

Article

On the Metric Dimension for Snowflake Graph

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Abstract. The concept of metric dimension is derived from the resolving set of a graph, that is measure the diameter among vertices in a graph. For its usefulness in diverse fields, it is interesting to find the metric dimension of various classes of graphs. In this paper, we introduce two new graphs, namely snowflake graph and generalized snowflake graph. After we construct these graphs, aided with a lemma about the lower bound of the metric dimension on a graph that has leaves, and manually recognized the pattern, we found that $\dim(\text{Snow}) = 24$ and $\dim(\text{Snow}(n,a,b,c)) = n(a+c+1)$.

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1. Introduction

It is known that graph theory is an intense region of combinatorics that has large variety of applications in diverse fields of science. Theoretical principles of graph theory are applied to practical fields, by determining graph invariants such as vertices, edges, diameter, and degree, then using them in real-life problems. One concept that spread among all the graph theory is that of distance, and distance is used in isomorphism testing, graph operations, maximal and minimal problems on connectivity and diameter. Several parameters related to distances in graphs are highly attracting the attention of several researchers. One of them, namely, the metric dimension, has specifically centered several investigations.

Slater in [1] initiated the concept of a resolving set for a connected graph G (and of a minimum resolving set) under the term *location set*. He called the cardinality of a minimum resolving set the *location number* of G . Independently, Harary and Melter [2] introduced the same concept but used the term *metric dimension*, rather than location number.

Gerey and Johnson showed that determining the metric dimension of a graph is an NP-complete problem that is reduced from 3-dimensional marching (3DM), while Khuller et al proved that it is reduced from 3 satisfiability (3SAT). However, Chartrand et al. [3] have obtained some results as follows.

Theorem 1.1. [3] Let G be a connected graph of order $n \geq 2$. Then

1. $\dim(G) = 1$ if only if $G = P_n$.
2. $\dim(G) = n - 1$ if only if $G = K_n$.
3. For $n \geq 3$, $\dim(C_n) = 2$
4. $\dim(G) = n - 2$ if only if G is either $K_{r,s}$ for $r, s \geq 1$, or $K_r + \bar{K}_s$ for $r > 1, s \geq 2$, or $K_r + (K_1 \cup K_s)$ for $r, s \geq 1$.

Some results for certain class of graphs have been obtained, such as cycles, trees, fans, wheels, complete n -partite graphs, unicyclic graphs, grids, honeycomb networks, circulant networks, Cayley graphs, graphs with pendants, Jahangir graphs, amalgamation of cycles, regular bipartite graphs, corona product of Graphs, lexicographic product of graphs, barycentric subdivision of Cayley graphs [4], windmill graphs [5], 2-connected graph [6], fullerene graphs [7], oriented graphs [8], generalized wheels [9], binary products [10], cartesian powers of a graph [11], and Generalized Petersen graphs [12].

The concept of metric dimension has been developed. There are several developments obtained such as edge metric dimensions [13] and local metric dimensions [14]. Some results for edge metric dimension have been acquired, such as 2-connected graphs [15], one-heptagonal carbon nanocone [16], hierarchical product [17], cacti [18], unicyclic graph [19], Generalized Petersen graphs [20], and some graph operation [21]. While, some results for local metric dimension have been acquired, such as lexicographic product of graphs [22], generalized hierarchical products [23], and graphs with small clique numbers [24].

Other developments related to the concept of metric dimensions are truncated metric dimensions [8], mixed metric dimension [25], [26], strong metric dimension [27], [28], [29], metric dimensions vs. cyclomatic number of graphs [30], fractional edge dimension of graphs [31], learning to compute metric dimensions [6] and [32], the dominant metric dimension [33], the complement metric dimension [34], deletion on the edge metric dimension [35], extremal results of bounded metric dimension [36], and robustness metric dimension [37].

A snowflake is a single ice crystal or collection of ice crystals that fall from the Earth's atmosphere. Snowflakes come in various sizes and shapes. Snowflakes are white even though they are made of pure ice. This is due to the diffuse reflection of the entire light spectrum on the small crystal. In Figure 1. An example of a snowflake is given [38].

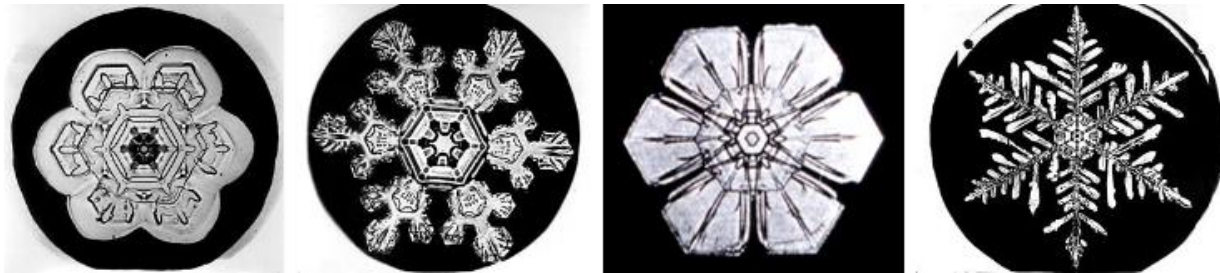


Figure 1. Photo of snowflakes by Wilson Bentley (1865–1931)

In this paper, we define some new graphs, that is a snowflake graph and a generalized snowflake graph, which are constructed by vertices and edges resembling one of the snowflake shapes. Then, we will determine the metric dimensions of the snowflake graph and the general snowflake graph.

2. Notation

The vertices and edges set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. We refer to the general graph theory notations and terminologies that are not described in this paper to the book *Graphs and Digraphs* [39]. The number of vertices adjacent to a vertex v is called degree of vertex v . A vertex of degree one is referred to as a leaf. An edge incident with an end-vertex is called a pendant edge.

The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G . For an ordered set $W = \{w_1, w_2, \dots, w_k\} \subseteq V(G)$ of vertices, we refer to the ordered k -tuple $r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$ as the (metric) representation of v with respect to W . The set W is called a resolving set for G if $r(u|W) = r(v|W)$ implies that $u = v$, for all $u, v \in G$. A resolving set of minimum cardinality for a graph G is called a minimum resolving set or a basis for G . The metric dimension $\dim(G)$ is the number of vertices in a basis for G .

3. Results and Discussion

In this section, we define a snowflake graph and a generalized snowflake graph, and then determine the metric dimensions of these graphs. To determine such metric dimensions, we obtain a lemma related to the lower bounds of the metric dimension on a graph that contains leaves. This lemma arose from the need of ensuring that every leaf adjacent to the same vertex must have distinct representation.

Lemma 3.2. Let G be any connected graph that has n vertices that are connected to pendant edges. If $v_i \in G$ connected with k_i many vertices of degree one, then $\dim(G) \geq \sum_{i=1}^n (k_i - 1)$, for all $i \in [1, n]$.

Proof. We will show that $\dim(G) \geq \sum_{i=1}^n (k_i - 1)$, for all $i \in [1, n]$. Let G is a connected graph and vertex $v_i \in G$ connected with k_i many vertices of degree one. Suppose the vertices of degree one connected with vertex v_i are $v_{1,j}$, where $i \in [1, n]$ and $j \in [1, k_i]$. Without loss of generality, consider vertex v_1 and $v_{1,j}$. Suppose the set $W = v(G) - \{v_1, v_{1,j}\}$ is a resolving set for G . Since $d(v_1, v_{1,k_1}) = 1$ and edge $v_1 v_{1,k_1}$ is the only edge connecting v_1 and v_{1,k_1} , then representation of $v_{1,j}$ with respect to W is $r(v_{1,1}|W) = r(v_{1,2}|W) = \dots = r(v_{1,k_1}|W) = r(v_1|W) + (\bar{1})$. So, the representation $r(v_{1,i}) \neq r(v_{1,j})$ for $i, j \in [1, k_1]$, and then any $k_1 - 1$ edges of v_{1,k_1} must be a member of resolving set W . As a result, in general, $\dim(G) \geq \sum_{i=1}^n (k_i - 1)$.

