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Quasi-Banach modulation spaces and localization operators on locally compact abelian groups

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Abstract

We introduce new quasi-Banach modulation spaces on locally compact abelian groups which coincide with the classical ones in the Banach setting and prove their main properties. Then, we study Gabor frames on quasi-lattices, significantly extending the original theory introduced by Gröchenig and Strohmer. These issues are the key tools in showing boundedness results for Kohn–Nirenberg and localization operators on modulation spaces and studying their eigenfunctions' properties. In particular, the results in the Euclidean space are recaptured.

Keywords Time–frequency analysis · Locally compact abelian groups · Localization operators · Short-time Fourier transform · Quasi-Banach spaces · Modulation spaces · Wiener amalgam spaces

Mathematics Subject Classification 42B35 · 46E35

1 Introduction

In the last decades, time–frequency analysis and pseudo-differential calculus on locally compact groups acquired increasing interest for both theoretical and practical reasons. The popularity gained in signal processing by time–frequency representations (see, e.g., [9] and references therein and [58]), led to the need for discrete versions of the techniques available on $\mathbb{R}^d$. Many works have been done on $\mathbb{Z}^d$, finite abelian and

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elementary groups [2, 10, 27, 46], and for the *p*-adic groups $\mathbb{Q}_p$ [16, 37, 42]. Since the group laws of the *p*-adic numbers resemble the computer arithmetic, the $\mathbb{Q}_p$ groups appear to be the natural settings for problems in computer science. On the other hand, the *p*-adic groups and the pseudo-differential calculus on them are essential for *p*-adic quantum theory [38, 53, 64]. More generally, group theory has caught the attention of many authors in the last 30 years, as it is witnessed by the huge production on the topic, see for example [1, 22, 24–26, 32, 34, 45, 48, 54–56, 61, 62, 66].

In this work, we shall focus on topological, locally compact abelian (LCA) groups $\mathcal{G}$. The very first motivation that led to this manuscript was the study of eigenfunctions' properties for localization operators on LCA groups, in the spirit of the results inferred in the Euclidean case [3]. Despite the many contributions on pseudo-differential operators acting on groups (cf., e.g., [56, 58, 66]), we believe this is the first work in this direction.

The function spaces used for both eigenfunctions and symbols are modulation spaces. For measuring the eigenfunctions' decay, it becomes necessary to extend the Banach cases of modulation spaces $M_m^{p,q}(\mathcal{G}) \, 1 \leq p, q \leq \infty$, originally defined by Feichtinger in his pioneering work [22], to the quasi-Banach setting. This is the first contribution of this paper. Although there is a well-established theory for $M_m^{p,q}(\mathbb{R}^d)$ including the quasi-Banach cases $0 < p, q \leq \infty$ [30], if we abandon the Euclidean setting for a general group $\mathcal{G}$ troubles arise. In fact, the Banach modulation spaces on groups $\mathcal{G}$ introduced in [22] cannot be adapted to the quasi-Banach case. We overcome this difficulty by getting inspiration from the idea of Feichtinger and Gröchenig in [24], and view modulation spaces on $\mathcal{G}$ as particular coorbit spaces over the Heisenberg group $\mathcal{G} \times \widehat{\mathcal{G}} \times \mathbb{T}$ (cf. Definition 3.1 below). Again, the coorbit theory proposed by Feichtinger and Gröchenig in their works [24–26] is not suitable for the quasi-Banach case. The right construction is provided by the new coorbit theory started by Rauhut in [49] and developed by Voigtlaender in his Ph.D. thesis [65], see also [63].

For a version of coorbit theory that does not need group representations, but only a continuous frame to start with, we refer to [29, 51].

Thanks to this new theory (see a brief summary in the Appendix below), we are able to give a definition of modulation spaces on LCA groups which recaptures Feichtinger's original one in [22] and deals with the quasi-Banach case. To explain the new modulation spaces, we first need to introduce the main notations.

We write $\widehat{\mathcal{G}}$ for the dual group of $\mathcal{G}$. Latin letters such as x, y and u denote elements in $\mathcal{G}$, whereas all the characters in $\widehat{\mathcal{G}}$, except the identity $\hat{e}$, are indicated by Greek letters like ξ, ω and η. For the evaluation of a character $\xi \in \widehat{\mathcal{G}}$ at a point $x \in \mathcal{G}$ we write

$$\langle \xi, x \rangle := \xi(x).$$

For $x \in \mathcal{G}, \xi \in \widehat{\mathcal{G}}$ and a function $f : \mathcal{G} \to \mathbb{C}$ we define the translation operator T_x, the modulation operator M_ξ and the time-frequency shift $\pi(x, \xi)$ as

$$T_x f(y) = f(y - x), \quad M_\xi f(y) = \langle \xi, y \rangle f(y), \quad \pi(x, \xi) = M_\xi T_x, \tag{1}$$

T_x and M_ξ fulfil the so-called commutation relations

$$M_\xi T_x = \langle \xi, x \rangle T_x M_\xi. \tag{2}$$

For $f, g \in L^2(\mathcal{G})$, the short-time Fourier transform (STFT) of f with respect to g is given by

$$V_g f(x, \xi) = \langle f, \pi(x, \xi)g \rangle = \int_{\mathcal{G}} f(y)\overline{\pi(x, \xi)g(y)}\, dy, \quad (x, \xi) \in \mathcal{G} \times \widehat{\mathcal{G}}. \tag{3}$$

To define modulation spaces $M_m^{p,q}(\mathcal{G})$, instead of considering the mixed Lebesgue space $L_m^{p,q}$ to measure the (quasi-)norm of the STFT as in [22], that is

$$\|f\|_{M_m^{p,q}} = \left\|V_g f\right\|_{L_m^{p,q}}, \tag{4}$$

we use the Wiener space $W(L^\infty, L_m^{p,q})$-norm (see Definition A.4 and subsequent comments):

$$\|f\|_{M_m^{p,q}} = \left\|V_g f\right\|_{W(L^\infty, L_m^{p,q})}. \tag{5}$$

We develop a new general theory which coincides with the classical one when

  (i) $p, q \geq 1$ and $\mathcal{G}$ is any LCA group;
  (ii) $0 < p, q \leq \infty$ and $\mathcal{G} = \mathbb{R}^d$.

Frame expansions and new convolution relations for $M_m^{p,q}(\mathcal{G})$ are obtained as well, see Theorem 4.17 and Proposition 4.18 below.

Galperin and Samarah proved in [30, Lemma 3.2] that for any $0 < p, q \leq \infty$ there exists constant $C > 0$ such that

$$\left\|V_g f\right\|_{W(L^\infty, L_m^{p,q})} \leq C \left\|V_g f\right\|_{L_m^{p,q}}, \quad \forall f \in M_m^{p,q}(\mathbb{R}^d), \tag{6}$$

g being the Gaussian. It is of course a natural question whether there are cases for which the quasi-norm (5) is equivalent to the more "natural" (4). To answer this question, one has to verify (6) for some suitable window function g. The techniques adopted in [30] to prove the above inequality rely on properties of entire functions on $\mathbb{C}^d$, which cannot be adopted for a general LCA group $\mathcal{G}$. Whether the inequality in (6) holds true whenever we replace $\mathbb{R}^d$ by any LCA group $\mathcal{G}$ is still an open problem and can be seen as a manifestation of a wider issue concerning coorbit theories, see Rauhut's observations in [49, Section 6].

In this work, we are able to give a positive answer when $\mathcal{G}$ is a discrete or compact group, see the subsequent Lemma 3.38.

Next, we focus on localization operators and their eigenfunctions.

The localization operator $A_a^{\psi_1, \psi_2}$ with symbol a and windows ψ_1, ψ_2 can be formally defined by

$$A_a^{\psi_1, \psi_2} f(x) = \int_{\mathcal{G} \times \widehat{\mathcal{G}}} a(u, \omega) V_{\psi_1} f(u, \omega) M_\omega T_u \psi_2(x)\, du d\omega.$$

In particular, if $a \in L^\infty(\mathcal{G} \times \widehat{\mathcal{G}})$ and the windows ψ_1, ψ_2 are in $L^2(\mathcal{G})$ then $A_a^{\psi_1,\psi_2}$ is bounded on $L^2(\mathcal{G})$, cf. [66].

For a linear bounded operator T on $L^2(\mathcal{G})$ we denote by $\sigma(T)$ the spectrum of T, that is the set $\{\lambda \in \mathbb{C} \mid T - \lambda I_{L^2}$ is not invertible$\}$; in particular, the set $\sigma_P(T)$ denotes the point spectrum of T, that is

$$\sigma_P(T) = \{\lambda \in \mathbb{C} \mid \exists f \in L^2(\mathcal{G}) \smallsetminus \{0\} \text{ such that } Tf = \lambda f\},$$

such an f is called eigenfunction of T associated to the eigenvalue λ.

Our main result in this framework can be formulated as follows:

If the symbol a belongs to the modulation space $M^{p,\infty}(\mathcal{G} \times \widehat{\mathcal{G}})$ for some $0 < p < \infty$, then any eigenfunction $f \in L^2(\mathcal{G})$ of the localization operator $A_a^{\psi_1,\psi_2}$ satisfies

$$f \in \bigcap_{\gamma > 0} M^\gamma(\mathcal{G}).$$

In particular, when $\mathcal{G} = \mathbb{Z}^d$, this means that any eigenfunction $f \in \ell^2(\mathbb{Z}^d)$ satisfies $f \in \bigcap_{\gamma > 0} \ell^\gamma(\mathbb{Z}^d)$, so the sequence f displays a fast decay at infinity.

The study of eigenfunctions of $A_a^{\psi_1,\psi_2}$ is pursued using the connection between localization and Kohn–Nirenberg operators $\mathrm{Op}_0(\sigma)$.

Let us first introduce the Rihaczek distribution. Given $f, g \in L^2(\mathcal{G})$, we define the (cross-)Rihaczek distribution of f and g by

$$R(f,g)(x,\xi) = f(x)\overline{\hat{g}(\xi)}\langle \xi, x \rangle, \quad (x,\xi) \in \mathcal{G} \times \widehat{\mathcal{G}}, \tag{7}$$

$\hat{g}$ being the Fourier transform of g (11). When $f = g$, $R(f,f)$ is called the Rihaczek distribution of f.

Then the pseudo-differential operator $\mathrm{Op}_0(\sigma)$ with Kohn–Nirenberg symbol σ is formally defined by

$$(\mathrm{Op}_0(\sigma)f)(x) = \int_{\widehat{\mathcal{G}}} \sigma(x,\xi)\hat{f}(\xi)\langle \xi, x \rangle \, d\xi. \tag{8}$$

Equivalently, we can define it weakly by

$$\langle \mathrm{Op}_0(\sigma)f, g \rangle = \langle \sigma, R(g,f) \rangle. \tag{9}$$

If $\sigma \in M^\infty(\mathcal{G} \times \widehat{\mathcal{G}})$ then

$$\mathrm{Op}_0(\sigma) : M^1(\mathcal{G}) \to M^\infty(\mathcal{G}) \tag{10}$$

is well defined, linear and continuous, see, e.g., [41, Corollary 4.2, Theorem 5.3]. A localization operator $A_a^{\psi_1,\psi_2}$ can be written in the Kohn–Nirenberg form:

$$A_a^{\psi_1,\psi_2} = \mathrm{Op}_0(a * R(\psi_2, \psi_1)),$$

that is, $A_a^{\psi_1,\psi_2}$ is a Kohn–Nirenberg operator with symbol

$$\sigma = a * R(\psi_2, \psi_1),$$

the convolution between the localization symbol a and the cross-Rihaczek distribution $R(\psi_2, \psi_1)$ of its windows ψ_2, ψ_1. It becomes then natural to study the properties of Kohn–Nirenberg pseudo-differential operators and convolution relations for modulation spaces on LCA groups.

We obtain new boundedness results for such operators in modulation spaces and describe the decay of their eigenfunctions in $L^2(\mathcal{G})$, see Theorem 4.21 and Proposition 4.22 below. The convolution properties are contained in Proposition 4.18.

We point out that Theorem 4.21 is not an easy generalization of the Euclidean case. It requires frame theory on quasi-lattices and proofs with high level of technicalities, cf. Sect. 4.2 below. Quasi-lattices were used by Gröchenig and Strohmer in [37] since not every $\mathcal{G}$ admits a lattice, e.g. the p-adic groups $\mathbb{Q}_p$.

They are the key issue in showing the boundedness properties for Kohn–Nirenberg operators in the subsequent Theorem 4.21, and we believe that these new techniques for Gabor frames on quasi-lattices can be valuable in and of themselves and applied in other contexts. Loosely speaking, the main insight (suggested in [37]) is "to consider the quotient group", cf. Sect. 3.2 for details.

The paper is organized as follows. In Sect. 1, we establish technical assumptions and notations. Section 2 is devoted to the new general theory for modulation spaces $M_m^{p,q}(\mathcal{G})$ with $0 < p, q \le \infty$. In Sect. 3, we study continuity properties on modulations spaces for the Rihaczek distribution and pseudo-differential operators with Kohn–Nirenberg symbols. Gabor frame over quasi-lattices, analysis and synthesis operators, convolution relations are investigated as well. Section 4 deals with localization operators and their eigenfunctions. In the Appendix we resume the coorbit theory presented in the thesis of Voigtlaender [65] and compare it with the one of Feichtinger and Gröchenig. We strongly recommend the reader who is not familiar with coorbit theory to read the Appendix, for it is heavily used in Sect. 2 and subsequent sections.

2 Preliminaries

We mainly follow the notations and assumptions of Gröchenig and Strohmer [37].

2.1 Notations

$\mathcal{G}$ denotes a LCA group with the Hausdorff property. $\widehat{\mathcal{G}}$ is the dual group of $\mathcal{G}$. The group operation on $\mathcal{G}$, and on any abelian group such as $\mathcal{G} \times \widehat{\mathcal{G}}$, is written additively. The unit in $\mathcal{G}$ and $\widehat{\mathcal{G}}$ are denoted by e and $\hat{e}$, respectively.

$\mathcal{G}$ is assumed second countable, which is equivalent to $L^2(\mathcal{G})$ separable (see [15, Theorem 2]) and implies the metrizability of the group [47, Pag. 34]. In order to avoid uncountable sets of indexes and sums we require $\mathcal{G}$ to be σ-compact; this last property is equivalent to σ-finiteness [28, Proposition 2.22], as observed in [65, Remark 2.3.2]. Note that, due to [52, Theorem 4.2.7] and Pontrjagin's duality, $\mathcal{G}$ is second countable and σ-compact if and only if $\widehat{\mathcal{G}}$ is second countable and σ-compact.

In the sequel, $A \lesssim B$ means that there exists a constant $c > 0$ independent of A and B such that $A \leq cB$; we write $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$. If $f : X \to \mathbb{C}, x \mapsto f(x)$ and $g : Z \to \mathbb{C}, z \mapsto g(z)$, then we define the tensor product of f g as $f \otimes g : X \times Z \to \mathbb{C}, (x, z) \mapsto f(x)g(z)$. We denote by $X \hookrightarrow Z$ the continuous injection of X into Z.

2.2 Fundamental operators, test functions, Rihaczek distribution

We adopt the space of special test functions $\mathcal{S}_{\mathcal{C}}(\mathcal{G})$ introduced in [37] and defined below. The definition is based on the structure theorem $\mathcal{G} \cong \mathbb{R}^d \times \mathcal{G}_0$ [39, Theorem 24.30], where $d \in \mathbb{N}_0$ and $\mathcal{G}_0$ is a LCA group containing a compact open subgroup $\mathcal{K}$. Consequently, we can identify $\widehat{\mathcal{G}}$ with $\mathbb{R}^d \times \widehat{\mathcal{G}}_0$, where the dual group $\widehat{\mathcal{G}}_0$ contains the compact open subgroup $\mathcal{K}^\perp$, see e.g. [32, Lemma 6.2.3]. We endow $\mathcal{G}$ and $\widehat{\mathcal{G}}$ with the Haar measures dx and $d\xi$, respectively, where $d\xi$ is the dual Haar measure. The Fourier transform is

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathcal{G}} f(x)\overline{\langle \xi, x \rangle}\, dx, \quad \xi \in \widehat{\mathcal{G}}. \tag{11}$$

$\mathcal{F}$ is an isometry from $L^2(\mathcal{G})$ onto $L^2(\widehat{\mathcal{G}})$.

On account of the structure theorem above, we define the generalized Gaussian on $\mathcal{G}$ as

$$\varphi(x_1, x_2) := e^{-\pi x_1^2} \chi_{\mathcal{K}}(x_2) =: \varphi_1(x_1)\varphi_2(x_2), \quad (x_1, x_2) \in \mathbb{R}^d \times \mathcal{G}_0, \tag{12}$$

and the set of special test functions

$$\mathcal{S}_{\mathcal{C}}(\mathcal{G}) := \mathrm{span}\Big\{ \pi(\mathbf{x})\varphi, \quad \mathbf{x} = (x, \xi) \in \mathcal{G} \times \widehat{\mathcal{G}} \Big\} \subseteq L^2(\mathcal{G}), \tag{13}$$

that is, the set of all time–frequency shifts of the Gaussian $\varphi = \varphi_1 \otimes \varphi_2$ in (12). For the main properties of this space we refer to [37, Section 2].

For $x = (x_1, x_2) \in \mathbb{R}^d \times \mathcal{G}_0$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^d \times \widehat{\mathcal{G}}_0$, the Rihaczek distribution of $\varphi = \varphi_1 \otimes \varphi_2$ in (12) is given by

$$\begin{aligned}
R(\varphi, \varphi)(x, \xi) &= R(\varphi_1, \varphi_1)(x_1, \xi_1)R(\varphi_2, \varphi_2)(x_2, \xi_2) \\
&= e^{-2\pi i \xi_1 x_1} e^{-\pi(x_1^2 + \xi_1^2)} \chi_{\mathcal{K}}(x_2) c(\mathcal{K}) \chi_{\mathcal{K}^\perp}(\xi_2)\overline{\langle \xi_2, x_2 \rangle} \\
&= c(\mathcal{K}) e^{-2\pi i \xi_1 x_1} e^{-\pi(x_1^2 + \xi_1^2)}\overline{\langle \xi_2, x_2 \rangle} \chi_{\mathcal{K} \times \mathcal{K}^\perp}(x_2, \xi_2) \\
&= c(\mathcal{K})\overline{\langle \xi, x \rangle} e^{-\pi(x_1^2 + \xi_1^2)} \otimes \chi_{\mathcal{K} \times \mathcal{K}^\perp}(x_2, \xi_2),
\end{aligned} \tag{14}$$

where $c(\mathcal{K}) > 0$ is a constant depending on the compact subgroup $\mathcal{K}$. Hence $R(\varphi, \varphi)(x, \xi)$ is up to a positive constant and a "chirp" a Gaussian on $\mathbb{R}^{2d} \times (\mathcal{G}_0 \times \widehat{\mathcal{G}}_0)$, where we fixed $\mathcal{K} \times \mathcal{K}^\perp$ as compact open subgroup of the not Euclidean component. We recall the following covariance property [37, Lemma 4.2 (i)]: for $\mathbf{x} = (x, \xi)$, $\mathbf{y} = (y, \eta) \in \mathcal{G} \times \widehat{\mathcal{G}}, f, g \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$,

$$R(\pi(\mathbf{x})f, \pi(\mathbf{y})g) = \langle \eta, x - y \rangle M_{\mathcal{J}(\mathbf{y}-\mathbf{x})} T_{(x,\eta)} R(f, g), \tag{15}$$

where $\mathcal{J}$ is the topological isomorphism

$$\mathcal{J} : \mathcal{G} \times \widehat{\mathcal{G}} \to \widehat{\mathcal{G}} \times \mathcal{G}, \ (x, \xi) \mapsto (-\xi, x), \tag{16}$$

and $\mathcal{J}^{-1}(\xi, x) = (x, -\xi)$. In what follows we shall need also the following identity:

$$V_\varphi \varphi(x, \xi) = c(\mathcal{K}) e^{-\frac{\pi}{2}(x_1^2 + \xi_1^2)} \otimes \chi_{\mathcal{K} \times \mathcal{K}^\perp}(x_2, \xi_2), \tag{17}$$

see [37] for calculations. Using a similar argument as in the estimate [37, formula (12)], one can show that $R(f, g)$ and $V_g f$ are in $L_m^p(\mathcal{G} \times \widehat{\mathcal{G}})$, $0 < p \le \infty$, for arbitrary moderate weight functions, which will be defined in the Appendix, and any $f, g \in \mathcal{S}_\mathcal{C}(\mathcal{G})$. Similarly, every function in $\mathcal{S}_\mathcal{C}(\mathcal{G})$ belongs to $L_m^p(\mathcal{G}), 0 < p \le \infty$. Recall that for any $f, g \in L^2(\mathcal{G})$ [37, formula (8)]

$$V_{M_\eta T_y g} M_\omega T_u f(x, \xi) = \overline{\langle \xi - \omega, u \rangle} \langle \eta, x - u \rangle T_{(u-y,\omega-\eta)} V_g f(x, \xi). \tag{18}$$

The previous formula, jointly with (15), allows us to write explicitly every STFT and cross-Rihaczek distribution of elements in $\mathcal{S}_\mathcal{C}(\mathcal{G})$.

Lemma 2.1 *Consider $f, g \in \mathcal{S}_\mathcal{C}(\mathcal{G})$, hence*

$$f = \sum_{k=1}^n a_k \pi(\mathbf{u}_k)\varphi, \quad g = \sum_{j=1}^m b_j \pi(\mathbf{y}_j)\varphi,$$

for some $n, m \in \mathbb{N}, a_k, b_j \in \mathbb{C}$ and $\mathbf{u}_k = (u_k, \omega_k), \mathbf{y}_j = (y_j, \eta_j) \in \mathcal{G} \times \widehat{\mathcal{G}}$. Then for every $(x, \xi) \in \mathcal{G} \times \widehat{\mathcal{G}}$:

$$V_g f(x, \xi) = \sum_{k=1}^n \sum_{j=1}^m a_k \overline{b_j} \langle \xi - \omega_k, u_k \rangle \langle \eta_j, x - u_k \rangle T_{\mathbf{u}_k - \mathbf{y}_j} V_\varphi \varphi(x, \xi), \tag{19}$$

$$R(f, g)(x, \xi) = \sum_{k=1}^n \sum_{j=1}^m a_k b_j \langle \eta_j, u_k - y_j \rangle M_{\mathcal{J}(\mathbf{y}_j - \mathbf{u}_k)} T_{(u_k, \eta_j)} R(\varphi, \varphi)(x, \xi). \tag{20}$$

Proof We write $\mathbf{x} = (x, \xi)$. The first claim follows from (18) after the following rephrasing:

$$V_g f(\mathbf{x}) = \langle f, \pi(\mathbf{x})g \rangle = \left\langle \sum_{k=1}^n a_k \pi(\mathbf{u}_k)\varphi, \pi(\mathbf{x}) \sum_{j=1}^m b_j \pi(\mathbf{y}_j)\varphi \right\rangle$$

$$= \sum_{k=1}^n \sum_{j=1}^m a_k \overline{b_j} \left\langle \pi(\mathbf{u}_k)\varphi, \pi(\mathbf{x})\pi(\mathbf{y}_j)\varphi \right\rangle$$

$$= \sum_{k=1}^n \sum_{j=1}^m a_k \overline{b_j} \left(V_{\pi(\mathbf{y}_j)\varphi} \pi(\mathbf{u}_k)\varphi \right)(\mathbf{x}).$$

For the second issue, we write

$$R(f, g)(\mathbf{x}) = \sum_{k=1}^{n} \sum_{j=1}^{m} a_k b_j \pi(\mathbf{u}_k)\varphi(x)\widehat{\pi(\mathbf{y}_j)\varphi}(\xi)\overline{\langle \xi, x \rangle}$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{m} a_k b_j R(\pi(\mathbf{u}_k)\varphi, \pi(\mathbf{y}_j)\varphi)(\mathbf{x})$$

and use (15).   □

3 Modulation spaces over a LCA group

A short survey of coorbit spaces on a locally compact Hausdorff (LCH) group with respect to a solid quasi-Banach function (QBF) space Y (developed by Voigtlaender in his Ph.D. thesis [65]) is contained in the Appendix. In particular, see the Appendix for the following concepts: left L_x and right R_x translations, relatively separated families, a discrete space Y_d associated to Y, BUPUs, maximal functions $\mathsf{M}_Q f$, Wiener amalgam spaces $W_Q(Y) = W_Q(L^\infty, Y)$ and their right-sided version $W_Q^R(Y) = W_Q^R(L^\infty, Y)$. Definition A.6 contains the hypothesis on weights and the class $\mathcal{M}_v$ used in what follows. Note that the coorbit space construction is listed in items **A**–**J** (unitary representation ρ, wavelet transform $W_g^\rho f$, assumptions on weights, sets $\mathbb{G}_v$, $\mathbb{A}_v^r$, $\mathcal{T}_v$, $\mathcal{R}_v$). Each of these items will be revisited in this section under specific choices, see list **A′**–**J′** below. The Appendix reports also some fundamental results of Voigtlaender [65] and a comparison with the earlier coorbit theory by Feichtinger and Gröchenig.

Relying on the theory in Appendix, we are able to give a definition of modulation spaces on LCA groups which covers Feichtinger's original one [22] and deals with the quasi-Banach case. The subsequent construction of $M_m^{p,q}(\mathcal{G})$ was suggested for the Banach case in [24, p. 67], although the coorbit theory applied here is different.

Since the group $\mathbb{H}_\mathcal{G}$ defined below is noncommutative, we adopt the multiplicative notation for its operation.

Definition 3.1 Let $\mathbb{T}$ be the torus with the complex multiplication. We define the Heisenberg-type group associated to $\mathcal{G}$, Heisenberg group for short, as

$$\mathbb{H}_\mathcal{G} := \mathcal{G} \times \widehat{\mathcal{G}} \times \mathbb{T}, \tag{21}$$

endowed with the product topology and the following operation:

$$(x, \xi, \tau)(x', \xi', \tau') = \left(x + x', \xi + \xi', \tau\tau'\langle \xi', x \rangle \right), \tag{22}$$

for $(x, \xi, \tau), (x', \xi', \tau') \in \mathbb{H}_\mathcal{G}$.

The group $\mathbb{H}_\mathcal{G}$ is also called Mackey obstruction group of $\mathcal{G} \times \widehat{\mathcal{G}}$, see [7, Section 4], in particular Example 4.6 therein.

Lemma 3.2 *The topological product space $\mathbb{H}_\mathcal{G}$ with the operation in (22) is a topological LCH, σ-compact, noncommutative, unimodular group with Haar measure the product measure $dx d\xi d\tau$, dx and $d\xi$ being dual Haar measures on $\mathcal{G}$ and $\widehat{\mathcal{G}}$ and $d\tau(\mathbb{T}) = 1$.*

Proof Hausdorff property, local compactness, σ-compactness and noncommutativity are trivial. For the proof that $\mathbb{H}_\mathcal{G}$ is a topological unimodular group, we refer to Theorem 3 in [40], for the bi-invariance of $dx d\xi d\tau$ see [40, p. 12] or, alternatively, [7, Lemma 4.3]. $\qquad\square$

The identity in $\mathbb{H}_\mathcal{G}$ is $(e, \hat{e}, 1)$ and the inverse of an element (x, ξ, τ) is

$$(x, \xi, \tau)^{-1} = (-x, -\xi, \overline{\tau}\langle \xi, x\rangle).$$

Lemma 3.3 *The mapping*

$$\varrho : \mathbb{H}_\mathcal{G} \to \mathcal{U}(L^2(\mathcal{G})), (x, \xi, \tau) \mapsto \tau M_\xi T_x \qquad (23)$$

is a unitary, strongly continuous, irreducible, integrable representation of $\mathbb{H}_\mathcal{G}$ on $L^2(\mathcal{G})$. We call ϱ Schrödinger representation.

Proof Well-posedness of ϱ is trivial, from the commutations relations (2), it is straightforward to see that ϱ is a group homomorphism. Observe that

$$\pi : \mathcal{G} \times \widehat{\mathcal{G}} \to \mathcal{U}(L^2(\mathcal{G})), (x, \xi) \mapsto M_\xi T_x$$

is a projective representation in the terminology of [7, Definition 4.1]. In fact, (i) $\pi(e, \hat{e}) = I_{L^2}$; (ii) from the commutation relations (2), we obtain

$$\pi\big((x, \xi) + (x', \xi')\big) = \overline{\langle \xi', x\rangle} \pi(x, \xi)\pi(x', \xi'),$$

where $\langle \cdot, \cdot \rangle$ is continuous on $\mathcal{G} \times \widehat{\mathcal{G}}$; (iii) the continuity of the STFT guarantees the required measurability. To verify that ϱ is strongly continuous, one can proceed as in the Euclidean case, see e.g. [13]. The result then follows from [7, Lemma 4.4 (ii)].

The fact that ϱ is irreducible was proved in [40], see page 14 before Sect. 5. For the integrability, consider the Gaussian $\varphi \in L^2(\mathcal{G})$ in (12) and observe that the torus is compact and $\left| W_\varphi^o \varphi \right| = \left| V_\varphi \varphi \right|$ (see (187) for the definition of $W_\varphi^o \varphi$). Then from (17) we have $V_\varphi \varphi \in L^1(\mathcal{G} \times \widehat{\mathcal{G}})$ and $W_\varphi^o \varphi \in L^1(\mathbb{H}_\mathcal{G})$. This concludes the proof. $\qquad\square$

Definition 3.4 We define the *extension of* $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ as

$$\tilde{m} : \mathbb{H}_{\mathcal{G}} \to (0, +\infty), (x, \xi, \tau) \mapsto m(x, \xi). \tag{24}$$

For $0 < p, q \leq \infty$, the space $L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}})$ consists of those equivalence classes of measurable complex-valued functions on $\mathbb{H}_{\mathcal{G}}$, where two functions are identified if they coincide a.e., for which the following application is finite

$$\|F\|_{L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}})} := \|F\|_{L_{\tilde{m}}^{p,q}} := \left(\int_{\widehat{\mathcal{G}} \times \mathbb{T}} \left(\int_{\mathcal{G}} |F(x, \xi, \tau)|^p m(x, \xi)^p \, dx \right)^{\frac{q}{p}} d\xi d\tau \right)^{\frac{1}{q}}, \tag{25}$$

obvious modifications for $p = \infty$ or $q = \infty$.

$(L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}}), \|\cdot\|_{L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}})})$ is a solid QBF space on $\mathbb{H}_{\mathcal{G}}$. If m is moderate with respect to a submultiplicative weight v on $\mathcal{G} \times \widehat{\mathcal{G}}$, then $\tilde{m}$ is left- and right-moderate w.r.t. $\tilde{v}$ on $\mathbb{H}_{\mathcal{G}}, \tilde{v}$ as in (24). Therefore $L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}})$ is left and right invariant, see Definition A.1.

Lemma 3.5 *Consider* $0 < p, q \leq \infty$. *Then* $\|\cdot\|_{L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}})}$ *is an r-norm on* $L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}})$ *with* $r := \min\{1, p, q\}$.

