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To cite this article: Vincent Daniel David *et al* 2021 *J. Phys.: Conf. Ser.* **1770** 012042

Critical flow over an uneven bottom topography using Forced Korteweg-de Vries (fKdV)

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Abstract. This research examined the critical flow over an uneven bump using forced Korteweg-de Vries (fKdV) model. The forced KdV model containing forcing term which represent an uneven bump is solved using Homotopy Analysis Method (HAM). HAM is a semi-analytic technique whereby its solution contains a series of approximated solution in which it converges immediately to the exact solution. A particular HAM solution is chosen with an appropriate convergence parameter by referring to horizontal line segment. The convergent HAM solution depicts that waves only exhibited over the sloping region and no rise of waves found on flat part of bottom topography.

1. Introduction

Water propagation over an obstacle is a vital problem in fluid mechanics. Since 80's, generation of solitary waves by seabed topography has gained attention since the experimental research [1,2]. Forced waves and the existence of wave trains for the solutions of same size moving ahead of the bottom topography were found numerically [3]. Linear theory usually used to explain over the wave when the flow is not critical. Solutions of linear theory applicable to cases such as subcritical or supercritical cases. Linear solutions usually fails at criticality condition as the energy is unable to propagate away from the obstacle [4]. Thus it is important to find a model to investigate the wave profile at critical flow.

One of suitable model is forced Korteweg-de Vries (fKdV) equation which identified as suitable model to study the free surface flow over a flatten bump. Standard form of forced KdV equation is given [5],

$$\frac{\partial \eta}{\partial t}+\Delta\frac{\partial \eta}{\partial x}+\alpha\eta\frac{\partial \eta}{\partial x}+\beta\frac{\partial^3 \eta}{\partial x^3}=\frac{\partial f}{\partial x} \tag{1}$$

with $\alpha = 3c/2h_0,\ \beta = ch_0^2/6$ and $f = -cz/2.$ (2)

where $\eta(x,t)$ refers to the water elevation, $z(x,t)$ represents the solid bottom, h_0 is the constant mean water depth, c is the long wave speed with g is acceleration due to gravity, and Δ represents critical parameter.

Forced KdV equation incorporated with variety of forcing term have been studied in the past years [6-8]. Forced KdV model admits external forcing disturbances when the surface pressure and bottom topography are entirely equivalent [9]. The effect of forcing length on wave amplitude was also studied over the years [11]. Zhang and Zhu [12] presented a nonlinear theory for different ranges of Froude numbers varying from subcritical, transcritical and supercritical. Zhang and Chwang [13] studied the generation of solitary waves at the critical velocity on different bottom topographies. Transcritical flow using fKdV model where numerical and asymptotic analytical solutions have shown upstream and downstream flows [4].

The Homotopy Analysis Method (HAM) introduced by Liao [14] is an analytical method to solve nonlinear partial differential problems. HAM has greater flexibility in the selection of a proper set of base functions for the solution and a much simpler way in the control of the convergence rate and region compared to perturbation approach [15-17]. This analytical technique does not have restriction of non-perturbation methods, such as Lyapunov's artificial small parameter method, the δ-expansion method and Adomian's decomposition method [18]. The analytical technique also has been applied successfully in many nonlinear problems in engineering and sciences [19] such as nonlinear progressive waves [20], free oscillations of positively damped systems with algebraically decaying amplitude [21], free oscillations of self-excited systems [22] and similarity boundary layer equations [23]. The HAM is applied to obtain the solitary solution of KdV equation and it shows excellent agreement with the exact solution [24]. Solutions of fKdV equation can only be obtained by numerical or perturbation series [25-27]. Recently, fKdV model with a specific choice of forcing term is successfully solved using HAM [28-29]. Analytical approximate solution for the fKdV model on critical flow over hole shaped bottom topography been investigated [30]. Recently, fKdV equation resemble critical flow over an inclination plane is solved using HAM [31].

Flow over a flatten bump is determined using shallow water fKdV model and it is solved using HAM. Objective of this research is (a) to describe the flatten obstacle (b) to find analytical approximate solution for fKdV model which incorporates flatten bump (c) to describe flow over the bump physically and discuss the new findings. It is found that HAM solution elaborated the flow of water over flatten bump. The bump is found to generate upstream and downstream flows. The bump also creates a uniform depth wave over the forcing region.