Q.E.D

Here we formally introduce our new graph. A snowflake graph is a graph obtained by resembling one of the snowflake shapes into vertices and edges so that it forms a simple graph denoted snowflake

graph. The structure of the snowflake graph consists of tree graph and circle graph as shown in Figure 2. which is defined by the set of vertices and edges as follows.

Definition 3.3.

$V(Snow) = \{v_0, v_i, v_{i,1,k,l}, v_{i,2,k}, v_{i,3,1,l}, v_{i,k5} | 1 \leq i \leq 6; 1 \leq k \leq 2; 0 \leq l \leq 2. \}$ and

$E(Snow) = \{v_0v_{i,3,1,0}, v_{i,3,1,0}v_{i,3,1,j}, v_{i,3,1,0}v_{i,2,2}, v_{i,2,2}v_{i,2,1}, v_{i,2,1}v_{i,1,2,0}, v_{i,1,2,0}v_{i,1,2,j}, v_{i,1,2,0}v_{i,1,1,0}, v_{i,1,1,0}v_{i,1,1,j}, v_{i,1,1,0}v_a, v_{i,2,j}v_{i,j}, v_{i,j}v_{i+1(mod6),2,j} | 1 \leq i \leq 6; 1 \leq j \leq 2\}$.

Theorem 3.4. If *Snow* is a graph that is defined in Definition 3.3, then $dim(Snow) = 24$.

Proof. Upper bound :

First, we will show that $dim(Snow) \leq 24$. Suppose an ordered set

$W = \{v_a, v_{1,1,1,1}, v_{1,1,2,1}, v_{1,3,1,1}, v_b, v_{2,1,1,1}, v_{2,1,2,1}, v_{2,3,1,1}, v_c, v_{3,1,1,1}, v_{3,1,2,1}, v_{3,3,1,1}, v_d, v_{4,1,1,1}, v_{4,1,2,1}, v_{4,3,1,1}, v_e, v_{5,1,1,1}, v_{5,1,2,1}, v_{5,3,1,1}, v_f, v_{6,1,1,1}, v_{6,1,2,1}, v_{6,3,1,1}\}$,

where $W \subseteq V(Snow)$ like figure 3.

Define the representation of $V(Snow)$ with respect to W as follow

$r(v|W) = (d(v, v_1), d(v, v_{1,1,1,1}), d(v, v_{1,1,2,1}), d(v, v_{1,3,1,1}), d(v, v_2), d(v, v_{2,1,1,1}), d(v, v_{2,1,2,1}), d(v, v_{2,3,1,1}), d(v, v_3), d(v, v_{3,1,1,1}), d(v, v_{3,1,2,1}), d(v, v_{3,3,1,1}), d(v, v_4), d(v, v_{4,1,1,1}), d(v, v_{4,1,2,1}), d(v, v_{4,3,1,1}), d(v, v_5), d(v, v_{5,1,1,1}), d(v, v_{5,1,2,1}), d(v, v_{5,3,1,1}), d(v, v_6), d(v, v_{6,1,1,1}), d(v, v_{6,1,2,1}), d(v, v_{6,3,1,1}))$.

for all $v \in V(snow)$.

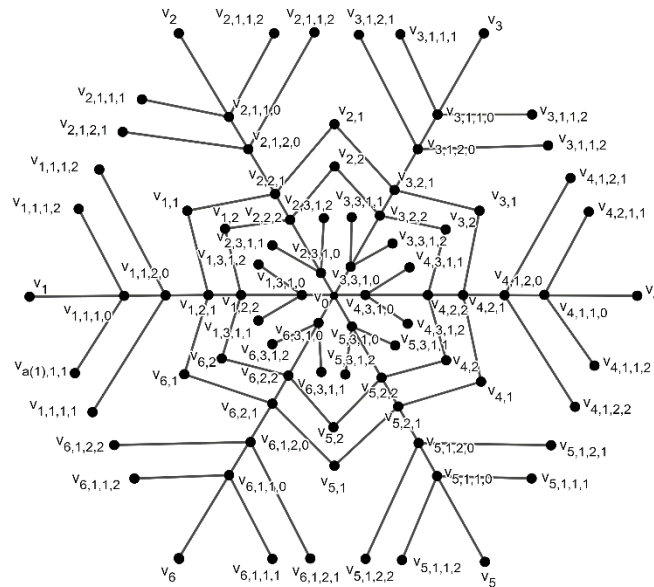


Figure 2. Snow Graph

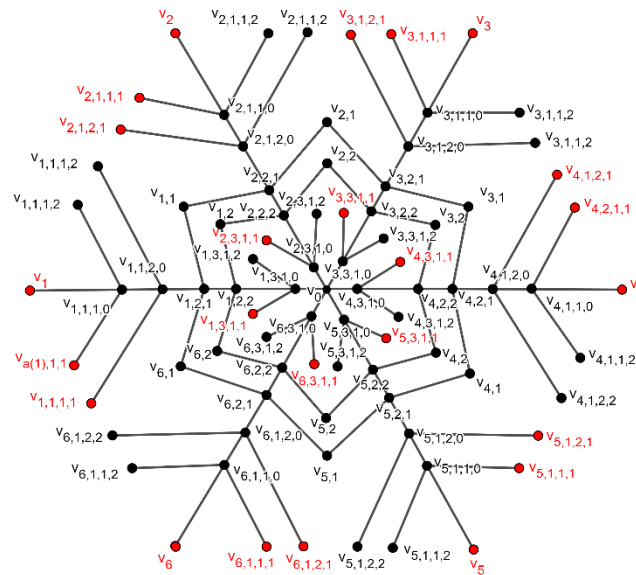


Figure 3. Partition of Snow Graph

The representation of each vertex in *Snow* is given as follows.

$$\begin{aligned}
 r(v_1|W) &= (0,2,3,6,8,8,7,8,10,10,9,8,12,12,11,8,10,10,9,8,8,7,8) \\
 r(v_{1,1,1,0}|W) &= (1,1,2,5,7,7,6,7,9,9,8,7,11,11,10,7,9,9,8,7,7,6,7) \\
 r(v_{1,1,1,1}|W) &= (2,0,3,6,8,8,7,8,10,10,9,8,12,12,11,8,10,10,9,8,8,7,8) \\
 r(v_{1,1,1,2}|W) &= (2,2,3,6,8,8,7,8,10,10,9,8,12,12,11,8,10,10,9,8,8,7,8) \\
 r(v_{1,1,2,0}|W) &= (2,2,1,4,6,6,5,6,8,8,7,6,10,10,9,6,8,8,7,6,6,5,6) \\
 r(v_{1,1,2,1}|W) &= (3,3,0,5,7,7,6,7,9,9,8,7,11,11,10,7,9,9,8,7,7,6,7) \\
 r(v_{1,1,2,2}|W) &= (3,3,2,5,7,7,6,7,9,9,8,7,11,11,10,7,9,9,8,7,7,6,7) \\
 r(v_{1,2,1}|W) &= (3,3,2,3,5,5,4,5,7,7,6,5,9,9,8,5,7,7,6,5,5,4,5) \\
 r(v_{1,2,2}|W) &= (4,4,3,2,6,6,5,4,8,8,7,4,8,8,7,4,8,8,7,4,6,6,5,4) \\
 r(v_{1,1}|W) &= (4,4,3,4,4,4,3,4,6,6,5,6,8,8,7,6,8,8,7,6,6,6,5,6) \\
 r(v_{1,2}|W) &= (5,5,4,4,5,5,4,3,7,7,6,5,9,9,8,5,9,9,8,5,7,7,6,5) \\
 r(v_{1,3,1,0}|W) &= (5,5,4,1,7,7,6,3,7,7,6,3,7,7,6,3,7,7,6,3,7,7,6,3) \\
 r(v_{1,3,1,1}|W) &= (6,6,5,0,8,8,7,4,8,8,7,4,8,8,7,4,8,8,7,4,8,8,7,4) \\
 r(v_{1,3,1,2}|W) &= (6,6,5,2,8,8,7,4,8,8,7,4,8,8,7,4,8,8,7,4,8,8,7,4) \\
 r(v_2|W) &= (8,8,7,8,0,2,3,6,8,8,7,8,10,10,9,8,12,12,11,8,10,10,9,8) \\
 r(v_{2,1,1,0}|W) &= (7,7,6,7,1,1,2,5,7,7,6,7,9,9,8,7,11,11,10,7,9,9,8,7) \\
 r(v_{2,1,1,1}|W) &= (8,8,7,8,2,0,3,6,8,8,7,8,10,10,9,8,12,12,11,8,10,10,9,8) \\
 r(v_{2,1,1,2}|W) &= (8,8,7,8,2,2,3,6,8,8,7,8,10,10,9,8,12,12,11,8,10,10,9,8) \\
 r(v_{2,1,2,0}|W) &= (6,6,5,6,2,2,1,4,6,6,5,6,8,8,7,6,10,10,9,6,8,8,7,6) \\
 r(v_{2,1,2,1}|W) &= (7,7,6,7,3,3,0,5,7,7,6,7,9,9,8,7,11,11,10,7,9,9,8,7) \\
 r(v_{2,1,2,2}|W) &= (7,7,6,7,3,3,2,5,7,7,6,7,9,9,8,7,11,11,10,7,9,9,8,7) \\
 r(v_{2,2,1}|W) &= (5,5,4,5,3,3,2,3,5,5,4,5,7,7,6,5,9,9,8,5,7,7,6,5) \\
 r(v_{2,2,2}|W) &= (6,6,5,4,4,4,3,2,6,6,5,4,8,8,7,4,8,8,7,4,8,8,7,4) \\
 r(v_{2,1}|W) &= (6,6,5,6,4,4,3,4,4,4,3,4,6,6,5,6,8,8,7,6,8,8,7,6) \\
 r(v_{2,2}|W) &= (7,7,6,5,5,5,4,4,5,5,4,3,7,7,6,5,9,9,8,5,9,9,8,5) \\
 r(v_{2,3,1,0}|W) &= (7,7,6,3,5,5,4,1,7,7,6,3,7,7,6,3,7,7,6,3,7,7,6,3)
 \end{aligned}$$

