

# Calculating geodesics: A computational approach on ellipsoids

*Cálculando geodésicas:  
Un enfoque computacional sobre el elipsoide*

*Calculando geodésicas:  
Uma abordagem computacional para o elipsóide*

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## Abstract

*Introduction:* The article is the product of the research "Connections on Semi-Riemannian Geometry and Christoffel Coefficients – Towards the study of the computational calculation of geodesics", developed at the Pascual Bravo University Institution in the year 2021.

*Problem:* Based on solutions of the Euler-Lagrange equations, the explicit calculation of geodesics on certain manifolds is possible. However, there are several cases in which it is impossible to continue calculating analytically and we have to resort to a numerical calculation. In this sense, several geometric and dynamic characteristics of geodesics, unexpectedly emerge.

*Objective:* The objective of the research is to calculate geodesics of a Riemannian or semi-Riemannian manifold using SageMath as software to more easily go beyond what intuition provides.

*Methodology:* First, some simple examples of characterizations of geodesics on certain manifolds, based on solutions of the Euler-Lagrange equations, are presented. Then, an ellipsoid is selected as a test subject with which to numerically calculate geodesics, observing how it changes depending on whether it is defined within a Spherical, Triaxial or Mercator coordinate system.

*Results:* With the flexibility of software like SageMath, an explicit expression of the differential equations was made possible along with, from numeric solutions for these equations, their corresponding simulations depending on the selected parameters.

*Conclusion:* These simulations confirm that great circles are not the only geodesics existing on the ellipsoid, but rather there are many other types of geodesic curves, some of which can be dense curves on the surface and others can be closed curves. At the same time, this shows a relationship between the existence of certain types of geodesic curves and the parameterization of the surface.

**Keywords:** Geodesics, Euler-Lagrange equations, simulations, SageMath, Ellipsoid

## Resumen

*Introducción:* El artículo es producto de la investigación Conexiones sobre Geometría Semi-Riemanniana y Coeficientes de Christoffel – Hacia el estudio del cálculo computacional de geodésicas desarrollada en la Institución Universitaria Pascual Bravo en el año 2021.

*Problema:* Gracias a las soluciones de las ecuaciones de Euler-Lagrange, es posible el cálculo explícito de geodésicas en ciertas variedades. Sin embargo, hay varios casos en los que es imposible seguir calculando analíticamente y tenemos que recurrir a un cálculo numérico. En este sentido, surgen inesperadamente varias características geométricas y dinámicas de las geodésicas.

*Objetivo:* El objetivo de la investigación es calcular geodésicas de una variedad Riemanniana o Semi-Riemanniana utilizando como software SageMath para ir más fácilmente más allá de lo que proporciona la intuición.

*Metodología:* Primero, se presentan algunos ejemplos simples de caracterizaciones de geodésicas en ciertas variedades, basados en soluciones de las ecuaciones de Euler-Lagrange. Luego, se selecciona el elipsoide como nuestro modelo de juguete para calcular geodésicas numéricamente y observar cómo cambian dependiendo de si está bajo un sistema de coordenadas esféricas, triaxiales o de Mercator.

*Resultados:* Con la flexibilidad de un software como SageMath, fue posible una expresión explícita de las ecuaciones diferenciales y a partir de las soluciones numéricas de estas ecuaciones sus correspondientes simulaciones en función de los parámetros seleccionados.

*Conclusiones:* Estas simulaciones confirman que los grandes círculos no son las únicas geodésicas existentes en el elipsoide sino que existen muchos otros tipos de curvas geodésicas, algunas de las cuales pueden ser

curvas densas en la superficie y otras pueden ser curvas cerradas. Al mismo tiempo, esto muestra una relación entre la existencia de ciertos tipos de curvas geodésicas y la parametrización de la superficie.

**Palabras clave:** geodésicas, ecuaciones de Euler-Lagrange, simulaciones, SageMath, elipsoide

## Resumo

*Introdução:* O artigo é produto da pesquisa Conexões sobre Geometria Semi-Riemanniana e Coeficientes de Christoffel – Rumo ao estudo do cálculo computacional de geodésicas desenvolvido na Instituição Pascual Bravo University no ano de 2021.

