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LIMIT CYCLES IN FILIPPOV SYSTEMS HAVING A CIRCLE AS SWITCHING MANIFOLD

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Abstract. It is known that planar discontinuous piecewise linear differential systems separated by a straight line have no limit cycles when both linear differential systems are centers.

Here we study the limit cycles of the planar discontinuous piecewise linear differential systems separated by a circle, when both linear differential systems are centers. Our main results show that such discontinuous piecewise differential systems can have 0, 1 or 2 limit cycles, but no more limit cycles than 2.

1. Introduction and statement of the main result

1.1. **Historical facts.** The problem of existence of limit cycles has been extensively treated in the literature since the early days of celestial mechanics. More recently, much work has been done on the rigorous mathematical foundation of nonsmooth dynamical systems problems, in particular, in the search of typical minimal sets that there are no counterparts in the smooth universe. It is worth to mention that, some existing smooth techniques are useful in solving many nonsmooth problems.

Some of the orbits of planar differential systems are difficult to study, this is the case of the limit cycles. Recall that a *limit cycle* of a differential system (S) is a periodic solution of (S) which is isolated in the set of all periodic solutions of (S). Concerning the nonsmooth universe and in the 2–dimensional case one can find many results on the existence of limit cycles when the switching set is an imbedded curve in $\mathbb{R}^2$, see [1–15, 18–29].

One of the main properties of smooth integrable systems in the plane $\mathbb{R}^2$ is that their periodic orbits usually appear in continuous one–parameter families, in contrast to the periodic orbits of piecewise nonsmooth integrable systems which typically are limit cycles, see [2–6, 18–28].

Andronov, Vitt and Khaikin [1] started the study of the discontinuous piecewise linear differential systems in the plane, mainly motivated for their applications to some mechanical problems. Recently the interest for this kind of differential systems increased due mainly to the fact that these differential systems model many processes appearing in mechanics, electronics, economy,... See for these applications the survey of Makarenkov and Lamb [27], and the books of Simpson [30] and of

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di Bernardo, Budd, Champneys and Kowalczyk [7], together with the hundreds of references quoted in these works.

In the plane the discontinuous piecewise linear differential systems separated by a straight line are the easiest possible discontinuous piecewise linear differential systems. For these systems a *crossing limit cycle* is a limit cycle which has exactly two points on the line of discontinuity. But for these easy systems there are open questions as the following: *Is three the maximum number of crossing limit cycles that a discontinuous piecewise linear differential systems separated by a straight line can have*? See for instance [6, 10–12, 14, 20, 23, 23].

1.2. **Setting the problem.** The main goal of this article is to discuss the existence of limit cycles of nonsmooth piecewise integrable systems in the plane $\mathbb{R}^2$ where the switching set is concentrated on a closed curve. Recently, the mentioned great interest in non-smooth dynamics leads us the need to explore new challenges in this subject. In the classical smooth case we find in the literature an endless number of papers involving the problem on the existence of limit cycles in the plane, see for instance the books [31, 32] and the hundreds of references therein. Moreover, problems involving the existence of limit cycles bifurcating from a smooth center are extensively studied in the last decades, see the book [8] and the references quoted there.

We recall that a *center* of a planar differential system is a singular point p of the system for which there is a neighborhood U such that all the orbits of $U \setminus \{p\}$ are periodic.

It is interesting to note that if we consider discontinuous piecewise linear differential systems formed by two centers separated by a straight line, then such systems have no crossing limit cycles, for a proof see Theorem 4 of [20].

In this paper we deal with discontinuous piecewise linear differential systems (C) formed by two centers separated by a circle, that without loss of generality we can assume that is the circle

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

A *crossing periodic orbit* for a system (C) is a periodic orbit which has exactly two points on the circle $\mathbb{S}^1$ of discontinuity, and of course a *crossing limit cycle* for a system (S) in (C) is a crossing periodic orbit isolated in the set of all crossing periodic orbits of system (S).

To provide a sharp upper bound on the number of limit cycles for a class of planar differential systems is in general a very difficult problem, consider for instance the 16-th Hilbert problem. Thus, in particular, for the class of quadratic polynomial differential systems we have systems with four limit cycles, but we do not know if there is a uniform bound for the maximum number of limit cycles for all the systems of this class. See for more details [17, 18].

We shall provide an upper bound for the number of limit cycles of the planar discontinuous piecewise linear differential systems separated by a circle, when both linear differential systems are centers. We stress that the respective proof is non-trivial although the expressions of the systems are very friendly. It should be noted that, as far as the authors know, there are few developed techniques or tools within the theory of discontinuous piecewise linear differential systems. So new techniques in this regard are, of course, notoriously welcomed.