2. Forced Korteweg-de Vries and Bottom Topography

The standard form of fKdV equation of (1) rewritten as

$$\left(\varphi'(t)/c_o\right)+\left(\Delta-3\varphi/2h_o\right)\varphi'(x)-\left(h_o^{\,2}\varphi'''(x)/6\right)=f'(x)/2 \tag{3}$$

where $\varphi(x,t)$ refers to the water elevation, $f(x)$ is the forcing term in which it represent the bottom topography, h_0 is the constant mean water depth, c is the long wave speed with g is acceleration due to gravity, and Δ represents critical parameter.

Critical parameter can be classified into three which are transcritical ($\Delta\approx 0$), subcritical ($\Delta<0$) and supercritical ($\Delta>0$). In this work, transcritical flow is considered.

3. Homotopy Analysis Method

We attempt to solve equation (3) for the $\Delta\approx 0$.Generalizing equation (3), then

$$\alpha\frac{\partial\varphi}{\partial t}+\beta\frac{\partial\varphi}{\partial x}+\lambda\varphi\frac{\partial\varphi}{\partial x}+\sigma\frac{\partial^3\varphi}{\partial x^3}+\varpi\frac{\partial f}{\partial x}=0 \tag{4}$$

where

$$\alpha=\frac{1}{c_o},\ \beta=Fr-1=\frac{U}{c_o}-1,\ \lambda=-\frac{3}{2h_o},\ \sigma=-\frac{1}{6}h_o^{\,2}\ \text{and}\ \varpi=-\frac{1}{2}f_m. \tag{5}$$

Consider the constant mean water depth, $h_0=1$, wave speed, $c_o\approx\sqrt{gh}=\sqrt{9.81}$, maximum height of topography chosen, $f_m=0.1$ and forcing term, $f=\frac{-z}{2}$, where $z=f_m\exp[-\frac{x^8}{4}]-1$. Below is the sketch of the bottom topography, z.

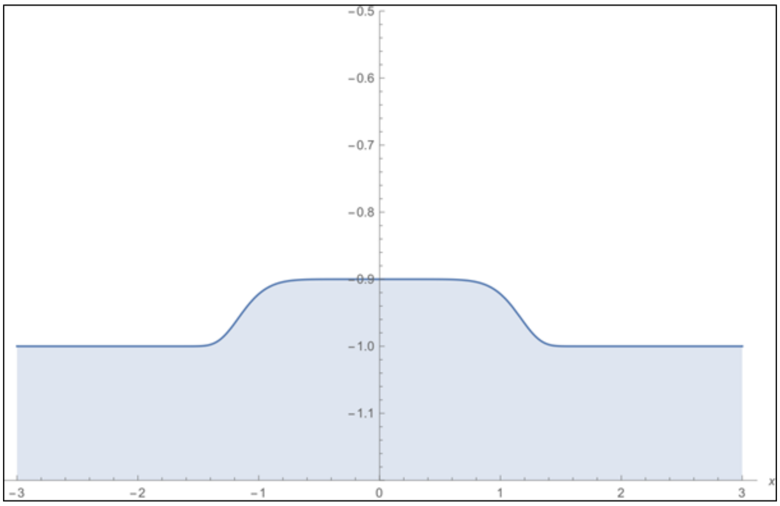

Figure 1 Bottom Topography, z for $f(x)=0.1*\exp[-\frac{x^8}{4}]-1$

From HAM,

$$(1-q)\ell[\varphi(x,t;q)-\varphi_0(x,t)]=qC_o\mathcal{H}(x,t)N[\varphi(x,t;q)] \tag{6}$$

we use

$$\varphi_0(x,t)=\frac{1}{4}(1+\sin[x]) \tag{7}$$

as the initial guess and

$$\ell[\varphi(x,t;q)]=\frac{\partial\varphi(x,t;q)}{\partial t} \tag{8}$$

as the auxiliary linear operator satisfying

$$\ell[g]=0 \tag{9}$$

where g is constant.
Considering

$$\mathcal{H}(x,t)=1 \tag{10}$$

$$N[\varphi(x,t;q)]=\alpha\frac{\partial\varphi(x,t;q)}{\partial t}+\beta\frac{\partial\varphi(x,t;q)}{\partial x}+\lambda\varphi(x,t;q)\frac{\partial\varphi(x,t;q)}{\partial x}+\sigma\frac{\partial^3\varphi(x,t;q)}{\partial x^3}+\varpi\frac{\partial f}{\partial x}, \tag{11}$$

