

Some new results of completion t^ω - Normed approach space

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ABSTRACT

In this paper, the completion of t^ω -approach spaces, isometric in t^ω -approach space and equivalent sequences are defined. Every t^ω -approach normed space can be embedded in t^ω -approach Banach space is proved, $\tilde{\chi}$ and an isometry ψ from χ onto the subspace F of $\tilde{\chi}$ which is dense in $\tilde{\chi}$ are introduced, as well as, the completion is shown uniquely up to an isometry. In addition, to what is mentioned above, some essential definitions, examples, and important theorems are included to illustrate some of our work.

Keywords: : t^ω -approach space, t^ω -approach normed space, t^ω –Cauchy sequence, t^ω –complete approach space, t^ω - Banach approach space, t^ω - app-isometric.

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1. Introduction

The first study on the distance between points and sets in a metric space was done by R. Lowen [4]. R. Baekeland and R. Lowen [7] defined and introduced the measure of Lindelof and separability in approach space in (1994). In (1996), R. Lowen [13] put the basic principles of the fundamental theory of approximation. R. Lowen, Y. Jinlee [2] explained the concepts of approach Cauchy structure and ultra-approach Cauchy structure in (1999). In (2000) and (2003) R. Lowen and M. Sioen [8,10] defined the important and essential definitions of many separation axioms in the approach spaces and found the relationship between them. In (2000), R. Lowen and B. Windels [14] showed the important notions of an approach groups spaces, semi-group spaces, and uniformly convergent. R. Lowen, M. Sion and D. Vaughan [3] explained “the complete theory for all approach spaces with an underlying topology which agrees with the usual metric completion theory for metric spaces in (2003). In (2004), R. Lowen and S. Verwulgen [5] introduced the concept of an approach vector space. In (2004), R. Lowen, C. Van Olmen, and T. Vroegrijk [9] found and showed a very essential relationship between functional ideas and topological theories. In (2006), G. C. L. Brümmer and M. Sion [16] defined and developed a bicompletion theory for the category of approach spaces in sense of Lowen [20] which is extended the completion theory obtained in [14]. In (2009), J. Martinez-Moreno, A. Roldan, and C. Roldan [17] studied the necessary and important notion of fuzzy approach spaces generalization of fuzzy metric spaces and proved several properties of fuzzy approach spaces. In (2009), some notions, definitions and relations in an approach theory were discussed by R. Lowen and C. Van Olmen [11]. In (2013), G. Gutierrez, D. Hofmann [12] introduced and studied the concept of cocompleteness for approach spaces so, proved some properties in a cocompleteness approach space. In (2013), K. Van Opdenbosch [18] defined and explained a new isomorphic characterizations of approach spaces, pre-approach spaces, convergence approach spaces, uniform gauge spaces, topological spaces, convergence spaces, topological spaces, metric spaces, and

uniform spaces. In (2014), R.Lowen and S.Sagiroglu [22] gave the possibility to weak the notion of approach spaces to incorporate not only topological and metric spaces but also closure spaces. In (2015), R.Lowen [6] discussed and showed two new types of numerically structured spaces which are required approach spaces on the local level and uniform gauge spaces on the uniform level. In (2016), many generalizations of known theorems of fixed point, and theorems for common fixed points of mapping to 2- Banach space were discussed by R. Malčeski and A. Ibrahim [21]. E. Colebunders and M. Sion [1] solved and proved several important consequences on real-valued contractions in (2017). As well as, in (2017) and (2019), M. Baran and M. Qasim [20,22] characterized the local distance-approach spaces, approach spaces, and gauge-approach spaces and compared them with usual approach spaces. In (2018), W. Li and Dexue Zhang [21] discussed the Smyth complete.

In this paper, we define t^ω -Cauchy sequence, t^ω - normed approach space, t^ω -Banach approach space structure on χ and t^ω - app-isometric, as well as we prove their properties. We obtain that t^ω - Banach approach spaces is a t^ω -complete normed vector approach spaces also, study the category-theoretic properties of t^ω - Banach approach spaces. The extension t^ω - Banach approach space by t^ω - complete normed approach space is explained and studied. In addition, we give an additional condition on the norm structure , that is $t_{\|\cdot\|}^\omega(\{x\}, A) := \sup_{x \in \chi} \inf_{a \in A} \|x - a\|$, this means that the distance generated by norm function between two subsets of power set in t^ω -approach space. In this case, we want to obtain any Cauchy sequence convergent in t^ω -approach space and the space has become t^ω - approach Banach space. Some properties of non-complete approach normed space are proved. The main aim of this paper is to introduce and discuss new results in convergent sequences in t^ω -approach spaces. In this work, we show that the completion of t^ω -approach spaces, that is complete t^ω - approach normed space in χ can be embedded as dense sub-space. t^ω -approach Banach space play essential and important role in functional analysis in a special case and many branches of mathematic in general with its applications. That is why we try and are able to find a structure for it in the t^ω -approach space using the convergence that we obtain in the t^ω -app-space of the t^ω -Cauchy sequences. We are able to make every t^ω -app-space is complete by embedding it in the t^ω -approach Banach space provided that it is dense in this space, which is called completion”, and we have worked on t^ω -approach normed space.

