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Article

Mathematical Connections and the Mathematics Teacher's Specialised Knowledge

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Abstract: This study seeks to explore the relationship between recent findings on mathematical connections and the Mathematics Teacher's Specialised knowledge model. From a qualitative approach, we seek to explore the specialised knowledge mobilised during the establishment of mathematical connections in different contexts. The study takes into consideration the results from previous studies that were conducted with both prospective and in service teachers. The results suggest the need to include the notion of complexity in the mathematical connections defined by the Mathematics Teacher's Specialised knowledge model. Similarly, they suggest that it is possible to find some ambiguities in the use of the model, so it would be enriching to consider the current classification of mathematical connections in order to improve the model as an analytical tool. Finally, the results suggest the need to include mathematical connections as part of the pedagogical content knowledge domain, since the knowledge of this domain could be mobilised during the establishment of mathematical connections.

Keywords: mathematics teacher's specialised knowledge; mathematical connections; complexity of mathematical connections

MSC: 97B50; 97F40; 97G30

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1. Introduction

Several models of teacher knowledge point to the importance of mathematical connections in describing different types of knowledge (Ball et al. [1]; Carrillo et al. [2]; Rowland [3]). For example, the MKT (Mathematics Knowledge for Teaching) model of Ball et al. [1] considers connections as a part of teachers' awareness, related to how mathematical ideas or concepts can be associated throughout the school years. In this sense, the model uses the notion of connection to define Horizon Content Knowledge, a subdomain of knowledge linked to knowing the relationships between content. In the Knowledge Quartet (Rowland [3]), connections define one of the dimensions of teacher knowledge and are linked to the establishment of connections between concepts, procedures, and to the use of these connections for sequencing content, or recognising the conceptual appropriateness of an activity. In the MTSK model (Mathematics teacher's specialised knowledge) Carrillo et al. ([2]) recognise two types of mathematical connections, intraconceptual and interconceptual. While intraconceptual connections are part of KoT (Knowledge of topics), interconceptual connections are recognised in the subdomain of KSM (knowledge of the structure of mathematics).

Although the previous models recognise and take into consideration the mathematical connections in the teacher's knowledge, they do not contemplate the characterisation of mathematical connections addressed in more recent research (Businskas [4]; de Gamboa et al. [5]; de Gamboa et al., [6]; Evitts [7]; Eli et al. [8]; Hatisaru [9]; Rodríguez-Nieto [10]; García-García and Dolores-Flores [11,12]). In this regard, this study aims to incorporate the results of this recent research into the analysis of knowledge mobilised by mathematics teachers, in line with other recent studies such as Hatisaru ([9]) who uses an extended model for mathematical connections as an analytical tool and in relation to mathematical knowledge for teaching.

In previous studies (de Gamboa et al. [5]; de Gamboa et al. [6]; Caviedes et al. [13]), we have used MTSK model to analyse the mathematical knowledge of both preservice and in service teachers. The results of using MTSK model in the analysis of teachers' knowledge show that the model is useful for identifying mathematical connections. However, the results of the aforementioned studies also show some difficulties in using MTSK model to classify the knowledge mobilised by teachers when establishing mathematical connections. The difficulties observed are linked to possible ambiguities in identifying the intra or inter-conceptual character of connections established by preservice and in service teachers. This, in turn, is associated to ambiguities in characterising the knowledge mobilised by them, e.g., KoT or KSM. Thus, in order to investigate such ambiguities in detail, the present study aims to answer the following questions: How does the classification of mathematical connections reported in recent research relate to the types of knowledge defined in MTSK model? What can research on mathematical connections contribute to MTSK model enrichment?

1.1. The Mathematics Teacher's Specialised Knowledge Model—MTSK

Taking as a reference the contributions of Shulman ([14]), it could be said that teacher knowledge has been a subject of research since the 1980s. The author points out that a deep knowledge of mathematical content must be complemented by a knowledge of learning and teaching mathematics, since a purely mathematical or purely didactic knowledge is not sufficient for the development of mathematical teaching skills (Shulman, 1986). In order to make mathematics comprehensible to students, Shulman ([14,15]) warns that it is necessary for teachers to have a problem-solving tendency and to have the necessary reasons to understand why they solve mathematical tasks in a certain way and not in another way. Thus, teachers also need to know different ways of solving and different ways of teaching the content to their students.

Since Shulman's work, other models have emerged seeking to address the demands of the task of teaching mathematics. One of these models is the MTSK model, which considers specialisation as the core of the mathematics teacher's knowledge in all its domains, subdomains, and categories (Aguilar et al. [16]). From this specialisation, the model understands mathematics as a "systemic knowledge network structured according to its own rules" (Carrillo et al. [2] (p. 6)). A good knowledge of this network, of the rules and of the characteristics of the process of creating mathematical knowledge, would allow a teacher to teach the contents in a connected way and to validate his own and the students' conjectures (Carrillo et al. [2]; Carrillo et al. [17]). The MTSK includes two main dimensions: Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK). Within MK are included KoT (knowledge of topics), KSM (knowledge of the structure of mathematics), and KPM (knowledge of mathematical practice). KoT describes what and how mathematics teachers know the content they teach, being a combination of the knowledge students are expected to learn and a deeper, formal and more rigorous understanding (Carrillo et al. [2]). KoT includes different categories of knowledge: definitions (e.g., what is an integer?); properties and their foundations (e.g., properties of powers); the phenomenology or contexts of use; procedures and their justifications (e.g., when, how and why to use formulas or decomposition procedures); and systems of representation. Intra-conceptual connections are also part of KoT "on the grounds that they are located in the proximity of a single concept, and can be regarded as enriched knowledge of a single content item" (Carrillo et al. [2] (p. 8)).

The KSM describes teachers' knowledge of connections between different mathematical topics (Montes et al. [18]). Four categories of connections are considered: connections of increasing complexity, simplification, auxiliary and transversal. The connections of increasing complexity (complexity) and simplification relate to elementary and advanced knowledge. Advanced knowledge allows teachers to deal with elementary mathematics from an advanced perspective. Elementary knowledge is treatment of advanced mathematics from an elementary perspective (Montes et al. [18]). For example, a simplification connection can be established when a teacher uses the Riemann sum and the surface grid in order to calculate areas by means of an elementary-advanced knowledge relationship (Flores-Medrano [19]). A complexity connection can be established when a teacher uses the calculation of the area of similar rectangles to introduce the comparison between linear and quadratic growth. Auxiliary connections are related to the involvement of a topic in a longer process (Carrillo et al. [2]). According to (Flores-Medrano [19]), a useful aspect for their identification, or even to generate examples of them, is the identification of prior knowledge of a concept and process, although " . . . not all prior knowledge would form an auxiliary connection with the object in question. Sometimes they may be examples of complexity or simplification connections (e.g., knowledge of whole number fractions as prior knowledge of algebraic fractions)" (Flores-Medrano [19] (p. 52)). Finally, transversal connections refer to knowledge that underlies the establishment of relationships between several topics with common features (Montes et al. [18]). For example, the connection between the measurement of a magnitude using different units of measurement, and the inversely proportional relationship between the size of the unit of measurement and the numerical value obtained from the measurement. The four categories of connections "do not intersect with each other and allow to have a global view on the possible interconceptual connections" (Flores-Medrano [19] (p. 53)).

Knowledge about defining, making conjectures, proving or solving problems are a part of the KPM. Thus, this subdomain acquires an organising role of the mathematical knowledge that makes up the KoT and the ways of operating them (Carrillo et al. [2]). For example, the different registers of representation that area calculation admits (geometric, symbolic) can be used to validate or demonstrate that a result is correct. In turn, the knowledge and use of registers of representation is related to knowledge of different procedures, properties, and justifications, directly impacting the relationships between KPM-KoT.

On the other hand, in PCK we find KFLM (knowledge of the features of student learning), KMT (knowledge of mathematics teaching) and KMLS (knowledge of mathematics learning standards). The former encompasses the knowledge associated with the inherent characteristics of learning mathematics, focusing on the mathematical content (as the object of learning) and not on the learner (Carrillo et al. [2] (p. 11)). It includes four categories: knowledge about theories of learning mathematics (e.g., Van Hiele's levels); strengths and weaknesses in learning mathematics; the way learners interact with mathematical content (e.g., often solving area tasks using formulas); knowledge of the motivations and expectations that learners have when confronted with particular content (e.g., how problem solving often generates rejection). For its part, KMT refers to knowledge about teaching that is intrinsically linked to content and considers three categories: knowledge about teaching theories (e.g., accumulated knowledge from experience or anthropological theory of didactics); knowledge about teaching resources (e.g., tangram or geoboard for teaching area), their potential and limitations (e.g., it is not possible to construct an equilateral triangle on an orthometric geoboard); knowledge about strategies, techniques, tasks and examples (e.g., the usefulness of using different representation registers for teaching content). Finally, KMLS includes knowledge about the expected learning outcomes for a given school level (through specific measurement instruments); knowledge about the content to be taught at a given school level; and the sequence that mathematical content follows throughout formal education.

1.2. Recent Results in the Characterisation of Mathematical Connections

In recent years, a growing line of research has been developed on the study and characterisation of mathematical connections (Businskas [4]; Evitts [7]; Eli et al. [8]; Rodríguez-Nieto [10]; Duval [20]; de Gamboa and Figueiras [21]; Rodríguez-Nieto et al. [22]). In the results of such research, two levels of complexity can be observed in terms of the types of connections described. In a broad sense, a differentiation is made between extra-mathematical connections and intra-mathematical connections. Extra-mathematical connections occur when mathematics is related to contexts outside of mathematics, as it is the case of modelling activities. Intra-mathematical connections occur when there is a relationship between representations, definitions, concepts, procedures, and propositions within the context of mathematics. Within the broad set of intra-mathematical connections, two types are differentiated (de Gamboa and Figueiras [19]). Intra-mathematical conceptual connections and intra-mathematical connections related to processes. The former connections refer to relationships between different conceptual elements. The latter are connections that explicitly emphasise transversal processes to mathematical activity, such as problem-solving heuristics, communication of mathematical information, argumentation and justification. In this context, this study considers three categories of mathematical connections: extra-mathematical connections (EMC), intra-mathematical conceptual connections (IMCC), and intra-mathematical connections related to processes (IMCRP).