$$\begin{aligned}
r(v_{2,3,1,1}|W) &= (8,8,7,4,6,6,5,0,8,8,7,4,8,8,7,4,8,8,7,4,8,8,7,4) \\
r(v_{2,3,1,2}|W) &= (8,8,7,4,6,6,5,2,8,8,7,4,8,8,7,4,8,8,7,4,8,8,7,4) \\
r(v_3|W) &= (10,10,9,8,8,8,7,8,0,2,3,6,8,8,7,8,10,10,9,8,12,12,11,8) \\
r(v_{3,1,1,0}|W) &= (9,9,8,7,7,7,6,7,1,1,2,5,7,7,6,7,9,9,8,7,11,11,10,7) \\
r(v_{3,1,1,1}|W) &= (10,10,9,8,8,8,7,8,2,0,3,6,8,8,7,8,10,10,9,8,12,12,11,8) \\
r(v_{3,1,1,2}|W) &= (10,10,9,8,8,8,7,8,2,2,3,6,8,8,7,8,10,10,9,8,12,12,11,8) \\
r(v_{3,1,2,0}|W) &= (8,8,7,6,6,6,5,6,2,2,1,4,6,6,5,6,8,8,7,6,10,10,9,6) \\
r(v_{3,1,2,1}|W) &= (9,9,8,7,7,7,6,7,3,3,0,5,7,7,6,7,9,9,8,7,11,11,10,7) \\
r(v_{3,1,2,2}|W) &= (9,9,8,7,7,7,6,7,3,3,2,5,7,7,6,7,9,9,8,7,11,11,10,7) \\
r(v_{3,2,1}|W) &= (7,7,6,5,5,5,4,5,3,3,2,3,5,5,4,5,7,7,6,5,9,9,8,5) \\
r(v_{3,2,2}|W) &= (8,8,7,4,6,6,5,4,4,4,3,2,6,6,5,4,8,8,7,4,8,8,7,4) \\
r(v_{3,1}|W) &= (8,8,7,6,6,6,5,6,4,4,3,4,4,4,3,4,6,6,5,6,8,8,7,6) \\
r(v_{3,2}|W) &= (9,9,8,5,7,7,6,5,5,5,4,4,5,5,4,3,7,7,6,5,9,9,8,5) \\
r(v_{3,3,1,0}|W) &= (7,7,6,3,7,7,6,3,5,5,4,1,7,7,6,3,7,7,6,3,7,7,6,3) \\
r(v_{3,3,1,1}|W) &= (8,8,7,4,8,8,7,4,6,6,5,0,8,8,7,4,8,8,7,4,8,8,7,4) \\
r(v_{3,3,1,2}|W) &= (8,8,7,4,8,8,7,4,6,6,5,2,8,8,7,4,8,8,7,4,8,8,7,4) \\
r(v_4|W) &= (12,12,11,8,10,10,9,8,8,8,7,8,0,2,3,6,8,8,7,8,10,10,9,8) \\
r(v_{4,1,1,0}|W) &= (11,11,10,7,9,9,8,7,7,6,7,1,1,2,5,7,7,6,7,9,9,8,7) \\
r(v_{4,1,1,1}|W) &= (12,12,11,8,10,10,9,8,8,8,7,8,2,0,3,6,8,8,7,8,10,10,9,8) \\
r(v_{4,1,1,2}|W) &= (12,12,11,8,10,10,9,8,8,8,7,8,2,2,3,6,8,8,7,8,10,10,9,8) \\
r(v_{4,1,2,0}|W) &= (10,10,9,6,8,8,7,6,6,6,5,6,2,2,1,4,6,6,5,6,8,8,7,6) \\
r(v_{4,1,2,1}|W) &= (11,11,10,7,9,9,8,7,7,6,7,3,3,0,5,7,7,6,7,9,9,8,7) \\
r(v_{4,1,2,2}|W) &= (11,11,10,7,9,9,8,7,7,6,7,3,3,2,5,7,7,6,7,9,9,8,7) \\
r(v_{4,2,1}|W) &= (9,9,8,5,7,7,6,5,5,5,4,5,3,3,2,3,5,5,4,5,7,7,6,5) \\
r(v_{4,2,2}|W) &= (8,8,7,4,8,8,7,4,6,6,5,4,4,4,3,2,6,6,5,4,8,8,7,4) \\
r(v_{4,1}|W) &= (8,8,7,6,8,8,7,6,6,6,5,6,4,4,3,4,4,4,3,4,6,6,5,6) \\
r(v_{4,2}|W) &= (9,9,8,5,9,9,8,5,7,7,6,5,5,5,4,4,5,5,4,3,7,7,6,5) \\
r(v_{4,3,1,0}|W) &= (7,7,6,3,7,7,6,3,7,7,6,3,5,5,4,1,7,7,6,3,7,7,6,3) \\
r(v_{4,3,1,1}|W) &= (8,8,7,4,8,8,7,4,8,8,7,4,6,6,5,0,8,8,7,4,8,8,7,4) \\
r(v_{4,3,1,2}|W) &= (8,8,7,4,8,8,7,4,8,8,7,4,6,6,5,2,8,8,7,4,8,8,7,4) \\
r(v_5|W) &= (10,10,9,8,12,12,11,8,10,10,9,8,8,8,7,8,0,2,3,6,8,8,7,8) \\
r(v_{5,1,1,0}|W) &= (9,9,8,7,11,11,10,7,9,9,8,7,7,6,7,1,1,2,5,7,7,6,7) \\
r(v_{5,1,1,1}|W) &= (10,10,9,8,12,12,11,8,10,10,9,8,8,8,7,8,2,0,3,6,8,8,7,8) \\
r(v_{5,1,1,2}|W) &= (10,10,9,8,12,12,11,8,10,10,9,8,8,8,7,8,2,2,3,6,8,8,7,8) \\
r(v_{5,1,2,0}|W) &= (8,8,7,6,10,10,9,6,8,8,7,6,6,6,5,6,2,2,1,4,6,6,5,6) \\
r(v_{5,1,2,1}|W) &= (9,9,8,7,11,11,10,7,9,9,8,7,7,6,7,3,3,0,5,7,7,6,7) \\
r(v_{5,1,2,2}|W) &= (9,9,8,7,11,11,10,7,9,9,8,7,7,6,7,3,3,2,5,7,7,6,7) \\
r(v_{5,2,1}|W) &= (7,7,6,5,9,9,8,5,7,7,6,5,5,5,4,5,3,3,2,3,5,5,4,5) \\
r(v_{5,2,2}|W) &= (8,8,7,4,8,8,7,4,8,8,7,4,6,6,5,4,4,4,3,2,6,6,5,4) \\
r(v_{5,1}|W) &= (6,6,5,6,8,8,7,6,8,8,7,6,6,6,5,6,4,4,3,4,4,4,3,4) \\
r(v_{5,2}|W) &= (7,7,6,5,9,9,8,5,9,9,8,5,7,7,6,5,5,5,4,4,5,5,4,3) \\
r(v_{5,3,1,0}|W) &= (7,7,6,3,7,7,6,3,7,7,6,3,7,7,6,3,5,5,4,1,7,7,6,3) \\
r(v_{5,3,1,1}|W) &= (8,8,7,4,8,8,7,4,8,8,7,4,8,8,7,4,6,6,5,0,8,8,7,4) \\
r(v_{5,3,1,2}|W) &= (8,8,7,4,8,8,7,4,8,8,7,4,8,8,7,4,6,6,5,2,8,8,7,4) \\
r(v_6|W) &= (6,8,8,7,8,10,10,9,8,12,12,11,8,10,10,9,8,8,8,7,8,0,2,3)
\end{aligned}$$

