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Unique Lifting to a Functor

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Abstract

We develop a functorial approach to quotient constructions, defining morphisms quotient relative to a functor and the dual concept of unique liftings relative to a functor. Various classes of epimorphism are given detailed analysis and their relationship to quotient morphisms characterized. The behavior of unique lifting morphisms with respect to products, equalizers, and general limits in a category are studied. Applications to generalized covering space theory, coreflective subcategories of topological spaces, topological groups and rings, and Galois theory are explored. Finally, we give conditions for the product of two quotient morphisms to be quotient in a braided monoidal closed category.

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Introduction

Many familiar areas of mathematics, such as the theory of groups, rings, modules, and topological spaces, allow the creation of new structures from preexisting ones via the notion of "quotienting out" or identifying underlying set points via an appropriate equivalence relation.[AHG09, p. 123] [Mun00, p. 136][DF04, pp. 15, 241, 348, 408]. In the case of topological spaces, for example, the quotient space construction involves an interplay between morphisms in the category of topological spaces (i.e. continuous maps) and an equivalence relation on the underlying point set X of the space (a surjective set function out of X.) The situation is analogous when forming a quotient group by passing to the collection of cosets. In both instances there is an equivalence relation defined on underlying set elements. To promote the collection of equivalence classes to the status of a topological space or group requires leaving the category of sets and "returning" to **Top** or **Grp**. In so doing, we also promote a surjective set function to a continuous map or group homomorphism. Thus there appears to be something inherently *functorial* about quotient constructions, as functors are the morphisms between categories. This thesis explores the consequences of this observation, found in [Bra14c, p. 1], and adopts a functorial perspective to regard various quotient constructions found in mathematics as examples of a single phenomenon, that of a morphism being *quotient relative to a functor.*

The question of how to characterize quotient constructions in terms of categorical structures does not have a single correct answer, and depending on context, one definition may be preferred over another [Bra14c, p. 2]. This may in part be due to the fact that the categorification of surjective set functions does not have a single correct formulation, as we will see when we consider the various types of epimorphism in Section 1.3. Although closely related to many known concepts, we give what appears to be a novel definition which has the advantage of precisely capturing the relationship between quotient constructions in a concrete category and their underlying set functions, while also allowing the generalization of quotients beyond the setting of concrete categories and faithful functors to that of arbitrary categories and functors. This is accomplished by recognizing how Grothendieck's *cocartesian morphisms* relate to qoutient constructions, a fruitful perspective not apparently explored in the previous literature. The first requirement of classical quotient constructions is that of surjectivity. Since equivalence relations are equipped with an associated surjection and surjections define an equivalence relation via the preimage [Bro06, p. 100] [AHG09, p. 124], it is natural to relate our definition to the categorical generalization of surjection, the epimorphisms [AHG09, p. 111], and the various classes thererof [AHG09, p. 121] [nLa21b]. Since the quotient topology is an example of a final topology, we also discuss the closely related notion of a strictly final lift and give several equivalent formulations of the idea of quotient morphism relative to a functor.

In chapter 2 we introduce the dual notion of *unique lifting relative to a functor* and show how this definition encodes the idea of a unique lifting along a map, examples of which include unique lifting of paths and homotopies in covering space theory [Mun00, pp. 342–343], Hensel's lemma in p-adic analysis [Lan94, p. 43] and the unique lifting of a representation of a Lie algebra to a representation of the associated Lie group when it is simply connected [FH91, p. 119]. We clarify the distinction between unique lifts and the presence merely of some lift, as in the definition of a fibration in algebraic topology, lifting of curves on a manifold M to a curve defined on the tangent bundle TM, and the lifting of module homomorphisms between projective modules.

Central to our definitions is the concept of cartesian and cocartesian morphism, used originally by Grothendieck [Gro71] to define fibered and cofibered categories in the context of descent theory. We use cocartesian morphisms to categorify topological quotient maps and cartesian morphisms to categorify unique liftings.

We explore the categorical properties of F-quotient and F-lifting morphisms, characterize the quotient morphisms relative to various forgetful functors that arise naturally in the study of topological groups and rings, relate unique liftings relative to the classical fundamental group functor to ideas from generalized covering space theory, and discuss how unique liftings can be used to give a partial characterization of Galois correspondence in Galois theory. We also show that in the context of Cartesian closed braided monoidal categories, under reasonable hypotheses, the tensor product of two quotient maps is also quotient, a fact not true in general [Mun00, p. 141].

Chapter 1

Quotient Morphisms, Epimorphisms, and Cartesian Morphisms

1.1 Epimorphisms, Monomorphisms and their Properties

1.1.1 Epimorphisms

Definition 1.1.1. A morphism $f : X \to Y$ in a category $\mathcal{C}$ is an *epimorphism* if $g_1 = g_2$ for any morphisms $g_1, g_2 : Y \to Z$ such that $g_1 \circ f = g_2 \circ f$.

Proposition 1.1.1. *Let $f : X \to Y$ and $g : Y \to Z$ be epimorphisms in the category $\mathcal{C}$. Then $g \circ f : X \to Z$ is also an epimorphism.*

Proof. Suppose $(g \circ f) \circ h = (g \circ f) \circ k$ for $h, k : Z \to W$. Then $g \circ (f \circ h) = g \circ (f \circ k)$. Since g is an epimorphism, $f \circ h = f \circ k$, and since f is an epimorphism, $h = k$, so that $g \circ f$ is an epimorphism. □

Proposition 1.1.2. *If $f : A \to B$ and $g : B \to C$ with $g \circ f$ an epimorphism, then f is an epimorphism.*

Proof. Suppose $f : A \to B$ and $g : B \to C$ with $g \circ f$ an epimorphism. Let $a, b : X \to A$ be such that $f \circ a = f \circ b$. Then $g \circ f \circ a = g \circ f \circ b$ so that $a = b$ since $g \circ f$ is an epimorphism, and therefore f is an epimorphism. □

Proposition 1.1.3. *A morphism $f : X \to Y$ in **Set** is an epimorphism if and only if f is surjective.*

Proof. Suppose f is a surjection. Let $g_1, g_2 : Y \to Z$ be such that $g_1 \circ f = g_2 \circ f$. Then for each $f(x) \in Y$ we have $g_1(f(x)) = g_2(f(x))$. Since f is surjective $f(x)$ describes an arbitrary element of Y, hence $g_1 = g_2$. Now suppose f is an epimorphism. Let Z denote be the set with two elements, 0 and 1. Suppose f is not a surjection. Then there is some $y_0 \in Y$ without premimage under f. Let $g_1 : Y \to Z$ be constant at 1, and let $g_2 : Y \to Z$ be defined by

$$g_2(y) = \begin{cases} 0, & y = y_0 \\ 1, & \text{otherwise.} \end{cases}$$

Then, since $f(x) \neq y_0$ for all $x \in X$, we have $g_1(f(x)) = g_2(f(x))$, but $g_1 \neq g_2$ since they disagree at the argument y_0. Therefore f is a surjection. □

Proposition 1.1.4. *[21c] In the category **Top** of topological spaces and continuous functions, the epimorphisms are the surjective continuous functions.*

Proof. By the same argument used above in the category **Set**, but restricting attention only to continuous set functions, the surjective continuous functions are epimorphisms. To see that epimorphisms in **Top** are surjections, give the two element set Z the indiscrete topology (in which only the entire set and empty set are open.) Then the set functions $g_1, g_2 : Y \to Z$ are both continuous, so again by the same argument used in the category **Set**, epimorphisms in **Top** must be surjective. □

Proposition 1.1.5. *[Lin70] [Mag] In **Grp**, the category of groups and group homomorphisms, the epimorphisms are precisely the surjective homomorphisms.*

Proof. Let $f : H \to K$ be an epimorphism in **Grp**. Let $X = K/f(H)$ be the set of right cosets of the image of f in K, which need not be a group since we do not know that $f(H)$ is normal in K. Let ∞ denote some set element not belonging to X. Let $Y = X \cup \{\infty\}$. Let S denote the permutation group associated to Y. K acts on X on the right by multiplication. That is, for $k_1 \in K$ and $f(H)k \in X$, we have $f(H)k.k_1 = f(H)kk_1$. This defines an action since $f(H)k.e = f(H)(ke) = f(H)k$ for the identity $e \in K$ and if $k_1, k_2 \in K$ we have

$$(f(H)k.k_1).k_2 = (f(H)kk_1).k_2 = f(H)kk_1k_2 = (f(H)k).(k_1k_2).$$

This action induces an embedding $g : K \hookrightarrow S$ given by $g(k) = \sigma_k : Y \to Y$ where $\sigma_k(f(H)k') = f(H)kk'$ and $\sigma_k(\infty) = \infty$. We check that σ_k defines a bijection $Y \to Y$. Suppose $\sigma_k(f(H)k_1) = \sigma_k(f(H)k_2)$. Then $f(H)k_1k = f(H)k_2k$ and it is clear that $f(H)k_1 = f(H)k_2$. Suppose $y \in Y$. If $y = \infty$ then $\sigma_k(\infty) = \infty$ for all $k \in K$. If $y = f(H)k$ then $\sigma_k(f(H)) = y$. Let $\sigma \in S$ satisfy $\sigma(f(H)) = \infty$ and $\sigma(\infty) = f(H)$ and fix all other elements of Y. Let $\phi_\sigma : S \to S$ denote conjugation by σ, $\phi_\sigma(\tau) = \sigma^{-1} \circ \tau \circ \sigma$. Let $h : K \to S = \phi_\sigma \circ g$. Then h defines a group homomorphism. For $x \in H$, $\sigma_{f(x)}$ fixes both $f(H)$ and ∞. The support of a permutation p is the subset of elements not left fixed by p. The support of $g(f(x)) = \sigma_{f(x)}$ is disjoint with the support of σ, as $\sigma_{f(x)}$ can only permute non-trivial cosets $f(H)k$ and σ fixes all such elements of Y. Then σ and $\sigma_{f(x)}$ commute, so that

$$h(f(x)) = \phi_\sigma \circ g(f(x)) = \sigma^{-1} \circ \sigma_{f(x)} \circ \sigma = \sigma_{f(x)} = g(f(x)).$$

and we have $h \circ f = g \circ f$. Since f is an epimorphism, we have $h = g$. Thus $h(k) = g(k)$ for all $k \in K$, and since $h(k) = g(k)$ for all $k \in K$,

$$h(k) = g(k) \implies \phi_k \circ g(k) = \sigma^{-1} \circ \sigma_k \circ \sigma = \sigma_k$$

and $\sigma_k \circ \sigma = \sigma \circ \sigma_k$ for all $k \in K$. As $\sigma_k = g(k)$ commutes with σ, we have

$$\begin{aligned} \sigma \circ g(k)(f(H)) &= g(k) \circ \sigma(f(H)) \\ \implies \sigma(f(H)k) &= \sigma_k(\infty) = \infty \\ \implies f(H)k &= f(H) \\ \implies g(k)(f(H)) &= f(H). \end{aligned}$$

Thus $f(H)$ is fixed by $g(k)$. This means $k \in f(H)$, and $K \subset f(H)$. Since $f(H) \subset K$, $K = f(H)$, making f a surjection.

Now suppose $f : G \to H$ is a surjective group homomorphism. Let $g, h : H \to K$ be group homomorphisms with $f \circ g = f \circ h$. Let $y \in H$. Since f is surjective there exists $x \in G$ with $f(x) = y$. Then $g(y) = g(f(x)) = h(f(x)) = h(y)$ and $g = h$. □

Definition 1.1.2. [Mun00][98] A topological space X is said to be *Hausdorff* if for each pair of distinct points x, y in X there exists disjoint open sets U, V such that $x \in U$ and $y \in V$.

Definition 1.1.3. [Mun00][187] A *Directed Set* J is a set equipped with a partial order $\preccurlyeq$ such that for any pair of elements $\alpha, \beta \in J$ there exists some $\gamma \in J$ such that $\alpha \preccurlyeq \gamma$ and $\beta \preccurlyeq \gamma$.

Definition 1.1.4. Let X be a topological space. A *net* in X is a function $f : J \to X$ from a directed set J. We write x_α for $f(\alpha)$ and $(x_\alpha)_{\alpha \in J}$ for f. A net $(x_\alpha)_{\alpha \in J}$ is said to converge to $x \in X$ (written $x_\alpha \to x$) if for every neighborhood U with $x \in U$ there exists $\alpha \in J$ such that $\alpha \preccurlyeq \beta$ implies $x_\beta \in U$.

Lemma 1.1.6. *Nets converge to at most one point in a Hausdorff space.*

Proof. Let X be a topological space and let J be a directed set. Let $(x_\alpha)_{\alpha \in J}$ be a net in X. Suppose $x_\alpha \to x$ and $x_\alpha \to y$. Since X is Hausdorff we can find disjoint open sets U, V with $x \in U$ and $y \in V$. Since $x_\alpha \to x$ we find α_1 such that $x_\beta \in U$ for $\alpha_1 \preccurlyeq \beta$. Since $x_\alpha \to y$ we can also find α_2 such that $x_\beta \in V$ for $\alpha_2 \preccurlyeq \beta$. Then since J is directed we can find $\gamma \in J$ such that $\alpha_1, \alpha_2 \preccurlyeq \gamma$. Thus when $\gamma \preccurlyeq \delta$ we have $x_\delta \in U \cap V = \emptyset$. □

Definition 1.1.5. [Mun00][191] Let X be a topological space. A subset A of X is *dense* in X if the closure $\overline{A} = X$.

Proposition 1.1.7. *In* **Haus***, the category of Hausdorff topological spaces and continuous maps between them, the maps with dense image are epimorphisms.*

Proof. Let $e : X \to Y$ be a morphism in **Haus** with $\overline{e(X)} = Y$. Let $g_1, g_2 : Y \to Z$ be morphisms in **Haus** such that $g_1 \circ e = g_2 \circ e$. Choose $y \in Y$. Since $e(X)$ is a dense subset of Y there exists a convergent net $(y_j)_{j \in J}$ such that $y_j \to y$ with $y_j \in e(X)$ for all $j \in J$. Then by the continuity of g_1 and g_2 we have nets $(g_1(y_j))_{j \in J}$ converging to $g_1(y)$ and $(g_2(y_j))_{j \in J}$ converging to $g_2(y)$. Since $y_j \in e(X)$ for all $j \in J$ we can find x_j such that $e(x_j) = y_j$ for all $j \in J$. Then we have $g_1(y_j) = g_1 \circ e(x_j) = g_2 \circ e(x_j) = g_2(y_j)$ for all $j \in J$. Therefore $(g_1(y_j))_{j \in J} = (g_2(y_j))_{j \in J}$ are equal convergent nets in the Hausdorff space Z, and must have the same limit, making $g_1(y) = g_2(y)$. Since y was arbitrary, $g_1 = g_2$ and e is an epimorphism. □

Proposition 1.1.8. *[HS73] [Sco] Epimorphisms in* **Haus** *have dense image.*

Proof. Let $f : A \to B$ be an epimorphism in **Haus**. Let C be the disjoint topological union of B with itself, with the identification $(b, 0) \sim (b, 1)$ for $b \in \overline{f(A)}$, so $C = (B \coprod B)/\sim$. Let $h, k : B \to C$ be defined by $h(b) = [(b, 0)]$ and $k(b) = [(b, 1)]$. We show that C is a Hausdorff space. Let $B_0 = B \times \{0\}$, $B_1 = B \times \{1\}$, $K = \overline{f(A)}$ and $K_i = K \times \{i\}$ for $i \in \{0, 1\}$. Let $q = B_0 \bigcup B_1 \to C$ be the quotient map. Let $h, k : B \to C$ be defined by $h(b) = (b, 0)$ and $k(b) = (b, 1)$. Suppose $x, y \in B$ with $x \neq y$. Since B is Hausdorff we have disjoint open sets $x \in U$ and $y \in V$ in B. Then $U_i = \{(u, i) \mid u \in U, i \in \{0, 1\}\}$ and $V_j = \{(v, j) \mid v \in V j \in \{0, 1\}\}$ are open in $B \coprod B$ since their preimages under h, k are U, V or $\emptyset$, all open in B. They are disjoint in $B \coprod B$ for all choices of i and j. Since q is an open map, $q(U_i)$ and $q(V_j)$ are open in C They are also disjoint, since if $q(u,i) = q(v,j)$ then $u = v$ and U, V are disjoint. If $x = y$ and $q((x,0)) = q((x,1))$ the $x \in B\backslash K$ so that $q(B_0\backslash K_0)$ and $q(B_1\backslash K_1)$ are disjoint open in C.

Since $h \circ f = k \circ f$ and f is an epimorphism, we have $h = k$, so that $[(b,1)] = [(b,0)] \implies (b,1) \sim (b,0) \implies b \in \overline{f(A)}$. Thus f has dense image in B. □

Example 1.1.9. Although surjections in concrete categories are always epimorphisms, the converse does not hold. Let **Ring** denote the category of (not necessarily unital) associative rings and ring homomorphisms. Consider the inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$. Suppose $f, g : \mathbb{Q} \to R$ are ring homomorphisms such that $f \circ i = g \circ i$. Then $f(n) = g(n)$ for each $n \in \mathbb{Z}$. Let $\frac{a}{b} \in \mathbb{Q}$. We have

$$\begin{aligned}
f\Big(\frac{a}{b}\Big) &= f\Big(\frac{1}{b} \cdot a\Big) \\
&= f\Big(\frac{1}{b}\Big) * f(a) \\
&= f\Big(\frac{1}{b}\Big) * g(a) \\
&= f\Big(\frac{1}{b}\Big) * g(b) * g\Big(\frac{a}{b}\Big) \\
&= f(1) * g\Big(\frac{a}{b}\Big) \\
&= g(1) * g\Big(\frac{a}{b}\Big) \\
&= g\Big(\frac{a}{b}\Big).
\end{aligned}$$

Therefore, $f = g$ and i is an epimorphism. But i is clearly not surjective, since not all rationals are integral.

In light of this example, the task of identifying which morphisms in a concrete category have underlying surjective set functions is not completely straightforward. This has led in part to the development of the various subclasses of epimorphisms to be discussed in the Section 1.3.

We may characterize the epimorphisms in C in terms of the contravariant hom functor as follows.

Proposition 1.1.10. *[nLa21b] A morphism $f : X \to Y$ in a category $\mathcal{C}$ is an epimorphism if and only if for each object $Z \in \mathcal{C}$, the image of f under the hom functor $Hom(-, Z) : \mathcal{C} \to \textbf{\textit{Set}}$ is an injective set function.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{C}$ and let $Z \in \mathcal{C}$. Denote $\mathrm{Hom}(f, Z) : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ by $f^{\#}$. Suppose f is an epimorphism. Let $\phi, \psi : Y \to Z$ be such that $f^{\#}(\phi) = f^{\#}(\psi)$. Then $\phi \circ f = \psi \circ f \implies \phi = \psi$, since f is an epimorphism. Thus $f^{\#}$ is an injection. If we suppose $f^{\#}$ to be injective for all objects Z in $\mathcal{C}$, then $f^{\#}(\phi) = f^{\#}(\psi)$ implies $\phi = \psi$ and we have $\phi \circ f = \psi \circ f$ implies $\phi = \psi$, making f is an epimorphism. □

This leads to the following:

Proposition 1.1.11. *[21c] Let $\mathcal{C}$ be a category. A morphism $f : X \to Y$ is an epimorphism in $\mathcal{C}$ if and only if the induced natural transformation*

$$\Psi : Hom(Y, -) \Rightarrow Hom(X, -)$$

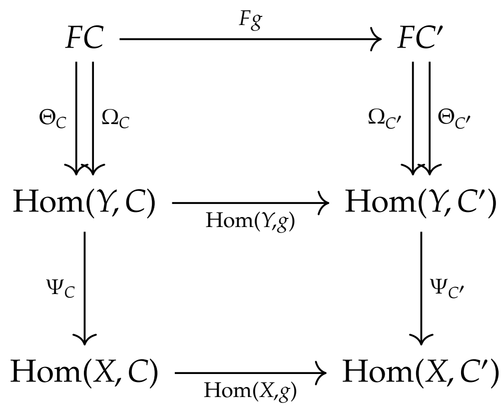
is a monomorphism in $\textbf{\textit{Set}}^{\mathcal{C}}$, the category of functors from $\mathcal{C}$ to $\textbf{\textit{Set}}$.

Proof. Suppose $f : X \to Y$ is an epimorphism in $\mathcal{C}$. Then for each Z in $\mathcal{C}$, $\Psi_Z : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$, $\Psi_Z(g) = g \circ f$ is an injection in **Set**, thus a monomorphism. These morphisms form the components of a natural transformation $\Psi : \mathrm{Hom}(Y, -) \to \mathrm{Hom}(X, -)$. To see this, let $g : C \to C'$ in $\mathcal{C}$. Consider the following diagram in **Set**:

$$\begin{array}{ccc}
\mathrm{Hom}(Y, C) & \xrightarrow[\mathrm{Hom}(Y,g)]{g\circ -} & \mathrm{Hom}(Y, C') \\
{\scriptstyle -\circ f}\big\downarrow{\scriptstyle \Psi_C} & & {\scriptstyle \Psi_{C'}}\big\downarrow{\scriptstyle -\circ f} \\
\mathrm{Hom}(X, C) & \xrightarrow[g\circ -]{\mathrm{Hom}(X,g)} & \mathrm{Hom}(X, C')
\end{array}$$

This square commutes by the associativity of morphism composition in $\mathcal{C}$. If $\Theta, \Omega : F \Rightarrow$

$\mathrm{Hom}(Y, -)$, then we have the following diagram at $g : C \to C'$:



If $\Psi \circ \Theta = \Psi \circ \Omega$, then we must have for each object in $C \in \mathcal{C}$ that $\Psi_C \circ \Theta_C = \Psi_C \circ \Omega_C$. But Ψ_C is a monomorphism since f is an epimorphism, so that $\Theta_C = \Omega_C$, making $\Theta = \Omega$ and Ψ a monomorphism in $\mathbf{Set}^{\mathcal{C}}$. If we assume Φ to be a monomorphism, we can reverse the argument to show that f must be an epimorphism. □

Proposition 1.1.12. *Coequalizers (see Appendix A.2) are epimorphisms.*

Proof. Let $f : X \to Y$ be the coequalizer of $g, h : Z \to X$. Let $j, k : Y \to W$ be such that $j \circ f = k \circ f$. Then $k \circ f \circ g = k \circ (f \circ g) = k \circ (f \circ h) = (k \circ f) \circ h$, so that $k \circ f = j \circ f$ coequalizes g and h. Then by the universal property of the coequalizer there exists a unique morphism $\phi : Y \to W$ such that $\phi \circ f = j \circ f = k \circ f$. Therefore, $j = \phi = k$ and f is an epimorphism. □

1.1.2 Monomorphisms

There is a dual notion to epimorphism which similarly characterizes and generalizes the injective set functions.

Definition 1.1.6. Let $\mathcal{C}$ be a category. A *monomorphism* is a morphism $f : X \to Y$ in $\mathcal{C}$ such that for all morphisms $g_1, g_2 : Z \to X$, if $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.

Proposition 1.1.13. *A composition of monomorphisms is a monomorphism.*

Proof. Let $f : A \to B$ and $g : B \to C$ be monomorphisms. Let $h, j : X \to A$ be such that $(g \circ f) \circ h = (g \circ f) \circ j$. Then $g \circ (f \circ h) = g \circ (f \circ j) \implies f \circ h = f \circ j$ since g is monic. Then, since f is monic we have $h = j$. □

Proposition 1.1.14. *Suppose $j \circ c = i$ with i a monomorphism. Then c is a monomorphism.*

Proof. If $c \circ x = c \circ y$ then $j \circ c \circ x = j \circ c \circ y$ and $i \circ x = i \circ y$ so that $x = y$. □

Proposition 1.1.15. *A morphism $f : X \to Y$ in **Set** is a monomorphism if and only if it is injective.*

Proof. Suppose $f : X \to Y$ is an injection and $f \circ g = f \circ h$ for $g, h : W \to X$. Then $f(g(w)) = f(h(w))$ for all $w \in W$ and since f is injective, $g(w) = h(w)$ and $g = h$, so that f is a monomorphism. Now suppose f is a monic. Suppose $f(x) = f(x')$. Define constant functions $g : W \to X$ and $h : W \to X$ by $g(w) = x, h(w) = x'$. Then $f(g(w) = f(x) = f(x') = f(h(w))$ for all $w \in W$. Thus $f \circ g = f \circ h$ and $g = h$, so $x = g(w) = h(w) = x'$ and f is an injection. □

The following shows that not all monomorphisms in categories where objects have underlying sets are injective.

Example 1.1.16. A group G is *divisible* if for every positive integer n and every $g \in G$ there exists $y \in G$ such that $ny = g$. Let **Div** be the category of divisible abelian groups and group homomorphisms. Note that $\mathbb{Q}$ is divisible since for any $\frac{a}{b} \in \mathbb{Q}$ and $n \in \mathbb{N}$ we have $n(\frac{a}{bn}) = \frac{a}{b}$. Also, $\frac{a}{b} + \mathbb{Z} = a(\frac{1}{b}) + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Let G be a divisible group and $f, g : G \to \mathbb{Q}$ such that $\pi \circ f = \pi \circ g$ for the projection $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. Then $\pi \circ f(x) = \pi \circ g(x) \implies \pi(f(x) - g(x)) = 0$ and $f(x) - g(x) = n$ for some integer n. We now show that for any $h : G \to \mathbb{Q}$ satisfying $\pi \circ h = 0$ we must have $h = 0$. To this end suppose $x \in G$ and $h(x) \geq 0$. If not choose $-x$. Let $n = h(x) + 1$. By the divisibiliy of G there is some $y \in G$ with $ny = x$ so that $h(x) = nh(y)$. Then $0 \leq \frac{h(x)}{h(x)+1} = h(y) < 1$. Since $h(y) \in \mathbb{Z}, h(y) = 0$ and therefore $h(x) = 0 = h(-x)$. Thus $h = 0$. Thus if $\pi \circ f = \pi \circ g, \pi \circ (f - g) = 0$ and $f - g = 0$ so that $f = 0 = g$ and π is a monomorphism.

Definition 1.1.7. An *isomorphism* in a category C is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.

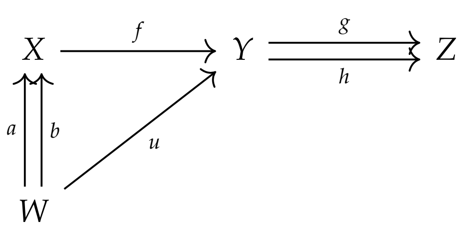
Proposition 1.1.17. *If $f : X \to Y$ is an isomorphism, it is both an epimorphism and a monomorphism.*

Proof. Suppose $f : X \to Y$ is an isomorphism. Then there exists $i : Y \to X$ such that $i \circ f = id_X$ and $f \circ i = id_Y$. Suppose $g, h : W \to X$ are such that $f \circ g = f \circ h$. Then $i \circ f \circ g = i \circ f \circ h$ and $g = h$. Thus f is a monomorphism. A symmetric argument involving $g', h' : Y \to Z$ shows that f is an epimorphism. □

Remark 1.1.18. A morphism may be a monomorphism and an epimorphism but not an isomorphism. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in **Ring** is an epimorphism as we have shown. It is also a monomorphism [21c]. It is not however an isomorphism, since it is not a surjection.

Example 1.1.19. Let **2** denote the category with two objects and one non-identity morphism $f : 0 \to 1$. Trivially, f is both a monomorphism and epimorphism, since, for example, if $g, h : 0 \to 0$ then $g = h = id_0$. But there is no morphism from 1 to 0, so f cannot have an inverse.