*Problema:* Graças às soluções das equações de Euler-Lagrange, o cálculo explícito de geodésicas em certas variedades é possível. No entanto, existem vários casos em que é impossível calcular mais analiticamente e temos que recorrer a um cálculo numérico. Nesse sentido, várias características geométricas e dinâmicas das geodésicas surgem inesperadamente.

*Objetivo:* O objetivo da pesquisa é calcular geodésicas de uma variedade Riemanniana ou Semi-Riemanniana usando o software SageMath para ir mais facilmente além do que é fornecido pela intuição.

*Metodologia:* Primeiramente, são apresentados alguns exemplos simples de caracterizações de geodésicas em certas variedades, com base em soluções das equações de Euler-Lagrange. O elipsóide é então selecionado como nosso modelo de brinquedo para calcular as geodésicas numericamente e observar como elas mudam dependendo de estar sob um sistema de coordenadas esférico, triaxial ou Mercator.

*Resultados:* Com a flexibilidade de softwares como o SageMath, foi possível uma expressão explícita das equações diferenciais e a partir das soluções numéricas dessas equações suas correspondentes simulações com base nos parâmetros selecionados.

*Conclusões:* Estas simulações confirmam que os grandes círculos não são as únicas geodésicas existentes no elipsóide, mas que existem muitos outros tipos de curvas geodésicas, algumas das quais podem ser curvas densas na superfície e outras podem ser curvas fechadas. Ao mesmo tempo, isso mostra uma relação entre a existência de certos tipos de curvas geodésicas e a parametrização da superfície.

**Palavras-chave:** geodésica, equações de Euler-Lagrange, simulações, SageMath, elipsóide

## 1. INTRODUCTION

Bernard Riemann, at the University of Göttingen in 1854, carried out the famous conference: On the hypotheses on which geometry is based; with the objective of defining an appropriate model of the universe. In it he would go through, with the mathematical tools of the time, concepts and relations of extended manifolds, tangent vectors, tangent space, metrics, curvature tensors, sectional curvature; all directed to applications in space. These would guide the results obtained in the following century by Ricci, Myers, Cartan, or Nash, but would also support Einstein's General Theory of Relativity, and would anticipate the essence of Quantum Mechanics: the intuition that in order to handle small distances it is necessary to work with discrete quantities (a favorable condition for computational implementations or simulations).

From the above perspective, this paper considers the geodesics of a Riemannian or semi-Riemannian variety, through the visualization achieved with its codification in free software, so as to facilitate the underlying geometric understanding beyond what intuition allows.

For this purpose, geodesics are studied. The concept of a geodesic as a curve with zero acceleration is then introduced. The central idea, then, is to consider a geodesic as a curve that minimizes the arc length for any pair of sufficiently close points.

As a consequence, the tangent bundle  $TM$  of a differentiable manifold reduces the local study of the geodesics over  $M$  to the study of the trajectories of a vector field (the geodesic field) over  $TM$  and then uses this fact to adapt the equations of a geodesic to a particular case of the Euler-Lagrange partial differential equations that serve as a support for the calculations.

## 1.1 Background

This article addresses calculating geodesics of a Riemannian or semi-Riemannian manifold using free software to more easily go beyond what intuition provides. Some simple examples are presented, and others including the use of computational methods.

The software used in the experiments is SageMath. Previously called SAGE, it is a free open-source computer algebra system with features that include applications in many fields of mathematics like algebra, combinatorics, graph theory, numerical analysis, number theory, calculus and statistics. Moreover, it builds on top of many open-source packages like NumPy, SciPy, matplotlib, Sympy, Maxima and many more. It has a strong package designed for differential geometry and tensor calculus which is suitable for our calculations and simulations.

After the basic terminology is determined, it proceeds to the study of geodesics. The concept of a geodesic is introduced as a curve with zero acceleration. A geodesic minimizes the arc length for points close enough (in the sense of making it accurate); if a curve minimizes the arc length between any pair of its points, this is a geodesic.

What follows is a short introduction of geodesics from the point of view of differential geometry.