Proof We present the proof for generic product measure space $X \times Y$, with product measure $d\mu(x)d\nu(y)$, instead of $\mathbb{H}_{\mathcal{G}} \cong \mathcal{G} \times (\widehat{\mathcal{G}} \times \mathbb{T})$. We tackle the unweighted case, the weighted one follows immediately. We recall that for $0 < p \leq \infty$ the application

$$\|f\|_{L^p(X)} := \left(\int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}},$$

with obvious modification for $p = \infty$, is an $\min\{1, p\}$-norm, see e.g. [65, Example 2.1.3]. Therefore, it is a $\min\{1, p, q\}$-norm also. Let us consider $f, g \in L^{p,q}(X \times Y)$ and $r := \min\{1, p, q\}$:

$$\|f + g\|_{L^{p,q}(X \times Y)}^r = \left(\int_Y \left(\int_X |f(x, y) + g(x, y)|^p d\mu(x) \right)^{\frac{q}{p}} d\nu(y) \right)^{\frac{r}{q}}$$

$$= \left(\int_Y \left(\int_X |f(x, y) + g(x, y)|^p d\mu(x) \right)^{\frac{r}{p}\frac{q}{r}} d\nu(y) \right)^{\frac{r}{q}}.$$

Using the fact that $\|\cdot\|_{L^p(X)}$ is an r-norm and $q/r \geq 1$:

$$\|f + g\|^r_{L^{p,q}(X \times Y)}$$

$$\leq \left(\int_Y \left(\left(\int_X |f(x,y)|^p d\mu(x) \right)^{\frac{r}{p}} + \left(\int_X |g(x,y)|^p d\mu(x) \right)^{\frac{r}{p}} \right)^{\frac{q}{r}} d\nu(y) \right)^{\frac{r}{q}}$$

$$\leq \left(\int_Y \left(\left(\int_X |f(x,y)|^p d\mu(x) \right)^{\frac{r}{p}} \right)^{\frac{q}{r}} d\nu(y) \right)^{\frac{r}{q}}$$

$$+ \left(\int_Y \left(\left(\int_X |g(x,y)|^p d\mu(x) \right)^{\frac{r}{p}} \right)^{\frac{q}{r}} d\nu(y) \right)^{\frac{r}{q}}$$

$$= \|f\|^r_{L^{p,q}(X \times Y)} + \|g\|^r_{L^{p,q}(X \times Y)}.$$

The proof is concluded. $\qquad\square$

Lemma 3.6 *Consider* $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ *and* $0 < p, q \leq \infty$*. Then there exists* $C = C(m, v) > 0$ *such that for any* $F \in L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})$ *and* $(x, \xi, \tau) \in \mathbb{H}_{\mathcal{G}}$

$$\left\| R_{(x,\xi,\tau)} F \right\|_{L^{p,q}_{\tilde{m}}} \leq C v(-x, -\xi) \|F\|_{L^{p,q}_{\tilde{m}}}, \quad \left\| L_{(x,\xi,\tau)} F \right\|_{L^{p,q}_{\tilde{m}}} \leq C v(x, \xi) \|F\|_{L^{p,q}_{\tilde{m}}}. \qquad (26)$$

Proof The claim is a straightforward calculation which follows by the bi-invariance of the Haar measure on $\mathbb{H}_{\mathcal{G}}$. For $p, q \neq \infty$,

$$\left\| R_{(x,\xi,\tau)} F \right\|^q_{L^{p,q}_{\tilde{m}}} = \int_{\widehat{\mathcal{G}} \times \mathbb{T}} \left(\int_{\mathcal{G}} |F((u, \omega, t)(x, \xi, \tau))|^p \tilde{m}(u, \omega, t)^p \, du \right)^{\frac{q}{p}} d\omega dt$$

$$\lesssim_{m,v} \int_{\widehat{\mathcal{G}} \times \mathbb{T}} \left(\int_{\mathcal{G}} |F(u', \omega', t')|^p \tilde{m}(u', \omega', t')^p \tilde{v}((x, \xi, \tau)^{-1})^p du' \right)^{\frac{q}{p}} d\omega' dt'$$

$$= v(-x, -\xi)^q \|F\|^q_{L^{p,q}_{\tilde{m}}}.$$

Left translations are treated similarly, as well as the cases $p = \infty$ or $q = \infty$. $\qquad\square$

Due to the symmetry of v (Definition A.6), the first inequality in (26) reads as

$$\left\| R_{(x,\xi,\tau)} F \right\|_{L^{p,q}_{\tilde{m}}} \leq C v(x, \xi) \|F\|_{L^{p,q}_{\tilde{m}}}.$$

Lemma 3.7 *Let* $0 < p, q \leq \infty$*. Fix* $V_{\mathcal{G}} \subseteq \mathcal{G}$ *and* $V_{\widehat{\mathcal{G}}} \subseteq \widehat{\mathcal{G}}$ *open, relatively compact, neighborhoods of* $e \in \mathcal{G}$ *and* $\hat{e} \in \widehat{\mathcal{G}}$, *respectively. Define*

$$V := V_{\mathcal{G}} \times V_{\widehat{\mathcal{G}}} \times \mathbb{T}. \qquad (27)$$

Consider $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$*. Then there exists* $C = C(m, v) > 0$ *such that for every* $(x, \xi, \tau) \in \mathbb{H}_{\mathcal{G}}$

$$\left|\left|\left|R_{(x,\xi,\tau)}\right|\right|\right|_{W_V(L_{\tilde{m}}^{p,q}) \to W_V(L_{\tilde{m}}^{p,q})} \leq Cv(-x,-\xi). \tag{28}$$

Proof V is a open, relatively compact, unit neighborhood and the set

$$V_{1,2} := V_{\mathcal{G}} \times V_{\widehat{\mathcal{G}}} \tag{29}$$

is also open, relatively compact, unit neighborhood in $\mathcal{G} \times \widehat{\mathcal{G}}$. For $F \in L_{\text{loc}}^\infty(\mathbb{H}_{\mathcal{G}})$

$$\mathsf{M}_V[R_{(x,\xi,\tau)}F] = \mathsf{M}_{V(x,\xi,\tau)}F, \tag{30}$$

see [65, Lemma 2.3.18, 1.]. For any $(x,\xi,\tau) \in \mathbb{H}_{\mathcal{G}}$

$$V(x,\xi,\tau) = \left(V_{\mathcal{G}} + x\right) \times \left(V_{\widehat{\mathcal{G}}} + \xi\right) \times \bigcup_{u \in V_{\mathcal{G}}} \mathbb{T}\tau\langle\xi,u\rangle = (x,\xi,\tau)V.$$

If $F \in W_V(L_{\tilde{m}}^{p,q})$ and $(x,\xi,\tau) \in \mathbb{H}_{\mathcal{G}}$, from what just observed we obtain:

$$\mathsf{M}_V[R_{(x,\xi,\tau)}F](u,\omega,t) = \operatorname*{ess\,sup}_{(y,\eta,s)\in(u,\omega,t)(x,\xi,\tau)V} |F(y,\eta,s)| = R_{(x,\xi,\tau)}[\mathsf{M}_V F(u,\omega,t)].$$

Eventually using (26)

$$\begin{aligned}
\left\|R_{(x,\xi,\tau)}F\right\|_{W_V(L_{\tilde{m}}^{p,q})} &= \left\|\mathsf{M}_V[R_{(x,\xi,\tau)}F]\right\|_{L_{\tilde{m}}^{p,q}} = \left\|R_{(x,\xi,\tau)}[\mathsf{M}_V F]\right\|_{L_{\tilde{m}}^{p,q}} \\
&\leq Cv(-x,-\xi)\left\|\mathsf{M}_V F\right\|_{L_{\tilde{m}}^{p,q}} = Cv(-x,-\xi)\|F\|_{W_V(L_{\tilde{m}}^{p,q})},
\end{aligned}$$

for some $C = C(m,v) > 0$. This concludes the proof. $\qquad\square$

As already highlighted, inequality (28) can be equivalently written with $v(x,\xi)$ in place of $v(-x,-\xi)$. Observe that the constant C involved in (26) and (28) is the one coming from the v-moderateness condition: $m((x,\xi) + (u,\omega)) \leq Cv(x,\xi)m(u,\omega)$.

Corollary 3.8 *Let* $0 < p,q \leq \infty$. *Consider* $Q \subseteq \mathbb{H}_{\mathcal{G}}$ *measurable, relatively compact, unit neighborhood and* $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$. *Then there exists* $C_Q = C(Q,m,v) > 0$ *such that for every* $(x,\xi,\tau) \in \mathbb{H}_{\mathcal{G}}$

$$\left|\left|\left|R_{(x,\xi,\tau)}\right|\right|\right|_{W_Q(L_{\tilde{m}}^{p,q}) \to W_Q(L_{\tilde{m}}^{p,q})} \leq C_Q v(-x,-\xi). \tag{31}$$

Proof The claim follows from the independence of the Wiener Amalgam space $W(L_{\tilde{m}}^{p,q})$ from the window subset (Lemma A.5) together with Lemma 3.7. $\qquad\square$

Remark 3.9 Consider the (generalized) wavelet transform induced by the Schrödinger representation ϱ in (189) taking $G = \mathbb{H}_{\mathcal{G}}$ and $f,g \in \mathcal{H} = L^2(\mathcal{G})$:

$$W_g^o f : \mathbb{H}_{\mathcal{G}} \to \mathbb{C}, (x,\xi,\tau) \mapsto \langle f, \tau M_\xi T_x g\rangle_{L^2(\mathcal{G})}. \tag{32}$$

This is a continuous and bounded function. It is straightforward to see that

$$W_g^o f(x, \xi, \tau) = \langle f, \tau M_\xi T_x g \rangle = \overline{\tau} V_g f(x, \xi), \quad \forall (x, \xi, \tau) \in \mathbb{H}_\mathcal{G}, \tag{33}$$

which implies

$$\left| W_g^o f(x, \xi, \tau) \right| = \left| V_g f(x, \xi) \right|, \quad \forall (x, \xi, \tau) \in \mathbb{H}_\mathcal{G}. \tag{34}$$

Therefore for $f, g \in L^2(\mathcal{G})$, being $\mathbb{T}$ compact,

$$W_g^o f \in L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G}) \quad \Leftrightarrow \quad V_g f \in L_m^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}}) \tag{35}$$

and

$$W_g^o f \in W(L^\infty(\mathbb{H}_\mathcal{G}), L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})) \Leftrightarrow V_g f \in W(L^\infty(\mathcal{G} \times \widehat{\mathcal{G}}), L_m^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}})). \tag{36}$$

We are now able to revisit steps **A–J** in the Appendix, as follows.

A'. For $G = \mathbb{H}_\mathcal{G}$ the Heisenberg group associated to $\mathcal{G}$, $\mathcal{H} = L^2(\mathcal{G})$ and $\rho = \varrho : \mathbb{H}_\mathcal{G} \to L^2(\mathcal{G})$ the Schrödinger representation, the requirements of **A** are fulfilled due to Lemmas 3.2 and 3.3.
B'. $W_g^o f$ was described in (32) and the integrability of ϱ was proved in Lemma 3.3 as well as that every element of $\mathcal{S}_\mathcal{C}(\mathcal{G})$ is admissible for ϱ.
C'. Take $Y = L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})$ (Definition 3.4) and $r = \min\{1, p, q\}$ (Lemma 3.5).

D'. The right invariance for each measurable, relatively compact, unit neighborhood $Q \subseteq \mathbb{H}_\mathcal{G}$ of $W_Q(L^\infty, L_{\tilde{m}}^{p,q})$ is guaranteed by the right invariance of $L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})$, Lemmas 3.6 and A.5. Since $\mathbb{H}_\mathcal{G}$ is unimodular, (188) and (189) can be summarized as

$$w(x, \xi, \tau) \gtrsim_Q \left|\left|\left| R_{(x,\xi,\tau)^{\pm 1}} \right|\right|\right|_{W_Q(L_{\tilde{m}}^{p,q}) \to W_Q(L_{\tilde{m}}^{p,q})}, \tag{37}$$

for some (hence every) measurable, relatively compact, unit neighborhood $Q \subseteq \mathbb{H}_\mathcal{G}$. Therefore, on account of (31) and the definition of $\mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$, we can take $w = \tilde{v}$ the extension of v defined as in (24).
E'. We take $\tilde{v}$ as control weight for $L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})$, see **E**.
F'. The class of good vectors we are considering is

$$\mathbb{G}_{\tilde{v}} := \left\{ g \in L^2(\mathcal{G}) \,|\, W_g^o g \in L_{\tilde{v}}^1 \right\}. \tag{38}$$

We shall prove that it is nontrivial.
G'. Our class of analysing vectors is

$$\mathbb{A}_{\tilde{v}}^r := \left\{ g \in L^2(\mathcal{G}) \,|\, W_g^o g \in W^R(L^\infty, W(L^\infty, L_{\tilde{v}}^r)) \right\}. \tag{39}$$

It is due to [65, Lemma 2.4.9] that $\mathbb{A}_{\tilde{v}}^r$ is a vector space, as observed in the proof of [65, Theorem 2.4.9], and that

$$W_h^o g \in W^R(L^\infty, W(L^\infty, L_{\check{v}}^r)) \tag{40}$$

for every $g, h \in \mathbb{A}_{\check{v}}^r$.

Lemma 3.10 *Let us define*

$$\mathscr{A}_{\check{v}} := \mathscr{A}_{\check{v}}(\mathcal{G}) := \bigcap_{0 < r \le 1} \mathbb{A}_{\check{v}}^r. \tag{41}$$

The following inclusions hold true:

$$\mathcal{S}_{\mathcal{C}}(\mathcal{G}) \subseteq \mathscr{A}_{\check{v}} \subseteq \mathbb{G}_{\check{v}}. \tag{42}$$

Proof The only inclusion to be shown is the first one, the second one was already mentioned in Remark A.8(ii). Fix $0 < r \le 1$. First, we show that the Gaussian $\varphi \in L^2(\mathcal{G})$ in (12) belongs to $\mathbb{A}_{\check{v}}^r$. From (17):

$$W_\varphi^o \varphi(x, \xi, \tau) = \overline{\tau} c(\mathcal{K}) e^{-\frac{\pi}{2}(x_1^2 + \xi_1^2)} \otimes \chi_{\mathcal{K} \times \mathcal{K}^\perp}(x_2, \xi_2) = \overline{\tau} V_\varphi \varphi(x, \xi),$$

for some $c(\mathcal{K}) > 0$. Take $V \subseteq \mathbb{H}_{\mathcal{G}}$ as in (27) and observe that if $F \in L_{\text{loc}}^\infty(\mathbb{H}_{\mathcal{G}})$

$$
\begin{aligned}
\mathsf{M}_V[\mathsf{M}_V^R F](x, \xi, \tau) &= \operatorname*{ess\,sup}_{(u,\omega,t) \in (x,\xi,\tau)V} \left| \mathsf{M}_V^R F(u, \omega, t) \right| \\
&= \operatorname*{ess\,sup}_{(u,\omega,t) \in (x,\xi,\tau)V} \left| \operatorname*{ess\,sup}_{(y,\eta,s) \in V(u,\omega,t)} |F(y, \eta, s)| \right| \\
&\le \operatorname*{ess\,sup}_{(y,\eta,s) \in V(x,\xi,\tau)V} |F(y, \eta, s)|.
\end{aligned}
$$

If $F = W_\varphi^o \varphi$, adopting notation of (29), we get

$$
\begin{aligned}
\mathsf{M}_V[\mathsf{M}_V^R W_\varphi^o \varphi](x, \xi, \tau) &\le \operatorname*{ess\,sup}_{(y,\eta,s) \in V(x,\xi,\tau)V} \left| \overline{s} V_\varphi \varphi(y, \eta) \right| \\
&= \operatorname*{ess\,sup}_{(y,\eta) \in V_{1,2} + (x,\xi) + V_{1,2}} \left| V_\varphi \varphi(y, \eta) \right| \\
&= \operatorname*{ess\,sup}_{(y,\eta) \in (x,\xi) + 2V_{1,2}} \left| V_\varphi \varphi(y, \eta) \right| = \mathsf{M}_{2V_{1,2}} V_\varphi \varphi(x, \xi),
\end{aligned}
$$

where $2V_{1,2} := V_{1,2} + V_{1,2}$ is a open, relatively compact, unit neighborhood in $\mathcal{G} \times \widehat{\mathcal{G}}$.

From the solidity of $L_{\check{v}}^r$,

$$\left\| W_\varphi^o \varphi \right\|_{W^R(W(L_{\check{v}}^r))} \asymp \left\| W_\varphi^o \varphi \right\|_{W_V^R(W_V(L_{\check{v}}^r))} \le \left\| \mathsf{M}_{2V_{1,2}} V_\varphi \varphi \right\|_{L_{\check{v}}^r(\mathcal{G} \times \widehat{\mathcal{G}})} \tag{43}$$

and we shall prove the right-hand side to be finite. Due to the arbitrariness of $V_{\mathcal{G}}$ and $V_{\widehat{\mathcal{G}}}$, we can assume that

$$V_{1,2} = V_{\mathcal{G}} \times V_{\widehat{\mathcal{G}}} \cong (E_1 \times D_1) \times (E_2 \times D_2) \cong (E_1 \times E_2) \times (D_1 \times D_2), \tag{44}$$

where $E_1, E_2 \subseteq \mathbb{R}^d$, $D_1 \subseteq \mathcal{G}_0$ and $D_2 \subseteq \widehat{\mathcal{G}}$ are open, relatively compact, unit neighborhoods. As done previously,

$$E_{1,2} := E_1 \times E_2 \subseteq \mathbb{R}^{2d}, \quad D_{1,2} := D_1 \times D_2 \subseteq \mathcal{G}_0 \times \widehat{\mathcal{G}}_0,$$
$$2E_{1,2} := E_{1,2} + E_{1,2}, \quad 2D_{1,2} := D_{1,2} + D_{1,2}.$$

Hence,

$$
\mathsf{M}_{2V_{1,2}} V_\varphi \varphi(x,\xi) = c(\mathcal{K}) \operatorname*{ess\,sup}_{\substack{((y_1,\eta_1),(y_2,\eta_2)) \in \\ ((x_1,\xi_1),(x_2,\xi_2)) + 2E_{1,2} \times 2D_{1,2}}} \left| e^{-\frac{\pi}{2}(y_1^2+\eta_1^2)} \chi_{\mathcal{K} \times \mathcal{K}^\perp}(y_2,\eta_2) \right|
$$
$$
= c(\mathcal{K}) \operatorname*{ess\,sup}_{(y_1,\eta_1) \in (x_1,\xi_1) + 2E_{1,2}} \left| e^{-\frac{\pi}{2}(y_1^2+\eta_1^2)} \right|
$$
$$
\times \operatorname*{ess\,sup}_{(y_2,\eta_2) \in (x_2,\xi_2) + 2D_{1,2}} \left| \chi_{\mathcal{K} \times \mathcal{K}^\perp}(y_2,\eta_2) \right|.
$$

Since $v(x,\xi)$ is submultiplicative, using the structure theorem we can majorize as follows:

$$v(x,\xi) = v((x_1,\xi_1),(x_2,\xi_2)) \le v((x_1,\xi_1),(e_0,\hat{e}_0))v((0,0),(x_2,\xi_2)),$$

where $x = (x_1,x_2) \in \mathbb{R}^d \times \mathcal{G}_0$, $\xi = (\xi_1,\xi_2) \in \mathbb{R}^d \times \widehat{\mathcal{G}}_0$. Let us define

$$v_1(x_1,\xi_1) := v((x_1,\xi_1),(e_0,\hat{e}_0)), \quad v_2(x_2,\xi_2) := v((0,0),(x_2,\xi_2)),$$

$(x_1,\xi_1) \in \mathbb{R}^{2d}$ and $(x_2,\xi_2) \in \mathcal{G}_0 \times \widehat{\mathcal{G}}_0$, which are still submultiplicative. Hence,

$$
\left\| \mathsf{M}_{2V_{1,2}} V_\varphi \varphi \right\|_{L_v^r(\mathcal{G} \times \widehat{\mathcal{G}})}^r
$$
$$
\le c(\mathcal{K})^r \underbrace{\int_{\mathbb{R}^{2d}} \operatorname*{ess\,sup}_{(y_1,\eta_1) \in (x_1,\xi_1) + 2E_{1,2}} \left| e^{-\frac{\pi}{2}(y_1^2+\eta_1^2)} \right|^r v_1(x_1,\xi_1)^r \, dx_1 d\xi_1}_{=:I_1}
$$
$$
\times \underbrace{\int_{\mathcal{G}_0 \times \widehat{\mathcal{G}}_0} \operatorname*{ess\,sup}_{(y_2,\eta_2) \in (x_2,\xi_2) + 2D_{1,2}} \left| \chi_{\mathcal{K} \times \mathcal{K}^\perp}(y_2,\eta_2) \right| v_2(x_2,\xi_2)^r \, dx_2 d\xi_2}_{=:I_2}.
$$

For $N > 2d$ and considering the weight $\langle \cdot \rangle := (1 + |\cdot|^2)^{1/2}$, we can write

$$
I_1 = \int_{\mathbb{R}^{2d}} \frac{\langle(x_1,\xi_1)\rangle^N}{\langle(x_1,\xi_1)\rangle^N} \operatorname*{ess\,sup}_{(y_1,\eta_1) \in (x_1,\xi_1) + 2E_{1,2}} \left| e^{-\frac{\pi}{2}(y_1^2+\eta_1^2)} \right|^r v_1(x_1,\xi_1)^r \, dx_1 d\xi_1
$$
$$
\le \int_{\mathbb{R}^{2d}} \frac{1}{\langle(x_1,\xi_1)\rangle^N} \operatorname*{ess\,sup}_{(y_1,\eta_1) \in (x_1,\xi_1) + 2E_{1,2}} \left[e^{-\frac{r\pi}{2}(y_1^2+\eta_1^2)} v_1(y_1,\eta_1)^r \langle(y_1,\eta_1)\rangle^N \right] dx_1 d\xi_1
$$
$$
\le \int_{\mathbb{R}^{2d}} \frac{1}{\langle(x_1,\xi_1)\rangle^N} \operatorname*{ess\,sup}_{(y_1,\eta_1) \in \mathbb{R}^{2d}} \left[e^{-\frac{r\pi}{2}(y_1^2+\eta_1^2)} v_1(y_1,\eta_1)^r \langle(y_1,\eta_1)\rangle^N \right] dx_1 d\xi_1.
$$

In fact,

$$
\operatorname*{ess\,sup}_{(y_1,\eta_1)\in(x_1,\xi_1)+2E_{1,2}} e^{-\frac{r\pi}{2}(y_1^2+\eta_1^2)}\langle(x_1,\xi_1)\rangle^N v_1(x_1,\xi_1)^r
$$

$$
\leq \operatorname*{ess\,sup}_{(y_1,\eta_1)\in(x_1,\xi_1)+2E_{1,2}} \left[e^{-\frac{r\pi}{2}(y_1^2+\eta_1^2)} v_1(y_1,\eta_1)^r \langle(y_1,\eta_1)\rangle^N \right]
$$

$$
\leq \| e^{\frac{-r\pi}{2}|\cdot|^2} v_1^r(\cdot)\langle\cdot\rangle^N \|_{L^\infty(\mathbb{R}^{2d})} < +\infty
$$

because v_1 is submultiplicative, so it can grow at most exponentially [13, Lemma 2.1.4]. Hence $I_1 < +\infty$.

We now study the integral I_2. Observe that the integrand is not equal to zero if and only if $(\mathcal{K}\times\mathcal{K}^\perp)\cap((x_2,\xi_2)+2D_{1,2})\neq\varnothing$, which means that there exist $k\in\mathcal{K}\times\mathcal{K}^\perp$ and $h\in 2D_{1,2}$, all depending on (x_2,ξ_2), such that $k=(x_2,\xi_2)+h$ if and only if $(x_2,\xi_2)=k-h$, which implies $(x_2,\xi_2)\in\mathcal{K}\times\mathcal{K}^\perp-2D_{1,2}$. Equivalently, $(x_2,\xi_2)\notin\mathcal{K}\times\mathcal{K}^\perp-2D_{1,2}$ if and only if $\operatorname*{ess\,sup}_{(y_2,\eta_2)\in(x_2,\xi_2)+2D_{1,2}}|\chi_{\mathcal{K}\times\mathcal{K}^\perp}(y_2,\eta_2)|=0$, that implies

$$
\operatorname*{ess\,sup}_{(y_2,\eta_2)\in(x_2,\xi_2)+2D_{1,2}} |\chi_{\mathcal{K}\times\mathcal{K}^\perp}(y_2,\eta_2)| \leq \chi_{\mathcal{K}\times\mathcal{K}^\perp-2D_{1,2}}(x_2,\xi_2).
$$

Note that $\mathcal{K}\times\mathcal{K}^\perp-2D_{1,2}$ is relatively compact, hence of finite measure. The local boundedness of the submultiplicative weight v_2, shown in [65, Theorem 2.2.22], ensures that the integral on $\mathcal{G}_0\times\widehat{\mathcal{G}}_0$ is finite.

So far we have shown $\varphi\in\mathbb{A}_{\tilde{v}}^r$. Now, consider $f=\sum_{k=1}^n a_k\pi(u_k,\omega_k)\varphi\in\mathcal{S}_\mathcal{C}(\mathcal{G})$ and apply (43), Lemma 2.1 and left/right invariance of $W_{2V_{1,2}}(L_v^r(\mathcal{G}\times\widehat{\mathcal{G}}))$:

$$
\left\| W_f^\varrho f \right\|_{W_V^R(W_V(L_{\tilde{v}}^r))}
$$

$$
\leq \left\| M_{2V_{1,2}} V_f f \right\|_{L_v^r(\mathcal{G}\times\widehat{\mathcal{G}})} = \left\| V_f f \right\|_{W_{2V_{1,2}}(L_v^r(\mathcal{G}\times\widehat{\mathcal{G}}))}
$$

$$
= \left\| \sum_{k,j=1}^n \|a_k \overline{a_j} \langle\xi-\omega_k,u_k\rangle \langle\omega_j,x-u_k\rangle T_{(u_k,\omega_k)-(u_j,\omega_j)} V_\varphi \varphi(x,\xi) \right\|
$$

$$
\underset{n,r}{\lesssim} \sum_{k,j=1}^n |a_k a_j| \left\| T_{(u_k,\omega_k)-(u_j,\omega_j)} V_\varphi \varphi(x,\xi) \right\|_{W_{2V_{1,2}}(L_v^r(\mathcal{G}\times\widehat{\mathcal{G}}))} < +\infty.
$$

This concludes the proof. $\qquad\square$

Of course, $\mathscr{A}_{\tilde{v}}$ is a vector space. We shall use the extended notation $\mathscr{A}_{\tilde{v}}(\mathcal{G})$ only when confusion may occur. It is also clear that writing $\mathscr{A}_{\tilde{v}}(\mathcal{G}\times\widehat{\mathcal{G}})$ we mean the weight v to be defined on $(\mathcal{G}\times\widehat{\mathcal{G}})\times(\widehat{\mathcal{G}}\times\mathcal{G})$, as done in the subsequent Corollary 3.11.

Corollary 3.11 *Let $f,g\in\mathcal{S}_\mathcal{C}(\mathcal{G})$, then $R(f,g)\in\mathscr{A}_{\tilde{v}}(\mathcal{G}\times\widehat{\mathcal{G}})$.*

Proof The proof follows the same arguments in Lemma 3.10, together with (14) and Lemma 2.1. □

H'. For a fixed $g \in \mathbb{G}_{\tilde{v}} \smallsetminus \{0\}$, the space of test vectors is

$$\mathcal{T}_{\tilde{v}} := \left\{ f \in L^2(\mathcal{G}) \mid W_g^o f \in L^1_{\tilde{v}}(\mathbb{H}_{\mathcal{G}}) \right\} \tag{45}$$

endowed with the norm

$$\|f\|_{\mathcal{T}_{\tilde{v}}} := \left\| W_g^o f \right\|_{L^1_{\tilde{v}}}. \tag{46}$$

$(\mathcal{T}_{\tilde{v}}, \|\cdot\|_{\mathcal{T}_{\tilde{v}}})$ is a ϱ-invariant Banach space which embeds continuously into $L^2(\mathcal{G})$ and it is independent from the choice of the window vector $g \in \mathbb{G}_{\tilde{v}} \smallsetminus \{0\}$, see [65, Lemma 2.4.7].

Lemma 3.12 *For any $g \in \mathcal{S}_C(\mathcal{G}) \smallsetminus \{0\}$, the following equality holds true*

$$\mathbb{G}_{\tilde{v}} = \mathcal{T}_{\tilde{v}} = \left\{ f \in L^2(\mathcal{G}) \mid V_g f \in L^1_v(\mathcal{G} \times \widehat{\mathcal{G}}) \right\}. \tag{47}$$

Proof The second equality is just Remark 3.9, for the first one the proof follows the pattern of [6, Proposition 3.6]. From [65, Lemma 2.4.7]: $\mathbb{G}_{\tilde{v}} \subseteq \mathcal{T}_{\tilde{v}}$. Being the Duflo–Moore operator [14, Theorem 3] the identity, the orthogonality relations for $f, h \in L^2(\mathcal{G})$ and $g, \gamma \in \mathbb{G}_{\tilde{v}}$ are

$$\langle W_g^o f, W_\gamma^o h \rangle_{L^2(\mathbb{H}_{\mathcal{G}})} = \langle \gamma, g \rangle_{L^2(\mathcal{G})} \langle f, h \rangle_{L^2(\mathcal{G})},$$

see [65, Theorem 2.4.3]. Fix $f \in \mathcal{T}_{\tilde{v}}$, take $\gamma = g \neq 0$, $h = \varrho(x, \xi, \tau) f$ and using Fubini's Theorem, symmetry and submultiplicativity of $\tilde{v}$ we compute

$$\left\| W_f^o f \right\|_{L^1_{\tilde{v}}}$$

$$= \int_{\mathbb{H}_{\mathcal{G}}} \left| W_f^o f(x, \xi, \tau) \right| \tilde{v}(x, \xi, \tau) \, dx d\xi d\tau$$

$$= \int_{\mathbb{H}_{\mathcal{G}}} |\langle f, \varrho(x, \xi, \tau) f \rangle| \tilde{v}(x, \xi, \tau) \, dx d\xi d\tau$$

$$= \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{H}_{\mathcal{G}}} \left| \langle W_g^o f, W_g^o [\varrho(x, \xi, \tau) f] \rangle \right| \tilde{v}(x, \xi, \tau) \, dx d\xi d\tau$$

$$\leq \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{H}_{\mathcal{G}}} \int_{\mathbb{H}_{\mathcal{G}}} \left| W_g^o f(y, \eta, s) W_g^o [\varrho(x, \xi, \tau) f](y, \eta, s) \right| dy d\eta ds \tilde{v}(x, \xi, \tau) dx d\xi d\tau$$

$$= \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{H}_{\mathcal{G}}} \left| W_g^o f(y, \eta, s) \right| \left(\int_{\mathbb{H}_{\mathcal{G}}} \left| W_g^o [\varrho(x, \xi, \tau) f](y, \eta, s) \right| \tilde{v}(x, \xi, \tau) dx d\xi d\tau \right) dy d\eta ds.$$

Observe

$$W_g^\varrho[\varrho(x,\xi,\tau)f](y,\eta,s) = \langle \varrho(x,\xi,\tau)f, \varrho(y,\eta,s)g \rangle = W_g^\varrho f\big((x,\xi,\tau)^{-1}(y,\eta,s)\big),$$

so that

$$
\begin{aligned}
\left\| W_f^\varrho f \right\|_{L_{\tilde{v}}^1} &\leq \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{H}_\mathcal{G}} \left| W_g^\varrho f(y,\eta,s) \right| \\
&\quad \times \left(\int_{\mathbb{H}_\mathcal{G}} \left| W_g^\varrho f\big((x,\xi,\tau)^{-1}(y,\eta,s)\big) \right| \tilde{v}(x,\xi,\tau)\, \mathrm{d}x\mathrm{d}\xi\mathrm{d}\tau \right) \mathrm{d}y\mathrm{d}\eta\mathrm{d}s \\
&\leq \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{H}_\mathcal{G}} \left| W_g^\varrho f(y,\eta,s) \right| \\
&\quad \times \left(\int_{\mathbb{H}_\mathcal{G}} \left| W_g^\varrho f(x',\xi',\tau') \right| \tilde{v}(x',\xi',\tau') \tilde{v}(y,\eta,s)\, \mathrm{d}x'\mathrm{d}\xi'\mathrm{d}\tau' \right) \mathrm{d}y\mathrm{d}\eta\mathrm{d}s \\
&= \frac{1}{\|g\|_{L^2}^2} \left(\int_{\mathbb{H}_\mathcal{G}} \left| W_g^\varrho f(x',\xi',\tau') \right| \tilde{v}(x',\xi',\tau')\, \mathrm{d}x'\mathrm{d}\xi'\mathrm{d}\tau' \right)^2 \\
&= \frac{1}{\|g\|_{L^2}^2} \left\| W_g^\varrho f \right\|_{L_{\tilde{v}}^1}^2 < +\infty.
\end{aligned}
$$

Hence, $f \in \mathbb{G}_{\tilde{v}}$ and the proof in concluded. $\qquad\square$

Lemma 3.13 $\mathcal{S}_\mathcal{C}(\mathcal{G})$ *is dense in* $(\mathcal{T}_{\tilde{v}}, \|\cdot\|_{\mathcal{T}_{\tilde{v}}})$.