Example 1.1.20. Let $f : X \to Y$ be the equalizer of $g, h : Y \to Z$ (see appendix A.2). Then f is a monomorphism. To see this, suppose $a, b : W \to X$ such that $f \circ a = f \circ b$. Then we have



where $u = f \circ a = f \circ b$. By the universal property of the equalizer (X, f) there exists a unique ϕ such that $f \circ \phi = u$, but a, b both satisfy this condition, so that $a = \phi = b$.

Lemma 1.1.21. *The pullback of a monomorphism is a monomorphism.*

Proof. Let the following diagram be a pullback square with $m : Z \to W$ a monomorphism.

$$\begin{array}{ccc} X \times_Y Y & \xrightarrow{p_1} & Y \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle v} \\ Z & \xrightarrow[m]{} & W \end{array}$$

Suppose $\phi, \psi : Z \to X \times_W Y$ satisfy $p_1 \circ \phi = p_1 \circ \psi$. then $v \circ p_1 \circ \phi = v \circ p_2 \circ \psi$ so that $m \circ p_2 \circ \phi = m \circ p_2 \circ \psi$. Since m is a monomorphism we have $p_2 \circ \phi = p_2 \circ \psi$. Let $g = p_1 \circ \phi = p_2 \circ \psi$ and $h = p_2 \circ \phi = p_2 \circ \psi$. Then $v \circ g = m \circ h$ and there exists a unique $\xi : Z \to X \times_W Y$ such that $p_1 \circ \xi = g$ and $p_2 \circ \xi = h$, so that $\phi = \psi$. □

Proposition 1.1.22. *Epimorphisms are dual to monomorphisms (See Section 2.2 for a discussion of duality.)*

Proof. Let $f : X \to Y$ be a monomorphism in C. Suppose we have $g^{op}, h^{op} : X \to W$ in C^{op} with $g^{op} \circ f^{op} = h^{op} \circ f^{op}$. Then $f \circ g = f \circ h$ in C and $g = h$ so that $g^{op} = h^{op}$, and f^{op} is an epimorphism in C^{op}. Since $(C^{op})^{op} = C$, if f^{op} is an epimorphism in C^{op} then f is a monomorphism in C. □

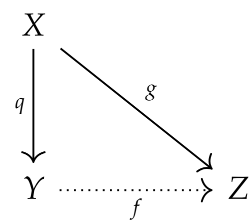
Proposition 1.1.23. *Let $f : X \to Y$ be a morphism in the category C. Then f is a monomorphism if and only if $f_* : Hom(Z, X) \to Hom(Z, Y)$ is an injection for all $Z \in C$.*

Proof. Suppose $f : X \to Y$ is a monomorphism. Suppose $f_*(\phi) = f_*(\psi)$ for $\phi, \psi : Z \to X$. Then $f \circ \phi = f \circ \psi \implies \phi = \psi$. If $f_*(\phi) = f_*(\psi)$ for all $Z \in C$ then $f \circ \phi = f \circ \psi$ for all $Z \in C$ and f is a monomorphism. □

1.2 Categorical Generalizations of "Quotient Morphism"

Definition 1.2.1. A *topological quotient map* is a continuous surjection $q : X \to Y$ between topological spaces such that for any continuous map $g : X \to Z$ constant on the fibers of

q, there is a unique continuous map $f : Y \to Z$ such that $f \circ q = g$.



Definition 1.2.2. Let X be a topological space and let $\sim$ be an equivalence relation on the underlying set of X. Then the *quotient space* $X/\sim$ is the collection of equivalence classes associated to $\sim$ equipped with the *quotient topology*. A set U in $X/\sim$ is open in the quotient topology if and only if its preimage under the canonical surjection $g : X \to X/\sim$ given by $g(x) = [x]$ is open.

Example 1.2.1. Let I denote the closed unit interval $[0, 1]$ equipped with the subspace topology inherited from $\mathbb{R}$. Let $\mathbb{S}^1$ denote the unit circle viewed as a subset of $\mathbb{R}^2$. Let $q : I \to \mathbb{S}^1$ be defined by $q(t) = (\cos\,(2\pi t)), \sin\,(2\pi t)$. Note that q is continuous, since products and compositions of continuous functions are continuous. It is also surjective, since for any $0 \leq \theta \leq 2\pi, 0 \leq \frac{\theta}{s\pi} \leq 1$. Suppose $g : I \to Z$ is constant on the fibers of q. Then we can define $f : \mathbb{S}^1 \to Z$ by $f(\cos\,\theta, \sin\,\theta) = g(\frac{\theta}{2\pi})$. Then $f \circ q = g$, since $g(\frac{2\pi t}{2\pi}) = g(t)$. As f is a composite of continuous functions, it is continuous. If h satisfies $h \circ q = g$, then $h \circ q = f \circ q$ and since q is an epimorphism, $h = f$.

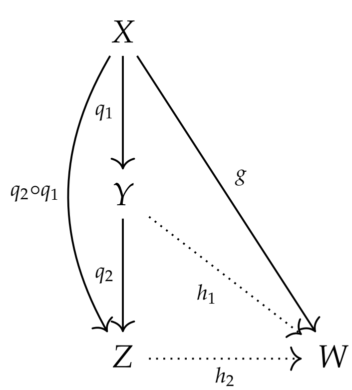
Proposition 1.2.2. *Let $f : X \to Y$ be an open surjection. Then f has the universal property of the quotient map.*

Proof. Let $g : X \to Z$ be constant on the fibers of f. Let $y \in Y$. By the axiom of choice we can select $x \in f^{-1}(y)$ (which is nonempty since q is a surjection) and set $h(y) = g(x)$. Since $g(x)$ is constant on the fibers of f, h is well-defined. Then $h \circ f(x) = g(x')$ where $x' \in f^{-1}(x)$, so that $g(x') = g(x)$. Now let V be open in Z. Then $g^{-1}(V)$ is open in X and since f is open, $f(g^{-1}(V))$ is open in Y. Then $f(g^{-1}(V)) = f((h \circ f)^{-1}(V)) = f(f^{-1}(h^{-1}(V)) = h^{-1}(V)$, since f

is a surjection. Therefore, h is continuous. It is unique in satisfying $h \circ f = g$ since f is a surjection and therefore an epimorphism. □

Proposition 1.2.3. *Let $q_1 : X \to Y$ and $q_2 : Y \to Z$ be quotient maps. Then $q_2 \circ q_1 : X \to Z$ is a quotient map.*

Proof. Let $g : X \to W$ be constant on the fibers of $q_2 \circ q_1$. Then g is constant on the fibers of q_1, since if $q_1(x_1) = q_1(x_2) = y$ we have $q_2(y) = q_2 \circ q_1(x_1) = q_2 \circ q_1(x_2)$ and g is constant on the fibers of $q_2 \circ q_1$. Since q_1 is a quotient map, there exists a unique continuous $h_1 : Y \to W$ such that $h_1 \circ q_1 = g$. We show that h_1 is constant on the fibers of q_2. Suppose $q_2(y_1) = q_2(y_2)$. By the surjectivity of q_1 there exist x_1 and x_2 in X such that $q_1(x_1) = y_1$ and $q_1(x_2) = y_2$. Then $q_2 \circ q_1(x_1) = q_2(y_1) = q_2(y_2) = q_2 \circ q_1(x_2)$. Since g is constant on the fibers of $q_2 \circ q_1$ we have $g(x_1) = g(x_2)$. Then $h_1 \circ q_1(x_1) = h_1 \circ q_1(x_2)$ and $h_1(y_1) = h_1(y_2)$, as was to be shown. Therefore, we have a unique continuous $h_2 : Z \to W$ such that $h_2 \circ q_2 = h_1$. Then $h_2 \circ q_2 \circ q_1 = h_1 \circ q_1 = g$. If $f : Z \to W$ satisfies $f \circ q_2 \circ q_1 = g$, we have $f \circ q_2 \circ q_1 = g = h_1 \circ q_1$ and $f \circ q_2 = h_1$ as q_1 is an epimorphism. Thus $f = h_2$ since h_2 uniquely satisfies $h_2 \circ q_2 = h_1$. Therefore, $q_2 \circ q_1$ satisfies the universal property of the quotient map.



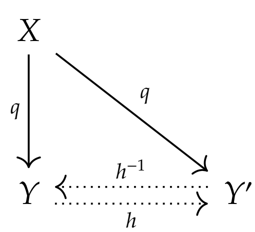
□

Theorem 1.2.4. *A morphism $q : X \to Y$ is a quotient map if and only if q is a surjection and whenever U is open (closed) in Y we have $q^{-1}(U)$ open (closed) in X.*

Proof. Let $q : X \to Y$ be a surjective map and let Y have the quotient topology relative to

q. Let $g : X \to Z$ be constant on the fibers of q. By the same argument used in Proposition 1.2.2, we can define f such that $f \circ q = g$. To show that f is continuous, let V be open in Z. Then $q^{-1}(f^{-1}(V)) = (f \circ q)^{-1}(V) = g^{-1}(V)$, which is open since g is continuous by assumption. Then since Y carries the quotient topology associated to q, $f^{-1}(V)$ is open, making f continuous. Suppose some continuous f' satisfies $f' \circ q = g$. Then $f' \circ q = f \circ q$. Since q is a surjection it is an epimorphism, hence $f' = f$.

Let $q : X \to Y$ be a surjective map between topological spaces and suppose q has the universal property of the quotient map. Let τ denote the topology on Y. We show that τ is in fact the quotient topology $Q = \{U \mid q^{-1}(U) \text{ open in } X\}$ induced by q. Let Y' have the same underlying set as Y but carry the quotient topology Q inherited from q. Then we may define the map $q : X \to Y'$. By the argument above, $q : X \to Y'$ has the universal property of the quotient map. Thus there exists a unique $h : Y \to Y'$ such that $h \circ q = q$. By a symmetric argument we get a continuous h^{-1} satisfying $h^{-1} \circ q = q$. Then h and h^{-1} are isomorphic by the uniqueness condition associated to the universal property of the quotient map. Since h and h^{-1} are continuous they are in fact homeomorphisms, thus are open and surjective, and by Proposition 1.2.2 it follows that h^{-1} is a quotient map. Since $q = h^{-1} \circ q$ and by Proposition 1.2.3 quotient maps are stable under composition, q is a quotient map.



□

Corollary 1.2.5. *[Etn20] If $f : X \to Y$ is a closed surjection, then f is a quotient map.*

Proof. Let f be a closed surjection. Suppose $f^{-1}(U)$ is open in X. Then $X - f^{-1}(U)$ is closed. Therefore $f(X - f^{-1}(U)) = f(f^{-1}(Y) - f^{-1}(U)) = f(f^{-1}(Y - U)) = Y - U$ is closed in Y, so U is open in Y. Since f is continuous, we can conclude f is a quotient map. □

Example 1.2.6. Not all surjective continuous functions are quotient [Ull18]. Consider $f : [0,1] \to S^1$ defined by $f(x) = (\cos 2\pi x, \sin 2\pi x)$ and the restriction $g = f|_{[0,1)} : [0,1) \to \mathbb{S}^1$. Then g is continuous and surjective. Let $U = \{(x,y) \mid y > 0\} \cup \{(0,0)\} \subset \mathbb{S}^1$. U is not open in the subspace topology inherited from $\mathbb{R}^2$, but $g^{-1}(U) = \{(0 \leq t < \frac{1}{2})\}$ is open in $[0,1)$. By theorem 1.2.4, quotient maps $q : X \to Y$ satisfy the property that U is closed in Y if and only if $q^{-1}(U)$ is closed in X, so g cannot be quotient.

Definition 1.2.3. [Bro06, p. 101] Let $X_\alpha, \alpha \in A$ be a family of topological spaces and let Y be a set. Let $f_\alpha : X_\alpha \to Y$ be a family of functions. A topology $\mathcal{F}$ on Y is said to be *final* with respect to f_α if for any topological space Z, a function $g : (Y, \mathcal{F}) \to Z$, is continuous if and only if $g \circ f_\alpha$ is continuous for all $\alpha \in A$.

Remark 1.2.7. Given a surjective morphism $q : X \to Y$ in **Top**, the final topology on Y relative to the single morphism q is the quotient topology associated to the quotient morphism q. That is, putting the final topology on Y with respect to a surjective map q makes q a quotient map.

Theorem 1.2.8. *Let X, Y be topological spaces. Let $q : X \to Y$ be a surjective function. Then q is a quotient map if and only if Y has the final topology with respect to q.*

Proof. Let $q : X \to Y$ be a quotient map. Let $g : Y \to Z$ be a function. Suppose g is continuous. Then since q is continuous, so is $g \circ q$. Now if $g \circ q$ is continuous, by the universal property of the quotient map there exists a unique continuous function $\phi : Y \to Z$ such that $\phi \circ q = g \circ q$, so that we must have $\phi = g$. Therefore, Y satisfies the characteristic property of the final topology with respect to q.

Now suppose the topology on Y is final with respect to the continuous function $q : X \to Y$. Let $g : X \to Z$ be continuous and constant on the fibers of q. By the same argument used to prove theorem 1.2.4 we can construct a function $f : Y \to Z$ by setting $f(y) = g(x)$, where $x \in q^{-1}(y)$ such that $f \circ q = g$. Since g is continuous, $f \circ q$ is also continuous and by the characteristic property of the final topology, f is continuous. The uniqueness of f follows from the fact that q is an epimorphism. □

Example 1.2.9. [Mun00][143] The product of continuous functions is continuous but the property of being a quotient map is not closed under taking products as the following example shows. Let $X = \mathbb{R}$ and X^* be the quotient space formed by identifying all elements in the subset $\mathbb{Z}_+$ of non-negative integers to a point b. Let $p : X \to X^*$ be the associated quotient map. Let $\mathbb{Q} \subset \mathbb{R}$ be the rational numbers. Let $i : \mathbb{Q} \to \mathbb{Q}$ be the identity map. Then $p \times i : X \to \mathbb{Q} \to X^* \times \mathbb{Q}$ is not a quotient map.

1.3 Types of Epimorphism and Their Comparison

The notion of epimorphism captures the property enjoyed by the surjections in **Set** of being cancellable on the right. But we have seen that the condition of surjectivity and right-cancellability are not equivalent in all categories (take **Ring**, for example.) It can be shown that the surjectivity of a morphism f in **Set** is equivalent to f being right-cancellable, having a right inverse, forming a quotient set, and having a canonical factorization into a surjection followed by an injection. These properties are not equivalent in a general categorical setting, so in order to characterize the morphisms in a concrete category that enjoy these properties, or to generalize such properties to an arbitrary categorical context, the various notions of split epimorphism, regular epimorphism, strong epimorphism, and extremal epimorphism have been defined [Fou17]. We provide a detailed comparison of these notions summarized in the following diagram of implications. Each double arrow indicates an inclusion of one epimorphism type into another. These arrows are decorated with information showing where the associated inclusion is proven in the present text and a brief description of the conditions required in the ambient category for the inclusion to hold. For example, we write

$$\begin{array}{c}\textbf{split epimorphism}\\ \text{kernel pairs}\ \Big\Downarrow\ \text{prop 1.3.33}\\ \textbf{effective epimorphism}\end{array}$$

to indicate that in a category with all kernel pairs, a split epimorphism is an effective epimorphism, which is proven in Proposition 1.3.33.

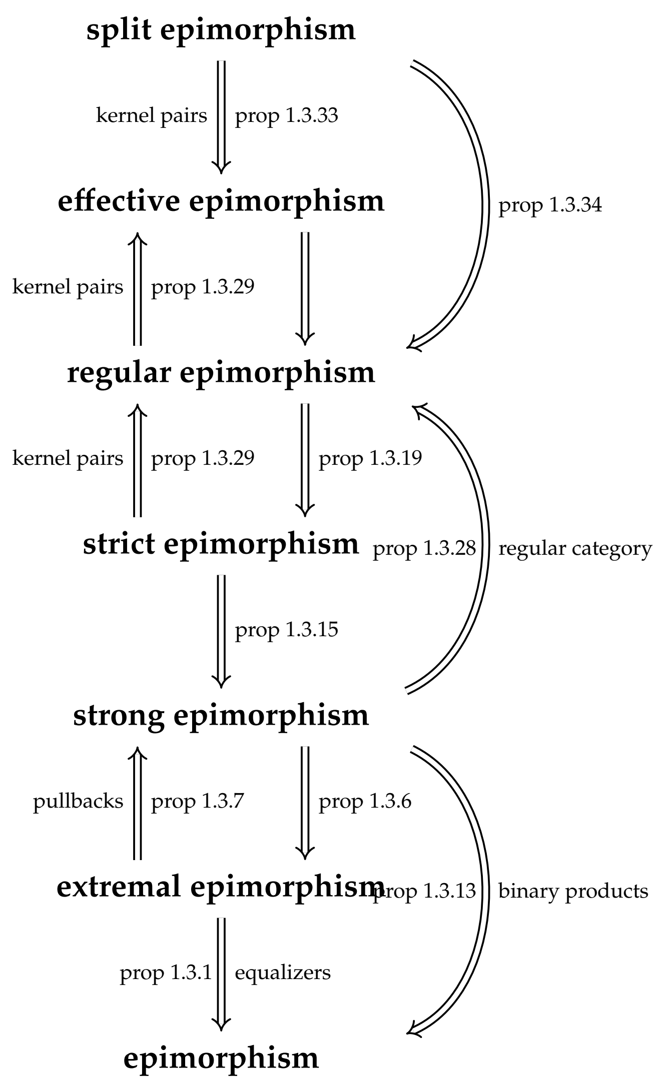


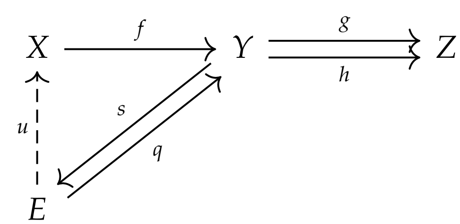
Figure 1.1: Comparison of Epimorphisms

1.3.1 Extremal Epimorphisms

Definition 1.3.1. A morphism $f : X \to Y$ in the category C is an *extremal epimorphism* if whenever $f = m \circ g$ with m a monomorphism, m must be an isomorphism. If C does not have all equalizers, we require the added condition that f is an epimorphism.

Proposition 1.3.1. *Let C be a category with all equalizers. Then an extremal epimorphism in C is an epimorphism.*

Proof. Let C have all equalizers. Let $f : X \to Y$ be an extremal epimorphism in C. Let $g, h : Y \to Z$ be such that $g \circ f = h \circ f$. Let (E, q) be the equalizer of g and h. Thus there exists a unique u such that $f = q \circ u$. By example 1.1.20 we know q is monic. Since f is an extremal epimorphism, q must be an isomorphism. Then there exists $s : Y \to E$ such that $q \circ s = id_Y$.



Therefore we have $g \circ q = h \circ q \implies g \circ q \circ s = h \circ q \circ s \implies g = h$. □

Example 1.3.2. [Bor94a][144] In the category **Rng** of unital commutative rings (which has all equalizers), an epimorphism $f : A \to B$ factors through its image $f(A)$ as $f = m \circ r$. For f to be extremal, we require $m : f(A) \to B$ to be an isomorphism. Thus we require m to be a surjection, making f a surjection. Thus the extremal epimorphisms in **Rng** are precisely the surjective ring homomorphisms.

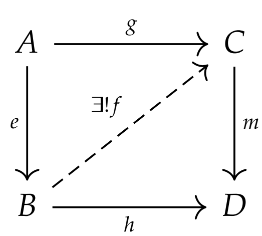
Although in **Top** the epimorphisms are the surjective continuous functions, in **Haus** the epimorphisms need only have dense image. We now show that extremal epimorphisms in **Haus** are surjective.

Proposition 1.3.3. *Let $e : X \to Y$ be an extremal epimorphism in **Haus**. Then e is a surjection.*

Proof. Let $e : X \to Y$ be an extremal epimorphism in **Haus**. Then since **Haus** has all equalizers, e is an epimorphism. therefore $e(X)$ is dense in Y. Suppose e is not surjective, $e(X) \neq Y$. Then $e = m \circ g$ where $g : X \to e(X)$ is e with codomain restricted to the $e(X)$ and $m : e(X) \to Y$ is inclusion. Since every subset of a Hausdorff space is Hausdorff, $e(X)$ is Hausdorff, and g is clearly continuous. Since m is continuous and injective it is a monomorphism. As e is an extremal epimorphism m must be an isomorphism, but the isomorphism in Haus are the homeomorphisms, which are surjective, and by assumption we have $e(X) \neq Y$, a contradiction. Therefore, e is a surjection. □

1.3.2 Strong Epimorphisms

Definition 1.3.2. A morphism $e : A \to B$ is said to be a *strong epimorphism* if for any monomorphism $m : C \to D$ and morphisms g, h such that the following square commutes, there exists a unique morphism $f : B \to C$ such that $f \circ e = g$ and $m \circ f = h$.



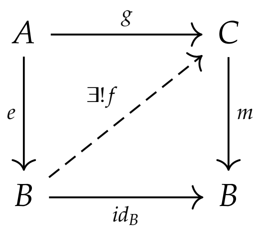
If such an f exists it is unique, since e is an epimorphism and m is a monomorphism.

Remark 1.3.4. Strong epimorphisms are a generalization of the property belonging to surjections in **Set** of having a canonical factorization into an epimorphism followed by a monomorphism [Fou17].

Example 1.3.5. in the category Graph of undirected graphs and graph homomorphisms, the strong epimorphisms are the graph homomorphisms that are surjective on both edges and vertices.

Proposition 1.3.6. *A strong epimorphism is an extremal epimorphism.*

Proof. Let $e : A \to B$ be a strong epimorphism. Let $e = m \circ g$ where m is a monomorphism. Then there exists a unique f such that both triangles in the following diagram commute



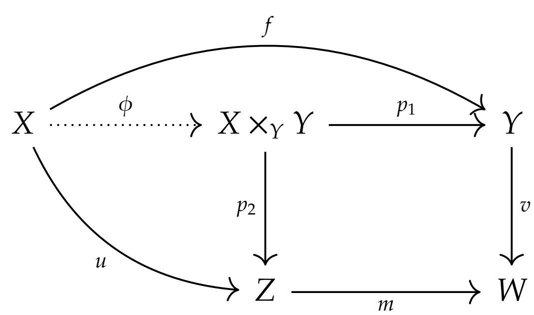
Thus $m \circ f = id_B$. Then we have

$$m \circ (f \circ m) = m \circ id_B = m = m \circ id_C$$

and since m is a monomorphism, $f \circ m = id_C$. Therefore, m is an isomorphism and e is extremal. □

Proposition 1.3.7. *Let $f : X \to Y$ be an extremal epimorphism in a category $\mathbf{C}$ with all pullbacks. Then f is a strong epimorphism.*

Proof. Let v, u, and m be such that $v \circ f = m \circ u$ with $m : Z \to W$ a monomorphism. Consider the following commuting diagram associated with the pullback $(Z \times_W Y, p_1, p_2)$ of m along v.



Since $v \circ v = m \circ u$ there exists a unique ϕ such that the above diagram commutes. Thus we have $f = p_1 \circ \phi$. By lemma 1.1.21, p_1 is a monomorphism, and as f is an extremal epimorphism, p_1 is an isomorphism. Thus there exists $s : Z \to Z \times_W Y$ such that $s \circ p_1 = id_Y$ and $p_1 \circ s = id_{Z \times_W Y}$. Set $t = p_2 \circ s$. For f to be a strong epimorphism, we require $t \circ f = u$

and $m \circ t = v$. We have

$$t \circ f = t \circ p_1 \circ \phi = p_2 \circ s \circ p_1 \circ \phi = p_2 \circ \phi = u$$

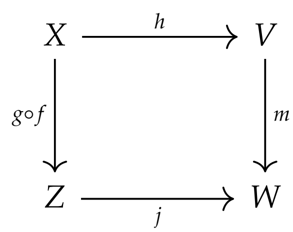
and

$$m \circ t = m \circ p_2 \circ s = v \circ p_1 \circ s = v.$$

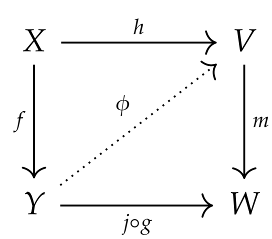
If some $t' : Y \to Z$ satisfies $t' \circ f = u$ and $m \circ t' = v$ then $m \circ t' = v = m \circ t$ and since m is a monomorphism, $t = t'$. □

Proposition 1.3.8. *A composition of strong epimorphisms is a strong epimorphism.*

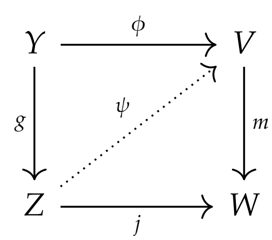
Proof. Let $f : X \to Y$ and $g : Y \to Z$ be strong epimorphisms in the category C. Assume we have h, j, m in the commuting diagram below, with m a monomorphism.



Then by the associativity of morphism composition in C we also have the commuting square



where $\phi : Y \to W$ is uniquely given since f is a strong epimorphism. We then have the diagram



The square commutes by the previous diagram. We have unique ϕ making the triangles commute since g is a strong epimorphism and m is a monomorphism. Then note that

$$\phi \circ g \circ f = \psi \circ f = h$$

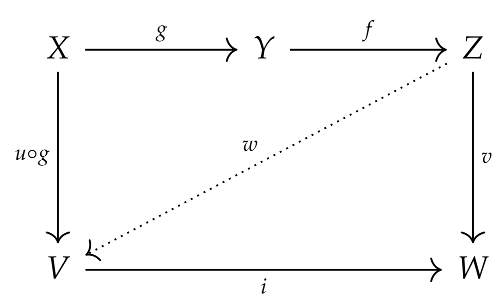
and

$$m \circ \psi = j$$

Therefore $g \circ f$ is a strong epimorphism. □

Proposition 1.3.9. *If $f \circ g$ is a strong epimorphism then f is a strong epimorphism.*

Proof. Let $f : Y \to Z$ and $g : X \to Y$ be such that $f \circ g$ is a strong epimorphism. Suppose we have $v : Z \to W$ and $i : V \to W$ and $u : X \to V$ such that $v \circ f = i \circ u$ with i a monomorphism. Then $v \circ f \circ g = i \circ u \circ g$ Since $f \circ g$ is a strong epimorphism there exists a unique w such that $i \circ w = v$. Since $v \circ f = i \circ u \implies i \circ w \circ f = i \circ u$ and since i is a monomorphism, $w \circ f = u$. Therefore, f is a strong epimorphism.

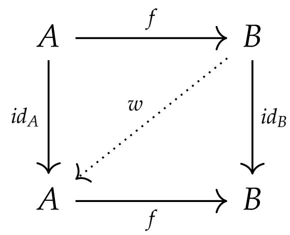


□

Lemma 1.3.10. *If $f : A \to B$ is a strong epimorphism and a monomorphism then f is an isomorphism.*

Proof. Let f be a strong epimorphism and a monomorphism. Then there exists $w : B \to A$

such that $w \circ f = id_A$ and $f \circ w = id_B$, making f an isomorphism.



□

Lemma 1.3.11. *Let $f = i \circ p$ where i is an isomorphism and p is a strong epimorphism. Then f is a strong epimorphism.*

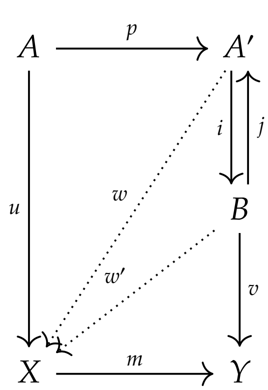
Proof. Let $f = i \circ p$ with i an isomorphism and p a strong epimorphism. Let u, v, m be such that $m \circ u = v \circ f$ with m a monomorphism. As p is a strong epimorphism, there exists a unique w such that $w \circ p = u$ and $m \circ w = v \circ i$. Set $w' = j \circ w$ where j is the inverse of the isomorphism i. Then

$$w' \circ f = w \circ j \circ f = w \circ j \circ i \circ p = w \circ p = u.$$

and

$$m \circ w' = m \circ w \circ j = v \circ i \circ j = v.$$

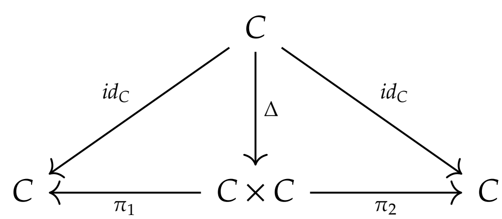
Thus f is a strong epimorphism.



