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Sparse non-smooth atomic decomposition of quasi-Banach lattices

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Abstract. A theory of non-smooth atomic decomposition is obtained for a large class of quasi-Banach lattices, including Morrey spaces, Lorentz spaces, mixed Lebesgue spaces as well as some related function spaces. As an application, an inequality comparing the fractional maximal operator and the fractional integral operator is considered. Some examples show that the restriction posed on quasi-Banach lattices are indispensable. This paper, which is a follow-up of the third author's paper in 2020, simplifies the proof of some existing results.

1. Introduction

In this note we consider the decomposition of functions in quasi-Banach lattices over $\mathbb{R}^n$ by the use of functions in $L^\infty_{\rm c}(\mathbb{R}^n)$, the space of all compactly supported essentially bounded functions, satisfying the moment conditions of arbitrary order. Here and below the space $L^0(\mathbb{R}^n)$ denotes the linear space of all Lebesgue measurable functions in $\mathbb{R}^n$ and $\mathbb{N}_0 := \{0, 1, \ldots\}$.

A quasi-Banach lattice over $\mathbb{R}^n$ is a quasi-Banach space $(\mathcal{X}(\mathbb{R}^n), \|\cdot\|_{\mathcal{X}})$ contained in $L^0(\mathbb{R}^n)$ such that, for all $g \in \mathcal{X}(\mathbb{R}^n)$ and $f \in L^0(\mathbb{R}^n)$, the implication "$|f| \le |g| \Rightarrow f \in \mathcal{X}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{X}} \le \|g\|_{\mathcal{X}}$" holds. A quasi-Banach lattice is said to satisfy the Fatou property if $\lim\limits_{j\to\infty} f_j \in \mathcal{X}(\mathbb{R}^n)$ and $\left\|\lim\limits_{j\to\infty} f_j\right\|_{\mathcal{X}} = \lim\limits_{j\to\infty} \|f_j\|_{\mathcal{X}}$ if we are given a sequence $\{f_j\}_{j=1}^\infty \subset \mathcal{X}(\mathbb{R}^n)$ such that $0 \le f_1 \le f_2 \le \cdots$ and $\sup\limits_{j\in\mathbb{N}} \|f_j\|_{\mathcal{X}} < \infty$.

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Next, we introduce the definition of the moment order of measurable functions. Let $L \in \mathbb{N}_0$. The set $\mathcal{P}_L(\mathbb{R}^n)$ stands for the linear space of all polynomials of degree less than or equal to L. The set $\mathcal{P}_L(\mathbb{R}^n)^\perp$ denotes the set of all $f \in L^0(\mathbb{R}^n)$ for which

$$\int_{\mathbb{R}^n} (1+|x|)^L |f(x)| \mathrm{d}x < \infty$$

and $\displaystyle\int_{\mathbb{R}^n} x^\alpha f(x) \mathrm{d}x = 0$ for all $\alpha \in {\mathbb{N}_0}^n$ with $|\alpha| \le L$.

To describe the distribution of the support of functions appearing in the decomposition we consider, we recall the definition of sparseness. By a "cube" we mean a compact cube whose edges are parallel to the coordinate axes. The symbol $\mathcal{Q}$ stands for the set of all such cubes. The set of all dyadic cubes is denoted by $\mathcal{D}$:

$$\mathcal{D} := \left\{ Q_{jm} := \prod_{l=1}^n [2^{-j}m_l, 2^{-j}(m_l+1)) \,:\, (j,m) \in \mathbb{Z} \times \mathbb{Z}^n \right\},$$

where $(j, m_1, m_2, \ldots, m_n) = (j, m)$. Recall also that a set $\mathfrak{A} \subset \mathcal{D}$ is sparse, if there exists a disjoint collection $\{K(Q)\}_{Q \in \mathfrak{A}}$ of measurable sets such that $K(Q)$ is contained in Q and that $2|K(Q)| \ge |Q|$ for each $Q \in \mathfrak{A}$. Each $K(Q)$ is called a nutshell of Q. Finally, denote by $M_{\mathcal{D}}$ the dyadic maximal operator, that is, for $f \in L^0(\mathbb{R}^n)$,

$$M_{\mathcal{D}} f(x) := \sup_{Q \in \mathcal{D}} \chi_Q(x) m_Q(|f|) \quad (x \in \mathbb{R}^n),$$

where $m_Q(g)$ stands for the average of a function g integrable over $Q \in \mathcal{Q}$. Recall that a quadrant is a set of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \,:\, (-1)^{k_j} x_j > 0, j = 1, 2, \ldots, n\}$$

for some $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$.

We formulate the main result in this paper as an extension of the one by Strömberg and Torchinsky [33]. Throughout this paper C is used for constants that may change from one occurence to another. When we need to emphasize that the constant C depends on some important parameters, we add them as subscripts. Constants with subscripts remain unchanged from one occurence to another.

Theorem 1.1. *Let $(\mathcal{X}(\mathbb{R}^n), \|\cdot\|_{\mathcal{X}})$ be a quasi-Banach lattice with the Fatou property such that*

$$\|M_{\mathcal{D}} f\|_{\mathcal{X}} \le C \|f\|_{\mathcal{X}} \quad (f \in \mathcal{X}(\mathbb{R}^n)) \tag{1.1}$$

and that $L^\infty_{\mathrm{c}}(\mathbb{R}^n) \subset \mathcal{X}(\mathbb{R}^n)$. Let $f \in \mathcal{X}(\mathbb{R}^n)$ and $L \in \mathbb{N}_0$.

Assume either $f \in L^1(\mathbb{R}^n)$ or that

$$\|\chi_F\|_{\mathcal{X}} = \infty \tag{1.2}$$

for all quadrants F.

Then f admits a decomposition:

$$f = \sum_{Q \in \mathfrak{A}} \lambda_Q a_Q \quad a.e.,$$

where $\mathfrak{A} \subset \mathcal{D}$ is a sparse set, $a_Q \in \mathcal{P}_L(\mathbb{R}^n)^\perp$ and $|a_Q| \le \chi_Q$ for all $Q \in \mathfrak{A}$ and $\{\lambda_Q\}_{Q \in \mathfrak{A}} \subset \mathbb{R}$ satisfies

$$0 \le \lambda_Q \le C_{n,L} m_Q(|f|), \quad \left\| \left(\sum_{Q \in \mathfrak{A}} (\lambda_Q \chi_Q)^v \right)^{\frac{1}{v}} \right\|_{\mathcal{X}} \le C_{v,n,L,\mathcal{X}} \|f\|_{\mathcal{X}} \quad (1.3)$$

for all $v > 0$.

If $f \in L^1(\mathbb{R}^n)$, then Theorem 1.1 is easy to prove (see Section 3). What is significant is that this integrability condition can be replaced by (1.1), a weak restriction of the lattice $\mathcal{X}(\mathbb{R}^n)$.