Proof In Lemma 3.10 we have shown that the Gaussian φ in (12) belongs to $\mathbb{G}_{\tilde{v}}$. Then from [65, Lemma 2.4.7, 5.], we have that

$$\mathcal{S}_\mathcal{C}^\varrho(\mathcal{G}) := \mathrm{span}\big\{ \varrho(x,\xi,\tau)\varphi \mid (x,\xi,\tau) \in \mathbb{H}_\mathcal{G} \big\}$$

is dense in $(\mathcal{T}_{\tilde{v}}, \|\cdot\|_{\mathcal{T}_{\tilde{v}}})$. The claim follows from the trivial fact that $\mathcal{S}_\mathcal{C}^\varrho(\mathcal{G}) = \mathcal{S}_\mathcal{C}(\mathcal{G})$. $\quad\square$

$\mathbf{I}'$. The reservoir is the Banach space

$$\mathcal{R}_{\tilde{v}} := \mathcal{T}_{\tilde{v}}^\neg := \big\{ f : \mathcal{T}_{\tilde{v}} \to \mathbb{C} \mid \text{antilinear and continuous} \big\}. \tag{48}$$

Remark 3.14 The Feichtinger algebra $S_0(\mathcal{G})$ [17–19] has numerous equivalent descriptions, see [41]. It can be seen as the vector space

$$S_0(\mathcal{G}) = \left\{ f \in L^2(\mathcal{G}) \mid V_g f \in L^1(\mathcal{G} \times \widehat{\mathcal{G}}) \right\}, \tag{49}$$

for some fixed non-zero window function $g \in L^2(\mathcal{G})$. Equipped with the norm

$$\|f\|_{S_0} = \left\| V_g f \right\|_{L^1}, \tag{50}$$

$S_0(\mathcal{G})$ is a Banach space. If $v \equiv 1$, then

$$\mathcal{T}_1 = \mathbb{G}_1 = S_0(\mathcal{G}), \quad \mathcal{R}_1 = S_0'(\mathcal{G}). \tag{51}$$

If v is not constant, then

$$\mathcal{T}_{\tilde{v}} = \mathbb{G}_{\tilde{v}} \hookrightarrow S_0(\mathcal{G}), \qquad \mathcal{R}_{\tilde{v}} \hookleftarrow S_0'(\mathcal{G}). \tag{52}$$

Corollary 3.15 *The following inclusion holds true*:

$$\mathscr{A}_{\tilde{v}} \subseteq C_0(\mathcal{G}), \tag{53}$$

the latter being the space of continuous complex-valued functions on $\mathcal{G}$ which vanish at infinity.

Proof Combining Lemma 3.10 and Remark 3.14 we have $\mathscr{A}_{\tilde{v}} \subseteq \mathbb{G}_{\tilde{v}} = \mathcal{T}_{\tilde{v}} \subseteq S_0(\mathcal{G})$. We conclude using the fact that $S_0(\mathcal{G}) \subseteq C_0(\mathcal{G})$, see, e.g., [41, Theorem 4.1]. $\qquad\square$

J'. We extend the wavelet transform to $f \in \mathcal{R}_{\tilde{v}}$ and $g \in \mathcal{T}_{\tilde{v}}$:

$$W_g^o f : \mathbb{H}_\mathcal{G} \to \mathbb{C}, (x, \xi, \tau) \mapsto {}_{\mathcal{R}_{\tilde{v}}}\langle f, \tau M_\xi T_x g\rangle_{\mathcal{T}_{\tilde{v}}}. \tag{54}$$

From now on, we shall simply write $\langle \cdot, \cdot \rangle$. Observe $W_g^o f \in C(\mathbb{H}_\mathcal{G}) \cap L_{1/\tilde{v}}^\infty(\mathbb{H}_\mathcal{G})$.

Remark 3.16 The class $\mathcal{S}_\mathcal{C}(\mathcal{G})$ defined in (13) actually depends on the compact open subgroup $\mathcal{K}$ in $\mathcal{G}_0$, where $\mathcal{G} \cong \mathbb{R}^d \times \mathcal{G}_0$. Then, we might write $\mathcal{S}_\mathcal{C}^\mathcal{K}$ in place of $\mathcal{S}_\mathcal{C}$. Observe that if $\mathcal{K}'$ is a compact open subgroup different from $\mathcal{K}$ Lemma 3.10 is still valid. More generally, if $\mathbb{K}$ is the class of all compact open subgroups in $\mathcal{G}_0$:

$$\mathbb{S}_\mathcal{C}(\mathcal{G}) := \bigcup_{\mathcal{K} \in \mathbb{K}} \mathcal{S}_\mathcal{C}^\mathcal{K}(\mathcal{G}) \subseteq \mathscr{A}_{\tilde{v}} \subseteq \mathbb{G}_{\tilde{v}}. \tag{55}$$

Therefore, coorbit spaces (defined in the subsequent (56)) are independent of the window $g \in \mathbb{S}_\mathcal{C}(\mathcal{G})$. Concretely, this gives us the freedom to chose the subgroup $\mathcal{K}$ which fits better to our purposes, as done in the proof of Lemma 3.38. Arguing similarly, we could replace $e^{-\pi x_1^2}$ in (12) with any $e^{-a x_1^2}$, $a > 0$. This fact will be used in Proposition 4.18.

From now on, for sake of simplicity, we shall only use the notation $\mathcal{S}_\mathcal{C}(\mathcal{G})$ with the convention that $\mathcal{K}$ and the coefficient of the Gaussian on $\mathbb{R}^d$ can be chosen freely, so that we shall never explicitly use the symbol $\mathbb{S}_\mathcal{C}(\mathcal{G})$.

K'. The coorbit space on $\mathbb{H}_\mathcal{G}$ with respect to $L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})$, $0 < p, q \leq \infty$, is, for some fixed non-zero window $g \in \mathcal{S}_\mathcal{C}(\mathcal{G})$,

$$\mathsf{Co}\,(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})) := \mathsf{Co}\,(L_{\tilde{m}}^{p,q}) := \Big\{ f \in \mathcal{R}_{\tilde{v}} \mid W_g^o f \in W(L^\infty, L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})) \Big\} \tag{56}$$

endowed with the quasi-norm

$$\|f\|_{\mathsf{Co}\,(L^{p,q}_{\tilde{m}})} := \left\|W^o_g f\right\|_{W(L^\infty, L^{p,q}_{\tilde{m}})}. \tag{57}$$

We stress that $\mathsf{Co}\,(L^{p,q}_{\tilde{m}})$ is independent of the window g and the space $(\mathsf{Co}\,(L^{p,q}_{\tilde{m}}), \|\cdot\|_{\mathsf{Co}\,(L^{p,q}_{\tilde{m}})})$ is a quasi-Banach space continuously embedded into $\mathcal{R}_{\tilde{v}}$. Moreover, $\|\cdot\|_{\mathsf{Co}\,(L^{p,q}_{\tilde{m}})}$ is a r-norm, with $r = \min\{1, p, q\}$. Notice that

$$\mathsf{Co}\,(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})) = \left\{f \in \mathcal{R}_{\tilde{v}} \mid V_g f \in W(L^\infty, L^{p,q}_m(\mathcal{G} \times \widehat{\mathcal{G}}))\right\}.$$

Remark 3.17 It is clear from the general coorbit theory (see the Appendix), that the set $\mathscr{A}_{\tilde{v}}$ defined in (41) is the maximal window space for all the coorbit spaces $\mathsf{Co}\,(L^{p,q}_{\tilde{m}})$, $0 < p, q \leq \infty$. For sake of simplicity we shall mainly work with window functions in the smaller class $\mathcal{S}_{\mathcal{C}}(\mathcal{G})$ and adopt the whole space $\mathscr{A}_{\tilde{v}}$ only when necessary, as done in Sect. 3.

The coorbit spaces are independent of the reservoir, in the sense shown below.

Proposition 3.18 *Fix a non-zero window $g \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$, then*

$$\mathsf{Co}\,(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})) = \left\{f \in S'_0(\mathcal{G}) \mid W^o_g f \in W(L^\infty, L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))\right\}, \tag{58}$$

in the sense that the restriction map

$$\left\{f \in S'_0(\mathcal{G}) \mid W^o_g f \in W(L^{p,q}_{\tilde{m}})\right\} \to \mathsf{Co}\,(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})), f \mapsto f|_{\mathcal{T}_{\tilde{v}}}$$

is a bijection.

Proof If $v \equiv 1$ the claim is trivial since $\mathcal{T}_1 = S_0$ and $\mathcal{R}_1 = S'_0$, with equal norms, see Remark 3.14. If v is not constant, then $v \gtrsim 1$ (since v is bounded from below), and the thesis follows from what observed in Remark 3.14 and [65, Theorem 2.4.9, 3.]. $\qquad\square$

Definition 3.19 Consider $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ and $0 < p, q \leq \infty$. The modulation space $M^{p,q}_m(\mathcal{G})$ is defined as

$$M^{p,q}_m(\mathcal{G}) := \mathsf{Co}\,(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}})), \tag{59}$$

endowed with the quasi-norm

$$\|\cdot\|_{M^{p,q}_m} := \|\cdot\|_{\mathsf{Co}\,(L^{p,q}_{\tilde{m}})}. \tag{60}$$

We adopt the notations $M^p_m = M^{p,p}_m$ and $M^{p,q} = M^{p,q}_1$.

Theorem 3.20 *For* $0 < p, q \leq \infty$, *the modulation spaces* $(M_m^{p,q}(\mathcal{G}), \|\cdot\|_{M_m^{p,q}})$ *are quasi-Banach spaces continuously embedded into* $\mathcal{R}_{\tilde{v}}$ *which do not depend on the window function* $g \in \mathcal{S}_C(\mathcal{G}) \setminus \{0\}$, *in the sense that different windows yield equivalent quasi-norms.*

Proof Since $(M_m^{p,q}(\mathcal{G}), \|\cdot\|_{M_m^{p,q}}) = (\mathsf{Co}\,(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})), \|\cdot\|_{\mathsf{Co}\,(L_{\tilde{m}}^{p,q})})$, the claim follows from the coorbit spaces theory, Lemma 3.10 and [65, Theorem 2.4.9]. □

Remark 3.21 If $g, h \in \mathcal{S}_C(\mathcal{G}) \setminus \{0\}$ (or $\mathscr{A}_{\tilde{v}} \setminus \{0\}$) and $f \in M_m^{p,q}(\mathcal{G})$, then from the proof in [65, Theorem 2.4.9] we see that

$$\left\| W_h^o f \right\|_{W_Q(L_{\tilde{m}}^{p,q})} \underset{Q,v,r}{\lesssim} \frac{\left\| W_g^o h \right\|_{W_Q(L_{\tilde{v}}^r)}}{\|g\|_{L^2}^2} \left\| W_g^o f \right\|_{W_Q(L_{\tilde{m}}^{p,q})}$$

$$= \frac{\|h\|_{M_v^r(\mathcal{G})}}{\|g\|_{L^2}^2} \left\| W_g^o f \right\|_{W_Q(L_{\tilde{m}}^{p,q})},$$

where $r = \min\{1, p, q\}$ as in **C′**; actually we could replace r with any r' such that $0 < r' \leq r$.

In the Banach case we have $r = 1$ and recapture [33, (11.33)], after taking into account Theorem 3.33 and Remark 3.37.

In order to prove the expected inclusion relations between modulation spaces, we need particular types of relatively separated families, BUPUs and discrete spaces. The proofs of some subsequent lemmas are omitted because well known or trivial.

Lemma 3.22 *Let* $Q, Q' \subseteq \mathbb{H}_\mathcal{G}$ *be relatively compact, unit neighborhoods and* $\mathfrak{F} = \{(x_l, \xi_l, \tau_l)\}_{l \in L} \subseteq \mathbb{H}_\mathcal{G}$ *relatively separated family, consider* $0 < p, q \leq \infty$ *and* $m \in \mathcal{M}_v(\mathcal{G} \times \hat{\mathcal{G}})$. *Then*

$$(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{F}, Q) = (L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{F}, Q')$$

with equivalent quasi-norms. Moreover, the equivalence constants depend only on Q, Q', m *and* v:

$$\left\| (\lambda_l)_{l \in L} \right\|_{(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{F}, Q)} \underset{Q,Q',m,v}{\asymp} \left\| (\lambda_l)_{l \in L} \right\|_{(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{F}, Q')}.$$

In particular, they do not depend on $\mathfrak{F}$ *or* p *and* q.

Proof From (26), we have that for every $0 < p, q \leq \infty$ and $(x, \xi, \tau) \in \mathbb{H}_\mathcal{G}$

$$\left| \left\| \left\| R_{(x,\xi,\tau)} \right\| \right\| \right|_{L_{\tilde{m}}^{p,q} \to L_{\tilde{m}}^{p,q}} \leq Cv(x, \xi),$$

where $C = C(m, v) > 0$ is the constant of v-moderateness for m. Since $L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})$ is right invariant, the proof goes like the one of [65, Lemma 2.3.16] applying the additional majorization above. □

Lemma 3.23 *Let* $Q, U \subseteq \mathbb{H}_\mathcal{G}$ *be relatively compact, unit neighborhoods,* $\Delta = \{\delta_l\}_{l \in L}$ *U-BUPU on* $\mathbb{H}_\mathcal{G}$ *with U-localizing family* $\mathfrak{F} = \{(x_l, \xi_l, \tau_l)\}_{l \in L} \subseteq \mathbb{H}_\mathcal{G}$, *consider* $0 < p, q \leq \infty$ *and* $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$. *Then*

$$\|f\|_{W_Q(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G}))} \underset{Q,U,\mathfrak{F},m,v}{\asymp} \left\|\left(\|\delta_l \cdot f\|_{L^\infty}\right)_{l \in L}\right\|_{(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{F},Q)}.$$

In particular, the equivalence constants do not depend on p and q.

Proof The result come from the proof [65, Theorem 2.3.17] (see (181) in the Appendix) together with Lemma 3.22.   □

Lemma 3.24 *Consider* $X = \{x_i\}_{i \in I} \subseteq \mathcal{G}, \Xi = \{\xi_j\}_{j \in J} \subseteq \widehat{\mathcal{G}}$ *and* $T = \{\tau_z\}_{z \in Z} \subseteq \mathbb{T}$ *relatively separated families. Then,* $\mathfrak{X} := X \times \Xi \times T$ *is a relatively separated family in* $\mathbb{H}_\mathcal{G}$.

We remark that if the group is σ-compact, then any relatively separated family is (at most) countable, [65, Lemma 2.3.10].

Lemma 3.25 *Let* $U \subseteq \mathcal{G}$ *and* $D \subseteq \widehat{\mathcal{G}}$ *be relatively compact, unit neighborhoods. Consider* $\Psi = \{\psi_i\}_{i \in I}$ *U-BUPU with localizing family* $X = \{x_i\}_{i \in I}$ *and* $\Gamma = \{\gamma_j\}_{j \in J}$ *D-BUPU with localizing family* $\Xi = \{\xi_j\}_{j \in J}$. *Then,*

$$\Psi \otimes \Gamma \otimes \mathbb{I} := \left\{\psi_i \otimes \gamma_j \otimes \chi_\mathbb{T}, (i,j) \in I \times J\right\} \tag{61}$$

is a $U \times D \times \mathbb{T}$-*BUPU in* $\mathbb{H}_\mathcal{G}$ *with localizing family* $\mathfrak{X} := X \times \Xi \times \{1\}$.

The following is a generalization of [65, Lemma 2.3.21] and we follow the pattern of its proof. Although we present it for the Heisenberg group $\mathbb{H}_\mathcal{G} \cong \mathcal{G} \times \left(\widehat{\mathcal{G}} \times \mathbb{T}\right)$, it can be easily adapted to any product group $G_1 \times G_2, G_1$ and G_2 even not abelian. A similar result for $1 \leq p = q \leq \infty$ had been stated in [21, Remark 4, p. 518] without proof.

Lemma 3.26 *Consider* $X = \{x_i\}_{i \in I} \subseteq \mathcal{G}$ *and* $\Xi = \{\xi_j\}_{j \in J} \subseteq \widehat{\mathcal{G}}$ *relatively separated families,* $\mathfrak{X}$ *as in Lemma 3.25, and* $V = V_\mathcal{G} \times V_{\widehat{\mathcal{G}}} \times \mathbb{T}$ *as in (27). For* $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ *and* $0 < p, q \leq \infty$,

$$\left(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})\right)_d(\mathfrak{X}, V) = \ell_{m_\mathbf{X}}^{p,q}(I \times J), \tag{62}$$

where

$$m_\mathbf{X} : I \times J \to (0, +\infty), (i,j) \mapsto m(x_i, \xi_j), \tag{63}$$

with equivalence of the relative quasi-norms depending on $X, \Xi, V_\mathcal{G}, V_{\widehat{\mathcal{G}}}, v, p$ *and* q.

Proof The proof is divided into four cases.

Case $p, q < \infty$. Consider a sequence $\left(\lambda_i\right)_{i \in I} \in \mathbb{C}^I$. For every $x \in \mathcal{G}$, we define I_x the subset of indexes

$$I_x = \{i \in I \mid \chi_{x_i + V_{\mathcal{G}}}(x) \neq \varnothing\} \subseteq \{i \in I \mid \left(x_i + \overline{V}_{\mathcal{G}}\right) \cap (x + \{e\}) \neq \varnothing\}. \quad (64)$$

From [65, Lemma 2.3.10], we have

$$\#\{i \in I \mid \left(x_i + V_{\mathcal{G}}\right) \cap (x + \{e\}) \neq \varnothing\} \leq C_{X, \overline{V}_{\mathcal{G}}} < +\infty, \qquad \forall x \in \mathcal{G}, \quad (65)$$

$C_{X, \overline{V}_{\mathcal{G}}} \in \mathbb{N}$ as in (173). Whence $\# I_x \leq C_{X, \overline{V}_{\mathcal{G}}}$ and

$$\begin{aligned}
\left(\sum_{i \in I} |\lambda_i| \chi_{x_i + V_{\mathcal{G}}}(x)\right)^p &\leq \left(\# I_x \cdot \max\{|\lambda_i| \mid i \in I_x\}\right)^p \\
&\leq C_{X, \overline{V}_{\mathcal{G}}}^p \max\{|\lambda_i|^p \mid i \in I_x\} \\
&\leq C_{X, \overline{V}_{\mathcal{G}}}^p \sum_{i \in I_x} |\lambda_i|^p = C_{X, \overline{V}_{\mathcal{G}}}^p \sum_{i \in I} |\lambda_i|^p \chi_{x_i + V_{\mathcal{G}}}(x).
\end{aligned}$$

Vice versa

$$\begin{aligned}
\left(\sum_{i \in I} |\lambda_i| \chi_{x_i + V_{\mathcal{G}}}(x)\right)^p &\geq \left(\max\{|\lambda_i| \mid i \in I_x\}\right)^p = \max\{|\lambda_i|^p \mid i \in I_x\} \\
&\geq C_{X, \overline{V}_{\mathcal{G}}}^{-p} \sum_{i \in I_x} |\lambda_i|^p = C_{X, \overline{V}_{\mathcal{G}}}^{-p} \sum_{i \in I} |\lambda_i|^p \chi_{x_i + V_{\mathcal{G}}}(x).
\end{aligned}$$

Hence, we have shown the equivalence

$$\left(\sum_{i \in I} |\lambda_i| \chi_{x_i + V_{\mathcal{G}}}(x)\right)^p \asymp \sum_{i \in I} |\lambda_i|^p \chi_{x_i + V_{\mathcal{G}}}(x). \quad (66)$$

Analogous equivalences hold for every relatively separated family and sequence on the corresponding set of indexes, which under our hypothesis are always countable. Due to the chosen V,

$$\chi_{(x_i, \xi_j, 1)V}(x, \xi, \tau) = \chi_{x_i + V_{\mathcal{G}}}(x) \chi_{\xi_j + V_{\hat{\mathcal{G}}}}(\xi) \qquad \forall (x, \xi, \tau) \in \mathbb{H}_{\mathcal{G}}.$$

Taking a sequence $\left(\lambda_{ij}\right)_{i \in I, j \in J} \in \mathbb{C}^{I \times J}$ and using twice the equivalence (66), we compute

$$\left\|(\lambda_{ij})_{i,j}\right\|_{(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\boldsymbol{\mathfrak{X}},V)}$$

$$= \left(\int_{\widehat{\mathcal{G}}\times\mathbb{T}}\left(\iint_{\mathcal{G}}\left(\sum_{i\in I,j\in J}\left|\lambda_{ij}\right|\chi_{x_i+V_{\mathcal{G}}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\right)^p m(x,\xi)^p\,\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}\xi\mathrm{d}\tau\right)^{\frac{1}{q}}$$

$$\asymp \left(\int_{\widehat{\mathcal{G}}}\left(\int_{\mathcal{G}}\sum_{i\in I,j\in J}\left|\lambda_{ij}\right|^p\chi_{x_i+V_{\mathcal{G}}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)m(x,\xi)^p\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}\xi\right)^{\frac{1}{q}}$$

$$= \left(\int_{\widehat{\mathcal{G}}}\left(\sum_{j\in J}\sum_{i\in I}\left|\lambda_{ij}\right|^p\int_{\mathcal{G}}m(x,\xi)^p\chi_{x_i+V_{\mathcal{G}}}(x)\mathrm{d}x\,\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\right)^{\frac{q}{p}}\mathrm{d}\xi\right)^{\frac{1}{q}}$$

$$\asymp \left(\int_{\widehat{\mathcal{G}}}\sum_{j\in J}\left(\sum_{i\in I}\left|\lambda_{ij}\right|^p\int_{\mathcal{G}}m(x,\xi)^p\chi_{x_i+V_{\mathcal{G}}}(x)\,\mathrm{d}x\right)^{\frac{q}{p}}\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\,\mathrm{d}\xi\right)^{\frac{1}{q}}$$

$$= \left(\sum_{j\in J}\int_{V_{\widehat{\mathcal{G}}}}\left(\sum_{i\in I}\left|\lambda_{ij}\right|^p\int_{V_{\mathcal{G}}}m(x+x_i,\xi+\xi_j)^p\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}\xi\right)^{\frac{1}{q}}.$$

The monotone convergence theorem justifies the interchanges of integration with summation performed. From [65, Corollary 2.2.23] we have

$$\left(\sup_{\overline{V}_{1,2}\cup-\overline{V}_{1,2}}v\right)^{-1}m((x,\xi)+(u,\omega))\leq m(u,\omega)\leq \left(\sup_{\overline{V}_{1,2}\cup-\overline{V}_{1,2}}v\right)m((x,\xi)+(u,\omega)),\quad (67)$$

for every $(u,\omega)\in\mathcal{G}\times\widehat{\mathcal{G}}$ and $(x,\xi)\in\overline{V}_{1,2}$, with $V_{1,2}$ defined in (29). Therefore, if $\xi\in V_{\widehat{\mathcal{G}}}$, we have

$$\int_{V_{\mathcal{G}}}m(x+x_i,\xi+\xi_j)^p\,\mathrm{d}x \underset{v,\overline{V}_{1,2}}{\asymp} \int_{V_{\mathcal{G}}}m(x_i,\xi_j)^p\,\mathrm{d}x = m(x_i,\xi_j)^p\mathrm{d}x(V_{\mathcal{G}}).\qquad (68)$$

Using the equivalences above,

$$\left\|(\lambda_{ij})_{i,j}\right\|_{(L^{p,q}_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\boldsymbol{\mathfrak{X}},V)}$$

$$\asymp \left(\sum_{j\in J}\int_{V_{\widehat{\mathcal{G}}}}\left(\sum_{i\in I}\left|\lambda_{ij}\right|^p\int_{V_{\mathcal{G}}}m(x+x_i,\xi+\xi_j)^p\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}\xi\right)^{\frac{1}{q}}$$

$$\asymp \left(\sum_{j\in J}\left(\sum_{i\in I}\left|\lambda_{ij}\right|^p m(x_i,\xi_j)^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} = \left\|(\lambda_{ij})_{i,j}\right\|_{\ell^{p,q}_{m_{\mathbf{X}}}(I\times J)}.$$

Case $p = q = \infty$. For $(x, \xi) \in \mathcal{G} \times \widehat{\mathcal{G}}$, we define

$$\mathbf{I}_{(x,\xi)} = \{(i,j) \in I \times J \mid \chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi) \neq \varnothing\}.$$

Arguing as for (64) and (65), we have that there exists $N = N(\mathbf{X}, V_{1,2}) = C_{\mathbf{X},\overline{V}_{1,2}} \in \mathbb{N}$ (see (173)) such that $\#\mathbf{I}_{(x,\xi)} \leq N$, where $\mathbf{X} := X \times \Xi$ and $V_{1,2}$ as in (29). Using (67), for $\left(\lambda_{ij}\right)_{i\in I, j\in J} \in \mathbb{C}^{I\times J}$,

$$\sum_{i\in I, j\in J} \left|\lambda_{ij}\right| \chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi) m(x,\xi) = \sum_{(i,j)\in\mathbf{I}_{(x,\xi)}} \left|\lambda_{ij}\right| m(x_i + u_i(x), \xi_j + \omega_j(\xi))$$

$$\asymp \sum_{(i,j)\in\mathbf{I}_{(x,\xi)}} \left|\lambda_{ij}\right| m(x_i, \xi_j) \qquad (69)$$

$$= \sum_{i\in I, j\in J} \left|\lambda_{ij}\right| m(x_i, \xi_j) \chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi),$$

where $(u_i(x), \omega_j(\xi)) \in V_{1,2}$ for every $(i,j) \in \mathbf{I}_{(x,\xi)}$. Consider now $\left(\lambda_{ij}\right)_{i\in I, j\in J} \in \ell^\infty_{m_{\mathbf{X}}}(I \times J)$. Then

$$\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{(L^\infty_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{X},V)}$$

$$= \left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{(L^\infty_m(\mathcal{G}\times\widehat{\mathcal{G}}))_d(\mathbf{X},V_{1,2})}$$

$$= \left\|\sum_{i\in I, j\in J} \left|\lambda_{ij}\right| \chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi) m(x,\xi)\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})}$$

$$\asymp \left\|\sum_{i\in I, j\in J} \left|\lambda_{ij}\right| m(x_i, \xi_j) \chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi)\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})}$$

$$\leq \left\|\sum_{i\in I, j\in J} \sup_{l,s} \left|\lambda_{ls}\right| m(x_l, \xi_s) \chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi)\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})}$$

$$\leq \left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell^\infty_{m_{\mathbf{X}}}(I\times J)} \left\|N\chi_{\mathcal{G}\times\widehat{\mathcal{G}}}\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})}$$

$$= N \left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell^\infty_{m_{\mathbf{X}}}(I\times J)}.$$

Vice versa, if $\left(\lambda_{ij}\right)_{i\in I, j\in J} \in (L^\infty_{\tilde{m}}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{X}, V)$,

$$\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell^\infty_{m_{\mathbf{X}}}(I\times J)} = \sup_{i\in I, j\in J} \left|\lambda_{ij}\right| m_{\mathbf{X}}(i,j)$$

$$= \sup_{i,j\in J} \left|\lambda_{i,j}\right| \chi_{(x_i,\xi_j)+V_{1,2}}(x_i,\xi_j) m(x_i,\xi_j)$$

$$\leq \sup_{i\in I, j\in J} \left\|\left|\lambda_{ij}\right| \chi_{(x_i,\xi_j)+V_{1,2}}(x,\xi) m(x,\xi)\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})}$$

$$\leq \sup_{i\in I, j\in J} \left\|\sum_{l\in I, s\in J} \left|\lambda_{ls}\right| \chi_{(x_l,\xi_s)+V_{1,2}}(x,\xi) m(x,\xi)\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})}$$

$$= \left\|\sum_{l\in I, s\in J} \left|\lambda_{ls}\right| \chi_{(x_l,\xi_s)+V_{1,2}}(x,\xi) m(x,\xi)\right\|_{L^\infty(\mathcal{G}\times\widehat{\mathcal{G}})}$$

$$= \left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{(L^\infty_{\tilde{m}}(\mathbb{H}_g))_d(\mathbf{X},V)}.$$

Case $p = \infty$ and $q < \infty$. We show the equivalence

$$\operatorname{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I, j\in J} \left|\lambda_{ij}\right| m_{\mathbf{X}}(i,j) \chi_{x_i+V_{\mathcal{G}}}(x) \chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)$$

$$\underset{\Xi,V_{\widehat{\mathcal{G}}}}{\asymp} \sum_{j\in J} \operatorname{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I} \left|\lambda_{ij}\right| m_{\mathbf{X}}(i,j) \chi_{x_i+V_{\mathcal{G}}}(x) \chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi). \tag{70}$$

In fact, arguing as in (64) and (65), for $\xi \in \widehat{\mathcal{G}}$ fixed and

$$J_\xi := \{j\in J \mid \chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi) \neq \varnothing\},$$

there exists $M = M(\Xi, V_{\widehat{\mathcal{G}}}) \in \mathbb{N}$ such that $\#J_\xi \leq M$. Therefore,

$$\operatorname{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I, j\in J} \left|\lambda_{ij}\right| m_{\mathbf{X}}(i,j) \chi_{x_i+V_{\mathcal{G}}}(x) \chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)$$

$$= \operatorname{ess\,sup}_{x\in\mathcal{G}} \sum_{j\in J_\xi} \sum_{i\in I} \left|\lambda_{ij}\right| \chi_{x_i+V_{\mathcal{G}}}(x) m_{\mathbf{X}}(i,j)$$

$$\leq \sum_{j\in J_\xi} \operatorname{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I} \left|\lambda_{ij}\right| \chi_{x_i+V_{\mathcal{G}}}(x) m_{\mathbf{X}}(i,j)$$

$$= \sum_{j\in J} \operatorname{ess\,sup}_{x\in\mathcal{G}} \sum_{i\in I} \left|\lambda_{ij}\right| \chi_{x_i+V_{\mathcal{G}}}(x) m_{\mathbf{X}}(i,j) \chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi).$$