This work is divided into three sections: Section one shows the introduction of the research. In section two, basic definitions with preliminaries are introduced. In section three, some results of t^ω - approach Banach space, non-completeness approach normed space and an isometry are given. Section four, we discuss and explain the important and the essential conclusions of the research.

2. Preliminaries

We will start by definition of essential notion of this paper, namely t^ω -approach distance.

Definition 1 [23]: Let χ be a non-empty set. A collection $(t^\omega)_{\omega < \infty}$ of functions $t^\omega : 2^\chi \times 2^\chi \rightarrow [0, \infty]$ is known as t^ω -approach distance on χ if this function satisfies the following properties:

- $(t_1) \forall \omega \in \mathbb{R}^+, \forall A, B \in 2^\chi : t^\omega(A, B) = 0 \Rightarrow A = B,$
- $(t_2) \forall \omega \in \mathbb{R}^+, \forall A \in 2^\chi, \text{ then } t^\omega(A, \emptyset) = \infty,$
- $(t_3) \forall \omega \in \mathbb{R}^+, \forall A, B, C \in 2^\chi : t^\omega(A, B \cap C) = \max\{t^\omega(A, B), t^\omega(A, C)\},$
- $(t_4) \forall \omega \in \mathbb{R}^+, \forall A, B \in 2^\chi, \forall \varepsilon < \omega : t^\omega(A_{(\varepsilon)}^\omega, B) \leq t^\omega(A, B) + \varepsilon$ where $A_{(\varepsilon)}^\omega = \{x \in \chi | t^\omega(\{x\}, B) \leq \omega - \varepsilon\}.$

A pair (χ, t^ω) where the function t^ω is a distance and this pair is called t^ω - approach space and denoted by t^ω -app-spaces.

Definition2 [24] : A set $A \in 2^X$ is said to be cluster point in t^ω - app- space (χ, t^ω) if there exists a sequence $\{x_n\}_{n=1}^\infty$ in χ , such that, $\inf_{x \in A} t^\omega(\{x_n\}, A) = 0$ which is written by $\{x_n\}_{n=1}^\infty \rightarrow A$,

we denoted the set of all cluster point in t^ω -approach space by $\Gamma(\chi)$.

Definition1: A sequence $\{m_n\}_{n=1}^\infty$ in t^ω -approach space is said to be t^ω -Cauchy sequence in t^ω -app-space or Cauchy distance (t^ω -Cauchy) if:

for every $\lim_{n \rightarrow \infty} \inf_{x_m \in A} t^\omega(\{m_n\}, A) = 0$, a sequence $\{m_n\}_{n=1}^\infty$ in χ is said to be t^ω -convergent sequence in t^ω -app-space if :

there exists $m \in \chi$, for any $M \in \Gamma(\chi)$, $: t^\omega(\{m_n\}, M) = 0$.

Remark1: If any sequence is a t^ω -convergent sequence then it is a t^ω -Cauchy sequence.

Definition3 [23]: Let F be a field and let χ be a non-empty power set with two binary operations :an addition and a scalar multiplication, $\forall \omega < \infty, t^\omega$ is an app- distance on 2^X , then, a quadruple $(\chi, t^\omega, +, \cdot)$ is said to be t^ω - approach vector space if satisfy the following:

- 1) $(\chi, t^\omega, +)$ is t^ω - app-group.
- 2) (χ, t^ω, \cdot) is t^ω - app-semi group.
- 3) $\mu \cdot (B + C) = \mu \cdot B + \mu \cdot C$, for all $\mu \in F$, for all $B, C \in 2^X$.
- 4) $(B + C) \cdot \mu = B \cdot \mu + C \cdot \mu$, for all $\mu \in F$, for all $B, C \in 2^X$.
- 5) $(\mu \cdot \vartheta) \cdot C = \mu(\vartheta C)$, for all $C \in 2^X$, for all $\mu, \vartheta \in F$.
- 6) $I \cdot A = A$, for all $A \in 2^X$.