## 2. MATERIALS AND METHODS

Let  $M$  be a  $n$ -dimensional smooth manifold and  $g$  a metric tensor on  $M$ , i.e.  $g$  is a symmetric, non-degenerate  $(0, 2)$  tensor field on  $M$  of constant index  $\iota$ . The pair

$(M, g)$  is known as a semi-Riemannian manifold and in the particular case that  $\epsilon=0$ , a Riemannian manifold. For every  $X, Y$  smooth vector field on  $M$ , the affine connection (or Koszul connection) denoted by  $\nabla_X Y$  which is in fact a vector field, on a specific coordinate system  $(U, x^1, \dots, x^n)$  is expressed as:

$$\nabla_X Y = \sum_k \left( \sum_{i,j} x^i y^j \Gamma_{ij}^k + X(y^k) \right) \frac{\partial}{\partial x^k}$$

where  $\Gamma_{ij}^k$  are the famous Christoffel symbols that appear as coefficients of  $\nabla_{\partial_i} \partial_j$ . Let  $\alpha: [a, b] \rightarrow M$  be a smooth curve. If  $X := \alpha'(t)$  and  $Y$  is a vector field along  $\alpha$  then the covariant derivative along  $\alpha(t)$  results:

$$\frac{DY}{dt} := \nabla_{\alpha'(t)} Y = \sum_k \left( \frac{dv^k}{dt} + \sum_{i,j} \Gamma_{ij}^k(\alpha(t)) v^j(t) \frac{dx^i}{dt} \right) \frac{\partial}{\partial x^k} \Big|_{\alpha(t)}$$

The curve  $\alpha$  is said to be a geodesic if  $\frac{D}{dt} \left( \frac{d\alpha}{dt} \right) = \mathbf{0}$  or, as it is defined in the literature, field  $\frac{d\alpha}{dt}$  is parallel along  $\alpha$ . Moreover, such a curve must locally satisfy that:

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt} \frac{dx_j}{dt} = 0, k = 1, \dots, n,$$

This is demonstrated in the example 3.2.

### 3. CALCULATING GEODESICS

#### 3.1. Example 1. Euler-Lagrange equations and Geodesics

The Euler-Lagrange equations are the conditions that a certain type of variational problem reaches an extreme. These appear in Physics and Differential Geometry.

The Euler-Lagrange equation is an equation which is satisfied with a function,  $q$ , with real argument  $t$ , which is a stationary point from the function

$$s(q) = \int_a^b L(t, q(t), q'(t)) dt$$

where

1.  $q$  is the function to obtain

$$q: [a, b] \subseteq \mathbb{R} \rightarrow M$$

$$t \rightarrow x = q(t)$$

such as  $q$  is distinguishable,  $q(a) = x_a \wedge q(b) = x_b$ ;

2.  $q'$  is the derivative of  $q$

$$q': [a, b] \rightarrow T_{q(t)} M$$

$$t \rightarrow x = q(t)$$

$T_{q(t)} M$  is the space tangent to  $M$  in  $q(t)$ .

3.  $L$  is a real function with partial derivatives of type  $C^1$

$$L: [a, b] \times TM \rightarrow \mathbb{R}$$

$$(t, x, v) \rightarrow L(t, x, v)$$

Where  $TM$  is the tangent bundle of  $M$ , and it is

$$TM = \bigcup_{x \in M} x \times T_x M$$

then the Euler-Lagrange equation is given by

$$L_x(t, q(t), q'(t)) - \frac{d}{dt} \frac{\partial L}{\partial q_i(t)}(t, q(t), q'(t)) = 0$$

for which  $L_x$  and  $L_v$  are the partial derivatives of  $L$  corresponding to the second and third arguments, respectively.

If the dimension of  $M$  is bigger than 1, it is a system of differential equations where each component takes the form:

$$\frac{\partial L}{\partial q_i}(t, q(t), q'(t)) - \frac{d}{dt} \frac{\partial L}{\partial q_i(t)}(t, q(t), q'(t)) = 0$$

Geodesics are particular solutions of the Euler-Lagrange equations for a Lagrangian based on the quadratic form associated with the metric tensor involved in the calculation of longitude in a Semi-Riemannian manifold. Thus,  $(U, x^1, \dots, x^n)$  is a coordinate part around a point  $p$  of a Semi-Riemannian manifold  $(M, g)$ , where the metric tensor is

$$g = \sum_{i,j} g_{i,j} dx^i \otimes dx^j$$

As geodesics minimize the length between two points  $M$ , the following variational problem of least action can be posed for the length of a curve:

$$s = \int_a^b \sqrt{\sum_{i,j} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt$$