We present an application of Theorem 1.1. Let I_α be the fractional integral operator of order $\alpha \in (0, n)$ given by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y \quad (x \in \mathbb{R}^n) \quad (1.4)$$

for a non-negative function $f \in L^0(\mathbb{R}^n)$. The fractional maximal operator M_α of order $\alpha \in [0, n)$ is defined by

$$M_\alpha f(x) := \sup_{Q \in \mathcal{Q}} \chi_Q(x) \ell(Q)^{\alpha-n} \int_Q |f(y)| \mathrm{d}y \quad (x \in \mathbb{R}^n)$$

for $f \in L^0(\mathbb{R}^n)$, where $\ell(S)$ denotes the side-length of $S \in \mathcal{Q} \cup \mathcal{D}$: $\ell(S) = |S|^{\frac{1}{n}}$. Note that $M := M_0$ is the Hardy–Littlewood maximal operator. A geometric observation shows that $M_\alpha f \le C_{\alpha,n} I_\alpha f$ for any non-negative function $f \in L^0(\mathbb{R}^n)$. As an application of Theorem 1.1, we obtain the following equivalence:

Theorem 1.2. *In addition to the Fatou property of $\mathcal{X}(\mathbb{R}^n)$, assume that there exist $1 < r < \infty$ and $C > 0$ such that*

$$\left\| \sum_{j=1}^\infty M f_j{}^r \right\|_{\mathcal{X}} \le C \left\| \sum_{j=1}^\infty |f_j|^r \right\|_{\mathcal{X}} \quad (1.5)$$

for all $\{f_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$. Then there exists $C > 0$ such that

$$C^{-1} \|M_\alpha f\|_{\mathcal{X}} \le \|I_\alpha f\|_{\mathcal{X}} \le C \|M_\alpha f\|_{\mathcal{X}}$$

for all non-negative functions $f \in L^0(\mathbb{R}^n)$.

A couple of remarks about condition (1.5) may be in order. First, either M or $M_{\mathcal{D}}$ does not have to be bounded on $\mathcal{X}(\mathbb{R}^n)$; it suffices to assume the

Fatou property of $\mathcal{X}(\mathbb{R}^n)$ and (1.5). Next, (1.5) is different from the usual vector-valued inequality:

$$\left\| \left(\sum_{j=1}^{\infty} M f_j{}^r \right)^{\frac{1}{r}} \right\|_{\mathcal{X}} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{X}}. \tag{1.6}$$

Needless to say, (1.6) is an extension of the well-known inequality which Fefferman and Stein proved for $\mathcal{X}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ in [10].

We note that (1.1) and (1.5) are independent conditions. In fact, $L^\infty(\mathbb{R}^n)$ satisfies (1.1) but fails (1.5), while $L^p(\mathbb{R}^n)$ with $0 < p \le 1$ satisfies (1.5) but fails (1.1). We claim that (1.5) is stronger than (1.2) since $L^\infty_{\rm c}(\mathbb{R}^n) \subset \mathcal{X}(\mathbb{R}^n)$. In fact, if $0 < r < \infty$, F is a quadrant and $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}$ is an increasing sequence such that 0 is a boundary point of Q_1 and that $\bigcup\limits_{j=1}^{\infty} Q_j = F$ almost everywhere, then

$$\infty \chi_{Q_1} \le \sum_{j=1}^{\infty} (M\chi_{Q_{j+1}\setminus Q_j})^r$$

and (1.5) implies

$$\infty \le C \left\| \sum_{j=1}^{\infty} \chi_{Q_{j+1}\setminus Q_j} \right\|_{\mathcal{X}} \le C\|\chi_F\|_{\mathcal{X}}.$$

Thus, (1.5) holds. It should be noted that (1.1) yields $\mathcal{X}(\mathbb{R}^n) \subset L^1_{\rm loc}(\mathbb{R}^n)$, as is seen from

$$m_Q(|f|)\|\chi_Q\|_{\mathcal{X}} \le \|M_{\mathcal{D}} f\|_{\mathcal{X}} \le C\|f\|_{\mathcal{X}} \tag{1.7}$$

for any $f \in \mathcal{X}(\mathbb{R}^n)$ and $Q \in \mathcal{Q}$.

There are many examples of quasi-Banach lattices satisfying (1.1), (1.2) and (1.5). At this moment, we content ourselves with Morrey spaces and give more examples in Section 6: We will use Morrey spaces to create a counterexample in Section 5. Let $0 < q \le p < \infty$. For $f \in L^q_{\rm loc}(\mathbb{R}^n)$ its Morrey quasi-norm is defined by

$$\|f\|_{\mathcal{M}^p_q} := \sup_{Q \in \mathcal{D}} \frac{|Q|^{\frac{1}{p}}}{\|\chi_Q\|_{L^q}} \left(\int_Q |f(y)|^q {\rm d}y \right)^{\frac{1}{q}}. \tag{1.8}$$

The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all $f \in L^q_{\rm loc}(\mathbb{R}^n)$ for which the quasi-norm $\|f\|_{\mathcal{M}^p_q}$ is finite. Chiarenza and Frasca established that M is bounded on $\mathcal{M}^p_q(\mathbb{R}^n)$ for $1 < q \le p < \infty$ [4], so that (1.1) is satisfied. Thus, we are in the position of using Theorem 1.1. Actually, Theorem 1.1 refines [18, Theorem 1.2]. Furthermore, the space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $0 < q \le 1 < p < \infty$ falls under the scope of Theorem 1.2. In this case, Theorem 1.2 recaptures a result by Tanaka [34]. In fact, (1.5) is equivalent to (1.6) for $\mathcal{X}(\mathbb{R}^n) = \mathcal{M}^{pr}_{qr}(\mathbb{R}^n)$. Thus, as long as $qr > 1$, or equivalently, $r > q^{-1}$, we have (1.5) according to [32, 35].

Theorem 1.1 can be located also as an extension of [33, Chapter VIII] in that the function space $L^p_w(\mathbb{R}^n)$ in [33, Chapter VIII] is replaced by general quasi-Banach lattices. In [30], assuming that (1.5) is true, we showed that many function spaces admit a decomposition as in Theorem 1.1 by the use of the grand maximal operator, so that the theory of Hardy spaces adapted to general Banach lattices had to be established. However, as our proof shows, we do not need to use it. Our proof significantly simplifies the one in [30]. Also, each a_Q is supported on the closure of Q instead of the one of its triple $3Q$. It is also remarkable that Theorem 1.1 can deal with the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with some discontinuous exponent (see Section 6.8) which does not fall within the scope of [30, 33]; see (6.5) for the precise definition of $L^{p(\cdot)}(\mathbb{R}^n)$.

Here we describe the structure of the remaining part of this paper. Section 2 deals with the preliminaries, while the proof of Theorem 1.1 is given in Section 3. We apply Theorem 1.1 in Section 4 to prove Theorem 1.2. We give examples of Banach lattices that we cannot drop conditions (1.1), (1.2) and (1.5) in Section 5. Finally, in Section 6 we conclude this paper with some function spaces we envisage and survey some results obtained earlier. We will see that Theorem 1.1 unifies many earlier results. We also supplement some auxiliary estimates for function spaces we list in Section 6.

2. A generalized Calderón–Zygmund decompsition

We follow [29] to recall some results. Fix $Q \in \mathcal{Q}$ and $L \in \mathbb{N}_0$.

Since $\{x^\alpha\}_{|\alpha|\le L}$ is linearly independent in $L^2(Q)$, for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ there uniquely exists $P^L_Q f \in \mathcal{P}_L(\mathbb{R}^n)$ such that $\chi_Q(f - P^L_Q f) \in \mathcal{P}_L(\mathbb{R}^n)^\perp$. The polynomial $P^L_Q f$ is called the Gram–Schmidt polynomial of order L for Q.