$$\begin{aligned}
r(v_{6,1,1,0}|W) &= (7,7,6,7,9,9,8,7,11,11,10,7,9,9,8,7,7,7,6,7,1,1,2,5) \\
r(v_{6,1,1,1}|W) &= (8,8,7,8,10,10,9,8,12,12,11,8,10,10,9,8,8,8,7,8,2,0,3,6) \\
r(v_{6,1,1,2}|W) &= (8,8,7,8,10,10,9,8,12,12,11,8,10,10,9,8,8,8,7,8,2,2,3,6) \\
r(v_{6,1,2,0}|W) &= (6,6,5,6,8,8,7,6,10,10,9,6,8,8,7,6,6,6,5,6,2,2,1,4) \\
r(v_{6,1,2,1}|W) &= (7,7,6,7,9,9,8,7,11,11,10,7,9,9,8,7,7,7,6,7,3,3,0,5) \\
r(v_{6,1,2,2}|W) &= (7,7,6,7,9,9,8,7,11,11,10,7,9,9,8,7,7,7,6,7,3,3,2,5) \\
r(v_{6,2,1}|W) &= (5,5,4,5,7,7,6,5,9,9,8,5,7,7,6,5,5,5,4,5,3,3,2,3) \\
r(v_{6,2,2}|W) &= (6,6,5,4,8,8,7,4,8,8,7,4,8,8,7,4,6,6,5,4,4,4,3,2) \\
r(v_{6,1}|W) &= (4,4,3,4,6,6,5,6,8,8,7,6,8,8,7,6,6,6,5,6,4,4,3,4) \\
r(v_{6,2}|W) &= (5,5,4,3,7,7,6,5,9,9,8,5,9,9,8,5,7,7,6,5,5,5,4,4) \\
r(v_{6,3,1,0}|W) &= (7,7,6,3,7,7,6,3,7,7,6,3,7,7,6,3,7,7,6,3,5,5,4,1) \\
r(v_{6,3,1,1}|W) &= (8,8,7,4,8,8,7,4,8,8,7,4,8,8,7,4,8,8,7,4,6,6,5,0) \\
r(v_{6,3,1,2}|W) &= (8,8,7,4,8,8,7,4,8,8,7,4,8,8,7,4,8,8,7,4,6,6,5,2)
\end{aligned}$$

Since $r(v_a|W) \neq r(v_b|W)$ implies that $v_a = v_b$, for all $v_a, v_b \in V(\text{Snow})$, then W is resolving set for Snow . Therefore, $\dim(\text{Snow}) \leq 24$.

Lower bound :

Second, we will show that $\dim(\text{Snow}) \geq 24$. Choose the set edges of $A = \{v_{i,1,1,0} | i \in [1,6] \text{ and } v_{i,1,1,0} \in E(\text{Snow})\}$ dan $B = \{v_{j,1,2,0}, v_{j,3,1,0} | j \in [1,6] \text{ and } \in E(\text{Snow})\}$. The sets A and B are, respectively, the set and edges connected by 3 and 2 edges of degree one, where $|A| = 6$ and $|B| = 12$. By Lemma 3.2, $\dim(\text{Snow}) \geq \sum_{i=1}^{|A|} (3-1) + \sum_{i=1}^{|B|} (2-1) = 6(2) + 12 = 24$, so $\dim(\text{Snow}) \geq 24$.

Putting the bounds together gives us, that $\dim(\text{Snow}) = 24$.

Q.E.D

A generalized snowflake graph, denoted by $\text{Snow}(n, a, b, c)$, is a Snow graph with n parts of the stem, a pair of outer leaves, b middle circle, and c pairs of inner leaves as shown in Figure 4. The set of vertices and edges of the graph is defined as follows.

Definition 3.5.

$$\begin{aligned}
V(\text{Snow}(n, a, b, c)) &= \{v_0, v_i, v_{i,1,j,k}, v_{i,2,l}, v_{i,3,p,k}, v_{i,l} | 1 \leq i \leq n; 1 \leq j \leq a; 0 \leq k \leq 2; 1 \leq l \leq b; 1 \leq \\
&\quad p \leq c\} \text{ and} \\
E(\text{Snow}(n, a, b, c)) &= \{v_0 v_{i,3,c,0}, v_{i,3,j,0} v_{i,3,j+1,0}, v_{i,3,k,0} v_{i,3,k,l}, v_{i,3,1,0} v_{i,2,b}, v_{i,2,p} v_{i,2,p+1}, v_{i,2,1} v_{i,1,a,0}, \\
&\quad v_{i,1,q,0} v_{i,1,1,0}, v_i, v_{i,2,r} v_{i,r}, v_{i,r} v_{i+1(\text{mod } n),2,r} | 1 \leq i \leq n; 1 \leq j \leq c-1; 1 \leq k \leq c; \\
&\quad 1 \leq l \leq 2; 1 \leq p \leq b-1; 1 \leq q \leq a-1; 1 \leq r \leq b.\}
\end{aligned}$$