The minimized expression above, being the square root of a monotonic function, is equivalent to minimizing an action integral for the Lagrangian (the point above a function indicates the derivative concerning the variable  $t$ )

$$L(x^i, \dot{x}^i) = \frac{1}{2} \sum_{i,j} g_{ij} \dot{x}^i \dot{x}^j$$

Hence, the differential equations of geodesics, as an application of the Euler-Lagrange equation, are given

$$\frac{\partial L}{\partial x^k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^k} = 0$$

that is,

$$\sum_{i,j} \left( \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j \right) - \frac{d}{dt} \sum_{k,j} g_{kj} \dot{x}^j = 0$$

it implies

$$\sum_{i,j} \left( \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j - \frac{\partial g_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j \right) - \sum_{k,j} g_{kj} \ddot{x}^j = 0$$

using the symmetry of the metric tensor and in terms of the Christoffel symbols, it can be expressed as

$$\sum_j g_{kj} \ddot{x}^j + \sum_{i,j} \Gamma_{k,ij} \dot{x}^i \dot{x}^j = 0$$

or

$$\sum_j \ddot{x}^j + \sum_{i,j} \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0$$

Therefore

$$\Gamma_{k,ij} = \left( \frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} \right), \Gamma_{ij}^k = \frac{1}{2} \sum_m g^{km} \Gamma_{k,ij}, g^{ik} g_{kj} = \delta_j^i$$

### 3.2. The Semi-Euclidean Space $R_v^n$

For every  $p$  in  $R^n$ , if  $v_p = (v_1, \dots, v_n)$ ,  $w_p = (w_1, \dots, w_n)$  a metric tensor over  $R^n$  is given by

$$v_p, w_p \geq - \sum_{i=1}^v v_i w_i + \sum_{i=v+1}^n v_i w_i; (r = 0, 1, 2, \dots, n)$$

of index  $v$ . The result is a Euclidean semi-space  $R_v^n$  which is reduced to the Euclidean space  $R^n$  if  $v=0$ . For  $n>1$ ,  $R_1^n$  is the Minkowski space, where  $n=4$  is the simplest example of a relativistic space-time.

In all cases of semi-Euclidean spaces  $R_v^n$ , the Christoffel coefficients  $\Gamma_{ij}^k$  equal to 0, thus, the geodesics are all of the form  $(a_1, b_1)t + (a_0, b_0)$  for all  $t$ .

### 3.3. The Poincare Half-Plane

Consider

$$H^2 = \{(x, y) \in R^2; y > 0\}$$

with a Riemannian metric

$$g_{11} = g_{22} = \frac{1}{y^2}, g_{12} = g_{21} = 0$$



Notice then, that the first fundamental form can be written as

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

(H,g), thus defined, is known as the Poincaré Half-plane and provides a non-Euclidean geometry; So the following is obtained:

$$\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0, \Gamma_{11}^2 = \frac{1}{y}, \Gamma_{12}^1 = \Gamma_{22}^2 = \frac{-1}{y}$$

Also, the geodesic equations for the Poincaré half-plane must satisfy

$$\frac{d^2x}{ds^2} - \frac{2}{y} \frac{dx}{ds} \frac{dy}{ds} = 0$$

$$\frac{d^2y}{ds^2} + \frac{2}{y} \left(\frac{dx}{ds}\right)^2 - \frac{1}{y} \left(\frac{dy}{ds}\right)^2 = 0$$

Geodesics are extreme values of

$$\int \left[ \frac{1}{y^2} \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right\} \right]^{\frac{1}{2}} dt$$

but, taking  $t=s$ , the arc length, the following is obtained

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = y^2$$

of the geodesic equations and the need to consider only one Euler-Lagrange equation; for example the simpler of the two

$$\frac{d^2x}{ds^2} - \frac{2}{y} \frac{dx}{ds} \frac{dy}{ds} = \frac{1}{y^2}$$

This has an integrating factor as  $\frac{1}{y^2}$ , thus

$$\frac{d}{ds} \left( \frac{1}{y^2} \frac{dx}{ds} \right) = 0$$

where there is a constant  $C$ , such as

$$\frac{dx}{ds} = Cy^2$$

- If  $C=0$ , then  $x=k$  (constant), which are Euclidean lines perpendicular to the  $x$ -axis.
- If  $C$  is not 0, then

$$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}} = \frac{\sqrt{y^2 - c^2y^4}}{cy^2} = \frac{1}{cy} \sqrt{1 - c^2y^2}$$