On the other hand,

$$\sum_{j \in J} \text{ess sup}_{x \in \mathcal{G}} \sum_{i \in I} \left| \lambda_{ij} \right| \chi_{x_i + V_\mathcal{G}}(x) m_{\mathbf{X}}(i,j) \chi_{\xi_j + V_{\widehat{\mathcal{G}}}}(\xi)$$

$$= \sum_{j \in J_\xi} \text{ess sup}_{x \in \mathcal{G}} \sum_{i \in I} \left| \lambda_{ij} \right| \chi_{x_i + V_\mathcal{G}}(x) m_{\mathbf{X}}(i,j)$$

$$\leq M \max \left\{ \text{ess sup}_{x \in \mathcal{G}} \sum_{i \in I} \left| \lambda_{ij} \right| \chi_{x_i + V_\mathcal{G}}(x) m_{\mathbf{X}}(i,j) \mid j \in J_\xi \right\}$$

$$\leq M \text{ess sup}_{x \in \mathcal{G}} \sum_{j \in J_\xi} \sum_{i \in I} \left| \lambda_{ij} \right| \chi_{x_i + V_\mathcal{G}}(x) m_{\mathbf{X}}(i,j)$$

$$= M \text{ess sup}_{x \in \mathcal{G}} \sum_{j \in J} \sum_{i \in I} \left| \lambda_{ij} \right| \chi_{x_i + V_\mathcal{G}}(x) m_{\mathbf{X}}(i,j) \chi_{\xi_j + V_{\widehat{\mathcal{G}}}}(\xi).$$

Finally, using the previous cases, the equivalences in (69) and (70), we can write

$$\left\| (\lambda_{ij})_{i,j} \right\|_{(L_{\tilde{m}}^{\infty,q}(\mathbb{H}_\mathcal{G}))_d(\mathbf{\mathcal{X}},V)}$$

$$= \left(\int_{\widehat{\mathcal{G}}} \left(\text{ess sup}_{x \in \mathcal{G}} \sum_{i \in I, j \in J} \left| \lambda_{ij} \right| \chi_{x_i + V_\mathcal{G}}(x) \chi_{\xi_j + V_{\widehat{\mathcal{G}}}}(\xi) m(x,\xi) \right)^q d\xi \right)^{\frac{1}{q}}$$

$$\asymp \left(\int_{\widehat{\mathcal{G}}} \left(\sum_{j \in J} \text{ess sup}_{x \in \mathcal{G}} \sum_{i \in I} \left| \lambda_{ij} \right| m_{\mathbf{X}}(i,j) \chi_{x_i + V_\mathcal{G}}(x) \chi_{\xi_j + V_{\widehat{\mathcal{G}}}}(\xi) \right)^q d\xi \right)^{\frac{1}{q}}$$

$$= \left(\int_{\widehat{\mathcal{G}}} \left(\sum_{j \in J} \left\| \sum_{i \in I} \left| \lambda_{ij} \right| m_{\mathbf{X}}(i,j) \chi_{x_i + V_\mathcal{G}}(\cdot) \right\|_{L^\infty(\mathcal{G})} \chi_{\xi_j + V_{\widehat{\mathcal{G}}}}(\xi) \right)^q d\xi \right)^{\frac{1}{q}}$$

$$= \left(\int_{\widehat{\mathcal{G}}} \left(\sum_{j \in J} \left\| (\lambda_{ij} m_{\mathbf{X}}(i,j))_{i \in I} \right\|_{(L^\infty(\mathcal{G}))_d(X,V_\mathcal{G})} \chi_{\xi_j + V_{\widehat{\mathcal{G}}}}(\xi) \right)^q d\xi \right)^{\frac{1}{q}}$$

so that

$$\left\| (\lambda_{ij})_{i,j} \right\|_{(L_{\tilde{m}}^{\infty,q}(\mathbb{H}_\mathcal{G}))_d(\mathbf{\mathcal{X}},V)}$$

$$\asymp \left(\int_{\widehat{\mathcal{G}}} \left(\sum_{j \in J} \left\| (\lambda_{ij} m_{\mathbf{X}}(i,j))_{i \in I} \right\|_{\ell^\infty(I)} \chi_{\xi_j + V_{\widehat{\mathcal{G}}}}(\xi) \right)^q d\xi \right)^{\frac{1}{q}}$$

$$= \left\| \left(\left\| (\lambda_{ij} m_{\mathbf{X}}(i,j))_{i \in I} \right\|_{\ell^\infty(I)} \right)_{j \in J} \right\|_{(L^q(\widehat{\mathcal{G}}))_d(\Xi, V_{\widehat{\mathcal{G}}})}$$

$$\asymp \left\| \left(\left\| (\lambda_{ij} m_{\mathbf{X}}(i,j))_{i \in I} \right\|_{\ell^\infty(I)} \right)_{j \in J} \right\|_{\ell^q(J)} = \left\| (\lambda_{ij})_{i,j} \right\|_{\ell_{m_{\mathbf{X}}}^{\infty,q}(I \times J)}.$$

Case $p < \infty$ and $q = \infty$. Similarly to what has been done before,

$$\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{(L_{\tilde{m}}^{p,\infty}(\mathbb{H}_{\mathcal{G}}))_d(\boldsymbol{\mathcal{X}},V)}$$

$$= \left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{(L_m^{p,\infty}(\mathcal{G}\times\widehat{\mathcal{G}}))_d(\mathbf{X},V_{1,2})}$$

$$= \operatorname{ess\,sup}_{\xi\in\widehat{\mathcal{G}}}\left(\int_{\mathcal{G}}\left(\sum_{i\in I,j\in J}\left|\lambda_{ij}\right|\chi_{x_i+V_{\mathcal{G}}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\right)^p m(x,\xi)^p \mathrm{d}x\right)^{\frac{1}{p}}$$

$$\asymp \operatorname{ess\,sup}_{\xi\in\widehat{\mathcal{G}}}\left(\int_{\mathcal{G}}\sum_{i\in I,j\in J}\left|\lambda_{ij}\right|^p \chi_{x_i+V_{\mathcal{G}}}(x)\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)m(x,\xi)^p \mathrm{d}x\right)^{\frac{1}{p}}$$

$$\asymp \operatorname{ess\,sup}_{\xi\in\widehat{\mathcal{G}}}\left(\sum_{i\in I,j\in J}\left|\lambda_{ij}\right|^p m(x_i,\xi_j)^p \chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)\right)^{\frac{1}{p}}$$

$$\asymp \operatorname{ess\,sup}_{\xi\in\widehat{\mathcal{G}}}\sum_{j\in J}\left(\sum_{i\in I}\left|\lambda_{ij}\right|^p m(x_i,\xi_j)^p\right)^{\frac{1}{p}}\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)$$

$$= \operatorname{ess\,sup}_{\xi\in\widehat{\mathcal{G}}}\sum_{j\in J}\left\|\left(\lambda_{ij}m_{\mathbf{X}}(i,j)\right)_{i\in I}\right\|_{\ell^p(I)}\chi_{\xi_j+V_{\widehat{\mathcal{G}}}}(\xi)$$

$$= \left\|\left(\left\|\left(\lambda_{ij}m_{\mathbf{X}}(i,j)\right)_{i\in I}\right\|_{\ell^p(I)}\right)_{j\in J}\right\|_{(L^{\infty}(\widehat{\mathcal{G}}))_d(\Xi,V_{\widehat{\mathcal{G}}})}$$

$$\asymp \left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell_{m_{\mathbf{X}}}^{p,\infty}(I\times J)}.$$

The proof is concluded. □

Remark 3.27 We want to state explicitly the equivalence constants involved in the previous lemma. We distinguish four cases, as done in the proof.

Case $p,q < \infty$. We have

$$A^{-1}B\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell_{m_{\mathbf{X}}}^{p,q}(I\times J)} \leq \left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{(L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathcal{G}}))_d(\boldsymbol{\mathcal{X}},V)} \leq AB\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell_{m_{\mathbf{X}}}^{p,q}(I\times J)},$$

where

$$A := A(X, \Xi, V_{\mathcal{G}}, V_{\widehat{\mathcal{G}}}, v, p) := C_{X,\overline{V}_{\mathcal{G}}}C_{\Xi,\overline{V}_{\widehat{\mathcal{G}}}}^{\frac{1}{p}+1}\left(\sup_{\overline{V}_{1,2}\cup-\overline{V}_{1,2}} v\right),$$

$$B := B(V_{\mathcal{G}}, V_{\widehat{\mathcal{G}}}, p, q) := \mathrm{d}x(V_{\mathcal{G}})^{\frac{1}{p}}\,\mathrm{d}\xi(V_{\widehat{\mathcal{G}}})^{\frac{1}{q}}.$$

Case $p = q = \infty$. The equivalence is

$$\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell^\infty_{m_{\mathbf{X}}}(I\times J)} \le \left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{(L^\infty_{\tilde{m}}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)}$$

$$\le \left(\sup_{\overline{V}_{1,2}\cup -\overline{V}_{1,2}} v\right) C_{\mathbf{X},\overline{V}_{1,2}}\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell^\infty_{m_{\mathbf{X}}}(I\times J)}.$$

Case $p = \infty$ and $q < \infty$. We got

$$D\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell^{\infty,q}_{m_{\mathbf{X}}}(I\times J)} \le \left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{(L^{\infty,q}_{\tilde{m}}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)} \le E\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell^{\infty,q}_{m_{\mathbf{X}}}(I\times J)},$$

where

$$D:=D(\Xi, V_\mathcal{G}, V_{\widehat{\mathcal{G}}}, v, q):=C^{-2}_{\Xi,\overline{V}_{\widehat{\mathcal{G}}}}\left(\sup_{\overline{V}_{1,2}\cup -\overline{V}_{1,2}} v\right)^{-1} \mathrm{d}\xi(V_{\widehat{\mathcal{G}}})^{\frac{1}{q}},$$

$$E:=B(X, \Xi, V_\mathcal{G}, V_{\widehat{\mathcal{G}}}, v, q):=C_{X,\overline{V}_\mathcal{G}} C_{\Xi,\overline{V}_{\widehat{\mathcal{G}}}}\left(\sup_{\overline{V}_{1,2}\cup -\overline{V}_{1,2}} v\right) \mathrm{d}\xi(V_{\widehat{\mathcal{G}}})^{\frac{1}{q}}.$$

Case $p < \infty$ and $q = \infty$. The last equivalence is given by

$$L\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell^{p,\infty}_{m_{\mathbf{X}}}(I\times J)} \le \left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{(L^{p,\infty}_{\tilde{m}}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)} \le M\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell^{p,\infty}_{m_{\mathbf{X}}}(I\times J)},$$

where

$$L:=L(X, \Xi, V_\mathcal{G}, V_{\widehat{\mathcal{G}}}, v, p):=C^{-1}_{X,\overline{V}_\mathcal{G}} C^{-1}_{\Xi,\overline{V}_{\widehat{\mathcal{G}}}}\left(\sup_{\overline{V}_{1,2}\cup -\overline{V}_{1,2}} v\right)^{-1} \mathrm{d}x(V_\mathcal{G})^{\frac{1}{p}},$$

$$M:=M(X, \Xi, V_\mathcal{G}, V_{\widehat{\mathcal{G}}}, v, p):=C_{X,\overline{V}_\mathcal{G}} C^2_{\Xi,\overline{V}_{\widehat{\mathcal{G}}}}\left(\sup_{\overline{V}_{1,2}\cup -\overline{V}_{1,2}} v\right) \mathrm{d}x(V_\mathcal{G})^{\frac{1}{p}}.$$

We recall that the definition of the constants $C_{X,\overline{V}_\mathcal{G}}, C_{\Xi,\overline{V}_{\widehat{\mathcal{G}}}}, C_{\mathbf{X},\overline{V}_{1,2}}$ is given in (173).

On account of the constants shown in the previous remark, we have the following corollary.

Corollary 3.28 *Fix $0 < \delta \le \infty$ and take p, q such that $0 < \delta \le p, q \le \infty$. Under the same assumptions of Lemma 3.26, there are two constants*

$$C_1:=C_1(X, \Xi, V_\mathcal{G}, V_{\widehat{\mathcal{G}}}, v, \delta) > 0 \quad \text{and} \quad C_1:=C_1(X, \Xi, V_\mathcal{G}, V_{\widehat{\mathcal{G}}}, v, \delta) > 0$$

such that

$$C_1\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell^{p,q}_{m_{\mathbf{X}}}(I\times J)} \le \left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{(L^{p,q}_{\tilde{m}}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X},V)} \le C_2\left\|\left(\lambda_{ij}\right)_{i,j}\right\|_{\ell^{p,q}_{m_{\mathbf{X}}}(I\times J)}$$

for every sequence $\left(\lambda_{ij}\right)_{i,j}$ in $(L^{p,q}_{\tilde{m}}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X}, V) = \ell^{p,q}_{m_{\mathbf{X}}}(I\times J)$.

Proof We notice that if $b \geq 1$, then $b^{\frac{1}{p}}$ is a strictly decreasing function of $p \in (0, \infty)$ and $b^{\frac{1}{p}} \geq 1$. Likewise $b^{-\frac{1}{p}}$ is strictly increasing and $0 < b^{-\frac{1}{p}} \leq 1$. The claim follows now from Remark 3.27. □

Remark 3.29 Although in Lemma 3.26 we considered $V = V_{\mathcal{G}} \times V_{\widehat{\mathcal{G}}} \times \mathbb{T}$ with $V_{\mathcal{G}}$ and $V_{\widehat{\mathcal{G}}}$ open, this last assumption can be relaxed into measurability. Even in this case the above lemma and the subsequent Corollary 3.30 hold true.

Corollary 3.30 *Consider* $0 < p_1 \leq p_2 \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$ *and* $m_1, m_2 \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ *such that* $m_2 \lesssim m_1$. *Let* V, X, Ξ *and* $\mathfrak{X}$ *be as in Lemma* 3.26. *Then,*

$$\left(L^{p_1,q_1}_{\tilde{m}_1}(\mathbb{H}_{\mathcal{G}}) \right)_d (\mathfrak{X}, V) \hookrightarrow \left(L^{p_2,q_2}_{\tilde{m}_2}(\mathbb{H}_{\mathcal{G}}) \right)_d (\mathfrak{X}, V). \tag{71}$$

Proof It is a straightforward consequence of Lemma 3.26 and the continuous inclusions

$$\ell^{p_1,q_1}_{m_{1,\mathbf{X}}}(I \times J) \hookrightarrow \ell^{p_2,q_2}_{m_{2,\mathbf{X}}}(I \times J), \tag{72}$$

since $m_{2,\mathbf{X}} \lesssim m_{1,\mathbf{X}}$. □

Proposition 3.31 *Consider* $0 < p_1 \leq p_2 \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$ *and* $m_1, m_2 \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ *such that* $m_2 \lesssim m_1$. *Then we have the following continuous inclusions*:

$$M^{p_1,q_1}_{m_1}(\mathcal{G}) \hookrightarrow M^{p_1,q_2}_{m_2}(\mathcal{G}). \tag{73}$$

Proof Under the hypothesis of Lemma 3.25, it is always possible to find a BUPU on $\mathbb{H}_{\mathcal{G}}$ of the type (61), see [65, Lemma 2.3.12]. For such a BUPU

$$\Psi \otimes \Gamma \otimes \mathbb{I} = \{\psi_i \otimes \gamma_j \otimes \chi_{\mathbb{T}}, (i,j) \in I \times J\},$$

the corresponding localizing family $\mathfrak{X} = X \times \Xi \times \{1\}$ fulfils the requirements of Corollary 3.30. To get the desired result we use the equivalence of quasi-norms shown in (181):

$$\|f\|_{M^{p_2,q_2}_{m_2}} \asymp \left\| W^o_g f \right\|_{W(L^{p_2,q_2}_{\tilde{m}_2})}$$

$$\asymp \left\| \left(\left\| (\psi_i \otimes \gamma_j \otimes \chi_{\mathbb{T}}) \cdot W^o_g f \right\|_{L^\infty} \right)_{i,j} \right\|_{(L^{p_2,q_2}_{\tilde{m}_2}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{X},V)}$$

$$\lesssim \left\| \left(\left\| (\psi_i \otimes \gamma_j \otimes \chi_{\mathbb{T}}) \cdot W^o_g f \right\|_{L^\infty} \right)_{i,j} \right\|_{(L^{p_1,q_1}_{\tilde{m}_1}(\mathbb{H}_{\mathcal{G}}))_d(\mathfrak{X},V)}$$

$$\asymp \left\| W^o_g f \right\|_{W(L^{p_1,q_1}_{\tilde{m}_1})} \asymp \|f\|_{M^{p_1,q_1}_{m_1}}.$$

This concludes the proof. □

If $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$, from the submultiplicativity and symmetry of v we have $1/m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$. This remark is implicitly used in the following issue.

Proposition 3.32 *If* $1 \leq p, q < \infty$, *then* $\left(M_m^{p,q}(\mathcal{G})\right)' = M_{1/m}^{p',q'}(\mathcal{G})$ *under the duality*

$$\langle f, h \rangle = \langle V_g f, V_g h \rangle_{L^2(\mathcal{G} \times \widehat{\mathcal{G}})}, \tag{74}$$

for all $f \in M_m^{p,q}(\mathcal{G}), h \in M_{1/m}^{p',q'}(\mathcal{G})$ *and some* $g \in \mathcal{S}_\mathcal{C}(\mathcal{G}) \smallsetminus \{0\}$.

Proof For $1 \leq p, q \leq \infty$, $L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})$ is a solid bi-invariant Banach function space continuously embedded into $L_{\mathrm{loc}}^1(\mathbb{H}_\mathcal{G})$. Therefore, from Theorem A.11 combined with Remark 3.9, we have

$$M_m^{p,q}(\mathcal{G}) = \mathrm{Co}\,(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})) = \mathrm{Co}_{FG}\,(L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G})) = \{f \in \mathcal{R}_{\tilde{v}} \mid V_g f \in L_m^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}})\} \tag{75}$$

with

$$\left\| V_g f \right\|_{W(L_m^{p,q})} \asymp \left\| V_g f \right\|_{L_m^{p,q}}. \tag{76}$$

The proof then goes as in [33, Theorem 11.3.6], after noticing that we can identify $(L_m^1)'$ with $L_{1/m}^\infty$ since under our assumptions $\mathcal{G} \times \widehat{\mathcal{G}}$ is σ-finite, similarly for mixed-norm cases. $\qquad\square$

Theorem 3.33

(i) *If* $0 < p, q < \infty$, *then* $\mathcal{S}_\mathcal{C}(\mathcal{G})$ *is quasi-norm-dense in* $M_m^{p,q}(\mathcal{G})$.

(ii) *If* $1 \leq p, q \leq \infty$ *and at least one between* p *and* q *is equal to* ∞, *then* $\mathcal{S}_\mathcal{C}(\mathcal{G})$ *is w-∗-dense in* $M_m^{p,q}(\mathcal{G})$.

Proof For any $0 \leq p, q \leq \infty$, $\mathcal{S}_\mathcal{C}(\mathcal{G})$ is a subspace of $M_m^{p,q}(\mathcal{G})$, cf. the computations in the proof of Lemma 3.10 and the inclusions in (73).

(i) Let φ be as in (12) and consider the relatively compact unit neighborhood U_0 coming from Theorem A.9. Without loss of generality we can assume $U_0 = V_\mathcal{G} \times V_{\widehat{\mathcal{G}}} \times \mathbb{T} = V$ as in (27), see the proofs of [65, Theorem 2.4.19] and [65, Lemma 2.4.17]. Then, there exists a U_0-BUPU with localizing family $\mathfrak{X} = \{(x_i, \xi_j, 1)\}_{(i,j) \in I \times J}$ such that any $f \in M_m^{p,q}(\mathcal{G})$ can be written as

$$f = \sum_{i \in I, j \in J} \lambda_{ij}(f) \varrho(x_i, \xi_j, 1)\varphi = \sum_{i \in I, j \in J} \lambda_{ij}(f) \pi(x_i, \xi_j)\varphi, \tag{77}$$

with unconditional convergence in $M_m^{p,q}(\mathcal{G})$ since the finite sequences are dense in $\ell_{m_\mathbf{x}}^{p,q}(I \times J) = (L_{\tilde{m}}^{p,q}(\mathbb{H}_\mathcal{G}))_d(\mathfrak{X}, V)$, $p, q < \infty$.

(ii) We show the case $p = q = \infty$, the remaining ones are analogous. From Proposition 3.32, $M_m^\infty(\mathcal{G})$ can be seen as the dual of $M_{1/m}^1(\mathcal{G})$. Therefore, with φ the Gaussian in (12),

$$^{\perp}\mathcal{S}_C(\mathcal{G}) = \{f \in M_{1/m}^1 \mid \langle V_\varphi f, V_\varphi h \rangle = 0, \quad \forall\, h \in \mathcal{S}_C(\mathcal{G})\}. \tag{78}$$

For fixed $(u, \omega) \in \mathcal{G} \times \widehat{\mathcal{G}}$ consider $h = \pi(u, \omega)\varphi \in \mathcal{S}_C(\mathcal{G})$. From Lemma 2.1

$$V_\varphi h(x, \xi) = \overline{\langle \xi - \omega, u \rangle} T_{(u,\omega)} V_\varphi \varphi(x, \xi). \tag{79}$$

In particular, from (17), $V_\varphi h(u, \omega) \neq 0$ and it is continuous. Therefore, if $f \in {}^{\perp}\mathcal{S}_C(\mathcal{G})$

$$\langle V_\varphi f, V_\varphi h \rangle = \int_{\mathcal{G} \times \widehat{\mathcal{G}}} V_\varphi f(x, \xi) \overline{V_\varphi h(x, \xi)} \, \mathrm{d}x \mathrm{d}\xi = 0 \quad \Rightarrow \quad V_\varphi f \overline{V_\varphi h} = 0 \text{ a.e.,}$$

but since $V_\varphi f \overline{V_\varphi h}$ is continuous this implies $V_\varphi f(x, \xi) \overline{V_\varphi h(x, \xi)} = 0$ for every $(x, \xi) \in \mathcal{G} \times \widehat{\mathcal{G}}$. Necessarily $V_\varphi f$ vanishes on a neighborhood of (u, ω). On account of the arbitrariness of the point $(u, \omega) \in \mathcal{G} \times \widehat{\mathcal{G}}$, we have $V_\varphi f \equiv 0$ which also means $W_\varphi^o f \equiv 0$. Since the application

$$W_\varphi^o : \mathcal{R}_{\tilde{v}} \to C(\mathbb{H}_{\mathcal{G}}) \cap L_{1/\tilde{v}}^\infty(\mathbb{H}_{\mathcal{G}})$$

is injective, see [65, Lemma 2.4.8], we infer $f = 0$. Therefore, $^{\perp}\mathcal{S}_C(\mathcal{G}) = \{0\}$ and

$$\overline{\mathcal{S}_C(\mathcal{G})}^{w-*} = \left(^{\perp}\mathcal{S}_C(\mathcal{G})\right)^{\perp} = (\{0\})^{\perp} = M_m^\infty(\mathcal{G}).$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Lemma 3.34 *For every*, $0 < p, q \leq \infty$ *and* $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$

$$\mathscr{A}_{\tilde{v}}(\mathcal{G}) \subseteq M_m^{p,q}(\mathcal{G}).$$

Proof We just need to show that for every $0 < r \leq 1$ the inclusion

$$\mathscr{A}_{\tilde{v}}(\mathcal{G}) \subseteq M_v^r(\mathcal{G}) \tag{80}$$

holds true, then the claim follows from the inclusion relations for modulation spaces. From (40) and the inclusion relations in [65, p. 113], if $g \in \mathscr{A}_{\tilde{v}} \subseteq \mathbb{A}_{\tilde{v}}^r$ and φ is the Gaussian as in (12), we get that

$$W_\varphi^o g \in W^R(L^\infty, W(L^\infty, L_{\tilde{v}}^r)) \hookrightarrow W(L^\infty, L_{\tilde{v}}^r).$$

Hence $g \in M_v^r(\mathcal{G})$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Corollary 3.35 *If* $0 < p, q < \infty$, *then* $\mathscr{A}_{\tilde{v}}$ *is quasi-norm-dense in* $M_m^{p,q}(\mathcal{G})$.

Proof The claim follows from the above theorem, the previous lemma and the inclusion $\mathcal{S}_C \subseteq \mathscr{A}_{\tilde{v}}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Corollary 3.36 *For every* $f \in S_0'(\mathcal{G})$ *there exists a net* $(f_\alpha)_{\alpha \in A} \subseteq \mathcal{S}_C(\mathcal{G})$ *such that*

$$\lim_{\alpha \in A} \langle f_\alpha, h \rangle_{L^2(\mathcal{G})} = {}_{S_0'}\langle f, h \rangle_{S_0}, \quad \forall\, h \in S_0(\mathcal{G}). \tag{81}$$

Proof From Lemma 3.13 we have that $\mathcal{S}_C(\mathcal{G})$ is norm-dense in $\mathcal{T}_1 = S_0(\mathcal{G})$. From [41, Proposition 6.15] there exists a bounded net $(f_\beta)_{\beta \in B} \subseteq S_0(\mathcal{G})$ such that

$$\lim_{\beta \in B} \langle f_\beta, h \rangle_{L^2(\mathcal{G})} = {}_{S_0'}\langle f, h \rangle_{S_0}, \qquad \forall\, h \in S_0(\mathcal{G}). \tag{82}$$

This concludes the proof. $\qquad\qquad\square$

Remark 3.37 From Theorem 3.33 and relations (75) and (76) it follows that the modulation spaces introduced in Definition 3.19 coincide with the classical ones in [22, 37]. This implies that

$$M_m^1(\mathcal{G}) \cong \left(\mathrm{clos}_{M_{1/m}^\infty} \left(\mathcal{S}_C(\mathcal{G}) \right) \right)', \tag{83}$$

the dual of the closure of $\mathcal{S}_C(\mathcal{G})$ with respect to the norm on $M_{1/m}^\infty(\mathcal{G})$. If $f \in M_m^{\infty,1}(\mathcal{G})$ and $g \in M_{1/m}^{1,\infty}(\mathcal{G})$, then for φ as in (12)

$$\left| \langle V_\varphi f, V_\varphi g \rangle_{L^2(\mathcal{G} \times \widehat{\mathcal{G}})} \right| \lesssim \|f\|_{M_m^{\infty,1}} \|g\|_{M_{1/m}^{1,\infty}}. \tag{84}$$

See [37, Proposition 2.2].

(ii) The theory for $\mathcal{G} = \mathbb{R}^d$ developed in [30] is recovered for every $0 < p, q \leq \infty$. In fact, it was observed in [49, Section 8] that from [30, Lemma 3.2] follows the equality

$$\mathrm{Co}\,(L_{\tilde{m}}^{p,q}(\mathbb{H}_{\mathbb{R}^d})) = \{ f \in \mathcal{S}' \mid V_g f \in L_m^{p,q}(\mathbb{R}^{2d}) \} \qquad 0 < p, q \leq \infty,$$

with equivalent quasi-norms.

For a general LCA group $\mathcal{G}$, it is an open problem whether a construction of the type

$$\{ f \in \mathcal{R}_{\tilde{v}} \mid V_g f \in L_m^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}}) \},$$

with obvious quasi-norm, could make sense or not when at least one between p and q is smaller than 1. However, we are able to answer affirmatively if $\mathcal{G}$ is discrete or compact, see the lemma and corollary below.

Lemma 3.38 *Let* $0 < p, q \leq \infty$. *Suppose* $\mathcal{G}$ *is discrete or compact. Then there exists* $C > 0$ *such that for every* $f \in M_m^{p,q}(\mathcal{G})$

$$\left\| W_g^o f \right\|_{W(L_{\tilde{m}}^{p,q})} \leq C \left\| W_g^o f \right\|_{L_{\tilde{m}}^{p,q}}, \tag{85}$$

for some $g \in \mathcal{S}_C(\mathcal{G}) \smallsetminus \{0\}$.

Proof If we prove for some suitable unit neighborhood $Q \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ that there exists $C > 0$ such that

$$\left\|V_g f\right\|_{W_Q(L_m^{p,q})} \leq C \left\|V_g f\right\|_{L_m^{p,q}}, \tag{86}$$

then (85) holds true, see Remark 3.9. Moreover, as shown in Proposition 3.18, we can consider the modulation spaces as subsets of $S_0'(\mathcal{G})$.

Case $\mathcal{G}$ discrete. $\widehat{\mathcal{G}}$ is compact and the structure theorem reads as $\mathcal{G} = \mathcal{G}_0$ and $\widehat{\mathcal{G}} = \widehat{\mathcal{G}}_0$. In the definition of the Gaussian function (12) we take, Remark 3.16, the open and compact subgroup $\mathcal{K} = \{e\}$, therefore

$$\varphi(x) := \chi_{\{e\}}(x) =: \delta_e(x).$$

We also choose $Q := \{e\} \times \widehat{\mathcal{G}}$, which is a measurable, relatively compact, unit neighborhood. Fix $f \in M_m^{p,q}(\mathcal{G})$, from [41, Proposition 6.15], we have that there exists a bounded net $(f_\alpha)_{\alpha \in A} \subseteq S_0(\mathcal{G})$ such that

$$\lim_{\alpha \in A} \langle f_\alpha, h \rangle_{L^2(\mathcal{G})} = {}_{S_0'} \langle f, h \rangle_{S_0}, \qquad \forall\, h \in S_0(\mathcal{G}). \tag{87}$$

Recall that $\mathcal{S}_\mathcal{C}(\mathcal{G}) \subseteq S_0(\mathcal{G})$, then adopting the widow function φ, we compute

$$V_\varphi f(x,\xi) = \langle f, \pi(x,\xi)\delta_e \rangle = \lim_{\alpha \in A} \langle f_\alpha, \pi(x,\xi)\delta_e \rangle = \lim_{\alpha \in A} \sum_{u \in \mathcal{G}} f_\alpha(u) \overline{\langle \xi, u \rangle \delta_x(u)}$$

$$= \lim_{\alpha \in A} f_\alpha(x) \overline{\langle \xi, x \rangle} = \overline{\langle \xi, x \rangle} \lim_{\alpha \in A} f_\alpha(x),$$

$$\mathsf{M}_Q V_\varphi f(x,\xi) = \operatorname*{ess\,sup}_{(y,\eta) \in (x,\xi) + \{e\} \times \widehat{\mathcal{G}}} \left| \langle \eta, y \rangle \lim_{\alpha \in A} f_\alpha(y) \right| = \operatorname*{ess\,sup}_{(y,\eta) \in \{x\} \times \widehat{\mathcal{G}}} \left| \lim_{\alpha \in A} f_\alpha(y) \right|$$

$$= \left| \lim_{\alpha \in A} f_\alpha(x) \right| = \left| V_\varphi f(x,\xi) \right|.$$

Therefore,

$$\left\|V_\varphi f\right\|_{W_Q(L_m^{p,q})} = \left\|\mathsf{M}_Q V_\varphi f\right\|_{L_m^{p,q}} = \left\|V_\varphi f\right\|_{L_m^{p,q}}.$$

Case $\mathcal{G}$ compact.