and the m^{th}-order deformation problem

$$\ell[\varphi_m(x,t)-\chi_m\varphi_{m-1}(x,t)]=qC_o[\alpha\frac{\partial\varphi_{m-1}}{\partial t}+\beta\frac{\partial\varphi_{m-1-i}}{\partial x}+\lambda(\sum_{i=0}^{m-1}\varphi_i\frac{\partial\varphi_{m-1-i}}{\partial x})+\sigma\frac{\partial^3\varphi_{m-1}}{\partial x^3}+\varpi\frac{\partial f_{m-1}}{\partial x}] \tag{12}$$

with

$$\varphi_m(x,0)=0 \text{ for } m>1 \tag{13}$$

4. HAM Solution of Critical Flow

Wolfram Mathematica Version 10 was used to solve the forced KdV equation. HAM solution of equation (3) is obtained at 5^{th}-order approximation. The solution is

$$\varphi(x,t)=\varphi_0(x,t)+\varphi_1(x,t)+\ldots\ldots\ldots\ldots+\varphi_5(x,t) \tag{14}$$

$$\begin{aligned}\varphi(x,t)&=0.25(1+\sin[x])-0.025e^{-\frac{x^8}{4}}t(\frac{8x^7}{5}-\frac{5}{6}e^{\frac{x^8}{4}}\cos[x]-\frac{3}{2}e^{\frac{x^8}{4}}\cos[x]\sin[x])\\&-0.00078125e^{-\frac{x^8}{4}}(358.4t^2x^4+\frac{1792tx^6}{5}+26.889999999999995t^2x^6+\ldots\ldots\ldots\end{aligned} \tag{15}$$

Journal of Physics: Conference Series **1770** (2021) 012042 doi:10.1088/1742-6596/1770/1/012042

The value of C_o is determined by plotting the derivatives of φ for a fixed point of x and time, t. It is to ensure the convergence of the HAM solution. Figure 2 shows the C_o- curves at 5th order approximation.

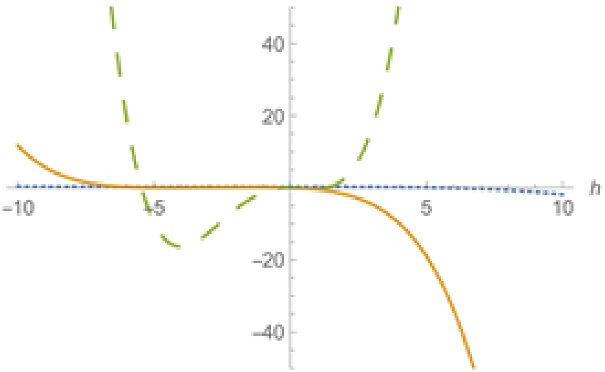

Figure 2 The C_o-curves according to the 5th order approximation. Dashed Point: $\varphi(0.01, 0.01)$, Solid Line: $\dot{\varphi}(0.01, 0.01)$ and Dashed Line: $\ddot{\varphi}(0.01, 0.01)$

It is pointed out that the valid region of C_o lies on the horizontal line segment. The permissible convergence interval of HAM solution is at C_o= - 4. HAM solution has been re-modified by adding a coefficient, K so that the solution presents the promising wave patterns that suits real water flow scenario.

$$\varphi_{Sol} = \mathrm{K} * \varphi(x,t) \tag{16}$$

This explains that the HAM solution contains a series of solution and C_o -values should be chosen accurately to achieve a reasonable solution. Perhaps, this is a new technique found in analyzing HAM solution. The following figures 3 and 4 are obtained by using equation (15) and equation (16) with a coefficient value, of $\mathrm{K} = 10^{-15}$ and $C_o = -4$.

Figure 3 depicts the flow of water waves over a flatten bump at t = 3. The bottom line over in Figure 3 represents the seabed topography which has height of 0.1 units from $-1 \le x \le 1$. The inclination (positive slope) of sea bed topography is at vicinity of $x = -1$. The declination (negative slope) of bump falls at $x = 1$. Both inclination and declination of bump can be observed by looking at the bottom line in Figure 3. Figure 4 depicts the water wave profile across the flatten bump over the period of $2.5 \le t \le 3$.

Initial guess function chosen in the analytical method is a sinusoidal function. This is to ensure wave travels from left to right and it is not concentrated at the centre of origin. Based on the shape of topography, it can be seen that sea bed is entirely flattening except these two-sloping regions which positive and negative slope. Waves patterns reveals that waves only exhibit over forcing vicinity. On the positive sea bed, it is found to have 3 peaks of waves and which 2 of them are identical. The centric waves of upstream peaked at height of 0.5 units. This proved that the waves profile reacted towards the sloping region. Forced Korteweg-de Vries (fKdV) is a model inclusive of nonlinearity and dispersion. Many researches attempted the solution of fKdV by reduction its dispersion order. In this research, fKdV were solved without reducing its order. The nonlinearity is very strong at the sloping part. This is shown by multi-solitary waves over the sloping region.