We follow [9, 26, 29] to recall the Calderón–Zygmund decomposition. Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$. Fix $A > 4^n$ and $k \in \mathbb{Z}$. Write $\Omega^k := \{x \in \mathbb{R}^n \,:\, M_{\mathcal{D}} f(x) > A^k\}$. Assume that Ω^k never contains any quadrant. We can find a disjoint collection $\{Q^k_j\}_{j\in J^k} \subset \mathcal{D}$ and a collection $\{g^k\} \cup \{b^k_j\}_{j\in J^k} \subset L^1(\mathbb{R}^n)$ such that:

(0) Each J^k is a countable index set.

(1) We have

$$\Omega^k = \sum_{j\in J^k} Q^k_j. \tag{2.1}$$

Furthermore,

$$A^k < m_{Q^k_j}(|f|) \le 2^n A^k.$$

(2) (Decomposition of f) f admits the following decomposition:

$$f = g^k + \sum_{j\in J^k} b^k_j \quad \text{a.e.}. \tag{2.2}$$

Here

$$g^k := f\chi_{\mathbb{R}^n\setminus\Omega^k} + \sum_{j\in J^k} P^L_{Q^k_j}(f)\chi_{Q^k_j}, \quad b^k_j := \chi_{Q^k_j}(f - P^L_{Q^k_j}(f)).$$

A direct consequence of this definition is that b^k_j satisfies the moment condition $b^k_j \in \mathcal{P}_L(\mathbb{R}^n)^\perp$ and the support condition $\mathrm{supp}(b^k_j) \subset \overline{Q^k_j}$.

We set

$$a^k_j := \chi_{Q^k_j}(g^k - g^{k+1}) \quad (k \in \mathbb{Z}, j \in J^k). \tag{2.3}$$

The proof of Theorem 1.1 uses the following facts:

Proposition 2.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $A > 4^n$. Assume that Ω^k never contains any quadrant for any $k \in \mathbb{Z}$.*

1. *The family $\mathfrak{A} := \{Q^k_j\}_{k\in\mathbb{Z},j\in J^k}$ is sparse. In fact, $K(Q^k_j) := Q^k_j \setminus \Omega^{k+1}$ is the nutshell.* [29, Proposition 1]
2. *There exists $C_{n,L} > 0$ such that $|g^k| \le C_{n,L}A^k$ for all $k \in \mathbb{Z}$.* [29, Proposition 2]
3. *Let $k \in \mathbb{Z}$ and $j \in J^k$. Then $a^k_j \in \mathcal{P}_L(\mathbb{R}^n)^\perp$ and there exists a constant $C_{n,L} > 0$ such that $|a^k_j| \le C_{n,L}A^k\chi_{Q^k_j}$.* [29, Proposition 3]
4. *Assume that $M_{\mathcal{D}}f$ is finite almost everywhere. Then $f = \lim_{k\to\infty}(g^k - g^{-k+1})$ in the sense of almost everywhere convergence.* [29, Corollary 1]

3. Proof of Theorem 1.1

We preserve all the notation in Section 2. We prove Theorem 1.1 for $f \in \mathcal{X}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ after we translate it into the following form:

Theorem 3.1. *Let $L \in \mathbb{N}_0$ and $A > 4^n$. Let $(\mathcal{X}(\mathbb{R}^n), \|\cdot\|_{\mathcal{X}})$ be a quasi-Banach lattice with the Fatou property such that* (1.1) *holds and that $L^\infty_{\mathrm{c}}(\mathbb{R}^n) \subset \mathcal{X}(\mathbb{R}^n)$. Let $f \in \mathcal{X}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then for $\{a^k_j\}_{k\in\mathbb{Z},j\in J^k}$ and $\mathfrak{A}$ as in* (2.3) *and Proposition* 2.1*, respectively, we have*

$$f = \sum_{k=-\infty}^{\infty}\sum_{j\in J^k} a^k_j \quad a.e., \quad A^k < m_{Q^k_j}(|f|) \le 2^n A^k$$

as well as the quasi-norm estimate

$$\left\| \left(\sum_{k=-\infty}^{\infty}\sum_{j\in J^k} (A^k\chi_{Q^k_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{X}} \le C_{v,n,L,\mathcal{X}}\|f\|_{\mathcal{X}}. \tag{3.1}$$

Proof. Since $f \in L^1(\mathbb{R}^n)$, we are in the position of using Proposition 2.1(4) thanks to the Hardy–Littlewood weak-(1,1) maximal inequality [9, Chapter 2]. What is not contained there is (3.1). Since the Ω^k's are nested,

$$\sum_{k=-\infty}^{\infty}\sum_{j\in J^k}(A^k\chi_{Q^k_j})^v = \sum_{k=-\infty}^{\infty}(A^k\chi_{\Omega^k})^v \le C_{v,A}\sup_{k\in\mathbb{Z}} A^{vk}\chi_{\Omega^k} \le C_{v,A}(M_{\mathcal{D}}f)^v.$$

Hence (3.1) holds from (1.1). □

We prove Theorem 1.1 for $\mathcal{X}(\mathbb{R}^n)$ satisfying (1.2). Let us drop the assumption $f \in L^1(\mathbb{R}^n)$; we suppose not only that $f \in \mathcal{X}(\mathbb{R}^n)$ but that (1.2) holds. Write $f^{(m)}(x) := \chi_{[0,m]}(\max(|x|, |f(x)|))f(x)$ for $x \in \mathbb{R}^n$ and $m \in \mathbb{N}$. Then, since $f^{(m)} \in L^\infty_{\rm c}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, according to what we have proved, each $f^{(m)}$ admits a decomposition as in Theorems 1.1 and 3.1. That is, with a slight change of notation, each $f^{(m)} \in \mathcal{X}(\mathbb{R}^n)$ admits a decomposition: there exist a sparse set $\mathfrak{A}^{(m)} := \{Q \in \mathcal{D} : \lambda_Q^{(m)} \neq 0\}$ with the nutshell $K^{(m)}(Q)$ for each $Q \in \mathfrak{A}^{(m)}$, $\{\lambda_Q^{(m)}\}_{Q\in\mathcal{D}} \subset \mathbb{R}$ and a collection $\{a_Q^{(m)}\}_{Q\in\mathcal{D}} \subset \mathcal{P}_L(\mathbb{R}^n)^\perp$ of functions satisfying $|a_Q^{(m)}| \leq \chi_Q$ for all $Q \in \mathcal{D}$ such that the following properties hold:

(1) There exists a constant $C_{n,L} > 0$ such that
$$0 \leq \lambda_Q^{(m)} \leq C_{n,L} m_Q(|f^{(m)}|). \tag{3.2}$$

(2) $f^{(m)} = \sum_{Q\in\mathcal{D}} \lambda_Q^{(m)} a_Q^{(m)}$ a.e..