1.3. **Statement of the main results.** The objective of this paper is to study the number of crossing limit cycles that systems in the class (C) can have, and to present an exotic center. Our main result is the following.

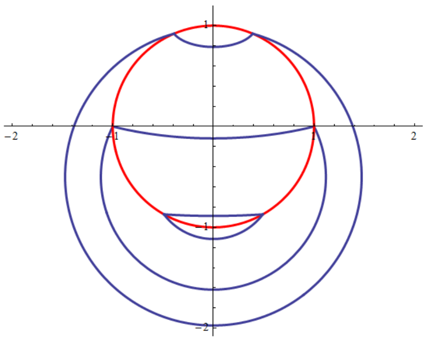

Figure 1: Three periodic solutions of the center $(0, -1)$ of the discontinuous piecewise linear differential system formed by the centers (1) and (2).

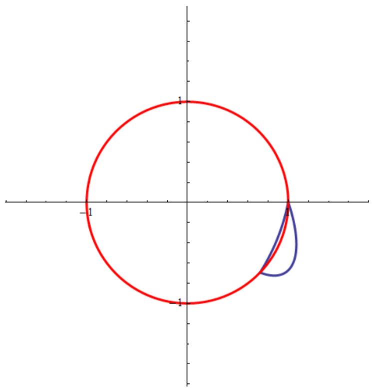

Figure 2: The unique limit cycle of the discontinuous piecewise linear differential system formed by the centers (3) and (4).

Theorem 1. *Let (C) be the class of planar discontinuous piecewise linear differential systems formed by two centers separated by the circle $\mathbb{S}^1$. Then the following statement hold.*

(a) *There are systems in (C) with infinitely many crossing periodic orbits, in fact with a continuum of crossing periodic orbits forming a center, see Figure 1.*

(*b*) *There are systems in* (C) *without crossing periodic orbits.*

(*c*) *There are systems in* (C) *having exactly one crossing limit cycle, see Figure 2.*

(*d*) *There are systems in* (C) *having exactly two crossing limit cycles, see Figure 3.*

(*e*) *Every system in* (C) *has at most two crossing limit cycles.*

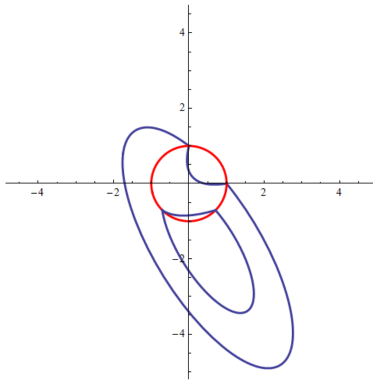

Figure 3: The two limit cycles of the discontinuous piecewise linear differential system formed by the centers (6) and (7).

Theorem 1 is proved in section 2.

The results of Theorem 1 show once again that the shape of the discontinuity line plays a main role on the number of crossing limit cycles that discontinuous planar piecewise linear differential systems can exhibit, see also [4, 5, 26, 28].

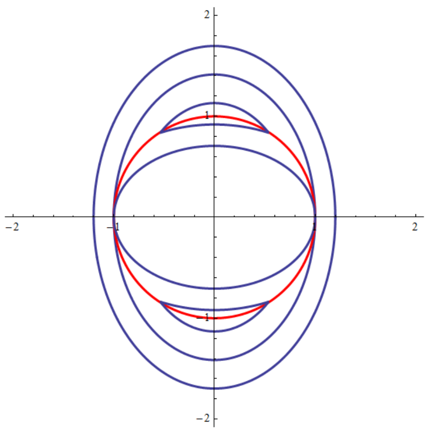

Figure 4: An exotic center.

We exhibit in section 3 a planar discontinuous piecewise linear differential systems formed by two centers separated by the circle $\mathbb{S}^1$ possessing an unusual center (exotic center). That means that its phase portrait contains the origin as a canonical smooth center and there are on the switching set two 2–fold singularities of the center type, see Figure 4.

2. Proof of Theorem 1

Proof of statement (*a*) *of Theorem 1.* We consider in the bounded region limited by the circle $\mathbb{S}^1$ the linear differential center

$$\dot{x} = 1 - y, \quad \dot{y} = \frac{1}{4}x. \tag{1}$$

In all this paper the dot denotes derivative with respect to the time t, except if we say explicitly another thing. This system has the first integral

$$H_1(x, y) = \frac{1}{4}x^2 + y^2 - 2y.$$

In the unbounded region of $\mathbb{R}^2$ with boundary the circle $\mathbb{S}^1$ we have the linear differential center

$$\dot{x} = -\frac{1}{2} - y, \quad \dot{y} = x. \tag{2}$$

This system has the first integral

$$H_2(x, y) = x^2 + y^2 + y.$$

For all $\alpha \in [0, 2\pi) \setminus \{\pm\pi/2\}$ the two ellipses

$$H_1(x, y) = H_1(\cos\alpha, \sin\alpha) \quad \text{and} \quad H_2(x, y) = H_2(\cos\alpha, \sin\alpha),$$