De Gamboa and Figueiras ([21]), when analysing in a classroom context the mathematical connections that emerged in practice, observed that these were produced as networks of other more elementary connections. They also observed that these connections were explicitly triggered by the teacher and student interventions. In this context, connections can be triggered by students' mathematically inconsistent ideas, which act as an indicator of the existence of a consistent mathematical connection that can lead to clarify these mathematically inconsistent ideas. Thus, mathematical connections are relations between two mathematical objects O_1 and O_2, and can be formed by one, two or more connections, which we will call elementary connections. In this study, a mathematical object is understood as any material or immaterial entity involved in the mathematical activity in the sense of OSA (Godino et al. ([23]). We focus on those types of connections that match our understanding of elementary connections and which derive from the recent literature related to mathematical connections. Figure 1 shows how elementary connections introduce relations with other mathematical objects (e.g., O_1 and O_3). The coordination between elementary connections defines the mathematical connection between O_1 and O_2. There are different types of elementary connections. In the following, we describe the types of elementary connections that are used both in this study and in others that characterise types of connections.

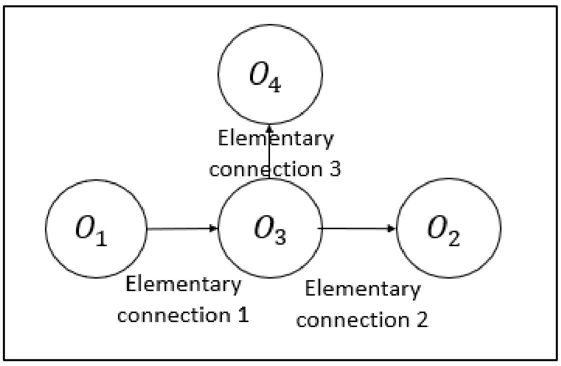


Figure 1. Example of the structure of a connection as a network of elementary connections (O: mathematical object).

Equivalent representation: The elementary connection is established between two representations of the same mathematical object ($R_1 \to R_2$). The connection relates equivalent representations (ER), when both representations belong to the same register (e.g., when a teacher connects -2^5 and $-(+2)^5$) to emphasise the base of the power) (Adu-Gyamfi [24]; Businskas [4]; Dolores-Flores and Garcia-Garcia [25]).

Alternate representation: The elementary connection is established between two representations of the same mathematical object ($R_1 \to R_2$). The connection relates alternate representations (AR), when there is a change in the register (e.g., representing numbers geometrically). It is proposed by Businskas ([4]) and is related to the notion of conversion proposed by Duval ([20]).

Common feature of representation: The elementary connection is established between two objects $O_1 \to O_2$ that share a common feature, without being equivalent. These elementary connections are triggered by erroneous and ambiguous interpretations of mathematical objects by the students. The common features that sustain the elementary connections can be related to commonalities in the representation (CFR), for instance, when $\frac{8}{32}$ and $\frac{32}{8}$ are connected. It is related to the category Categorical proposed by Eli et al. ([26]).

Common feature of definition: As in the previous case, the elementary connection is established between two objects $O_1 \to O_2$ that share a common feature, without being equivalent. The common features that sustain the connection can be related to their definition (CFD), for instance when a student makes a mistake such as $(-1)^{36} = -1 \times 36$ and the teacher connects both operations by stressing their commonalities and differences. It is related to the Categorical category proposed by Eli et al. ([26]).

Metaphorical projection: The elementary connection is established between two objects $O_1 \to O_2$ and they can be related to some metaphorical projection (MP) in which the metaphor is used literally. For instance, when $-4 - (-8)$ is interpreted as a metaphor of being in basement 8 and going up 4 floors. It corresponds to the Metaphor category proposed by Rodríguez-Nieto ([10,22])

Procedure: The elementary connection is established between a concept and a procedure (C→P) that can be used when dealing with the concept. For instance, when a teacher suggests that using the rules for operating with integers to obtain a notation without double symbols is a good resource to make the calculation (Dolores-Flores and Garcia-Garcia [25]; Businskas [4]).

Equivalent procedure: The elementary connection is established between two procedures ($P_1 \to P_2$) that are useful for solving the same task. For instance, when a teacher conducts a discussion on the different procedures that can be used to solve $\frac{(-5)^7}{(-5)^7}$ (de Gamboa et al. [6]).

Justification: The elementary connection is established between two propositions ($PR_1 \to PR_2$), where PR_1 stands for a premise and PR_2 stands for a conclusion. It refers to justifications (JU) of practices in mathematics. For instance, when a teacher discusses the difference between inductive reasoning and deductive reasoning emphasizing the meaning of a mathematical proof (de Gamboa et al. [6]).

Implication: The elementary connection is also established between two propositions ($PR_1 \to PR_2$); it refers to implications (IM) or if-then arguments (Businskas [4]).

Particular case: The elementary connection is also established between two propositions ($PR_1 \to PR_2$), it can refer to the application of a property to a particular case (PC). It corresponds with the category Part-Whole proposed by Businskas ([4]) and Dolores-Flores and Garcia-Garcia ([25]).

Generalisation: The elementary connection is also established between two propositions ($PR_1 \to PR_2$), it can refer to generalisations (GE) of a property to a broader set. For instance, when a teacher generalises $\frac{a^p}{a^q}$ when $p \geq q$ to the case when $p < q$. (Businskas, [4]).

The above-mentioned classification of elementary connections corresponds to a reinterpretation of the extended model for mathematical connections proposed by (Rodriguez-Nieto. [22]), which includes some of the results of our previous research. In the case of procedural connections, we also propose the equivalent procedure. The reason for this is that we have found some examples of connections that do not correspond to the definition of procedural connection, but refer to explicit relations between two different procedures that can be used in the same activity. In the case of the Part-Whole connections proposed by Businskas ([4]) and Dolores-Flores and García-García ([25]) we have decided to separate them into two different types, *particular case* and *generalisation*. This is because we have observed in previous studies (de Gamboa, [6]) that generalisation connections can be more complex in nature than particular case connections, and therefore we believe that they should be considered separately.

2. Methods

The present study has an exploratory and comprehensive purpose on the connections established and the type of knowledge mobilised by in service and preservice teachers. Therefore, the study is situated in an interpretative paradigm and follow a qualitative approach (Cohen et al. [27]; Bryman [28]). Qualitative research is concerned with interpretation and exploration that guides researchers toward understanding and explaining different phenomena and events (Creswell and Garrett [29]). In this case, the aim is to explore how the classification of mathematical connections reported in the literature relates to the types of knowledge described in MTSK model.

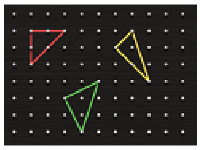
A content analysis (Krippendorff [30]) is carried out which contemplates data that were collected in two broader studies, and which relate to mathematics teachers' knowledge. In one study (Caviedes et al. [13]), the knowledge that preservice teachers (PSTs) bring into play when solving area tasks that require making mathematical connections is characterised. In another study (de Gamboa [5], de Gamboa et al. [6]), the knowledge in use of an in service teacher is characterised when she favours the establishment of mathematical connections in the teaching of a unit on integers. The decision to consider two different studies is motivated by two reasons. First in both studies difficulties were observed when characterising the teacher's knowledge during the establishment of connections using the MTSK tools. Mainly, difficulties were observed in the classification of intra and inter conceptual connections; specifically, in KoT-KSM subdomains. Second, the above-mentioned studies allow for a complementary analysis of the mathematical connections established by in service and preservice teachers at different educational stages (Primary and Secondary). Moreover, the mathematical topics taught by in service and preservice teachers are of a different nature. In study 1, a topic corresponding to a geometrical context is taught, while in study 2, a topic corresponding to a numerical context is taught. Finally, the types of data analysed are also different. While in the first case they correspond to written resolutions of questionnaires, in the second case they correspond to recordings of classroom sessions. All of the above diversity, in terms of data, contexts, topics, levels and teachers' experience allows for a complementary analysis that leads to more robust results in terms of the relationship between the establishment of connections and the specialised knowledge of teachers who teach mathematics.

The first study analysed the responses to a written questionnaire, on the area of flat figures, from a group of 147 preservice teachers (PST) in the third year of the Primary Education Degree. PSTs had one week to answer the questionnaire and send it in pdf or word format. The questionnaire consisted of a total of eight tasks: three tasks that respond to contexts of equal distribution or comparison and reproduction of shapes, and where the use of formulas was prohibited (Tasks 1, 2 and 3); two measurement tasks involving the use of different procedures (Tasks 4 and 5); a task of classification of statements and another of definition of the concept of area (Tasks 6 and 7); finally, a task of analysis of students' answers (Task 8). In this manuscript, tasks 4 and 8 of the questionnaire are analysed. Task 4 (Figure 2) was designed by the researchers and PSTs had to solve it using two or three different procedures. Task 8 (Figure 3) is based on a previous study (Caviedes et al. [31]) and PSTs had to analyse a high school student's resolution of an area task. In task 4 PSTs established a greater number of mathematical connections, as the task required to be answered using different procedures. This allowed PSTs to use procedures of a geometric and numerical nature in a joint manner. In the case of task 8, when PSTs were asked to analyse students' answers, mathematical connections linked to their mathematical and pedagogical knowledge were established. This did not happen in the rest of the tasks.

Formulation

Task 4: Look at the triangles constructed on the geoboard. **What is the area of each triangle? Which one has the largest area? Justify your answers using two or three different procedures.**

Graphic representation of the Task



(Compiled by authors)

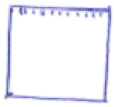
Figure 2. Task 4 of the questionnaire.

