mathbf{A}$. Then I_F is an ideal of A, where*

$$I_F := \{\lambda_a(x) : x \in F, \exists a \in \mathcal{I}(\mathbf{A}), x \leqslant a\}.$$

Proof. For all $x \in A$, we have

$$x \in I_F \iff \exists a \in \mathcal{I}(\mathbf{A}) \text{ s.t. } x \leqslant a, \lambda_a(x) \in F.$$

It is obvious that $0 \in I_F$. Suppose $x, y \in I_F$. There exist $a, b \in \mathcal{I}(\mathbf{A})$ such that $x \leqslant a$ and $y \leqslant b$. It follows $\lambda_a(x)$, $\lambda_b(y) \in F$. Let $c \in \mathcal{I}(\mathbf{A})$ such that $a, b \leqslant c$. Then $\lambda_c(x)$, $\lambda_c(y) \in F$ by Proposition 3.1 (iv). That implies $\lambda_c(x) \otimes \lambda_c(y) \in F$. Since $\lambda_c(x)$, $\lambda_c(y) \leqslant c$ and $\lambda_c(x) \otimes \lambda_c(y) = \lambda_c(x \oplus y)$, we have $x \oplus y \in I_F$.

Suppose $x, y \in A$ with $x \in I_F$ and $y \leqslant x$. There exists $a \in \mathcal{I}(\mathbf{A})$ such that $x \leqslant a$ and $\lambda_a(x) \in F$. Since $x, y \in [0, a]$ and $y \leqslant x$, we have $\lambda_a(x) \leqslant \lambda_a(y)$. It implies $\lambda_a(y) \in F$ and $y \in I_F$. □

In the following, we give an equivalent condition of maximal filters.

Proposition 5.2. *Let $\mathbf{A}$ be an Equasi-MV algebra and F be a proper filter of $\mathbf{A}$.*

(i) For all $x \in A$, $\langle F \cup \{x\}\rangle = \{z \in A : z \geqslant y \otimes x^n, \exists n \in \mathbb{N}, y \in F\}$;

(ii) F is a maximal filter iff for all $x \notin F$, there exist $n \in \mathbb{N}$ and $b \in \mathcal{I}(\mathbf{A})$ with $x \leqslant b$ such that $\lambda_b(x^n) \in F$.

Proof. (i) It is obvious.

(ii) Let F be a maximal filter and $x \notin F$. We have $0 \in \langle F \cup \{x\}\rangle$ by (i) and so there exist $n \in \mathbb{N}$ and $y \in F$ such that $0 = y \otimes x^n$. There exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leqslant b$. Then $b = \lambda_b(y \otimes x^n) = \lambda_b(y) \oplus \lambda_b(x^n)$, it follows that $y \leqslant \lambda_b(x^n)$ and $\lambda_b(x^n) \in F$. Conversely, for any $x \in A \setminus F$, there exist $n \in \mathbb{N}$, $b \in \mathcal{I}(\mathbf{A})$ such that $\lambda_b(x^n) \in F$. Then $0 = \lambda_b(x^n) \otimes x^n$ and $0 \in \langle F \cup \{x\}\rangle$. It follows that $\langle F \cup \{x\}\rangle = A$ and F is a maximal filter.□

Lemma 5.1. *Let F be a proper filter of an Equasi-MV algebra $\mathbf{A}$.*

(i) If $a \in F \cap \mathcal{I}(\mathbf{A})$, we have $a \notin I_F$.

(ii) If $a \in F \cap \mathcal{I}(\mathbf{A})$, then for all $b \in \mathcal{I}(\mathbf{A})$ with $a < b$, we have $\lambda_b(a) \in I_F$.

(iii) If F is a maximal filter of A, then for all $a \in \mathcal{I}(\mathbf{A})$, $a \notin I_F$ implies $a \in F$.

(iv) If J is a maximal ideal of A, then

$$\forall a \in \mathcal{I}(\mathbf{A}) \setminus J \Longrightarrow \lambda_b(a) \in J, \text{ where } b \in \mathcal{I}(\mathbf{A}) \text{ and } a < b. \qquad (*)$$

(v) If J is an ideal of A satisfying $(*)$, then F_J is a filter of A, where

$$F_J := \{\lambda_a(x) : x \in J, a \in \mathcal{I}(\mathbf{A}) \setminus J, x < a\}.$$

Proof. (i) Suppose $a \in F \cap \mathcal{I}(\mathbf{A})$ and $a \in I_F$. There exists $b \in \mathcal{I}(\mathbf{A})$ such that $a \leqslant b$ and $\lambda_b(a) \in F$. It follows from $\lambda_b(a), a \in F$ that $0 = a \otimes \lambda_b(a) \in F$, which is a contradiction.