The argument is identical to the previous one, take $\mathcal{K} = \mathcal{G}$ and $Q := \mathcal{G} \times \{\hat{e}\}$.   □

Corollary 3.39 *Suppose $\mathcal{G}$ is discrete or compact. Consider $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ and $0 < p, q \leq \infty$. Then,*

$$M_m^{p,q}(\mathcal{G}) = \{f \in S_0'(\mathcal{G}) \mid V_g f \in L_m^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}})\}$$

and

$$\|f\|_{M_m^{p,q}} \asymp \left\|V_g f\right\|_{L_m^{p,q}}, \tag{88}$$

for some $g \in \mathcal{S}_\mathcal{C}(\mathcal{G}) \smallsetminus \{0\}$.

Proof We consider $M_m^{p,q}(\mathcal{G})$ as a subspace of $S_0'(\mathcal{G})$ instead of $\mathcal{R}_{\tilde{v}}$ (Proposition 3.18). The claim then follows from the continuous embedding $W(L_{\tilde{m}}^{p,q}) \hookrightarrow L_{\tilde{m}}^{p,q}$, Lemma 3.38 and Remark 3.9. $\qquad\square$

4 Continuity of the Rihaczek and Kohn–Nirenberg operators

In this section, we investigate the continuity of the Rihaczek distribution (7) on modulation spaces and infer boundedness results for the Kohn–Nirenberg operators, defined in (8).

4.1 Boundedness results

We first study the boundedness of the Rihaczek distribution on modulation spaces. The techniques are mainly borrowed from [12, Theorem 3.1] and [11, Theorem 4] for the Wigner distribution on $\mathbb{R}^d$.

From now on, we shall mainly work with $S_0(\mathcal{G})$ and $S_0'(\mathcal{G})$ instead of $\mathcal{T}_{\tilde{v}}$ and $\mathcal{R}_{\tilde{v}}$ (Proposition 3.18). Preliminary, we exhibit a proof for Young's inequality in $L^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}})$ and some generalizations. This result is folklore, but no explicit proof is available according to authors' knowledge.

Proposition 4.1 *Consider* $1 \le p_i, q_i, r_i \le \infty, i = 1, 2,$ *such that*

$$\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i}, \quad i = 1, 2. \tag{89}$$

If $F \in L^{p_1,p_2}(\mathcal{G} \times \widehat{\mathcal{G}})$ *and* $H \in L^{q_1,q_2}(\mathcal{G} \times \widehat{\mathcal{G}})$, *then* $F * H \in L^{r_1,r_2}(\mathcal{G} \times \widehat{\mathcal{G}})$ *with*

$$\|F * H\|_{L^{r_1,r_2}} \le \|F\|_{L^{p_1,p_2}} \|H\|_{L^{q_1,q_2}}. \tag{90}$$

Proof We follow the pattern of [5, Part II, Theorem 1, b)]. It suffices to prove the claim for $F, H \ge 0$. Given a measurable function $W : \mathcal{G} \times \widehat{\mathcal{G}} \to \mathbb{C}$ and $1 \le s \le \infty$, we define the (measurable) function on $\widehat{\mathcal{G}}$

$$\|W\|_{(s)}(\xi) := \begin{cases} \left(\int_{\mathcal{G}} |W(x,\xi)|^s \, dx \right)^{\frac{1}{s}} & \text{if } s < \infty, \\ \text{ess sup}_{x \in \mathcal{G}} |W(x,\xi)| & \text{if } s = \infty. \end{cases} \tag{91}$$

We show the case $r_1 < \infty$, the case $r_1 = \infty$ is done similarly. In the following we shall use Minkowski's integral inequality (see [57, Appendix A.1]):

$$\|F * H\|_{(r_1)}(\xi) = \left(\int_{\mathcal{G}} \left[\int_{\mathcal{G} \times \widehat{\mathcal{G}}} F((x, \xi) - (u, \omega)) H(u, \omega) \, du d\omega \right]^{r_1} dx \right)^{\frac{1}{r_1}}$$

$$= \left(\int_{\mathcal{G}} \left[\int_{\widehat{\mathcal{G}}} \left(\int_{\mathcal{G}} F((x, \xi) - (u, \omega)) H(u, \omega) \, du \right) d\omega \right]^{r_1} dx \right)^{\frac{1}{r_1}}$$

$$\leq \int_{\widehat{\mathcal{G}}} \left(\int_{\mathcal{G}} \left[\int_{\mathcal{G}} F((x, \xi) - (u, \omega)) H(u, \omega) \, du \right]^{r_1} dx \right)^{\frac{1}{r_1}} d\omega$$

$$= \int_{\widehat{\mathcal{G}}} \left(\int_{\mathcal{G}} [[F(\cdot, \xi - \omega) * H(\cdot, \omega)](x)]^{r_1} dx \right)^{\frac{1}{r_1}} d\omega$$

$$= \int_{\widehat{\mathcal{G}}} \|F(\cdot, \xi - \omega) * H(\cdot, \omega)\|_{L^{r_1}(\mathcal{G})} d\omega := I.$$

Using Young's inequality (see [39, Theorem 20.18]) with indexes p_1, q_1, r_1 as in (89), we majorize as

$$I \leq \int_{\widehat{\mathcal{G}}} \|F(\cdot, \xi - \omega)\|_{L^{p_1}(\mathcal{G})} \|H(\cdot, \omega)\|_{L^{q_1}(\mathcal{G})} d\omega$$

$$= \int_{\widehat{\mathcal{G}}} \left(\int_{\mathcal{G}} F(x, \xi - \omega)^{p_1} dx \right)^{\frac{1}{p_1}} \left(\int_{\mathcal{G}} H(x, \omega)^{q_1} dx \right)^{\frac{1}{q_1}} d\omega$$

$$= \int_{\widehat{\mathcal{G}}} \|F\|_{(p_1)}(\xi - \omega) \|H\|_{(q_1)}(\omega) d\omega$$

$$= \left(\|F\|_{(p_1)} * \|H\|_{(q_1)} \right)(\xi).$$

Using Young's inequality with indices p_2, q_2, r_2 in (89), we obtain the desire result. Namely,

$$\|F * H\|_{L^{r_1, r_2}(\mathcal{G} \times \widehat{\mathcal{G}})} = \left(\int_{\widehat{\mathcal{G}}} \left[\|F * H\|_{(r_1)}(\xi) \right]^{r_2} d\xi \right)^{\frac{1}{r_2}}$$

$$\leq \left(\int_{\widehat{\mathcal{G}}} \left[\left(\|F\|_{(p_1)} * \|H\|_{(q_1)} \right)(\xi) \right]^{r_2} d\xi \right)^{\frac{1}{r_2}}$$

$$= \left\| \|F\|_{(p_1)} * \|H\|_{(q_1)} \right\|_{L^{r_2}(\widehat{\mathcal{G}})}$$

$$\leq \left\| \|F\|_{(p_1)} \right\|_{L^{p_2}(\widehat{\mathcal{G}})} \left\| \|H\|_{(q_1)} \right\|_{L^{q_2}(\widehat{\mathcal{G}})}$$

$$= \|F\|_{L^{p_1, p_2}(\mathcal{G} \times \widehat{\mathcal{G}})} \left\| \|H\|_{(q_1)} \right\|_{L^{q_1, q_2}(\mathcal{G} \times \widehat{\mathcal{G}})}.$$

This concludes the proof. $\qquad \square$

A straightforward consequence is the weighted Young's inequality below.

Corollary 4.2 *Consider* $1 \le p_i, q_i, r_i \le \infty, i = 1, 2$, *such that*

$$\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i}, \quad i = 1, 2. \tag{92}$$

Consider $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$. *If* $F \in L_m^{p_1, p_2}(\mathcal{G} \times \widehat{\mathcal{G}})$ *and* $H \in L_v^{q_1, q_2}(\mathcal{G} \times \widehat{\mathcal{G}})$, *then* $F * H \in L_m^{r_1, r_2}(\mathcal{G} \times \widehat{\mathcal{G}})$ *with*

$$\|F * H\|_{L_m^{r_1, r_2}} \le \|F\|_{L_m^{p_1, p_2}} \|H\|_{L_v^{q_1, q_2}}. \tag{93}$$

Note that Proposition 4.1 can be easily generalized to N indices, $N \ge 2$, as in [5, Part II, Theorem 1, b)]:

Proposition 4.3 *Consider* $N \in \mathbb{N}$ *and let* $\mathcal{G}_i$ *be a LCA, σ-finite group with Haar measure* $\mathrm{d}x_i, i = 1, \ldots, N$. *Consider* $1 \le p_i, q_i, r_i \le \infty, i = 1, \ldots, N$, *such that*

$$\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i}, \quad i = 1, \ldots, N.$$

If $F \in L^{p_1, \ldots, p_N}(\mathcal{G}_1 \times \cdots \times \mathcal{G}_N)$ *and* $H \in L^{q_1, \ldots, q_N}(\mathcal{G}_1 \times \cdots \times \mathcal{G}_N)$, *then* $F * H \in L^{r_1, \ldots, r_N}(\mathcal{G}_1 \times \cdots \times \mathcal{G}_N)$ *with*

$$\|F * H\|_{L^{r_1, \ldots, r_N}(\mathcal{G}_1 \times \cdots \times \mathcal{G}_N)} \le \|F\|_{L^{p_1, \ldots, p_N}(\mathcal{G}_1 \times \cdots \times \mathcal{G}_N)} \|H\|_{L^{q_1, \ldots, q_N}(\mathcal{G}_1 \times \cdots \times \mathcal{G}_N)},$$

where the product LCA σ-finite group $\mathcal{G}_1 \times \cdots \times \mathcal{G}_N$ *is endowed with the product Haar measure* $\mathrm{d}x_1 \ldots \mathrm{d}x_N$.

We need to extend [37, formula (51)] to wider classes of functions. Namely,

Lemma 4.4 *Consider* $\psi \in \mathcal{S}_\mathcal{C}(\mathcal{G})$ *and* $f, g \in S_0'(\mathcal{G})$. *Then*

$$V_{R(\psi, \psi)} R(g, f)((x, \xi), (\omega, u)) = \overline{\langle \xi, u \rangle} V_\psi g(x, \xi + \omega) \overline{V_\psi f(x + u, \xi)}, \tag{94}$$

with $x, u \in \mathcal{G}$ *and* $\xi, \omega \in \widehat{\mathcal{G}}$.

Proof For $f, g, \psi \in \mathcal{S}_\mathcal{C}(\mathcal{G})$ formula (94) is proved in [37, formula (51)]. Consider now $f, g \in S_0'(\mathcal{G})$. From Corollary 3.36 there exist nets $(f_\alpha)_{\alpha \in A}, (g_\alpha)_{\alpha \in A} \in S_0(\mathcal{G})$ which converge pointwisely to f and g in $S_0'(\mathcal{G})$. Therefore for every $x, u \in \mathcal{G}$ and $\xi \in \widehat{\mathcal{G}}$,

$$\lim_{\alpha \in A} V_\psi f_\alpha(x + u, \xi) = \lim_{\alpha \in A} \langle f_\alpha, \pi(x + u, \xi)\psi \rangle = \langle f, \pi(x + u, \xi)\psi \rangle = V_\psi f(x + u, \xi),$$

and similarly for $V_\psi g$. For the left-hand side of (94), observe that

$$R(f_\alpha, g_\alpha)(x, \xi) = \overline{\langle \xi, x \rangle} \mathcal{F}_2(f_\alpha \otimes \overline{g}_\alpha)(x, \xi).$$

The partial Fourier transform $\mathcal{F}_2$ is a topological isomorphism from $S_0(\mathcal{G} \times \mathcal{G})$ onto $S_0(\mathcal{G} \times \widehat{\mathcal{G}})$ and from $S_0'(\mathcal{G} \times \mathcal{G})$ onto $S_0'(\mathcal{G} \times \widehat{\mathcal{G}})$. Write $\mathbf{x} = (x, \xi)$ and $\boldsymbol{\omega} = (\omega, u)$,

$$\lim_{\alpha \in A} V_{R(\psi,\psi)} R(f_\alpha, g_\alpha)(\mathbf{x}, \boldsymbol{\omega}) = \lim_{\alpha \in A} \langle \overline{\langle \cdot, \cdot \rangle} \mathcal{F}_2(f_\alpha \otimes \overline{g}_\alpha), \pi(\mathbf{x}, \boldsymbol{\omega}) R(\psi, \psi) \rangle$$

$$= \langle \overline{\langle \cdot, \cdot \rangle} \mathcal{F}_2(f \otimes \overline{g}), \pi(\mathbf{x}, \boldsymbol{\omega}) R(\psi, \psi) \rangle$$

$$= V_{R(\psi,\psi)} R(f, g)(\mathbf{x}, \boldsymbol{\omega}),$$

being $R(\psi, \psi) \in S_0(\mathcal{G} \times \widehat{\mathcal{G}})$. This concludes the proof. $\qquad \square$

Proposition 4.5 *Consider* $p, q, p_i, q_i \in (0, \infty], i = 1, 2,$ *such that*

$$p_i, q_i \leq q, \quad i = 1, 2; \tag{95}$$

$$\min \left\{ \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q_1} + \frac{1}{q_2} \right\} \geq \frac{1}{p} + \frac{1}{q}. \tag{96}$$

Let v be a even submultiplicative weight bounded from below on $\mathcal{G} \times \widehat{\mathcal{G}}$, and $\mathcal{J}$ the isomorphism in (16). *For* $g \in M_v^{p_1,q_1}(\mathcal{G})$ *and* $f \in M_v^{p_2,q_2}(\mathcal{G})$, *we have* $R(g, f) \in M_{1 \otimes v \circ \mathcal{J}^{-1}}^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}})$, *with*

$$\|R(g,f)\|_{M_{1 \otimes v \circ \mathcal{J}^{-1}}^{p,q}} \lesssim \|g\|_{M_v^{p_1,q_1}} \|f\|_{M_v^{p_2,q_2}}. \tag{97}$$

Proof Consider $\psi \in \mathcal{S}_\mathcal{C}(\mathcal{G})$, $f \in M_v^{p_2,q_2}(\mathcal{G})$, $g \in M_v^{p_1,q_1}(\mathcal{G})$. By Lemma 4.4 the STFT of the Rihaczek distribution is given by

$$V_{R(\psi,\psi)} R(g,f)((x,\xi),(\omega,u)) = \overline{\langle \xi, u \rangle} V_\psi g(x, \xi + \omega) \overline{V_\psi f(x + u, \xi)}, \tag{98}$$

with $x, u \in \mathcal{G}$ and $\xi, \omega \in \widehat{\mathcal{G}}$. Corollary 3.11 shows that $R(\psi, \psi) \in \mathscr{A}_{1 \otimes v \circ \mathcal{J}^{-1}}(\mathcal{G} \times \widehat{\mathcal{G}})$. Consider $V_\mathcal{G} \subseteq \mathcal{G}$ and $V_{\widehat{\mathcal{G}}} \subseteq \widehat{\mathcal{G}}$ open, relatively compact, unit neighborhoods. According to the notation in (29), we define

$$V_{1,2} = V_\mathcal{G} \times V_{\widehat{\mathcal{G}}}, \quad V_{2,1} := V_{\widehat{\mathcal{G}}} \times V_\mathcal{G}, \quad O := V_{1,2} \times V_{2,1} \times \mathbb{T}. \tag{99}$$

Set

$$H_g((x,\xi),(\omega,u),\tau) := V_\psi g(x, \xi + \omega) \quad \text{and} \quad H_f((x,\xi),(\omega,u),\tau) := \overline{V_\psi f(x + u, \xi)},$$

which are functions on the Heisenberg group associated with $\mathcal{G} \times \widehat{\mathcal{G}}$. Notice

$$\mathsf{M}_O[\overline{\tau} V_{R(\psi,\psi)} R(g,f)] = \mathsf{M}_O[H_g \cdot H_f] \leq \mathsf{M}_O[H_g] \cdot \mathsf{M}_O[H_f].$$

We compute

$$M_O[H_g]((x,\xi),(\omega,u),\tau) = \operatorname*{ess\,sup}_{\substack{((y,\eta),(v,z),s)\,\in \\ ((x,\xi),(\omega,u),\tau)O}} \Big|V_\psi g(y,\eta+v)\Big|$$

$$= \operatorname*{ess\,sup}_{v\in\omega+V_{\hat{\mathcal{G}}}} \operatorname*{ess\,sup}_{(y,\eta)\in(x,\xi)+V_{1,2}} \Big|T_{(e,-v)}V_\psi g(y,\eta)\Big|$$

$$= \operatorname*{ess\,sup}_{v\in\omega+V_{\hat{\mathcal{G}}}} \Big(M_{V_{1,2}}[T_{(e,-v)}V_\psi g](x,\xi)\Big)$$

$$= \operatorname*{ess\,sup}_{v\in\omega+V_{\hat{\mathcal{G}}}} \Big(T_{(e,-v)}[M_{V_{1,2}}V_\psi g(x,\xi)]\Big)$$

$$= \operatorname*{ess\,sup}_{v\in\omega+V_{\hat{\mathcal{G}}}} \Big(M_{V_{1,2}}V_\psi g(x,\xi+v)\Big).$$

Similarly,

$$M_O[H_f]((x,\xi),(\omega,u),\tau) = \operatorname*{ess\,sup}_{z\in u+V_{\mathcal{G}}} \Big(T_{(-z,\hat{e})}[M_{V_{1,2}}V_\psi f(x,\xi)]\Big)$$

$$= \operatorname*{ess\,sup}_{z\in u+V_{\mathcal{G}}} \Big(M_{V_{1,2}}V_\psi f(x+z,\xi)\Big).$$

By the modulation spaces independence of the window in $\mathscr{A}_{1\otimes v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\hat{\mathcal{G}})$, we can write

$$\|R(g,f)\|_{M^{p,q}_{1\otimes v\circ\mathcal{J}^{-1}}}$$

$$\asymp \left(\int_{\hat{\mathcal{G}}\times\mathcal{G}\times\mathbb{T}} \left(\int_{\mathcal{G}\times\hat{\mathcal{G}}} \Big|M_O[\bar{\tau}V_{R(\psi,\psi)}R(g,f)]((x,\xi),(\omega,u),\tau)\Big|^p \mathrm{d}x\mathrm{d}\xi\right)^{\frac{q}{p}}\right.$$

$$\left. \times v^q\circ\mathcal{J}^{-1}(\omega,u)\,\mathrm{d}\omega\mathrm{d}u\mathrm{d}\tau\right)^{\frac{1}{q}}$$

$$\le \left(\int_{\hat{\mathcal{G}}\times\mathcal{G}} \left(\int_{\mathcal{G}\times\hat{\mathcal{G}}} \operatorname*{ess\,sup}_{v\in\omega+V_{\hat{\mathcal{G}}}} \Big(M_{V_{1,2}}V_\psi g(x,\xi+v)\Big)^p\right.\right.$$

$$\left.\left. \times \operatorname*{ess\,sup}_{z\in u+V_{\mathcal{G}}} \Big(M_{V_{1,2}}V_\psi f(x+z,\xi)\Big)^p \mathrm{d}x\mathrm{d}\xi\right)^{\frac{q}{p}} v^q\circ\mathcal{J}^{-1}(\omega,u)\,\mathrm{d}\omega\mathrm{d}u\right)^{\frac{1}{q}}$$

$$= \left(\int_{\hat{\mathcal{G}}\times\mathcal{G}} \left(\operatorname*{ess\,sup}_{(v,z)\in(\omega,u)+V_{2,1}} \int_{\mathcal{G}\times\hat{\mathcal{G}}} M_{V_{1,2}}V_\psi g(x,\xi+v)^p M_{V_{1,2}}V_\psi f(x+z,\xi)^p \mathrm{d}x\mathrm{d}\xi\right)^{\frac{q}{p}}\right.$$

$$\left. \times v^q\circ\mathcal{J}^{-1}(\omega,u)\,\mathrm{d}\omega\mathrm{d}u\right)^{\frac{1}{q}}.$$

The inner integral can be <u>rephrased</u> using the left-right invariance of Haar measure and the involution $h^*(\cdot):=\overline{h(-\cdot)}$ as follows:

$$\int_{\mathcal{G}\times\widehat{\mathcal{G}}} \mathsf{M}_{V_{1,2}} V_\psi g(x, \xi + \nu)^p \mathsf{M}_{V_{1,2}} V_\psi f(x + z, \xi)^p \, \mathrm{d}x\mathrm{d}\xi$$

$$= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} \mathsf{M}_{V_{1,2}} V_\psi g(x', \xi')^p \mathsf{M}_{V_{1,2}} V_\psi f((x', \xi') + (z, -\nu))^p \, \mathrm{d}x'\mathrm{d}\xi'$$

$$= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} (\mathsf{M}_{V_{1,2}} V_\psi g)^*(x'', \xi'')^p \mathsf{M}_{V_{1,2}} V_\psi f((z, -\nu) - (x'', \xi''))^p \, \mathrm{d}x''\mathrm{d}\xi''$$

$$= (\mathsf{M}_{V_{1,2}} V_\psi g)^{*\, p} * (\mathsf{M}_{V_{1,2}} V_\psi f)^p(z, -\nu)$$

$$= (\mathsf{M}_{V_{1,2}} V_\psi g)^{*\, p} * (\mathsf{M}_{V_{1,2}} V_\psi f)^p \circ \mathcal{J}^{-1}(\nu, z).$$

Whence, using [65, Lemma 2.3.23], we majorize

$$\underset{(\nu,z)\in(\omega,u)+V_{1,2}}{\mathrm{ess\ sup}} (\mathsf{M}_{V_{1,2}} V_\psi g)^{*\, p} * (\mathsf{M}_{V_{1,2}} V_\psi f)^p \circ \mathcal{J}^{-1}(\nu, z)$$

$$= \underset{(z',\nu')\in\mathcal{J}^{-1}(\omega,u)+\mathcal{J}^{-1}V_{2,1}}{\mathrm{ess\ sup}} (\mathsf{M}_{V_{1,2}} V_\psi g)^{*\, p} * (\mathsf{M}_{V_{1,2}} V_\psi f)^p(z', \nu')$$

$$= \mathsf{M}_{\mathcal{J}^{-1}V_{2,1}} [(\mathsf{M}_{V_{1,2}} V_\psi g)^{*\, p} * (\mathsf{M}_{V_{1,2}} V_\psi f)^p](\mathcal{J}^{-1}(\omega, u))$$

$$\leq [\mathsf{M}_{\mathcal{J}^{-1}V_{2,1}} [(\mathsf{M}_{V_{1,2}} V_\psi g)^{*\, p}] * (\mathsf{M}_{V_{1,2}} V_\psi f)^p](\mathcal{J}^{-1}(\omega, u))$$

$$= [\mathsf{M}_{\mathcal{J}^{-1}V_{2,1}} [(\mathsf{M}_{V_{1,2}} V_\psi g)^{*\, p}] * (\mathsf{M}_{V_{1,2}} V_\psi f)^p] \circ \mathcal{J}^{-1}(\omega, u).$$

Setting $U := -\mathcal{J}^{-1}V_{2,1} + V_{1,2}$, which is an open, relatively compact, unit neighborhood, we obtain

$$\mathsf{M}_{\mathcal{J}^{-1}V_{2,1}} [(\mathsf{M}_{V_{1,2}} V_\psi g)^{*\, p}](u, \omega)$$

$$= \underset{(y,\eta)\in(u,\omega)+\mathcal{J}^{-1}V_{2,1}}{\mathrm{ess\ sup}} (\mathsf{M}_{V_{1,2}} V_\psi g)^{*\, p}(y, \eta)$$

$$= \underset{(y,\eta)\in(u,\omega)+\mathcal{J}^{-1}V_{2,1}}{\mathrm{ess\ sup}} (\mathsf{M}_{V_{1,2}} V_\psi g)^p(-y, -\eta)$$

$$= \underset{(y,\eta)\in(u,\omega)+\mathcal{J}^{-1}V_{2,1}}{\mathrm{ess\ sup}} \ \underset{(x,\xi)\in-(y,\eta)+V_{1,2}}{\mathrm{ess\ sup}} \left| V_\psi g(x, \xi) \right|^p$$

$$\leq \underset{(y,\eta)\in(u,\omega)+\mathcal{J}^{-1}V_{2,1}}{\mathrm{ess\ sup}} \ \underset{(x,\xi)\in(u,\omega)-\mathcal{J}^{-1}V_{2,1}+V_{1,2}}{\mathrm{ess\ sup}} \left| V_\psi g(x, \xi) \right|^p$$

$$= \underset{(x,\xi)\in(u,\omega)-\mathcal{J}^{-1}V_{2,1}+V_{1,2}}{\mathrm{ess\ sup}} \left| V_\psi g(x, \xi) \right|^p$$

$$= \left(\mathsf{M}_U V_\psi g(-u, -\omega) \right)^p = \left([\mathsf{M}_U V_\psi g(u, \omega)]^* \right)^p.$$

Observe that for positive functions h, l on $\mathcal{G} \times \widehat{\mathcal{G}}$ and v a submultiplicative weight, we can write

$$((h * l)v)(x, \xi) \leq (hv * lv)(x, \xi), \quad (x, \xi) \in \mathcal{G} \times \widehat{\mathcal{G}}, \tag{100}$$

moreover, v^p is submultiplicative as well. Therefore,

$$\|R(g,f)\|_{M^{p,q}_{1\otimes v\circ\mathcal{J}^{-1}}}$$

$$\lesssim \left(\int_{\widehat{\mathcal{G}}\times\mathcal{G}} \left([([\mathsf{M}_U V_\psi g]^*)^p * (\mathsf{M}_{V_{1,2}} V_\psi f)^p]\circ\mathcal{J}^{-1}(\omega,u) \right)^{\frac{q}{p}} \right.$$

$$\left. \times v^q\circ\mathcal{J}^{-1}(\omega,u)\,\mathrm{d}\omega\mathrm{d}u \right)^{\frac{1}{q}}$$

$$\leq \left(\int_{\widehat{\mathcal{G}}\times\mathcal{G}} \left(([\mathsf{M}_U V_\psi g]^* \cdot v)^p * (\mathsf{M}_{V_{1,2}} V_\psi f \cdot v)^p \right)^{\frac{q}{p}}\circ\mathcal{J}^{-1}(\omega,u)\,\mathrm{d}\omega\mathrm{d}u \right)^{\frac{1}{q}}$$

$$= \left\| ([\mathsf{M}_U V_\psi g]^* \cdot v)^p * (\mathsf{M}_{V_{1,2}} V_\psi f \cdot v)^p \right\|_{L^{q/p}(\mathcal{G}\times\widehat{\mathcal{G}})}^{1/p}.$$

By Young's convolution inequality and following the same arguments as in the proofs in [12, Theorem 3.1] and [11, Theorem 4] for the Euclidean case (replacing the Wigner distribution with the Rihaczek), we infer the estimate

$$\|R(g,f)\|_{M^{p,q}_{1\otimes v\circ\mathcal{J}^{-1}}} \lesssim \|g\|_{M^{p_1,q_1}_v} \|f\|_{M^{p_2,q_2}_v}, \tag{101}$$

with indices satisfying the conditions (95) and (96). Following the patterns of [11, 12] the same result is obtained when $p = \infty$ or $q = \infty$. $\qquad\square$

The boundedness properties of the Rihaczek distributions enter the study of Kohn–Nirenberg pseudo-differential operators $\mathrm{Op}_0(\sigma)$, defined in (8) and (9), in the same fashion of [37].

The boundedness result for Weyl operators in the Euclidean setting [12, Theorem 5.1] can be written for Kohn–Nirenberg operators on groups as follows.

Theorem 4.6 *Consider* $p, q, p_i, q_i \in [1, \infty]$, $i = 1, 2$, *such that*:

$$q \leq \min\{p_1', q_1', p_2, q_2\}; \tag{102}$$

$$\min\left\{ \frac{1}{p_1} + \frac{1}{p_2'}, \frac{1}{q_1} + \frac{1}{q_2'} \right\} \geq \frac{1}{p'} + \frac{1}{q'}. \tag{103}$$

Consider v *submultiplicative weight even and bounded from below on* $\mathcal{G}\times\widehat{\mathcal{G}}$. *If* $\sigma \in M^{p,q}_{1\otimes\frac{1}{v}\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$, *then* $\mathrm{Op}_0(\sigma)$ *is a bounded operator from* $M^{p_1,q_1}_v(\mathcal{G})$ *into* $M^{p_2,q_2}_{1/v}(\mathcal{G})$ *with estimate*

$$\left\| \mathrm{Op}_0(\sigma)f \right\|_{M^{p_2,q_2}_{1/v}} \lesssim \|\sigma\|_{M^{p,q}_{1\otimes\frac{1}{v}\circ\mathcal{J}^{-1}}} \|f\|_{M^{p_1,q_1}_v}. \tag{104}$$

Proof It follows by duality using Proposition 4.5 and the weak definition of $\mathrm{Op}_0(\sigma)$ in (9). $\qquad\square$

4.2 Gabor frames on quasi-lattices

The key tool in the boundedness properties of Kohn–Nirenberg operators on quasi-Banach modulation spaces is the Gabor frame theory. For a detailed treatment of frame theory see, e.g., [8].

In what follows we shall recall and prove new properties for Gabor frames on a LCA group. As a byproduct, we generalize the convolution relations for modulation spaces firstly given in [3, Proposition 3.1], see Proposition 4.18.