Definition4[24]: Let χ be t^ω - app- vector space. A triple $(\chi, \|\cdot\|, t^\omega_{\|\cdot\|})$ said

to be t^ω - approach normed space if satisfy the following :

- (1) $\|x\| = 0$ if and only if $x = 0$, for all $x \in X$.
- (2) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\| \forall \lambda \in F, x \in X$
- (3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$.
- (4) $\|x\| \geq 0, \forall x \in X$
- (5) $t^\omega_{\|\cdot\|}(\{x\}, A) = \sup_{x \in X} \inf_{a \in A} \|x - a\|$.

Definition5 [24]: t^ω - approach space is called t^ω -complete app- space if every t^ω -Cauchy is t^ω -convergent in (χ, t^ω) .

Definition6 [23]: A triple $(\chi, t^\omega, *)$ is known as t^ω - app- semi-group if and only if

1. (χ, t^ω) is t^ω - app- space.
2. $(\chi, *)$ is a semi-group.
3. $*$: $2^X \otimes 2^X \rightarrow 2^X: (A, B) \mapsto A * B$ is a t^ω -contraction such that $A * B = \{x * y: x \in A, y \in B\}$.

Definition7 [23]: The triple $(\chi, t^\omega, *)$ is known as t^ω - app- group if satisfy the following:

- a) (χ, t^ω) is t^ω - app- space.
- b) $(\chi, *)$ is a group.

c) $*$: $2^X \otimes 2^X \rightarrow 2^X: (A, B) \mapsto A + B$ is the t^ω -contraction such that $A + B = \{x + y: x \in A, y \in B\}$.

d) $-$: $2^X \rightarrow 2^X: A \rightarrow -A$ is t^ω -contraction such that $x \in A, -x \in -A$.

Remark2: Let $(\mathcal{X}, \|\cdot\|, t^\omega_{\|\cdot\|})$ is a t^ω - app-normed space then it is a normed space.

Definition8 [24]: t^ω - app-Banach space is t^ω - app-complete normed space.

Definition 9: Let $(X, t^\omega_{\|\cdot\|}, \|\cdot\|_\mathcal{X})$ and $(Y, t^{\omega'}_{\|\cdot\|}, \|\cdot\|_Y)$ be t^ω -approach normed space :

(1) t^ω - app-isometric is a mapping ψ from t^ω - app-normed space into another t^ω - app-normed space satisfy the condition :

for all $m, n \in X$, for all $M \subseteq X, \|t^{\omega'}(\psi(\{m\}), \psi(M)) - t^{\omega'}(\psi(\{n\}), \psi(M))\|_Y = \|t^\omega(\{m\}, M) - t^\omega(\{n\}, M)\|_\mathcal{X}$.

(2) The t^ω -app- space \mathcal{X} isometric to another t^ω -app-space if there exists a bijective t^ω -app-isometry from \mathcal{X} into F .

Definition 10: Equivalent t^ω -Cauchy sequence m_n and $\{m'_n\}$ in t^ω -normed app-space if:

$$\lim_{n \rightarrow \infty} \|t^\omega(\{m_t\}, M) - t^\omega(\{m'_t\}, M)\| = 0, \text{ for all } t \in \mathbb{Z}^+, \text{ for all } M \in 2^X \text{ denoted by } \{m_t\} \sim \{m'_t\}.$$

Some Properties of t^ω - approach Banach space

In this section, we have obtained a new structure which is a unique t^ω - approach complete space for every no-completeness t^ω - approach normed space.

Proposition1: For every non- completeness t^ω - approach normed space $(\mathcal{X}, t^\omega_{\|\cdot\|}, \|\cdot\|)$, there exists t^ω - approach Banach space.

Proof: A new structure of t^ω - approach Banach space $(\tilde{\mathcal{X}}, \tilde{t}^\omega_{\|\cdot\|_{\tilde{\mathcal{X}}}}, \|\cdot\|_{\tilde{\mathcal{X}}})$ is given as follows: Consider $\tilde{\mathcal{X}}$ be the collection of all equivalence classes of t^ω - Cauchy sequence $\{\tilde{m}\}$ in \mathcal{X} under the following equivalence relation:

If $\{\mathfrak{S}_t\}$ and $\{\mathfrak{S}'_t\}$ be t^ω - Cauchy sequence in \mathcal{X} .