□

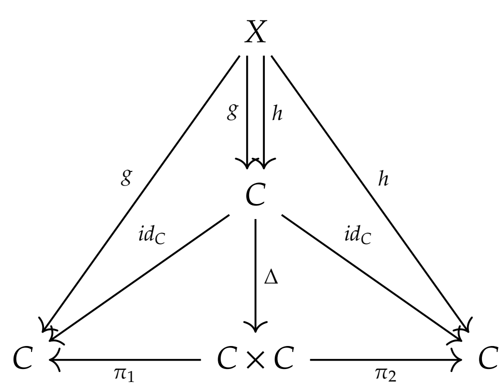
In a category with binary products we do not require the hypothesis that strong epimorphisms are epimorphisms, as we now show.

Definition 1.3.3. Let $\mathcal{C}$ be a category with binary products and let C be an object of $\mathcal{C}$. Then we call the morphism (id_C, id_C) induced by the universal property of the product $C \times C$ the *diagonal morphism* $\Delta : C \to C \times C$. Thus the following diagram commutes



Lemma 1.3.12. *A diagonal morphism is a monomorphism.*

Proof. Let $g, h : X \to C$ be such that $\Delta \circ g = \Delta \circ h$. Consider the following diagram



Thus we have

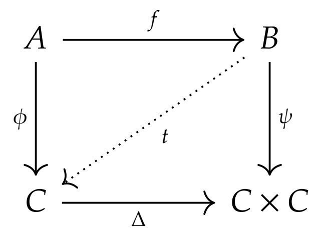
$$g = id_C \circ g = \pi_1 \circ \Delta \circ g = \pi_1 \circ \Delta \circ h = id_C \circ h = h.$$

□

Proposition 1.3.13. *Let $\mathcal{C}$ be a category with binary products. Then a strong epimorphism in $\mathcal{C}$ is an epimorphism.*

Proof. Let $\Delta : C \to C \times C$ be the diagonal morphism. Let $f : A \to B$ be a strong epimorphism. Let $u, v : B \to C$ satisfy $u \circ f = \phi = v \circ f$. Let ψ denote the morphism $\psi : B \to C \times C$

induced by the universal property of the product $(C \times C, \pi_1, \pi_2)$ with respect to the family of functions $u, v : B \to C$. Then we have the following



where, since Δ is a monomorphism, there exists a unique $t : B \to C$ such that $t \circ f = \phi$ and $\Delta \circ t = \psi$. Then

$$u = \pi_1 \circ \psi = \pi_1 \circ \Delta \circ t = t = \pi_2 \circ \Delta \circ t = \pi_2 \circ \psi = v.$$

□

1.3.3 Strict Epimorphisms

Definition 1.3.4. [DS][4] Let $f : X \to Y$ be a morphism in the category $\mathcal{A}$. A morphism $g : X \to Y$ is said to be *compatible* with f if whenever $f \circ h_1 = f \circ h_2$ we have $g \circ h_1 = g \circ h_2$.

Definition 1.3.5. [DS][4] A morphism $f : X \to Y$ is a *strict epimorphism* if given any $g : X \to Z$ compatible with f there exists a unique $k : Y \to Z$ such that $k \circ f = g$.

Proposition 1.3.14. *A strict epimorphism is an epimorphism.*

Proof. Let $f : A \to B$ be a strict epimorphism. Let $g_1, g_2 : B \to C$ be such that $g_1 \circ f = g_2 \circ f$. Then $g_1 \circ f$ also jointly coequalizes all the morphisms f does, so by the universal property of f as a coequalizer, there exists a unique morphism $u : A \to C$ such that $u \circ f = g_1 \circ f = g_2 \circ f \implies g_1 = u = g_2$ and f is an epimorphism. □

Proposition 1.3.15. *A strict epimorphism is a strong epimorphism.*

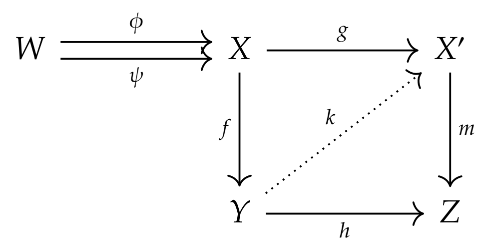
Proof. Let $f : X \to Y$ be a strict epimorphism. Suppose we have g, h and monic m such that $m \circ g = h \circ f$. Let $\phi, \psi : W \to X$ be coequalized by f. Then

$$m \circ g \circ \phi = h \circ f \circ \phi = h \circ f \circ \psi = m \circ g \circ \phi$$

and since m is a monomorphism, $g \circ \phi = g \circ \psi$. Since f is a strict epimorphism there exists a unique $k : Y \to X'$ such that $k \circ g = f$. Then

$$h \circ f = m \circ g = m \circ k \circ f$$

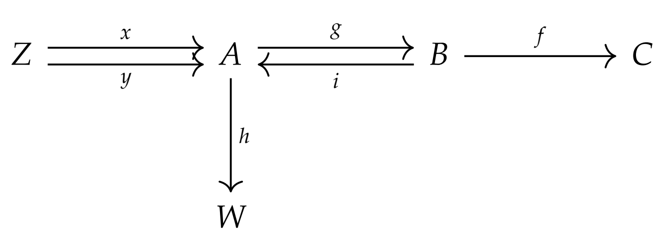
and $m \circ k = h$ since f is an epimorphism. Therefore, f is a strong epimorphism.



□

Proposition 1.3.16. *[Kel69][127] If $g : A \to B$ is a retraction and $f : B \to C$ is a strict epimorphism then $f \circ g$ is a strict epimorphism.*

Proof. Let $f : B \to C$ be a strict epimorphism and let $i : B \to A$ be such that $g \circ i = id_B$. We verify $f \circ g$ has the universal property of the coequalizer by first assuming $h : A \to W$ coequalizes $x, y : Z \to A$ whenever $f \circ g$ does.



we have

$$f \circ g \circ id_A = f \circ id_B \circ g = f \circ g \circ i \circ g \implies h \circ id_A = h \circ i \circ g$$

and for any suitable $u, v : D \to B$,

$$f \circ u = f \circ v \implies f \circ g \circ i \circ u = f \circ g \circ i \circ v \implies h \circ i \circ u = h \circ i \circ v$$

As f is a strict epimorphism, $h \circ i$ factors uniquely through f, so there exists $k : C \to W$ with $h \circ i = k \circ f$. Then

$$h = h \circ i \circ g = k \circ f \circ g$$

and h factors uniquely through $f \circ g$. □

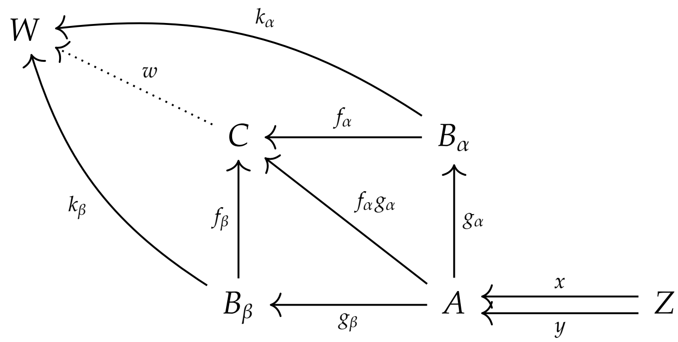
Proposition 1.3.17. *[Kel69][127] If $f \circ g$ is a strict epimorphism and g is an epimorphism, f is a strict epimorphism.*

Proof. If we have $h \circ x = h \circ y$ whenever $f \circ x = f \circ y$ then $h \circ g \circ u = h \circ g \circ v$ whenever $f \circ g \circ u = f \circ g \circ v$. Since $f \circ g$ is a strict epimorphism by assumption, there exists unique k such that $k \circ f \circ g = h \circ g$. Since g is an epimorphism, $k \circ f = g$. Thus f is a universal coequalizer of any pair it coequalizes, making f a strict epimorphism. □

Proposition 1.3.18. *[Kel75][127] Let $g_\alpha : A \to B_\alpha$ be strict epimorphisms indexed by $\alpha \in I$. Let $f_\alpha : B_\alpha \to C$ be thier fibred coproduct (pushout.) Let $f_\alpha g_\alpha$ denote the common value of $f_\alpha \circ g_\alpha$ $\forall \alpha \in I$. Then $f_\alpha g_\alpha$ is a strict epimorphism and f_α is a strict epimorphism for all $\alpha \in I$.*

Proof. Let $h : A \to W$ be such that $h \circ x = h \circ y$ if $f_\alpha g_\alpha \circ x = f_\alpha g_\alpha \circ y$. If for some $\gamma \in I$ we have $g_\gamma \circ x = g_\gamma \circ y$, then $f_\gamma \circ g_\gamma \circ x = f_\gamma \circ g_\gamma \circ y$ and therefore $f_\alpha g_\alpha \circ x = f_\alpha g_\alpha \circ y$, so that $h \circ x = h \circ y$. Since g_α is a strict epimorphism it is the coequalizer of x, y and there exists a unique k_α such that $h = k_\alpha \circ g_\alpha$. Since $k_\alpha \circ g_\alpha = h$, it is independent of α. Then by the universal property of the fibred coproduct, there exists a unique $w : C \to W$ such that

$k_\alpha = w \circ f_\alpha$. Therefore $h = w \circ f_\alpha \circ g_\alpha = w \circ f_\alpha g_\alpha$, as desired.



□

1.3.4 Regular Epimorphisms and Regular Categories

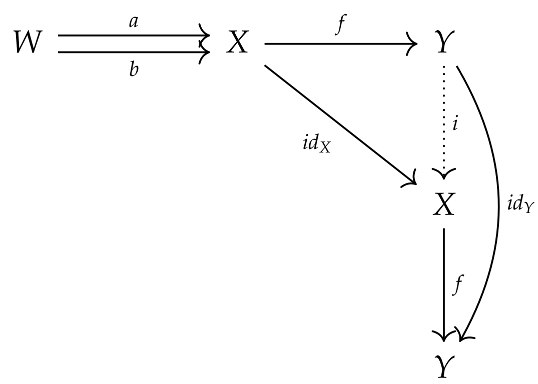
Definition 1.3.6. A morphism $f : X \to Y$ is a *regular epimorphism* if f is the coequalizer of some parallel pair of morphisms $a, b : W \to X$.

Proposition 1.3.19. *A regular epimorphism is a strict epimorphism.*

Proof. Suppose $f : X \to Y$ is regular. Then f is the coequalizer of some parallel pair $a, b : C \to X$. Suppose $g : X \to Z$ is compatible with f. Then $g \circ a = g \circ b$. Therefore, there exists a unique $h : Y \to Z$ such that $h \circ f = g$, so f is a strict epimorphism. □

Proposition 1.3.20. *If $f : X \to Y$ is a monomorphism and a regular epimorphism, it is an isomorphism.*

Proof. Let $f : X \to Y$ be a monomorphism and a regular epimorphism. Then there exists $a, b : W \to X$ coequalized by f. Since f is a monomorphism, $a = b$, so that id_X also satisfies $id_X \circ a = id_X \circ b$. Then by the universal property of the coequalizer f there exists a unique $i : Y \to X$ such that $i \circ f = id_X$. Since f is a regular epimorphism, it is an epimorphism, so that $f \circ (i \circ f) = f \implies (f \circ i) \circ f = id_Y \circ f$ and $f \circ i = id_Y$.



□

Example 1.3.21. [Hal16] In **Set** every epimorphism is a regular epimorphism, since every epimorphism in **Set** is effective.

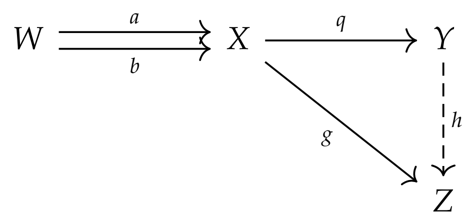
Proposition 1.3.22. *The regular epimorphisms in* ***Ring*** *are precisely the surjective ring homomorphisms.*

Proof. Let $f : A \to B$ be a regular epimorphism in **Ring**. Consider the forgetful functor to the category of abelian groups $F : \mathbf{Ring} \to \mathbf{Ab}$. Since F is left adjoint to the free functor, it preserves colimits [Bor94a][106], hence coequalizers, so that $Ff : FA \to FB$ must also be a regular epimorphism. Since the regular epimorphisms in **Grp** are precisely the surjective group homomorphisms, this also holds for the subcategory $\mathbf{Ab} \subset \mathbf{Grp}$. Therefore, Ff is a surjection. Since the forgetful functor $F : \mathbf{Ring} \to \mathbf{Ab}$ only forgets multiplicative structure, f must also be a surjection. □

Proposition 1.3.23. *A morphism $q : X \to Y$ in* ***Top*** *is a quotient map if and only if it is a regular epimorphism.*

Proof. Suppose $q : X \to Y$ is a regular epimorphism in **Top**. Then q coequalizes some $a, b : W \to X$. If $g : X \to Z$ is constant on the fibers of q, then for any $w \in W$, $g(a(w)) = g(b(w))$ since $q(a(w)) = q(b(w))$, and g coequalizes a, b. Thus there exists a unique map $h : Y \to Z$

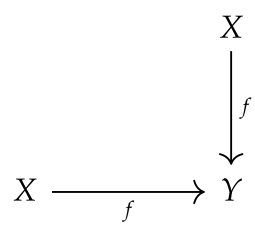
such that the following diagram commutes in **Top**



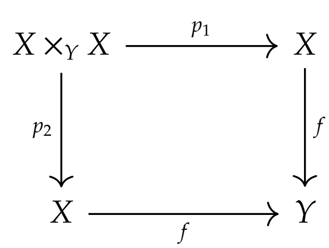
Then since q is an epimorphism, q is a topological quotient map.

Note that since **Top** has kernel pairs, if $q : X \to Y$ is a regular epimorphism then it is an effective epimorphism by Proposition 1.3.29. If $g : X \to Z$ is constant on the fibers of q then g also coequalizes the kernel pair of f, so by the universal property of the coequalizer, q is a topological quotient map.

Definition 1.3.7. Let $f : X \to Y$ be a morphism in the category C. Then f is said to have a *kernel pair* if the diagram



has a pullback, typically written $(X \times_Y X, p_1, p_2)$. That is to say, the following is a pullback square in C:



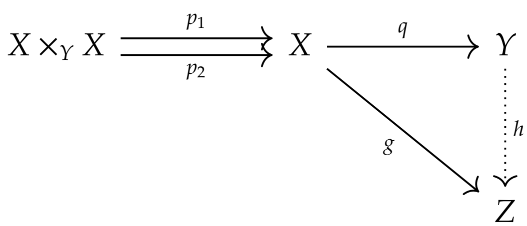
When f has an underlying set function we refer to $X \times_Y X = \{(x, x') \in X \times X \mid f(x) = f(x')\}$ as a *fiber product*, but in the case that morphisms in C lack underlying set functions, the kernel pair will not be a fiber product as such.

Now suppose q is a topological quotient map. Since **Top** is complete, the kernel pair of q exists. It is defined by the fiber product. If g coequalizes the kernel pair of f, then for

$(x_1, x_2) \in X \times_Y X$,

$$g \circ p_1(x_1, x_2) = g \circ p_2(x_1, x_2) \implies g(x_1) = g(x_2).$$

Since $q(x_1) = q(x_2)$ by the definition of the fiber product, we see that g respects the fibers of q. Therefore by the universal property of the quotient map, there exists a unique $h : Y \to Z$ such that $h \circ q = g$. Thus q coequalizes its kernel pair, is effective, and therefore regular.



□

Remark 1.3.24. In many concrete categories, regular epimorphism is the correct strengthening of the concept of epimorphism needed to characterize those morphisms with surjective underlying set function. We have already seen in example 1.1.9 that not all epimorphisms in **Ring** are surjections. But all regular epimorphisms in **Ring** are surjective by Proposition 1.3.22. In any *algebraic category* (see [AHG09]), of which familiar examples are the categories **Grp**, **Ring**, and the category **Vect** of vector spaces and linear transformations, the morphisms with surjective underlying set function are the regular epimorphisms [nLa21c]. For a morphism $f : X \to Y$ in a concrete category, the coequalizing condition means in a non-trivial case that some argument $w \in W$ must exist such that $a(w) \neq b(w)$ but $f(a(w)) = f(b(w))$. Thus $a(w)$ and $b(w)$ are distinct points in X which are identified by f in Y. In this way, the regular epimorphisms also generalize the quotient morphisms, and it will be the case in several categories considered in later chapters, coincide with the quotient morphisms relative to a forgetful functor to **Set**. However, regular epimorphisms are not in general stable under composition [AHG09], a property shared by quotient maps and morphisms quotient relative to a functor.

Example 1.3.25. [Sot] We show there exist composable regular epimorphisms such that the composite is not a regular epimorphism. A category C is *small* if the collection of objects and all hom sets in C form sets rather than proper classes. Let **Cat** denote the category of small categories and functors between them. We may consider the additive monoid of natural numbers $\mathbb{N}$ a category having a single object as follows. Denote the single object by $*$ and the identity $id_* : * \to *$ by $0 : * \to *$. Let the non-identity morphisms in $\mathbb{N}$ be the powers of a distinguished morphism $1 : * \to *$. Addition in the monoid $\mathbb{N}$ is then modeled by composition in the category $\mathbb{N}$, as this composition is associative by definition and composition on the right and left with the identity 0 models addition with additive identity. Thus we interpret $n \circ m$ as $m + n$ where a number n is regarded as the composition $1^n : * \to *$. Let $\mathbf{2} = \{f : 0 \to 1\}$ be the category with two objects and one non-identity morphism. Let $F : \mathbf{2} \to \mathbb{N}$ be the unique functor from $\mathbf{2}$ to $\mathbb{N}$ satisfying $F(f) = 1$. Let $G : \mathbb{N} \to \mathbb{Z}$ be the inclusion functor into the single object category $\mathbb{Z}$ in which every morphism is now an isomorphism and we have for each $n : * \to *$ a morphism $-n : * \to *$ such that $n \circ (-n) = (-n) \circ n = 0$. Let $\mathbb{Z}/2\mathbb{Z}$ denote the category with one object $*$ and one non-identity morphism $1 : * \to *$ satisfying $1 \circ 1 = 1$. Let $H : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ be defined by $H(1) = 1$. Then $H(2) = H(1) \circ H(1) = 1 \circ 1 = 1$ and so on.

We show F is a regular epimorphism in **Cat**. Note that this category has all finite limits and thus all kernel pairs. By Proposition 1.3.29, regular epimorphisms in **Cat** are effective epimorphisms (coequalizers of their kernel pair. See Definition 1.3.9.) Let $\mathbf{0}$ denote the trivial category with one object, 0, and one morphism id_0. Let $A, B : \mathbf{0} \to \mathbf{2}$ be the constant functors defined by $A(0) = 0$ and $B(0) = 1$. Let $J : \mathbf{2} \to C$ be a functor such that $J \circ A = J \circ B$. Then $J(0) = J(1) = X$ and f is mapped to an endomorphism $Jf : X \to X$. Let $\Phi : \mathbb{N} \to C$ be given by $\Phi(1) = Jf$. This specifies Φ as a functor, since the morphisms in $\mathbb{N}$ are generated by 1 and functors respect composition. Clearly Φ uniquely satisfies $\Phi \circ F = J$, and F is the coequalizer of A, B.

We now show $H \circ G$ is a regular epimorphism. Let $\Gamma, \Delta : \mathbf{2} \to \mathbb{N}$ be such that $\Gamma(f) = 1$ and $\Delta(f) = 2$. Then $H \circ G \circ \Gamma(f) = H \circ G(1) = H(1) = 1$ and $H \circ G \circ \Delta(f) =$

$H \circ G(2) = H(2) = H(1) \circ H(1) = 1$. If $K : \mathbb{N} \to \mathcal{D}$ satisfies $K \circ \Gamma = K \circ \Delta$ we have $K(1) = K(2) = K(1) \circ K(1)$. Therefore we can define $\Psi : \mathbb{Z}/2\mathbb{Z} \to \mathcal{D}$ by $\Psi(1) = K(1)$ which uniquely satisfies $\Psi \circ (H \circ G) = K$.

To see that $L = H \circ G \circ F$ is not a regular epimorphism first note that regular epimorphisms in **Cat** must be effective. Let C denote the subcategory of $\mathbf{2} \times \mathbf{2}$ with all four objects (i, j) and one non-identity morphism $f \times f : (0,0) \to (1,1)$. Let $Ver : C \to \mathbf{2}$ and $Hor : C \to \mathbf{2}$ be the functors that project onto the first and second factor respectively, so that $Ver(g \times h : (i, j) \to (k,l)) = g : i \to k$. Then (C, Ver, Hor) is the kernel pair of L. Note the morphisms in C are precisely the products $g \times h$ satisfying $L(g) = L(h)$. Thus $L \circ Hor(g \times h) = L \circ Ver(g \times h)$. If we have a pair of functors $\Phi, \Psi : \mathcal{D} \to \mathbf{2}$ satisfying $L \circ \Phi = L \circ \Psi$ then for $d : D \to D$ in C we must have $L(\Phi(d)) = L(\Psi(d))$ and $\Phi(d) \times \Psi(d)$ is in C. Define $\Theta : \mathcal{D} \to C$ by $\Theta(d) = \Phi(d) \times \Psi(d)$. This functor uniquely satisfies $Hor \circ \Theta = \Phi$ and $Vert \circ \Theta = \Psi$.

We now show that $F : \mathbf{2} \to \mathbb{N}$ is the coequalizer of the kernel pair $Hor, Vert$. By inspection we see that $F \circ Hor = F \circ Vert$. Both compositions send all identities in C to 0 and $f \times f$ to 1. If we have some functor $\Lambda : \mathbf{2} \to \mathcal{E}$ such that $\Lambda \circ Hor = \Lambda \circ Vert$ then $\Lambda(id_0) = \Lambda(id_1)$ and Λ sends f to an endomorphism in $\mathcal{E}$. We send 1 to $\Lambda(f)$ to define a unique functor M satisfying $M \circ F = \Lambda$. Coequalizers are unique up to isomorphism so that if L coequalizes its kernel pair then there must be an isomorphism of categories $\Xi : \mathbb{N} \to \mathbb{Z}/2\mathbb{Z}$, but no such functor can be bijective on hom sets. Hence L is not an effective epimorphism and therefore not a regular epimorphism.

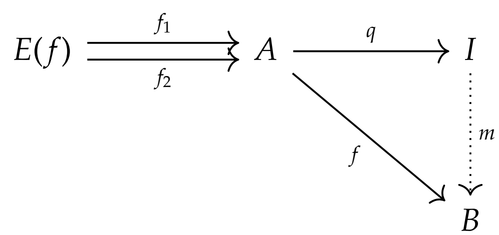
Definition 1.3.8. [Gra14, p. 8] If C is a category in which coequalizers of kernel pairs exist and regular epimorphisms are stable under pullback, then we call C a *regular category*.

Lemma 1.3.26. *For a morphism $p : A \to B$ with kernel pair $(E(p), p_1, p_2)$, p is a monomorphism if and only if $p_1 = p_2$.*

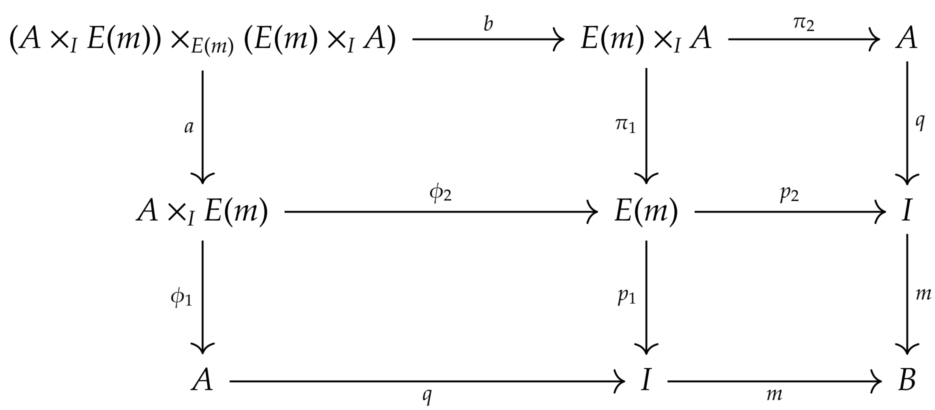
Proof. If p is a monomorphism then since $p \circ p_1 = p \circ p_2$, we have $p_1 = p_2$. If $p_1 = p_2$ then if $p \circ x = p \circ y$, there exists a unique $\phi : Z \to E(p)$ such that $x = p_1 \circ \phi = p_2 \circ \phi = y$. □

Theorem 1.3.27. *[Gra14][7] In a regular category $\mathcal{C}$, a morphism $f : A \to B$ has a (unique up to isomorphism) factorization $f = m \circ q$, where m is a monomorphism and q is a regular epimorphism.*

Proof. Let $\mathcal{C}$ be regular and let $f : A \to B$ be an arbitrary morphism in $\mathcal{C}$. Since $\mathcal{C}$ is regular the kernel pair $(E(f), f_1, f_2)$ of f has a coequalizer $q : A \to I$. Then there exists a unique $m : I \to B$ such that $m \circ q = f$.



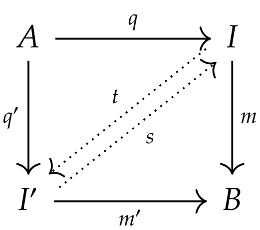
Let all the squares in the following diagram be pullbacks



By the Pullback Pasting Lemma (Lemma 2.1.13), the top and bottom rectangles are pullbacks. Note that the resulting adjacent pullbacks are of the same form only rotated and reflected, so that we can apply the same lemma to show the large outer square is also a pullback square. Since pullbacks are unique up to isomorphism, there exist $\alpha : E(f) \to (A \times_I E(m)) \times_{E(m)} (E(m) \times_I A)$ and $\beta : (A \times_I E(m)) \times_{E(m)} (E(m) \times_I A) \to E(f)$ such that $\alpha \circ \beta = id_{E(f)}$ and $\beta \circ \alpha = id_{(A \times_I E(m)) \times_{E(m)} (E(m) \times_I A)}$. Since q is a regular epimorphism and $\mathcal{C}$ is a regular category, $\pi_1 \circ b = \phi_2 \circ a$ is an epimorphism. We then have

$$p_1 \circ (\phi_2 \circ a) = q \circ \phi_1 \circ a = q \circ f_1 \circ \alpha = q \circ \pi_2 \circ b = p_2 \circ \pi_1 \circ b = p_2 \circ (\phi_2 \circ a)$$

and since $\phi_2 \circ a$ is an epimorphism, $p_1 = p_2$, making m a monomorphism by Lemma 1.3.26. We now show this factorization is unique up to isomorphism. Suppose we have another factorization $f = m' \circ q'$. Then since q and q' are regular epimorphisms and therefore strong epimorphisms, there exist unique s and t such that all triangles in the following diagram commute.