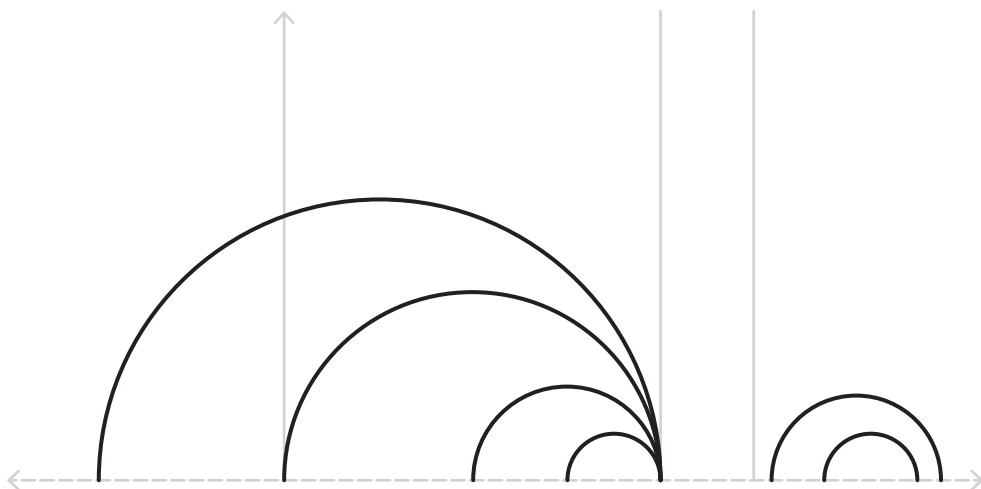
and solving this differential equation for separable variables, obtains

$$\sqrt{1 - c^2y^2} = c(x + x_1)$$

with  $x_1$  constant and thus

$$(x + x_1)^2 + y^2 = \frac{1}{C^2}$$

which are semi-circumferences in the Poincaré Half-plane, centered and perpendicular to the  $x$ -axis.



**Figure 1.** Geodesics on  $H^2$

## 4. RESULTS:

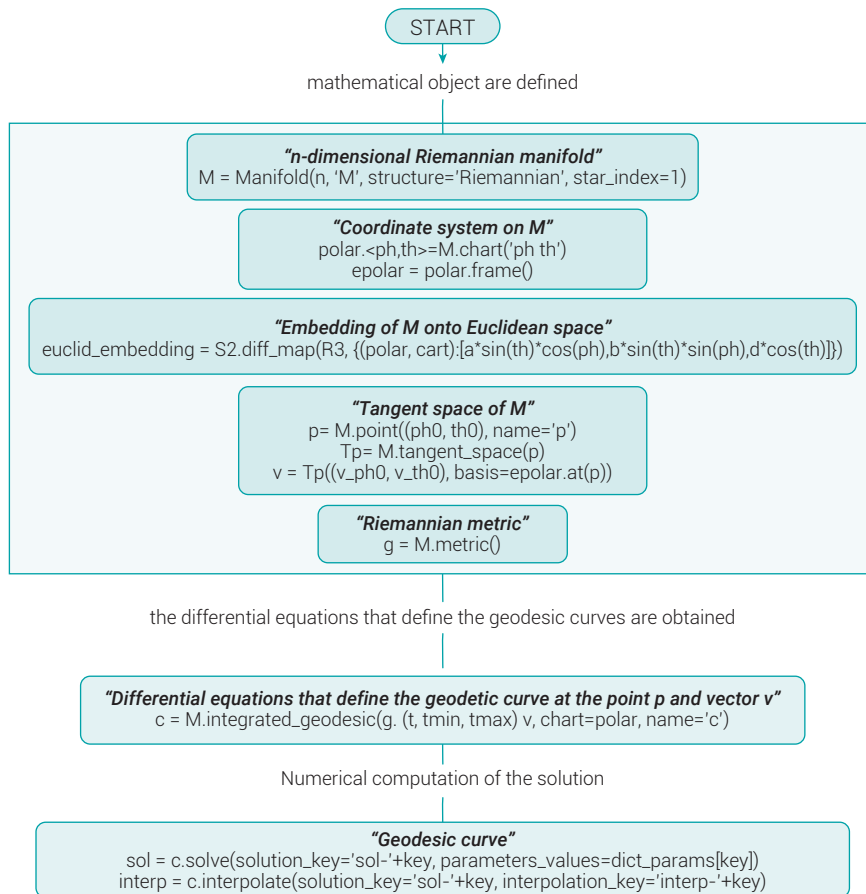
Some simulations associated with the ellipsoid will be presented in this section. In them, the interest is to show non-canonical geodesics from different coordinate systems and from a numerical calculation provided by the powerful SAGE code.