A sequences $\{\mathfrak{S}_t\}$ and $\{\mathfrak{S}'_t\}$ are equivalent ,written $\{\mathfrak{S}_t\} \sim \{\mathfrak{S}'_t\}$,if and only if

$$\lim_{t \rightarrow \infty} \|t^\omega(\{\mathfrak{S}_t\}, M) - t^\omega(\{\mathfrak{S}'_t\}, M)\|_M = 0 \dots \dots \dots (1).$$

Where $\{\mathfrak{S}_t\} \in \{\tilde{m}\}$ to express $\{\mathfrak{S}_t\}$ is an element of $\tilde{\mathcal{X}}$ and representative of the class $\tilde{\mathcal{X}}$.

Step 1: To prove that $(\tilde{\mathcal{X}}, \tilde{t}^\omega)$ is an app- space . Let (\mathcal{X}, t^ω) is t^ω - app-space, then we define $\tilde{t}^\omega: 2^{\tilde{\mathcal{X}}} \times 2^{\tilde{\mathcal{X}}} \rightarrow [0, \infty]$ by $\tilde{t}^\omega(\{\tilde{m}\}, \tilde{M}) = \inf_{m \in \tilde{\mathcal{X}}, M \in \tilde{M}} t^\omega(\{m\}, M), \tilde{M} \subseteq 2^{\tilde{\mathcal{X}}}$,

for each $\tilde{m} \in \tilde{\mathcal{X}}$.

We will prove \tilde{t}^ω is satisfying the four conditions of distance in def.[23]

(t_1) Each $m \in \mathcal{X}$, so $t^\omega(\{m\}, \{m\}) = 0$. Then, for all $\tilde{m} \in \tilde{\mathcal{X}}$, then, $\inf_{\tilde{m} \in \tilde{\mathcal{X}}} t^\omega(\{\tilde{m}\}, \{\tilde{m}\}) = 0$, Hence $\tilde{t}^\omega(\{\tilde{m}\}, \{\tilde{m}\}) = 0$.

(t_2) Since , for all $x \in \mathcal{X}, t^\omega(\{m\}, \emptyset) = \infty$. Then, $\inf_{\tilde{m} \in \tilde{\mathcal{X}}} t^\omega(\{m\}, \emptyset) = \infty$.

Hence $\tilde{t}^\omega(\tilde{m}, \emptyset) = \infty$, for all $\tilde{m} \in \tilde{\mathcal{X}}$.

(t_3) Firstly, for all $\tilde{m} \in \tilde{\chi}, \forall \tilde{M}, \tilde{N} \in 2^{\tilde{\chi}}, \tilde{t}^\omega(\{m\}, \tilde{M} \cup \tilde{N}) = \inf_{\tilde{m} \in \tilde{\chi}, M \in \tilde{M}, N \in \tilde{N}} t^\omega(\{m\}, M \cup N)$
 $= \inf_{\tilde{m} \in \tilde{\chi}, M \in \tilde{A}, N \in \tilde{B}} (\min\{t^\omega(\{m\}, M), t^\omega(\{m\}, N)\})$, Then,

$$\min \{ \inf_{\tilde{m} \in \tilde{\chi}, M \in \tilde{A}} t^\omega(\{m\}, M), \inf_{\tilde{m} \in \tilde{\chi}, N \in \tilde{B}} (t^\omega(\{m\}, N)) \} = \min \{ \tilde{t}^\omega(\{\tilde{m}\}, \tilde{M}), (\tilde{t}^\omega(\{\tilde{m}\}, \tilde{N})) \}.$$

(t_4) Each $\tilde{m} \in \tilde{\chi}$ and $\tilde{M} \in 2^{\tilde{\chi}}$, for all $\varepsilon \in [0, \infty]$

$$\tilde{t}^\omega(\{\tilde{m}\}, \tilde{M}) = \inf_{\tilde{m} \in \tilde{\chi}, M \in \tilde{A}} t^\omega(\{m\}, M) (t^\omega(\{m\}, (M^\varepsilon)) + \varepsilon) \leq \inf_{\tilde{m} \in \tilde{\chi}, M \in \tilde{A}} (t^\omega(\{m\}, (M^\varepsilon)) + \varepsilon) \leq \inf_{\tilde{m} \in \tilde{\chi}, M \in \tilde{A}} \tilde{t}^\omega(\{\tilde{m}\}, \tilde{M}) \leq \tilde{t}^\omega(\{\tilde{m}\}, (\tilde{M}^\varepsilon)) + \varepsilon.$$

A pair $(\tilde{\chi}, \tilde{t}^\omega)$ is t^ω -approach space and \tilde{t}^ω is a distance.