Task 8 proposed the following situation: "Tània and Laia are in the 2nd year of Secondary Education (ESO). In maths class, the teacher asks them to calculate the area of two squares using more than one procedure. Tània has difficulty for calculating the area of one of the squares. Laia has been able to solve the whole task using different procedures. What knowledge do you think Tània needs to be able to solve the task? How could the teacher help her? Justify your answer". Figure 3 illustrates the students' answers.

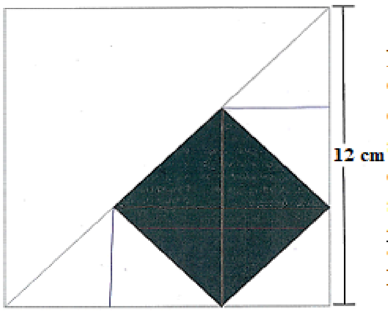
a- What is the area of the 12 cm square? **why?**

$A = L^2 = 12^2 = 144$ cm

"If we form small squares of 1 cm^2 we would get 12 rows and 12 columns. If we multiply the number of rows by the number of columns, we would get 144 cm^2"



b- **How many possible ways** can you calculate the area of the black square. **Explain it.**



"If you draw one of the diagonals of the square you can see that it represents twice the number of triangles on its side and if you draw the other one you see that it represents four times the smallest of all. If you draw the height of the other triangles, you can see that half of the square represents 9 triangles... the black square $\frac{4}{9}$ of the half. Therefore $\frac{4}{18}$ o $\frac{2}{9}$ of the total because I also divided into 9 parts the other half"

"By measuring the sides of the square and using Pythagoras to calculate the diagonal of the square and dividing it by three to find the side of the black square"

"Measuring the side of the square"

"$\frac{2}{9}$ of $12^2 = \frac{2}{9} \times 144 = \frac{288}{9} = 32$ cm^2"

"The side of the square is made up of three small triangles. Therefore, each sides measures 4cm → $\frac{b \times h}{2} = \frac{4 \times 4}{2} = \frac{16}{2} = 8$ cm^2. As the square is four times this → 8 cm^2 × 4 = 32 cm^2".

TÀNIA

LAIA

a- It is 144 cm^2 since the squares have equal sides and the formula to calculate it is as follows: A □ = $12^2 = 144$ cm^2

b- Visually the side of the square measures 8 cm, 1/3 part of the side of the triangle → 8×8 = 64 cm

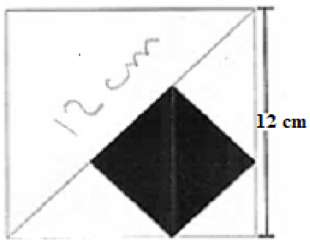


Figure 3. Figure accompanying Task 8.

The analysis was carried out with the use of MAXQDAplus software in order to identify the elementary connections established by preservice teachers (Figure 4). These elementary connections are established in written justification and in the resolution process followed by PSTs. Connections are stablished between representations, procedures and properties (KoT), and can be auxiliary (KSM) and linked to knowledge about teaching strategies (KMT) and students' difficulties/strengths (KFLM). Once the connections have been identified, a frequency count is carried out. This is done using the aforementioned software.

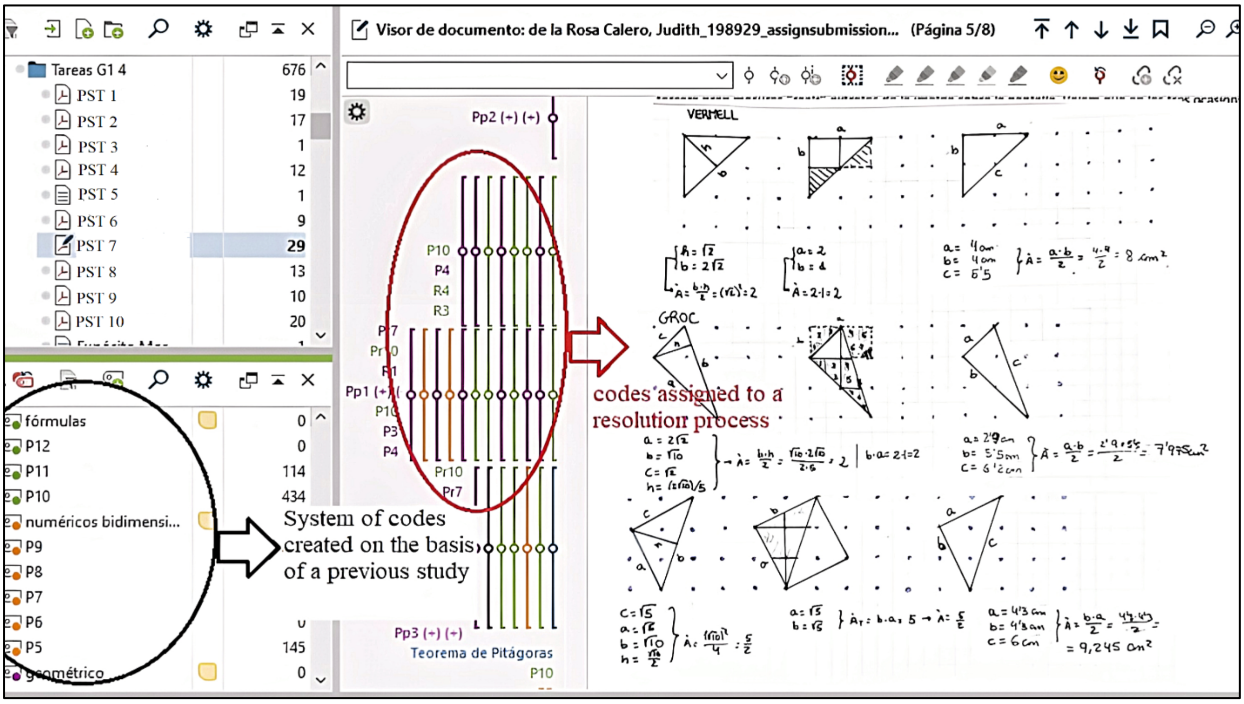


Figure 4. Example of the analysis performed in study 1.

The second study analysed the discursive interactions between the teacher and a class of 12–13 year-old students during the teaching of powers and integers. The data used to perform the analysis are video and audio recordings of eight regular class sessions of 60 min and their corresponding transcriptions. To collect the information, non-participant observation was used (Caldwell and Atwal [32]; Cohen et al. [27]). Therefore, the class group was observed, listened to and video-recorded, and notes were taken, without intervening in the design of the sessions, the activities proposed, or the development of the sessions. The data-processing consists of viewing all eight class sessions from the teaching unit on integers. The aim of this first viewing is to identify and define episodes in which connections appear. Each session was viewed at least twice by two researchers that took notes of relevant moments in the videos in which potential connections may emerge. The average duration of the episodes was 5 min and 55 s.

In both studies, the analysis is divided into two phases (Figure 5). The first phase is divided into three stages. The first stage consists of the identification and characterisation of mathematical connections, following the categories of elementary connections presented in Section 1.2. These elementary connections can form more complex connections if the relationship between them is of an explicit nature: $O_1 \xrightarrow{relation} O_2$.

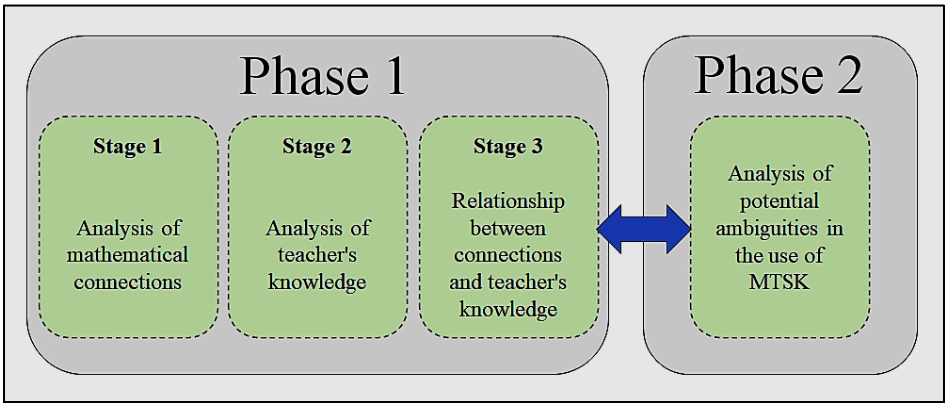


Figure 5. Phases of the analysis process.

In the case of the first study, given the nature of the data (only written justifications) it is not possible to look for evidence of complex connections (formed by more than one elementary connection). The elementary connections identified in PST justifications do not present explicit relations between them, but are established separately. In the second study, the three elements (O_1, O_2 and relation) are identified from explicit interventions of the teacher and students, so it was possible to identify more complex connections. The connections in study 1 are named QC (questionnaire connections) and the connections in study 2 are named CC (classroom connections).

The second stage consists of characterising the knowledge mobilised by PSTs and the teacher when these connections are established, using the tools of the MTSK model. Finally, the third stage identifies the relationships between the types of knowledge mobilised when mathematical connections are established. The results of the first phase of analysis showed that the classification given by MTSK model presents ambiguities when analysing teachers' written and oral discourse. In the second phase, these ambiguities are analysed, emphasising the characterisation of connections related to more than one subdomain of knowledge, e.g., KoT or KSM.

3. Results

3.1. Global Results of the First and Second Study

In the first study (characterisation of PSTs' knowledge mobilised in the resolution and analysis of area tasks), a total of 169 elementary intra-mathematical conceptual connections are identified, in the first stage of Phase 1 of the analysis. These connections are identified in the resolution process followed by PSTs in Task 4, and in the analysis process of student responses followed by PSTs in Task 8 of the questionnaire. Only 5 out of 11 types of elementary intra-mathematical connections are identified. No complex connections are identified, as the PSTs' written justifications did not explicitly coordinate elementary connections to form more complex connections. Table 1 shows that the smallest number of connections established by PSTs corresponds to particular case connections. This means that connections associated with the use of different geometrical properties of the figures were more difficult to establish by PSTs. In addition, it can be observed that the rest of the intra-mathematical connections of conceptual type present a similar frequency.