What follows is the architecture that describes the computational construction for each of the simulations associated with the ellipsoid.

An ellipsoid consists of quadratic surfaces defined as  $(x,y,z)$  in  $R^3$  such that it satisfies

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{d^2} = 1$$

where  $a, b, d > 0$  are the semi-axis lengths.



**Figure 2.** Flow Diagram with algorithm for ellipsoid.

Source: own work

## 4.1. Spherical Coordinates

The parametric equations are given by

$$X = a \sin(\theta) \cos(\varphi)$$

$$Y = b \sin(\theta) \sin(\varphi)$$

$$Z = d \cos(\theta)$$

the metric components are

$$g_{\varphi\varphi} = (a^2 \cos^2(\varphi) + b^2 \sin^2(\varphi)) \sin^2(\theta)$$

$$g_{\theta\theta} = ([a^2 \cos^2(\varphi) + b^2 \sin^2(\varphi)] \cos^2(\theta) + d^2 \sin^2(\theta))$$

$$g_{\varphi\theta} = g_{\theta\varphi} = (b^2 - a^2) \cos(\varphi) \sin(\varphi) \cos(\theta) \sin(\theta)$$

If  $a=b=2$  and  $d=1$ , the geodesic in  $(M, g)$  integrated over the interval  $(t_{min}, t_{max})$  with respect to the chart  $(M, (\varphi, \theta))$  is the solution for the following differential equation system

$$\frac{d^2\varphi}{dt^2} = \frac{-2\cos(\theta)}{\sin(\theta)} \left( \frac{d\varphi}{dt} \right) \left( \frac{d\theta}{dt} \right)$$

$$\frac{d^2\theta}{dt^2} = \left[ 4 \left( \frac{d\varphi}{dt} \right)^2 + 3 \left( \frac{d\theta}{dt} \right)^2 \right] \frac{\cos(\theta) \sin(\theta)}{3\cos^2(\theta) + 1}$$

## 4.2. Triaxial coordinate systems

The parametric equations are given by

$$X = a \cos(\theta) \frac{\sqrt{a^2 - b^2 \sin^2(\varphi) - d^2 \cos^2(\varphi)}}{\sqrt{a^2 - d^2}}$$

$$Y = b \sin(\theta) \cos(\varphi)$$

$$Z = d \sin(\varphi) \frac{\sqrt{a^2 \sin^2(\theta) + b^2 \cos^2(\theta) - d^2}}{\sqrt{a^2 - d^2}}$$

the metric components are

$$g_{\varphi\varphi} = \frac{b^2 \sin^2(\varphi) + d^2 \cos^2(\varphi)((a^2 - b^2)\sin^2(\theta) + (b^2 - d^2)\cos^2(\varphi))}{a^2 - b^2 \sin^2(\varphi) - d^2 \cos^2(\varphi)}$$

$$g_{\theta\theta} = \frac{a^2 \sin^2(\theta) + d^2 \cos^2(\theta)((a^2 - b^2)\sin^2(\theta) + (b^2 - d^2)\cos^2(\varphi))}{a^2 \sin^2(\theta) + b^2 \cos^2(\theta) - c^2}$$

$$g_{\varphi\theta} = g_{\theta\varphi} = 0$$

If  $a=b=2$  and  $d=1$ , the geodesic in  $(M,g)$  integrated over the interval  $(t_{min}, t_{max})$  with respect to the chart  $(M, (\varphi, \theta))$  is the solution for the following differential equation system

$$\frac{d^2\varphi}{dt^2} = \left[ 3 \left( \frac{d\varphi}{dt} \right)^2 + 4 \left( \frac{d\theta}{dt} \right)^2 \right] \frac{\cos(\varphi)\sin(\varphi)}{3\cos^2(\varphi) - 4}$$

$$\frac{d^2\theta}{dt^2} = \frac{\sin(\varphi)}{\cos(\varphi)} \left( \frac{d\varphi}{dt} \right) \left( \frac{d\theta}{dt} \right)$$