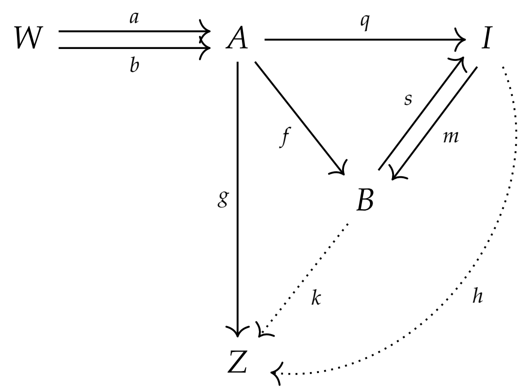
Then $m' \circ s = m$ and $m \circ t = m'$, so that $m' \circ s \circ t = m' \circ (s \circ t) = m'$ and $(m \circ t) \circ s = m \circ (t \circ s) = m$. Thus s and t are isomorphisms. □

Proposition 1.3.28. *If C is a regular category then*

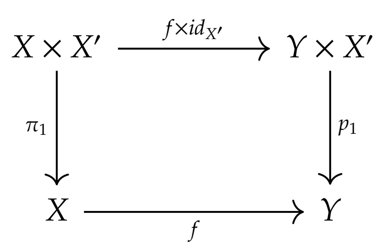
1. *a morphism $f : X \to Y$ is a regular epimorphism if and only if it is a strong epimorphism;*
2. *if $g \circ f$ is a regular epimorphism, then g is a regular epimorphism;*
3. *if g and f are regular epimorphisms, then $g \circ f$ is a regular epimorphism;*
4. *if $f : X \to Y$ and $g : X' \to Y'$ are regular epimorphisms, then $f \times g : X \times X' \to Y \times Y'$ is a regular epimorphism.*

Proof. Let C be a regular category. For (1), regular epimorphisms are strict epimorphisms by proposition 1.3.19 and strict epimorphisms are strong epimorphisms by Proposition 1.3.15. We show strong epimorphisms are regular epimorphisms in C. Let $f : X \to Y$ be a strong epimorphism in C. Then we have $f = m \circ q$ where m is a monomorphism and q is a regular epimorphism by Proposition 1.3.27. Then by Proposition 1.3.9, m is a strong epimorphism, and so by lemma 1.3.10, also an isomorphism. Therefore there exists isomorphism s such that $s \circ f = q$. If $q \circ a = q \circ b$, then $s \circ f \circ a = s \circ f \circ b$. Since s is an isomorphism, it is an epimorphism, and $f \circ a = f \circ b$. Given $g : A \to Z$ such that

$g \circ a = g \circ b$, there exists a unique $h : I \to Z$ such that $h \circ q = g$. Thus $k = h \circ s$ satisfies $k \circ f = k \circ m \circ q = h \circ s \circ f = h \circ q = g$. If $g = k' \circ f$ for some k' then $k' \circ f = k \circ f$ so that $k' \circ m \circ q = k \circ m \circ q$ and since q is an epimorphism, $k' \circ m = k \circ m$. Thus $k' \circ m \circ s = k \circ m \circ s$ and $k' = k$. Therefore, f is a regular epimorphism.



For (2), since by (1) regular epimorphisms in C are strong epimorphisms, the result follows from Proposition 1.3.9. We have (3) again by (1) and by Proposition 1.3.8. For (4) let $f : X \to Y$ be a regular epimorphism. Consider the following diagram



this square defines a pullback by the universal property of the product $Y \times X'$. Therefore, $f \times id_{X'}$ is a regular epimorphism. By a symmetric argument, $id_Y \times g$ is a regular epimorphism. Then $f \times g = (id_Y) \times g) \circ (f \times id_{X'})$ is also a regular epimorphism. □

1.3.5 Effective and Split Epimorphisms

Definition 1.3.9. A morphism $f : X \to Y$ in category C is an *effective epimorphism* if f is the coequalizer of its kernel pair.

Proposition 1.3.29. *In a category C with all kernel pairs, $f : X \to Y$ is a strict epimorphism if and only if f is effective (and therefore regular.)*

Proof. Let $f : X \to Y$ be a regular epimorphism in C where C has all kernel pairs. Suppose f is the coequalizer of $a, b : Z \to X$. Let $p_1, p_2 : X \times_Y X \to X$ be the kernel pair of f. Since $f \circ a = f \circ b$ by the universal property of the kernel pair there exists a unique $\phi : Z \to X \times_Y X$ such that $p_1 \circ \phi = a$ and $p_2 \circ \phi = b$. Suppose $g \circ p_1 = g \circ p_2$. Then $g \circ p_1 \circ \phi = g \circ p_2 \circ \phi$, so that $g \circ a = g \circ b$. Therefore, there exists a unique h such that $h \circ f = g$ and f is an effective epimorphism.

Effective epimorphisms are by definition regular epimorphisms and by Proposition 1.3.19, regular epimorphisms are strict epimorphisms. We complete the proof by showing that strict epimorphisms in C are regular epimorphisms.

Let $f : X \to Y$ be a strict epimorphism in C. A morphism $g : X \to Z$ is compatible with f if $f \circ x = f \circ y$ implies $g \circ x = g \circ y$. Suppose $g \circ p_1 = g \circ p_2$. We claim g is compatible with f. Since $f \circ x = f \circ y$ there exists a unique ϕ such that $p_1 \circ \phi = x$ and $p_2 \circ \phi = y$ by the universal property of the kernel pair. Then $g \circ p_1 = g \circ p_2 \implies g \circ p_1 \circ \phi = g \circ p_2 \circ \phi$ so that $g \circ x = g \circ y$. Then, since f is a strict epimorphism we have $h \circ f = g$, so that f is an effective epimorphism and therefore a regular epimorphism. □

Definition 1.3.10. [AHG09, p. 107]A morphism $f : A \to B$ is a *split epimorphism* or *retraction* if f has a right inverse. That is, there exists $g : B \to A$ such that $f \circ g = id_B$.

Example 1.3.30. Let $f : X \to Y$ be a surjective set function. By the Axiom of Choice we may select for each $y \in Y$, $x \in f^{-1}(y)$ and set $g(y) = x$. This defines a funciton $g : Y \to X$ which clearly satisfies $f \circ g(y) = y$.

Example 1.3.31. Let X, A be topological spaces with $A \subset X$. A *retraction* is a continuous function $r : X \to A$ such that the restriction $r|_A$ is the identity on A. To see that a retraction is a split epimorphism, consider the inclusion map $i : A \hookrightarrow X$ that sends $a \in A$ to $a \in X$. Then $r \circ i(a) = r(a) = a$ and i is a right inverse of r.

Proposition 1.3.32. *The split epimorphisms are stable under composition.*

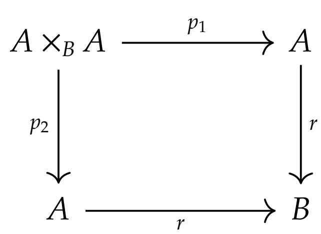
Proof. Let $f : A \to B$ and $g : B \to C$ be split epimorphisms in a category $\mathcal{C}$. Then there exists $i : B \to A$ such that $f \circ i = id_B$ and $j : C \to B$ with $g \circ j = id_C$. Then

$$g \circ f \circ i \circ j = g \circ id_B \circ j = g \circ j = id_C$$

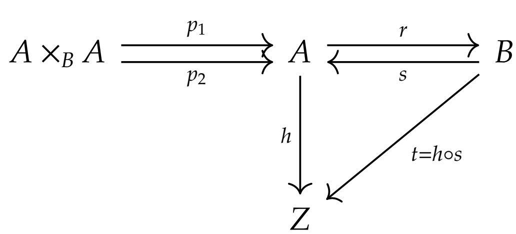
and $i \circ j$ is a right inverse for $g \circ f$, making $g \circ f$ a split epimorphism. $\square$

Proposition 1.3.33. *[Smi] Let $\mathcal{C}$ be a category with kernel pairs. Then a split epimorphism $r : A \to B$ in $\mathcal{C}$ is an effective epimorphism.*

Proof. Let $\mathcal{C}$ have all kernel pairs. Let $r : A \to B$ be a retraction, so that we have $s : B \to A$ with $r \circ s = id_B$. Let $A \times_B A, p_1, p_2$ be the kernel pair of r. Then the following square commutes

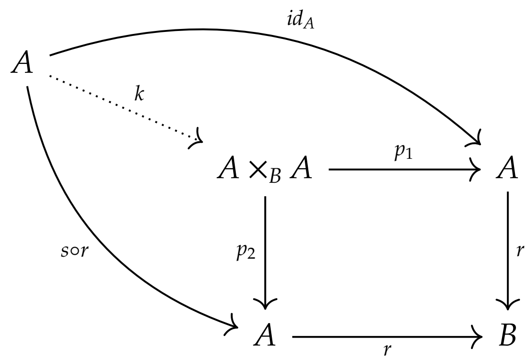


To show that r is effective we must show it coequalizes its kernel pair, so for any $h : A \to Z$ satisfying $h \circ p_1 = h \circ p_2$ we must find a t uniquely satisfying $t \circ r = h$. If we set $t = h \circ s$ we must show $h = t \circ r$ as in the following diagram:



Since $r \circ s \circ r = r$ by the universal property of the kernel pair, there must exist a unique

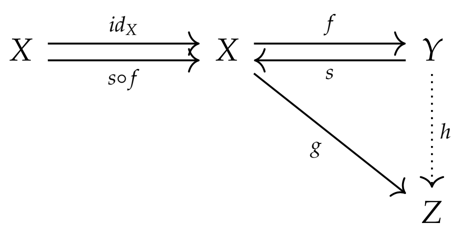
$k : A \to A \times_B A$ such that $p_1 \circ k = id_A$ and $p_2 \circ k = s \circ r$ as in the following diagram:



Therefore $h \circ s \circ r = h \circ p_2 \circ k = h \circ id_A = h$. Then $h \circ s$ is unique in satisfying this condition by the uniqueness of k. □

Proposition 1.3.34. *A split epimorphism is a regular epimorphism.*

Proof. Suppose $f : X \to Y$ is a split epimorphism, so that we have $s : Y \to X$ with $f \circ s = id_Y$. Then f is the coequalizer of $s \circ f$ and id_X. Suppose $g : X \to Z$ satisfies $g \circ id_X = g = g \circ (s \circ f)$. Let $h : Y \to Y$ be given by $h = g \circ s$. Then $h \circ f = g \circ s \circ f = g$.



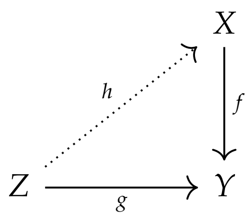
□

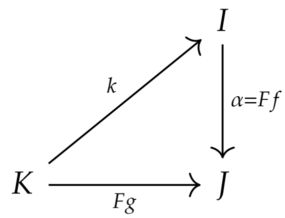
1.4 Cartesian and Cocartesian Morphisms

Definition 1.4.1. [Bor94b, p. 375] Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\alpha : I \to J$ be a morphism in $\mathcal{D}$. A morphism $f : X \to Y$ in $\mathcal{C}$ is *cartesian* over α if

1. $F(f) = \alpha$;

2. given a morphism $g : Z \to Y$ in C with $Fg = \alpha \circ k$ there exists a unique $h : Z \to X$ in C such that $Fh = k$ and $g = f \circ h$

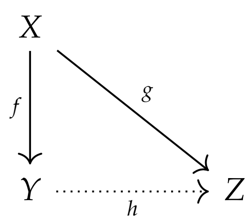


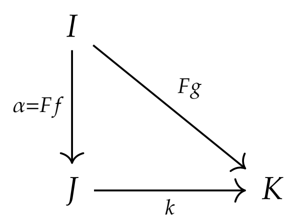


Definition 1.4.2. Let $F : C \to \mathcal{D}$ be a functor. Let $\alpha : I \to J$ be a morphism in $\mathcal{D}$. A morphism $f : X \to Y$ in C is *cocartesian* over α if

1. $F(f) = \alpha$;

2. given a morphism $g : X \to Z$ with $Fg = k \circ \alpha$ there exists unique $h : Y \to Z$ such that $Fh = k$ and $h \circ f = g$.





Remark 1.4.1. These definitions are categorically *dual* to one another. This idea is explored further in Section 2.2.

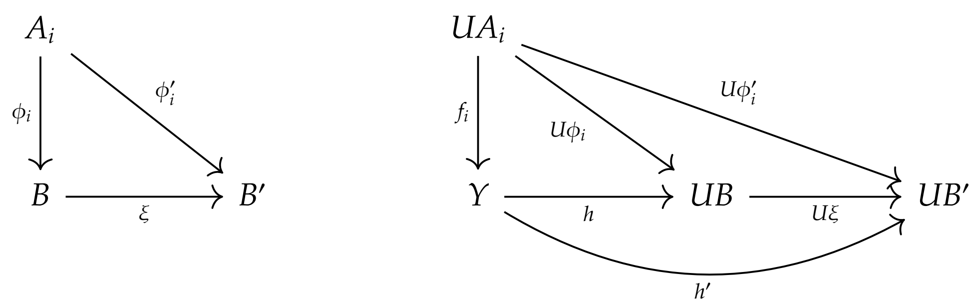
1.4.1 Strictly Final Lifts

Definition 1.4.3. A *sink* in the category C is an object A and collection of morphisms $\{f_i : A_i \to A\}$ into A.

Definition 1.4.4. Let $U : C \to \mathcal{D}$ be a functor. A U-structured sink is a sink $\mathcal{F} := \{f_i : U(A_i) \to Y\}, i \in I$ in $\mathcal{D}$. A *lift* of Y along $\mathcal{F}$ is given by an object $B \in C$ together with a sink $\Phi := \{\phi_i : A_i \to B\}$ in C and morphism $h : Y \to U(B)$ in $\mathcal{D}$ with $U(\phi_i) = h \circ f_i$ for all $i \in I$.

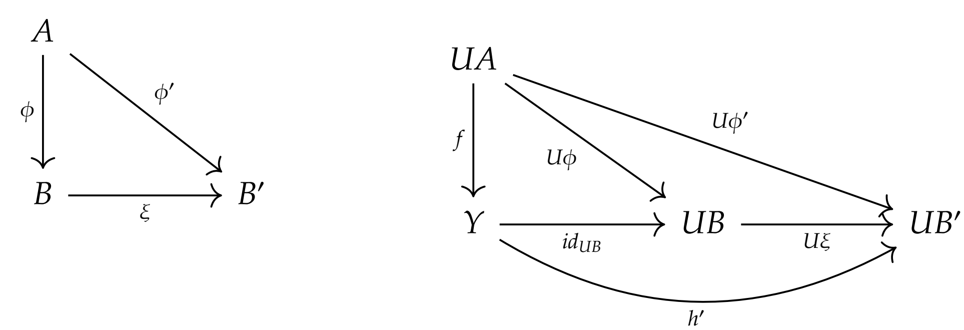
Definition 1.4.5. Let $\mathcal{F}$ be a sink as above. A *morphism of lifts* $\Phi \to \Phi'$ is a morphism $\xi : B \to B'$ in C such that Φ' factors through Φ as $\phi'_i = \xi \circ \phi_i$ and such that the morphism

$h' : Y \to U(B')$ associated to the lift Φ' factors as $h' = U(\xi) \circ h$. Thus for each $i \in I$ the following diagrams commute

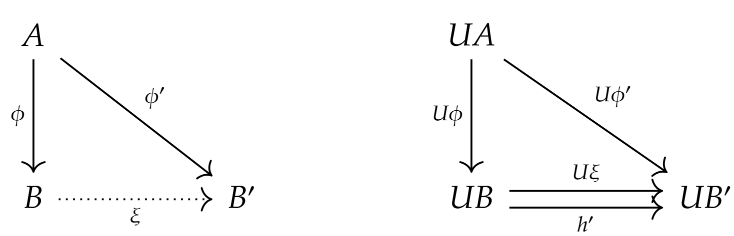


Definition 1.4.6. A *semi-final lift* of a U-structured sink $\mathcal{F}$ is a lift (B, Φ, h) admitting a unique morphism of lifts to any other lift (B', Φ', h') of $\mathcal{F}$. In the case that h is an isomorphism we call Φ a *final lift*. If h is an identity we call Φ a *strictly final lift*.

Example 1.4.2. In the case that we have a U-structured sink consisting of a single morphism $f : UA \to Y$ and Φ is a strictly final lift we have



which gives



We may restate the definition of quotient morphism in terms of strictly final lifts as follows. Let U : **Top** $\to$ **Set** be the forgetful functor. Given a topological space A and surjective set function $UA \to Y$, a strictly final lift of f is a continuous map $\phi : A \to B$ such that $U\phi = f$ and $UB = Y$ having the property that given a continuous map $\phi' : A \to B'$ such that there exists a morphism $h' : UB \to UB'$ satisfying $h' \circ U\phi = U\phi'$, there exists a unique continuous map $\xi : B \to B'$ such that $\xi \circ \phi = \phi'$ and $U\xi = h'$ (since f is surjective.) Therefore, to say ϕ is a strictly final lift of f is to say ϕ (which is surjective since $U\phi = f$) is a quotient morphism with underlying set function f. Thus given a functor $F : \mathcal{C} \to \mathcal{D}$ and morphism $f : FA \to Y$ in $\mathcal{D}$, we have generalized the idea of a final topology with respect to a single function to an arbitrary categorical setting. However, we have not completely characterized quotient maps, since the requirement of surjectivity has not been addressed.

Chapter 2

F-Quotient Morphisms

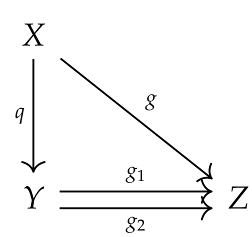
2.1 Introduction to F-Quotients

We now introduce our first main object of study.

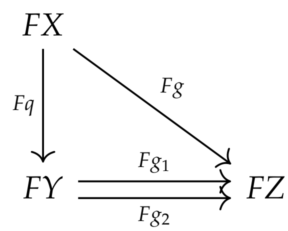
Definition 2.1.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then f is said to be *quotient relative to* F or *F-quotient* if f is cocartesian with respect to F and Ff is an epimorphism.

Proposition 2.1.1. *If $q : X \to Y$ is quotient relative to $F : \mathcal{C} \to \mathcal{D}$, then q is an epimorphism.*

Proof. Suppose we have $q : X \to Y$ and $g_1, g_2 : Y \to Z$ such that $g_1 \circ q = g_2 \circ q$. Let q be $F-$quotient. Then let $g = g_1 \circ q = g_2 \circ q$ and we have the following in $\mathcal{C}$:



and by the functoriality of F, the following in $\mathcal{D}$:



Since q is $F-$quotient, Fq is an epimorphism, and $Fg_1 = Fg_2 = k$. This ensures the existence of a *unique* h with $h \circ q = g$ and therefore $h = g_1 = g_2$, making q an epimorphism. □

Proposition 2.1.2. *A morphism $q : X \to Y$ in* **Top** *is a topological qoutient map if and only if it is quotient relative the forgetful functor U :* **Top** $\to$ **Set***.*

Proof. Suppose $q : X \to Y$ is a quotient map. If $g : X \to Z$ is a map constant on the fibers of q then using the Axiom of Choice we can define $k : Y \to Z$ such that $k \circ q = g$ in the category **Set**. As q is a quotient map, there exists unique continuous $f : Y \to Z$ such that $f \circ q = g$ in **Top**. Then $f \circ q = g$ also in **Set**, so that $f \circ q = k \circ q$. Since q is surjective its underlying set function is surjective and thus an epimorphism in **Set**, so we conclude $f = k$, or $U(f) = k$.

Now let $q : X \to Y$ be quotient relative the forgetful functor U : **Top** $\to$ **Set**. Suppose $g : X \to Z$ is constant on the fibers of q. There there exists a set function $k : U(X) \to U(Y)$ such that $k \circ U(q) = U(g)$, and therefore there exists a unique continuous $f : Y \to Z$ such that $f \circ q = g$, making q a topological quotient map. □

Remark 2.1.3. Recall that a topological quotient map (Definition 1.2.1) is guaranteed to factor uniquely through $g : X \to Z$ in **Top** when g is constant on the fibers of q. This condition is verified on the underlying set function of g. Viewing a topological qoutient map as a morphism quotient relative to U : **Top** $\to$ **Set** makes this relationship clearer from a categorical perspective by moving the context from **Top** to **Set** and in doing so, stating the condition in terms of morphism composition.

Proposition 2.1.4. *Let $U : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor. Then the U-quotient morphisms are precisely the epimorphisms (which are the surjective group homomorphisms by Proposition 1.1.5.)*

Proof. If $f : X \to Y$ in **Grp** is U-quotient then it is an epimorphism by definition, and a surjective group homomorphism. Let $f : X \to Y$ be a surjective group homomorphism. Let $g : X \to Z$ be a group homomorphism. Suppose we have a set function $k : Y \to Z$ such that $k \circ f = g$. Then for $y_1, y_2 \in Y$ we have $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in X$. Then in Z we have

$$\begin{aligned} k(y_1)k(y_2) &= k(f(x_1))k(f(x_2)) \\ &= g(x_1)g(x_2) = g(x_1x_2) \\ &= k \circ f(x_1x_2) = k(f(x_1)f(x_2)) \\ &= k(y_1y_2) \end{aligned}$$

and k is a group homomorphism. Since f is an epimorphism, k uniquely satisfies $k \circ f = g$, and clearly $Uk = k$. □

Proposition 2.1.5. *Let $q : X \to Y$ be quotient relative to $F : \mathcal{C} \to \mathcal{D}$. then for all $Z \in \mathcal{C}$ the following is a pullback diagram in* **Set**.

$$\begin{array}{ccc} \mathcal{C}(Y,Z) & \xrightarrow{\quad q^\# \quad} & \mathcal{C}(X,Z) \\ {\scriptstyle F_{Y,Z}}\downarrow & & \downarrow{\scriptstyle F_{X,Z}} \\ \mathcal{D}(FY,FZ) & \xrightarrow[\quad Fq^\# \quad]{} & \mathcal{D}(FX,FZ) \end{array}$$

where $f^\#$ denotes $- \circ f$.

Proof. First note that the above square commutes for all $f : Y \to Z$ by functoriality of F.

To see this, let $k : Y \to Z$ be a morphism in $\mathcal{C}$. Chasing k around the above square gives

$$\begin{array}{ccc} k & \longrightarrow & k \circ q \\ \downarrow & & \downarrow \\ Fk & \longrightarrow & Fk \circ Fq = F(k \circ q) \end{array}$$

In fact we have a natural transformation $Hom_{\mathcal{C}}(-,-) \Rightarrow Hom_{\mathcal{D}}(-,-) \circ (F \times F)$. Suppose we have (W, p, π) such that the following square commutes

$$\begin{array}{ccc} W & \xrightarrow{\;p\;} & \mathcal{C}(X, Z) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle F_{X,Z}} \\ \mathcal{D}(FY, FZ) & \xrightarrow[Fq^{\#}]{} & \mathcal{D}(FX, FZ) \end{array}$$

Let $w \in W$. Let $p(w) = p_w : X \to Z$ and $\pi(w) = \pi_w : FY \to FZ$. Then

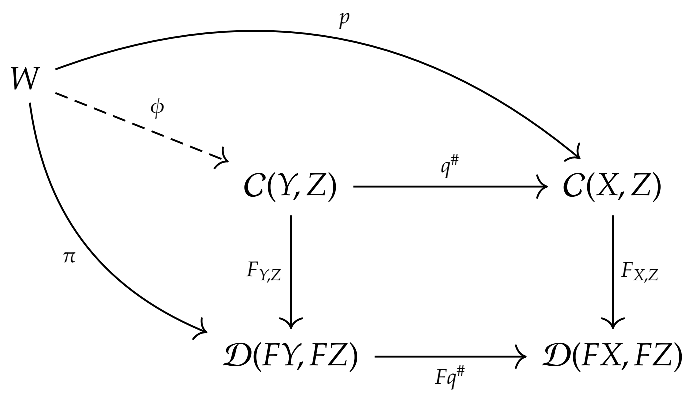
$$F_{X,Z} \circ p_w = Fp_w = Fq^{\#}(\pi_w) = \pi_w \circ Fq.$$

Therefore we have

$$\begin{array}{ccc} X & & \\ {\scriptstyle q}\downarrow & \searrow^{p_w} & \\ Y & & Z \end{array} \qquad\qquad \begin{array}{ccc} FX & & \\ {\scriptstyle Fq}\downarrow & \searrow^{Fp_w} & \\ FY & \xrightarrow[\pi_w]{} & FZ \end{array}$$

Since q is $F-$quotient there exists a unique $\phi_w : Y \to Z$ such that $\phi_w \circ q = p_w$ and $F\phi_w = \pi_w$.

Let $\phi : W \to \mathcal{C}(Y, Z)$ be such that $\phi(w) = \phi_w$. Consider the diagram in **Set**



We have for all w in W

$$q^{\#} \circ \phi(w) = q^{\#}(\phi_w) = \phi_w \circ q = p_w = p(w) \implies q^{\#} \circ \phi = p$$

while

$$F_{Y,Z} \circ \phi(w) = F_{Y,Z}(\phi_w) = F\phi_w = \pi_w \implies F_{Y,Z} \circ \phi = \pi.$$

Therefore both triangles in the diagram commute. If $\phi : W \to \mathcal{C}(Y, Z)$ also makes both triangles commute, then for each $w \in W$

$$\psi_w \circ q = p_w$$

and $F\psi_w = \pi_w$, which implies $\psi_w = \phi_w$ by the universal property of the F-quotient morphism q, and $\psi = \phi$. Therefore, the right-hand commuting square is a pullback diagram. Along the way we have also shown that ψ is well-defined, since $\phi(w)$ exists by virtue of the existence of the morphisms p and π and is unique due to the commutativity hypotheses upon them and the fact that q is $F-$quotient. □

This fact implies (and is equivalent to) there being a canonical bijection

$$\mathcal{C}(Y, Z) \cong \{(k, g) \in \mathcal{C}(X, Z) \times \mathcal{D}(FX, FZ) \mid Fq^{\#} \circ k = Fg)\}.$$

Definition 2.1.2. [AHG09][46] Let $\mathcal{C}$ be a category and let X be an object in $\mathcal{C}$. Then we define the under category or comma category $(X \downarrow \mathcal{C})$ of objects under X by setting morphisms with domain X as objects and commuting triangles as morphisms, so that if $f : X \to Y$ and $g : X \to Z$ with $h : Y \to Z$ satisfying $h \circ f = g$, then we have $h : f \to g$ in $(X \downarrow \mathcal{C})$.

The result of Proposition 2.1.5 can be restated in terms of hom sets of comma categories as follows.

Proposition 2.1.6. *Given a functor $F : \mathcal{C} \to \mathcal{D}$ and morphism $q : X \to Y \in \mathcal{C}$, q is quotient relative to F if and only if the induced morphism Φ on hom sets between comma categories*

$$\Phi : (X \downarrow \mathcal{C})(q, g) \to (FX \downarrow \mathcal{D})(Fq, Fg)$$

is a bijection for all objects Z in $\mathcal{C}$ and morphisms $g : X \to Z$.

Proof. Suppose q is $F-$quotient. Let $k \in (FX \downarrow \mathcal{D})(Fq, Fg)$. Then k satisfies $k \circ Fq = Fg$ and therefore we have a unique morphism $h : X \to Z$ which satisfies $h \circ q = g$ and $Fh = k$. Thus h defines a morphism $h : q \to g$ in $(X \downarrow \mathcal{C})$ and $\Phi(h) = Fh = k$, so Φ is a surjection. If $\Phi(h) = \Phi(h')$ then $Fh \circ Fq = Fg = Fh' \circ Fq$, so that $h \circ q = g = h' \circ q$ and since q is $F-$quotient we must have $h = h'$ as morphisms in $\mathcal{C}$ and therefore the commuting triangles associated to them are equal and thus define a unique morphism in $(X \downarrow \mathcal{C})$, making Φ an injection, and thus bijective. If Φ is bijective by hypothesis, the above arguments may be run in the opposite direction to show that q is an $F-$quotient morphism. □

Remark 2.1.7. Note that only by passing to the comma category can we ensure such a bijection exists, since here F is an arbitrary functor and may not be full or faithful.