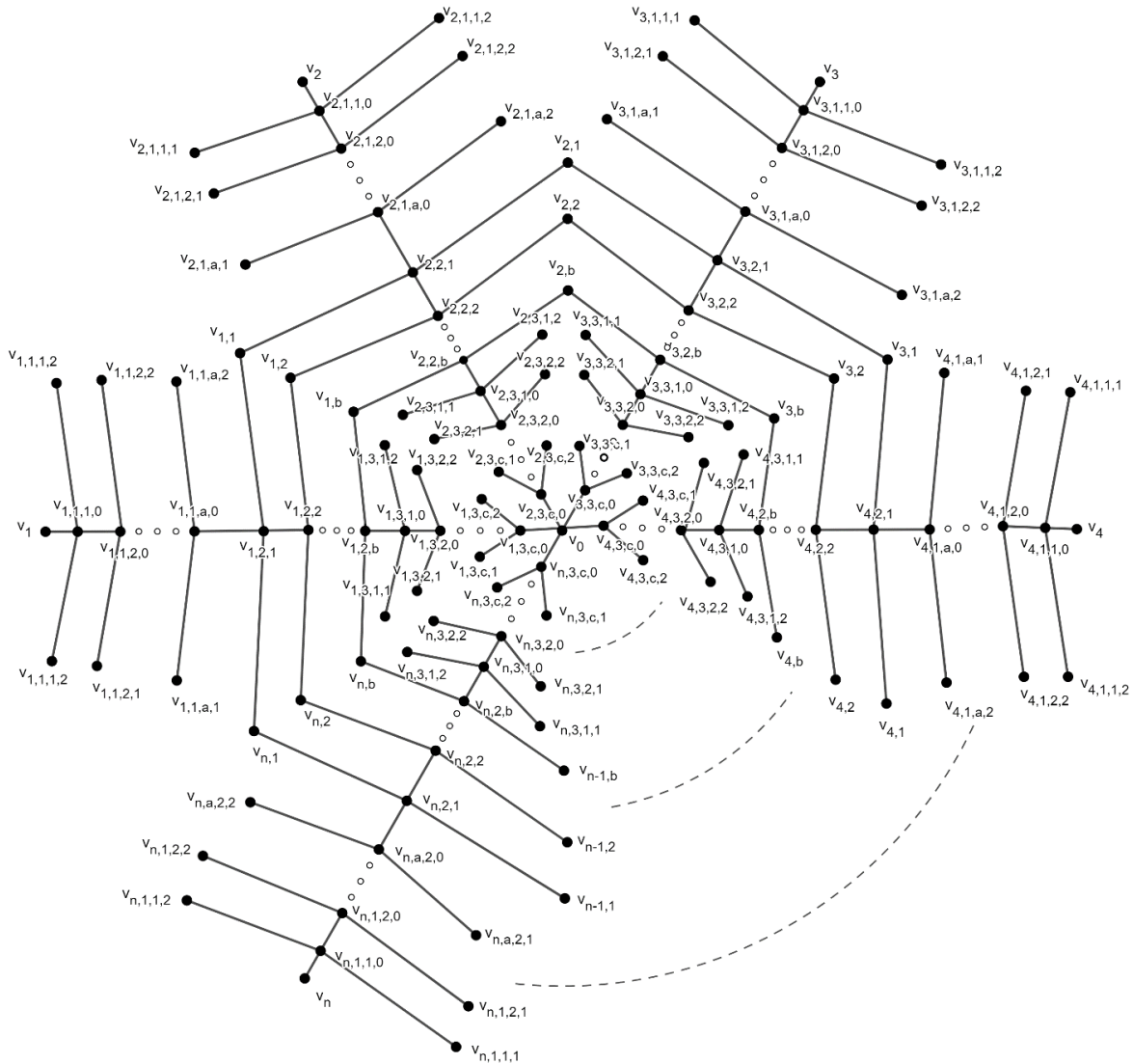


Figure 4. $Snow(n, a, b, c)$

Theorem 3.6. If $Snow(n, a, b, c)$ is a graph defined in Definition 3.5, then $dim(Snow) = n(a + c + 1)$ for all $n, a, b, c \in \mathbb{N}$ and $n \geq 3$.

Proof. Upper Bound :

First, we will show that $dim(Snow(n, a, b, c)) \leq n(a + c + 1)$. Suppose an ordered set $W = \{v_1, v_{1,1,1,1}, v_{1,1,2,1}, \dots, v_{1,1,a,1}, v_{1,3,1,1}, v_{1,3,2,1}, \dots, v_{2,3,c,1}, v_2, v_{2,1,1,1}, v_{2,1,2,1}, \dots, v_{2,1,a,1}, v_{2,3,1,1}, v_{2,3,2,1}, \dots, v_{2,3,c,1}, \dots, v_n, v_{n,1,1,1}, v_{n,1,2,1}, \dots, v_{n,1,a,1}, v_{n,3,1,1}, v_{n,3,2,1}, \dots, v_{n,3,c,1}\}$, Where $W \subseteq V(Snow(n, a, b, c))$ and $|W| = n(a + c + 1)$.

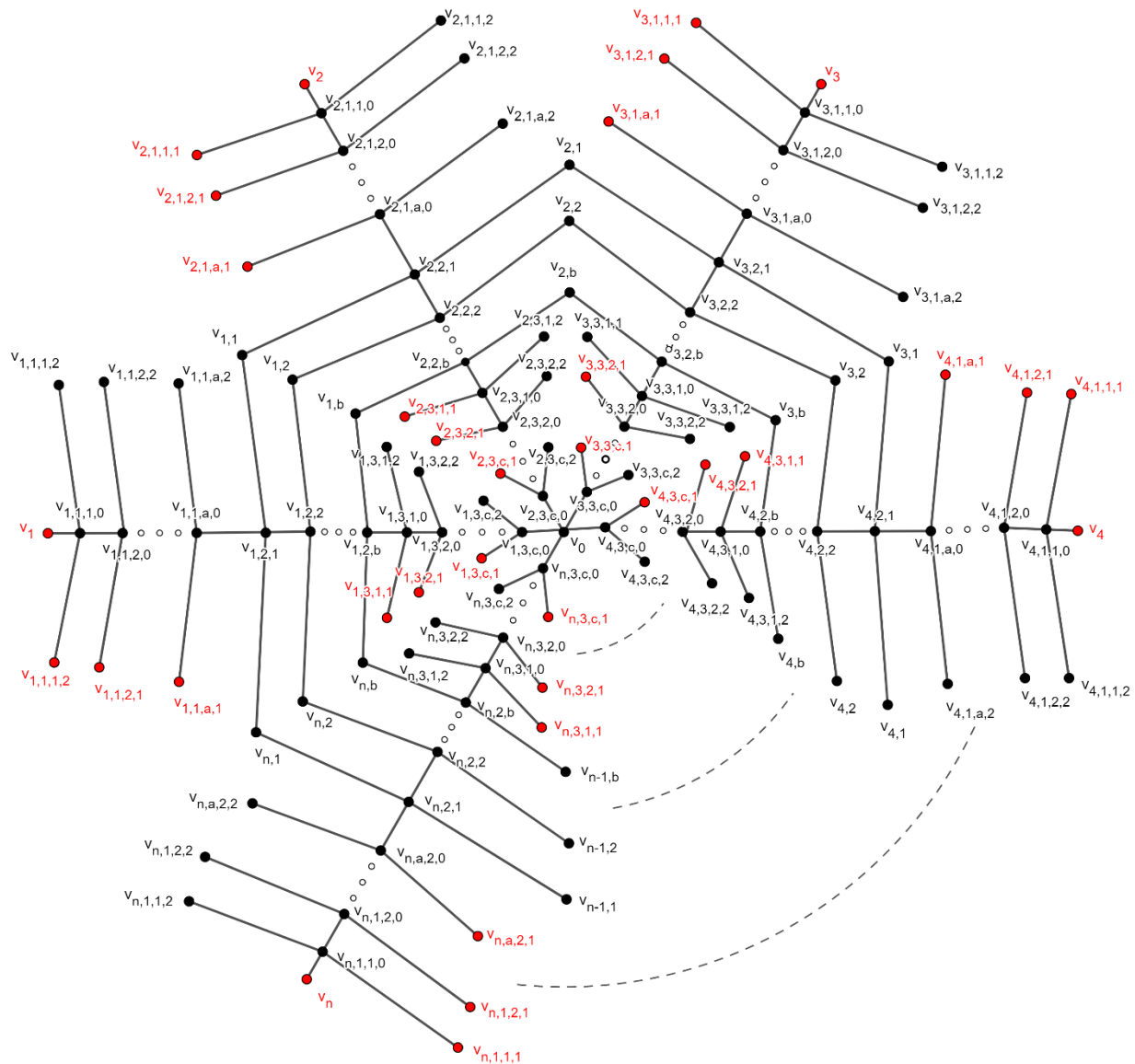


Figure 4. Partition of $Snow(n, a, b, c)$ Graph

Define the representation of $V(Snow(n, a, b, c))$ with respect to W as follow

$$r(v|W) = (d(v, v_1), d(v, v_{1,1,1,1}), d(v, v_{1,1,2,1}), \dots, d(v, v_{1,1,a,1}), d(v, v_{1,3,1,1}), d(v, v_{1,3,2,1}), \dots, d(v, v_{2,3,c,1}), d(v, v_2), d(v, v_{2,1,1,1}), d(v, v_{2,1,2,1}), \dots, d(v, v_{2,1,a,1}), d(v, v_{2,3,1,1}), d(v, v_{2,3,2,1}), \dots, d(v, v_{2,3,c,1}), \dots, \dots, d(v, v_n), d(v, v_{n,1,1,1}), d(v, v_{n,1,2,1}), \dots, d(v, v_{n,1,a,1}), d(v, v_{n,3,1,1}), d(v, v_{n,3,2,1}), \dots, d(v, v_{n,3,c,1})),$$

for all $v \in V(Snow(n, a, b, c))$.