### 4.3. Mercator parametrization

The parametric equations are given by

$$X = a \operatorname{sech}(\theta) \cos(\varphi)$$

$$Y = b \operatorname{sech}(\theta) \sin(\varphi)$$

$$Z = d \tanh(\theta)$$

the metric components are

$$g_{\varphi\varphi} = (a^2 \sin^2(\varphi) + b^2 \cos^2(\varphi)) \operatorname{sech}^2(\theta)$$

$$g_{\theta\theta} = ([a^2 \cos^2(\varphi) \tanh^2(\theta) \operatorname{sech}^2(\theta) + b^2 \sin^2(\varphi) \tanh^2(\theta) \operatorname{sech}^2(\theta)] + d^2 \operatorname{sech}^4(\theta))$$

$$g_{\varphi\theta} = g_{\theta\varphi} = (a^2 - b^2) \cos(\varphi) \sin(\varphi) \tanh(\theta) \operatorname{sech}^2(\theta)$$

If  $a=b=2$  and  $d=1$ , the geodesic in  $(M,g)$  integrated over the interval  $(t_{min}, t_{max})$  with respect to the chart  $(M, (\varphi, \theta))$  is the solution for the following differential equation system

$$\frac{d^2\varphi}{dt^2} = \frac{-2\operatorname{sech}(\theta)}{\cosh(\theta)} \left(\frac{d\varphi}{dt}\right) \left(\frac{d\theta}{dt}\right)$$

$$\frac{d^2\theta}{dt^2} = -2 \left( 2 \left[ \left(\frac{d\varphi}{dt}\right)^2 - \left(\frac{d\theta}{dt}\right)^2 \right] \cosh(\theta) + 3 \left(\frac{d\theta}{dt}\right)^2 \right) \frac{\sinh(\theta)}{4\cosh(\theta)\sinh^2(\theta) + \cosh(\theta)}$$

## 5. DISCUSSION AND CONCLUSIONS

An ellipsoid is a quadratic surface that may be obtained from a sphere by deforming it by means of an affine transformation. Due to the combined effects of gravity and rotation, the shape of the Earth and of all planets is slightly flattened in the direction of its axis of rotation. For this reason, in cartography and geodesy the Earth is approximated by an ellipsoid, instead of a sphere

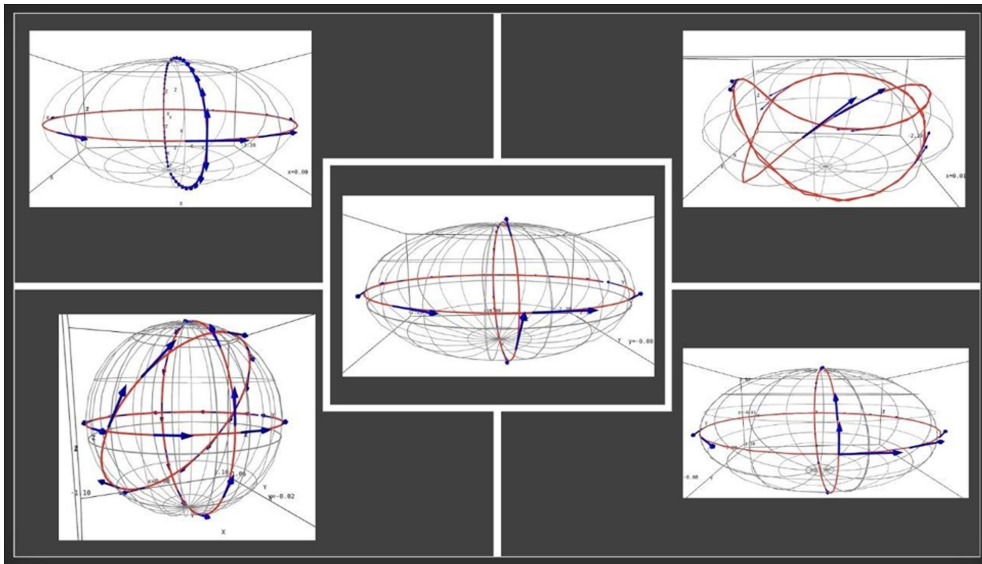
Geodesics have several uses in several fields: air navigation, Geographic Information System (3D-GIS), or cartography.