Table 1. Intra-mathematical questionnaire connections (QC) identified in study 1.

Type of IMCC Connections	Number of Connections	Total
ER	2, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169	39
AR	1, 6, 91, 92, 93, 94, 95, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120,121, 122, 1238, 124, 125, 126, 127, 128, 129, 130, 131	33
P	7, 8, 12, 50, 51, 52, 53, 54, 55,56, 57, 58, 59, 60, 61, 62,63, 64, 65, 66, 67,68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 103	45
EP	3, 4, 5, 9, 13, 14, 15,16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 104, 105	43
PC	10, 11, 96, 97, 98, 99, 100, 101, 102	9

In the second study (characterisation of the knowledge a teacher uses in a classroom context when making mathematical connections), 34 complex connections triggered and constructed by discursive interactions between the teacher and students were identified. These complex connections consisted of elementary connections that could vary from 1 to 6. For example, CC13 and CC31 were made up of one elementary connection, while CC12 was made up of six elementary connections. Table 2 shows 34 complex connections involving a total of 84 elementary connections in their formation. It can be seen that most of the connections identified were intra-mathematical conceptual connections (23), followed by process-related connections (8) and extra-mathematical connections. The frequency of elementary connections is detailed in Table 3.

Table 2. Complex classroom connections (CC) identified in study 2 and number of elementary connections that form them.

		Number of Elementary Connections						
		1	2	3	4	5	6	Total
Categories of connections	EMC	13, 31			4			3
	IMCRP	27, 33, 29	5, 20	7 17		1		8
	IMCC	10, 28, 34	2, 3, 6, 8, 11, 16, 18, 19, 21, 23, 24, 25, 26, 30, 32		15	14	9, 12, 22	23
Total		7	18	2	2	2	3	34

Table 3. Types of elementary connections identified in study 2.

Type of Elementary Connection	ER	CFR	CFD	MP	P	EP	JU	IM	PC	GE
Frequency of connections	8	9	8	4	17	3	15	2	3	15

3.2. Mathematical Knowledge Mobilised by PSTs and Teachers

In the second stage of Phase 1 of the analysis, PSTs knowledge mobilised during the establishment of mathematical connections, in the resolution and analysis of area tasks (Study 1), is characterised. Five types of elementary intra-mathematical conceptual connections are distinguished. For each connection, the knowledge mobilised by PSTs is identified. For this, the categories defined by each of the subdomains of MTSK model and described in Section 1.1 are considered.

Table 4 shows that the subdomain that is most mobilised is KoT. Likewise, it is possible to observe that when connections related to procedure-concepts (P), equivalent procedures (EP) and properties (PC) are stablished, the knowledge mobilised by PSTs is mainly KoT and KFLM. Furthermore, it is observed that KMT is mobilised when connections with procedures-concepts (P) and with equivalent procedures (EP) occur. Since all KMT connections are also included in KoT, it is possible to consider a relationship between conceptual-procedural knowledge (P) and KMT mobilisation. Something similar occurs with KSM, since two of the three connections present in that subdomain are found in KoT, specifically in equivalent procedural knowledge (EP) and procedural-conceptual knowledge (P). The evidence of KFLM mobilisation, during the establishment of connections related to procedures and properties, allowed us to identify that PSTs mobilise their KFLM associated with knowledge about students' strengths/difficulties (Figure 5). Thus, connections for KFLM (Table 4) are related to procedural and conceptual knowledge that can be useful for the resolution of the task. For example, in connections QC10 and QC3, PST 3 points out that the student who solves the task by making mistakes needs to acquire knowledge about the properties of squares and right triangles, as well as the decomposition of figures into other figures (Figure 6). Although it is true that PST 3 does not make explicit the properties of the square, we can infer that it refers to the fact that this figure can be divided into two isosceles right triangles, by tracing one of its diagonals, as this is what was shown in the resolution to be analysed.

Table 4. Specialised knowledge mobilised during the establishment of mathematical connections (QC) in study 1.

IMCC Connections	Categories of Specialised Knowledge and Number of Mathematical Connections Related to Them			
	KoT	KSM	KMT	KFLM
ER	2, 132, 133, 134, 135, 136, 137, 138, 39, 40, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169			
AR	1, 6, 91, 92, 93, 94, 95, 106, 107, 108, 109, 110, 111, 112, 113, 114, 125, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131			
P	8, 7, 50, 51, 52, 53, 53, 55,56, 57, 58, 59, 60, 61, 62,63, 64, 65, 66, 67,68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89. 90	8, 12, 103	8	7, 8, 60, 61, 62,63, 64
EP	3, 4, 9, 13, 14, 15,16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49	3	3, 5, 13, 14, 15,16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 104, 105	3, 9, 13, 14, 15,16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49
PC	10, 11, 96, 97, 98, 99, 100, 101, 102			10, 11, 96, 97, 98, 99

"The knowledge that Tània needs to internalise is diverse…we need to understand the properties of the square…In addition, one concept that will be very useful for us are the spins, to observe that the black geometric figure is also a square…another would be the Pythagorean theorem, as the formula is necessary to discover the diagonal and to find out the area of the black square. … fractions, as they are a key element to be able to solve this exercise… the properties of triangles, more specifically right triangles… these can help us when splitting the figure, in order to know the area of the black square".

Figure 6. Analysis of PST 3 to a student's resolution (first part).

In the case of connection QC8, the PST mentions the need to know and operate with the Pythagorean theorem, it is inferred that this is related to the properties of triangles (QC10), since when these are rectangles, the length of the hypotenuse can be obtained by means of this theorem. In connection QC7, the PST points out that the student needs to acquire knowledge about fractions due to their usefulness in dividing a whole into parts of equal area and in calculating the fraction of a number (1/3 of 12). Finally, in connection QC14, the PST refers to "spins", it is inferred that she refers to rotational movements that allow to recompose a figure into a different one.

Figure 7 shows the KMT mobilised by PST 3. Thus, the connections for KMT (Table 4) are mainly associated with teaching strategies to guide a student. For example, connection QC8 relates to conceptual-procedural knowledge about Pythagorean theorem, which can be used as a teaching strategy. Connection QC16 relates to the identification of errors about the diagonal and side of a square. Connection QC17 relates to the use of various procedures to help students to correct errors.

"To help her, we have to concretise all those concepts that we are not clear about…to do this, we will observe how the diagonal of the square does not measure the same as one of its sides (as justified in her answer) …we have several options, one…is to have her position the side of the square parallel to observe that the diagonal is longer. Another would be to give her a ruler to measure the side… when dealing with the Pythagorean theorem, the first option is better… then the measure of the diagonal can be found through the formula… she makes an incorrect fraction. In order for her to observe the error, we will make her divide the side into three segments and put in each segment how many centimetres there are… She will then be able to observe visually, or by means of a sum, that 1/3 of 12 cm is not 8 cm. Finally, we will intervene in the division of figures into small triangles… we will ask a good question: What is the geometric figure with which we can divide the square (only using that figure)? From here on, we can see that Tània will try to give an answer by trial and error".

Figure 7. Analysis of PST 3 to a student's resolution (part two).

It is inferred that PST 3 proposes a manipulative type of strategy, as she points out that the student can compare the side and the diagonal of the square by placing both of them in parallel, in order to observe their lengths. PST 3 also points out that such a procedure would help to work on the Pythagorean theorem. Connections QC3 and QC5 are associated with procedural knowledge about decomposition of figures, as PST 3 indicates that it is possible to guide the student through a procedure that involves measuring each of the three segments into which the side of the large square is divided. In turn, connection QC18 relates to the use of questions that aim to evoke alternative procedures. In this respect, PST 3 asks a question that aims to evoke procedures for decomposing figures.

Regarding KoT, it is possible to observe that PSTs mobilise this subdomain in relation to representations, properties and procedures. Thus, the knowledge mobilised in the connections corresponding to KoT (Table 4) is associated with the use of geometric and symbolic representations, either jointly or separately. For example, connection QC1 is related to decomposing and recompositing procedures performed by PST 7 (Figure 8) on triangles 1 and 2, in order to subsequently apply the area formula. Connection QC6 would be related to the use of the decomposition of figures into congruent units, in order to obtain the area in an additive way. Connection QC10, with the use of KoT associated with the properties of polygons, for example, when PST 7 uses the area formula, it is inferred that she recognises that a triangle is half of a square with the same base and height that contains it. Another example is evident in connections QC3, QC4 and QC9, which are related to PST 7's strategic use of procedures, e.g., cutting two-dimensional space to compare areas, fractioning the unit of measurement to facilitate the process of measuring areas, and rearranging a figure into a new figure. From this, it is inferred that QC11 connection is associated with the properties involved in area measurement processes (area conservation). Likewise, it is possible to observe that connection QC3 is related to more elementary procedures for the calculation of area, in this sense it is linked to KSM.

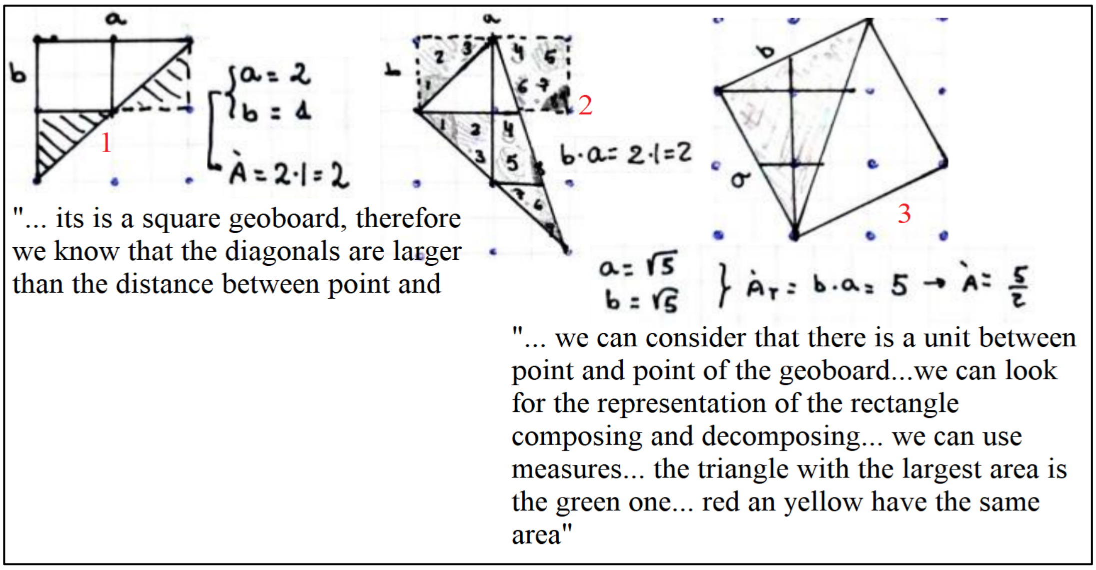


Figure 8. Resolution of PST 7 to Task 4.