Let $W_1 = \{v_1, v_2, \dots, v_n\}$, $W_2 = \{v_1, v_{1,1,j,1}, v_2, v_{2,1,j,1}, \dots, v_n, v_{n,1,j,1}\}$, and $W_3 = \{v_1, v_{1,3,j,1}, v_2, v_{2,3,j,1}, \dots, v_n, v_{n,3,j,1}\}$, be an ordered sets, where $W_1 \subseteq W_2 \subseteq W$ and $W_1 \subseteq W_3 \subseteq W$. So,

$r(v|W_1) = (d(v, v_1), d(v, v_2), \dots, d(v, v_n))$, $W_2 = \{v_1, v_{1,1,j,1}, v_2, v_{2,1,j,1}, \dots, v_n, v_{n,1,j,1}\}$, and $W_3 = \{v_1, v_{1,3,j,1}, v_2, v_{2,3,j,1}, \dots, v_n, v_{n,3,j,1}\}$.

Representation of each vertex in $Snow(n, a, b, c)$ is given as follows.

1. Representation of v_0 with respect to $W_1 \subseteq W$ is

$$r(v_0|W_1) = (a + b + c + 1, a + b + c + 1, \dots, a + b + c + 1).$$

2. Representation of $v_i, i \in [1, n]$, with respect to $W_1 \subseteq W$ are

$$r(v_1, W_2) = \left(0, \min\{2a + 4, 2(a + b + c + 1)\}, \min\{2a + 6, 2(a + b + c + 1)\}, \dots, \min\left\{2\left(a + 1 + \left\lfloor \frac{n}{2} \right\rfloor\right), 2(a + b + c + 1)\right\}, \dots, \min\{2a + 6, 2a + 4, 2(a + b + c + 1)\}, \min\{2a + 4, 2(a + b + c + 1)\}\right)$$

$$r(v_2, W_2) = \left(\min\{2a + 4, 2(a + b + c + 1)\}, 0, \min\{2a + 4, 2(a + b + c + 1)\}, \min\{2a + 6, 2(a + b + c + 1)\}, \dots, \min\left\{2\left(a + 1 + \left\lfloor \frac{n}{2} \right\rfloor\right), 2(a + b + c + 1)\right\}, \dots, \min\{2a + 6, 2a + 4, 2(a + b + c + 1)\}\right)$$

⋮

$$r(v_n, W_2) = \left(\min\{2a + 4, 2(a + b + c + 1)\}, \min\{2a + 6, 2(a + b + c + 1)\}, \dots, \min\left\{2\left(a + 1 + \left\lfloor \frac{n}{2} \right\rfloor\right), 2(a + b + c + 1)\right\}, \dots, \min\{2a + 6, 2a + 4, 2(a + b + c + 1)\}, \min\{2a + 4, 2(a + b + c + 1)\}, 0\right)$$

Since $r(v_r, W_1) \neq r(v_s, W_1)$ and $W_1 \subseteq W$, then $r(v_r, W) \neq r(v_s, W)$, for all $r, s \in [1, n]$.

3. Representation $v_{i,1,j,0}, i \in [1, n]$ and $j \in [1, a]$ with respect to $W_1 \subseteq W$ are

$$r(v_{1,1,j,0}, W_1) = \left(j, \min\{2a + 4 - j, 2(a + b + c + 1) - j\}, \min\{2a + 6 - j, 2(a + b + c + 1) - j\}, \dots, \min\left\{2\left(a + 1 + \left\lfloor \frac{n}{2} \right\rfloor\right) - j, 2(a + b + c + 1) - j\right\}, \dots, \min\{2a + 6 - j, 2(a + b + c + 1) - j\}, \min\{2a + 4 - j, 2(a + b + c + 1) - j\}\right)$$

$$r(v_{1,1,j,0}, W_1) = \left(\min\{2a + 4 - j, 2(a + b + c + 1) - j\}, j, \min\{2a + 4 - j, 2(a + b + c + 1) - j\}, \min\{2a + 6 - j, 2(a + b + c + 1) - j\}, \dots, \min\left\{2\left(a + 1 + \left\lfloor \frac{n}{2} \right\rfloor\right) - j, 2(a + b + c + 1) - j\right\}, \dots, \min\{2a + 6 - j, 2(a + b + c + 1) - j\}\right)$$

⋮

$$r(v_{1,1,j,0}, W_1) = \left(\min\{2a + 4 - j, 2(a + b + c + 1) - j\}, \min\{2a + 6 - j, 2(a + b + c + 1) - j\}, \dots, \min\left\{2\left(a + 1 + \left\lfloor \frac{n}{2} \right\rfloor\right) - j, 2(a + b + c + 1) - j\right\}, \dots, \min\{2a + 6 - j, 2(a + b + c + 1) - j\}, \min\{2a + 4 - j, 2(a + b + c + 1) - j\}, j\right)$$

Since $r(v_{r,1,j,0}, W_1) \neq r(v_{s,1,j,0}, W_1)$ and $W_1 \subseteq W$, then $r(v_{r,1,j,0}, W) \neq r(v_{s,1,j,0}, W)$, for all $r, s \in [1, n]$.

4. Representation $v_{i,1,j,k}, \forall i \in [1, n], j \in [1, a]$, and $k \in [1, 2]$ with respect to $W_2 \subseteq W$ as follow

$$r(v_{1,1,j,1}|W_2) = \left(j + 1, 0, \min\{2a - j + 5, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 6, 2(a + b + c - j) + 2\}, \min\{2a - j + 7, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 8, 2(a + b + c - j) + \right.$$

$$\begin{aligned}
& 2\}, \dots, \min\left\{2a - j + 3 + 2\left\lfloor\frac{n}{2}\right\rfloor, 2(a + b + c) - 3 - j\right\}, \min\{2a - 2j + 4 + \\
& 2\left\lfloor\frac{n}{2}\right\rfloor, 2(a + b + c - j) + 2\}, \dots, \min\{2a - j + 7, 2(a + b + c) - 3 - j\}, \min\{2a - \\
& 2j + 8, 2(a + b + c - j) + 2\}, \min\{2a - j + 5, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + \\
& 6, 2(a + b + c - j) + 2\}) \\
r(v_{2,1,j,1}|W_2) &= \left(\min\{2a - j + 5, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 6, 2(a + b + c - j) + 2\}, j + \right. \\
& 1, 0, \min\{2a - j + 5, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 6, 2(a + b + c - j) + \\
& 2\}, \min\{2a - j + 7, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 8, 2(a + b + c - j) + \\
& 2\}, \dots, \min\left\{2a - j + 3 + 2\left\lfloor\frac{n}{2}\right\rfloor, 2(a + b + c) - 3 - j\right\}, \min\{2a - 2j + 4 + \\
& 2\left\lfloor\frac{n}{2}\right\rfloor, 2(a + b + c - j) + 2\}, \dots, \min\{2a - j + 7, 2(a + b + c) - 3 - j\}, \min\{2a - \\
& 2j + 8, 2(a + b + c - j) + 2\}) \\
& \vdots \\
r(v_{n,1,j,1}|W_2) &= \left(\min\{2a - j + 5, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 6, 2(a + b + c - j) + \right. \\
& 2\}, \min\{2a - j + 7, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 8, 2(a + b + c - j) + \\
& 2\}, \dots, \min\left\{2a - j + 3 + 2\left\lfloor\frac{n}{2}\right\rfloor, 2(a + b + c) - 3 - j\right\}, \min\{2a - 2j + 4 + \\
& 2\left\lfloor\frac{n}{2}\right\rfloor, 2(a + b + c - j) + 2\}, \dots, \min\{2a - j + 7, 2(a + b + c) - 3 - j\}, \min\{2a - \\
& 2j + 8, 2(a + b + c - j) + 2\}, \min\{2a - j + 5, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + \\
& 6, 2(a + b + c - j) + 2\}, j + 1, 0) \\
r(v_{1,1,j,2}|W_2) &= \left(j + 1, 2, \min\{2a - j + 5, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 6, 2(a + b + c - j) + \right. \\
& 2\}, \min\{2a - j + 7, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 8, 2(a + b + c - j) + \\
& 2\}, \dots, \min\left\{2a - j + 3 + 2\left\lfloor\frac{n}{2}\right\rfloor, 2(a + b + c) - 3 - j\right\}, \min\{2a - 2j + 4 + \\
& 2\left\lfloor\frac{n}{2}\right\rfloor, 2(a + b + c - j) + 2\}, \dots, \min\{2a - j + 7, 2(a + b + c) - 3 - j\}, \min\{2a - \\
& 2j + 8, 2(a + b + c - j) + 2\}, \min\{2a - j + 5, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + \\
& 6, 2(a + b + c - j) + 2\}) \\
r(v_{2,1,j,2}|W_2) &= \left(\min\{2a - j + 5, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 6, 2(a + b + c - j) + 2\}, j + \right. \\
& 1, 2, \min\{2a - j + 5, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 6, 2(a + b + c - j) + \\
& 2\}, \min\{2a - j + 7, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 8, 2(a + b + c - j) + \\
& 2\}, \dots, \min\left\{2a - j + 3 + 2\left\lfloor\frac{n}{2}\right\rfloor, 2(a + b + c) - 3 - j\right\}, \min\{2a - 2j + 4 + \\
& 2\left\lfloor\frac{n}{2}\right\rfloor, 2(a + b + c - j) + 2\}, \dots, \min\{2a - j + 7, 2(a + b + c) - 3 - j\}, \min\{2a - \\
& 2j + 8, 2(a + b + c - j) + 2\}) \\
& \vdots \\
r(v_{n,1,j,2}|W_2) &= \left(\min\{2a - j + 5, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 6, 2(a + b + c - j) + \right. \\
& 2\}, \min\{2a - j + 7, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 8, 2(a + b + c - j) +
\end{aligned}$$