(3) There exists a constant $C_A > 0$ such that
$$\sum_{Q\in\mathcal{D}} \lambda_Q^{(m)} \chi_Q^{(m)} \leq C_A M_{\mathcal{D}} f^{(m)} \leq C_A M_{\mathcal{D}} f \in L^1_{\rm loc}(\mathbb{R}^n). \tag{3.3}$$

(4) There exists a constant $C_{v,n,L,\mathcal{X}} > 0$ such that
$$\left\| \left(\sum_{Q\in\mathcal{D}} (\lambda_Q^{(m)} \chi_Q)^v \right)^{\frac{1}{v}} \right\|_{\mathcal{X}} \leq C_{v,n,L,\mathcal{X}} \|f^{(m)}\|_{\mathcal{X}} \leq C_{v,n,L,\mathcal{X}} \|f\|_{\mathcal{X}}. \tag{3.4}$$

By the diagonal argument, there exists an increasing sequence $\{m_l\}_{l=1}^\infty \subset \mathbb{N}$ such that for each $Q \in \mathcal{D}$,
$$a_Q := \lim_{l\to\infty} a_Q^{(m_l)} \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_L(\mathbb{R}^n)^\perp, \quad \lambda_Q := \lim_{l\to\infty} \lambda_Q^{(m_l)}$$
exist in the weak-* topology of $L^\infty(\mathbb{R}^n)$ and in $\mathbb{R}$, respectively. Since the weak-* topology preserves the moment, the support and the size of functions, we see that $|a_Q| \leq \chi_Q$ and that $a_Q \in \mathcal{P}_L(\mathbb{R}^n)^\perp$ for each $Q \in \mathcal{D}$. By the Fatou property of $\mathcal{X}(\mathbb{R}^n)$ and (3.4) with $v = 1$,
$$g := \sum_{Q\in\mathcal{D}} \lambda_Q a_Q \in \mathcal{X}(\mathbb{R}^n).$$
We claim that $f = g$. Once this is achieved, we obtain a candidate of the decomposition of f. Observe that the Lebesgue differentiation theorem reduces matters to
$$\int_R f(x){\rm d}x = \int_R g(x){\rm d}x \tag{3.5}$$
for all $R \in \mathcal{D}$.

Keeping in mind (1.7) and (3.3), we deduce

$$\int_R f(x)\mathrm{d}x = \lim_{l\to\infty}\int_R f^{(m_l)}(x)\mathrm{d}x = \lim_{l\to\infty}\sum_{Q\in\mathcal{D}}\int_R \lambda_Q^{(m_l)} a_Q^{(m_l)}(x)\mathrm{d}x \tag{3.6}$$

by using the Lebesgue convergence theorem twice.

We deal with

$$\sum_{Q\in\mathcal{D}}\int_R \lambda_Q^{(m_l)} a_Q^{(m_l)}(x)\mathrm{d}x.$$

First of all, $a_Q^{(m_l)} \in \mathcal{P}_L(\mathbb{R}^n)^\perp \subset \mathcal{P}_0(\mathbb{R}^n)^\perp$. Thus

$$\sum_{Q\in\mathcal{D}}\int_R \lambda_Q^{(m_l)} a_Q^{(m_l)}(x)\mathrm{d}x = \sum_{Q\in\mathcal{D}, Q\supset R}\int_R \lambda_Q^{(m_l)} a_Q^{(m_l)}(x)\mathrm{d}x.$$

Observe that for each $\mu\in\mathbb{N}$, there exists uniquely $R_\mu\in\mathcal{D}$ such that $R\subset R_\mu$ and that $\ell(R_\mu)=2^\mu\ell(R)$. Let $\varepsilon>0$ be fixed. Then there exists $M_\varepsilon \gg 1$ such that

$$\|\chi_{R_\mu}\|_{\mathcal{X}} > \frac{1+\|f\|_{\mathcal{X}}}{C_{1,n,L,\mathcal{X}}\varepsilon}$$

for all $\mu\ge M_\varepsilon$ thanks to (1.2). Here $C_{1,n,L,\mathcal{X}}$ is a constant in (3.4) with $v=1$. Thus, for all $l\in\mathbb{N}$,

$$\sum_{\mu=M_\varepsilon}^\infty \lambda_{R_\mu}^{(m_l)} \le \frac{1}{\|\chi_{R_{M_\varepsilon}}\|_{\mathcal{X}}}\left\|\sum_{\mu=M_\varepsilon}^\infty \lambda_{R_\mu}^{(m_l)}\chi_{R_\mu}\right\|_{\mathcal{X}} \le \frac{C_{1,n,L,\mathcal{X}}}{\|\chi_{R_{M_\varepsilon}}\|_{\mathcal{X}}}\|f\|_{\mathcal{X}} \le \varepsilon.$$

Consequently,

$$\left|\sum_{Q\in\mathcal{D},Q\supset R}\int_R \lambda_Q a_Q(x)\mathrm{d}x - \lim_{l\to\infty}\sum_{Q\in\mathcal{D},Q\supset R}\int_R \lambda_Q^{(m_l)} a_Q^{(m_l)}(x)\mathrm{d}x\right| \le 2|R|\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we deduce from (3.6)

$$\begin{aligned}\int_R f(x)\mathrm{d}x &= \lim_{l\to\infty}\sum_{Q\in\mathcal{D}}\int_R \lambda_Q^{(m_l)} a_Q^{(m_l)}(x)\mathrm{d}x\\ &= \sum_{Q\in\mathcal{D}}\lim_{l\to\infty}\int_R \lambda_Q^{(m_l)} a_Q^{(m_l)}(x)\mathrm{d}x\\ &= \sum_{Q\in\mathcal{D}}\int_R \lambda_Q a_Q(x)\mathrm{d}x\\ &= \int_R g(x)\mathrm{d}x.\end{aligned}$$

It remains to find a sparse family $\mathfrak{A}$ and establish the norm estimate (1.3). Let $\mathfrak{A} := \{Q\in\mathcal{D} : \lambda_Q\ne 0\}$. We claim that $\mathfrak{A}$ is a sparse family. In fact, if $\lambda_Q\ne 0$, then $\lambda_Q^{(m_l)}\ne 0$ for large l, say $l\ge L_Q$. Let $Q\in\mathfrak{A}$. Then since $f\in L^1_{\rm loc}(\mathbb{R}^n)$ thanks to (1.7), $[\log_A m_Q(|f^{(m_l)}|)]$ does not depend on l as long as l is large. Thus, we may assume $\{K^{(m_l)}(Q)\}_{l=L_Q}^{\infty}$ is decreasing if we replace L_Q by a larger number. With this in mind, we define

$$K(Q) := \bigcap_{l=L_Q}^{\infty} K^{(m_l)}(Q).$$

Let us check that $\{K(Q)\}_{Q\in\mathfrak{A}}$ is a family of the nutshells. Since $2|K^{(m_l)}(Q)| \geq |Q|$, we see that $2|K(Q)| \geq |Q|$. Let $Q, Q' \in \mathfrak{A}$ be different cubes. Since $K^{(m_l)}(Q) \cap K^{(m_l)}(Q') = \emptyset$ as long as $l \geq L_Q + L_{Q'}$, we see that $K(Q) \cap K(Q') = \emptyset$. Thus, $K(Q)$ is a nutsell of each $Q \in \mathfrak{A}$. Finally, (1.3) follows from (3.2) and (3.4). Thus the proof of Theorem 1.2 is complete.