Connection QC8 is linked to conceptual-procedural knowledge. Thus, QC8 connection is auxiliary in KSM and procedural in KoT. It is observed that PST 7 identifies that the legs of triangle 3 correspond to hypotenuses of other right triangles (not given by the exercise). This allows her to use her knowledge of Pythagorean theorem to calculate the length of the legs (seen as hypotenuses of other triangles). Thus, the calculation of areas of right triangles allows to evoke content that serves as a support for the resolution of the task. In the case of connection QC12, the right triangles located on the geoboard allow PST 107 to evoke knowledge of Pick's theorem (Figure 9), through a conceptual-procedural relationship, and which is linked to KoT-KSM. Something similar occurs with connection QC103, referring to the use of the semiperimeter formula for the calculation of areas.

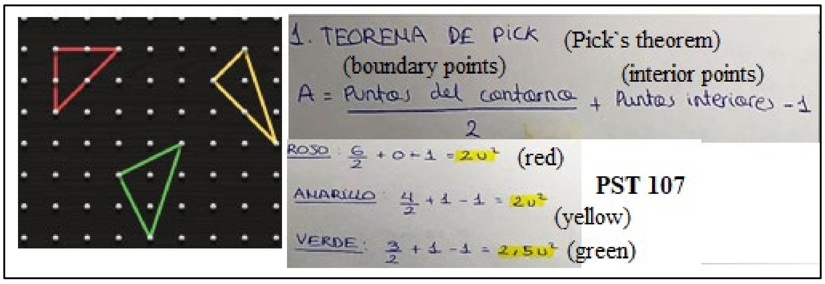


Figure 9. Resolution of PST 107 to Task 4.

In the second study, MTSK model is used to characterise the knowledge mobilised by a teacher when making mathematical connections in a classroom practice context. From the complex connections triggered by the teacher, the knowledge she mobilises in making such connections is characterised. Table 5 shows the results of the types of knowledge mobilised by the teacher in the establishment of each complex connection.

Table 5. Relationship between the establishment of mathematical connections (CC) and types of knowledge mobilised by the teacher in study 2.

Type of Connection	Categories of Specialised Knowledge and Number of Mathematical Connections Related to Them					
	KOT	KSM	KPM	KMT	KFLM	KMLS
EMC	4			4, 31	13, 31	13, 31
IMCC	3, 6, 8, 9, 10, 11, 12, 14, 16, 19, 21, 22, 23, 24, 25, 30, 34		9, 10, 16, 20, 22	2, 6, 10, 11, 12, 14, 15, 16, 18, 21, 22, 23, 24, 26, 30, 32	6, 12, 16, 21, 23, 25, 26, 28, 30	
IMCRP	1, 5, 7, 17, 20, 29	1	1, 5, 17, 20, 27, 29, 33	7	5, 33	

About 57 out of 84 elementary connections identified were triggered by teacher's interventions and 27 were triggered by students' interventions. Table 6 shows the types of knowledge mobilised in the establishment of the elementary connections that were triggered by the teacher interventions.

Table 6. Relationship between the establishment of elementary connection and the specialised knowledge mobilised by the teacher in study 2.

Type of Elementary Connection	Categories of Specialised Knowledge and Number of Mathematical Connections Related to Them					
	KOT	KSM	KPM	KMT	KFLM	KMLS
ER	27, 33, 36, 39, 40, 43			35, 36, 37, 39, 40, 43	33	
CFR	84			34, 44, 45, 46, 47		
CFD	78			7, 54, 74, 78	74, 76, 78	
MP				11, 80	38, 80	38, 80
P	9, 10, 70			10, 70		
EP	26, 56, 57, 65, 71		66	66		71
JU	5, 15, 20, 21, 49, 52, 60, 68, 77		5, 15, 49, 52, 58, 75, 77	20, 22, 49, 60, 68	15, 49, 68	60
IM	41			41		
PC	1, 18					
GE	4, 8, 12, 13, 17, 19, 28, 29, 31	4	4, 28, 29, 83	12, 13, 17, 29, 31, 82	17, 83	

The analysis of the knowledge mobilised by the teacher in each connection allows us to observe some patterns in the mobilisation of this knowledge. First, it can be observed that the two most mobilised subdomains are KoT and KMT. Second, it can be observed that when connections related to processes (CRP) are established, the knowledge mobilised by the teacher is mostly KoT and KPM. Likewise, it can be observed that KPM is mainly mobilised when intra mathematical conceptual connections (IMCC) and connections related to processes (CRP) occur. If we take into account that the number of CRP connections is one-third of the IMCC, we can infer a relationship between the establishment of CRP connections and the mobilisation of KPM type of knowledge by the teacher.

The evidence of the teacher's mobilisation of KPM during the establishment of CRP connections allowed us to identify two patterns of knowledge. First, the knowledge mobilised by the teacher during the establishment of connections CC1, CC5, CC27 and CC33 emphasised general characteristics of mathematical practice (KPM). During connection CC1 the teacher emphasised the difference between demonstration and inductive intuition (Figure 10). During connection CC5 the teacher emphasises the importance of knowing how to explain where mathematical results come from (Figure 11). During connection CC27, the teacher points out the importance of knowing how to explain things, while in connection CC33 the teacher emphasises the importance of thoroughness in communication by stating "we should answer in the same terms that the problem asks us to. When the problem asks us to do divisions that are whole quotient then we speak that language. You haven't told us anything about decimals".

> Because suspicion is one thing, and being sure is another […] we are going to make our first mathematical demonstration. We are going to prove that our suspicions are true. [Suspicion proven. Now, a conclusion that I like very much. If something happens once, twice or four times, it doesn't necessarily always happen.

Figure 10. Fragment of the teacher's intervention during the connection CC1.

> Igor: What I don't understand is why two minuses make a plus. […]
> Laia: Because, yes, there's no explanation.
> Teacher: That's where I come in, it does have a bit of an explanation, it's just that it's not easy, but it does have an explanation.

Figure 11. Fragment of the sequence of interventions that defined connection CC5.

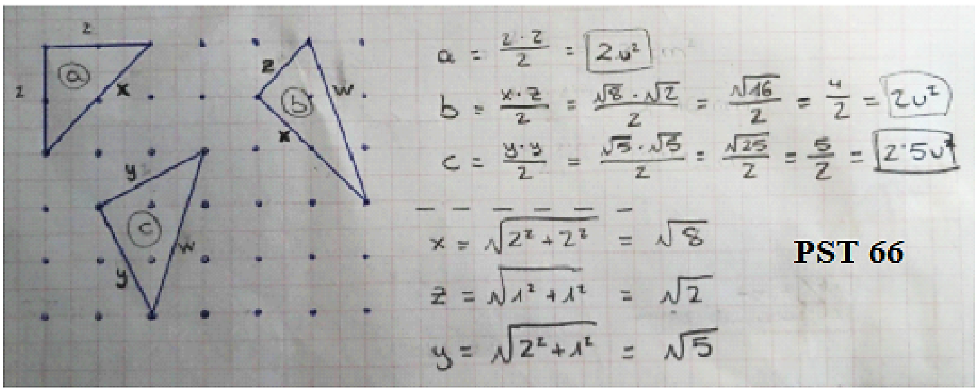
Second, the knowledge mobilised in connections CC17, CC20 and CC29 is related to the strategic use of procedures. In the case of connections CC17 and CC29 the teacher emphasised the importance of being able to decide why one procedure is more appropriate than another in a given context. For example, during connection CC17, the teacher assessed the appropriateness of a procedure based on matching negative factors when performing natural powers of negative base: "It's true, so with Ivan's trick, it turns out that hey . . . What Pedro says is true, but it's tougher". In the same way, during connection CC29, the teacher valued the appropriateness of a specific use of the associative property by stating "you and Clara should associate in a different way [. . .] I find it a bit easier and that is why I have shown it to you, but it is not necessary". Finally, during the CC20 connection, the teacher emphasises the importance of looking for alternative procedures in case of blocking by commenting "You have to be resolute, and if you don't remember the powers, fine, nothing happens, but imagine".

The analysis the teacher knowledge mobilised when her interventions trigger elementary connections allowed us to observe some patterns in the mobilisation of mathematical knowledge, related to the establishment of elementary connections in the classroom. First, it is identified that the types of knowledge mobilised when establishing common feature connections (CFR and CFD) were mostly linked to pedagogical content knowledge (KMT and KFLM). Second, it can be observed that the mobilisation of KPM is linked to elementary connections of justification and generalisation. Finally, it can be observed that the elementary connections of representation are related to the mobilisation of two types of knowledge that, in general, were mobilised at more points in time, namely KoT and KMT. Finally, it can be observed that in the establishment of the elementary connections of justification all types of knowledge were mobilised except KMLS.

A Critical View on Mathematical Connections within MTSK Model

We seek to observe whether the nature of the connections identified allows us to pinpoint the need for a refinement of MTSK model. To do so, we identified connections that involved knowledge that could admit different classifications, according to the current definition of MTSK model.