$$2\}, \dots, \min\left\{2a - j + 3 + 2\left\lfloor\frac{n}{2}\right\rfloor, 2(a + b + c) - 3 - j\right\}, \min\left\{2a - 2j + 4 + 2\left\lfloor\frac{n}{2}\right\rfloor, 2(a + b + c - j) + 2\right\}, \dots, \min\{2a - j + 7, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 8, 2(a + b + c - j) + 2\}, \min\{2a - j + 5, 2(a + b + c) - 3 - j\}, \min\{2a - 2j + 6, 2(a + b + c - j) + 2\}, j + 1, 2\}$$

Since $r(v_{r,1,j,k}, W_2) \neq r(v_{s,1,j,k}, W_2)$ and $W_2 \subseteq W$, then $r(v_{r,1,j,k}, W) \neq r(v_{s,1,j,k}, W)$, for all $r, s \in [1, n]$.

5. Representation $v_{i,3,j,0}, \forall i \in [1, n]$ for $j \in [1, c]$ with respect to $W_1 \subseteq W$ are

$$r(v_{1,3,j,0}, W_1) = \left(a + b + j, \min\{a + b + j + 2, a + b + 2c - j + 2\}, \min\{a + b + j + 4, a + b + 2c - j + 2\}, \dots, \min\left\{a + b + j + 2\left\lfloor\frac{n}{2}\right\rfloor, a + b + 2c - j + 2\right\}, \dots, \min\{a + b + j + 4, a + b + 2c - j + 2\}, \min\{a + b + j + 2, a + b + 2c - j + 2\}\right)$$

$$r(v_{2,3,j,0}, W_1) = \left(\min\{a + b + j + 2, a + b + 2c - j + 2\}, a + b + j, \min\{a + b + j + 2, a + b + 2c - j + 2\}, \min\{a + b + j + 4, a + b + 2c - j + 2\}, \dots, \min\left\{a + b + j + 2\left\lfloor\frac{n}{2}\right\rfloor, a + b + 2c - j + 2\right\}, \dots, \min\{a + b + j + 4, a + b + 2c - j + 2\}\right)$$

⋮

$$r(v_{n,3,j,0}, W_1) = \left(\min\{a + b + j + 2, a + b + 2c - j + 2\}, \min\{a + b + j + 4, a + b + 2c - j + 2\}, \dots, \min\left\{a + b + j + 2\left\lfloor\frac{n}{2}\right\rfloor, a + b + 2c - j + 2\right\}, \dots, \min\{a + b + j + 4, a + b + 2c - j + 2\}, \min\{a + b + j + 2, a + b + 2c - j + 2\}, a + b + j\right)$$

Since $r(v_{r,3,j,0}, W_1) \neq r(v_{s,3,j,0}, W_1)$ and $W_1 \subseteq W$, then $r(v_{r,3,j,0}, W) \neq r(v_{s,3,j,0}, W)$, for all $r, s \in [1, n]$.

6. Representation $v_{i,3,j,k}, \forall i \in [1, n], j \in [1, c]$, and $k \in [1, 2]$ with respect to $W_3 \subseteq W$ are

$$r(v_{1,3,j,1}|W_3) = \left(a + b + j + 1, 0, \min\{a + b + j + 3, a + b + 2c - j + 3\}, \min\{2j + 4, 2(c - j + 1)\}, \min\{a + b + j + 5, a + b + 2c - j + 3\}, \min\{2j + 6, 2(c - j + 1)\}, \dots, \min\left\{a + b + j + 1 + 2\left\lfloor\frac{n}{2}\right\rfloor, a + b + 2c - j + 3\right\}, \min\left\{2j + 4 + 2\left\lfloor\frac{n}{2}\right\rfloor, 2(c - j + 1)\right\}, \dots, \min\{a + b + j + 5, a + b + 2c - j + 3\}, \min\{2j + 6, 2(c - j + 1)\}, \min\{a + b + j + 3, a + b + 2c - j + 3\}, \min\{2j + 4, 2(c - j + 1)\}\right)$$

$$r(v_{2,3,j,1}|W_3) = \left(\min\{a + b + j + 3, a + b + 2c - j + 3\}, \min\{2j + 4, 2(c - j + 1)\}, a + b + j + 1, 0, \min\{a + b + j + 3, a + b + 2c - j + 3\}, \min\{2j + 4, 2(c - j + 1)\}, \min\{a + b + j + 5, a + b + 2c - j + 3\}, \min\{2j + 6, 2(c - j + 1)\}, \dots, \min\left\{a + b + j + 1 + 2\left\lfloor\frac{n}{2}\right\rfloor, a + b + 2c - j + 3\right\}, \min\left\{2j + 4 + 2\left\lfloor\frac{n}{2}\right\rfloor, 2(c - j + 1)\right\}, \dots, \min\{a + b + j + 5, a + b + 2c - j + 3\}, \min\{2j + 6, 2(c - j + 1)\}\right)$$

⋮

$$r(v_{n,3,j,1}|W_3) = (\min\{a + b + j + 3, a + b + 2c - j + 3\}, \min\{2j + 4, 2(c - j + 1)\}, \min\{a + b + j + 5, a + b + 2c - j + 3\}, \min\{2j + 6, 2(c - j + 1)\}, \dots, \min\{a + b + j + 1 + 2\lfloor \frac{n}{2} \rfloor, a + b + 2c - j + 3\}, \min\{2j + 4 + 2\lfloor \frac{n}{2} \rfloor, 2(c - j + 1)\}, \dots, \min\{a + b + j + 5, a + b + 2c - j + 3\}, \min\{2j + 6, 2(c - j + 1)\}, \min\{a + b + j + 3, a + b + 2c - j + 3\}, \min\{2j + 4, 2(c - j + 1)\}, a + b + j + 1, 0)$$

$$r(v_{1,3,j,2}|W_3) = (a + b + j + 1, 2, \min\{a + b + j + 3, a + b + 2c - j + 3\}, \min\{2j + 4, 2(c - j + 1)\}, \min\{a + b + j + 5, a + b + 2c - j + 3\}, \min\{2j + 6, 2(c - j + 1)\}, \dots, \min\{a + b + j + 1 + 2\lfloor \frac{n}{2} \rfloor, a + b + 2c - j + 3\}, \min\{2j + 4 + 2\lfloor \frac{n}{2} \rfloor, 2(c - j + 1)\}, \dots, \min\{a + b + j + 5, a + b + 2c - j + 3\}, \min\{2j + 6, 2(c - j + 1)\}, \min\{a + b + j + 3, a + b + 2c - j + 3\}, \min\{2j + 4, 2(c - j + 1)\})$$

$$r(v_{(2,3,j,2)}|W_3) = (\min\{a + b + j + 3, a + b + 2c - j + 3\}, \min\{2j + 4, 2(c - j + 1)\}, a + b + j + 1, 2, \min\{a + b + j + 3, a + b + 2c - j + 3\}, \min\{2j + 4, 2(c - j + 1)\}, \min\{a + b + j + 5, a + b + 2c - j + 3\}, \min\{2j + 6, 2(c - j + 1)\}, \dots, \min\{a + b + j + 1 + 2\lfloor n/2 \rfloor, a + b + 2c - j + 3\}, \min\{2j + 4 + 2\lfloor n/2 \rfloor, 2(c - j + 1)\}, \dots, \min\{a + b + j + 5, a + b + 2c - j + 3\}, \min\{2j + 6, 2(c - j + 1)\})$$