intersect exactly with the circle $\mathbb{S}^1$ in the two points $(\pm\cos\alpha, \sin\alpha)$. Therefore the discontinuous piecewise linear differential system formed by the centers (1) and (2) has a continuum of crossing periodic solutions, which intersect the circle at the two points $(\pm\cos\alpha, \sin\alpha)$ for all $\alpha \in [0, 2\pi) \setminus \{\pm\pi/2\}$. See three crossing periodic solutions of this continuum of crossing periodic solutions corresponding to the values of α equal to 0, 1.0572619009906146.. and 5.114273122257285.. in Figure 1. □

Proof of statement (*b*) *of Theorem 1.* This statement has an easy proof it is sufficient to consider the discontinuous piecewise linear differential system having the linear center $\dot{x} = -y$, $\dot{y} = x$ in the bounded region limited by the circle $\mathbb{S}^1$, and an arbitrary center different from the previous one in the unbounded region limited by $\mathbb{S}^1$. Clearly a such discontinuous piecewise linear differential system has no crossing periodic orbits. □

Proof of statement (*c*) *of Theorem 1.* In the unbounded region limited by the circle $\mathbb{S}^1$ we consider the linear differential center

$$\dot{x} = \frac{1}{3} - \frac{x}{2} - \frac{y}{2}, \quad \dot{y} = x + \frac{y}{2} - \frac{1}{2}. \tag{3}$$

This system has the first integral

$$H_1(x,y) = 4\left(x+\frac{y}{2}\right)^2 - 8\left(\frac{x}{2}+\frac{y}{3}\right) + y^2.$$

In the bounded region of $\mathbb{R}^2$ with boundary the circle $\mathbb{S}^1$ we have the linear differential center

$$\dot{x} = \frac{2}{5} - y, \quad \dot{y} = 1 + x. \tag{4}$$

This system has the first integral

$$H_2(x,y) = (x+1)^2 + \left(\frac{2}{5} - y\right)^2.$$

This discontinuous piecewise differential system formed by the linear differential systems (3) and (4) has exactly one crossing limit cycle, because the unique real solutions $(\alpha,\beta,\gamma,\delta)$ of the system

$$\begin{array}{l} H_1(\alpha,\beta) = H_1(\gamma,\delta), \\ H_2(\alpha,\beta) = H_2(\gamma,\delta), \\ \alpha^2+\beta^2 = 1, \\ \gamma^2+\delta^2 = 1, \end{array} \tag{5}$$

where $(\alpha,\beta,\gamma,\delta)$ satisfying $(\alpha,\beta) \neq (\gamma,\delta)$ are

$$\left(\frac{10\left(232-7\sqrt{29}\right)}{2697}, \frac{-928-175\sqrt{29}}{2697}, \frac{10\left(232+7\sqrt{29}\right)}{2697}, \frac{175\sqrt{29}-928}{2697}\right),$$

and the one interchanging (α,β) by (γ,δ), which defines the same crossing limit cycle. See this crossing limit cycle in Figure 2. $\square$

Proof of statement (d) of Theorem 1. In the unbounded region limited by the circle $\mathbb{S}^1$ we consider the linear differential center

$$\dot{x} = -x - y - \frac{1}{8}\left(9+5\sqrt{3}\right), \quad \dot{y} = 2x + y + \frac{5}{8}\left(1+\sqrt{3}\right). \tag{6}$$

This system has the first integral

$$H_1(x,y) = 2x^2 + x\left(2y+\sqrt{2}\right) + y\left(y+\sqrt{2}+1\right).$$

In the bounded region of $\mathbb{R}^2$ with boundary the circle $\mathbb{S}^1$ we have the linear differential center

$$\dot{x} = x - \frac{5y}{4} + \frac{1}{\sqrt{2}} + \frac{1}{8}, \quad \dot{y} = x - y - \frac{1}{\sqrt{2}}. \tag{7}$$

This system has the first integral

$$H_2(x,y) = 4x^2 - 4x\left(2y+\sqrt{2}\right) + y\left(5y - 4\sqrt{2} - 1\right).$$

This discontinuous piecewise differential system formed by the linear differential systems (6) and (7) has exactly two crossing limit cycles, because the unique real solutions $(\alpha,\beta,\gamma,\delta)$ of the system (10) are the four solutions $(1,0,0,1)$, $(0,1,1,0)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Hence, the pairs $(1,0)$,

$(0,1)$ and $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ are the intersection points of the two crossing limit cycles with the circle $\mathbb{S}^1$. See these two crossing limit cycles in Figure 3. □

We shall use the next result which is well known, see for instance [24, 26].