4. Application of Theorem 1.1–Proof of Theorem 1.2

Let $f \in L^0(\mathbb{R}^n)$ be a non-negative function. By the truncation and by the monotone convergence theorem, we may assume $f \in L^p(\mathbb{R}^n)$ with $1 < p < \frac{n}{\alpha}$. Apply Theorem 1.1 for $f \in L^p(\mathbb{R}^n)$. Let $L \gg 1$. Then there is a decomposition:

$$f = \sum_{Q\in\mathfrak{A}} \lambda_Q a_Q \quad \text{a.e.}, \tag{4.1}$$

where $\mathfrak{A} \subset \mathcal{D}$ is a sparse set, $a_Q \in \mathcal{P}_L(\mathbb{R}^n)^{\perp}$ and $|a_Q| \leq \chi_Q$ for all $Q \in \mathfrak{A}$ and $\{\lambda_Q\}_{Q\in\mathfrak{A}} \subset \mathbb{R}$ satisfies

$$\left\| \sum_{Q\in\mathfrak{A}} \lambda_Q \chi_Q \right\|_{L^p} \leq C_{1,n,L,L^p} \|f\|_{L^p}, \quad 0 \leq \lambda_Q \leq C_{n,L} m_Q(f).$$

Note that (4.1) takes place in $L^p(\mathbb{R}^n)$. Since I_α maps $L^p(\mathbb{R}^n)$ boundedly to $L^q(\mathbb{R}^n)$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$,

$$I_\alpha f = \sum_{Q\in\mathfrak{A}} \lambda_Q I_\alpha a_Q$$

in $L^q(\mathbb{R}^n)$. As a result, using [18, Lemma 4.2], we have

$$I_\alpha f \leq C \sum_{Q\in\mathfrak{A}} \lambda_Q \ell(Q)^\alpha (M\chi_Q)^r$$

since $L \gg 1$. Consequently, by (1.5),

$$\|I_\alpha f\|_{\mathcal{X}} \leq C \left\| \sum_{Q\in\mathfrak{A}} \lambda_Q \ell(Q)^\alpha (M\chi_Q)^r \right\|_{\mathcal{X}} \leq C \left\| \sum_{Q\in\mathfrak{A}} \lambda_Q \ell(Q)^\alpha \chi_Q \right\|_{\mathcal{X}}.$$

Since $\mathfrak{A}$ is a sparse family, by using (1.5) once again we obtain

$$\|I_\alpha f\|_{\mathcal{X}} \leq C \left\| \sum_{Q\in\mathfrak{A}} \lambda_Q \ell(Q)^\alpha (M\chi_{K(Q)})^r \right\|_{\mathcal{X}} \leq C \left\| \sum_{Q\in\mathfrak{A}} \lambda_Q \ell(Q)^\alpha \chi_{K(Q)} \right\|_{\mathcal{X}}. \tag{4.2}$$

Since $\{K(Q)\}_{Q\in\mathfrak{A}}$ is disjoint,

$$\sum_{Q\in\mathfrak{A}} \lambda_Q \ell(Q)^\alpha \chi_{K(Q)} \le CM_\alpha f. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain the desired result.

5. Examples of function spaces showing that (1.1), (1.2) and (1.5) are necessary

Here we collect some counterexamples.

5.1. Condition (1.1)

As the example of $L^1(\mathbb{R}^n)$ shows, we cannot drop condition (1.1).

Proposition 5.1. *We have $f \in \mathcal{P}_0(\mathbb{R}^n)^\perp$ if f admits a decomposition:*

$$f = \sum_{Q\in\mathcal{D}} \lambda_Q a_Q \quad a.e.,$$

where $a_Q \in \mathcal{P}_0(\mathbb{R}^n)^\perp$ and $|a_Q| \le \chi_Q$ for all $Q \in \mathcal{D}$ and $\{\lambda_Q\}_{Q\in\mathcal{D}} \subset [0,\infty)$ satisfies

$$\left\| \sum_{Q\in\mathcal{D}} \lambda_Q \chi_Q \right\|_{L^1} < \infty.$$

Proof. Simply apply the Lebesgue convergence theorem. □

5.2. Condition (1.2)

We cannot require the moment condition for $L^\infty(\mathbb{R}^n)$-functions.

Proposition 5.2. *There do not exist $\{a_Q\}_{Q\in\mathcal{D}} \subset \mathcal{P}_0(\mathbb{R}^n)^\perp$ and $\{\lambda_Q\}_{Q\in\mathcal{D}} \subset [0,\infty)$ such that $|a_Q| \le \chi_Q$ for all $Q \in \mathcal{D}$, that*

$$1 = \sum_{Q\in\mathcal{D}} \lambda_Q a_Q$$

almost everywhere and that

$$\left\| \sum_{Q\in\mathcal{D}} \lambda_Q \chi_Q \right\|_{L^\infty} < \infty.$$

Proof. Assume to the contrary that there exist such a couple $\{a_Q\}_{Q\in\mathcal{D}} \subset \mathcal{P}_0(\mathbb{R}^n)^\perp$ and $\{\lambda_Q\}_{Q\in\mathcal{D}} \subset [0,\infty)$. Then there exists $K \in \mathbb{Z}$ such that

$$\left\| \sum_{Q\in\mathcal{D}, Q\supset Q_{K0}} \lambda_Q \chi_Q \right\|_{L^\infty} \le \frac{1}{2},$$

since

$$\sum_{Q\in\mathcal{D}, Q\supset[0,1)^n} \lambda_Q \le \left\| \sum_{Q\in\mathcal{D}, Q\supset[0,1)^n} \lambda_Q \chi_Q \right\|_{L^\infty} < \infty.$$

Thus, if we take the average of this expansion over $Q_{K0} \in \mathcal{D}$, then

$$1 = \sum_{Q \in \mathcal{D}, Q \subset Q_{K0}} \lambda_Q m_{Q_{K0}}(a_Q) + \sum_{Q \in \mathcal{D}, Q \supset Q_{K0}} \lambda_Q m_{Q_{K0}}(a_Q)$$
$$= \sum_{Q \in \mathcal{D}, Q \supset Q_{K0}} \lambda_Q m_{Q_{K0}}(a_Q).$$

Meanwhile

$$\left| \sum_{Q \in \mathcal{D}, Q \supset Q_{K0}} \lambda_Q m_{Q_{K0}}(a_Q) \right| \le \frac{1}{2}$$

by the choice of K. This is a contradiction. □

5.3. Condition (1.5)

Let $1 < p < \infty$. Set $\alpha := \frac{n}{p}$. Then

$$C^{-1} \|f\|_{\mathcal{M}^p_1} \le \|M_\alpha f\|_{L^\infty} \le C \|f\|_{\mathcal{M}^p_1}$$

for all $f \in L^0(\mathbb{R}^n)$. Thus, if the conclusion of Theorem 1.2 were true for $\mathcal{X}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$, then I_α would be bounded from $\mathcal{M}^p_1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ and hence $L^p(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. This is impossible since $|\cdot|^{\alpha-n} \notin L^{p'}(\mathbb{R}^n)$.

6. Examples of $\mathcal{X}(\mathbb{R}^n)$

Here are some examples other than Morrey spaces to which Theorem 1.1 is applicable.