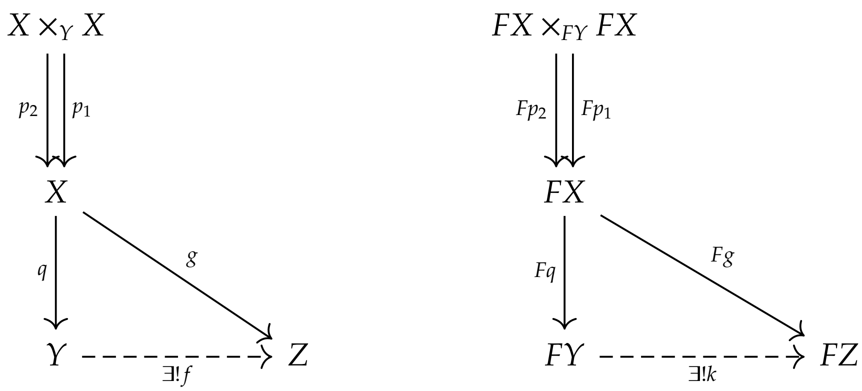
2.1.1 F-Quotients and Epimorphisms

Note that in several commonly encountered categories, such as **Set**, **Grp**, and any category **G**-Set of sets acted on by a group G and G-equivariant maps (and more generally in any category that defines a topos [nLa22a]), every epimorphism is effective.

Proposition 2.1.8. *Let C and $\mathcal{D}$ be categories with all pullbacks. Let $\mathcal{D}$ be such that all epimorphisms in $\mathcal{D}$ are effective (respectively regular, strict.)*

1. *If $F : C \to \mathcal{D}$ preserves pullbacks, then every morphism in C quotient relative to F is an effective (respectively regular, strict) epimorphism.*
2. *If F is a faithful functor, then all effective (respectively regular, strict) epimorphisms are quotient relative to F.*

Proof. Let C and $\mathcal{D}$ be categories with all pullbacks and let all epimorphisms in $\mathcal{D}$ be effective. For (1), let $F : C \to \mathcal{D}$ preserve pullbacks and let $q : X \to Y$ be a quotient morphism relative to F. Let $(X \times_Y X), p_1, p_2)$ be the kernel pair of q. Suppose $g : X \to Y$ is such that $g \circ p_1 = g \circ p_2$. As $\mathcal{D}$ has all pullbacks and they are preserved by F, we can form the kernel pair $FX \times_{FY} FX$ of Fq and its projections are $Fp_1, Fp_2 : FX \times_{FY} FX :\to FX$. Since q is quotient relative to F and all epimorphism in $\mathcal{D}$ are by hypothesis effective epimorphisms, Fq is an effect epimorphism. Since $Fg \circ Fp_1 = F(g \circ p_1) = F(g \circ p_2) = Fg \circ Fp_2$ and Fq is effective, there exists a unique $k : FY \to FZ$ such that $k \circ Fq = Fg$. Since q is F-quotient, there exists a unique $f : Y \to Z$ such that $f \circ q = f$ (and $Ff = k$.) Therefore, q is the coequalizer of p_1, p_2, and is an effective epimorphism.



For (2) suppose F is faithful and let q be an effective epimorphism. Suppose $g : X \to Z$ and $k : FX \to FZ$ such that $k \circ Fq = Fg$. Since

$$F(g \circ p_1) = Fg \circ Fp_1 = (k \circ Fq) \circ Fp_1 = k \circ (Fq \circ Fp_1)$$
$$= k \circ (Fq \circ Fp_2) = (k \circ Fq) \circ Fp_2 = Fg \circ Fp_2 = F(g \circ p_2)$$

and F is faithful, we have $g \circ p_1 = g \circ p_2$. Since q is effective, by the universal property of the coequalizer there exists a unique f such that $f \circ q = g$. Then since $f \circ q = g$ and k is unique in satisfying $k \circ Fq = Fg$, we must have $Ff = k$, and q is an F-quotient morphism.

As $\mathcal{C}$ and $\mathcal{D}$ have all pullbacks and therefore kernel pairs, by Proposition 1.3.29, the effective, regular, and strict epimorphisms coincide in these categories. Therefore if we assume all epimorphisms in $\mathcal{D}$ are strict or regular, then Fq will be a strict or regular epimorphism, hence an effective epimorphism, and the above argument shows q is an effective epimorphism in $\mathcal{C}$, and therefore regular or strict. For (2), if F is faithful and q is a strict or regular epimorphism, then it is effective, and the above argument shows q is F-quotient. □

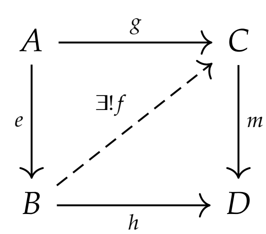
Corollary 2.1.9. *If $\mathcal{C}$ has pullbacks and $F : \mathcal{C} \to$ **Set** is faithful and has a left adjoint, then the quotient morphisms relative to F coincide with the effective epimorphisms in $\mathcal{C}$.*

Proof. Note that since $\mathcal{C}$ has pullbacks, morphisms in $\mathcal{C}$ have kernel pairs. If F is right adjoint then it preserves limits, hence pullbacks, so by the previous proposition quotient relative F morphisms in $\mathcal{C}$ are effective epimorphisms. Since F is faithful, every effective epimorphism is quotient relative F. □

Proposition 2.1.10. *Let $F : \mathcal{C} \to \mathcal{D}$ and let $q : X \to Y$ be quotient relative to F. Then q has a left inverse if and only if Fq does.*

Proof. If q has a left inverse then by the functoriality of F, so does Fq. Now suppose there exists $k : FY \to FX$ such that $k \circ Fq = id_{FX}$. Consider $id_X : X \to X$. Since $k \circ Fq = F(id_X) = id_{FX}$, as q is quotient relative to F there exists a unique $h : Y \to X$ such that $h \circ q = id_X$ and $Fh = k$. Therefore h is a left inverse of q. □

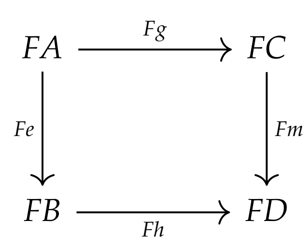
Recall from Definition 1.3.2 an epimorphism $e : A \to B$ is a strong epimorphism if for any monomorphism $m : C \to D$ and morphisms g, h such that the following square commutes, there exists a unique morphism $f : B \to C$ such that $f \circ e = g$ and $m \circ f = h$.



If such an f exists it is unique, since e is an epimorphism and m is a monomorphism.

Proposition 2.1.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories such that all monomorphisms in $\mathcal{D}$ are sections and let $F : \mathcal{C} \to \mathcal{D}$ be a functor that preserves monomorphisms. Let $e : A \to B$ in $\mathcal{C}$ be quotient relative to F. Then e is a strong epimorphism.*

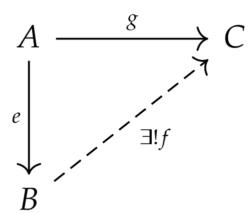
Proof. Applying F to the commuting square above gives



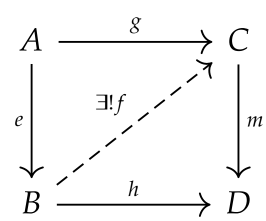
Since F preserves monomorphisms and all monomorphisms in $\mathcal{D}$ are sections, there exists a retraction $r : FD \to FC$ such that $r \circ Fm = 1_{FC}$. Then we have

$$(r \circ Fh) \circ Fe = (r \circ Fm) \circ Fg = 1_{F(C)} \circ Fg = Fg.$$

Setting $k = r \circ Fh$ we have $k \circ Fe = Fg$. Then since e is an F-quotient morphism, there exists a unique f making the following triangle commute in $\mathcal{C}$:



and satisfying $Ff = k$. Thus we have $f \circ e = g$ and $m \circ g = h \circ e$ so that $m \circ f = h$ since e is an epimorphism. If f' also satisfies $m \circ f' = h$ then $f' = f$ since m is a monomorphism. Thus f uniquely makes both triangles in following diagram commute:

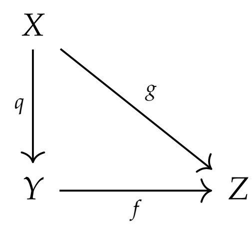


and e is a strong epimorphism. □

2.1.2 Compositions of F-Quotient Morphisms

We have the following result for commuting triangles involving $F-$quotient morphisms:

Lemma 2.1.12. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and*



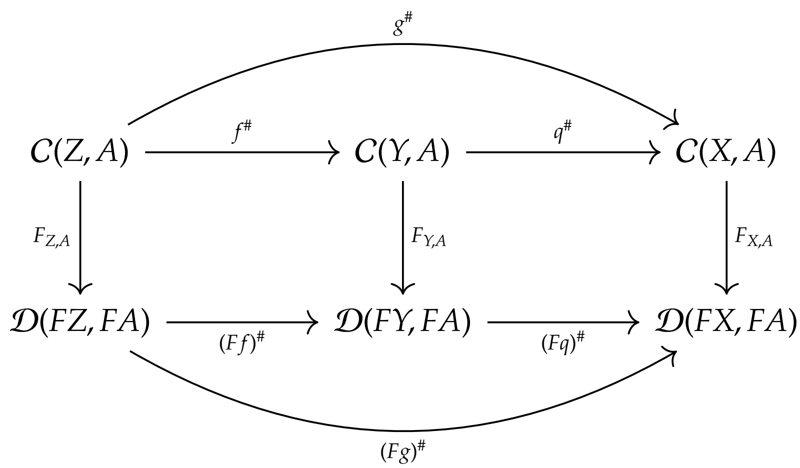
be a commuting triangle in $\mathcal{C}$.

1. *If q and f are quotient relative to F, then so is g.*

2. *If q and g are quotient relative to F, then so is f.*

3. *If f and g are quotient relative to F, then q need not be a quotient morphism.*

Proof. Let A be an object in $\mathcal{C}$ and consider the following diagram in **Set**:



(2.1)

All cells of the diagram commute by the functoriality of F. In detail, if $\phi : Z \to A$, then

$$(F_{Y,A} \circ f^{\#})(\phi) = F_{Y,A}(\phi \circ f) = F(\phi \circ f) = F(\phi) \circ F(f) = (Ff)^{\#}(F\phi) = (Ff)^{\#} \circ F_{Z,A}(\phi).$$

and the left hand square commutes. If $\psi : Y \to A$, then

$$(F_{X,A} \circ q^{\#})(\psi) = F(\psi \circ q) = F(\psi) \circ F(q) = (Fq)^{\#}(F\psi) = ((Fq)^{\#} \circ F_{Y,A})(\psi).$$

and the right hand square commutes. For the top triangle,

$$(q^\# \circ f^\#)(\phi) = q^\#(\phi \circ f) = (\phi \circ f) \circ q = \phi \circ (f \circ q) = \phi \circ g = g^\#(\phi).$$

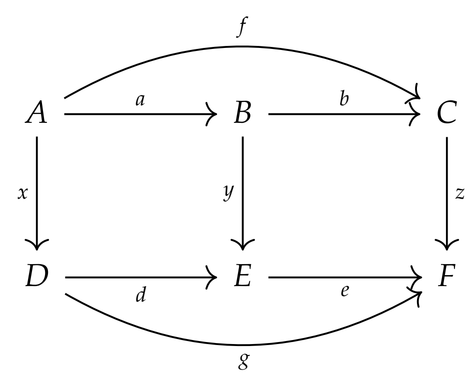
Now let $\xi : FZ \to FA$.

$$((Fq)^\# \circ (Ff)^\#)(\xi) = (Fq)^\#(\xi \circ Ff) = (\xi \circ Ff) \circ Fq =$$

$$\xi \circ (Ff \circ Fq) = \xi \circ F(f \circ q) = \xi \circ Fg = (Fg)^\#(\xi).$$

We have the following elementary result:

Lemma 2.1.13. *If the right hand and left hand squares of the following commuting diagram are pullbacks, so is the outer square.*



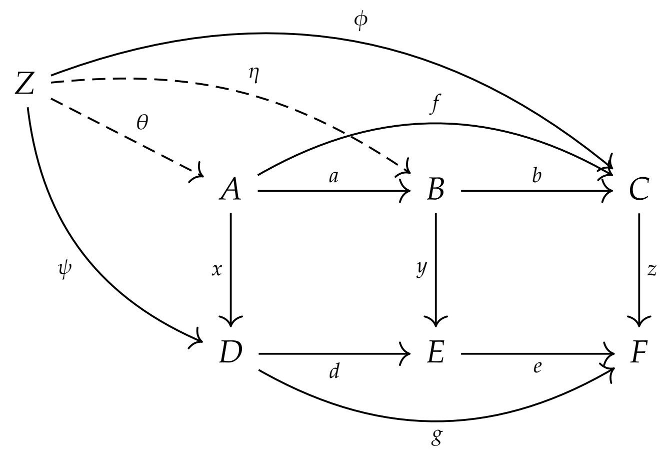
Proof. Suppose $z \circ \phi = g \circ \psi$. Then

$$z \circ \phi = (e \circ d) \circ \psi = e \circ (d \circ \psi).$$

By the universal property of the pullback (B, b, y) there exists a unique $\eta : Z \to B$ such that $b \circ \eta = \phi$ and $d \circ \psi = y \circ \eta$. Then by the universal property of the pullback (A, x, a) there exists a unique $\theta : Z \to A$ such that $a \circ \theta = \eta$ and $x \circ \theta = \psi$. Then

$$f \circ \theta = (b \circ a) \circ \theta = b \circ \eta = \phi.$$

as in the following diagram



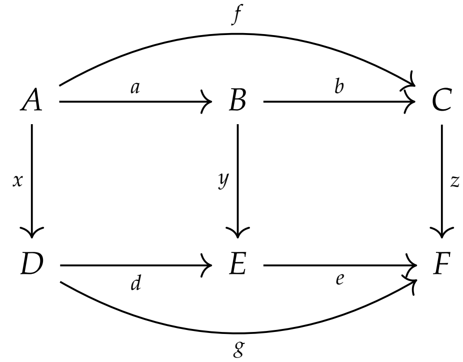
It remains to check that θ is unique in satisfying $f \circ \theta = \phi$ and $x \circ \theta = \psi$.

Suppose $\zeta : Z \to A$ satisfies $f \circ \zeta = \phi$ and $x \circ \zeta = \psi$. Then $(b \circ a) \circ \zeta = \phi$ and $b \circ (a \circ \zeta) = \phi$ and $y \circ (a \circ \zeta) = d \circ \psi$. Then $y \circ (a \circ \zeta) = (d \circ x) \circ \zeta = d \circ (x \circ \zeta) = d \circ \psi$. Therefore, $a \circ \zeta$ satisfies the universal property of η and $a \circ \zeta = \eta$. Now, since $a \circ \zeta = \eta$ and $x \circ \zeta = \psi$, ζ satisfies the universal property of θ associated with the pullback (A, a, x), so that $\zeta = \theta$. □

For (1), since the right and left squares in figure 2.1 are both pullback diagrams, by the above lemma, the outer rectangle is a pullback diagram, and so by Proposition 2.1.5, g is quotient relative F.

We have another elementary result:

Lemma 2.1.14. *If the right hand square and outer rectangle of the following commuting diagram are pullbacks, then the left hand square is also a pullback.*



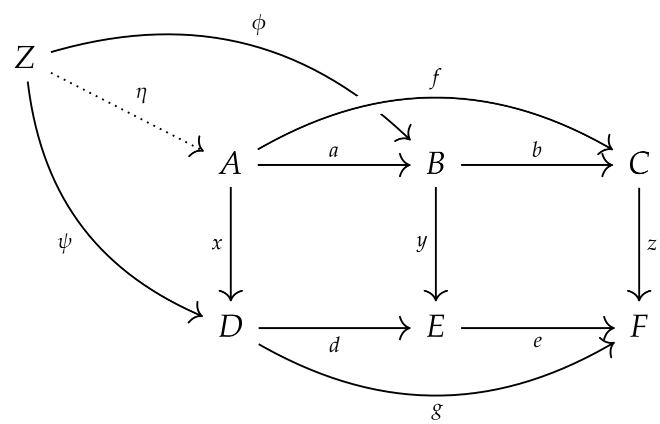
Proof. Suppose $\phi : Z \to B$ and $\psi : Z \to D$ satisfy $d \circ \psi = y \circ \phi$. Then we have

$$z \circ (b \circ \phi) = (z \circ b) \circ \phi = (e \circ y) \circ \phi = e \circ (y \circ \phi) = e \circ (d \circ \psi)$$

and by the universal propery of the pullback (B, b, y), there exists a unique morphism $\theta : Z \to B$ such that $b \circ \theta = b \circ \phi$ so that $\theta = \phi$. Then we have

$$g \circ \psi = e \circ d \circ \psi = e \circ (d \circ \psi) = e \circ (y \circ \phi) = z \circ (b \circ \phi)$$

and so by the universal property of the pullback (A, f, g), there exists a unique $\eta : Z \to A$ satisfying $f \circ \eta = b \circ \phi$ and $x \circ \eta = \psi$. By the universal property of ϕ we have $a \circ \eta = \phi$. The uniqueness of η follows from the uniqueness of ϕ and η with respect to (A, f, x).



□

Now for (2), if q and g are quotient relative to F, the outer and right-hand squares in figure 2.1 are pullbacks, making the left-hand square a pullback, and so f is also quotient relative to F.

For (3) let $F : \mathbf{Top} \to \mathbf{Set}$ be the forgetful functor. Let Z be the one-point space. Then $f : X \to Z$ and $g : Y \to Z$ must be constant. Thus for any $q : X \to Y$ we have $f \circ q = g$, but not all continuous functions are topological quotient maps. □

Theorem 2.1.15. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors and $f : X \to Y$ be a morphism in $\mathcal{C}$.*

1. *If f is quotient relative to F and $F(f)$ is quotient relative G, then f is quotient $G \circ F$.*

2. *If f is quotient relative $G \circ F$ and $F(f)$ is quotient relative G, then f is quotient relative F.*

3. *If F is full and essentially surjective, f is quotient relative F, and quotient relative $G \circ F$, then $F(f)$ is quotient relative G.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. We first show that the image of each prospective quotient morphisms is an epimorphism. For (1), Since $F(f)$ is quotient relative to G its image $G(F(f))$ is an epimorphism. For (2), $F(f)$ is quotient relative G so $F(f)$ is an epimorphism by definition. In (3) f is quotient relative F so $F(f)$ is an epimorphism.

Recall Proposition 2.1.6. We have three functors, F, G and $G \circ F$. For pairs of morphisms $f : X \to Y$ and $g : X \to Z$ in $\mathcal{C}$ and $h : A \to B$, $k : A \to C$ in $\mathcal{D}$ we have three associated induced morphisms on hom sets $\Phi : (X \downarrow \mathcal{C})(f, g) \to (FX \downarrow \mathcal{D})(Ff, Fg)$, $\Psi : (A \downarrow \mathcal{D})(h, k) \to (GA \downarrow \mathcal{E})(Gh, Gk)$ and $\Omega : (X \downarrow \mathcal{C})(f, g) \to (G \circ F(X) \downarrow \mathcal{E})(G \circ F(f), G \circ F(g))$, respectively. Consider the following diagram induced by an arbitrary morphism $g : X \to Z$ in $\mathcal{C}$.

$$
\begin{array}{ccc}
(X \downarrow \mathcal{C})(f, g) & \xrightarrow{\quad\Omega\quad} & \\
\Big\downarrow{\scriptstyle\Phi} & \searrow & \\
(FX \downarrow \mathcal{D})(Ff, Fg) & \xrightarrow[\Psi]{} & (G \circ F(X) \downarrow \mathcal{E})(G \circ F(f), G \circ F(g))
\end{array}
$$

We check that the relevant set functions are bijective. For (1), let f be quotient rel. F and $F(f)$ be quotient rel. G. Fix $g : X \to Z$ in $\mathcal{C}$. Then Φ and Ψ are bijections and $\Omega = \Psi \circ \Phi$ is also a bijection, making f quotient rel. $G \circ F$ by Proposition 2.1.6. For (2) suppose $F(f)$ is qoutient rel. G and f is qoutient rel. $G \circ F$. Fix $g : X \to Z$ in $\mathcal{C}$. The set functions Ω and Ψ in the above triangle are bijections. If for some h, j we have $\Phi(h) = \Phi(j)$, then $\Psi \circ \Phi(h) = \Psi \circ \Phi(j)$ and $\Omega(h) = \Omega(j)$, making $h = j$. If Φ fails to be surjective then we have some $k \in (FX \downarrow \mathcal{D})(Ff, Fg)$ without preimage under Φ. Then k has no preimage under $\Psi \circ \Phi = \Omega$, but Ω is surjective, a contradiction. Thus Φ is a bijection and f is therefore quotient relative to F.

For (3), let F be full and essentially surjective on objects (see Appendix A.1). Let $g' : FX \to Z'$ be arbitrary in $\mathcal{D}$. Then since F is essentially surjective there exists Z in C and isomorphism $\xi : Z' \to F(Z)$. Then setting $k = \xi \circ g'$ we have $k : F(X) \to F(Y)$ and since F is full there exists some $g : X \to Y$ such that $k = F(g)$. Then since f is quotient rel. F and $G \circ F$, Ω and Φ are bijections, making Ψ a bijection. Let $j : G \circ F(Y) \to G(Z')$ be such that $j \circ (G \circ F)(f) = G(g')$. Set $j' = G(\xi \circ j)$. Then

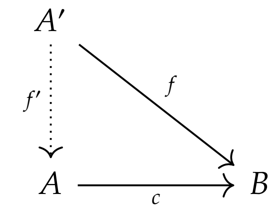
$$j' \circ (G \circ F)(f) = G(\xi) \circ j \circ (G \circ F)(f) = G(\xi) \circ G(g') = G(\xi \circ g') = G(k) = (G \circ F)(g)$$

Therefore, by the bijectivity of Ψ there exists a unique $h : F(Y) \to F(Z)$ satisfying $h \circ F(f) = k$ and $G(h) = j'$. Note that $G(\xi^{-1} \circ h) = G(\xi^{-1}) \circ G(h) = G(\xi^{-1}) \circ j' = G(\xi^{-1}) \circ G(\xi) \circ j = j$. Thus if $h' = \xi^{-1} \circ h$ then $h' \circ F(f) = \xi^{-1} \circ h \circ F(f) = \xi^{-1} \circ k = \xi^{-1} \circ \xi \circ g' = g'$. Since $F(f)$ is an epimorphism, h' uniquely satisfies $h' \circ F(f) = g'$. Then since $G(h') = j$, $F(f)$ is quotient rel. G. □

2.1.3 F-Quotient Objects and Coreflective Subcategories

Definition 2.1.3. Let $F : C \to \mathcal{D}$ be a functor. An object $Y \in C$ is an *F-quotient object* if it is the target of an F-quotient morphism $q : X \to Y$. A subcategory $C' \subset C$ is said to be *closed under F-quotients* if $q : X \to Y$ being quotient relative to F and $X \in C'$ implies $Y \cong Y'$ for some $Y' \in C'$.

Definition 2.1.4. [AHG09][56] Let $\mathcal{B}$ be a category and let $\mathcal{A}$ be a subcategory of $\mathcal{B}$. An *$\mathcal{A}$-coreflection* for an object B of $\mathcal{B}$ is a $\mathcal{B}$-morphism $c : A \to B$ with $A \in \mathcal{A}$ such that for any $\mathcal{B}$-morphism $f : A' \to B$ with A' in $\mathcal{A}$ there exists a unique $\mathcal{A}$-morphism $f' : A' \to A$ such that $f = c \circ f'$.

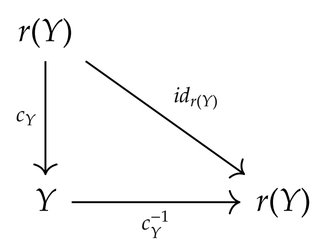


Definition 2.1.5. A subcategory $\mathcal{A}$ of $\mathcal{B}$ is called coreflective if every object in $\mathcal{B}$ has an $\mathcal{A}$-coreflection.

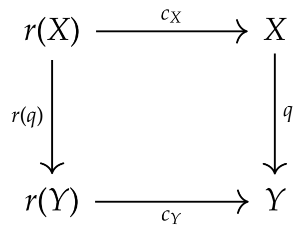
Remark 2.1.16. A coreflective subcategory $\mathcal{A}$ of $\mathcal{B}$ defines a functor $R : \mathcal{B} \to \mathcal{A}$ which is right adjoint to the inclusion functor $i : \mathcal{A} \to \mathcal{B}$. Here c is called the universal morphism from R to B and is the component at B of the counit natural transformation associated to the adjunction (see Appendix A.3).

Proposition 2.1.17. *Let C' be a coreflective subcategory of C. Let $r : C \to C'$ be the coreflection functor and $c : r \to 1_C$ be the counit with component $c_X : r(X) \to X$ at $X \in C$. If $F : C \to \mathcal{D}$ is a faithful functor and $F(c_X)$ is an isomorphism for all $X \in C$, then C' is closed under F-quotient objects.*

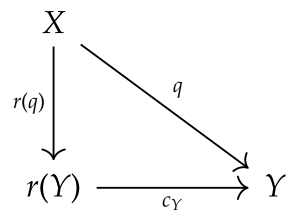
Proof. Let $q : X \to Y$ be quotient relative to $F : C \to \mathcal{D}$ and let X be in the coreflective subcategory C'. It suffices to show that $c_Y : r(Y) \to Y$ is an isomorphism, since if so, by the universal property of the coreflection we have the following commuting triangle



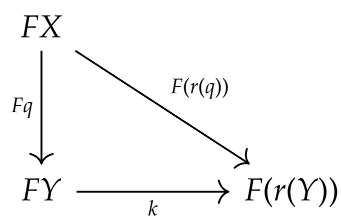
where $Y \in C'$. By hypothesis $F(c_Y) : F(r(Y)) \to F(Y)$ is an isomorphism. Let $k = F(c_Y)^{-1}$. By the unit-counit adjunction associated to $i_{C'} \dashv r$, and noting that $i_{C'} \circ r(X) = r(X)$ as an object in C, the following square in C commutes:



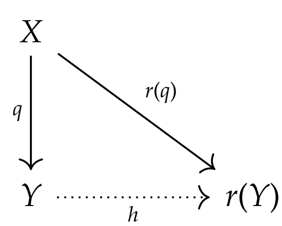
Since $X \in \mathcal{C}'$ by hypothesis, $r(X) = X$ and $c_X = id_X$, so that we have the commuting triangle



Now applying F we have $F(c_Y \circ r(q)) = F(c_Y) \circ F(r(q)) = Fq$. Therefore, $k \circ Fq = F(r(q))$ and the followind triangle commutes in $\mathcal{D}$.



Since q is $F-$quotient, there exists unique $h : Y \to c(Y)$ with $h \circ q = r(q)$ and $Fh = k$:

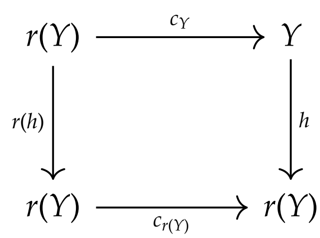


Since $c_Y \circ r(q) = q$ and $r(q) = h \circ q$, we have $c_Y \circ r(q) \circ q = q$. Then

$$F(c_Y \circ h \circ q) = Fc_Y \circ k \circ Fq = Fq.$$

Then since F is faithful, $c_Y \circ h \circ q = q = id_Y$ and $c_Y \circ h = id_Y$ since q is an epimorphism.