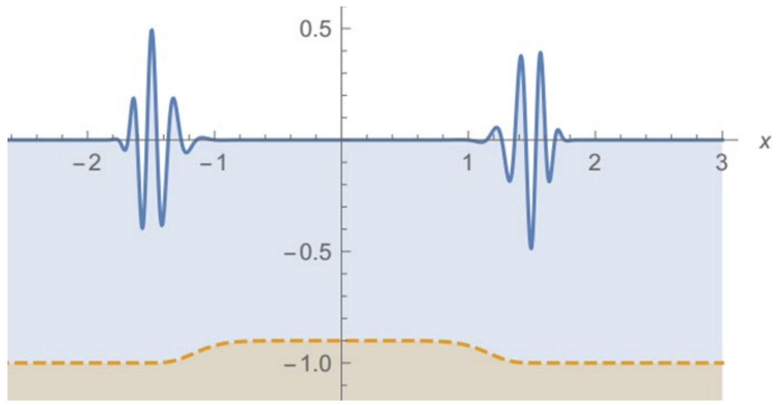

Figure 3 Wave Profile of HAM solution for equation (16) at t = 3.
Bottom line: bump, z and Upper line: water elevation, $\varphi(x,3)$

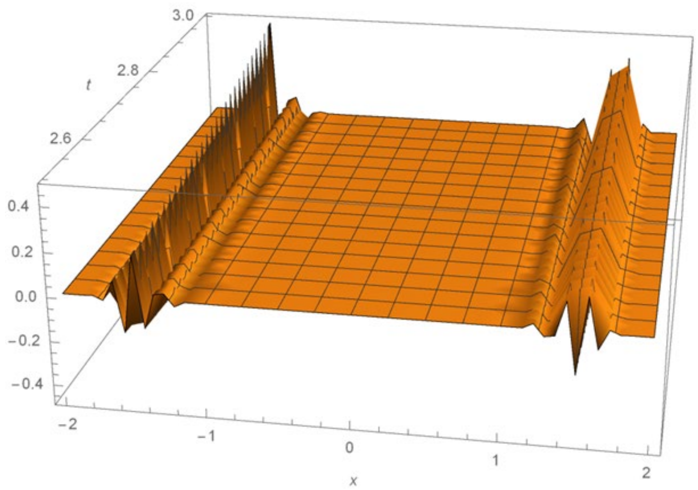

Figure 4 Wave Profile of HAM solution corresponds with distance and time for equation (16)

There is no excitement of waves over the centric part of flatten bump. No evidence found here that disturbance occurs over flat part of bump which agreed with Grimshaw et. al, 2007 [4]. Water wave rise again at the downstream part over the negative slope. There are 2 high and 2 small waves exhibits over the sloping region. But the height of highest peak is not similar with the peaked waves at the upstream. This means, the water waves over downstream is smaller in height with the upstream waves. This is a sign of depression which were similarly found in Grimshaw et. al, 2007 [4]. Nonlinearity of waves at the downstream is weaker compared to the upstream waves. Grimshaw et. al, 2007 [4] and Zhang et. al, 2001 [13] concluded that upstream and downstream wave trains generated by transcritical flow over an obstacle could be generated by separate process. This shows waves radiated upstream strongly and since no activity found in middle, waves move with a weaker downstream. Downstream nonlinearity is found to be weak and it could be an act of dispersion. This provide evidence that higher order forced KdV model could be a key to reduce the nonlinearity of waves. The outcome of this research has good agreement with Samuel Shen, 1993 [7] where the forced KdV admits solitons generated periodically and radiated upstream at transcritical regime.

5. Conclusion

In this work, forced Korteweg-de Vries (fKdV) model been used to examined transcritical flow over a flatten bump which consist of positive and negative sloping regions. An appropriate flatten bump is analysed using the forcing term over fKdV model. An analytic approximate solution is obtained by solving fKdV model. The HAM solution shows water waves exhibit over positive and negative sloping region. The result depicts when water flows over the bump its radiated strong upstream, no activity on flatten bump and finally water flows downstream and exhibit weaker waves which is due to dispersion effect. It can be concluded that the effect of dispersion cannot be neglected although the effect is found to be weak.

6. Acknowledgements

The first author is thankful to Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA for the financial funding.

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