6.1. Lorentz spaces

Let $f \in L^0(\mathbb{R}^n)$. Then its decreasing rearrangement f^* is the function defined on $(0, \infty)$ by

$$f^*(t) := \inf(\{s > 0 \, : \, |\{x \in \mathbb{R}^n \, : \, |f(x)| > s\}| \le t\} \cup \{\infty\}) \quad (t > 0).$$

Let $0 < p < \infty$ and $0 < q \le \infty$. Then the Lorentz space $L^{p,q}(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{L^{p,q}} := \left\{ \int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{\mathrm{d}t}{t} \right\}^{\frac{1}{q}}$$

is finite. It is well known that $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ [3, Theorem 5.2.1]. Thanks to the famous result by Hunt [17], $L^{p,q}(\mathbb{R}^n)$ is normable when $p > 1$ and $q \ge 1$. In fact,

$$\|f\|_{L^{p,q,*}} := \left\{ \int_0^\infty \left(t^{\frac{1}{p}-1} \int_0^t f^*(s) \mathrm{d}s \right)^q \frac{\mathrm{d}t}{t} \right\}^{\frac{1}{q}}$$

is a norm equivalent to $\|\cdot\|_{L^{p,q}}$. As a special case, we define $\mathrm{W}L^p(\mathbb{R}^n) = L^{p,\infty}(\mathbb{R}^n)$ and this space is called the weak Lebesgue space. Ariño and Muckenhoupt established that M is bounded on $\mathrm{W}L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ [1]. In this case, (1.2) is satisfied. Thus, Theorem 1.1 is applicable in this case. According to the extrapolation theorem in [8], (1.6) is satisfied as long as

$1 < p < \infty$ and $1 < q \le \infty$. In addition, using the real interpolation theory, we can also establish (1.6) for $0 < q \le 1$.

Theorem 6.1. *Let $1 < p < \infty$, $0 < q \le \infty$ and $1 < r \le \infty$. Then* (1.6) *holds for $\mathcal{X}(\mathbb{R}^n) = L^{p,q}(\mathbb{R}^n)$.*

To prove Theorem 6.1, we invoke a result from the textbook of Bergh and Löfström. We denote by $L^{p,q}(\ell^r, \mathbb{R}^n)$ the set of all sequences $\{f_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$ for which

$$\|\{f_j\}_{j=1}^\infty\|_{L^{p,q}(\ell^r)} := \left\| \left(\sum_{j=1}^\infty |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^{p,q}} < \infty.$$

The space $L^p(\ell^r, \mathbb{R}^n)$ stands for $L^{p,p}(\ell^r, \mathbb{R}^n)$.

Lemma 6.2. [3, Theorem 5.3.1] *Let p_0, p_1, q, $r \in (0, \infty]$ and $0 < \eta < 1$ satisfy $p_0 \ne p_1$. Define $p \in (0, \infty]$ by*

$$\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}. \tag{6.1}$$

Then

$$(L^{p_0}(\ell^r, \mathbb{R}^n), L^{p_1}(\ell^r, \mathbb{R}^n))_{\eta,q} \cong L^{p,q}(\ell^r, \mathbb{R}^n)$$

with equivalence of norms.

Proof of Theorem 6.1. We resort to a technique in [12]. Fix $f \in L^0(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ for a while. By the density of $\mathbb{Q}$ in $\mathbb{R}$, we have

$$Mf(x) = \sup_{\substack{y \in \mathbb{Q}^n, \\ r \in \mathbb{Q} \cap (0,\infty)}} \chi_{Q(y,r)}(x) m_{Q(y,r)}(|f|).$$

Let $r_1, r_2, \dots$ be an enumeration of $\mathbb{Q} \cap (0, \infty)$, and let $y_1, y_2, \dots$ be the one of $\mathbb{Q}^n$. Then

$$Mf(x) = \lim_{J \to \infty} \sup_{k,l \in \{1,2,\dots,J\}} \chi_{Q(y_k,r_l)}(x) m_{Q(y_k,r_l)}(|f|).$$

Here and below, we fix such enumerations and write

$$M_J f(x) := \sup_{k,l \in \{1,2,\dots,J\}} \chi_{Q(y_k,r_l)}(x) m_{Q(y_k,r_l)}(|f|)$$

for each $J \in \mathbb{N}$. We have only to show that

$$\|\{M_J f_j\}_{j=1}^\infty\|_{L^{p,q}(\ell^r)} \le C \|\{f_j\}_{j=1}^\infty\|_{L^{p,q}(\ell^r)} \tag{6.2}$$

with the constant C independent of J.

By the definition of the maximal operator M_J, we can find $k(x), l(x) \in \{1, 2, \dots, J\}$ so that

$$M_J f(x) \le 2\chi_{Q(y_{k(x)},r_{l(x)})}(x) m_{Q(y_{k(x)},r_{l(x)})}(|f|). \tag{6.3}$$

We may assume that such $(k(x), l(x))$ is the smallest couple in the lexicographic order of $\{1, 2, \dots, J\}^2$ among (k, l) satisfying (6.3), so that the mapping $x \mapsto (k(x), l(x))$ is measurable. Write

$$E_{k,l}(f) := \{x \in \mathbb{R}^n : k(x) = k,\, l(x) = l\} \quad ((k,l) \in \{1, 2, \dots, J\}^2).$$

Then by the definition of $E_{k,l}(f)$, we have

$$M_J f(x) \le 2 \sum_{k,l=1}^{J} \chi_{E_{k,l}(f) \cap Q(y_k, r_l)}(x) m_{Q(y_k, r_l)}(|f|).$$

We fix parameters $p_0 \in (1, p)$, $p_1 \in (p, \infty)$ and $\eta \in (0, 1)$ satisfying (6.1). Write

$$\Phi(\{h_j\}_{j=1}^{\infty}) = \{\Phi_j(h_j)\}_{j=1}^{\infty} := \left\{ \sum_{k,l=1}^{J} \chi_{E_{k,l}(f_j) \cap Q(y_k, r_l)} m_{Q(y_k, r_l)}(h_j) \right\}_{j=1}^{\infty}$$

for $\{h_j\}_{j=1}^{\infty} \subset L^1_{\rm loc}(\mathbb{R}^n)$. Since Φ is a linear operator and $|\Phi_j(h_j)| \le M h_j$, Φ is bounded on $L^{p_0}(\ell^r, \mathbb{R}^n)$ and $L^{p_1}(\ell^r, \mathbb{R}^n)$ thanks to (1.6). Consequently, thanks to Lemma 6.2, Φ is bounded on $L^{p,q}(\ell^r, \mathbb{R}^n)$, that is, (6.2) holds. The proof of Theorem 6.1 is therefore complete. □

6.2. Weak Morrey spaces

Let $0 < q \le p < \infty$. The weak Morrey space $\mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{\mathrm{W}\mathcal{M}^p_q} := \sup_{\lambda > 0} \lambda \|\chi_{[\lambda, \infty]}(|f|)\|_{\mathcal{M}^p_q} \tag{6.4}$$

is finite. Condition (1.1) is trivial since $\|\chi_Q\|_{\mathrm{W}\mathcal{M}^p_q} = |Q|^{\frac{1}{p}}$ for all $Q \in \mathcal{Q}$. Ho proved (1.5) in [16, Theorem 3.2] so as to include generalized Morrey spaces considered in [20].

6.3. Lorentz–Morrey spaces

Let $0 < q \le p < \infty$ and $0 < r \le \infty$. By replacing the L^q-quasi-norm $\|\cdot\|_{L^q}$ by the Lorentz quasi-norm $\|\cdot\|_{L^{q,r}}$ in the definition of the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q}$ (see (1.8)), we obtain the Lorentz–Morrey quasi-norm $\|\cdot\|_{\mathcal{M}^p_{q,r}}$ considered by Ragusa [25]. As a special case of $q = r$, the Lorentz–Morrey space $\mathcal{M}^p_{q,q}(\mathbb{R}^n)$ coincides with the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ endowed with the quasi-norm (1.8) and as a speical case of $r = \infty$, the Lorentz–Morrey space $\mathcal{M}^p_{q,\infty}(\mathbb{R}^n)$ coincides with the weak Morrey space $\mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$ endowed with the quasi-norm (6.4).