A lattice in $\mathcal{G}$ is a discrete subgroup Λ such that the quotient group $\mathcal{G}/\Lambda$ is compact. In this case, there is a relatively compact $U \subseteq \mathcal{G}$, called fundamental domain for Λ, such that

$$\mathcal{G} = \bigcup_{w \in \Lambda} (w + U), \qquad (w + U) \cap (u + U) = \varnothing \quad \text{for } w \neq u.$$

Not every LCA admits a lattice, for example the p-adic groups $\mathbb{Q}_p$, therefore we adopt the following construction of Gröchenig and Strohmer [37]. Recall from the structure theorem $\mathcal{G} \cong \mathbb{R}^d \times \mathcal{G}_0$. Consider $D \subseteq \mathcal{G}_0$ a collection of coset representatives of $\mathcal{G}_0/\mathcal{K}$ and $A \in \mathrm{GL}(\mathbb{R}^d)$. We define $U := A[0,1)^d \times \mathcal{K}$. The discrete set $\Lambda := A\mathbb{Z}^d \times D$ is called quasi-lattice with fundamental domain U. Observe that we have the following partition

$$\mathcal{G} = \bigcup_{w \in \Lambda} (w + U).$$

According to the above theory, a quasi-lattice on the phase-space $\mathcal{G} \times \widehat{\mathcal{G}}$ is of the type:

$$\Lambda := \Lambda_1 \times \Lambda_2 := \left(A_1 \mathbb{Z}^d \times D_1\right) \times \left(A_2 \mathbb{Z}^d \times D_2\right) \cong A_{1,2} \mathbb{Z}^{2d} \times D_{1,2} \qquad (105)$$

with fundamental domain

$$U := U_1 \times U_2 := \left(A_1[0,1)^d \times \mathcal{K}\right) \times \left(A_2[0,1)^d \times \mathcal{K}^\perp\right) \cong A_{1,2}[0,1)^{2d} \times \left(\mathcal{K} \times \mathcal{K}^\perp\right), \tag{106}$$

where $D_2 \subseteq \widehat{\mathcal{G}}_0$ is a set of coset representatives of $\widehat{\mathcal{G}}_0/\mathcal{K}^\perp$ and

$$A_{1,2} := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad D_{1,2} := D_1 \times D_2. \tag{107}$$

We shall denote elements of a quasi-lattice Λ in $\mathcal{G} \times \widehat{\mathcal{G}}$ as

$$\mathbf{w} = (w, \mu) = ((w_1, w_2), (\mu_1, \mu_2)) \in \Lambda = \Lambda_1 \times \Lambda_2 \subseteq \mathcal{G} \times \widehat{\mathcal{G}}.$$

Lemma 4.7 *Let $\Lambda \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ be a quasi-lattice as in* (105). *Then, Λ is a relatively separated family.*

Proof We use Lemma 3.24. The fact that $A_{1,2}\mathbb{Z}^{2d}$ is relatively separated in $\mathbb{R}^{2d}$ is trivial. We only have to show that D_1 is relatively separated in $\mathcal{G}_0$; D_2 is treated similarly. For a fixed compact set $Q_0 \subseteq \mathcal{G}_0$, we have to show that

$$C_{D_1,Q_0} = \sup_{x \in D_1} \#\{y \in D_1 \mid (x + Q_0) \cap (y + Q_0) \neq \emptyset\} < +\infty.$$

Since Q_0 is compact and $\mathcal{K}$ is an open subgroup, there exist $q_1, \dots, q_n \in Q_0$ such that

$$Q_0 \subseteq \bigcup_{i=1}^{n} (q_i + \mathcal{K}) =: Q_0'.$$

For $x, y \in D_1$, if $(x + Q_0) \cap (y + Q_0) \neq \emptyset$ then $(x + Q_0') \cap (y + Q_0') \neq \emptyset$, hence $C_{D_1,Q_0} \leq C_{D_1,Q_0'}$. Assume that $(x + Q_0') \cap (y + Q_0') \neq \emptyset$, then there are $i_0, j_0 \in \{1, \dots, n\}$ and $k_{i_0}, k_{j_0} \in \mathcal{K}$ such that

$$x + q_{i_0} + k_{i_0} = y + q_{j_0} + k_{j_0} \quad \Leftrightarrow \quad y = x + q_{i_0} - q_{j_0} + k_{i_0} - k_{j_0}.$$

Fix $x \in D_1$, quotienting by $\mathcal{K}$,

$$[y]^\bullet = [x + q_{i_0} - q_{j_0}]^\bullet \quad \Rightarrow \quad \#\{y \in D_1 \mid (x + Q_0') \cap (y + Q_0') \neq \emptyset\} \leq n^2, \tag{108}$$

where $[y]^\bullet$ denotes the projection of $y \in \mathcal{G}_0$ onto the quotient $\mathcal{G}_0/\mathcal{K}$. This proves $C_{D_1,Q_0} \leq C_{D_1,Q_0'} < +\infty$. The desired result follows now from Lemma 3.24. $\qquad\square$

Corollary 4.8 *Let $\Lambda \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ be a quasi-lattice. Then Λ is at most countable.*

Proof We use the fact that Λ is a relatively separated family and [65, Lemma 2.3.10]. $\qquad\square$

In the following issue about the existence of a particular BUPU, we use the quasi-lattice Λ both as localizing family and as indexes' set. The argument was presented in [23, Remark 2.5].

Lemma 4.9 *Let $\Lambda \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ be a quasi-lattice as in (105) with fundamental domain U as in (106). Then, there exist two open, relatively compact, unit neighborhoods Q and $V_{1,2}$ in $\mathcal{G} \times \widehat{\mathcal{G}}$, where $V_{1,2}$ as in (29), such that $Q \subsetneq V_{1,2}$ and there is a $V_{1,2}$-BUPU*

$$\{\psi_w \otimes \gamma_\mu\}_{(w,\mu) \in \Lambda}$$

with localizing family Λ and such that for every $(w, \mu) \in \Lambda$

$$\psi_w \otimes \gamma_\mu \equiv 1 \quad \text{on } (w, \mu) + Q.$$

Remark 4.10 Without loss of generality, the unit neighborhood Q of the previous lemma can be chosen such that

$$\left(\{0_{\mathbb{R}^d}\} \times \mathcal{K}\right) \times \left(\{0_{\mathbb{R}^d}\} \times \mathcal{K}^\perp\right) \subsetneq Q. \tag{109}$$

Therefore for every $(w, \mu) \in \Lambda = \Lambda_1 \times \Lambda_2$ we have

$$\psi_w \equiv 1 \quad \text{on } w + \left(\{0_{\mathbb{R}^d}\} \times \mathcal{K}\right), \qquad \gamma_\mu \equiv 1 \quad \text{on } \mu + \left(\{0_{\mathbb{R}^d}\} \times \mathcal{K}^\perp\right).$$

Definition 4.11 Given a quasi-lattice $\Lambda \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ and windows $g, h \in L^2(\mathcal{G})$, the Gabor system generated by g is

$$\{\pi(\mathbf{w})g \mid \mathbf{w} \in \Lambda\} = \{\pi(\mathbf{w})g\}_{\mathbf{w} \in \Lambda}.$$

The coeffcient or analysis operator is given by

$$\mathcal{C}_g : L^2(\mathcal{G}) \to \ell^2(\Lambda), f \mapsto (\langle f, \pi(\mathbf{w})g \rangle)_{\mathbf{w} \in \Lambda}. \tag{110}$$

Its adjoint is called reconstruction or synthesis operator and has the form

$$\mathcal{C}_g^* : \ell^2(\Lambda) \to L^2(\mathcal{G}), \left(c_{\mathbf{w}}\right)_{\mathbf{w} \in \Lambda} \mapsto \sum_{\mathbf{w} \in \Lambda} c_{\mathbf{w}} \pi(\mathbf{w}) g. \tag{111}$$

The Gabor frame operator $S_{h,g}$ is given by

$$S_{h,g} f = \mathcal{C}_h^* \mathcal{C}_g f = \sum_{\mathbf{w} \in \Lambda} \langle f, \pi(\mathbf{w})g \rangle \pi(\mathbf{w})h. \tag{112}$$

We say that $\{\pi(\mathbf{w})g\}_{\mathbf{w} \in \Lambda}$ is a Gabor frame for $L^2(\mathcal{G})$ if there exist $A, B > 0$ such that

$$A\|f\|_{L^2}^2 \leq \sum_{\mathbf{w} \in \Lambda} |\langle f, \pi(\mathbf{w})g \rangle|^2 \leq B\|f\|_{L^2}^2, \quad \forall f \in L^2(\mathcal{G}); \tag{113}$$

this is equivalent to saying that $S_{g,g}$ is invertible on $L^2(\mathcal{G})$. If $A = B$ the frame is called tight. Moreover, if $h \in L^2(\mathcal{G})$ is such that

$$S_{h,g} = S_{g,h} = I_{L^2}, \tag{114}$$

then h is named dual window for the frame $\{\pi(\mathbf{w})g\}_{\mathbf{w} \in \Lambda}$.

We note that Theorem 2.7 in [37] is till valid for the case of the Gaussian φ and considering a Gabor frame not tight. Namely,

Theorem 4.12 *Let* $\Lambda := \alpha \mathbb{Z}^{2d} \times D_{1,2}$, $\alpha \in (0, 1)$, *be a quasi-lattice in* $\mathcal{G} \times \widehat{\mathcal{G}}$. *Consider the Gaussian* φ *in* (12). *Then*

$$\{\pi(\mathbf{w})\varphi \mid \mathbf{w} \in \Lambda\} \tag{115}$$

is a Gabor frame for $L^2(\mathcal{G})$.

Corollary 4.13 *There exists* $\alpha \in (0, 1)$ *such that the Gabor frame* $\{\pi(\mathbf{w})\varphi \mid \mathbf{w} \in \Lambda\}$ *defined in* (115) *admits a dual window* $h \in \mathscr{A}_{\tilde{v}}$.

Proof We first tackle the problem of finding a dual window. The proof is similar to that in [37, Theorem 2.7]. We distinguish three cases.

Case $\mathcal{G} = \mathbb{R}^d$. In this case the frame we are considering is

$$\{\pi(w_1, \mu_1)e^{-\pi x_1^2}, (w_1, \mu_1) \in \alpha \mathbb{Z}^{2d}\}, \quad \alpha \in (0, 1). \tag{116}$$

We fix α such that $\alpha^{2d} < (d + 1)^{-1}$. Then the existence of a dual window γ_0 for the Gabor frame generated by the first Hermite function H_0 (the Gaussian) was proved by Gröchenig and Lyubarskii, see [35, 36]. In particular, in [36, Remarks 2] was observed that γ_0 belongs to the Gelfand–Shilov space $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$, cf. [31].

Case $\mathcal{G} = \mathcal{G}_0$. In this case the frame that we are dealing with is the orthonormal basis for $L^2(\mathcal{G}_0)$

$$\{\pi(w_2, \mu_2)\chi_{\mathcal{K}}(x_2), (w_2, \mu_2) \in D_{1,2}\}. \tag{117}$$

Therefore, $\chi_{\mathcal{K}}$ is a dual window itself.

Case $\mathcal{G} \cong \mathbb{R}^d \times \mathcal{G}_0$. The frame in this case is the tensor product of the previous ones:

$$\{\pi(\mathbf{w})\varphi, \ \mathbf{w} = ((w_1, w_2), (\mu_1, \mu_2)) \in \Lambda = \left(\alpha\mathbb{Z}^d \times D_1\right) \times \left(\alpha\mathbb{Z}^d \times D_2\right)\}, \tag{118}$$

where $\varphi(x_1, x_2) = e^{-\pi x_1^2}\chi_{\mathcal{K}}(x_2) = (\varphi_1 \otimes \varphi_2)(x_1, x_2)$. Recall that the functions of the type $f_1 \otimes f_2$, with $f_1 \in L^2(\mathbb{R}^d)$ and $f_2 \in L^2(\mathcal{G}_0)$, are dense in $L^2(\mathbb{R}^d \times \mathcal{G}_0)$. Let us show that

$$h(x_1, x_2) := (\gamma_0 \otimes \chi_{\mathcal{K}})(x_1, x_2) \tag{119}$$

is a dual window. In fact,

$$\sum_{\mathbf{w} \in \Lambda} \langle f_1 \otimes f_2, \pi(\mathbf{w})\varphi\rangle \pi(\mathbf{w})\gamma_0 \otimes \chi_{\mathcal{K}}$$
$$= \sum_{(w_1, \mu_1)} \langle f_1, \pi(w_1, \mu_1)\varphi_1\rangle \pi(w_1, \mu_1)\gamma_0 \sum_{(w_2, \mu_2)} \langle f_2, \pi(w_2, \mu_2)\varphi_2\rangle \pi(w_2, \mu_2)\chi_{\mathcal{K}}$$
$$= f_1 \otimes f_2;$$

similarly,

$$\sum_{\mathbf{w} \in \Lambda} \langle f_1 \otimes f_2, \pi(\mathbf{w})\gamma_0 \otimes \chi_{\mathcal{K}}\rangle \pi(\mathbf{w})\varphi = f_1 \otimes f_2.$$

The claim follows by density argument.

We now prove that $h \in \mathscr{A}_{\tilde{v}}$ in the general case $\mathcal{G} \cong \mathbb{R}^d \times \mathcal{G}_0$. Similarly to the wavelet transform of the generalized Gaussian φ in (12), see (17), we obtain

$$W_h^o h(x, \xi, \tau) = \overline{\tau}c(\mathcal{K})V_{\gamma_0}\gamma_0 \otimes \chi_{\mathcal{K} \times \mathcal{K}^\perp}(x, \xi). \tag{120}$$

Since $V_{\gamma_0}\gamma_0 \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^{2d})$, see, e.g., [4, Theorem 2.13], calculations similar to the ones performed in Lemma 3.10 yield the desired result. $\qquad\square$

Lemma 4.14 *Let $\Lambda = \alpha\mathbb{Z}^{2d} \times D_{1,2}, \alpha \in \mathbb{R}$, be a quasi-lattice in $\mathcal{G} \times \widehat{\mathcal{G}}$. Consider the function*

$$\varphi^\circ(x) := \varphi^\circ(x_1, x_2) := 2^{-\frac{d}{2}}\mathrm{meas}(\mathcal{K})e^{-\frac{\pi}{2}x_1^2} \otimes \chi_{\mathcal{K}}(x_2) \in \mathscr{A}_{\tilde{v}} \tag{121}$$

for $x = (x_1, x_2) \in \mathbb{R}^d \times \mathcal{G}_0 \cong \mathcal{G}$, where $\mathrm{meas}(\mathcal{K})$ is the (finite) measure of the compact open closed subgroup $\mathcal{K}$ in $\mathcal{G}_0$. Then there exist $\alpha \in (0, 1)$ and a function $h^\circ \in \mathscr{A}_{\tilde{v}}$ such that

$$\{\pi(\mathbf{w})\varphi^\circ \mid \mathbf{w} \in \Lambda\} \tag{122}$$

is a Gabor frame for $L^2(\mathcal{G})$ with dual window h°.

Proof The result is obtained using the same arguments as in Theorem 4.12 and Corollary 4.13, combined with [13, Lemma 3.2.2]. $\qquad\square$

Theorem 4.15 *Let $\Lambda \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ be a quasi-lattice with fundamental domain U. Consider $0 < p, q \leq \infty$, $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ and $g \in \mathscr{A}_{\tilde{v}}$. Then the coefficient operator $\mathcal{C}_g$ admits a unique continuous and linear extension*

$$\mathcal{C}_g : M_m^{p,q}(\mathcal{G}) \to \ell_{m_\Lambda}^{p,q}(\Lambda), \tag{123}$$

where m_Λ is the restriction of m to Λ. Moreover, if $0 < \delta \leq \infty$ is such that $0 < \delta \leq \min\{p, q\} \leq \infty$, then there is a constant $C = C(\delta) > 0$, such that

$$\left\|\left\|\mathcal{C}_g\right\|\right\|_{M_m^{p,q} \to \ell_{m_\Lambda}^{p,q}} \leq C$$

for all $p, q \geq \delta$. The constant $C = C(\delta)$ may depend on other elements, but not on p and q.

Proof Consider $f \in M_m^{p,q}(\mathcal{G})$. Let $\{\psi_w \otimes \gamma_\mu\}_{(w,\mu) \in \Lambda}$ be the BUPU on $\mathcal{G} \times \widehat{\mathcal{G}}$ constructed in Lemma 4.9. Since tensor product of BUPUs is a BUPU (Lemma 3.25) it follows that $\{\psi_w \otimes \gamma_\mu \otimes \chi_\mathbb{T}\}_{(w,\mu) \in \Lambda}$ is a V-BUPU on $\mathbb{H}_\mathcal{G}$, V as in (27), with localizing family $\mathfrak{X} = \Lambda \times \{1\}$ and such that

$$(\psi_w \otimes \gamma_\mu \otimes \chi_\mathbb{T})(w, \mu, 1) = 1 \qquad \forall (w, \mu) \in \Lambda. \tag{124}$$

Hence

$$\begin{aligned}
|\langle f, \pi(w, \mu)g\rangle| &= \left|V_g f(w, \mu)\right| \\
&= \left|(\psi_w \otimes \gamma_\mu \otimes \chi_\mathbb{T})(w, \mu, 1) \cdot W_g^o f(w, \mu, 1)\right| \\
&\leq \left\|(\psi_w \otimes \gamma_\mu \otimes \chi_\mathbb{T}) \cdot W_g^o f\right\|_{L^\infty}.
\end{aligned}$$

By Lemma 3.26,

$$\left\|\mathcal{C}_g f\right\|_{\ell^{p,q}_{m_\Lambda}(\Lambda)} = \left\|(\langle f, \pi(w,\mu)g\rangle)_{(w,\mu)\in\Lambda}\right\|_{\ell^{p,q}_{m_\Lambda}(\Lambda)}$$

$$\leq \left\|\left(\left\|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot W_g^o f\right\|_{L^\infty}\right)_{(w,\mu)\in\Lambda}\right\|_{\ell^{p,q}_{m_\Lambda}(\Lambda)}$$

$$\asymp \left\|\left(\left\|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot W_g^o f\right\|_{L^\infty}\right)_{(w,\mu)\in\Lambda}\right\|_{(L^{p,q}_{\tilde{m}})_d(\mathfrak{X},V)}$$

$$\asymp \left\|W_g^o f\right\|_{W(L^{p,q}_{\tilde{m}})} = \|f\|_{M^{p,q}_m},$$

where in the last equivalence we used Lemma 3.23, see also (181) in the Appendix. The last claim comes from Lemma 3.23 and Corollary 3.28. $\square$

Theorem 4.16 *Let $\Lambda \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ be a quasi-lattice with fundamental domain U. Consider $0 < p, q \leq \infty$, $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ and $g \in \mathscr{A}_{\tilde{v}}$. Then the synthesis operator $\mathcal{C}_g^*$ admits a unique continuous and linear extension*

$$\mathcal{C}_g^* : \ell^{p,q}_{m_\Lambda}(\Lambda) \to M^{p,q}_m(\mathcal{G}), \tag{125}$$

where m_Λ is the restriction of m to Λ. If $p, q \neq \infty$, then the series representing $\mathcal{C}_g^*(c)$ converges unconditionally in $M^{p,q}_m(\mathcal{G})$. Otherwise $\mathcal{C}_g^*(c)$ w--converges in $M^\infty_{1/v}(\mathcal{G})$. Moreover, if $0 < \delta \leq \infty$ is such that $0 < \delta \leq \min\{p,q\} \leq \infty$, then there is a constant $C = C(\delta) > 0$, such that*

$$\left\|\left\|\left\|\mathcal{C}_g^*\right\|\right\|\right\|_{\ell^{p,q}_{m_\Lambda} \to M^{p,q}_m} \leq C$$

for all $p, q \geq \delta$. The constant $C = C(\delta)$ may depend on other elements, but not on p and q.

Proof The proof follows the pattern displayed in [30]. Let $(x, \xi, \tau) \in \mathbb{H}_{\mathcal{G}}$ and $c = (c_{\mathbf{w}})_{\mathbf{w}\in\Lambda} \in \ell^{p,q}_{m_\Lambda}(\Lambda)$, then we write

$$\left|W_g^o[\mathcal{C}_g^*(c)](x,\xi,\tau)\right| = \left|V_g\left[\sum_{\mathbf{w}\in\Lambda} c_{\mathbf{w}} \pi(\mathbf{w})g\right](x,\xi)\right| = \left|\sum_{\mathbf{w}\in\Lambda} c_{\mathbf{w}} V_g \pi(\mathbf{w})g(x,\xi)\right|$$

$$\leq \sum_{\mathbf{w}\in\Lambda} |c_{\mathbf{w}}|\left|T_{\mathbf{w}} V_g g(x,\xi)\right| =: F_c^g(x,\xi,\tau).$$

Let $\{\psi_w \otimes \gamma_\mu\}_{(w,\mu)\in\Lambda}$ be the $V_{1,2}$-BUPU on $\mathcal{G} \times \widehat{\mathcal{G}}$ constructed in Lemma 4.9. Then, $\{\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}\}_{(w,\mu)\in\Lambda}$ is a V-BUPU, V as in (27), on $\mathbb{H}_{\mathcal{G}}$ with localizing family $\mathfrak{X} = \Lambda \times \{1\}$. Using the norm equivalence in (181) (Appendix) and Lemma 3.26

$$\left\|\mathcal{C}_g^*(c)\right\|_{M_m^{p,q}} \asymp \left\|\mathcal{C}_g^*(c)\right\|_{W_V(L_{\tilde{m}}^{p,q})}$$

$$\lesssim \left\|\left(\left\|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot F_c^g\right\|_{L^\infty}\right)_{\mathbf{w} \in \Lambda}\right\|_{(L_{\tilde{m}}^{p,q})_d(\mathfrak{X},V)}$$

$$\asymp \left\|\left(\left\|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot F_c^g\right\|_{L^\infty}\right)_{\mathbf{w} \in \Lambda}\right\|_{\ell_{m_\Lambda}^{p,q}(\Lambda)}.$$

We control the latter sequence as follows:

$$\left\|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot F_c^g\right\|_{L^\infty} \leq \sum_{\mathbf{u} \in \Lambda} |c_{\mathbf{u}}| \operatorname{ess\,sup}_{(x,\xi) \in \mathbf{w}+V_{1,2}} \left|T_{\mathbf{u}} V_g g(x,\xi)\right|$$

$$= \sum_{\mathbf{w} \in \Lambda} |c_{\mathbf{w}}| \mathsf{M}_{V_{1,2}} V_g g(\mathbf{w} - \mathbf{u})$$

$$= \left((|c_{\mathbf{u}}|)_{\mathbf{u}} * (\mathsf{M}_{V_{1,2}} V_g g(\mathbf{u}))_{\mathbf{u}}\right)(\mathbf{w}).$$

We set $t = \min\{1,p\}$ and $s = \min\{1,p,q\}$. Using the convolution relations for the sequences' spaces in [30, Lemma 2.7], we obtain

$$\left\|\mathcal{C}_g^*(c)\right\|_{M_m^{p,q}} \lesssim \left\|\left(\left((|c_{\mathbf{u}}|)_{\mathbf{u}} * (\mathsf{M}_{V_{1,2}} V_g g(\mathbf{u}))_{\mathbf{u}}\right)(\mathbf{w})\right)_{\mathbf{w} \in \Lambda}\right\|_{\ell_{m_\Lambda}^{p,q}(\Lambda)}$$

$$\lesssim \|c\|_{\ell_{m_\Lambda}^{p,q}(\Lambda)} \left\|(\mathsf{M}_{V_{1,2}} V_g g(\mathbf{w}))_{\mathbf{w} \in \Lambda.}\right\|_{\ell_{v_\Lambda}^{t,s}(\Lambda)}.$$

Arguing as in the proof of Theorem 4.15 and using Lemma 3.26 and (181) again

$$\left\|(\mathsf{M}_{V_{1,2}} V_g g(\mathbf{w}))_{\mathbf{w} \in \Lambda}\right\|_{\ell_{v_\Lambda}^{a,b}(\Lambda)} \leq \left\|\left(\left\|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot \mathsf{M}_V W_g^o g\right\|_{L^\infty}\right)_{\mathbf{w} \in \Lambda}\right\|_{\ell_{v_\Lambda}^{t,s}(\Lambda)}$$

$$\asymp \left\|\left(\left\|(\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot \mathsf{M}_V W_g^o g\right\|_{L^\infty}\right)_{\mathbf{w} \in \Lambda}\right\|_{(L_{\tilde{v}}^{t,s})_d(\mathfrak{X},V)}$$

$$\asymp \left\|\mathsf{M}_V W_g^o g\right\|_{W_V(L_{\tilde{v}}^{t,s})} = \left\|\mathsf{M}_V \mathsf{M}_V W_g^o g\right\|_{L_{\tilde{v}}^{t,s}}$$

$$\leq \left\|\mathsf{M}_{V^2} W_g^o g\right\|_{L_{\tilde{v}}^{t,s}} = \left\|W_g^o g\right\|_{W_{V^2}(L_{\tilde{v}}^{t,s})},$$

where we set $V^2 := VV$ (multiplicative notation in $\mathbb{H}_G$). As reported in Remark A.8, for any $0 < r \leq 1$ we have the continuous inclusion

$$W^R(L^\infty, W(L^\infty, L_{\tilde{v}}^r)) \hookrightarrow W(L^\infty, L_{\tilde{v}}^r). \tag{126}$$

Arguing as in Proposition 3.31 and taking $r < \min\{t,s\}$, we obtain

$$W(L^\infty, L_{\tilde{v}}^r) \hookrightarrow W(L^\infty, L_{\tilde{v}}^{t,s}). \tag{127}$$

The fact that g is in $\mathscr{A}_{\tilde{v}}$ (defined in (41)) implies then

$$\left\| W_g^o g \right\|_{W_{V^2}(L_{\tilde{v}}^{t,s})} < +\infty$$

and

$$\left\| C_g^*(c) \right\|_{M_m^{p,q}} \lesssim \|c\|_{\ell_{m_\Lambda}^{p,q}(\Lambda)}. \tag{128}$$

Unconditional convergence for the series defining $C_g^*(c)$ in $M_m^{p,q}(\mathcal{G})$ if $p, q \neq \infty$, and w-$*$-convergence in $M_{1/v}^\infty(\mathcal{G})$ otherwise, is inferred as in [33, Theorem 12.2.4]. The last claim comes from Lemma 3.23 and Corollary 3.28. □

Theorem 4.17 *Let* $0 < p, q \leq \infty$, $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ *and* φ *as in* (12). *Consider* $h \in \mathscr{A}_{\tilde{v}}$ *such that*

$$S_{h,\varphi} = S_{\varphi,h} = I_{L^2}, \tag{129}$$

for a suitable quasi-lattice $\Lambda = \Lambda_1 \times \Lambda_2 \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$. *Then*

$$f = \sum_{\mathbf{w} \in \Lambda} \langle f, \pi(\mathbf{w})\varphi \rangle \pi(\mathbf{w})h = \sum_{\mathbf{w} \in \Lambda} \langle f, \pi(\mathbf{w})h \rangle \pi(\mathbf{w})\varphi \tag{130}$$

with unconditional convergence in $M_m^{p,q}(\mathcal{G})$ *if* $p, q \neq \infty$, *and w-$*$-convergence in* $M_{1/v}^\infty(\mathcal{G})$ *otherwise. Moreover, for every* $f \in M_m^{p,q}(\mathcal{G})$ *we have the following quasi-norm equivalences*:

$$\begin{aligned}
\|f\|_{M_m^{p,q}} &\asymp \left(\sum_{\mu \in \Lambda_2} \left(\sum_{w \in \Lambda_1} \left| V_\varphi f(w,\mu) \right|^p m(w,\mu)^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&= \left\| \left(V_\varphi f(\mathbf{w}) \right)_{\mathbf{w} \in \Lambda} \right\|_{\ell_{m_\Lambda}^{p,q}(\Lambda)}, \\
\|f\|_{M_m^{p,q}} &\asymp \left(\sum_{\mu \in \Lambda_2} \left(\sum_{w \in \Lambda_1} \left| V_h f(w,\mu) \right|^p m(w,\mu)^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&= \left\| \left(V_h f(\mathbf{w}) \right)_{\mathbf{w} \in \Lambda} \right\|_{\ell_{m_\Lambda}^{p,q}(\Lambda)},
\end{aligned} \tag{131}$$

and similarly if $p = \infty$ *or* $q = \infty$.

Proof The proof is based on the continuity of C_φ, C_φ^*, C_h and C_h^*. The pattern is the same of [33, Corollary 12.2.6]. □

Expansions and equivalences analogous to (130) and (131) hold for φ° and h° defined in Lemma 4.14.

Proposition 4.18 *Consider $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$, define for $x \in \mathcal{G}$ and $\xi \in \widehat{\mathcal{G}}$*

$$m_1(x) := m(x, \hat{e}), \quad v_1(x) := v(x, \hat{e}), \quad v_2(\xi) := v(e, \xi). \tag{132}$$

Let $v(\xi) > 0$ be an arbitrary weight function on $\widehat{\mathcal{G}}$ such that

$$m_1 \otimes v, v_1 \otimes v_2 v^{-1} \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}}). \tag{133}$$

Let $0 < p, q, r, t, u, \gamma \le \infty$, with

$$\frac{1}{u} + \frac{1}{t} = \frac{1}{\gamma}, \tag{134}$$

and

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \quad \text{for } 1 \le r \le \infty \tag{135}$$

whereas

$$p = q = r, \quad \text{for } 0 < r < 1. \tag{136}$$

Then,

$$M_{m_1 \otimes v}^{p,u}(\mathcal{G}) * M_{v_1 \otimes v_2 v^{-1}}^{q,t}(\mathcal{G}) \hookrightarrow M_m^{r,\gamma}(\mathcal{G}) \tag{137}$$

with quasi-norm inequality

$$\|f * g\|_{M_m^{r,\gamma}} \lesssim \|f\|_{M_{m_1 \otimes v}^{p,u}} \|g\|_{M_{v_1 \otimes v_2 v^{-1}}^{q,t}}. \tag{138}$$

Proof We follow the patter displayed in [3, Proposition 3.1]. A direct computation gives $\varphi * \varphi = \varphi^\circ$, where φ is defined in (12) and φ° in (121). Similarly, the following identities can be easily checked:

$$V_h f(x, \xi) = \overline{\langle \xi, x \rangle}(f * M_\xi[h^*])(x), \quad M_\xi[\varphi^{\circ *}](x) = \left(M_\xi[\varphi^*] * M_\xi[\varphi^*]\right)(x)$$

(recall the involution $h^*(x) = \overline{h(-x)}$). Using associativity and commutativity of the convolution product, we can write

$$V_{\varphi^\circ}(f * g)(x, \xi) = \overline{\langle \xi, x \rangle}\left(\left(f * M_\xi[\varphi^*]\right) * \left(g * M_\xi[\varphi^*]\right)\right)(x). \tag{139}$$

In what follows we will use the frame expansions in Theorem 4.17 with φ° in place of φ, see Lemma 4.14. We majorize the weight m by

$$m(\mathbf{w}) = m(w, \mu) \lesssim m(w, \hat{e})v(e, \mu) = m_1(w)v_2(\mu) \qquad \mathbf{w} = (w, \mu) \in \Lambda,$$

use Young's convolution inequality for sequences in the w-variable and Hölder's one in the μ-variable. The indices p, q, r, γ, t, u fulfil the equalities in the assumptions. We show in details the case when $r, \gamma, t, u < \infty$. The others are similar. Namely,

$$
\begin{aligned}
&\|f * g\|_{M_m^{r,\gamma}} \\
&\asymp \left\| \left((V_{\varphi^\circ}(f * h))(\mathbf{w})m(\mathbf{w}) \right)_{\mathbf{w} \in \Lambda} \right\|_{\ell^{r,\gamma}(\Lambda)} \\
&= \left(\sum_{\mu \in \Lambda_2} \left(\sum_{w \in \Lambda_1} \left| V_{\varphi^\circ}(f * g)(w, \mu) \right|^r m(w, \mu)^r \right)^{\frac{\gamma}{r}} \right)^{\frac{1}{\gamma}} \\
&\lesssim \left(\sum_{\mu \in \Lambda_2} \left(\sum_{w \in \Lambda_1} \left| (f * M_\mu[\varphi^*]) * (g * M_\mu[\varphi^*])(w) \right|^r m_1(w)^r \right)^{\frac{\gamma}{r}} v_2(\mu)^\gamma \right)^{\frac{1}{\gamma}} \\
&= \left(\sum_{\mu \in \Lambda_2} \left\| \left((f * M_\mu[\varphi^*]) * (g * M_\mu[\varphi^*])(w) \right)_{w \in \Lambda_1} \right\|_{\ell_{m_1}^r(\Lambda_1)}^\gamma v_2(\mu)^\gamma \right)^{\frac{1}{\gamma}} \\
&\lesssim \left(\sum_{\mu \in \Lambda_1} \left\| \left((f * M_\mu[\varphi^*])(w) \right)_{w \in \Lambda_1} \right\|_{\ell_{m_1}^p(\Lambda_1)}^\gamma \left\| \left((g * M_\mu[\varphi^*])(w) \right)_{w \in \Lambda_1} \right\|_{\ell_{v_1}^q(\Lambda_1)}^\gamma \right. \\
&\quad \left. \times v_2(\mu)^\gamma \frac{v(\mu)^\gamma}{v(\mu)^\gamma} \right)^{\frac{1}{\gamma}};
\end{aligned}
$$

hence

$$
\begin{aligned}
\|f * g\|_{M_m^{r,\gamma}} &\leq \left(\sum_{\mu \in \Lambda_2} \left\| \left((f * M_\mu[\varphi^*])(w) \right)_{w \in \Lambda_1} \right\|_{\ell_{m_1}^p(\Lambda_1)}^u v(\mu)^u \right)^{\frac{1}{u}} \\
&\quad \times \left(\sum_{\mu \in \Lambda_2} \left\| \left((g * M_\mu[\varphi^*])(w) \right)_{w \in \Lambda_1} \right\|_{\ell_{m_1}^q(\Lambda_1)}^t \frac{v_2(\mu)^t}{v(\mu)^t} \right)^{\frac{1}{t}} \\
&= \left\| (V_\varphi f(\mathbf{w}))_{\mathbf{w} \in \Lambda} \right\|_{\ell_{m_1 \otimes v}^{p,u}(\Lambda)} \left\| (V_\varphi g(\mathbf{w}))_{\mathbf{w} \in \Lambda} \right\|_{\ell_{m_1 \otimes v_2 v^{-1}}^{q,t}(\Lambda)} \\
&\asymp \|f\|_{M_{m_1 \otimes v}^{p,u}} \|g\|_{M_{v_1 \otimes v_2 v^{-1}}^{q,t}},
\end{aligned}
$$

the last equivalence is (131). This concludes the proof. $\qquad\square$

Let us introduce the closed and compact subgroups of $\mathcal{G} \times \widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}} \times \mathcal{G}$, respectively:

$$
\mathbb{U}(\mathcal{G}) := \left(\{0_{\mathbb{R}^d}\} \times \mathcal{K} \right) \times \left(\{0_{\mathbb{R}^d}\} \times \mathcal{K}^\perp \right), \quad \mathbb{U}(\widehat{\mathcal{G}}) := \left(\{0_{\mathbb{R}^d}\} \times \mathcal{K}^\perp \right) \times \left(\{0_{\mathbb{R}^d}\} \times \mathcal{K} \right). \tag{140}
$$

Given $\mathbf{x} \in \mathcal{G} \times \widehat{\mathcal{G}}$, we will denote its projection on $(\mathcal{G} \times \widehat{\mathcal{G}})/\mathbb{U}(\mathcal{G})$ by

$$\dot{\mathbf{x}} \quad \text{or} \quad [\mathbf{x}]^{\bullet},$$

and similarly for the projection of $\xi \in \widehat{\mathcal{G}} \times \mathcal{G}$ onto $(\widehat{\mathcal{G}} \times \mathcal{G})/\mathbb{U}(\widehat{\mathcal{G}})$.