Step 2: Now, to prove $(\tilde{\chi}, \|\cdot\|_{\tilde{\chi}})$ is t^ω -approach normed space by defined : $\|\tilde{m}\|_{\tilde{\chi}} = \inf_{\tilde{m} \in \tilde{\chi}} \{\|m\|\}$

Clearly, $\|\cdot\|_{\tilde{\chi}}$ satisfies the conditions 1 – 4 in definition [24]

Since $\|\cdot\|_{\tilde{\chi}}$ is approach norm on χ , $\|\tilde{m} + \tilde{n}\| \leq (\|\tilde{m}\|_{\tilde{\chi}} + \|\tilde{n}\|_{\tilde{\chi}})$ for all $\tilde{m}, \tilde{n} \in \tilde{\chi}$.

$$\text{Since } \tilde{t}^\omega_{\|\cdot\|_{\tilde{\chi}}}(\{\tilde{\mathfrak{S}}\}, \tilde{M}) = \inf_{\tilde{\mathfrak{S}} \in \tilde{\chi}, a \in \tilde{M}} t^\omega(\{\tilde{\mathfrak{S}}\}, a) = \inf_{\tilde{\mathfrak{S}} \in \tilde{\chi}, a \in \tilde{M}} \sup_{\mathfrak{S} \in \chi} \inf_{a \in M} \|\mathfrak{S} - a\|$$

$$= \sup_{\mathfrak{S} \in \chi} \inf_{a \in M} \inf_{\tilde{\mathfrak{S}} \in \tilde{\chi}, a \in \tilde{M}} \|\mathfrak{S} - a\|, \text{ for all } M \subset \chi.$$

Then, $t^\omega_{\|\cdot\|_{\tilde{\chi}}}(\{\tilde{\mathfrak{S}}\}, M) = \sup_{\mathfrak{S} \in \chi} \inf_{a \in M} \|\mathfrak{S} - a\|$, for all $M \subset \tilde{\chi}$.

So $\|\cdot\|_{\tilde{\chi}}$ is an app norm on $\tilde{\chi}$.

$$\text{We put, } \|\tilde{\mathfrak{S}}_t - \tilde{n}\|_{\tilde{\chi}} = \lim_{t \rightarrow \infty} \|\mathfrak{S}_t - \mathfrak{h}_t\|_{\chi} \dots \dots \dots (2)$$

Where $\{\mathfrak{S}_n\} \in \tilde{\mathfrak{S}}_n$ and $\{\mathfrak{h}_n\} \in \tilde{N}$.

Now, we prove the existence of limit

$$\|\mathfrak{S}_n - \mathfrak{h}_n\|_{\chi} \leq \|\mathfrak{S}_n - \mathfrak{S}_k\|_{\chi} + \|\mathfrak{S}_k - \mathfrak{h}_k\|_{\chi} + \|\mathfrak{h}_k - \mathfrak{h}_n\|_{\chi}.$$

$$(\|\mathfrak{S}_n - \mathfrak{h}_n\|_{\chi} - \|\mathfrak{S}_k - \mathfrak{h}_k\|_{\chi}) \leq \|\mathfrak{S}_n - \mathfrak{S}_k\|_{\chi} + \|\mathfrak{h}_k - \mathfrak{h}_n\|_{\chi}.$$

And similarity in the previous inequality with k and t reciprocal, that is

$$(\|\mathfrak{S}_k - \mathfrak{h}_k\|_{\chi} - \|\mathfrak{S}_n - \mathfrak{h}_t\|_{\chi}) \leq \|\mathfrak{S}_k - \mathfrak{S}_n\|_{\chi} + \|\mathfrak{h}_t - \mathfrak{h}_k\|_{\chi}.$$

$$\text{Hence, } |\|\mathfrak{S}_t - \mathfrak{h}_t\|_{\chi} - \|\mathfrak{S}_k - \mathfrak{h}_k\|_{\chi}| \leq \|\mathfrak{S}_t - \mathfrak{h}_k\|_{\chi} + \|\mathfrak{S}_k - \mathfrak{h}_t\|_{\chi} \dots \dots \dots (3).$$

Since $\{\mathfrak{S}_t\}$ and $\{\mathfrak{h}_t\}$ are t^ω -Cauchy ,

Since χ is complete. Then, the limit exists in (2).

We must show that the limit in (2) is unique.