According to [13, Theorem 10], (1.6) is satisfied if $q, r > 1$, while (1.2) is satisfied if $q, r > 0$. In addition, using Theorem 6.1, we can verify that (1.6) is also true in the case of $r \le 1$ by the same method as [13, Theorem 10] or going through the argument above. Since $\||\cdot|^\delta\|_{\mathcal{M}^p_{q,r}} = \|\cdot\|^\delta_{\mathcal{M}^{p\delta}_{q\delta, r\delta}}$ for $\delta > 0$, $\mathcal{X}(\mathbb{R}^n) = \mathcal{M}^p_{q,r}(\mathbb{R}^n)$ satisfies (1.5) for all $0 < q \le p < \infty$ and $0 < r \le \infty$. Theorem 1.2 thus recaptures a result for Morrey–Lorentz spaces by the first author [13].

6.4. Orlicz spaces

Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function, that is, a convex homeomorphism. Then define the Luxemburg–Nakano norm $\| \cdot \|_{L^\Phi}$ by

$$\|f\|_{L^\Phi} := \inf \left(\left\{ \lambda > 0 \,:\, \int_{\mathbb{R}^n} \Phi \left(\frac{|f(x)|}{\lambda} \right) \mathrm{d}x \le 1 \right\} \cup \{\infty\} \right)$$

for $f \in L^0(\mathbb{R}^n)$. The Orlicz space $L^\Phi(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{L^\Phi}$ is finite.

We impose some standard conditions on the Young functions. A Young function $\Phi : [0, \infty) \to [0, \infty)$ is said to satisfy the Δ_2-condition or the doubling condition, denoted by $\Phi \in \Delta_2$, if there exists a constant $k > 1$ called the doubling constant such that $\Phi(2\cdot) \le k\Phi$. In this case, we also say that Φ satisfies the doubling condition. A Young function $\Phi : [0, \infty) \to [0, \infty)$ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if there exists a constant $k > 1$, called the ∇_2-constant, such that $2k\Phi \le \Phi(k\cdot)$. According to [19], M is bounded on $L^\Phi(\mathbb{R}^n)$, if $\Phi \in \nabla_2$. According to [22, Corollary 2.8], (1.5) is satisfied as long as Φ additionally satisfies the doubling condition, that is, $\Phi \in \Delta_2 \cap \nabla_2$. It is noteworthy that (1.2) is satisfied if we merely assume that $\Phi : [0, \infty) \to [0, \infty)$ is a Young function.

Moreover, according to [14], $\Phi \in \Delta_2$ implies that the Orlicz space $L^\Phi(\mathbb{R}^n)$ satisfies condition (1.5). In fact, to check this, we write

$$\|\{f_j\}_{j=1}^\infty\|_{L^\Phi(\ell^r)} := \left\| \left(\sum_{j=1}^\infty |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^\Phi}$$

for $\{f_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$. Then, for all $\theta > 1$, $\Phi_\theta : [0, \infty) \to [0, \infty)$, which is defined by

$$\Phi_\theta(r) := \int_0^{r^\theta} \frac{\Phi(t)}{t} \, \mathrm{d}t,$$

is a Young function in $\Delta_2 \cap \nabla_2$ according to [14], and we obtain

$$\begin{aligned} \left(\left\| \sum_{j=1}^\infty M f_j{}^r \right\|_{L^{\Phi_1}} \right)^{\frac{1}{r}} &= \|\{Mf_j\}_{j=1}^\infty\|_{L^{\Phi_r}(\ell^r)} \\ &\le C \|\{f_j\}_{j=1}^\infty\|_{L^{\Phi_r}(\ell^r)} \\ &= C \left(\left\| \sum_{j=1}^\infty |f_j|^r \right\|_{L^{\Phi_1}} \right)^{\frac{1}{r}} \end{aligned}$$

for any $\{f_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$. Here we employed [22, Theorem 2.6] for the above inequality. Since $\|\cdot\|_{L^{\Phi_1}}$ and $\|\cdot\|_{L^\Phi}$ are equivalent according to [14], inequality (1.5) holds.

6.5. Generalized Orlicz–Morrey spaces

Let $\mathcal{G}_1$ be the set of all non-decreasing functions $\varphi : [0,\infty) \to [0,\infty)$ such that $t \in (0,\infty) \mapsto \frac{\varphi(t)}{t} \in (0,\infty)$ is non-increasing. We give two definitions:

Definition 6.3. *Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function.*

(1) [21] *Let $\varphi \in \mathcal{G}_1$. For a cube $Q \in \mathcal{Q}$, define the (φ,Φ)-average over Q of $f \in L^0(\mathbb{R}^n)$ by*

$$\|f\|_{(\varphi,\Phi);Q} := \inf\left(\left\{\lambda > 0 \,:\, \varphi(|Q|)m_Q\left(\Phi\left(\frac{|f|}{\lambda}\right)\right) \le 1\right\} \cup \{\infty\}\right).$$

The generalized Orlicz-Morrey space $\mathcal{L}^{\varphi,\Phi}(\mathbb{R}^n)$ of the first kind is defined to be the Banach space equipped with the norm

$$\|f\|_{\mathcal{L}^{\varphi,\Phi}} := \sup_{Q\in\mathcal{Q}} \|f\|_{(\varphi,\Phi);Q}.$$

(2) [31] *For a cube $Q \in \mathcal{Q}$, define the Φ-average over Q of $f \in L^0(\mathbb{R}^n)$ by*

$$\|f\|_{\Phi;Q} := \inf\left(\left\{\lambda > 0 \,:\, m_Q\left(\Phi\left(\frac{|f|}{\lambda}\right)\right) \le 1\right\} \cup \{\infty\}\right).$$

Let $\varphi \in \mathcal{G}_1$. Then the generalized Orlicz-Morrey space $\mathcal{M}^{\varphi}_{\Phi}(\mathbb{R}^n)$ of the second kind is defined to be the Banach space equipped with the norm

$$\|f\|_{\mathcal{M}^{\varphi}_{\Phi}} := \sup_{Q\in\mathcal{Q}} \varphi(|Q|)\|f\|_{\Phi;Q}.$$

Let $\Phi(t) = t^p$ with $1 \le p < \infty$. Then $\mathcal{M}^{\varphi}_{\Phi}(\mathbb{R}^n) = \mathcal{L}^{\varphi^p}_{\Phi}(\mathbb{R}^n)$ with coincidence of norms and the above spaces $\mathcal{M}^{\varphi}_{\Phi}(\mathbb{R}^n)$ and $\mathcal{L}^{\varphi^p}_{\Phi}(\mathbb{R}^n)$ boil down to the generalized Morrey space $\mathcal{M}^{\varphi}_{p}(\mathbb{R}^n)$ defined by Nakai [20].