Let $\Lambda = A_{1,2}\mathbb{Z}^{2d} \times D_{1,2} \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$ and $\Gamma = A_{3,4}\mathbb{Z}^{2d} \times D_{3,4} \subseteq \widehat{\mathcal{G}} \times \mathcal{G}$ be quasi-lattices, then their projections

$$\mathscr{D}(\mathcal{G}) := \mathscr{D}(\mathcal{G}, A_{1,2}) := \dot{\Lambda} \qquad \text{and} \qquad \mathscr{D}(\widehat{\mathcal{G}}) := \mathscr{D}(\widehat{\mathcal{G}}, A_{3,4}) := \dot{\Gamma} \tag{141}$$

are discrete and at most countable LCA groups. Given a distribution f in $\mathcal{R}_{\tilde{v}}$, or S_0', and a window $g \in \mathscr{A}_{\tilde{v}}$, the function

$$\dot{V}_g f(\dot{\mathbf{x}}) := \sup_{\mathbf{z} \in \mathbb{U}(\mathcal{G})} \left| V_g f(\mathbf{x} + \mathbf{z}) \right| = \mathsf{M}_{\mathbb{U}(\mathcal{G})} V_g f(\mathbf{x}) \tag{142}$$

is well defined on the quotient group $(\mathcal{G} \times \widehat{\mathcal{G}})/\mathbb{U}(\mathcal{G})$. In fact, if $\mathbf{u}$ is such that $\dot{\mathbf{x}} = \dot{\mathbf{u}}$, then there exists $\mathbf{n} \in \mathbb{U}(\mathcal{G})$ such that $\mathbf{u} = \mathbf{x} + \mathbf{n}$. Setting $\mathbf{y} = \mathbf{n} + \mathbf{z} \in \mathbb{U}(\mathcal{G})$ we have

$$\sup_{\mathbf{z} \in \mathbb{U}(\mathcal{G})} \left| V_g f(\mathbf{u} + \mathbf{z}) \right| = \sup_{\mathbf{z} \in \mathbb{U}(\mathcal{G})} \left| V_g f(\mathbf{x} + \mathbf{n} + \mathbf{z}) \right| = \sup_{\mathbf{y} \in \mathbb{U}(\mathcal{G})} \left| V_g f(\mathbf{x} + \mathbf{y}) \right|.$$

Similarly, given a weight $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$, the function

$$\dot{m}(\dot{\mathbf{x}}) := \sup_{\mathbf{z} \in \mathbb{U}(\mathcal{G})} m(\mathbf{x} + \mathbf{z}) \tag{143}$$

is well defined on the quotient.

Lemma 4.19 *Consider a quasi-lattice Λ in $\mathcal{G} \times \widehat{\mathcal{G}}$. Let $g \in \mathscr{A}_{\tilde{v}}$, $0 < p, q \leq \infty$, $m \in \mathcal{M}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ and define the mapping*

$$\dot{\mathcal{C}}_g : M_m^{p,q}(\mathcal{G}) \to \dot{\ell}_m^{p,q}(\mathscr{D}(\mathcal{G})), f \mapsto \left(\dot{V}_g f(\dot{\mathbf{w}}) \right)_{\mathbf{w} \in \mathscr{D}(\mathcal{G})}, \tag{144}$$

where the weight $\dot{m}$ is understood to be restricted on $\mathscr{D}(\mathcal{G})$. Then, there exists a constant $C > 0$ such that for every $f \in M_m^{p,q}(\mathcal{G})$ we have

$$\left\| \dot{\mathcal{C}}_g f \right\|_{\dot{\ell}_m^{p,q}(\mathscr{D}(\mathcal{G}))} \leq C \|f\|_{M_m^{p,q}}. \tag{145}$$

Proof The BUPU $\{\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}, \mathbf{w} = (w, \mu) \in \Lambda\}$ coming from Lemma 4.9 is such that

$$\psi_w \otimes \gamma_\mu \equiv 1 \quad \text{on } \mathbf{w} + \mathbb{U}(\mathcal{G}).$$

Noticing that the projection of Λ onto $\mathscr{D}(\mathcal{G})$ is one-to-one we have without ambiguity

$$\dot{V_g}f(\dot{\mathbf{w}}) \leq \left\| (\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot V_g f \right\|_{L^\infty} = \left\| (\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot W_g^o f \right\|_{L^\infty},$$

where (w, μ) is the only representative of $\dot{\mathbf{w}}$ in the quasi-lattice. Since $\mathbb{U}(\mathcal{G})$ is compact there exists a constant $C = C(\mathbb{U}(\mathcal{G}), v) > 0$ such that

$$\frac{1}{C}m(\mathbf{x} + \mathbf{z}) \leq m(\mathbf{x}) \leq Cm(\mathbf{x} + \mathbf{z}), \tag{146}$$

for every $\mathbf{x} \in \mathcal{G} \times \widehat{\mathcal{G}}$ and $\mathbf{z} \in \mathbb{U}(\mathcal{G})$, see [65, Corollary 2.2.23]. For $\mathbf{x} = \mathbf{w} \in \Lambda$, taking the supremum over $\mathbf{z}$ in $\mathbb{U}(\mathcal{G})$ we can unambiguously write

$$\dot{\dot{m}}(\dot{\mathbf{w}}) \asymp m(\mathbf{w}). \tag{147}$$

All together we have

$$\left\| \dot{\mathcal{C}_g}f \right\|_{\ell_{\dot{m}}^{p,q}(\mathscr{D}(\mathcal{G}))} = \left\| \left(\dot{V_g}f(\dot{\mathbf{w}}) \cdot \dot{m}(\dot{\mathbf{w}}) \right)_{\dot{\mathbf{w}} \in \mathscr{D}(\mathcal{G})} \right\|_{\ell^{p,q}(\mathscr{D}(\mathcal{G}))}$$

$$\lesssim \left\| \left(\left\| (\psi_w \otimes \gamma_\mu \otimes \chi_{\mathbb{T}}) \cdot W_g^o f \right\|_{L^\infty} \cdot m(\mathbf{w}) \right)_{\mathbf{w} \in \Lambda} \right\|_{\ell^{p,q}(\Lambda)}.$$

Then, we conclude as in the proof of Theorem 4.15. $\qquad\square$

4.3 Eigenfunctions of Kohn–Nirenberg operators

We have now all the instruments to study the eigenfunctions for Kohn–Nirenberg operators. Let us first introduce the Gabor matrix of $\mathrm{Op}_0(\sigma)$.

Definition 4.20 Consider $g \in \mathcal{S}_C(\mathcal{G})$ and $\sigma \in S_0'(\mathcal{G} \times \widehat{\mathcal{G}})$. The Gabor matrix of the Kohn–Nirenberg operator $\mathrm{Op}_0(\sigma)$ (with respect to g) is defined by

$$[M(\sigma)]_{\mathbf{x},\mathbf{y}} := \langle \mathrm{Op}_0(\sigma)\pi(\mathbf{y})g, \pi(\mathbf{x})g \rangle, \qquad \mathbf{x}, \mathbf{y} \in \mathcal{G} \times \widehat{\mathcal{G}}. \tag{148}$$

The machinery developed in the previous subsection let us generalize what stated in [3, Thereom 3.3 (i)] for Weyl operators on $\mathbb{R}^d$ and proved separately in [59, Theorem 4.3] and [60, Theorem 3.1]. We will then obtain properties for the eigenfunctions in $L^2(\mathcal{G})$ of $\mathrm{Op}_0(\sigma)$ similar to the ones for Weyl operators on the Euclidean space, cf. [3, Proposition 3.5].

We start with the boundedness properties of Weyl operators.

Theorem 4.21 *Consider* $0 < p, q, \gamma \leq \infty$ *such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{\gamma} \tag{149}$$

and a symbol $\sigma \in M^{p,\min\{1,\gamma\}}(\mathcal{G} \times \widehat{\mathcal{G}})$. Then Kohn–Nirenberg operator $\mathrm{Op}_0(\sigma)$: $S_0(\mathcal{G}) \to S_0'(\mathcal{G})$ admits a unique linear continuous extension

$$\mathrm{Op}_0(\sigma) \,:\, M^q(\mathcal{G}) \to M^\gamma(\mathcal{G}).$$

Proof We distinguish two cases: $\gamma \leq 1$ and $\gamma > 1$.

Case $\gamma \leq 1$. Let φ be as in (12) and consider $h \in \mathscr{A}_{\tilde{v}}$ and a quasi-lattice Λ such that $S_{h,\varphi} = S_{\varphi,h} = I_{L^2}$. Write

$$\mathrm{Op}_0(\sigma) = \mathcal{C}_h^* \circ \mathcal{C}_\varphi \circ \mathrm{Op}_0(\sigma) \circ \mathcal{C}_\varphi^* \circ \mathcal{C}_h =: \mathcal{C}_h^* \circ M(\sigma) \circ \mathcal{C}_h. \tag{150}$$

We shall prove that the Gabor matrix $M(\sigma)$ is linear and continuous from $\ell^q(\Lambda)$ into $\ell^\gamma(\Lambda)$. It is sufficient to prove that the diagram

$$
\begin{array}{ccc}
M^q & \xrightarrow{\ \mathrm{Op}_0(\sigma)\ } & M^\gamma \\
{\scriptstyle \mathcal{C}_h}\Big\downarrow & & \Big\uparrow{\scriptstyle \mathcal{C}_h^*} \\
\ell^q & \xrightarrow[\ M(\sigma)\ ]{} & \ell^\gamma
\end{array}
$$

is commutative. We show in detail the cases $p < +\infty$ and $q < +\infty$, the others are similar. For $f \in M^q(\mathcal{G})$, using the decomposition in (150) and the notation for the Gabor matrix (148), we have

$$
\begin{aligned}
\mathrm{Op}_0(\sigma)f &= \sum_{\mathbf{w}\in\Lambda}\sum_{\mathbf{u}\in\Lambda} \langle \mathrm{Op}_0(\sigma)\pi(\mathbf{u})\varphi, \pi(\mathbf{w})\varphi \rangle \langle f, \pi(\mathbf{u})h \rangle \pi(\mathbf{w})h \\
&= \sum_{\mathbf{w}\in\Lambda}\sum_{\mathbf{u}\in\Lambda} [M(\sigma)]_{\mathbf{w},\mathbf{u}} \langle f, \pi(\mathbf{u})h \rangle \pi(\mathbf{w})h,
\end{aligned}
$$

so that

$$M(\sigma) \,:\, \ell^q(\Lambda) \to \ell^\gamma(\Lambda), \quad \left(c_{\mathbf{w}}\right)_{\mathbf{w}\in\Lambda} \mapsto \left(\sum_{\mathbf{u}\in\Lambda} [M(\sigma)]_{\mathbf{w},\mathbf{u}} c_{\mathbf{u}} \right)_{\mathbf{w}\in\Lambda}.$$

From the weak definition (9) and (15) we can write each entry of the (discrete) Gabor matrix of $\mathrm{Op}_0(\sigma)$ as follows:

$$
\begin{aligned}
[M(\sigma)]_{\mathbf{w},\mathbf{u}} &= \langle \mathrm{Op}_0(\sigma)\pi(\mathbf{u})\varphi, \pi(\mathbf{w})\varphi \rangle \\
&= \langle \sigma, R(\pi(\mathbf{w})\varphi, \pi(\mathbf{u})\varphi) \rangle \\
&= \langle \sigma, \langle v, w - u \rangle M_{\mathcal{J}(\mathbf{u}-\mathbf{w})} T_{(w,v)} R(\varphi, \varphi) \rangle \\
&= \overline{\langle v, w - u \rangle} V_\Phi \sigma((w,v), \mathcal{J}(\mathbf{u} - \mathbf{w})),
\end{aligned}
$$

where $\mathbf{w} = (w, \mu)$, $\mathbf{u} = (u, v)$ and $\Phi := R(\varphi, \varphi) \in \mathscr{A}_{\tilde{v}}(\mathcal{G} \times \widehat{\mathcal{G}})$. We introduce the mapping

$$\mathsf{T}_0 : \left(\mathcal{G} \times \widehat{\mathcal{G}}\right) \times \left(\mathcal{G} \times \widehat{\mathcal{G}}\right) \to \mathcal{G} \times \widehat{\mathcal{G}}, \quad ((w, \mu), (u, v)) \mapsto (w, v) \tag{151}$$

and write

$$\left|[M(\sigma)]_{\mathbf{w},\mathbf{u}}\right| = \left|V_\Phi \sigma\left(\mathsf{T}_0(\mathbf{w}, \mathbf{u}), \mathcal{J}(\mathbf{u} - \mathbf{w})\right)\right|. \tag{152}$$

Since $\gamma \leq 1$, we have $\|c\|_{\ell^1} \leq \|c\|_{\ell^\gamma}$ and we estimate

$$
\begin{aligned}
\|M(\sigma)c\|_{\ell^\gamma(\Lambda)} &= \left(\sum_{\mathbf{w} \in \Lambda} \left|\sum_{\mathbf{u} \in \Lambda} [M(\sigma)]_{\mathbf{w},\mathbf{u}} c_{\mathbf{u}}\right|^\gamma\right)^{\frac{1}{\gamma}} \\
&\leq \left(\sum_{\mathbf{w} \in \Lambda} \left(\sum_{\mathbf{u} \in \Lambda} \left|[M(\sigma)]_{\mathbf{w},\mathbf{u}}\right| |c_{\mathbf{u}}|\right)^\gamma\right)^{\frac{1}{\gamma}} \\
&\leq \left(\sum_{\mathbf{w} \in \Lambda} \sum_{\mathbf{u} \in \Lambda} \left|[M(\sigma)]_{\mathbf{w},\mathbf{u}}\right|^\gamma |c_{\mathbf{u}}|^\gamma\right)^{\frac{1}{\gamma}} \\
&= \left(\sum_{\mathbf{w} \in \Lambda} \sum_{\mathbf{u} \in \Lambda} \left|V_\Phi \sigma\left(\mathsf{T}_0(\mathbf{w}, \mathbf{u}), \mathcal{J}(\mathbf{u} - \mathbf{w})\right)\right|^\gamma |c_{\mathbf{u}}|^\gamma\right)^{\frac{1}{\gamma}}.
\end{aligned}
$$

Let us majorize each entry of the matrix as follows:

$$
\begin{aligned}
\left|V_\Phi \sigma\left(\mathsf{T}_0(\mathbf{w}, \mathbf{u}), \mathcal{J}(\mathbf{u} - \mathbf{w})\right)\right| &\leq \sup_{\mathbf{z} \in \mathbb{U}(\mathcal{G}), \delta \in \mathbb{U}(\widehat{\mathcal{G}})} \left|V_\Phi \sigma\left(\mathsf{T}_0(\mathbf{w}, \mathbf{u}) + \mathbf{z}, \mathcal{J}(\mathbf{u} - \mathbf{w}) + \delta\right)\right| \\
&= \overset{\bullet}{V}_\Phi \sigma\left(\left[\mathsf{T}_0(\mathbf{w}, \mathbf{u})\right]^\bullet, \left[\mathcal{J}(\mathbf{u} - \mathbf{w})\right]^\bullet\right),
\end{aligned}
\tag{153}
$$

where the function on the quotient group was introduced in (142). Fix $\mathbf{w}, \mathbf{u} \in \Lambda$ and consider $\mathbf{x} = (x, \xi)$, $\mathbf{y} = (y, \eta)$ such that $\overset{\bullet}{\mathbf{w}} = \overset{\bullet}{\mathbf{x}}$ and $\overset{\bullet}{\mathbf{u}} = \overset{\bullet}{\mathbf{y}}$. Then there exist unique $\mathbf{z} = (z, \zeta) = ((0, z_2), (0, \zeta_2))$, $\mathbf{n} = (n, \iota) = ((0, n_2), (0, \iota_2)) \in \mathbb{U}(\mathcal{G})$ such that

$$\mathbf{x} = \mathbf{w} + \mathbf{z}, \qquad \mathbf{y} = \mathbf{u} + \mathbf{n}.$$

Therefore,

$$
\begin{aligned}
\mathsf{T}_0(\mathbf{x}, \mathbf{y}) &= \mathsf{T}_0(\mathbf{w} + \mathbf{z}, \mathbf{u} + \mathbf{n}) = ((w_1, w_2 + z_2), (v_1, v_2 + \iota_2)) \\
&= \mathsf{T}_0(\mathbf{w}, \mathbf{u}) + ((0, z_2), (0, \iota_2))
\end{aligned}
$$

where $((0, z_2), (0, \iota_2)) \in \mathbb{U}(\mathcal{G})$, so that we have shown

$$\overset{\bullet}{\mathbf{w}} = \overset{\bullet}{\mathbf{x}}, \ \overset{\bullet}{\mathbf{u}} = \overset{\bullet}{\mathbf{y}} \qquad \Rightarrow \qquad \left[\mathsf{T}_0(\mathbf{w}, \mathbf{u})\right]^\bullet = \left[\mathsf{T}_0(\mathbf{x}, \mathbf{y})\right]^\bullet. \tag{154}$$

Similarly,

$$\mathcal{J}(\mathbf{y} - \mathbf{x}) = \mathcal{J}(\mathbf{u} + \mathbf{n} - \mathbf{w} - \mathbf{z}) = \mathcal{J}(\mathbf{u} - \mathbf{w}) + \mathcal{J}(\mathbf{n} - \mathbf{z})$$

and being $\mathcal{J}(\mathbf{n} - \mathbf{z}) \in \mathbb{U}(\widehat{\mathcal{G}})$ we have proved

$$\dot{\mathbf{w}} = \dot{\mathbf{x}}, \ \dot{\mathbf{u}} = \dot{\mathbf{y}} \qquad \Rightarrow \qquad [\mathcal{J}(\mathbf{u} - \mathbf{w})]^{\bullet} = [\mathcal{J}(\mathbf{y} - \mathbf{x})]^{\bullet}. \tag{155}$$

Hence the function in (153) depends only on the cosets of $\mathbf{w}$ and $\mathbf{u}$, so that the application

$$\dot{H}(\dot{\mathbf{u}}, \dot{\mathbf{w}}) := \dot{V}_\Phi \sigma \left([\mathsf{T}_0(\mathbf{w}, \mathbf{u})]^{\bullet}, [\mathcal{J}(\mathbf{u} - \mathbf{w})]^{\bullet} \right) \tag{156}$$

is well defined. A sequence $c = (c_\mathbf{w})_{\mathbf{w}\in\Lambda}$ on the quasi-lattice Λ uniquely determines a sequence on $\mathscr{D}(\mathcal{G}) = \Lambda$ simply by

$$\dot{c} := \left(c_{\dot{\mathbf{w}}} := c_\mathbf{w} \right)_{\dot{\mathbf{w}}\in\mathscr{D}(\mathcal{G})} \tag{157}$$

with

$$\|c\|_{\ell^q(\Lambda)} = \left\| \dot{c} \right\|_{\ell^q(\mathscr{D}(\mathcal{G}))}.$$

Using Hölder's inequality in the $\dot{\mathbf{u}}$ variable (observe $1/(p/\gamma) + 1/(q/\gamma) = 1$) and the consideration above:

$$\|M(\sigma)c\|_{\ell^\gamma(\Lambda)}$$

$$\leq \left(\sum_{\mathbf{w}\in\Lambda} \sum_{\mathbf{u}\in\Lambda} \dot{H}(\dot{\mathbf{u}}, \dot{\mathbf{w}})^\gamma |c_\mathbf{u}|^\gamma \right)^{\frac{1}{\gamma}}$$

$$= \left(\sum_{\dot{\mathbf{w}}\in\mathscr{D}(\mathcal{G})} \sum_{\dot{\mathbf{u}}\in\mathscr{D}(\mathcal{G})} \dot{H}(\dot{\mathbf{u}}, \dot{\mathbf{w}})^\gamma \left| c_{\dot{\mathbf{u}}} \right|^\gamma \right)^{\frac{1}{\gamma}}$$

$$\leq \left(\sum_{\dot{\mathbf{w}}\in\mathscr{D}(\mathcal{G})} \left(\sum_{\dot{\mathbf{u}}\in\mathscr{D}(\mathcal{G})} \dot{H}(\dot{\mathbf{u}}, \dot{\mathbf{w}})^{\gamma\frac{p}{\gamma}} \right)^{\frac{\gamma}{p}} \left(\sum_{\dot{\mathbf{u}}\in\mathscr{D}(\mathcal{G})} \left| c_{\dot{\mathbf{u}}} \right|^{\gamma\frac{q}{\gamma}} \right)^{\frac{\gamma}{q}} \right)^{\frac{1}{\gamma}}$$

$$= \|c\|_{\ell^q(\Lambda)} \left(\sum_{\dot{\mathbf{w}}\in\mathscr{D}(\mathcal{G})} \left(\sum_{\dot{\mathbf{u}}\in\mathscr{D}(\mathcal{G})} \dot{V}_\Phi \sigma \left([\mathsf{T}_0(\mathbf{w}, \mathbf{u})]^{\bullet}, [\mathcal{J}(\mathbf{u} - \mathbf{w})]^{\bullet} \right)^p \right)^{\frac{\gamma}{p}} \right)^{\frac{1}{\gamma}}.$$

Let us perform the following change of variables:

$$\dot{\theta} := [\mathcal{J}(\mathbf{u} - \mathbf{w})]^{\bullet} \in \mathscr{D}(\widehat{\mathcal{G}}) = [\mathcal{J}\Lambda]^{\bullet}. \tag{158}$$

Notice that $\mathcal{J}\Lambda \subseteq \widehat{\mathcal{G}} \times \mathcal{G}$ is a quasi-lattice. Then, there exists $\delta \in \mathbb{U}(\widehat{\mathcal{G}})$ such that $\theta + \delta = \mathcal{J}(\mathbf{u} - \mathbf{w})$ and

$$\mathbf{u} - \mathbf{w} = \mathcal{T}^{-1}(\theta + \delta) \Rightarrow \mathbf{w} = \mathbf{u} - \mathcal{T}^{-1}(\theta) - \mathcal{T}^{-1}(\delta) \Rightarrow \dot{\mathbf{w}} = [\mathbf{u} - \mathcal{T}^{-1}(\theta)]^{\bullet},$$

since $-\mathcal{T}^{-1}(\delta) \in \mathbb{U}(\mathcal{G})$. Recalling (154) and writing

$$\boldsymbol{\theta} = (\theta, s) = ((\theta_1, \theta_2), (s_1, s_2)) \in \mathcal{J}\Lambda,$$

we have

$$\begin{aligned}
[\mathsf{T}_0(\mathbf{w}, \mathbf{u})]^{\bullet} &= [\mathsf{T}_0(\mathbf{u} - \mathcal{T}^{-1}(\theta), \mathbf{u})]^{\bullet} = [\mathsf{T}_0((u - s, v + \theta), (u, v))]^{\bullet} \\
&= [(u - s, v)]^{\bullet} = [\mathbf{u} - (s, \hat{e})]^{\bullet}.
\end{aligned}$$

In the above calculation we can choose as representative of $\dot{\theta}$ the only one in $\mathcal{J}\Lambda$ without loss of generality. In fact, write $\Lambda = (\alpha\mathbb{Z}^d \times D_1) \times (\alpha\mathbb{Z}^d \times D_2)$, $\mathcal{J}\Lambda = (\alpha\mathbb{Z}^d \times -D_2) \times (\alpha\mathbb{Z}^d \times D_1)$, and consider $\boldsymbol{\eta} = (\eta, l) = ((\eta_1, \eta_2), (l_1, l_2))$ such that $\boldsymbol{\theta} = \dot{\boldsymbol{\eta}}$ and $\boldsymbol{\eta} \notin \mathcal{J}\Lambda$. Being $\mathbb{U}(\widehat{\mathcal{G}}) = (\{0_{\mathbb{R}^d}\} \times \mathcal{K}^{\perp}) \times (\{0_{\mathbb{R}^d}\} \times \mathcal{K})$, it necessarily follows that $\theta_1 = \eta_1$ and $s_1 = l_1$ in $\alpha\mathbb{Z}^d$, $[\theta_2]^{\bullet} = [\eta_2]^{\bullet}$ in $\widehat{\mathcal{G}}_0/\mathcal{K}^{\perp}$, $[s_2]^{\bullet} = [l_2]^{\bullet}$ in $\mathcal{G}_0/\mathcal{K}$ and $[(l, \hat{e})]^{\bullet} \in \Lambda$.

Eventually, we set

$$\dot{\mathbf{z}} := \dot{\mathbf{u}} - [(s, \hat{e})]^{\bullet} \in \mathscr{D}(\mathcal{G}) = \dot{\Lambda} \tag{159}$$

and using Lemma 4.19

$$\begin{aligned}
\left(\sum_{\dot{\mathbf{w}} \in \mathscr{D}(\mathcal{G})} \left(\sum_{\dot{\mathbf{u}} \in \mathscr{D}(\mathcal{G})} \dot{V}_{\Phi}\sigma\left([\mathsf{T}_0(\mathbf{w}, \mathbf{u})]^{\bullet}, [\mathcal{J}(\mathbf{u} - \mathbf{w})]^{\bullet} \right)^p \right)^{\frac{\gamma}{p}} \right)^{\frac{1}{\gamma}} \\
= \left(\sum_{\dot{\theta} \in \mathscr{D}(\widehat{\mathcal{G}})} \left(\sum_{\dot{\mathbf{z}} \in \mathscr{D}(\mathcal{G})} \dot{V}_{\Phi}\sigma(\dot{\mathbf{z}}, \dot{\theta})^p \right)^{\frac{\gamma}{p}} \right)^{\frac{1}{\gamma}} \\
= \left\| \dot{\mathcal{C}}_{\Phi}\sigma \right\|_{\ell^{p,\gamma}(\mathscr{D}(\mathcal{G}) \times \mathscr{D}(\widehat{\mathcal{G}}))} \\
\lesssim \|\sigma\|_{M^{p,\gamma}(\mathcal{G} \times \widehat{\mathcal{G}})} < +\infty.
\end{aligned}$$

Case $\gamma > 1$. Observe that $p \geq \gamma > 1$ and $q \geq \gamma > 1$. Consider first $p \neq \infty$. The desired result is obtained by duality. By Proposition 3.32 $M^{\gamma}(\mathcal{G}) \cong (M^{\gamma'}(\mathcal{G}))'$, we hence show that if $f \in M^q(\mathcal{G})$ then $\mathrm{Op}_0(\sigma)f$ is a continuous linear functional on $M^{\gamma'}(\mathcal{G})$. Let $g \in M^{\gamma'}(\mathcal{G})$, from the weak definition (9) and the fact that $M^{p,1}(\mathcal{G} \times \widehat{\mathcal{G}}) \cong (M^{p',\infty}(\mathcal{G} \times \widehat{\mathcal{G}}))'$ we get:

$$\left| \langle \mathrm{Op}_0(\sigma)f, g \rangle \right| = |\langle \sigma, R(g, f) \rangle| \leq \|\sigma\|_{M^{p,1}} \|R(g, f)\|_{M^{p',\infty}}.$$

The indexes' conditions in (95) and (96) become

$$\gamma', q \le \infty, \tag{160}$$

$$\frac{1}{\gamma'} + \frac{1}{q} \ge \frac{1}{p}. \tag{161}$$

The first one is trivial, the second follows from the assumption (149).

Therefore

$$\|R(g,f)\|_{M^{p',\infty}} \lesssim \|g\|_{M^{\gamma'}} \|f\|_{M^q}$$

and the boundedness of $\mathrm{Op}_0(\sigma)$ from $M^q(\mathcal{G})$ into $M^\gamma(\mathcal{G})$ follows.

If $p = \infty$ the argument is similar, we use the duality (84) between $M^{\infty,1}$ and $M^{1,\infty}$. □

Proposition 4.22 *Consider a symbol σ on the phase space such that for some $0 < p < \infty$*

$$\sigma \in \bigcap_{\gamma > 0} M^{p,\gamma}(\mathcal{G} \times \widehat{\mathcal{G}}). \tag{162}$$

If $\lambda \in \sigma_P(\mathrm{Op}_0(\sigma)) \smallsetminus \{0\}$, then any eigenfunction $f \in L^2(\mathcal{G})$ with eigenvalue λ satisfies $f \in \bigcap_{\gamma > 0} M^\gamma(\mathcal{G})$.

Proof We use Theorem 4.21 and follow the proof pattern of [3, Proposition 3.5].