Regarding the first study, potential ambiguities were identified in three intra-mathematical conceptual connections. All the ambiguities identified are associated with difficulties in using the MTSK to classify the knowledge mobilised by PSTs. Specifically, in the three cases that present ambiguities, difficulties are observed when using the KoT- KSM subdomains, which indicates connections associated with a conceptual-procedural knowledge could be part of more than one subdomain of knowledge in the MTSK. This is the case of connections QC3, QC8 and QC12. In this sense, knowledge about the properties of right triangles, as well as the procedures that allow their area to be calculated, would be linked to what and how mathematics teachers know the content they teach (Carrillo et al. [2]), and would therefore be part of the KoT. Similarly, if we consider, for example, that the properties of right triangles are prior knowledge that allows us to evoke another type of procedure (Flores-Medrano [17]) we could classify the use of the Pythagorean theorem (QC8) as an auxiliary connection within KSM (Figure 12). Similarly, if we consider that the characteristics of the triangles constructed in geoboard can be a previous knowledge that allows us to evoke, later on, Pick's theorem (QC12), this theorem could be positioned as an auxiliary connection, or as a complexity one if we take into account the temporality and complexity of it. However, given that in both cases the action of "evoking" is taken into consideration, both theorems are positioned in the category of auxiliary connections. Likewise, if it is taken into consideration that both theorems are mobilised at a procedural level, the associated knowledge can be linked to KoT, since it can be interpreted as being related to properties and their underlying principles, definitions and procedures (Carrillo et al. ([2]).



"As the triangles are rectangles, we can use the Pythagorean theorem to measure all the sides and then calculate the area, considering that the distance between point and point of the geoboard is 1, the red triangle (a) has 2u^2; the yellow one (b) has 2 u^2; and the green one (c) 2.5 u^2, so it is the biggest one".

Figure 12. Example of a resolution process followed by PST 66.

Another ambiguous example is the procedure that involves decomposing surfaces (Figure 13) to simplify the area calculation process (QC3). In this case, if such a procedure is considered to be of a prior and elementary nature in the calculation of areas, it could be positioned as a simplification connection within KSM. If it is considered that such a procedure is a prior knowledge that allows, subsequently, to evoke the area formula, it could well be an auxiliary connection. In this case, we have taken it as prior and elementary knowledge. Moreover, it is also possible to interpret that knowing different ways of decomposing figures, in order to calculate their area, is part of a knowledge of characteristic procedures for the calculation of areas and therefore could be linked to the KoT.

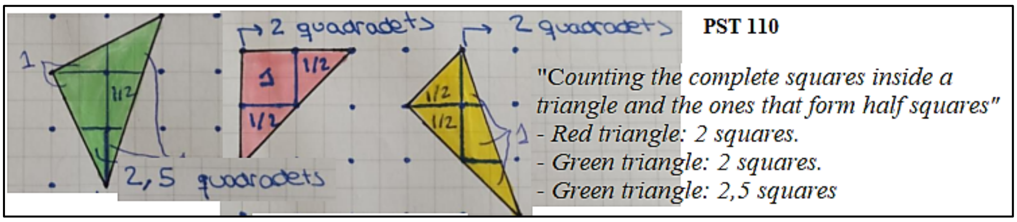


Figure 13. Example of a resolution process followed by PST 110.

With regard to the second study, potential ambiguities are identified in 13 of the 34 connections identified. These ambiguities are related to difficulties in using MTSK to classify the knowledge mobilised by the teacher. In particular, in the 13 cases with ambiguities, difficulties are observed when using KSM subdomain, especially when possible inter-conceptual connections were observed. In three complex connections (CC9, CC17 and CC29), elementary connections were identified between operations with integers (powers and multiplication) and with the use of the associative property (Figure 14). At first, it was considered that the teacher was deepening connections of inter-conceptual type, and therefore knowledge belonging to KSM was mobilised. This type of connection can be interpreted as an auxiliary type of connection, since a mathematical property (associative property) is used with the calculation of negative base powers. However, it was observed that, although the teacher made detailed explanations related to the connection, the use made of the associative property was of a procedural type, and was therefore more related to properties and their underlying principles, than to the interlinking systems which bind the subject (Carrillo et al. [2]), so it was decided to classify the knowledge mobilised as KoT. For example, during the interactions defining the CC9 connection, while calculating $(-2)^5$, the teacher tries to organise the students' proposals to introduce a property of negative base powers, which aims to justify a property whose use will be mostly procedural.

Let's go first to the associative property and then see if there are other, easier ways. […] I can group them 2 by 2 as I want. So then I can put it like this or I can put it like this. That is, we can associate the signs as we want. […] I have a base with a minus sign, […] and an exponent that is odd, in the end, I don't need to go through all this trouble, in the end it's going to give me a minus sign, all of them?

Figure 14. Excerpt from the teacher's intervention during the connection CC9.

The same ambiguity between KoT and KSM occurs in connection CC17, where the teacher introduces an assessment of the use of a procedure proposed by a student for the calculation of $(-1)^{36}$ compared to the use of the associative property to group the product of the 36 factors into 18 pairs of products that would give a positive result (Figure 15). As in the previous case, this could be interpreted as an auxiliary inter-conceptual connection indicating the mobilisation of KSM. However, the procedural role played by the connection relates more to knowledge of the properties that apply when operating with integers, and can therefore be classified as KoT.

S: For example, we put the thirty six -1. And for example it would be − times − equals +, + times − equals −, − times − equals +, and so on.
T: What Peter says is true, but it's like much heavier. […] So with these two (pointing to the first two brackets) we make two + signs […] And with these other two—(pointing to the 3rd and 4th) I'll make another + sign […] Well, we have 18, and as they are all +, what we can do is the associative rule of Pedro signs and then we only have + left.

Figure 15. Fragment of the sequence of interactions that defined the connection CC17.

In the case of connection CC29, when performing the operation $(-(+9):(+3))((+3) -(-7))$ the teacher again suggests the use of the associative property by stating that "At this point I think we have one in which you and Clara (student) would like to associate differently" and therefore the same ambiguity occurs as before. In the case of connections CC21, CC23 and CC26, there are connections between different operations that could be interpreted as inter-conceptual connections related to the generative role of mathematical items in the construction of other items (Carrillo et al. [2]). However, it was identified that the knowledge mobilised by the teacher in these connections was not part of KSM, as the teacher's interventions had a descriptive and procedural character. In the case of connections CC21 and CC26, there were connections between multiplication and empowerment of integers. In connection CC21, the teacher answered a student's question (Figure 16) by showing an example of how a power of negative exponent can be obtained. In this case, it can be interpreted that the teacher's intervention establishes a connection of increasing complexity, as positive exponent integer powers are related to a more complex idea, such as negative exponent integer powers. However, the knowledge mobilised by the teacher was classified as KoT, as she limits herself to showing an example without delving into how powers of integer base and positive exponent are related to those of negative exponent.

S: If p is smaller than q, then it would be a negative number?
T: If I had for example $2^3/2^7$, hey Victoria, is that really 2^{-4}? right? Is that what you are saying? And my answer to you is yes. Do you want me to prove it to you?

Figure 16. Fragment of the sequence of interactions that defined the connection CC21.

In the case of the CC23 connection, the difference between operations of the type $\frac{a}{a} = 1$ y $a - a = 0$ was discussed in class. This discussion was triggered by a mistake made by a student when simplifying a fraction $\frac{a}{a}$ as $\frac{0}{0}$. In this case, the teacher responded to the student by saying "Anna, it's ones, because actually what you're getting … If I would leave a 0 here, it means it's kind of hidden, right? Yes, and you'll have zero for this whole wagon $(-5\cdot -5\cdot -5\cdot -5\ldots)$ and zero for whatever it is?". It could be interpreted that in this case there is a transversal inter-conceptual connection between division and subtraction (more specifically between their neutral elements), since the idea (in this case intuitively developed) of a neutral element is common to many mathematical operations. However, given the procedural nature of the context and of the teacher's intervention, it was decided to classify this knowledge as KoT.

In connections CC11, CC25 and CC32 there were connections between different numerical sets: natural, integers and decimals; and in connection CC2 there was a connection between the concepts of greatest common divisor and least common multiple. In the same way as in the previous cases, it can be interpreted that these were inter-conceptual connections and that the knowledge mobilised by the teacher in these cases was of KSM type. However, in these last four cases it was also observed that the teacher's interventions were focused on a procedimental treatment of the contents, which is why it was decided to classify the knowledge mobilised by the teacher as KoT. Moreover, the classification of ambiguity between KSM and KoT was in favour of the use of KSM. In the case of the CC1 connection, there is a connection between the least common multiple and the greatest common divisor. In this case, what determined the classification of the knowledge mobilised by the teacher was the depth with which the connection was made (Figure 17), by means of a mathematical demonstration that in the case of a number b being a multiple of a number a, it will always be the case that $m.c.m(a,b) = b$ y que $m.c.d(a,b) = a$. The demonstration of the two results shows how the two concepts are related, which goes beyond the knowledge of properties and procedures and is related to a deep understanding of the multiplicative structure of integers.

Why? Because suspicion is one thing, being sure is another. Now everybody put down your notebooks, because we are going to do our first mathematical demonstration. We are going to prove that our suspicions are true. Let's start with the *m.c.d.* […] There can be no more common divisors, that is, a is the maximum of the common divisors. Suspect proven.

Figure 17. Fragment of the teacher's interventions in the connection CC1.

In the case of connection CC16, the difference between the operations -2^5 and $(-2)^5$ was discussed. The discussion was triggered by the intervention of a student who proposed that both expressions were equivalent representations of the same operation, since the result of both is the same. In her response, the teacher emphasised that the mere fact of obtaining equal results does not imply that the operations are equal. Although the connection can be classified as inter-conceptual, the teacher's intervention shows knowledge referred to establishing criteria of equivalence between mathematical operations, which relates to how different operations are linked, and therefore relates more to KSM than to KoT.