(ii) It is obvious.

(iii) Let $a \in \mathcal{I}(\mathbf{A})$ and $a \notin I_F$. For all $b \in \mathcal{I}(\mathbf{A})$ with $a \leqslant b$, we have $\lambda_b(a) \notin F$ by Proposition 5.1. Suppose $a \notin F$. Since F is a maximal filter, we have $\langle F \cup \{a\}\rangle = A$. By Proposition 5.2, there exist $n \in \mathbb{N}$ and $x \in F$ such that $0 = x \otimes a^n$. We have $u \in \mathcal{I}(\mathbf{A})$ such that $x, a \leqslant u$ and $0 = x \otimes a^n = x \otimes_u a^n$. Since $a \in \mathcal{I}(\mathbf{A})$, we get $a^n = a$ and so $u = \lambda_u(x) \oplus \lambda_u(a)$. It follows that $x \leqslant \lambda_u(a)$ and $\lambda_u(a) \in F$, which is a contradiction.

(iv) Suppose $a \in \mathcal{I}(\mathbf{A})$ and $a \notin J$. For any $b \in \mathcal{I}(\mathbf{A})$ and $a < b$, we have $\lambda_b(a) \in \langle J \cup \{a\}\rangle = A$. By Lemma 4.4, there exist $n \in \mathbb{N}$ and $x \in J$ such that $\lambda_b(a) \leqslant x \oplus n.a$. Since $a, \lambda_b(a) \in [0, b]$, we have

$$\begin{aligned} \lambda_b(a) &= \lambda_b(a) \oplus 0 \\ &= \lambda_b(a) \Cap (x \oplus n.a) \\ &\leqslant (\lambda_b(a) \Cap x) \oplus (\lambda_b(a) \Cap n.a) \text{ (Lemma 2.4 (iii))} \\ &= \lambda_b(a) \Cap x. \end{aligned}$$

It follows $\lambda_b(a) \leqslant x \in J$ and so $\lambda_b(a) \in J$.

(v) Suppose $x, y \in A$ with $x \leqslant y$ and $x \in F_J$. There exists $a \in \mathcal{I}(\mathbf{A}) \setminus J$ such that $x < a$ and $\lambda_a(x) \in J$. Let $b \in \mathcal{I}(\mathbf{A})$ and $a, y \leqslant b$. We have $\lambda_b(y) \leqslant \lambda_b(x) \leqslant \lambda_a(x) \oplus \lambda_b(a)$. By (iv), we have $\lambda_b(a) \in J$ and $\lambda_a(x) \oplus \lambda_b(a) \in J$. That implies $\lambda_b(y) \in J$ and $y \in F_J$.

Let $x, y \in F_J$. There exist $a, b \in \mathcal{I}(\mathbf{A}) \setminus J$ such that $x \leqslant a$, $y \leqslant b$ and $\lambda_a(x), \lambda_b(y) \in J$. Let $c \in \mathcal{I}(\mathbf{A})$ and $a, b \leqslant c$. We have $\lambda_c(a), \lambda_c(b) \in J$ by (iv) and $\lambda_c(x) \leqslant \lambda_c(x) \oplus 0 = \lambda_a(x) \oplus \lambda_c(a) \in J$, $\lambda_c(y) \leqslant \lambda_c(y) \oplus 0 = \lambda_b(y) \oplus \lambda_c(b) \in J$ by Proposition 3.1. It follows that $\lambda_c(x), \lambda_c(y) \in J$ and $\lambda_c(x) \oplus \lambda_c(y) \in J$. Thus $\lambda_c(\lambda_c(x) \oplus \lambda_c(y)) \in F_J$. That is, $x \otimes y = x \otimes_c y \in F_J$.$\square$

Definition 5.3. *Let $\mathbf{A}$ be an Equasi-MV algebra and I be an ideal of $\mathbf{A}$. We call I to be prime if for all $x, y \in A$, $x \Cap y \in I$ implies that $x \in I$ or $y \in I$.*

Proposition 5.3. *Let I be an ideal of an Equasi-MV algebra $\mathbf{A}$. Then I is prime iff* *for any $x, y \in A$, there exists $a \in \mathcal{I}(\mathbf{A})$ with $x, y \leqslant a$ such that $\lambda_a(\lambda_a(x) \oplus y) \in I$ or $\lambda_a(\lambda_a(y) \oplus x) \in I$.*