If $\{\mathfrak{S}_t\} \sim \{\mathfrak{S}'_t\}$ and $\{\mathfrak{h}_t\} \sim \{\mathfrak{h}'_t\}$.

$$\text{Then, by (1) } |\|\mathfrak{S}_t - \mathfrak{h}_t\|_{\chi} - \|\mathfrak{S}'_k - \mathfrak{h}'_k\|_{\chi}| \leq \|\mathfrak{S}_t - \mathfrak{S}'_k\|_{\chi} + \|\mathfrak{h}'_k - \mathfrak{h}_t\|_{\chi}.$$

$$\text{Thus, } \lim_{t \rightarrow \infty} |\|\mathfrak{S}_t - \mathfrak{h}_t\|_{\chi} - \|\mathfrak{S}'_k - \mathfrak{h}'_k\|_{\chi}| \leq 0.$$

Which converge to zero as $t \rightarrow \infty$.

$$\text{Then } \lim_{t \rightarrow \infty} \|\mathfrak{S}_t - \mathfrak{h}_t\|_{\chi} = \lim_{t \rightarrow \infty} \|\mathfrak{S}'_t - \mathfrak{h}'_t\|_{\chi}.$$

Proposition 2: Let $(\chi, \tau^\omega_{\|\cdot\|, \|\cdot\|})$ be non-complete approach normed space, then there exists an isometric ψ from χ onto a dense subspace F of $\tilde{\chi}$.

Proof: We construct an isometry $\psi: \chi \rightarrow F \subset \tilde{\chi}$.

The class $\tilde{b} \in \tilde{\chi}$, for each $b \in \chi$, contains (b, b, \dots) which is constant τ^ω -Cauchy sequence. Define a mapping $\psi: \chi \rightarrow F \subset \tilde{\chi}$ which is subspace of $\tilde{\chi}$, ψ is given by: $\psi(b) = \tilde{b}$, For $(b, b, \dots) \in \tilde{b}$.

We show that ψ is an isometry, note that (2) becomes

$$\|\tau^{\omega'}(\psi(\{\mathfrak{S}_n\}), \psi(M)) - \tau^{\omega'}(\psi(\{\mathfrak{h}_n\}), \psi(M))\|_{\tilde{\chi}} = \|\tau^\omega(\{\mathfrak{S}_n\}, M) - \tau^\omega(\{\mathfrak{h}_n\}, M)\|_{\chi}.$$

Where \tilde{b} is the class of $\{\mathfrak{h}_t\}$ where $\mathfrak{h}_t = b$, for all $t \in N$. Any τ^ω -app-isometry is one to one, and $\psi: \chi \rightarrow Y$ is onto.

Hence Y and χ are isometric.

Now, to prove that F is dense in $\tilde{\chi}$. Assume $\{\mathfrak{S}_n\} \in \tilde{m}$ for any $\tilde{m} \in \tilde{\chi}$ and for every $\epsilon > 0$, exists positive number such that $\|\tau^\omega(\{\mathfrak{S}_t\}, M) - \tau^\omega(\{\mathfrak{S}_L\}, M)\|_{\chi} < \frac{\epsilon}{2}$, for all $t > N$.

Let $(\mathfrak{S}_n, \mathfrak{S}_L, \dots) \in \tilde{\chi}_L$.

Then $\tilde{m}_L \in Y$.

That is $\|\tau^{\omega'}(\psi(\{\mathfrak{S}_n\}), \psi(M)) - \tau^{\omega'}(\psi(\{\mathfrak{S}_L\}), \psi(M))\|_{\tilde{\chi}} = \lim_{t \rightarrow \infty} \|\tau^\omega(\{\mathfrak{S}_t\}, M) - \tau^\omega(\{\mathfrak{S}_L\}, M)\|_{\chi} \leq \frac{\epsilon}{2} < \epsilon$.

Then $\|\tau^{\omega'}(\psi(\{\tilde{m}\}), \psi(M)) - \tau^{\omega'}(\psi(\tilde{m}_L), \psi(M))\|_{\tilde{\chi}} = \lim_{t \rightarrow \infty} \|\tau^\omega(\{\mathfrak{S}_t\}, M) - \tau^\omega(\{\mathfrak{S}_L\}, M)\|_{\chi} \leq \frac{\epsilon}{2} < \epsilon$, for all $\mathfrak{S}_t \in \tilde{m}$ and $\mathfrak{S}_L \in \tilde{m}_L$

We see every ϵ -open ball of the arbitrary $\tilde{m} \in \tilde{\chi}$ contains member of F . Hence F is τ^ω -dense in $\tilde{\chi}$.