One of the most common applications is related to air navigation where the main objective is to reduce the distance travelled between places and address long-range air navigation. The development of models and computational tools using geodesics allow airlines, airports, and passengers to save on fuel consumption, reduce travel times, calculate and administrate routes, and reduce CO2 emissions.

Regarding 3D-GIS, geodesics are applied to improve the understanding and development of spatial and geographical areas and facilitate mapping processes. Moreover, the elaboration of models and computational tools have numerous applications on terrain; for instance, applying geodesics to engineering, city and country planning, transportation, logistics, among others. Geodesics are even crucial for exploring other planets, such as the current exploration of the Martian surface.

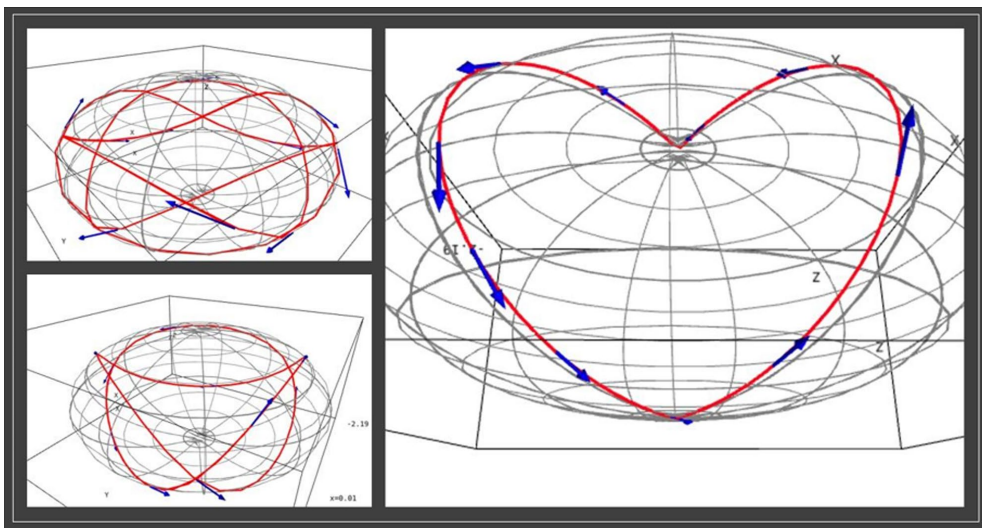
In cartography, for instance, the usage of geodesics has helped with the definition of ground and maritime borders among states. Internally, different authorities have used geodesics to calculate administrative territories and define areas for particular purposes.

In contrast to the geodesics regarding spheres, consisting only of closed curves with the shape of circumferences:



Source: own work

An ellipsoid, depending on the parametrization, can have meridians and parallels like those of the sphere. However, it gives rise to other types of closed or dense curves on the ellipsoid in other directions, as we can see in the previous graphs.



Source: own work

Geodesic curves might even be able to offer insight into love; the most incredible feeling, the most potent force, for which the most passionate stories, dramas, and experiences have been written. Constantly sought by a human being, love is complex to prove mathematically because there is no formula to define it.

However, now we can be more convincing because a long-awaited mathematical argument through functions, equations of curves, and surfaces confirm what we feel, and it can be shown with numbers. It is conclusive proof about love, managing to merge romanticism with mathematics to demonstrate how great and perfect the most fantastic human feeling turns out to be with geodesic curves.

Mathematics is, for Carl Friedrich Gauss, the most sensitive expression of love; for René Descartes, the science of order and measure. Equations, curves, poetry, and mathematics converge to reveal love in a geodesic curve. Therefore, all those who are passionate about science have one more reason to profess love for mathematics and its infinite applications.

The study aimed to find parametrizations for the ellipsoid of non-canonical geodesic curves. In this sense, and how it was explained in this paper, it is possible to demonstrate that tight or even dense curves can be displayed on the same ellipsoid.

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