⋮

$$r(v_{n,3,j,2}|W_3) = (\min\{a + b + j + 3, a + b + 2c - j + 3\}, \min\{2j + 4, 2(c - j + 1)\}, \min\{a + b + j + 5, a + b + 2c - j + 3\}, \min\{2j + 6, 2(c - j + 1)\}, \dots, \min\{a + b + j + 1 + 2\lfloor \frac{n}{2} \rfloor, a + b + 2c - j + 3\}, \min\{2j + 4 + 2\lfloor \frac{n}{2} \rfloor, 2(c - j + 1)\}, \dots, \min\{a + b + j + 5, a + b + 2c - j + 3\}, \min\{2j + 6, 2(c - j + 1)\}, \min\{a + b + j + 3, a + b + 2c - j + 3\}, \min\{2j + 4, 2(c - j + 1)\}, a + b + j + 1, 2)$$

Since $r(v_{r,3,j,k}, W_3) \neq r(v_{s,3,j,k}, W_3)$ and $W_3 \subseteq W$, then $r(v_{r,3,j,k}, W) \neq r(v_{s,3,j,k}, W)$, for all $r, s \in [1, n]$.

7. Representation $v_{i,2,j}, \forall i \in [1, n]$ and $j \in [1, b]$ with respect to $W_1 \subseteq W$ are

$$r(v_{1,2,j}, W_1) = (j + a, \min\{a + j + 2, a + 2b + 2c + 2 - j\}, \min\{a + j + 4, a + 2b + 2c + 2 - j\}, \dots, \min\{a + j + 2\lfloor \frac{n}{2} \rfloor, a + 2b + 2c + 2 - j\}, \dots, \min\{a + j + 4, a + 2b + 2c + 2 - j\}, \min\{a + j + 2, a + 2b + 2c + 2 - j\})$$

$$r(v_{2,2,j}, W_1) = (\min\{a + j + 2, a + 2b + 2c + 2 - j\}, j + a, \min\{a + j + 2, a + 2b + 2c + 2 - j\}, \min\{a + j + 4, a + 2b + 2c + 2 - j\}, \dots, \min\{a + j + 2\lfloor \frac{n}{2} \rfloor, a + 2b + 2c + 2 - j\}, \dots, \min\{a + j + 4, a + 2b + 2c + 2 - j\})$$

⋮

$$r(v_{n,2,j}, W_1) = \left(\min\{a + j + 2, a + 2b + 2c + 2 - j\}, \min\{a + j + 4, a + 2b + 2c + 2 - j\}, \dots, \min\left\{a + j + 2 \left\lfloor \frac{n}{2} \right\rfloor, a + 2b + 2c + 2 - j\right\}, \dots, \min\{a + j + 4, a + 2b + 2c + 2 - j\}, \min\{a + j + 2, a + 2b + 2c + 2 - j\}, j + a \right)$$

Since $r(v_{r,2,j}, W_1) \neq r(v_{s,2,j}, W_1)$ and $W_1 \subseteq W$, then $r(v_{r,2,j}, W) \neq r(v_{s,2,j}, W)$, for all $r, s \in [1, n]$.

8. Representation $v_{i,j}, \forall i \in [1, n]$ and $j \in [1, b]$ with respect to $W_1 \subseteq W$ are

$$r(v_{1,j}, W_1) = (a + j + 1, \min\{a + j + 3, a + 2b + 2c + 3 - j\}, \min\{a + j + 5, a + 2b + 2c + 3 - j\}, \dots, \min\left\{a + j + 1 + 2 \left\lfloor \frac{n}{2} \right\rfloor, a + 2b + 2c + 3 - j\right\}, \dots, \min\{a + j + 5, a + 2b + 2c + 3 - j\}, \min\{a + j + 3, a + 2b + 2c + 3 - j\})$$

$$r(v_{2,j}, W_1) = (\min\{a + j + 3, a + 2b + 2c + 3 - j\}, a + j + 1, \min\{a + j + 3, a + 2b + 2c + 3 - j\}, \min\{a + j + 5, a + 2b + 2c + 3 - j\}, \dots, \min\left\{a + j + 1 + 2 \left\lfloor \frac{n}{2} \right\rfloor, a + 2b + 2c + 3 - j\right\}, \dots, \min\{a + j + 5, a + 2b + 2c + 3 - j\})$$

⋮

$$r(v_{n,j}, W_1) = (\min\{a + j + 3, a + 2b + 2c + 3 - j\}, \min\{a + j + 5, a + 2b + 2c + 3 - j\}, \dots, \min\left\{a + j + 1 + 2 \left\lfloor \frac{n}{2} \right\rfloor, a + 2b + 2c + 3 - j\right\}, \dots, \min\{a + j + 5, a + 2b + 2c + 3 - j\}, \min\{a + j + 3, a + 2b + 2c + 3 - j, a + j + 1\})$$

since $r(v_{r,j}, W_1) \neq r(v_{s,j}, W_1)$ and $W_1 \subseteq W$, then $r(v_{r,j}, W) \neq r(v_{s,j}, W)$, for all $r, s \in [1, n]$.

Since $W_1 \subseteq W_2 \subseteq W$ and $W_1 \subseteq W_3 \subseteq W$, then the representation of each vertex with respect to W_1 , W_2 , and W_3 will correspond to representation with respect to W . Since $r(v_a|W) \neq r(v_b|W)$ implies that $v_a = v_b$, for all $v_a, v_b \in V(\text{Snow}(n, a, b, c))$, then W is resolving set for Snow . Therefore, $\dim(\text{Snow}(n, a, b, c)) \leq n(a + c + 1)$.

Lower Bound :

Second, we will show that $\dim(\text{Snow}(n, a, b, c)) \geq n(a + c + 1)$. Choose the set edges of $A = \{v_{i,1,1,0} \in V(\text{Snow}(n, a, b, c)) | 1 \leq i \leq n\}$ and $B = \{v_{i,1,j,0}, v_{i,3,k,0} \in V(\text{Snow}(n, a, b, c)) | 1 \leq i \leq 0; 2 \leq j \leq a; 1 \leq k \leq c\}$. The sets A and B are, respectively, the set of edges connected by 3 and 2 edges of degree one, where $|A| = n$ and $|B| = n(a + c)$. By Lemma 3.2, $\dim(\text{Snow}(n, a, b, c)) \geq \sum_{i=1}^{|A|} (3 - 1) + \sum_{i=1}^{|B|} (2 - 1) = n(2) + n(a - 1 + c) = n(a + c + 1)$, so $\dim(\text{Snow}(n, a, b, c)) \geq n(a + c + 1)$.

Putting the bounds together gives us that $\dim(\text{Snow}(n, a, b, c)) = n(a + c + 1)$.

4. Conclusion

In this paper, we have defined a snowflake graph in Definition 3.3 and generalized snowflake graph in Definition 3.5. Supported with a lemma about the lower bound of the metric dimension of a such graph, we find $\dim(\text{Snow}) = 24$ proved in Theorem 3.4 and $\dim(\text{Snow}(n, a, b, c)) = n(a + c + 1)$ proved in Theorem 3.6. However, it is interesting to find other measures of these graphs such as edge metric dimension, total metric dimension, partition dimension, and chromatic location number. In addition, it is worth considering that there are many other graphs that could be formed based on various snowflakes shapes other than the shape we define in this paper.

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