Proposition3: Let $(\chi, \tau^\omega_{\|\cdot\|, \|\cdot\|})$ be τ^ω - approach normed space, then there exists a unique τ^ω - approach Banach space $\tilde{\chi}$ of χ .

Proof: We must prove the completeness of $\tilde{\chi}$ and uniqueness of $\tilde{\chi}$ up to isometry

Step 1: Let $\{\tilde{m}_n\}$ that any τ^ω -Cauchy sequence in $\tilde{\chi}$. We can have from proposition (2) that

$$F \text{ is dense in } \tilde{\chi}, \text{ for any } \tilde{m}_t \text{ There is an } \tilde{z}_t \in F \text{ such that } \|\tau^{\omega'}(\psi(\{\tilde{z}_t\}), \psi(M)) - \tau^{\omega'}(\psi(\{\tilde{z}_t\}), \psi(M))\|_{\tilde{m}} < \frac{1}{t} \dots\dots\dots(4).$$

Hence by definition of τ^ω -app-norm (4), $\|\tau^{\omega'}(\psi(\{\tilde{z}_t\}), \psi(M)) - \tau^{\omega'}(\psi(\{\tilde{z}_t\}), \psi(M))\|_{\tilde{\chi}} \leq \|\tau^{\omega'}(\psi(\{\tilde{z}_t\}), \psi(M)) - \tau^{\omega'}(\psi(\{\tilde{m}_k\}), \psi(M))\|_{\tilde{\chi}} + \|\tau^{\omega'}(\psi(\{\tilde{m}_k\}), \psi(M)) - \tau^{\omega'}(\psi(\{\tilde{m}_k\}), \psi(M))\|_{\tilde{\chi}} + \|\tau^{\omega'}(\psi(\{\tilde{m}_k\}), \psi(M)) - \tau^{\omega'}(\psi(\{\tilde{z}_t\}), \psi(M))\|_{\tilde{\chi}} < \frac{1}{k} + \|\tau^{\omega'}(\psi(\{\tilde{m}_k\}), \psi(M)) - \tau^{\omega'}(\psi(\{\tilde{m}_k\}), \psi(M))\|_{\tilde{\chi}} + \frac{1}{t}$, for any $\epsilon > 0$ for large k and t since $\{\tilde{m}_k\}$ is τ^ω -Cauchy seq.. Hence $\{\tilde{z}_t\}$ is τ^ω -Cauchy sequence.

Since $\psi: \chi \rightarrow F$ isometric and $\tilde{z}_t \in F$, Let $\{\mathfrak{E}_t\}$, where $\mathfrak{E}_t = \psi^{-1}(\tilde{z}_t)$, is τ^ω -Cauchy sequence in χ .

Suppose $\tilde{m} \in \tilde{\chi}$ be the class to which $\{\mathfrak{E}_t\}$ belongs.

We see that \tilde{m} is the limit of $\{\tilde{m}_t\}$.

By definition of \mathcal{t}^ω -app-norm space and (4)

$$\begin{aligned} & \|\mathcal{t}^{\omega'}(\psi(\{\tilde{m}_t\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{\tilde{m}\}), \psi(M))\|_{\tilde{\chi}} \leq \|\mathcal{t}^{\omega'}(\psi(\{\tilde{m}_t\}), \psi(M)) - \\ & \mathcal{t}^{\omega'}(\psi(\{\tilde{z}_t\}), \psi(M))\|_{\tilde{\chi}} + \|\mathcal{t}^{\omega'}(\psi(\{\tilde{z}_t\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{\tilde{m}\}), \psi(M))\|_{\tilde{\chi}} < \frac{1}{t} + \\ & \|\mathcal{t}^{\omega'}(\psi(\{\tilde{z}_t\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{\tilde{m}\}), \psi(M))\|_{\tilde{\chi}} \dots \dots \dots (5) \end{aligned}$$

Since $\{\mathcal{E}_t\} \in \tilde{m}$ and $\tilde{z}_t \in n$, so that $(\mathcal{E}_k, \mathcal{E}_k, \mathcal{E}_k, \dots \dots) \in \tilde{z}_t$.

The inequality in (5) becomes $\|\mathcal{t}^{\omega'}(\psi(\{\tilde{m}_t\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{\tilde{m}\}), \psi(M))\|_{\tilde{\chi}} < \frac{1}{t} + \lim_{t \rightarrow \infty} \|\mathcal{t}^\omega(\{\mathcal{E}_t\}, M) - \mathcal{t}^\omega(\{\mathcal{E}_k\}, M)\|_{\chi}$. The right side is less than zero.