Proof. $\Longleftarrow$: Let π: $A \longrightarrow A/I$ be the canonical projection and θ be an ideal congruence. If $x \Cap y \in I$, we have $(x \Cap y)/\theta = x/\theta \Cap y/\theta \in \pi(I)$. Let $x/\theta = [i]$ or $y/\theta = [j]$, where $i, j \in I$. There exists $a \in \mathcal{I}(\mathbf{A})$ such that $x, y, i, j \leqslant a$, $\lambda_a(x) \otimes i \in I$, $\lambda_a(i) \otimes x \in I$ or $\lambda_a(y) \otimes j \in I$, $\lambda_a(j) \otimes y \in I$. It follows from Proposition 4.1 that $x \in I$ or $y \in I$.

$\Longrightarrow$: For any $x, y \in A$, there exists $a \in \mathcal{I}(\mathbf{A})$ such that $x, y \leqslant a$. We have

$$
\begin{aligned}
&(\lambda_a(x)\oplus y)\Cup(\lambda_a(y)\oplus x)\\
=&\lambda_a(x)\oplus y\oplus\lambda_a(\lambda_a(x)\oplus y\oplus\lambda_a(\lambda_a(y)\oplus x))\\
=&\lambda_a(x)\oplus\lambda_a(\lambda_a(x)\oplus\lambda_a(\lambda_a(y)\oplus x))\oplus\lambda_a(\lambda_a(\lambda_a(x)\oplus\lambda_a(\lambda_a(y)\oplus x))\oplus\lambda_a(y))\\
=&\lambda_a(y)\oplus x\oplus\lambda_a(\lambda_a(y)\oplus x\oplus x)\oplus\lambda_a(\lambda_a(\lambda_a(x)\oplus\lambda_a(\lambda_a(y)\oplus x))\oplus\lambda_a(y))\\
=&\lambda_a(x)\oplus\lambda_a(\lambda_a(y)\oplus x))\oplus\lambda_a((\lambda_a(x)\oplus\lambda_a(\lambda_a(y)\oplus x))\oplus y)\oplus x\oplus\lambda_a(\lambda_a(y)\oplus x\oplus x)\\
=&a.
\end{aligned}
$$

It follows $\lambda_a((\lambda_a(x) \oplus y) \Cup (\lambda_a(y) \oplus x)) = 0 \in I$. That is, $\lambda_a(\lambda_a(x) \oplus y) \Cap \lambda_a(\lambda_a(y) \oplus x) = 0 \in I$. Therefore, $\lambda_a(\lambda_a(x) \oplus y) \in I$ or $\lambda_a(\lambda_a(y) \oplus x) \in I$.$\square$

Example 5.1. *Let $A \times M$ be an Equasi-MV algebra mentioned in Example 3.6. It can be easily proved that $P = \{0, b\}$ is a prime ideal of a quasi-MV algebra A. Now we show that $P \times M$ is a prime ideal of an Equasi-MV algebra $A \times M$. Obviously, $\langle 0, 0\rangle \in P \times M$ and $\langle 0, M\rangle \oplus \langle b, M\rangle = \langle b, M\rangle \in P \times M$. And for any $\langle x, M\rangle \leqslant \langle b, M\rangle$, we have $\langle x, M\rangle \in A \times M$. Then $P \times M$ is an ideal of $A \times M$. For any $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in A \times M$, suppose $\langle x_1, y_1\rangle \Cap \langle x_2, y_2\rangle = \langle x_1 \Cap x_2, y_1 \wedge y_2\rangle \in P \times M$, we have $x_1 \in P$ or $x_2 \in P$. That is, $\langle x_1, y_1\rangle \in P \times M$ or $\langle x_2, y_2\rangle \in P \times M$.*

Let $\mathbf{A}$ be a proper Equasi-MV algebra and $a \in \mathcal{I}(\mathbf{A}) \setminus \{0\}$. We define

$$\uparrow a = \{x \in A : x > a\}.$$

Then $\uparrow a$ is a filter of A. Moreover, $\uparrow a$ is a proper filter of A.

Proposition 5.4. *Let F be a maximal filter of an Equasi-MV algebra $\mathbf{A}$. Then*

$$I_F = \{\lambda_a(x) : x \in F, \exists a \in \mathcal{I}(\mathbf{A}), x \leqslant a\}$$

is a maximal ideal of A.

Proof. We know that I_F is an ideal of A by Proposition 5.1. As $F \neq \emptyset$, we have $a \in \mathcal{I}(\mathbf{A}) \cap F$ and so $a \notin I_F$ by Lemma 5.1 (i).