By the counit natural transformation, we have the following commuting square:



Since $r(Y) \in C'$, $r(r(Y)) = r(Y)$ and $c_{r(Y)} = id_{r(Y)}$, so that $h \circ c_Y = r(h)$. Then

$$F(r(h)) = k \circ F(c_Y) = F(c_Y)^{-1} \circ F(c_Y) = id_{F(r(Y))}.$$

As F is faithful, $r(h) = id_{r(Y)}$. Thus $h \circ c_Y = r(h) = id_{r(Y)}$, making c_Y an isomorphism. □

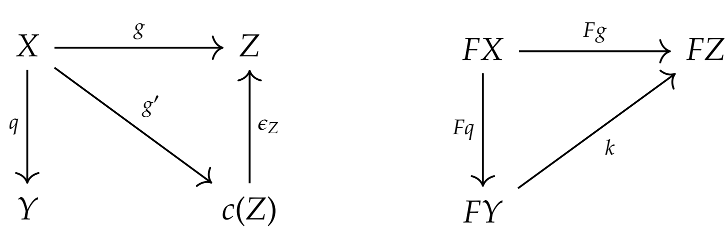
Proposition 2.1.18. *Let C' be coreflective in C with coreflection functor $c : C \to C'$ and counit $\epsilon_Z : c(Z) \to Z$. Let $\mathcal{D}'$ be coreflective in $\mathcal{D}$ with coreflection functor $d : \mathcal{D} \to \mathcal{D}'$ and counit $\epsilon_A : d(A) \to A$. Let $F : C \to \mathcal{D}$ be a functor such that $F(C') \subset \mathcal{D}'$ and*

1. *$d \circ F(\epsilon_Z) : d(F(Z)) \to F(Z)$ is a retraction for all $Z \in C$.*

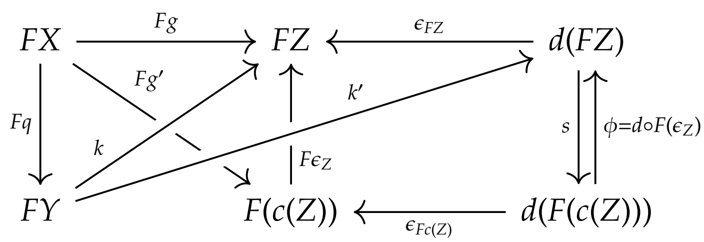
2. *$F(\epsilon_Z)$ is monic for all $Z \in C$.*

Then if $q : X \to Y$ is quotient relative to $F|_{C'} : C' \to \mathcal{D}'$, q is quotient relative to F.

Proof. Suppose $q : X \to Y$ is quotient relative to $F|_{C'} : C' \to \mathcal{D}'$. Then q is a morphism in C. Suppose we have $k : FY \to FZ$ such that $k \circ Fq = Fg$. Let $g : X \to Z$ be a morphism in C. Since X is in C', by the universal property of the coreflection, there exists $g' : X \to c(Z)$ such that $\epsilon_Z \circ g' = g$:



Note we have applied $F : \mathcal{C} \to \mathcal{D}$ here, since Z is in $\mathcal{C}$. Now we have the morphism $F\epsilon_Z : F(c(Z)) \to FZ$ in $\mathcal{D}$ and apply the coreflection functor $d : \mathcal{D} \to \mathcal{D}'$. We then have components of the associated counit natural transformation at FZ and $F(c(Z))$ making the square on the right involving ϕ in the following diagram commute.



By assumption, $\phi = d(F(\epsilon_Z))$ is a retraction with associated section $s : d(FZ) \to d(F(c(Z)))$ satisfying $\phi \circ s = id_{d(FZ)}$. Since $FY \in \mathcal{D}'$ (as q is in $\mathcal{C}'$), by the universal property of the coreflection d there exists a unique $k' : FY \to d(FZ)$ such that $\epsilon_{FZ} \circ k' = k$. Then

$$
\begin{aligned}
F\epsilon_Z \circ \epsilon_{F(c(Z))} \circ s \circ k' \circ Fq &= \epsilon_{FZ} \circ \phi \circ s \circ k' \circ Fq \\
&= \epsilon_{FZ} \circ \epsilon_{F(c(Z))} \circ id_{d(FZ)} \circ k' \circ Fq \\
&= \epsilon_{FZ} \circ k' \circ Fq \\
&= k \circ Fq \\
&= Fg \\
&= F(\epsilon_Z \circ g') = F\epsilon_Z \circ Fg'.
\end{aligned}
$$

Since $F\epsilon_Z$ is a monomorphism by hypothesis, we have

$$\epsilon_{F(c(Z))} \circ s \circ k' \circ Fq = Fg'.$$

Let $\psi = \epsilon_{F(c(Z))} \circ s \circ k'$. Then by the above we have $\psi \circ Fq = Fg'$. By assumption, $\phi : FY \to F(c(Z))$ is in $\mathcal{D}'$. Therefore since q is quotient (rel.) $F|_{\mathcal{C}'}$, there exists a unique $f : Y \to c(Z)$ such that $Ff = \psi$ and $f \circ q = g'$. Let $f' = \epsilon_Z \circ f : Y \to Z$. Then $f' \circ q = \epsilon_Z \circ f \circ q = g' \circ \epsilon_Z = g$, so $f' \circ q = g$. Then

$$\begin{aligned} Ff' = F\epsilon_Z \circ Ff = F\epsilon_Z \circ \psi &= F\epsilon_Z \circ \epsilon_{F(c(Z))} \circ s \circ k' \\ &= \epsilon_{FZ} \circ \phi \circ s \circ k' \\ &= \epsilon_{FZ} \circ k' \\ &= k. \end{aligned}$$

Thus given k such that $k \circ Fq = Fg$ we have f' such that $f' \circ q = g$ and $Ff' = k$. The uniqueness of f' follows from the uniqueness of f. □

Proposition 2.1.19. *If C' is a full subcategory of C and $q : X \to Y$ is in C' and quotient rel. $F : C \to \mathcal{D}$ then q is quotient rel. $F|_{C'} : C' \to \mathcal{D}$.*

Proof. Let $q : X \to Y$ be a morphism in $C' \subseteq C$ that is quotient rel. $F : C \to \mathcal{D}$. Let $g : X \to Z$ be a morphism in C' and let $k : FY \to FZ$ be such that $k \circ Fq = Fg$. Then there exists a unique $h : Y \to Z$ in C such that $h \circ q = g$ and $Fh = k$. As q and g are in C' and C' is full, h is also in C'. Since h is unique in C it is unique in C'. □

Lemma 2.1.20. *If $f : X \to Y$ and $f' : X \to Y'$ are quotient relative to F and $Ff = Ff'$, then there exists a unique isomorphism $h : Y \to Y'$ such that $h \circ f = f'$.*

Proof. Since f is quotient relative to F and $FY = FY'$ we let $k = id_{FY} = id_{FY'}$. Then since $Ff = Ff'$ we have $k \circ Ff = Ff'$ and there exists a unique $h : Y \to Y'$ such that $h \circ f = f'$. By a symmetric argument there exists unique $h' : Y' \to Y$ with $h' \circ f' = f$. Then $h \circ h' \circ f' = f'$. Since $id_{Y'} \circ f' = f'$ and f' is an epimorphism, we conclude that $h \circ h' = id_{Y'}$. By a similar argument $h' \circ h = id_Y$ and h is an isomorphism. □

2.1.4 F-Quotients and Adjunctions

Proposition 2.1.21. *Suppose $f : X \to Y$ is quotient relative to $F : C \to \mathcal{D}$ and suppose $L : C \to \mathcal{E}$ is left adjoint to $R : \mathcal{E} \to C$. Then $Lf : LX \to LY$ is quotient relative to $F \circ R$.*

Proof. Let $g : LX \to Z$ be a morphism in $\mathcal{E}$. Let $g' : X \to RZ$ be adjoint to g and $(Lf)' : X \to RLY$ adjoint to Lf. If $\eta_X : X \to RLX$ is the component at X of the unit natural transformation $\eta : id_{\mathcal{C}} \Rightarrow R \circ L$, then the following square commutes in **Set** by the hom-set adjunction $L \dashv R$.

$$\begin{CD} \mathcal{E}(LX, LX) @>{\Phi_{X,LX}}>> \mathcal{C}(X, RLX) \\ @V{g\circ -}VV @VV{Rg\circ -}V \\ \mathcal{E}(LX, Z) @>>{\Phi_{X,Z}}> \mathcal{C}(X, RZ) \end{CD}$$

For $1_{LX} : LX \to LX$ this gives

$$Rg \circ \Phi_{X,LX}(1_{LX}) = Rg \circ \eta_X = \Phi_{X,Z}(g \circ 1_{LX}) = g'.$$

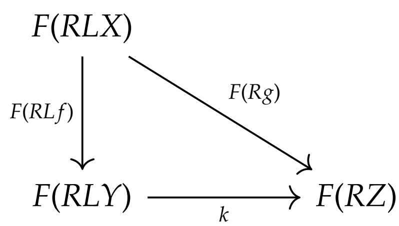
Similarly, if $\epsilon_Z : LRZ \to Z$ is the component of the counit natural transformation $\epsilon : L \circ R \to id_{\mathcal{E}}$, we have the commuting square

$$\begin{CD} \mathcal{C}(RZ, RZ) @>{\Phi_{RZ,RZ}}>> \mathcal{E}(LX, Z) \\ @V{-\circ g'}VV @VV{-\circ L(g')}V \\ \mathcal{C}(X, RZ) @>>{\Phi_{X,RZ}}> \mathcal{E}(LX, Z) \end{CD}$$

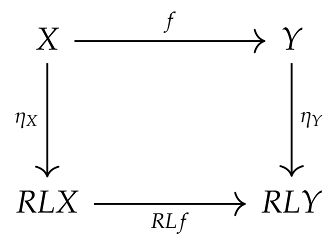
For $1_{RZ} : RZ \to RZ$ this gives

$$\Phi_{RZ,RZ}(1_{RZ}) \circ L(g') = \epsilon_Z \circ L(g') = \Phi_{X,RZ}(1_{RZ} \circ g') = g.$$

Therefore, $Rg \circ \eta_X = g'$ and $\epsilon_Z \circ L(g') = g$. Here $\Phi_{X,Z}$ denotes the bijection natural in X and Z that defines the hom-set adjunction. Let $k : F(RLY) \to F(RZ)$ be a morphism in $\mathcal{D}$ such that $k \circ F(RLf) = F(Rg)$. Then the following diagram commutes in $\mathcal{D}$

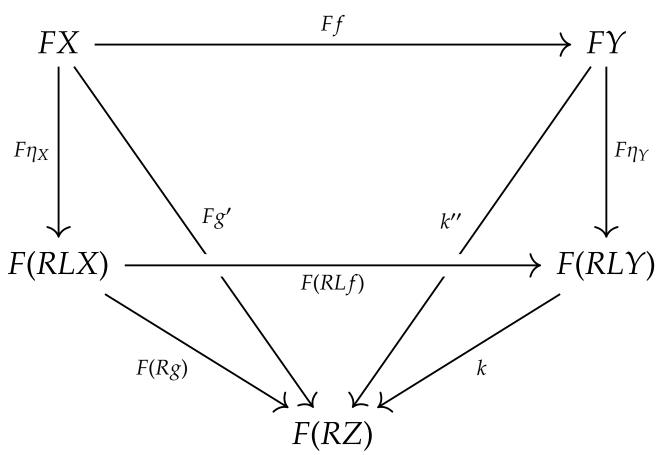


Since the unit is a natural transformation $\eta : id_C \Rightarrow R \circ L$ we have the commuting square



so that $\eta_Y \circ f = RLf \circ \eta_X$. Set $k'' = k \circ F(\eta_Y) : FY \to F(RZ)$. Then

$$k \circ F(\eta_Y \circ f) = k \circ F\eta_Y \circ Ff = k \circ F(FRf) \circ F\eta_X = F(Rg) \circ F\eta_X = Fg'.$$



So that $k'' \circ Ff = Fg'$. As f is an F-quotient morphism there exists a unique $h' : Y \to RZ$ such that $h' \circ f = g'$ with $Fh' = k''$. By the hom-set adjunction the following square commutes:

$$\begin{array}{ccc} \mathcal{C}(Y, RZ) & \xrightarrow{\Phi_{Y,RZ}} & \mathcal{E}(LY, Z) \\ \Big\downarrow{\scriptstyle -\circ f} & & \Big\downarrow{\scriptstyle -\circ Lf} \\ \mathcal{C}(X, RZ) & \xrightarrow[\Phi_{X,RZ}]{} & \mathcal{E}(LX, Z) \end{array}$$

Then for $h' : Y \to RZ$ we have $\widetilde{h' \circ f} = \widetilde{g'} = g = \widetilde{h'} \circ Lf = h \circ Lf$. Thus h' has a unique (since Φ is a bijection) adjoint $h : LY \to Z$ satisfying $h \circ Lf = g$. □

2.1.5 Quotient Morphisms Relative to Faithful Functors

A category $\mathcal{C}$ is called *concrete* if there exists a faithful functor $F : \mathcal{C} \to \mathbf{Set}$.

Lemma 2.1.22. *If $F : \mathcal{C} \to \mathcal{D}$ is faithful (see Appendix A.1), then every retraction in $\mathcal{C}$ is quotient relative to F.*

Proof. Let $r : X \to Y$ be a retraction and $s : Y \to X$ be the associated section. Let $g : X \to Y$ and $k : FY \to FZ$ satisfy $k \circ Fr = Fg$. Then since $g \circ s \circ r = g$ so $k = Fg \circ Fs$ and since F is faithful, $h = g \circ s$ is unique in satisfying $h \circ r = g$, making r an F-quotient morphism. □

Definition 2.1.6. A morphism $f : X \to Y$ in a concrete category $(\mathcal{C}, F)$ where F is faithful to Set, is *quotient* if f is quotient relative to F.

Corollary 2.1.23. *Every retraction in a concrete category is quotient.*

Lemma 2.1.24. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is faithful. If $f : X \to Y$ is a regular epimorphism such that Ff is an epimorphism, then f is quotient relative to F.*

Proof. If f is a regular epimorphism then f coequalizes some parallel pair $a, b : A \to X$. Let $g : X \to Z$ and $k : FY \to FZ$ be morphisms such that $k \circ Ff = Fg$. Since $f \circ a = f \circ b$ we have $f(g \circ a) = k \circ F(f \circ a) = k \circ F(f \circ b) = F(g \circ b)$. Since F is faithful, $g \circ a = g \circ b$. Therefore by the universal property of the coequalizer there exists a unique map $h : Y \to Z$ such that $h \circ f = g$. Since Ff is an epimorphism and $Fh \circ Ff = Fg = k \circ Ff$ we have $Fh = k$. □

2.2 Duality and F-Liftings

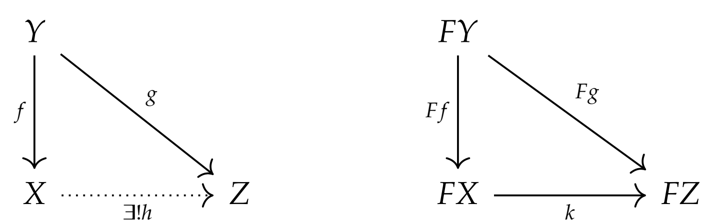
Definition 2.2.1. Let C be a category. We denote by C^{op} the *opposite category* or *dual* category associated to C, having the same objects as C but in which we reverse arrows (more precisely, interchange the source and target of an arrow) and reverse the order of composition, so that if $f \circ g$ is defined in C then $g^{op} \circ f^{op}$ is defined in C^{op}, as is consistent with the intuition of "reversing arrows." We write $f^{op} : Y \to X$ for the arrow in C^{op} that we associate to f, but this superscript is often omitted in practice.

Remark 2.2.1. The dual of a category C is a formal construction. For instance, the opposite category $\mathbf{Grp}^{op}$ has groups for its objects but need not have group homomorphisms for its morphisms. The morphisms in $\mathbf{Grp}^{op}$ exist and behave as they do with respect to composition because their opposites exist and behave as they do with respect to composition in **Grp**.

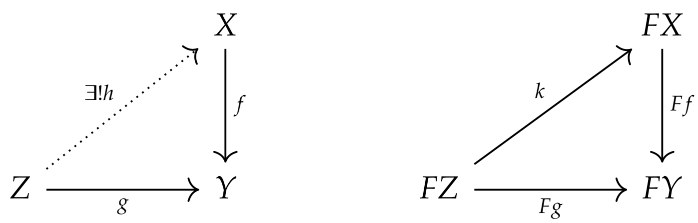
Crucially, if $g \circ f = h$ in C, then $f^{op} \circ g^{op} = h^{op}$ in C^{op}, so that commuting diagrams are preserved by the dual operation. Since C^{op} is a category we can form its dual $(C^{op})^{op}$. A morphism f^{op} exists in C^{op} if and only if f exists in C so that $(f^{op})^{op} = f$ and $(C^{op})^{op} = C$.

If we are given a functor $F : C \to \mathcal{D}$ then we may define $F^{op} : C^{op} \to \mathcal{D}^{op}$ by $F^{op}(X) = FX$ for all $X \in C$ and for $f : X \to Y$ in C, $F^{op}(f) = op(Ff : FX \to FY) = (Ff)^{op} : FY \to FX$ in $\mathcal{D}^{op}$.

We now use the dual operation to define unique liftings relative to a functor. Suppose we have a functor $F : C \to \mathcal{D}$. Let $f : Y \to X$ be quotient relative to F. Then given $k : FX \to FZ$ such that $k \circ Fg = Fg$, we have a unique h such that $h \circ f = g$ and $Fh = k$.



Let us now consider the dual categories to C and $\mathcal{D}$ and the functor $F^{op} : C^{op} \to \mathcal{D}^{op}$. Then we have the following situation, where we have omitted superscripts.

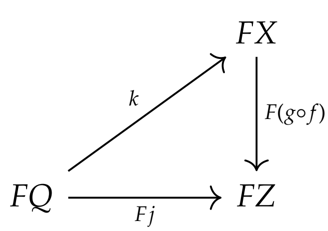


These diagrams commute in $\mathcal{C}^{op}$ and $\mathcal{D}^{op}$ if and only if their duals commute in $\mathcal{C}$ and $\mathcal{D}$. By dualizing an $F-$quotient morphism it satisfies the dual condition of being a *unique* $F^{op}-$*lifting morphism*. Thus we start with a cocartesian morphism and dualizing it arrive at the definition of a cartesian morphism. A cocartesian morphism f with Ff an epimorphism is quotient relative to F and a cartesian morphism with Ff a monomorphism is called a *unique F-lifting*, or simply *F-lifting*. Note that a unique F-lifting is itself a monomorphism, as monomorphisms are dual to epimorphisms.

2.3 Basic Properties of F-Liftings

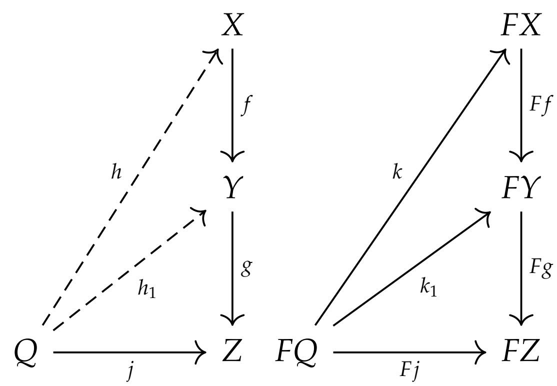
Proposition 2.3.1. *Let $f : X \to Y$ and $g : Y \to Z$ be unique $F-$liftings to the functor $F : \mathcal{C} \to \mathcal{D}$. Then $g \circ f : X \to Z$ is a unique lifting to F.*

Proof. Note this result follows immediately by dualizing Lemma 2.1.12. We include a direct argument for the sake of completeness. Let $j : Q \to Z$ be a morphism in $\mathcal{C}$. Suppose we are given k such that the following diagram in $\mathcal{D}$ commutes:



Set $k_1 = F(f) \circ k$ so that $Fj = Fg \circ k_1$. Since g is an $F-$lifting, there exists a unique h_1 such that $Fh_1 = k_1$ and $g \circ h_1 = f$. Then as f is an $F-$lifting there exists a unique morphism h

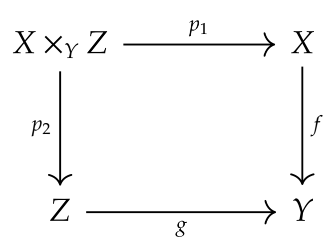
such that $f \circ h = h_1$ and $Fh = k$. We then have



Since $(g \circ f) \circ h = g \circ (f \circ h) = g \circ h_1 = j$, the large triangle in $\mathcal{C}$ commutes. If some h' satisfies $(g \circ f) \circ h' = j$ and $Fh' = k$ then $g \circ (f \circ h') = j = g \circ h_1$ and since g is a monomorphism, $f \circ h' = h_1$. By assumption we have $Fh' = k$ and since f is a unique F-lifting, $h' = h$. Therefore, $g \circ f$ is an $F-$lifting morphism. □

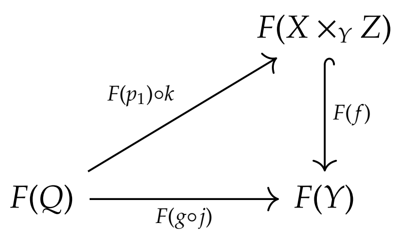
Proposition 2.3.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $f : X \to Y$ and $g : Z \to Y$ be morphisms in $\mathcal{C}$ with f an F-lifting morphism. If F preserves pullbacks, the pullback $p_2 : X \times_Y Z \to Z$ of g along f is an F-lifting morphism.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ b a functor and let $f : X \to Y$ be an $F-$lifting morphism. Suppose we have the following pullback square in $\mathcal{C}$:

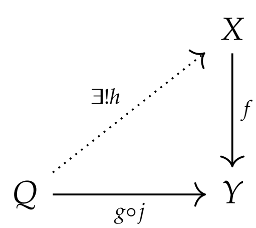


Suppose we are given $j : Q \to Z$ and $k : F(Q) \to F(X \times_Y Z)$ such that the triangle on the right commutes:

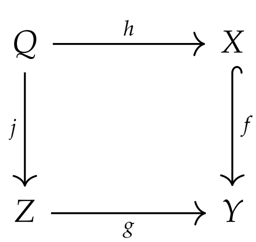
Then we have the commuting triangle:



Since f is an $F-$lifting morphism, there exists a unique h such that the following triangle commutes in C

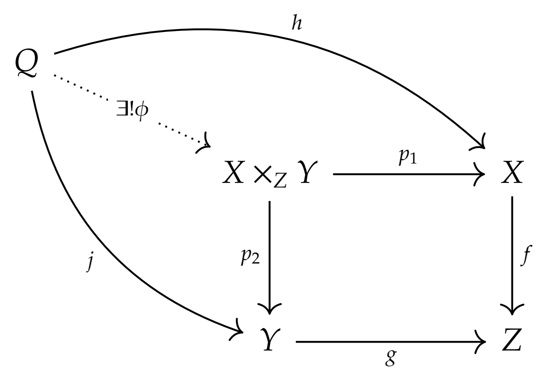


And $F(h) = F(p_1) \circ k$. Then the following square commutes:

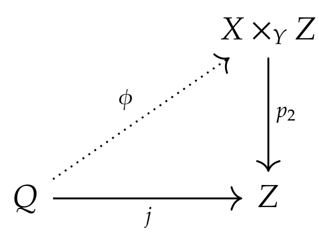


Therefore by the universal property of the pullback, there exists a unique morphism ϕ

such that the following diagram commutes:



and we have



Note that since $F(\phi)$ and k both satisfy the universal property of the pullback $(F(X \times_Y Z), F(p_1), F(p_2))$ of $F(f)$ along $F(g)$, we have $F(\phi) = k$. We also have that $F(p_2)$ is a monomorphism since $F(f)$ is a monomorphism and pullbacks preserve monomorphisms by Proposition 1.1.21. So we can argue

$$F(p_2) \circ F(\phi) = F(p_2 \circ \phi) = F(j) = F(p_2) \circ k \implies F(\phi) = k.$$

Therefore, p_2 is an F-lifting morphism.

□

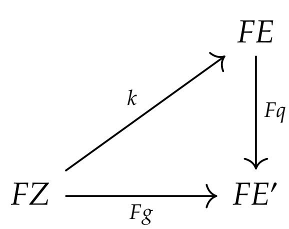
Chapter 3

Categorical Properties of F-Lifting Morphisms

3.1 Unique F-liftings and Equalizers

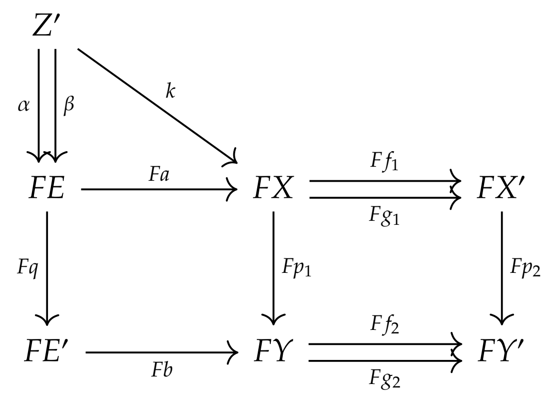
Proposition 3.1.1. *Let $a : E \to X$ be the equalizer of $f_1, g_1 : X \to X'$ and $b : E' \to Y$ be the equalizer of $f_2, g_2 : Y \to Y'$ in the category $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor preserving equalizers. Suppose $p_1 : X \to Y$ and $p_2 : X' \to Y'$ are unique F-liftings such that $p_2 \circ f_1 = f_2 \circ p_1$ and $p_2 \circ g_1 = g_2 \circ p_1$. Let $q : E \to E'$ be any morphism such that $b \circ q = p_1 \circ a$. Then q is a unique $F-$lifting.*

Proof. Let $g : Z \to E'$ be a morphism in $\mathcal{C}$. Suppose we are given $k : FZ \to FE$ such that $Fq \circ k = Fg$ in $\mathcal{D}$:



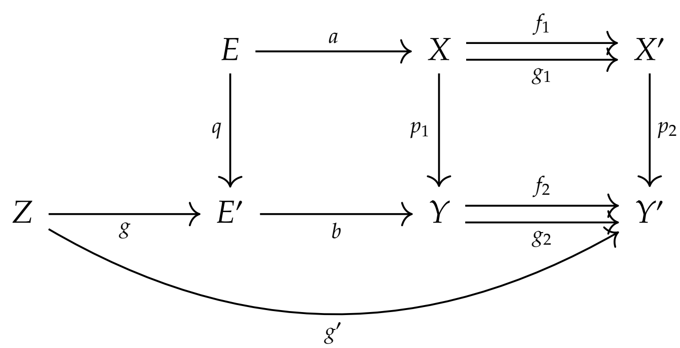
We first establish that Fq is a monomorphism. Let $\alpha, \beta : Z' \to FE$ be such that $Fq \circ \alpha = Fq \circ \beta$.

Consider the following commuting diagram in $\mathcal{D}$

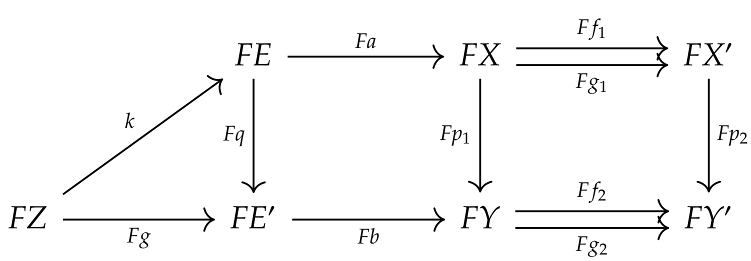


By assumption we have $Fq \circ \alpha = Fq \circ \beta$, so that $Fb \circ Fq \circ \alpha = Fb \circ Fq \circ \beta$ and therefore $Fp_1 \circ Fa \circ \alpha = Fp_1 \circ Fa \circ \beta$. Then since Fp_1 is a monomorphism, $Fa \circ \alpha = Fa \circ \beta$. Set $k = Fa \circ \alpha = Fa \circ \beta$. Then $Ff_1 \circ k = Ff_1 \circ Fa \circ \alpha = Fg_1 \circ Fa \circ \alpha = Fg_1 \circ k$ and k equalizes Ff_1 and Fg_1. Since F preserves equalizers, Fa is the equalizer of Ff_1 and Fg_1, so there exists a unique ϕ satisfying $Fa \circ \phi = k$. Thus $\alpha = \beta$ and Fq is a monomorphism.