In the case of connection CC22 (Figure 18), there was a discussion on how to solve operations of type $\frac{a^n}{a^n}$. During the discussion, the students proposed different procedures related to two properties of the integers, $\frac{a}{a} = 1$ and $a^0 = 1$. The teacher explains that all the proposed procedures are correct and makes some comments. Although it could be considered to be a connection that emphasises the properties of powers of integers, it was decided to classify it as a KSM, since it not only connects the concepts of multiplication and division of integers, but also makes explicit how they are connected, emphasising a property that may not be very transparent for the students, such as $a^0 = 1$.

Because there are many operations in maths that give the same result and come from different places, shall I tell you one? 2 and 2? 4? 10 − 6? 4. Hey, you have the same result, but they come from signs, from different things. Well, this is an example. Dimitri, and the last one. But at the end of the day it's the same as doing the above. No, no, no…. I totally disagree, because, here (b) the base is? −2, and on the other hand, here the base is…? 2. They're two totally different powers, which, well, for some reason give the same result.

Figure 18. Fragment of the teacher's interventions in the connection CC22.

4. Discussion

The above results show that the establishment of mathematical connections by the teacher and PSTs is linked to the mobilisation of specific types of specialised knowledge. First, it can be observed that knowledge associated with mathematical topics (KoT), mathematics teaching (KMT), and features of students' learning (KFLM) is mobilised in most of the mathematical connections established. This is relevant, since the characterisation of KMT and KFLM subdomains does not include the establishment of mathematical connections, but within the MTSK, these connections are only part of mathematical knowledge (MK). For mathematical connections to be included in the practice of PSTs, and used and promoted by teachers, it is necessary that such connections are interpreted from pedagogical content knowledge (PCK). In particular, common feature connections (CFR and CFD) and connections involving equivalent procedures (EP).

The analysis of the data from the second study allowed us to observe specific relationships between connections related to process (CRP) and the teacher's mobilisation of KPM.

In this relationship, it is observed the way in which KPM can be linked to the development of transversal competences in students, such as those related to problem solving, justification and communication of mathematical results. Likewise, the mobilisation of KPM, when the teacher establishes CRP connections, represents a clear example of relevant situations in teaching practice where KPM can play a leading role. The identification of these situations and connections can be useful for the development of training activities for teachers, aimed at the development of KPM with emphasis on its specific dimension for teaching practice.

The analysis of ambiguities in the characterisation of mathematical knowledge (mobilised by the teacher and PSTs during the establishment of mathematical connections), allowed us to observe difficulties in the use of MTSK model, when identifying the knowledge mobilised during the establishment of connections. First, the proposed differentiation between intra- and inter-conceptual connections, contained in KoT and KSM respectively, is linked to certain difficulties that emerge when classifying the knowledge mobilised by the teacher and PSTs. In this sense, when comparing the different ambiguities identified, it can be observed that the evidence of the knowledge mobilised by the teacher and PSTs when making inter-conceptual connections is more linked to the knowledge of procedures associated with both, the calculation of areas and whole numbers (KoT) and not to the structure of mathematics itself (KSM). On the contrary, the evidence of the knowledge mobilised in connections CC1 and CC22 does reach a level of complexity and elaboration that allows us to affirm that the teacher in study 2 has a knowledge of the structure of mathematics (KSM). This is due to the fact that the connections are detailed from their internal structure.

Thus, it is possible that mentioning or describing the emergence of a connection is not sufficient evidence of the mobilisation of knowledge related to the structure of mathematics. It is possible that the interpretation of inter-conceptual connections does not necessarily imply the mobilisation of KSM, as the inter-conceptual character could be determined by a procedural competence on a particular concept. For knowledge about the structure of mathematics to be mobilised, it would be necessary to detail and characterise the connections at a structural level (de Gamboa and Figueiras ([21]). For example, the explicit mobilisation of knowledge about how concepts are related. Thus, the analysis shows that KSM subdomain would have an associated understanding of the structure of concepts and how they are related, which coincides with the definition proposed in Carrillo et al. ([2]), although not with the characterisation of the KSM subdomain.

The differentiation between intra- and inter-conceptual connections, and the separation of concepts as disjoint units, contradicts the idea of connection and the perspective of mathematical knowledge in which the MTSK is positioned: "network of systemic knowledge structured according to its own rules" (Carrillo et al. [2] (p. 6)). Thus, including intra-conceptual and extra-mathematical connections in the KoT subdomain, leaving inter-conceptual connections in the KSM domain, could be problematic. This is because in many cases the difference between intra- and inter-conceptual connections is very subtle, which can lead to an inappropriate use of the KSM tools. For example, in teaching area, when introducing the idea of a square unit of measurement to measure the area of a rectangle, multiplication is used to efficiently count the number of square units of measurement that cover the rectangle. Therefore, it can be argued that there is an inter-conceptual connection between area concept and the concept of multiplication, which could be classified as an auxiliary connection, within KSM sub-domain. However, the use of multiplication in the calculation of areas is more procedural than structural in nature, so it could be argued that it is an intra-conceptual connection in which the characteristic procedure for the calculation of areas of flat figures is introduced, and this would be an example of KoT.

In the case of simplification and complexity connections, it is possible that the knowledge linked to their establishment is of two types, KoT or KSM. Flores-Medrano ([19]) mentions, as an example, that knowledge about complexity connections would correspond to a teacher's interpretation of the meanings associated with fractions of whole numbers in basic education, and when working with fractions of differentials in secondary and higher education. However, this knowledge associated with meanings could be somewhat confusing, since when talking about meanings, reference is also being made to phenomenology or contexts of use of a given mathematical object (KoT). In the case of simplification connections, the example proposed in Carrillo et al. ([2]) of relating algebraic expressions to numerical expressions can be part of KoT when it allows a teacher to assess the equivalence between two algebraic expressions, or it can be part of KSM if there is an elaborate justification of the relationship between algebraic work, with indeterminate quantities, and arithmetic work with real numbers. In addition, it could be argued that KFLM is also mobilised when the teacher uses this numerical substitution strategy to show students a first approach to assessing equivalence between algebraic expressions, which can be useful for students to avoid common errors when manipulating algebraic expressions, such as $(a+b)^2 = a^2 + b^2$.

On the other hand, auxiliary connections present two problems. From the point of view of classification of connections, the use of one mathematical concept (e.g., equations) to develop another mathematical concept (e.g., roots of polynomials), can be interpreted as procedural in nature, and therefore more like knowledge related to KoT than to KSM. From a teacher's knowledge point of view, knowing different mathematical ideas that can be used when working with a concept is more related to KoT -understood as a "thoroughgoing knowledge of mathematical content (e.g., concepts, procedures, facts, rules and theorems) and their meanings" (Carrillo et al. [2] (p. 7)) than to KSM -understood as the teacher's knowledge of connections between mathematical items (Carrillo et al. [2] (p. 8)). In this sense, and in accordance with the above, it is possible that what determines the type of knowledge that is mobilised is not the inter- or intra-conceptual character of the connections that are produced, but the level of elaboration of the connection by the teacher. This elaboration provides evidence of a knowledge that goes beyond knowing that relationships exist, as it includes a knowledge of how such relationships are produced.

Therefore, the analysis of ambiguities in characterising the knowledge mobilised by the teacher and PSTs when establishing mathematical connections shows examples of the way in which recent research on mathematical connections (e.g., Businskas, [4]; Ro-dríguez-Nieto et al. [10]; de Gamboa and Figueiras [21]; Rodríguez-Nieto et al. [22]; Dolores-Flores and Garcia-Garcia [25]) can contribute to a future refinement of MTSK categories. First, the research does not distinguish between intra- and inter-conceptual connections, which is consistent with the difficulties in differentiating between them, as in the case of the connections discussed above. This suggests the need to refine the distinction that MTSK model makes between KoT and KSM. In the definitions of KSM that can be read in recent publications (Carrillo et al. [2]; Carrillo et al. [17]; Flores-Medrano [19]), KSM is defined by knowledge of inter-conceptual connections. As noted in the previous paragraphs, this definition can be problematic. Mainly because the delimitation of concepts can be fuzzy, making it difficult to differentiate between intra- and inter-conceptual. Second, the characterisation of complex connections as networks of elementary connections, as well as the different types of these elementary connections (representational, common feature, procedural, justification or generalisation) point to different levels of complexity for connections (de Gamboa et al. [5]). In turn, this implies different levels of complexity in the mobilised knowledge.

5. Conclusions

The performed analysis allowed us to identify relationships between the subdomains of MTSK model and various types of mathematical connections. The identification of these relationships shows the great usefulness of MTSK model to analyse the knowledge mobilised by teachers when they participate in situations rich in mathematical connections, which confirms the results of de Gamboa et al. ([33,34]). Moreover, these relationships show that recent results in the characterisation of mathematical connections can be useful for better understanding some of the subdomains of MTSK model.

The relationship between the mobilisation of KPM and the emergence of IMCRP-type connections is particularly relevant, as these connections give explicit examples of how KPM can be mobilised in classroom practice. This can be useful to identify classroom situations that can contribute to the characterisation of KPM, which is still under construction (Carrillo et al. [2]). Furthermore, the relationship identified between the establishment of elementary connections (of the EP, CFR and CFD types), and the mobilisation of knowledge linked to KMT and KFLM subdomains is also relevant. This is because this relationship points to the importance, for teachers, of knowing the types of connections that are linked to student's recurring errors and possible teaching strategies.

The problem identified in the intra- and inter-conceptual distinction, in the analysis of the knowledge mobilised by teachers, suggests that it is necessary that the characterisation of KSM can explicitly include knowledge about the way in which concepts are related. The predominance of a procedural use of connections is identified as an indicator of KoT subdomain, although they may be interpreted as inter-conceptual connections. For its part, KSM is associated with the use of mathematical connections aimed at understanding how mathematical ideas are related in a teaching-learning context, which is consistent with Hatisaru's findings ([9,34]).