Let J be an ideal of A and $I_F \subseteq J$. Suppose $a \notin J$ and $a \in \mathcal{I}(\mathbf{A})$, we have $a \notin I_F$ and so $a \in F$ by Lemma 5.1 (iii). Then for any $b \in \mathcal{I}(\mathbf{A})$ with $a \leqslant b$, we have $\lambda_b(a) \in I_F \subseteq J$. Hence, J satisfies condition $(*)$ in Lemma 5.1 (iv). It follows from Lemma 5.1 (iv) that F_J is a filter of A.

Suppose $x \in F$ and $w \in \mathcal{I}(\mathbf{A}) \setminus J$. There exists $u \in \mathcal{I}(\mathbf{A})$ such that $x, w \leqslant u$. Since J is a proper ideal, we have $u \notin J$. It follows from the definition of I_F that $\lambda_u(x) \in I_F \subseteq J$ and then $x \in F_J$. That implies $F \subseteq F_J$.

Since F is a maximal filter, we have $F_J = F$ or $F_J = A$. If $F_J = A$, then there exist $x \in J$ and $a \in \mathcal{I}(\mathbf{A})$ such that $x < a$ and $\lambda_a(x) = 0$, which is a contradiction. Thus $F_J = F$. By Lemma 5.1 (v), for all $x \in J$, there exists $a \in \mathcal{I}(\mathbf{A}) \setminus J$ such that $x < a$ and $\lambda_a(x) \in F_J = F$. Hence, we have $x \in I_F$. That is, $J \subseteq I_F$. Thus $J = I_F$. This proves that I_F is a maximal ideal of A.□

Theorem 5.1. *Let* $\mathbf{A}$ *be a proper Equasi-MV algebra. Then* $\mathbf{A}$ *has at least one maximal ideal.*

Proof. Suppose $0 \neq a \in A$. Note that $\uparrow a$ is a filter and $\{0\} \neq \uparrow a$. By Zorn's lemma, we know that the set of all filters that does not contain 0 has a maximal element, which is a maximal filter of A, denoted by F. It follows from Proposition 5.4 that I_F is a maximal ideal.□

The following statement gives the prime theorem on Equasi-MV algebras.

Theorem 5.2. *Let I be a proper ideal of an Equasi-MV algebra* $\mathbf{A}$ *and $a \in A \setminus I$. Then there exists a maximal ideal P which contains I and $a \in A \setminus P$. Moreover, P is prime.*

Proof. Let $M = \{J : I \subseteq J, a \notin J\}$ where I, J are ideals of A. By Zorn's lemma, M has a top element P. It follows from $I \in M$ that $M \neq \emptyset$. We claim that P is prime. Suppose $x \Cap y \in P$ and $x, y \notin P$. We have $a \in \langle P \cup \{x\} \rangle$ and $a \in \langle P \cup \{y\} \rangle$. Then there exist $n \in \mathbb{N}$ and $u, v \in P$ such that $a \leqslant u \oplus n.x$ and $a \leqslant v \oplus n.y$. It follows that

$$a \leqslant (u \oplus n.x) \Cap (v \oplus n.y) \leqslant (u \oplus v \oplus n.x) \Cap (u \oplus v \oplus n.y).$$

By Lemma 2.4 (iii), we have

$$a \leqslant (u \oplus v \oplus n.x) \Cap (u \oplus v \oplus n.y) = (u \oplus v) \oplus (n.x \Cap n.y) \leqslant (u \oplus v) \oplus n^2.(x \Cap y) \in P.$$

It follows that $a \in P$, which is a contradiction. Thus, we have $x \in P$ or $y \in P$.□

6 Conclusion

In this paper, we introduce the notion of Equasi-MV algebras, which are generalizations of quasi-MV algebras. We study some basic properties of Equasi-MV algebras, such as ideals, ideal congruences and filters and investigate their mutual relationships. We show that there is a one-to-one correspondence between the set of all ideals and the set of all ideal congruences on an Equasi-MV algebra. And we also studied some results on maximal ideals and prime ideals.

There are many topics that deserve further study. For example, (1) can any Equasi-MV algebra be embedded into an Equasi-MV algebra with a top element?

(2) Does any simple Equasi-MV algebra have a top element? (3) The author introduced ME-algebras and studied the categorical equivalence between equality algebras and abelian lattice-ordered groups in Liu [2019]. We will study the relationships between monadic Equasi-MV algebras and monadic equality algebras.

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