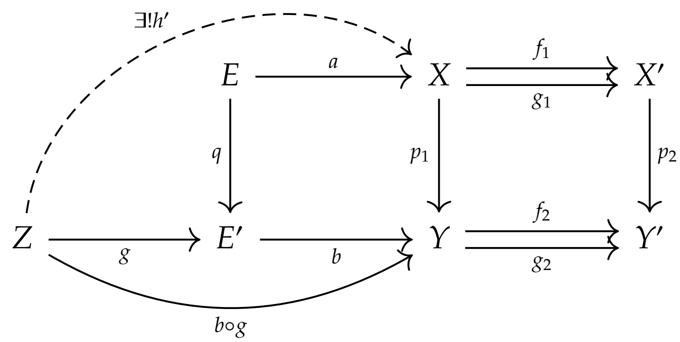
Now consider the following commuting diagram in $\mathcal{C}$:



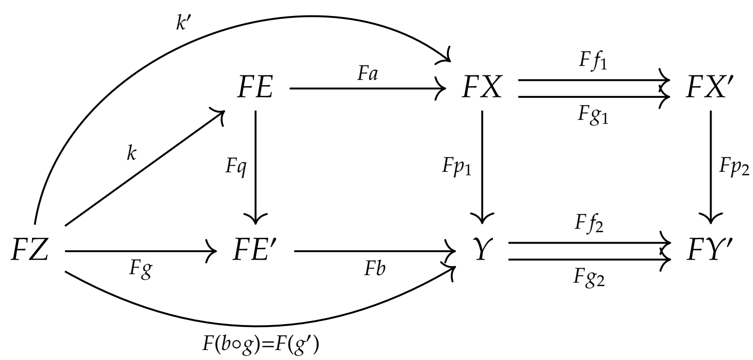
where $g' = f_2 \circ b \circ g = g_2 \circ b \circ g$. Applying F gives the commuting diagram in $\mathcal{D}$:



Therefore, letting $k' = Fa \circ k$, since p_1 is a unique F–lifting there exists a unique $h' : Z \to X$ such that $p_1 \circ h' = b \circ g$. So in $\mathcal{C}$ we have

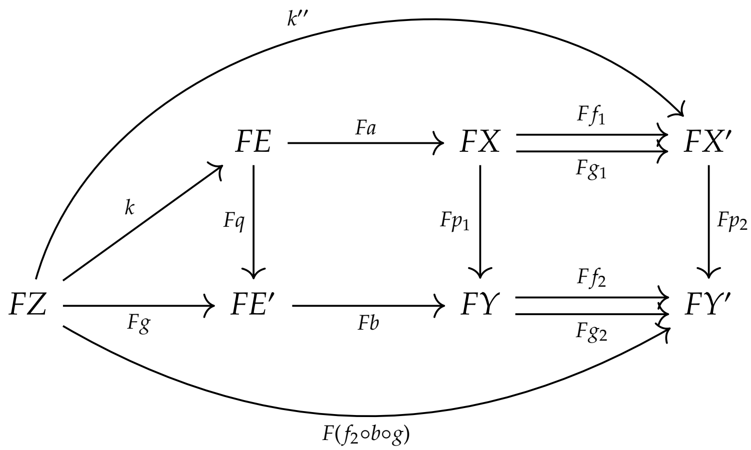


and in $\mathcal{D}$ we have

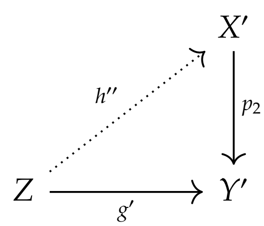


Then since (E', b) is the equalizer of f_2 and g_2, $f_2 \circ b = g_2 \circ b$ and the commutativity of the diagram in $\mathcal{C}$ gives $p_2 \circ (f_1 \circ h') = p_2 \circ (g_1 \circ h')$.

Let $k'' = Ff_1 \circ k' = Ff_1 \circ Fa \circ k = F(f_1 \circ a) \circ k = F(g_1 \circ a) \circ k = Fg_1 \circ k$. Then $Fp_2 \circ k'' = F(g')$.



Since p_2 is a unique F−lifting, there is a unique h'' such that $p_2 \circ h'' = g'$ and $F(g') = Fp_2 \circ k''$

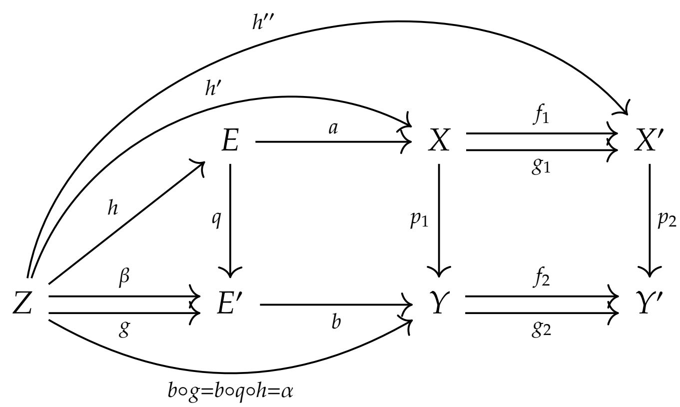


Then since $p_2 \circ (f_1 \circ h') = p_2 \circ (g_1 \circ h') = g'$ and $F(f_1 \circ h') = F(g_1 \circ h') = k''$, we must have $h'' = f_1 \circ h' = g_1 \circ h'$. Therefore, h' is an equalizer of f_1 and g_1, and by the universal property of (E, a), there exists a unique morphism $h : Z \to E$ such that $a \circ h = h'$. We have

$$b \circ q \circ h = p_1 \circ a \circ h = p_1 \circ h' = b \circ g.$$

Now $b \circ g = b \circ q \circ h = \alpha$ equalizes f_2 and g_2, so by the universal property of (E', b), there exists a unique morphism β : such that $b \circ \beta = \alpha$, and we have $b \circ g = b \circ q \circ h = \alpha$ and

$g = q \circ h$.



Lastly, we have $Fq \circ Fh = Fg = Fq \circ k$. Thus, since Fq is a monomorphism, $Fh = k$. □

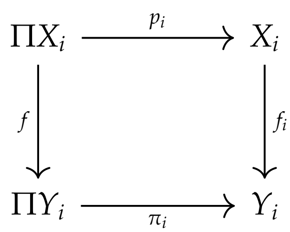
3.2 Unique F-Liftings and Products

Lemma 3.2.1. *The product of a family of monomorphisms is a monomorphism.*

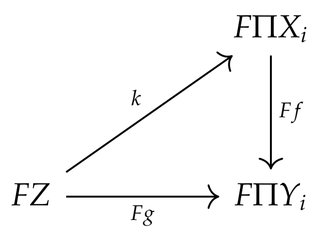
Proof. Let $\{f_i : Z \to X_i\}_{i \in I}$ be a family of monomorphisms and let $(f_i) : Z \to \Pi X_i$ be the product of the f_i. Suppose $\alpha, \beta : W \to Z$ are such that $(f_i) \circ \alpha = (f_i) \circ \beta$. Let $p_i : \Pi X_i \to X_i$ be the projection morphisms. Then for each $i \in I$ we have $p_i \circ (f_i) \circ \alpha = p_i \circ (f_i) \circ \beta$ and therefore $f_i \circ \alpha = f_i \circ \beta$, so that $\alpha = \beta$. □

Proposition 3.2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with all products. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\{f_i : X_i \to Y_i\}_{i \in I}$ be a family of $F-$lifting morphisms indexed by the (possibly infinite) set I. If the canonical morphism $(Fp_i) : F\Pi X_i \to \Pi F X_i$ is a monomorphism, then the product morphism $f : \Pi X_i \to \Pi Y_i$ is also a unique $F-$lifting morphism.*

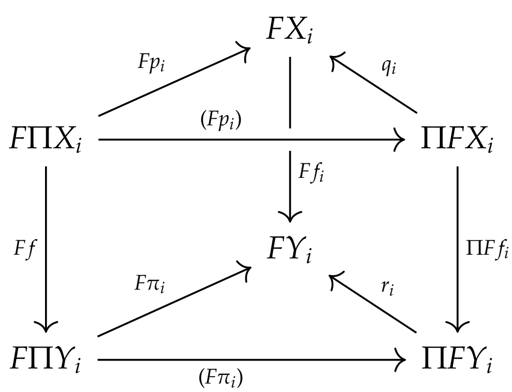
Proof. Let $p_i : \Pi X_i \to X_i$ be the projection morphisms associated to the product ΠX_i and $\pi_i : \Pi Y_i \to Y_i$ the projection morphisms associated to the product ΠY_i. By the universal property of the product $(\Pi Y_i, \pi_i)$, there exists a unique $f : \Pi X_i \to \Pi Y_i$ such that the following square commutes for all $i \in I$.



Suppose we are given $g : Z \to \Pi Y_i$ and $k : FZ \to F\Pi X_i$ such that the following diagram commutes:

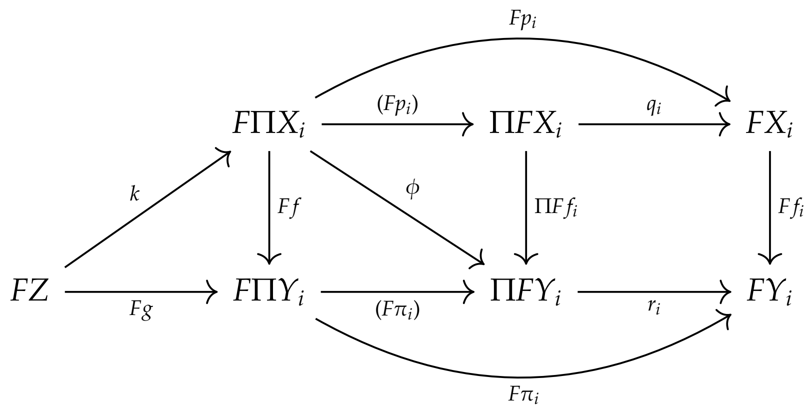


Consider the following diagram in $\mathcal{D}$ for each $i \in I$, where $q_i : \Pi FX_i \to FX_i$ and $r_i : \Pi FY_i \to FY_i$ are the projection morphisms:

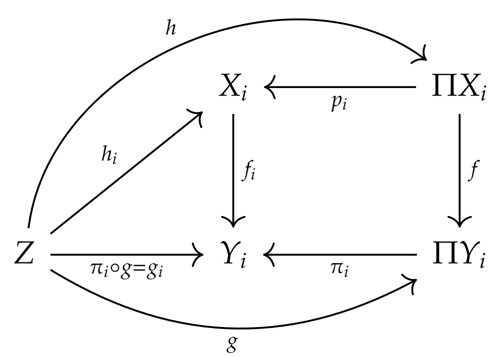


□

By the universal property of the product ΠFY_i there exists $\phi : F\Pi X_i \to \Pi FY_i$ such that $r_i \circ \phi = Ff_i \circ Fp_i$ for all $i \in I$. In the diagram, ϕ becomes the diagonal of the front face. Now we have $r_i \circ (F\pi_i) \circ Ff = F\pi_i \circ Ff = Ff_i \circ Fp_i$ and $(F\pi_i) \circ Ff = \phi$. Similarly, $r_i \circ \Pi Ff_i \circ (Fp_i) = Ff_i \circ q_i \circ (Fp_i) = Fp_i \circ Ff_i = \phi$. Therefore, the front square of the diagram commutes. Then for each $i \in I$ we have in $\mathcal{D}$:



By the functoriality of F the square on the right commutes. This combined with the above shows the entire diagram commutes. Let $k_i = q_i \circ (Fp_i) \circ k = Fp_i \circ k$. Since f_i is a unique F-lifting, there exists a unique $h_i : Z \to X_i$ such that $f_i \circ h_i = \pi_i \circ g$ for each $i \in I$. In this way we get a family of morphisms $\{h_i : Z \to X_i\}$ and by the universal property of the product ΠX_i, there exists a unique $h : Z \to \Pi X_i$ such that for all $i \in I$, $p_i \circ h = h_i$. We then have the commuting diagram in C for all $i \in I$



Then $\pi_i \circ f \circ h = f_i \circ p_i \circ h = f_i \circ h_i$, so that $f \circ h$ is the product of the family $\{f_i \circ h_i : Z \to Y_i\}$. Therefore, since $f_i \circ h_i = \pi_i \circ g = g_i$, we have $f \circ h = (f_i \circ h_i) = (g_i) = g$.

We now show Ff is a monomorphism. Suppose we are given $\alpha, \beta : Z \to F\Pi X_i$ such that $Ff \circ \alpha = Ff \circ \beta$. Consider the following diagram in $\mathcal{D}$:

$$\begin{array}{ccccc} Z & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & F\Pi X_i & \xrightarrow{(Fp_i)} & \Pi FX_i \\ & & \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle \Pi Ff_i} \\ & & F\Pi Y_i & \xrightarrow[(F\pi_i)]{} & \Pi FY_i \end{array}$$

If $Ff \circ \alpha = Ff \circ \beta$ then $(F\pi_i) \circ Ff \circ \alpha = \pi_i) \circ Ff \circ \beta$ and $\Pi Ff_i \circ (Fp_i) \circ \alpha = \Pi Ff_i \circ (Fp_i) \circ \beta$. Since ΠFf_i is a monomorphism by the previous lemma, $(Fp_i) \circ \alpha = (Fp_i) \circ \beta$. By hypothesis, (Fp_i) is a monomorphism, so that $\alpha = \beta$.

Finally, $F(f \circ h) = Ff \circ Fh = Fg = Ff \circ k$ and $Fh = k$, since Ff is a monomorphism.

3.3 F-Liftings and Diagonal Morphisms

Proposition 3.3.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with products, X be an object in $\mathcal{C}$, and J be an indexing set. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and suppose the canonical morphism $(Fq_i) : F\Pi_{j\in J}X \to \Pi_{j\in J}FX$ is a monomorphism. Then the diagonal morphism $\Delta : X \to \prod X$ is an F-lifting morphism if and only if for all families $\{f_j : X \to Y_j\}_{j\in J}$ of $F-liftings$, the induced map $f : X \to \prod Y_j$ is an F-lifting.*

Proof. Suppose for all families $\{f_j : X \to Y_j\}_{j\in J}$ of F-lifting morphisms we have that $f = (f_j)$ is an F-lifting morphism. Set $Y_j = X$ and $f_j = id_X$ for all $j \in J$. This is a family of F-liftings, so that $(id_X) = \Delta$ is an F-lifting.

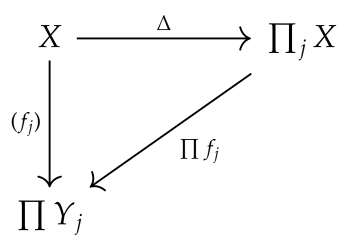
Now let $\Delta : X \to \prod_j X$ be an F-lifting. Let $\{f_j : X \to Y_j\}_{j\in J}$ be a family of F-lifting morphisms. We first show that Ff is a monomorphism. Consider the following diagram:

$$\begin{array}{ccccccc} Z & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & FX & \xrightarrow{F\Delta} & F\Pi_i X & \xrightarrow{(Fq_i)} & \Pi_i FX \\ & & \downarrow{\scriptstyle Ff} & & & & \downarrow{\scriptstyle \Pi Ff_i} \\ & & F\Pi Y_i & & \xrightarrow[(Fr_i)]{} & & \Pi FY_i \end{array}$$

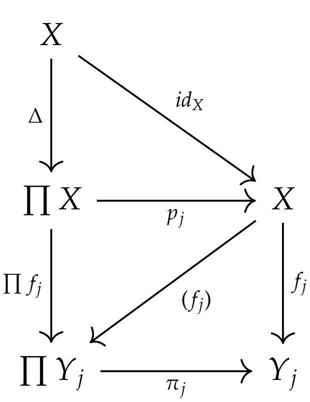
Suppose $\alpha, \beta : Z \to FX$ such that $Ff \circ \alpha = Ff \circ \beta$. Then since each f_i is F-lifting and a product of monomorphisms is a monomorphism,and (Fq_i), $F\Delta$ are monic by hypothesis, we have

$$\begin{aligned}
&Ff \circ \alpha = Ff \circ \beta \\
&\implies (Fr_i) \circ Ff \circ \alpha = (Fr_i) \circ Ff \circ \beta \\
&\implies \Pi F f_i \circ (Fq_i) \circ F\Delta \circ \alpha = \Pi F f_i \circ (Fq_i) \circ F\Delta \circ \beta \\
&\implies (Fq_i) \circ F\Delta \circ \alpha = (Fq_i) \circ F\Delta \circ \beta \\
&\implies F\Delta \circ \alpha = F\Delta \circ \beta \implies \alpha = \beta
\end{aligned}$$

and Ff is a monomorphism. We now show f is a cartesian morphism. Consider the following diagram in C:



By Proposition 3.2.2, $\prod f_j$ is an F-lifting. Thus if the above diagram commutes, (f_j) is an $F-$lifting morphism, since F-liftings are preserved by composition. We have the following commutative diagram



Then

$$\Pi f_j \circ \Delta = (f_j) \circ p_j \circ \Delta = (f_j) \circ id_X = (f_j)$$

therefore, (f_j) is an F-lifting morphism. □

Our results on products and equalizers lead to the following result for all limits:

Theorem 3.3.2. *Suppose $\mathcal{C}, \mathcal{D}$ are complete categories and $F : \mathcal{C} \to \mathcal{D}$ 1) preserves equalizers and 2) is such that for all families X_j, $j \in J$ in $\mathcal{C}$ the canonical morphism $(Fq_i) : F\Pi X_i \to \Pi FX_i$ is a monomorphism. Then the full subcategory $FQ(\mathcal{C})$ of the arrow category $Arr(\mathcal{C})$ of F-quotient morphisms is complete.*

Proof. We know $FQ(\mathcal{C})$ is closed under products by Proposition 3.2.2 and closed under equalizers by Proposition 3.1 and is therefore closed under all limits [Bor94a]. □

Chapter 4

Examples and Applications

4.1 Coreflective Subcategories of Top

4.1.1 $\mathcal{C}$-Covering Morphisms

Definition 4.1.1. [Bra15] A map $p : E \to X$ is a *disk-covering* if E is non-empty, path-connected, and if for each $e \in E$ and map $f : (D^2, d) \to (X, p(e))$ there exists a unique map $\widetilde{f} : (D^2, d) \to (E, e)$ such that $p \circ \widetilde{f} = f$.

Lemma 4.1.1. *Let $p : E \to X$ be a disk-covering. Let $e \in E$. Then each path $\alpha : ([0,1], 0) \to (X, p(e))$ has a unique lift $\widetilde{\alpha}_e$ satisfying $p \circ \widetilde{\alpha}_e = \alpha$.*

Proof. Let $p : E \to X$ be a disk-covering. Let $r : (D^2, d) \to ([0,1], 0)$ be a retraction. Let $\alpha : ([0,1], 0) \to (X, p(e))$ be continuous. Since p is a disk-covering there exists unique $g = \widetilde{\alpha \circ r} : (D^2, d) \to (E, e))$ such that $p \circ g = \alpha \circ r$. For $(t, 0) \in [0,1] \subseteq (D^2, d)$ we have $\alpha \circ r((t,0)) = \alpha(t)$. Then $(\alpha \circ r)|_{[0,1]} = \alpha$ and $p \circ \widetilde{\alpha \circ r}|_{[0,1]} = \alpha$. Set $\widetilde{\alpha}_e = \widetilde{\alpha \circ r}|_{[0,1]}$. □

Corollary 4.1.2. *Let $p : E \to X$ be a disk-covering. Then p is surjective.*

Proof. Since all paths lift uniquely across p, for each $x \in (X, p(x))$ the constant path c_x has unique lift $\widetilde{c_x}$ satisfying $p \circ \widetilde{c_x} = c_x$, so that $p(\widetilde{c_x}(t)) = x$ for all $t \in [0,1]$ and p is surjective. □

Definition 4.1.2. [Bra15] Let $\mathcal{C} \subseteq \mathbf{Top}_0$ be a full subcategory of non-empty path-connected spaces such that the unit disk D^2 is an object of $\mathcal{C}$. A *$\mathcal{C}$-covering map* is a map $p : \widetilde{X} \to X$ such that

1. $\widetilde{X} \in \mathcal{C}$.

2. For any $Y \in \mathcal{C}$, $\widetilde{x} \in \widetilde{X}$, and based map $f : (Y, y) \to (X, p(\widetilde{x}))$ such that $f_*(\pi_1(Y, y)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}))$ there is a unique map $\widetilde{f} : (Y, y) \to (\widetilde{X}, \widetilde{x})$ such that $p \circ \widetilde{f} = f$.

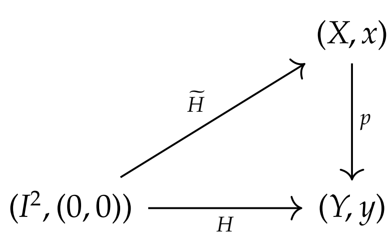
The map p is called a *universal $\mathcal{C}$-covering* if $\widetilde{X}$ is simply connected. If p satisfies only the second condition above, p is then called a *weak $\mathcal{C}$-covering*.

Proposition 4.1.3. *A $\mathcal{C}$-covering morphism is precisely a π_1-lifting morphism where $\pi_1 : \mathcal{C} \to$* **Grp** *is the restriction of the usual fundamental group functor to $\mathcal{C}$.*

Proof. We begin with a preliminary lemma.

Lemma 4.1.4. *If $\mathcal{C} \subseteq \mathbf{Top}_0$ contains $[0, 1] = I$ and I^2, then a $\mathcal{C}$−covering morphism $p : (X, x) \to (Y, y)$ induces an injection on π_1. That is, $\pi_1(p) = p_* : \pi_1(X, x) \to \pi_1(Y, y)$ is an injection.*

Proof. Let $[\gamma], [\delta] \in \pi_1(X, x)$ with $p_*([\gamma]) = p_*([\delta])$. Then $[p \circ \gamma] = [p \circ \delta]$ in $\pi_1(Y, y)$ where $p(x) = y$. Thus $p \circ \gamma$ and $p \circ \delta$ are homotopic, and there exists a based homotopy $H : (I^2, (0, 0)) \to (Y, y)$ such that $H_0 = \gamma$ and $H_1 = \delta$. Note that $\pi_1(I^2, (0, 0))$ is the trivial group. Therefore, $H_*(\pi_1(I^2, (0, 0)) \subseteq p_*(\pi_1(X, x))$. As I^2 is an object of $\mathcal{C}$ and p is a $\mathcal{C}$-covering morphism, there exists $\widetilde{H}$ such that the following triangle commutes



Note that $\gamma(t) = H(0, t)$. Then $p \circ \widetilde{H}(0, t) = H(0, t) = p \circ \gamma(t)$, so that $\widetilde{H}(0, t) = \gamma(t)$ by unique path lifting. Likewise, $\widetilde{H}(1, t) = \delta(t)$, and $\widetilde{H}$ is a path-homotopy between γ and δ, so that $[\gamma] = [\delta]$, making p_* an injection. □

We now continue with the proof of Proposition 4.1.3.Let C be a full subcategory of $\mathbf{Top}_0$. Suppose $p : (X, x) \to (Y, y)$ is a π_1-lifting morphism, where we restrict π_1 to C. Consider $f : Z \to Y$ with Z in C such that $f_*(\pi_1(Z, z)) \subseteq p_*(\pi_1(X, x))$ with $f(z) = p(x)$. By lemma 4.1.4 p_* is an injection. Since $f_*(\pi_1(Z, z)) \subseteq p_*(\pi_1(X, x))$ there exists $k : \pi_1(Z, z) \to \pi_1(X, x)$ such that $p_* \circ k = f_*$, which must be a homomorphism since p_* and f_* are homomorphisms. Then as p is a π_1-lifting morphism, there exists a unique $\widetilde{f}$ such that $p \circ \widetilde{f} = f$. We conclude that p is a C-covering morphism.

Now let $p : (X, x) \to (Y, y)$ be a C-covering morphism. Consider $f : (Z, z) \to (Y, y)$ and $k : \pi_1(Z, z) \to \pi_1(X, x)$ such that $p_* \circ k = f_*$. Then $f_*(\pi_1(Z, z)) = p_*(k(\pi_1(Z, z) \subseteq p_*(\pi_1(X, x))$ and so by assumption there exists a unique $\widetilde{f}$ such that $p \circ \widetilde{f} = f$. Since p is a C-covering morphism, again by lemma 4.1.4, p_* is an injection, so that $p_* \circ \widetilde{f_*} = f_* = p_* \circ k$ implies $\pi_1(\widetilde{f}) = \widetilde{f_*} = k$. □

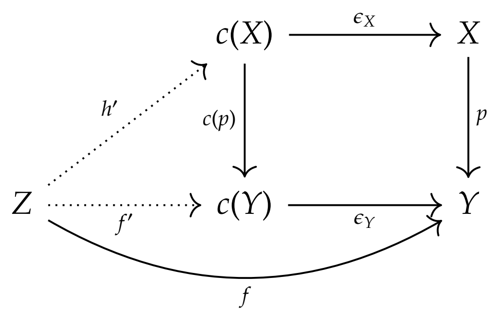
Note that the codomain of a C-covering is not required to belong to C but merely $\mathbf{Top}_0$.

Definition 4.1.3. [AHG09] Let C be a subcategory of **Top**. The *coreflective hull* of C is the full subcategory $H(C)$ consisiting of all quotients and topological sums (disjoint unions) of objects in C.

Remark 4.1.5. The coreflective hull is so-named because it defines a coreflection functor $c : \mathbf{Top} \to H(C)$ which takes a space X to the space $c(X)$ having the same underlying set as X but which is equipped with the final topology with respect to all maps into X from objects in C. So $U \subseteq c(X)$ is open if and only if for any $f : Z \to X$ with Z in C, we have $f^{-1}(U)$ open in Z. This functor is right adjoint to the inclusion $i : H(C) \to \mathbf{Top}$ [Bra14b].

Proposition 4.1.6. *Let C be a subcategory of **Top** containing D^2. Let $p : X \to Y$ be a morphism in $\mathbf{Top}_0$ with X in the subcategory $C \subseteq \mathbf{Top}_0$ but with Y not necessarily in C. Then p is a C-covering morphism if and only if $c(p) : c(X) \to c(Y)$ is a $\pi_1 : H(C) \to \mathbf{Grp}$ lifting morphism, where $c : \mathbf{Top} \to H(C)$ is the coreflection functor.*

Proof. Suppose $c(p) : c(X) \to c(Y)$ is a π_1–lifting morphism. Let $f : (Z, z) \to (Y, y)$ where $f(z) = y$ is a based map with Z in $\mathcal{C}$. Let $f_*(\pi_1(Z, z)) \subseteq p_*(\pi_1(X, x))$. Let ϵ denote the counit of the adjunction $i \dashv c$ with component at X denoted ϵ_X. Then we have the following commuting diagram in **Top**



where we have unique f' such that $\epsilon_Y \circ f' = f$ by the universal property of the coreflection, since $Z \in \mathcal{C} \subseteq H(\mathcal{C})$. As $\mathcal{C}$ contains D^2 we have up to homeomorphism that $H(\mathcal{C})$ contains I and I^2, so that by lemma 4.1.4, $c(p)_*$ is an injection.Therefore there exists a homomorphism $k : \pi_1(Z, z) \to \pi_1(c(X), x)$ such that $c(p)_* \circ k = f'_*$. Thus there exists a unique h' such that $c(p) \circ h' = f'$ and $f'_* = k$. Setting $h = \epsilon_X \circ h'$ we then have

$$f = \epsilon_Y \circ f' = \epsilon_Y \circ c(p) \circ h' = p \circ \epsilon_X \circ h' = p \circ h.$$

If $g : Z \to X$ such that $f = p \circ g$ then $g = h$ since p is a surjection and thus an epimorphism.