It is noteworthy that the generalized Orlicz-Morrey space $\mathcal{M}^{\varphi}_{\Phi}(\mathbb{R}^n)$ appears naturally in the context of the Calderón–Lozanovskiĭ product. Let $\mathcal{U}$ denote the set of all non-zero positive concave and positively homogeneous continuous functions defined over $[0,\infty)^2$. Then the Calderón–Lozanovskiĭ product $\varphi(E) = \varphi(E_0, E_1)$ consists of all $f \in L^0(\mathbb{R}^n)$ such that $|f| \le \lambda\varphi(|f_0|, |f_1|)$ a.e. for some $\lambda > 0$ and $f_j \in E_j$ with norm 1, $j = 0, 1$. Its norm is given by

$$\|f\|_{\varphi(E)} := \inf\{\lambda > 0 \,:\, |f| \le \lambda\varphi(|f_0|, |f_1|), f_j \in E_j, \|f_j\|_{E_j} = 1, j = 0, 1\}.$$

Let $\varphi \in \mathcal{U}$ and denote by Φ the inverse of $\varphi(\cdot, 1)$. Then Φ is a convex function since $\varphi(\cdot, 1)$ is concave. We can say that $\mathcal{L}^{\varphi}_{\Phi}(\mathbb{R}^n)$ is a natural function space in view of the following lemma:

Lemma 6.4. *With concidence of norms, $\varphi(\mathcal{M}^{\varphi}_{1}(\mathbb{R}^n), L^\infty(\mathbb{R}^n)) = \mathcal{L}^{\varphi}_{\Phi}(\mathbb{R}^n)$.*

Proof. Simply observe

$$\|f\|_{\varphi(\mathcal{M}^{\varphi}_{1}, L^\infty)} = \inf\{\lambda > 0 \,:\, |f| \le \lambda\varphi(|f_0|, 1), f_0 \in \mathcal{M}^{\varphi}_{1}(\mathbb{R}^n), \|f_0\|_{\mathcal{M}^{\varphi}_{1}} = 1\}.$$

□

Another important thing to note is that $L^2(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$ is realized as a special case of $\mathcal{L}^{\varphi}_{\Phi}(\mathbb{R}^n)$. Although $L^2(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$ cannot be realized as a special case of $\mathcal{M}^{\varphi}_{\Phi}(\mathbb{R}^n)$ [11], $\mathcal{M}^{\varphi}_{\Phi}(\mathbb{R}^n)$ is important. In fact, letting $\Phi(t) := t\log(3+t)$ for $t > 0$, we learn that M maps $\mathcal{M}^{\varphi}_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n) := \mathcal{M}^{\varphi}_{\Phi}(\mathbb{R}^n)$ to $\mathcal{M}^{\varphi}_{1}(\mathbb{R}^n)$ and that

$$C^{-1}\|f\|_{\mathcal{M}^{\varphi}_{\mathrm{L}\log\mathrm{L}}} \le \|Mf\|_{\mathcal{M}^{\varphi}_{1}} \le C\|f\|_{\mathcal{M}^{\varphi}_{\mathrm{L}\log\mathrm{L}}}$$

for all $f \in \mathcal{M}^{\varphi}_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ [31].

If $\Phi \in \nabla_2$, then M is bounded on both $\mathcal{L}^{\varphi}_{\Phi}(\mathbb{R}^n)$ and $\mathcal{M}^{\varphi}_{\Phi}(\mathbb{R}^n)$ according to [21] and [31], respectively. According to [27, Theorem 4.1] and [28], or by using a technique in [14], we can check that (1.5) is satisfied as long as $\Phi \in \Delta_2$.

6.6. Mixed Lebesgue spaces

Let $0 < p_1, p_2, \dots, p_n \le \infty$ be constants. We abbreviate $\mathbf{p} := (p_1, p_2, \dots, p_n)$. Then define the mixed Lebesgue norm $\|\cdot\|_{L^{\mathbf{p}}}$ by

$$\|f\|_{L^{\mathbf{p}}} := \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, \dots, x_n)|^{p_1} \mathrm{d}x_1\right)^{\frac{p_2}{p_1}} \mathrm{d}x_2\right)^{\frac{p_3}{p_2}} \cdots \mathrm{d}x_n\right)^{\frac{1}{p_n}}.$$

A natural modification for x_i is made when $p_i = \infty$. The mixed Lebesgue space $L^{\mathbf{p}}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^0(\mathbb{R}^n)$ with $\|f\|_{L^{\mathbf{p}}} < \infty$. According to Bagby [2], (1.1) is satisfied for $p_1, p_2, \dots, p_n \in (1, \infty)$, while (1.5) is satisfied for $p_1, p_2, \dots, p_n \in (0, \infty)$ according to [23].

6.7. Mixed Morrey spaces

Let $\mathbf{q} = (q_1, q_2, \dots, q_n) \in (0, \infty]^n$. By replacing the L^q-quasi-norm by the mixed Lebesgue quasi-norm $\|\cdot\|_{L^{\mathbf{q}}}$ in the definition of the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q}$ (see (1.8)), we obtain the mixed Morrey space $\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$. Nogayama proved (1.5) in [23], while (1.2) follows from $\mathcal{M}^p_{q_0}(\mathbb{R}^n) \hookleftarrow \mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$ if $q_0 \le \min\limits_{i=1,2,\dots,n} q_i$. In this case, Theorem 1.1 refines [24, Theorem 3].

6.8. Variable Lebesgue spaces

For a measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \bigcup_{\lambda > 0} \{f \in L^0(\mathbb{R}^n) \,:\, \rho_{p(\cdot)}(\lambda^{-1}f) < \infty\},$$

where

$$\rho_{p(\cdot)}(f) := \|\,|f|^{p(\cdot)}\,\|_{L^1}.$$

Moreover, for $f \in L^{p(\cdot)}(\mathbb{R}^n)$ one defines the variable Lebesgue quasi-norm by

$$\|f\|_{L^{p(\cdot)}} := \inf\left(\{\lambda > 0 \,:\, \rho_{p(\cdot)}(\lambda^{-1}f) \le 1\} \cup \{\infty\}\right).$$

There exists a discontinuous exponent $p(\cdot)$ such that (1.1) and (1.5) hold. In fact, the following function is one of such functions:

$$p(\cdot) = \sum_F c_F \chi_F, \tag{6.5}$$

where F moves over all quadrants and each $c_F \in (1, \infty)$ is a fixed constant.

We recall a sufficient condition for (1.5). The exponent $p(\cdot)$ satisfies the local log-Hölder continuity condition if

$$|p(x) - p(y)| \le \frac{c_*}{\log(|x-y|^{-1})} \quad \text{for} \quad |x-y| \le \frac{1}{2}, \ x, y \in \mathbb{R}^n, \tag{6.6}$$

while the exponent $p(\cdot)$ satisfies the log-Hölder-type decay condition at infinity if

$$|p(x) - p_\infty| \le \frac{c^*}{\log(e + |x|)} \quad \text{for} \quad x \in \mathbb{R}^n. \tag{6.7}$$

Here c_*, c^* and p_∞ are positive constants independent of x and y.

If $p(\cdot)$ satisfies these conditions and $0 < p_- := \operatorname{essinf}_{x \in \mathbb{R}^n} p(x) \le p_+ := \operatorname{esssup}_{x \in \mathbb{R}^n} p(x) < \infty$, then we have (1.5) according to [7, 5, 6]. Finally, (1.2) follows from the definition.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Authors' contributions

The three authors contributed equally to the paper. They read the whole paper and approved it.

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