□

5 Localization operators on groups

The aim of this section is to infer a result for L^2 eigenfunctions of localization operators which extends the one obtained in the Euclidean setting in [3, Theorem 3.7].

We address the reader to Wong's book [66] for a detailed treatment of localization operators on locally compact Hausdorff groups and point out the recent works [43, 44]. Let us recall their definition.

Definition 5.1 Consider windows $\psi_1, \psi_2 \in S_0(\mathcal{G}) = \mathbb{G}_1$ and symbol $a \in S_0'(\mathcal{G} \times \widehat{\mathcal{G}})$. Then the localization operator with symbol a and windows ψ_1, ψ_2 in $S_0(\mathcal{G})$ is formally defined as

$$A_a^{\psi_1,\psi_2} f(x) = \int_{\mathcal{G} \times \widehat{\mathcal{G}}} a(u,\omega) V_{\psi_1} f(u,\omega) M_\omega T_u \psi_2(x) \, du d\omega. \tag{163}$$

Equivalently, its weak definition is

$$\langle A_a^{\psi_1,\psi_2} f, g \rangle = \langle a, \overline{V_{\psi_1} f} V_{\psi_2} g \rangle \qquad \forall g \in S_0(\mathcal{G}). \tag{164}$$

It is straightforward computation to check that

$$A_a^{\psi_1,\psi_2} : S_0(\mathcal{G}) \to S_0'(\mathcal{G}) \tag{165}$$

is well defined, linear and continuous (cf. [41, Theorem 5.3]). Concretely, we shall mainly consider windows $\psi_1, \psi_2 \in \mathcal{S}_\mathcal{C}(\mathcal{G})$ rather than in the whole Feichtinger algebra. Notice that if $a \in L^p(\mathcal{G} \times \widehat{\mathcal{G}})$, for any $1 \le p \le \infty$, then $A_a^{\psi_1,\psi_2} \in B(L^2(\mathcal{G}))$, cf. [66, Proposition 12.1, 12.2, 12.3].

Given a function F on $\mathcal{G} \times \mathcal{G}$, we introduce the operator $\mathfrak{T}_b$:

$$\mathfrak{T}_b F(x, u) = F(x, u - x). \tag{166}$$

Recall that $\mathcal{F}_2$ stands for the partial Fourier transform with respect to the second variable of measurable functions σ defined on $\mathcal{G} \times \widehat{\mathcal{G}}$. We shall consider $\mathcal{F}_2 \sigma$ to be defined on $\mathcal{G} \times \mathcal{G}$, instead of $\mathcal{G} \times \widehat{\mathcal{G}}$, due to the Pontryagin's duality. $\mathfrak{T}_b$ and $\mathcal{F}_2$ are automorphisms of $S_0(\mathcal{G} \times \mathcal{G})$ and $S_0(\mathcal{G} \times \widehat{\mathcal{G}})$, respectively, which extend to automorphisms of $S_0'(\mathcal{G} \times \mathcal{G})$ and $S_0'(\mathcal{G} \times \widehat{\mathcal{G}})$ by transposition.

Lemma 5.2 *Consider* $\sigma \in S_0'(\mathcal{G} \times \widehat{\mathcal{G}})$ *and* $f, g \in S_0(\mathcal{G})$. *Then*

$$\langle \operatorname{Op}_0(\sigma)f, g \rangle_{L^2(\mathcal{G})} = \left\langle k_\sigma, g \otimes \overline{f} \right\rangle_{L^2(\mathcal{G} \times \mathcal{G})}, \tag{167}$$

where the kernel k_σ *is given by*

$$k_\sigma(x, u) = \int_{\widehat{\mathcal{G}}} \sigma(x, \xi) \overline{\langle u - x, \xi \rangle} \, \mathrm{d}\xi = \mathfrak{T}_b(\mathcal{F}_2 \sigma(x, u)). \tag{168}$$

Proof The proof carries over from the Euclidean case almost verbatim, see, e.g., [13, formula (4.3)]. $\qquad \square$

The following issue presents the connection between localization and Kohn–Nirenberg operators on LCA groups, extending the Euclidean case proved in [4, Proposition 2.16].

Proposition 5.3 *Consider windows* $\psi_1, \psi_2 \in S_0(\mathcal{G})$ *and a symbol* $a \in S_0'(\mathcal{G} \times \widehat{\mathcal{G}})$. *Then, we have*

$$A_a^{\psi_1,\psi_2} = \operatorname{Op}_0(a * R(\psi_2, \psi_1)). \tag{169}$$

Proof The proof is similar to the Euclidean case. We detail it for sake of clarity. We first compute the kernel k of $A_a^{\psi_1,\psi_2}$:

$$\left\langle A_a^{\psi_1,\psi_2} f, g \right\rangle$$
$$= \int_{\mathcal{G} \times \widehat{\mathcal{G}}} a(x, \xi) \left(\int_{\mathcal{G}} f(u) \overline{\pi(x, \xi)\psi_1(u)} \, \mathrm{d}u \right) \left(\int_{\mathcal{G}} \overline{g(y)} \pi(x, \xi)\psi_2(y) \, \mathrm{d}y \right) \mathrm{d}x \mathrm{d}\xi$$
$$= \int_{\mathcal{G} \times \mathcal{G}} f(u) \overline{g(y)} k(y, u) \, \mathrm{d}y \mathrm{d}u,$$

with

$$k(y, u) = \int_{\mathcal{G} \times \widehat{\mathcal{G}}} a(x, \xi) \overline{\pi(x, \xi)\psi_1(u)} \pi(x, \xi)\psi_2(y) \, dxd\xi.$$

Using Lemma 5.2, we set $\mathfrak{T}_b \circ \mathcal{F}_2(\sigma) = k$ and compute σ using (15) as follows:

$$
\begin{aligned}
\mathcal{F}_2^{-1} \circ \mathfrak{T}_b^{-1}(k) &= \int_{\mathcal{G} \times \widehat{\mathcal{G}}} a(x, \xi) \mathcal{F}_2^{-1} \circ \mathfrak{T}_b^{-1} \left(\pi(x, \xi)\psi_2 \otimes \overline{\pi(x, \xi)\psi_1}(y, u) \right) dxd\xi \\
&= \int_{\mathcal{G} \times \widehat{\mathcal{G}}} a(x, \xi) \mathcal{F}_2^{-1} \left(\pi(x, \xi)\psi_2(y) \cdot \overline{\pi(x, \xi)\psi_1(u + y)} \right) dxd\xi \\
&= \int_{\mathcal{G} \times \widehat{\mathcal{G}}} a(x, \xi)\pi(x, \xi)\psi_2(y) \int_{\mathcal{G}} \overline{\pi(x, \xi)\psi_1(u + y)} \langle \omega, u \rangle \, du \, dxd\xi \\
&= \int_{\mathcal{G} \times \widehat{\mathcal{G}}} a(x, \xi)\pi(x, \xi)\psi_2(y) \overline{\langle \omega, y \rangle \mathcal{F}(\pi(x, \xi)\psi_1)(\omega)} \, dxd\xi \\
&= \int_{\mathcal{G} \times \widehat{\mathcal{G}}} a(x, \xi) R(\pi(x, \xi)\psi_2, \pi(x, \xi)\psi_1)(y, \omega) \, dxd\xi \\
&= \int_{\mathcal{G} \times \widehat{\mathcal{G}}} a(x, \xi) R(\psi_2, \psi_1)((y, \omega) - (x, \xi)) \, dxd\xi \\
&= a * R(\psi_2, \psi_1)(y, \omega).
\end{aligned}
$$

We then infer the thesis from the kernels' theorem [18, Theorem B3]. □

Theorem 5.4 *Let $0 < p < \infty$ and $a \in M^{p,\infty}(\mathcal{G} \times \widehat{\mathcal{G}})$. Consider $\psi_1, \psi_2 \in \mathcal{S}_{\mathcal{C}}(\mathcal{G}) \smallsetminus \{0\}$. Suppose that $\sigma_P(A_a^{\psi_1,\psi_2}) \smallsetminus \{0\} \neq \varnothing$ and $\lambda \in \sigma_P(A_a^{\psi_1,\psi_2}) \smallsetminus \{0\}$. Then, any eigenfunction $f \in L^2(\mathcal{G})$ with eigenvalue λ satisfies*

$$f \in \bigcap_{\gamma > 0} M^\gamma(\mathcal{G}). \tag{170}$$

Proof Observe that for $\psi_1, \psi_2 \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$ we have $R(\psi_2, \psi_1) \in \mathscr{A}_{\tilde{v}}(\mathcal{G} \times \widehat{\mathcal{G}})$, by Corollary 3.11. Therefore, $R(\psi_2, \psi_1)$ belongs to every modulation space on the phase space; this is easily seen using (41), the inclusion relations (195) and the inclusion between modulation spaces in Proposition 3.31. Then, the argument is the same as in [3, Theorem 3.7]: we write $A_a^{\psi_1,\psi_2}$ in the Kohn–Nirenberg form (Proposition 5.3)

$$A_a^{\psi_1,\psi_2} = \mathrm{Op}_0 \left(a * R(\psi_2, \psi_1) \right), \tag{171}$$

use the convolution relations in Proposition 4.18 and infer the thesis applying Proposition 4.22. □

Appendix

We summarize the construction of coorbit spaces $\mathsf{Co}(Y)$, when Y is a solid quasi-Banach function space on a locally compact Hausdorff group G, even not abelian. This theory was first developed by Rauhut in [49] and technically fixed and deepened by Voigtlaender in his Ph.D. thesis [65]. In the end we shall highlight the differences with the original theory for Banach spaces by Feichtinger and Gröchenig, see [24–26].

We mention that an exposition and treatment of the named coorbit theory is now available also in the recent article [63] from van Velthoven and Voigtlaender, where the requirements on the weights are lightened up. However, due to the time when this work was written, we shall stick to the first version presented in [65]. Moreover, on account of the objects of our particular setting, this makes no difference.

In this section, we deal with a locally compact, Hausdorff, σ-compact group G; the notation is intended to distinguish from the abelian case $\mathcal{G}$. The group operation on G will be expressed as multiplication; whenever a measure on G is involved, it is understood to be the left Haar measure. We shall not list systematically the known properties for the spaces introduced in the sequel, but rather recall them when necessary. The reader is invited to consult [65, Chapter 2] for an exhaustive treatment.

Given $x \in G$ and a function f on G, we define the left and right translation operators as

$$L_x f(y) = f(x^{-1}y), \quad R_x f(y) = f(yx). \tag{172}$$

Definition A.1 We say that $\left(Y, \|\cdot\|_Y\right)$ is a function space on G if it is a quasi-normed space consisting of equivalence classes of measurable complex-valued functions on G, where two functions are identified if they coincide a.e..

A function space $\left(Y, \|\cdot\|_Y\right)$ on G is said to be left invariant if $L_x : Y \to Y$ is well defined and bounded for every $x \in G$, similarly we define the right invariance. We say that Y is bi-invariant if it is both left and right invariant.

A function space $(Y, \|\cdot\|_Y)$ on G is said solid if given $g \in Y$ and $f : G \to \mathbb{C}$ measurable the following holds true:

$$|f| \leq |g| \quad \text{a.e.} \quad \Rightarrow \quad f \in Y, \quad \|f\|_Y \leq \|g\|_Y;$$

Y is called quasi-Banach function (QBF) space on G if it is complete.

Without loss of generality, we can assume $\|\cdot\|_Y$ to be a r-norm, $0 < r \leq 1$, i.e.,

$$\|f + g\|_Y^r \leq \|f\|_Y^r + \|g\|_Y^r, \quad \forall f, g \in Y.$$

This is due to the Aoki–Rolewicz Theorem and the fact that equivalent quasi-norms induce the same topology [65, Theorem 2.1.4, Lemma 2.1.5].

It can be useful to describe Wiener Amalgam spaces, defined below, in terms of sequences. To this end, the so-called BUPUs and a particular space of sequences Y_d associated to Y are introduced. We present the space Y_d under specific hypothesis fitting our framework, nevertheless a more general theory is possible, see [50] and [65, Chapter 2].

Definition A.2 A family $X = \{x_i\}_{i \in I}$ in G is called relatively separated if for all compact sets $K \subseteq G$ we have

$$C_{X,K} := \sup_{i \in I} \#\{j \in I \mid x_i K \cap x_j K \neq \varnothing\} < +\infty, \tag{173}$$

where #S is the cardinality of a set S. Consider $X = \{x_i\}_{i \in I}$ relatively separated family in G, $Q \subseteq G$ measurable, relatively compact set of positive measure and $(Y, \|\cdot\|_Y)$ solid QBF space on G. Then, the discrete sequence space associated to Y is the set

$$Y_d(X, Q) = \left\{ (\lambda_i)_{i \in I} \in \mathbb{C}^I \mid \sum_{i \in I} |\lambda_i| \chi_{x_i Q} \in Y \right\} \tag{174}$$

endowed with the quasi-norm

$$\left\| (\lambda_i)_{i \in I} \right\|_{Y_d(X,Q)} = \left\| \sum_{i \in I} |\lambda_i| \chi_{x_i Q} \right\|_Y; \tag{175}$$

$\mathbb{C}^I$ is the space of functions from I into $\mathbb{C}$.

If G is σ-compact, then any relatively separated family X is (at most) countable, see [65, Lemma 2.3.10]. In the setting presented so far, $Y_d(X, Q)$ is a quasi-Banach space. Moreover, if Y is right invariant then $Y_d(X, Q)$ is independent of Q in the sense that another $U \subseteq G$ measurable, relatively compact and with non empty interior yields the same space with an equivalent quasi-norm (cf. [50, Lemma 2.2] and [65, Lemma 2.3.16]).

Definition A.3 Let $U \subseteq G$ be a relatively compact, unit neighborhood. A family $\Psi = \{\psi_i\}_{i \in I}$ of continuous functions on G is called a bounded uniform partition of unity of size U (U-BUPU) if

  (i) $0 \leq \psi_i(x) \leq 1$ for all $x \in G$ and every $i \in I$;
  (ii) there exists $X = \{x_i\}_{i \in I}$ U-localizing family for ψ i.e., X is a relatively separated family in G such that

$$\operatorname{supp} \psi_i \subseteq x_i U \quad \forall i \in I;$$

  (iii) $\sum_{i \in I} \psi_i \equiv 1$.

Given any relatively compact unit neighborhood U in G, there always exists a family Ψ which is a U-BUPU with some U-localizing family X ([20, Theorem 2],

[65, Lemma 2.3.212]) and, since G is σ-compact, the indexes' set is (at most) countable.

We introduce the Wiener Amalgam spaces not in their full generality, but restrict ourselves to cases which ensure "good" properties.

Definition A.4 Consider $Q \subseteq G$ measurable, relatively compact, unit neighborhood and $f : G \to \mathbb{C}$ measurable. We call maximal function of f with respect to Q the following application

$$\mathsf{M}_Q f : G \to [0, +\infty], \quad x \mapsto \operatorname*{ess\,sup}_{y \in xQ} |f(y)|. \tag{176}$$

We fix a solid QBF space $(Y, \|\cdot\|_Y)$ on G and define the Wiener Amalgam space with window Q, local component $L^\infty = L^\infty(G)$ and global component Y as

$$W_Q(Y) := W_Q(L^\infty, Y) = \left\{ f \in L^\infty_{\text{loc}}(G) \mid \mathsf{M}_Q f \in Y \right\} \tag{177}$$

and endow it with

$$\|f\|_{W_Q(Y)} := \|f\|_{W_Q(L^\infty, Y)} = \left\| \mathsf{M}_Q f \right\|_Y. \tag{178}$$

It was proven in [65, Lemma 2.3.4] that the maximal function $\mathsf{M}_Q f$ is measurable. Under the assumptions of the above definition, the Wiener Amalgam space $(W_Q(Y), \|\cdot\|_{W_Q(Y)})$ is a solid QBF space on G, in particular, $\|\cdot\|_{W_Q(Y)}$ is a r-norm, $0 < r \le 1$, if $\|\cdot\|_Y$ is. For each $f \in L^\infty_{\text{loc}}(G)$ we have

$$|f(x)| \le \mathsf{M}_Q f(x) \quad \text{a.e.,} \tag{179}$$

which together with the solidity of Y gives the continuous embedding

$$W_Q(L^\infty, Y) \hookrightarrow Y. \tag{180}$$

In general the definition of $W_Q(Y)$ may depend on the chosen subset Q. However, we shall require some further properties in order to make the Wiener space independent of it. We collect some of the results of [65, Lemma 2.3.16, Theorem 2.3.17] in the following lemma (which holds under milder assumptions).

Lemma A.5 *Under the hypothesis presented so far, if the solid QBF space Y on G is right invariant, then the following equivalent facts hold true:*

(i) *The Wiener Amalgam space $W_Q(L^\infty, Y)$ is right invariant for each measurable, relatively compact, unit neighborhood $Q \subseteq G$;*

(ii) *The Wiener Amalgam space $W_Q(L^\infty, Y)$ is independent of the choice of the measurable, relatively compact, unit neighborhood $Q \subseteq G$, in the sense that different choices yield the same set with equivalent quasi-norms. The equivalence constants depend only on the two sets $Q, Q' \subseteq G$ and on Y.*

If these conditions are fulfilled, $\Psi = \{\psi_i\}_{i \in I}$ is a U-BUPU for some localizing family $X = \{x_i\}_{i \in I}$ and $U \subseteq G$ relatively compact unit neighborhood, then

$$\|f\|_{W_Q(L^\infty, Y)} \underset{X, Q, Y}{\asymp} \left\| \left(\|\psi_i \cdot f\|_{L^\infty} \right)_{i \in I} \right\|_{Y_d(X, Q)} \tag{181}$$

for every $f \in W_Q(L^\infty, Y)$ and the constants involved in the above equivalence depend only on X, Q and Y.

We remark that the right invariance of Y is sufficient for conditions (i) or (ii) but not necessary; the existence of an U-BUPU Ψ is always guaranteed. When one of the above conditions is satisfied, we suppress the index Q in the Wiener space and simply write $W(L^\infty, Y)$ or $W(Y)$.

By considering Qx instead of xQ in the definition of the maximal function, we obtain the "right-sided" version of the Wiener spaces. So that we set the right-sided maximal function to be

$$\mathsf{M}_Q^R f : G \to [0, +\infty], \quad x \mapsto \operatorname*{ess\,sup}_{y \in Qx} |f(y)| \tag{182}$$

and define the right-sided Wiener Amalgam space $W_Q^R(Y)$ similarly as before. Analogous considerations hold for $W_Q^R(Y)$, with the proper cautions about Lemma A.5. In particular, the independence of $W_Q^R(Y)$ from Q is guaranteed if Y is left invariant, see [65, Lemma 2.3.29].

Definition A.6 A weight on G is a measurable function $m : G \to (0, +\infty)$. A weight v is said to be submultiplicative if

$$v(xy) \leq v(x)v(y), \qquad \forall x, y \in G. \tag{183}$$

Given two weights m and v on G, m is said to be left-moderate *w.r.t.* v if

$$m(xy) \lesssim v(x)m(y), \qquad \forall x, y \in G, \tag{184}$$

it is right-moderate *w.r.t.* v if

$$m(xy) \lesssim m(x)v(y), \qquad \forall x, y \in G. \tag{185}$$

If a weight m is both left- and right-moderate w.r.t. v, we simply say that it is moderate *w.r.t.* v or *v-moderate*.

Consider v submultiplicative weight on G which is also even, bounded from below and satisfies the Gelfand–Raikov–Shilov (GRS) condition, i.e.,

$$v(x) = v(-x) \quad \forall x \in G,$$
$$\exists c > 0 : v(x) \geq c \quad \forall x \in G,$$
$$\lim_{n \to +\infty} v(x^n)^{\frac{1}{n}} = 1 \quad \forall x \in G,$$

then the class of weights on G moderate *w.r.t.* v is denoted as follows:

$$\mathcal{M}_v(G) = \{m \text{ weight on } G \mid m \text{ is } v-\text{moderate}\}. \tag{186}$$

Remark A.7 The GRS condition will be used in this paper only in the subsection dealing with Gabor frames, see Theorem 4.12. In this framework, v is a weight on the abelian group $\mathcal{G} \times \widehat{\mathcal{G}}$, hence the GRS condition has the form

$$\lim_{n \to +\infty} v(n\mathbf{x})^{\frac{1}{n}} = 1 \quad \forall\, \mathbf{x} \in \mathcal{G} \times \widehat{\mathcal{G}}.$$

We are now able to state the coorbit theory in [65, Assumption 2.4.1], see items **A–G** and **H –J**.

A. We assume G to be a LCH, σ-compact group. We consider $\rho : G \to \mathcal{U}(\mathcal{H})$ a strongly continuous, unitary, irreducible representation of G for some non-trivial complex Hilbert space $\mathcal{H}$. $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators on $\mathcal{H}$ (see e.g., [28, 66]).

B. Given $f, g \in \mathcal{H}$, we define the (generalized) wavelet transform induced by ρ, or voice transform, of f w.r.t. g as

$$W_g^\rho f : G \to \mathbb{C}, \quad x \mapsto \langle f, \rho(x)g \rangle_{\mathcal{H}}, \tag{187}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, also denoted by $\langle \cdot, \cdot \rangle$, is the inner product on $\mathcal{H}$ supposed antilinear in the second component. $W_g^\rho f$ is always a continuous and bounded function on G, see [66]. We assume the representation ρ to be integrable, i.e. there exists $g \in \mathcal{H} \smallsetminus \{0\}$ such that $W_g^\rho g \in L^1(G)$; this implies that ρ is also square-integrable: there exists $g \in \mathcal{H} \smallsetminus \{0\}$ such that $W_g^\rho g \in L^2(G)$. Such a g is said to be admissible.

C. $(Y, \|\cdot\|_Y)$ will be supposed to be a solid QBF space on G with $\|\cdot\|_Y$, or some equivalent quasi-norm, r-norm with $0 < r \le 1$.

D. The Wiener Amalgam space $W_Q(L^\infty, Y)$ is assumed right invariant for each measurable, relatively compact, unit neighborhood $Q \subseteq G$. We consider a submultiplicative weight $w : G \to (0, +\infty)$ such that for some (and hence each) measurable, relatively compact, unit neighborhood $Q \subseteq G$

$$w(x) \gtrsim_Q \left|\left|\left| R_x \right|\right|\right|_{W_Q(Y) \to W_Q(Y)} \tag{188}$$

and

$$w(x) \gtrsim_Q \Delta(x^{-1}) \left|\left|\left| R_{x^{-1}} \right|\right|\right|_{W_Q(Y) \to W_Q(Y)}, \tag{189}$$

where $\Delta(x)$ is the modular function on G. We also require the weight w to be bounded from below, i.e. there exists $c > 0$ such that $w(x) \ge c$ for every $x \in G$.

If the condition on $W_Q(Y)$ in **D** is satisfied, then the Wiener space is independent of Q, so that we can omit the lower index. Moreover, this is ensured if Y is right invariant (Lemma A.5).

E. We fix a submultiplicative weight $v : G \to (0, +\infty)$, which will be called control weight for Y, such that

$$v \geq w, \qquad v \geq w_{\vee, r}, \tag{190}$$

where w is defined in **D** and

$$w_{\vee, r}(x) = w(x^{-1})\big[\Delta(x^{-1})\big]^{1/r}. \tag{191}$$

F. The class of good vectors is defined to be

$$\mathbb{G}_v := \Big\{ g \in \mathcal{H} \mid W_g^\rho g \in L_v^1(G) \Big\} \tag{192}$$

and supposed nontrivial, $\{0\} \subsetneq \mathbb{G}_v$.

G. The class of analyzing vectors is defined as

$$\mathbb{A}_v^r := \Big\{ g \in \mathcal{H} \mid W_g^\rho g \in W^R(L^\infty, W(L^\infty, L_v^r)) \Big\} \tag{193}$$

and supposed nontrivial, $\{0\} \subsetneq \mathbb{A}_v^r$.

Remark A.8 (i) Observe that, since v is submultiplicative, $L_v^r(G)$ is bi-invariant. This implies that $W(L_v^r)$ is independent of the window Q and it is left invariant, hence also $W^R(W(L_v^r))$ is independent of the window subset. Concretely, this allows us to work with the same Q:

$$\Big\| W_g^\rho g \Big\|_{W^R(W(L_v^r))} \asymp \Big\| \mathsf{M}_Q \mathsf{M}_Q^R W_g^\rho g \Big\|_{L_v^r}, \tag{194}$$

(see in Lemma 3.10). (ii) From the continuous embeddings for $0 < r \leq 1$

$$W^R(L^\infty, W(L^\infty, L_v^r)) \hookrightarrow W(L^\infty, L_v^r) \hookrightarrow W(L^\infty, L_v^1) \hookrightarrow L_v^1, \tag{195}$$

see [65, p. 113], follows the inclusion $\mathbb{A}_v^r \subseteq \mathbb{G}_v$.

H. For a fixed $g \in \mathbb{G}_v \smallsetminus \{0\}$, the space of test vectors is the set

$$\mathcal{T}_v := \Big\{ f \in \mathcal{H} \mid W_g^\rho f \in L_v^1(G) \Big\} \tag{196}$$

endowed with the norm

$$\|f\|_{\mathcal{T}_v} := \Big\| W_g^\rho f \Big\|_{L_v^1}. \tag{197}$$

$(\mathcal{T}_v, \|\cdot\|_{\mathcal{T}_v})$ is a ρ-invariant Banach space which embeds continuously and with density into $\mathcal{H}$ and it is independent from the choice of the window vector $g \in \mathbb{G}_v \smallsetminus \{0\}$, see [65, Lemma 2.4.7]. Recall that often the notation $\mathcal{H}_v^1$ is used in place of $\mathcal{T}_v$, see e.g. [24–26, 49].

I. We call reservoir the Banach space

$$\mathcal{R}_v := \mathcal{T}_v := \left\{ f : \mathcal{H}_v^1 \to \mathbb{C} \mid \text{antilinear and continuous} \right\}. \tag{198}$$

J. We can extend the wavelet transform to $f \in \mathcal{R}_v$ and $g \in \mathcal{T}_v$:

$$W_g^\rho f : G \to \mathbb{C}, \quad x \mapsto {}_{\mathcal{R}_v}\langle f, \rho(x)g\rangle_{\mathcal{T}_v}, \tag{199}$$

where ${}_{\mathcal{R}_v}\langle\cdot,\cdot\rangle_{\mathcal{T}_v}$ is the duality between $\mathcal{R}_v$ and $\mathcal{T}_v$ that will be denoted simply by $\langle\cdot,\cdot\rangle$. We have that $W_g^\rho f \in C(G) \cap L_{1/v}^\infty(G)$.

K. For a fixed vector window $g \in \mathbb{A}_v^r \smallsetminus \{0\}$, the coorbit space on G with respect to Y is defined as

$$\mathsf{Co}\,(Y) := \left\{ f \in \mathcal{R}_v \mid W_g^\rho f \in W(L^\infty, Y) \right\} \tag{200}$$

endowed with the quasi-norm

$$\|f\|_{\mathsf{Co}\,(Y)} := \left\| W_g^\rho f \right\|_{W(L^\infty, Y)}. \tag{201}$$

The coorbit space $\mathsf{Co}\,(Y)$ is independent of $g \in \mathbb{A}_v^r \smallsetminus \{0\}$, in the sense that different windows yield equivalent quasi-norms. Moreover, $(\mathsf{Co}\,(Y), \|\cdot\|_{\mathsf{Co}\,(Y)})$ is a quasi-Banach space continuously embedded into $\mathcal{R}_v$ and $\|\cdot\|_{\mathsf{Co}\,(Y)}$ is a r-norm, $0 < r \leq 1$, if $\|\cdot\|_Y$ is. We refer to [65, Theorem 2.4.9].

In the following theorem, we collect [65, Theorem 2.4.19, Remark 2.4.20].

Theorem A.9 *For every $g \in \mathbb{A}_v^r \smallsetminus \{0\}$ there exists $U_0 \subseteq G$ relatively compact unit neighborhood such that for each U_0-BUPU $\Psi = \{\psi_i\}_{i\in I}$ with localizing family $X = \{x_i\}_{i\in I}$ the following hold true:*

(i) *for each $i \in I$ there exists a continuous linear functional*

$$\lambda_i : \mathcal{R}_v \to \mathbb{C}$$

such that $\left(\lambda_i(f)\right)_{i\in I} \in Y_d(X)$ for every $f \in \mathcal{R}_v$ and

$$f = \sum_{i\in I} \lambda_i(f)\rho(x_i)g, \quad \forall f \in \mathsf{Co}\,(Y), \tag{202}$$

where the sum converges unconditionally in the w-∗-topology of $\mathcal{R}_v$. If the finite sequences are dense in $Y_d(X)$, then the series converges unconditionally in $\mathsf{Co}\,(Y)$;

(ii) *for all $\lambda = \left(\lambda_i\right)_{i\in I} \in Y_d(X)$ the series*

$$\mathsf{S}_g^X(\lambda) := \sum_{i\in I} \lambda_i \rho(x_i)g \tag{203}$$

is an element of $\mathsf{Co}\,(Y)$. The above sum converges unconditionally in the w-∗-topology of $\mathcal{R}_v$ (pointwise). If the finite sequences are dense in $Y_d(X)$, then

the series converges unconditionally in $\mathsf{Co}\,(Y)$ *and there exists* $C > 0$ *such that*

$$\left\|\mathsf{S}_g^X(\lambda)\right\|_{\mathsf{Co}\,(Y)} \le C\left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X)}, \quad \forall\, \lambda \in Y_d(X); \tag{204}$$

(iii) *for* $f \in \mathcal{R}_v$ *we have*

$$f \in \mathsf{Co}\,(Y) \Leftrightarrow (\lambda_i(f))_{i\in I} \in Y_d(X) \tag{205}$$

and for every $f \in \mathsf{Co}\,(Y)$

$$\|f\|_{\mathsf{Co}\,(Y)} \asymp \left\|(\lambda_i(f))_{i\in I}\right\|_{Y_d(X)}. \tag{206}$$

Remark A.10 Let us remark the main differences with the Banach setting considered by Feichtinger and Gröchenig [24]:

(i) in [24] a solid Banach function space Y on G is considered and supposed continuously embedded in $L^1_{\text{loc}}(G)$. In particular, we observe how the condition $Y \hookrightarrow L^1_{\text{loc}}(G)$ is restrictive, in fact even if one would allow Y to be quasi-Banach, all the spaces $L^p(G)$ with $0 < p < 1$ would be excluded;

(ii) the window space considered in the construction of the coorbit space is larger than the one presented so far, namely it is sufficient a non-zero $g \in \mathcal{A}_v := \mathbb{G}_v$ and

$$\mathsf{Co}_{FG}\,(Y) := \left\{ f \in \mathcal{R}_v \mid W_g^\rho f \in Y \right\} \tag{207}$$

with obvious norm. Hence, $\mathsf{Co}_{FG}\,(Y)$ is a Banach space independent of the chosen window $g \in \mathcal{A}_v \setminus \{0\}$.

It is a natural question whether the two constructions coincide. In the Banach case the answer is positive, see [26, Theorem 8.3] and [49, Theorem 6.1].

Theorem A.11 *Consider a solid Banach function space* Y *such that it is bi-invariant and continuously embedded in* $L^1_{\text{loc}}(G)$. *Then,*

$$\mathsf{Co}_{FG}\,(Y) = \mathsf{Co}\,(Y)$$

with equivalent norms.

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