Lemma 2. *Doing a rescaling of the independent variable and a linear change of variables every linear differential system in $\mathbb{R}^2$ having a center can be written as*

$$\dot{x} = -bx - \frac{4b^2+\omega^2}{4a}y + d, \quad \dot{y} = ax + by + c, \tag{8}$$

with $a > 0$ and $\omega > 0$.

Proof of statement (e) of Theorem 1. In the bounded region limited by the circle $\mathbb{S}^1$ we consider the arbitrary linear differential center (8), which has the first integral

$$H_1(x,y) = 4(ax+by)^2 + 8a(cx-dy) + y^2\omega^2.$$

In the unbounded region limited by the circle $\mathbb{S}^1$ we consider the arbitrary linear differential center

$$\dot{x} = -Bx - \frac{4B^2+\Omega^2}{4A}y + dD, \quad \dot{y} = Ax + By + C, \tag{9}$$

with $A > 0$ and $\Omega > 0$. This system has the first integral

$$H_2(x,y) = 4(Ax+By)^2 + 8A(Cx-Dy) + y^2\Omega^2.$$

If this discontinuous piecewise differential system formed by the linear differential systems (8) and (9) has a crossing limit cycle which intersect the circle $x^2+y^2=1$ in the pair of points (α,β) and (γ,δ), then these points must satisfy the system

$$\begin{aligned} &H_1(\alpha,\beta) = H_1(\gamma,\delta),\\ &H_2(\alpha,\beta) = H_2(\gamma,\delta),\\ &\alpha^2+\beta^2 = 1,\\ &\gamma^2+\delta^2 = 1, \end{aligned}$$

with $(\alpha,\beta) \neq (\gamma,\delta)$, or equivalently the system

$$\begin{aligned} &H_1(\alpha,\beta) = h_1,\\ &H_2(\alpha,\beta) = h_2,\\ &H_1(\gamma,\delta) = h_1,\\ &H_2(\gamma,\delta) = h_2,\\ &\alpha^2+\beta^2 = 1,\\ &\gamma^2+\delta^2 = 1. \end{aligned} \tag{10}$$

The two first equations of system (10) has at most four real solutions for (α,β), because these solutions are the intersection of two ellipses. These four solutions are also the four solutions of the third and fourth equations of system (10). Since these four solutions on the circle $x^2+y^2=1$ must be the intersection points of the crossing limit cycles with the circle, we can have at most two crossing limit cycles. □

This completes the proof of Theorem 1.

3. Exotic Center

Let $f(x,y) = x^2 + y^2 - 1$, $H_0(x,y) = x^2 + \dfrac{y^2}{2}$ and

$$H_1(x,y;\theta) = (x\cos\theta + y\sin\theta)^2 + \frac{(-x\sin\theta + y\cos\theta)^2}{2}.$$

Notice that the Hamiltonian vector field X_0 originated by H_0 has four tangential contacts with $\Sigma = f^{-1}(0)$ at $p_1 = (0,1)$, $p_2 = (1,0)$, $p_3 = (0,-1)$ and $p_4 = (-1,0)$. Since the level curves of H_1 are the same level curves of H_0 after a rotation of angle θ, we obtain that the Hamiltonian vector field Y_θ generated by H_1 has also four contact points at $q_1(\theta) = (\sin\theta, \cos\theta)$, $q_2(\theta) = (\cos\theta, -\sin\theta)$ $q_3(\theta) = (-\sin\theta, -\cos\theta)$, and $q_4(\theta) = (-\cos\theta, \sin\theta)$.

Of course the orbits of X_0 (resp. Y_θ) are symmetric with respect to the lines containing the segments $\overline{p_1p_2}$ and $\overline{p_3p_4}$ (resp. $\overline{q_1q_2}$ and $\overline{q_3q_4}$).

If $\theta = \pi/2$ or $\theta = 3\pi/2$, we have that all the tangency points of the Hamiltonian systems X_0 and Y_θ coincide generating four fold–fold points. Since the phase portrait of this discontinuous piecewise linear differential system is symmetric with respect to the x-axis and the y-axis, we have that p_1 and p_3 are parabolic fold–fold points surrounded by crossing periodic orbits presenting a behavior similar to a singularity of type center. Also, the singularities p_2 and p_4 are connected by four orbits giving rise to pseudo–separatrices for the discontinuous piecewise linear system formed by X_0 and Y_θ.

If $\theta = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$, then it follows from symmetry that the orbits of Y_θ give rise to two connections between the fold–regular points p_1 and p_2.

If θ is different from the values mentioned in items 1 and 2 the two previous paragraphs, then we have that the system presents a polycycle having two parabolic fold–fold singularities.

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