Then the arbitrary \mathcal{t}^ω -Cauchy sequence $\{\tilde{m}_t\}$ in $\tilde{\chi}$ has the limit $\tilde{m} \in \tilde{\chi}$, and $\tilde{\chi}$ is complete. **Step2:** We will prove the uniqueness of $\tilde{\chi}$ up to isometry. If another complete \mathcal{t}^ω -normed app-space $(\chi^*, \|\cdot\|_*)$ with a subspace D dense in $\tilde{\chi}$ and isometric with χ , then there is $m^*, n^* \in \chi^*$ and the sequence $\{\mathfrak{S}_t^*\}, \{\mathfrak{h}_t^*\}$ in D such that $\mathfrak{S}_t^* \rightarrow m^*$ and $\mathfrak{h}_t^* \rightarrow n^*$.

so $\|\mathcal{t}^{\omega'}(\psi(\{m^*\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{n^*\}), \psi(M))\|_{\chi^*} = \lim_{t \rightarrow \infty} \|\mathcal{t}^{\omega'}(\psi(\{\mathfrak{S}_t^*\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{\mathfrak{h}_t^*\}), \psi(M))\|_{\chi^*}$. Follows from

$$\begin{aligned} & \left| \|\mathcal{t}^{\omega'}(\psi(\{m^*\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{n^*\}), \psi(M))\|_{\chi^*} - \|\mathcal{t}^{\omega'}(\psi(\{\mathfrak{S}_t^*\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{\mathfrak{h}_t^*\}), \psi(M))\|_{\chi^*} \right| \\ & \leq \|\mathcal{t}^{\omega'}(\psi(\{m^*\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{\mathfrak{S}_t^*\}), \psi(M))\|_{\chi^*} + \|\mathcal{t}^{\omega'}(\psi(\{n^*\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{\mathfrak{h}_t^*\}), \psi(M))\|_{\chi^*} \\ & \rightarrow 0. \end{aligned}$$

$$\begin{aligned} & \lim_{t \rightarrow \infty} \left| \|\mathcal{t}^{\omega'}(\psi(\{m^*\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{n^*\}), \psi(M))\|_{\chi^*} - \|\mathcal{t}^{\omega'}(\psi(\{\mathfrak{S}_t^*\}), \psi(M)) - \right. \\ & \left. \mathcal{t}^{\omega'}(\psi(\{\mathfrak{h}_t^*\}), \psi(M))\|_{\chi^*} \right| \leq \lim_{t \rightarrow \infty} \|\mathcal{t}^{\omega'}(\psi(\{m^*\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{\mathfrak{S}_t^*\}), \psi(M))\|_{\chi^*} + \\ & \lim_{t \rightarrow \infty} \|\mathcal{t}^{\omega'}(\psi(\{n^*\}), \psi(M)) - \mathcal{t}^{\omega'}(\psi(\{\mathfrak{h}_t^*\}), \psi(M))\|_{\chi^*} \rightarrow 0. \end{aligned}$$

This is equivalent the inequality in to (3).

Because D is isometric with $F \subset \tilde{\chi}$ and $F = \tilde{\chi}$, then the norms on χ^* and $\tilde{\chi}$ must be the same.

Then χ^* and $\tilde{\chi}$ are isometric. ■

Theorem1: Let $(\chi, \mathcal{t}^\omega_{\|\cdot\|}, \|\cdot\|)$ be \mathcal{t}^ω - app- normed space, then there exists \mathcal{t}^ω -Banach approach space $\tilde{\chi}$ and an isometry ψ from χ onto a dense subspace F of $\tilde{\chi}$ and $\tilde{\chi}$ is unique up to isometry.

Proof: This direct from Proposition (1), Proposition(2), and Proposition (3).

Conclusion

In this paper, we defined \mathcal{t}^ω -approach normed space, \mathcal{t}^ω -convergence app-space, \mathcal{t}^ω -complete normed space that is \mathcal{t}^ω - approach Banach space. Also, we proved a \mathcal{t}^ω -completion approach space by exists \mathcal{t}^ω -approach Banach space $\tilde{\chi}$ and an isometry ψ from χ onto a dense sub space D of $\tilde{\chi}$. The approach normed space $\tilde{\chi}$ is unique up to isometric. Some other results that related to \mathcal{t}^ω - approach Banach space and \mathcal{t}^ω -normed approach space are proved.

Declaration of competing interest

The authors declare that they have no known financial or non-financial competing interests in any material discussed in this paper.

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