The analysis carried out in the two studies shows that the notion of complexity associated to connections should also be considered in the characterisation of KSM, in order to enrich and facilitate its use as an analytical tool. The enrichment of KSM subdomain can help to understand the relationship between different subdomains of MTSK model, in line with those proposed by Badillo et al. ([35]). In this way, the connections between concepts, characterised in KSM, would benefit from taking into account the notion of complexity in their characterisation. Primarily, because it would be possible to further clarify the knowledge mobilised by teachers during the establishment of mathematical connections. The levels of complexity of the connections would be related, for example, to the way in which different concepts and properties can form a mathematical connection. In this sense, the connections that mention or describe procedures would be insufficient to characterise the complexity that underlies the structure of a complex mathematical connection. In any case, we consider that further studies are needed to problematise the use of MTSK in the establishment of mathematical connections.

Finally, we consider that in order to refine the characterisation of the subdomains of MTSK model, in terms of mathematical connections, more studies are required with a variety of data collection instruments that allow us to account for the complexity of the knowledge mobilised by in service and preservice teachers when establishing mathematical connections. For example, it is necessary to incorporate the use of clinical interviews that, triangulated with the data available to us, allow us to deepen the subdomains of knowledge that teachers use when establishing mathematical connections, both in the resolution of professional tasks and in classroom practice.

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References

1. Ball, D.L.; Thames, M.; Phelps, G. Content Knowledge for Teaching: What Makes It Special? *J. Teach. Educ.* **2008**, *59*, 389–407. [CrossRef]
2. Carrillo-Yañez; Climent, N.; Montes, M.; Contreras, L.C.; Flores-Medrano, E.; Escudero-Ávila, D.; Vasco, D.; Rojas, N.; Flores, P.; Aguilar-González, Á.; et al. The Mathematics Teacher's Specialised Knowledge (MTSK) Model. *Res. Math. Educ.* **2018**, *20*, 236–253. [CrossRef]
3. Rowland, T. The knowledge quartet: The genesis and application of a framework for analysing mathematics teaching and deepening teachers' mathematics knowledge. *Sisyphus J. Educ.* **2013**, *1*, 15–43.
4. Businskas, A.M. Conversations about Connections: How Secondary Mathematics Teachers Conceptualize and Contend with Mathematical Connections. Ph.D. Thesis, Simon Fraser University, Burnaby, BC, Canada, 2008.
5. De Gamboa, G.; Badillo, E.; Ribeiro, M.; Montes, M.; Sánchez-Matamoros, G. The Role of Teachers' Knowledge in the Use of Learning Opportunities Triggered by Mathematical Connections. In *Professional Development and Knowledge of Mathematics Teachers*; Zehetmaier, S., Potari, D., Ribeiro, M., Eds.; Routledge: London, UK, 2020; pp. 24–43.
6. De Gamboa, G. Aproximación a la Relación Entre el Conocimiento del Profesor y el Establecimiento de Conexiones en el Aula. Ph.D. Thesis, Universitat Autònoma de Barcelona, Barcelona, Spain, 2015.
7. Evitts, T. Investigating the Mathematical Connections that Preservice Teachers Use and Develop while Solving Problems from Reform Curricula. Ph.D. Thesis, Pennsylvania State University, State College, PA, USA, 2004.
8. Eli, J.A.; Mohr-Schroeder, M.J.; Lee, C.W. Exploring mathematical connections of prospective middle-grades teachers through card-sorting tasks. *Math. Educ. Res. J.* **2011**, *23*, 297–319. [CrossRef]
9. Hatisaru, V. Mathematical Connections Established in the Teaching of Functions. *Teach. Math. Appl. Int. J. IMA* **2022**, hrac013. [CrossRef]
10. Rodríguez-Nieto, C.A.; Font Moll, V.; Borji, V.; Rodríguez-Vásquez, F.M. Mathematical Connections from a Networking of Theories between Extended Theory of Mathematical Connections and Onto-Semiotic Approach. *Int. J. Math. Educ. Sci. Technol.* **2021**, *53*, 2364–2390. [CrossRef]
11. García-García, J.; Dolores-Flores, C. Intra-mathematical connections made by high school students in performing calculus tasks. *Int. J. Math. Educ. Sci. Technol.* **2018**, *49*, 227–252. [CrossRef]
12. García-García, J.; Dolores-Flores, C. Pre-university students' mathematical connections when sketching the graph of derivative and antiderivative functions. *Math. Educ. Res. J.* **2021**, *33*, 1–22. [CrossRef]
13. Caviedes, S.; de Gamboa, G.; Badillo Jiménez, E.R. Conocimiento movilizado por estudiantes para maestro al comparar áreas de figuras 2D. *Uniciencia* **2022**, *36*, 1–20. [CrossRef]
14. Shulman, L.S. Those Who Understand: Knowledge Growth in Teaching. *Educ. Res.* **1986**, *15*, 4–14. [CrossRef]
15. Shulman, L.S. Knowledge and Teaching: Foundations of the new reforms. *Harv. Educ. Rev.* **1987**, *57*, 1–22. [CrossRef]
16. Aguilar-González, Á.; Muñoz-Catalán, M.C.; Carrillo, J. An Example of Connections between the Mathematics Teacher's Conceptions and Specialised Knowledge. *EURASIA J. Math. Sci. Technol. Educ.* **2018**, *15*, em1664. [CrossRef]
17. Carrillo-Yañez, J.; Climent, N.; Contreras, L.; y Muñoz-Catalán, M. Determining specialised knowledge for mathematics teaching. In Proceedings of the 8th Conference of the European Society for Research in Mathematics Education (CERME), Antalya, Turkey, 6–10 February 2013; Ubuz, B., Haser, C., Mariotti, M., Eds.; pp. 2985–2994.
18. Montes, M.; Aguilar, A.; Carrillo, J.; Muñoz-Catalán, M. MTSK: From common and horizon knowledge to knowledge of topics and structures. In Proceedings of the 8th Conference of the European Society for Research in Mathematics Education (CERME), Antalya, Turkey, 6–10 February 2013; Ubuz, B., Haser, C., Mariotti, M., Eds.; pp. 3185–3194.
19. Flores-Medrano, E. Conocimiento de la estructura de las matemáticas. In *Investigación sobre Conocimiento Especializado del Profesor de Matemáticas (MTSK): 10 Años de Camino*; Carrillo-Yañez, J., Montes, M.A., Climent, N., Eds.; Dykinson: Madrid, Spain, 2022; pp. 47–55. [CrossRef]
20. Duval, R. A cognitive analysis of problems of comprehension in a learning of mathematics. *Educ. Stud. Math.* **2006**, *61*, 103–131. [CrossRef]
21. De Gamboa, G.; Lourdes, F. Connections in the mathematics teacher's knowledge: A proposal of an analysis model [Conexiones en el conocimiento matemático del profesor: Propuesta de un modelo de análisis]. In Proceedings of the Investigación en Educación Matemática XVIII, Salamanca, Spain, 1–4 September 2014; González, M.T., Codes, M., Arnau, D., Ortega, T., Eds.; SEIEM: Salamanca, Spain, 2014; pp. 337–344.
22. Rodríguez-Nieto, C.A.; Rodríguez-Vásquez, F.M.; Moll, V.F. A New View about Connections: The Mathematical Connections Established by a Teacher when Teaching the Derivative. *Int. J. Math. Educ. Sci. Technol.* **2022**, *53*, 1231–1256. [CrossRef]
23. Godino, J.D.; Font, V.; Wilhelmi, M.R.; Lurduy, O. Why Is the Learning of Elementary Arithmetic Concepts Difficult? Semiotic Tools for Understanding the Nature of Mathematical Objects. *Educ. Stud. Math.* **2011**, *77*, 247–265. [CrossRef]
24. Adu-Gyamfi, K.; Bossé, M.J.; Chandler, K. Student connections between algebraic and graphical polynomial representations in the context of a polynomial relation. *Int. J. Sci. Math. Educ.* **2017**, *15*, 915–938. [CrossRef]
25. Dolores-Flores, C.; García-García, J. Intra-mathematical and extra-mathematical connections that occur when solving Calculus' problems in context: A case study at a higher level. *Bol. Educ. Mat.* **2017**, *31*, 158–180.
26. Eli, J.A.; Mohr-Schroeder, M.J.; Lee, C.W. Mathematical connections and their relationship to mathematics knowledge for teaching geometry. *Sch. Sci. Math.* **2013**, *113*, 120–134. [CrossRef]

27. Cohen, L.; Manion, L.; Morrison, K. *Research Methods in Education*; Routledge: London, UK, 2011.
28. Bryman, A. *Social Research Methods*, 5th ed.; Oxford University Press: Oxford, UK; New York, NY, USA, 2016.
29. University of Nebraska-Lincoln; Creswell, J.W.; Garrett, A.L. The "Movement" of Mixed Methods Research and the Role of Educators. *SAJE* **2008**, *28*, 321–333. [CrossRef]
30. Krippendorff, K. *Content Analysis: An Introduction to Its Methodology*; Sage: Newcastle upon Tyne, UK, 2011.
31. Caviedes, S.; de Gamboa, G.; Badillo Jiménez, E.R. Mathematical Objects That Configure the Partial Area Meanings Mobilized in Task-Solving. *Int. J. Math. Educ. Sci. Technol.* **2021**, 1–20. [CrossRef]
32. Caldwell, K.; Atwal, A. Non-participant observation: Using video tapes to collect data in nursing research. *Nurse Res.* **2005**, *13*, 42–54. [CrossRef] [PubMed]
33. De Gamboa, G.D.; Badillo, E.; Couso, D.; Márquez, C. Connecting Mathematics and Science in Primary School STEM Education: Modeling the Population Growth of Species. *Mathematics* **2021**, *9*, 2496. [CrossRef]
34. Hatisaru, V.; Erbas, A.K. Mathematical Knowledge for Teaching the Function Concept and Student Learning Outcomes. *Int. J. Sci. Math. Educ.* **2017**, *15*, 703–722. [CrossRef]
35. Badillo, E.; González, M.T.; Moreno, M. Una mirada crítica al MTSK. In *Investigación sobre Conocimiento Especializado del Profesor de Matemáticas (MTSK): 10 Años de Camino*; Carrillo-Yañez, J., Montes, M.A., Climent, N., Eds.; Dykinson: Madrid, Spain, 2022; pp. 317–335.