Now suppose p is a $\mathcal{C}$-covering morphism. Let $Z \in H(\mathcal{C})$ and let $f : (Z, z_0) \to c((Y, y))$ be a map such that $f_*(\pi_1(Z, z_0)) \subseteq c(p)_*(\pi_1(c(X), x_0)$. We define $\widetilde{f} : Z \to c(X)$ for each $z \in Z$ by first letting $\alpha : [0, 1] \to Z$ be a path from z_0 to z. Then $f \circ \alpha$ is a path from y_0 to $f(z)$ and since p satisfies the unique path lifting property, there exists a unique lift $\widetilde{f \circ \alpha} : [0, 1] \to c(X)$ with $\widetilde{f \circ \alpha}(0) = x_0$ and $p \circ \widetilde{f \circ \alpha} = f \circ \alpha$. Since $[0, 1] \in H(\mathcal{C})$, the lift $\widetilde{f \circ \alpha}$ is continuous. Here we view the unit interval as a based space $([0, 1], 0)$ and note that since $f_*(\pi_1(Z, z_0)) \subseteq c(p)_*(\pi_1(c(X), x_0)$ and $\alpha_*(\pi_1([0, 1], 0) \subseteq f_*(\pi_1(Z, z_0))$ we have $(f \circ \alpha)_*(\pi_1([0, 1], 0)) \subseteq c(p)_*(\pi_1(c(X), x_0)$ (this also follows from the fact that $\pi_1([0, 1], 0)$ is the trivial group.) Now we set $\widetilde{f}(z) = \widetilde{f \circ \alpha}(1)$. A standard argument shows that $\widetilde{f} : Z \to X$ is well-defined [Bra19]. Then $\widetilde{f} : Z \to c(X)$ is also independent of the continuous path α from z_0 tp z. Since p is a C-covering morphism and has the unique lifting property, the uniqueness of $\widetilde{f}$ follows once we establish that $\widetilde{f}$ is indeed continuous. A morphism $g : A \to B$ with $A, B \in H(C)$ is continuous if and only if for all continuous functions $h : D \to A$ with $D \in C$, the composite $g \circ h$ is continuous. Let $h : (A, a_0) \to (Z, z_0)$ be continuous with $A \in C$. Since we have

$$(f \circ h)_*(\pi_1(A, a_0) \subseteq f_*(\pi_1(Z, z_0) \subseteq p_*(\pi_1(X, x_0)$$

and p is a C-covering morphism, there exists unique $k : (A, a_0) \to (X, x_0)$ such that $p \circ k = f \circ h$. As $A \in C$, $k : A \to c(X)$ is continuous. We then have

$$c(p) \circ k = p \circ k = f \circ h = (c(p) \circ \widetilde{f}) \circ h = c(p) \circ (\widetilde{f} \circ h).$$

Therefore by the unique path-lifting property of p we have $k = \widetilde{f} \circ h$. Since k is continuous, $\widetilde{f} \circ h$ is continuous and we conclude $\widetilde{f} : Z \to c(X)$ is continuous. □

Corollary 4.1.7. *Let C be a coreflective subcategory of* $\mathbf{Top_0}$. *Then $p : X \to Y$ in C is a C-covering morphism if and only if p is π_1-lifting with respect to $\pi_1 : C \to$* **Grp**.

Proof. For a coreflective subcategory C of $\mathbf{Top_0}$ we have $H(C) = C$. The result follows from applying the above proposition. □

4.1.2 Lpc-Coverings

We now apply the results from the previous section to the special case of locally path-connected spaces.

Lemma 4.1.8. *Let X be a locally path-connected space and let U be open in X. Then the path components of U are open in X.*

Proof. Let C be a path component of U. Let $x \in C$. Since X is locally path-connected, we have open path-connected set V such that $x \in V \subseteq U$. Since V is path-connected and $x \in V$, if $y \in V$ then there is path from x to y hence $y \in C$. Thus $V \subseteq C$ and C is open in X. □

Definition 4.1.4. Let $\{X_j : j \in J\}$ be a family of topological spaces. The *disjoint union* or *topological sum* of $\{X_j\}$, denoted $\coprod_{j \in J} X_j$, is the disjoint union of the underlying sets $\{X_j\}$ equipped with the final topology relative to the canonical inclusion maps $i_j : X_j \to \coprod X_j$ given by $i_j(x) = (x, j)$. That is, $U \subseteq \coprod X_j$ is open iff $i_j^{-1}(U)$ is open in X_j for all $j \in J$.

Lemma 4.1.9. *The canonical inclusions $i_j : X_j \to \coprod X_j$ are open maps.*

Definition 4.1.5. A topological space X is *locally path-connected* if it has a basis of path-connected sets. That is, for each open set $U \subseteq X$ and $x \in U$ there exists open $V \subseteq U$ with $x \in V$ and V path-connected as a topological space endowed with the subspace topology.

Lemma 4.1.10. *The disjoint union of a family of locally path-connected topological spaces is locally path-connected.*

Proof. Let $x \in U \subseteq \coprod X_j$ with each X_j path-connected. Then $x = (x, j_0)$ for some $j_0 \in J$. Then $i_{j_0}^{-1}(U)$ is open in X_{j_0} and there exists $V \subseteq i_{j_0}^{-1}(U)$ open, path-connected, and containing $i_{j_0}^{-1}(x)$. Then $i_{j_0}(V)$ is open and path-connected, with $i_{j_0}(V) \subseteq U$, hence it is open in the subspace topology on U, and $\coprod X_j$ is locally path-connected. □

Lemma 4.1.11. *The quotient of a locally path-connected space is locally path-connected.*

Proof. Let X be locally path-connected. Let $q : X \to Y$ be a quotient map. Let U be open in Y and let C be a non-empty path component of U. If C is open in Y then for any $y \in U$ we will have $y \in C \subseteq U$ for some path component C open in Y, and Y will be locally path-connected. since q is quotient and therefore open, if $q^{-1}(C)$ is open, we are done. If $x \in q^{-1}(C) \subseteq q^{-1}(U)$ then since X is locally path-connected, there exists open set V such that $x \in V \subseteq q^{-1}(U)$. Let $v \in V$ and let $\gamma : [0, 1] \to V$ be a path with $\gamma(0) = x$ and $\gamma(1) = v$. Then $q \circ \gamma : [0, 1] \to q(V) \subseteq U$ is a path with $(q \circ \gamma)(0) = q(x)$ and $(q \circ \gamma)(1) = q(v)$. Since C

is a path component of $q(x)$ in U, $(q \circ \gamma)([0,1]) \subseteq C$ and γ has image in $q^{-1}(C)$. Therefore, $\gamma(1) = v \in q^{-1}(C)$ and $V \subseteq q^{-1}(C)$. Thus $q^{-1}(C)$ is open making C open and Y is locally path-connected. □

Definition 4.1.6. Let (X, τ) be a topological space. Define $lpc(X)$ to be the topological space (X, τ') where τ' is the topology on X generated by the collection $\{C\}$ of path components of open sets $U \in \tau$. Let $\mathcal{B}_C$ be the collection of these sets.

Proposition 4.1.12. *[Bra14a] The collection $\mathcal{B}_C$ indeed forms a basis for a topology on X.*

Proof. Let $x \in X$ have path component C' in X. Let U, V be open in X with $x \in U \cap V$. Let C_U and C_V be the path components of x in U and V respectively. Let C be the path component of x in $C_U \cap C_V$. If $C \subseteq C_U \cap C_V$, then since C_U and C_V are both open in X, the collection of path components of open sets in X will form the basis of a topology on X. Let $c \in C$. Then there exists path $\gamma : [0,1] \to C_U \cap C_V$ with $\gamma(0) = c$ and $\gamma(1) = x$. The path γ therefore lies entirely in C_U and C_V. Therefore $c \in C_U \cap C_V$ and $C \subseteq C_U \cap C_V$ □

Proposition 4.1.13. *[Bra14a] Let (X, τ) be a topological space. The topology τ' on X given by $lpc(X)$ is finer than τ. This is equivalent to the continuity of the identity function $id : lpc(X) \to X$.*

Proof. Let $U \in \tau$. Then U is the union of its path components, since if $x \in U$, we have $x \in C_i$ for some path component C_i and $\bigcup C_i = U$. These path components are basic open sets in $lpc(X)$, making U a union of basic open sets in $lpc(X)$. Thus $U \in \tau \implies U \in \tau'$ and the topology of $lpc(X)$ is finer than the topology of X. This is equivalent to the continuity of the identity function $id : lpc(X) \to X$ since all open sets U are then pulled back to open sets in $lpc(X)$ and U is the union of its path components. □

Theorem 4.1.14. *[Bra14a] Let Y be a locally path-connected space and let $f : Y \to X$ be continuous. Then $f : Y \to lpc(X)$ is also continuous.*

Proof. Let C be the path component of an open set U in X. Let $y \in Y$ such that $f(y) \in C$. As U is open and f is continuous, there is some open set V in Y with $y \in V$ and $f(V) \subseteq U$. As

Y is locally path-connected, there exists some path-connected open set W with $y \in W \subseteq V$. For each $w \in W$ we have a path γ from y to w. Then $f \circ \gamma : [0,1] \to f(W) \subseteq f(V) \subseteq U$ is a path from $f(y)$ to $f(w)$. As C is the path component of $f(y)$, we have $f(w) \in C$ and $f(W) \subseteq C$. Thus we have shown for an arbitrary $y \in Y$ and basic open neighborhood C of $f(y)$ in $lpc(X)$ there exists open set W containing y such that $f(W) \subseteq C$, making f continuous. □

Corollary 4.1.15. *[Bra14a] Let Y be a locally path-connected space. Then the continuous identity function* $id : lpc(X) \to X$ *induces a bijection* η *between hom sets* $\eta : \mathbf{Top}(Y, lpc(X)) \to \mathbf{Top}(Y, X)$.

Proof. If $\eta(f) = \eta(g)$ then $id \circ f = id \circ g$ and $f = g$. If $h : Y \to X$ is continuous, then by the above theorem, $\hat{h} : Y \to lpc(X)$ is continuous and $\eta(\hat{h}) = id \circ h = h$. □

Theorem 4.1.16. *[Bra14a] Let X be a topological space. Then* $lpc(X)$ *is locally path-connected. We have* $X = lpc(X)$ *if and only if X is locally path-connected.*

Proof. Let $x \in lpc(X)$ and let C be a basic open neighborhood in $lpc(X)$ with $x \in C$. Then C is a path component of some open set U in X Note that although C is path-connected by definition as a subspace of X, we must prove C is path-connected as a subspace of $lpc(X)$. Let $x, y \in C$. Then there exists some path $\gamma : [0,1] \to X$ satisfying $\gamma([0,1]) \subseteq C$ with $\gamma(0) = x$, $\gamma(1) = y$. By the above argument, $\gamma : [0,1] \to lpc(X)$ is continuous, and $\gamma([0,1]) \subseteq C$, so there is a path from x to y in $C \subseteq lpc(X)$. Thus $lpc(X)$ is locally path-connected foll all X.

Thus if $X = lpc(X)$ we know X must be locally path-connected. If X is locally path-connected, we can apply theorem 4.1.14 to the identify function $id : X \to X$ to conclude $id : X \to lpc(X)$ is continuous. As $id : lpc(X) \to X$ is also continuous. Therefore the topologies on X and $lpc(X)$ are each finer than the other, and thus equal. □

Theorem 4.1.17. *[Bra14a] The construction* $lpc(X)$ *is functorial and lpc is right adjoint to the inclusion functor.*

Proof. To establish the functoriality of lpc, a rule of assignment on morphisms is required. Note that if $f : X \to Y$ is continuous then $f \circ id : lpc(X) \to Y$ is continuous and since $lpc(X)$

is locally path-connected, by theorem 4.1.14 the function $lpc(f) : lpc(X) \to lpc(Y)$ defined by $lpc(f)(x) = f(x)$ is continuous. Thus we send f to $lpc(f)$. This assignment respects morphism composition, since as a rule of assignment we have

$$lpc(g) \circ lpc(f)(x) = lpc(g) \circ (id \circ f)(x) = lpc(g)(f(x)) = id \circ g(f(x)) = id \circ (g \circ f)(x) = lpc(g \circ f)(x).$$

Since the rule of assignment is unchanged, we also have that lpc preserves identities, and $id_X = id_{lcp(X)}$.

We need to establish a natural bijection

$$\mathbf{lcpTop}(Y, lcp(X)) \cong \mathbf{Top}(i(Y), X) = \mathbf{Top}(Y, X).$$

Let $X \to X'$ be a continuous map between topological spaces. Fix a locally path-connected space Y. Then the following square commutes:

$$\begin{array}{ccc} \mathbf{lpcTop}[Y, lpc(X)] & \xrightarrow{lpc(f)\circ -} & \mathbf{lpcTop}[Y, lpc(X')] \\ \downarrow \eta & & \downarrow \eta \\ \mathbf{Top}[Y, X] & \xrightarrow[f\circ -]{} & \mathbf{Top}[Y, X'] \end{array}$$

since for $\psi : Y \to lpc(X)$ we have

$$f \circ (\eta(\psi)) = f \circ (id \circ \psi) = f \circ \psi = lpc(f) \circ \psi = id \circ (lpc(f) \circ \psi) = \eta(lpc(f) \circ \psi)$$

Now fix a topological space X and let $g : Y \to Y'$ be a map between locally path-connected spaces. Then for $\psi : Y' \to lpc(X)$ the following square commutes

$$\begin{array}{ccc} \mathbf{lpcTop}[Y', lpc(X)] & \xrightarrow{-\circ g} & \mathbf{lpcTop}[Y, lpc(X)] \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\ \mathbf{Top}[Y', X] & \xrightarrow[-\circ g]{} & \mathbf{Top}[Y, X] \end{array}$$

since we have

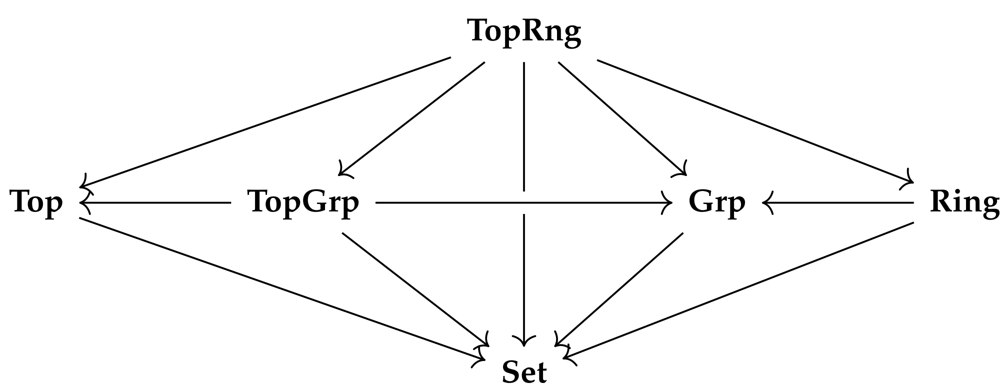
$$\eta(\psi) \circ g = (id \circ \psi) \circ g = \psi \circ g = id \circ (\psi \circ g) = \eta(\psi \circ g).$$

Thus the bijection η is natural in X and Y and defines an adjunction $i \dashv lpc$. $\square$

We have shown that lpc defines a coreflective subcategory of **Top**. Therefore by Proposition 4.1.6, a morphism $p : X \to Y$ is an $lpc-$Covering morphism if and only if it is a $\pi_1|_{lpc}-$lifting morphism (since $H(lpc) = lpc$).

4.2 Topological Groups and Rings

Consider the following diagram of forgetful functors:



We wish to characterize the quotient morphisms relative to the functors indicated above. First we recall some facts about the relevant categories.

Definition 4.2.1. [Mun00][145] A *topological group* is a topological space G equipped with a group operation $\mu : G \times G \to G$ which is continuous when $G \times G$ is equipped with the product topology. We also require $i : G \to G$ given by $i(g) = g^{-1}$ to be continuous.

Definition 4.2.2. [War93][1] A *topological ring* is a topological space R equipped with continuous addition $+ : R \times R \to R$ and multiplication $\cdot : R \times R \to R$ operations and continuous multiplicative inverse $i : R \to R$ such that $+$ and $\cdot$ satisfy the axioms of a ring.

It is well-known that **TopGrp** and **TopRng** have all limits and colimits, hence have all pullbacks. [Pro21b].

Proposition 4.2.1. *Let* U : **Grp** $\to$ **Set** *be the forgetful functor. The morphisms quotient rel.* U *are precisely the epimorphisms.*

Proof. Since **Grp** and **Set** have pullbacks and every epimorphisms in **Set** is regular, by Proposition 2.1.8.1, the quotient rel. U morphisms in **Grp** are regular epimorphisms. All epimorphisms in the category of groups are regular epimorphisms [Mac71][21], so the quotient rel. U morphisms are epimorphisms. Since U : **Grp** $\to$ **Set** has a left adjoint [21a], U preserves pullbacks, and the regular epimorphisms (i.e., all epimorphisms) in **Grp** are quotient rel. U by Proposition 2.1.8.1. □

Proposition 4.2.2. *Let* U : **Ring** $\to$ **Set** *be the forgetful functor. The surjective ring homomorphisms are precisely the quotient rel.* U *morphisms.*

Proof. Note that **Ring** has all limits [21b] as does Set. Since U : **Ring** $\to$ **Set** has a left adjoint [MS21], U is right adjoint and preserves limits, so U preserves pullbacks and the quotient rel. U morphisms in **Ring** are regular epimorphisms (i.e., surjective ring homomorphisms) by Proposition 2.1.8.1. Since U is faithful, regular epimorphisms in **Ring** are quotient rel. U by Proposition 2.1.8.2. Thus the quotient rel. U morphisms are precisely the surjective ring homomorphisms. □

Recall that by Proposition 2.1.2 the quotient morphisms relative U : **Top** $\to$ **Set** are the topological quotient maps and by Proposition 1.3.23 these are precisely the regular epimorphisms in **Top**.

Proposition 4.2.3. *Let U :* **TopGrp** $\to$ **Set** *be the forgetful functor. Then the morphisms quotient rel. U are precisely the surjective open homomorphisms.*

Proof. Since **TopGrp** and **Set** both have pullbacks, every epimorphism in **Set** is a regular epimorphism, and U : **TopGrp** $\to$ **Set** is faithful, by Proposition 2.1.8.2 every regular epimorphism in **TopGrp** is qoutient rel. U. Since the regular epimorphisms in **TopGrp** are precisely the surjective open maps [LH17], the surjective open maps are quotient rel. U morphisms. If we view U : **TopGrp** $\to$ **Set** as the composite $V \circ W$ where W : **TopGrp** $\to$ **Top** and V : **Top** $\to$ **Set** are the functors forgetting group structures and topologies respectively, then since both V and W have left adjoints [Pro21a], so does $U = V \circ W$ [Mac71][101]. Therefore, U preserves all limits and pullbacks in particular, so by Proposition 2.1.8.1, The quotient rel. U morphisms in **TopGrp** are regular epimorphisms, thus are precisely the surjective open maps. □

Proposition 4.2.4. *Let U :* **TopRng** $\to$ **Set** *be the forgetful functor. Then the surjective open maps are precisely the quotient rel. U morphisms.*

Proof. The category **TopRng** has pullbacks [Pro21b] as does **Set**. Note that all epimorphisms in **Set** are effective and since **Set** has all kernel pairs, by Proposition 1.3.29, the effective, strict, and regular epimorphisms coincide in **Set**. Therefore by Proposition 2.1.8.2, the regular epimorphisms in **TopRng** are quotient rel. U. The regular epimorphisms in **TopRng** are precisely the surjective open maps [Usp89].

The categories **TopRng** and **Set** both have pullbacks and all epimorphisms in **Set** are regular. Since V : **TopRng** $\to$ **Top** and W : **Top** $\to$ **Set** both have left adjoints, $W \circ V = U$ has a left adjoint and therefore U preserves pullbacks. So by Proposition

2.1.8.1, the quotient morphisms rel. U are regular epimorphisms, thus surjective open maps in **TopRng**. □

Proposition 4.2.5. *Let U : **Ring** $\to$ **Grp** be the forgetful functor. Then the surjective ring homomorphisms are precisely the morphisms quotient rel. U.*

Proof. Every epimorphism in **Grp** is regular and **Ring** and **Grp** have pullbacks. We may write $U = W \circ V$ where V : **Ring** $\to$ **Ab** and W : **Ab** $\to$ **Grp**. Both W and V have left adjoints [21a] [21b], so U has a left adjoint [Mac71][101]. Therefore, U preserves pullbacks, and so by Proposition 2.1.8.1, the morphisms quotient rel. U are regular epimorphisms (surjective ring homomorphisms) in **Ring**. Since U is faithful, by Proposition 2.1.8.2, the regular epimorphisms in **Ring** are quotient rel. U. □

Proposition 4.2.6. *Let U : **TopGrp** $\to$ **Grp** be the forgetful functor. The surjective open maps in **TopGrp** are precisely the morphisms quotient relative to U.*

Proof. All epimorphisms in the category of groups are regular epimorphisms [Mac71][21]. Since **TopGrp** and **Grp** both have pullbacks and U is faithful, again by Proposition 2.1.8.2, all regular epimorphisms in **TopGrp** are quotient relative U, thus the surjective open maps in **TopGrp** are quotient relative U. Since U has a left adjoint [Pro21a], U preserves pullbacks and the quotient rel. U morphisms are surjective open maps by Proposition 2.1.8.1. □

Proposition 4.2.7. *Let U : **TopGrp** $\to$ **Top** be the forgetful functor. The morphisms quotient rel. U are precisely the continuous surjective homomorphisms.*

Proof. By definition, quotient rel. U morphisms are epimorphisms, which are surjective in **TopGrp**. Suppose $q : G \to H$ is a continuous surjective group homomorphism. Suppose we have a continuous function $k : H \to K$ such that $k \circ q = g$. We show k is a group homomorphism. Let $h_1 h_2 = h_3$ in H. Then since q is a surjecive homomorphism there exist

x_1, x_2 and x_3 in G such that $x_1x_2 = x_3$ and

$$k(h_1h_2) = k(h_3) = k(q(x_3)) = g(x_3) = g(x_1x_2) = g(x_1)g(x_2) = k(q(x_1)k(q(x_2) = k(h_1)k(h_2),$$

and k is a (continuous) group homomorphism. Letting $f = k$ we then have $Uf = k$ uniquely since U is faithful, and q is U-quotient. □

Proposition 4.2.8. *Let U : **TopRng** → **Top** be the forgetful functor. The morphisms quotient rel. U are precisely the surjective ring homomorpisms.*

Note that if $q : R \to S$ is U-quotient then Uq is an epimorphism in **Top**, hence a surjection, making q surjective in **TopRng** since U does not change the rule of assignment of q. Suppose $q : R \to S$ is a surjective ring homomorphism. Let $g : R \to T$ be a continuous ring homomorphism and $k : S \to T$ be a continuous function such that $k \circ q = g$. Then by the same argument used in the case of U : **TopGrp** → **Top**, we can show that k preserves ring operations. Since $k(1_S) = k(q(1_R) = g(1_R) = 1_T$, k preserves multiplicative units. Therefore k is a ring homomorphism and we again set $f = k$ so that $Uf = k$. Again by the faithfulness of U, f uniquely satisfies $Uf = k$ and clearly $k \circ q = g$, making q quotient rel. U.

Proposition 4.2.9. *Let U : **TopRng** → **Ab** be the forgetful functor. Then the morphisms quotient rel. U are precisely the surjective open ring homomorphisms.*

Proof. Note that V : **TopRng** → **Ring** preserves limits [Pro21c]. Then since W : **Ring** → **Ab** has a left adjoint [21b], $W \circ V = U$: **TopRng** → **Ab** preserves limits and therefore pullbacks. Since all epimorphisms in **Ab** are regular epimorphisms, the quotient morphisms rel. U are regular epimorphisms in **TopRng**, i.e., surjective open maps by Proposition 2.1.8.1. Since U is faithful, by Proposition 2.1.8.2, the surjective open maps in **TopRng** are precisely the morphisms quotient rel. U. □

Definition 4.2.3. Let I be an indexing set, $\{Y_i\}_{i \in I}$ a family of topological spaces indexed by I, and $\{f_i : X \to Y_i\}_{i \in I}$ be a family of functions indexed by I. The *initial topology* on X is the coarsest topology such that f_i is continuous for all $i \in I$. Note this is dual to Definition 1.2.3. The initial topology with respect to $\{f_i\}_{i \in I}$ has the following characteristic (but not universal) property: For a topological space Z a function $g : Z \to X$ is continuous if and only if $f_i \circ g$ is continous for all $i \in I$.

Lemma 4.2.10. *Let R be a ring, $\{S_i\}_{i \in I}$ a family of topological rings, and $\{f_i : R \to S_i\}$ a family of functions. If R is given the initial topology with respect to $\{f_i\}_{i \in I}$, then R is a topological ring.*

Proof. Let $p : R \times R \to R$ and $p_i : S_i \times S_i \to S_i$ define addition in R and S_i respectively. Consider the following square where products are given the product topology:

$$\begin{array}{ccc} R \times R & \xrightarrow{f_i \times f_i} & S_i \times S_i \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p_i} \\ R & \xrightarrow[f_i]{} & S_i \end{array}$$

Let $(r_1, r_2) \in R \times R$. Then $p_i \circ (f_i \times f_i)(r_1, r_2) = p_i(f_i(r_1), f_i(r_2)) = f_i(r_1) + f_i(r_2) = f_i(r_1 + r_2) = f_i \circ p(r_1, r_2)$. Since a products and compositions of continuous functions are continuous, $f_i \circ p$ is continuous for all $i \in I$, making p continuous by the characteristic property of the initial topology. Exactly the same argument holds for multiplication since the f_i are continuous ring homomorphisms (as R has the initial topology). Therefore, addition and multiplication in R are continuous. Since $f_i(-r) = -f_i(r)$ for all $r \in R$ and f_i a similar argument shows that the additive inverse in R is continuous. Therefore, R with the initial topology is a topological ring. □

Proposition 4.2.11. *Let U : **TopRng** $\to$ **Ring** be the forgetful functor. Then $q : R \to S$ is U-quotient in **TopRng** if and only if the topology on S is the finest topology on the underlying ring of S such that q is continuous and S is a topological ring.*

Proof. Let R be a topological ring and let $q : R \to S$ be a morphism of underlying rings. Let $\mathcal{T}$ be the collection of topologies on S such that q is continuous and S is a topological ring.

Note $\mathcal{T}$ is nonempty, since it contains the indiscrete topology. Give S the initial topology $\tau_{\rangle\backslash}$ with respect to the family of homomorphisms $\{id_\tau : S \to (S, \tau)\}$ where τ ranges over $\mathcal{T}$. Since $q = id_\tau \circ q$ for all $\tau \in \mathcal{T}$, by the universal property of the initial topology, q is continuous and by the previous lemma, S is a topological ring. In fact, we have shown the initial topology on S is the finest topology for which q is continuous and S is a topological ring. We show q is U-quotient if and only if $S = (S, \tau_{\rangle\backslash})$ as a topological ring.

We show $q : R \to (S, \tau')$ is U-quotient if and only if the identity $id : (S, \tau') \to (S, \tau_{in})$ is a homeomorphism.

Suppose $q : R \to S$ is U-quotient where S has topology τ'. Then since q is in **TopRng**, $\tau' \in \mathcal{T}$. It follows that $\tau' \subseteq \tau_{in}$ since τ_{in} is the finest topology in $\mathcal{T}$. Therefore, $id : (S, \tau_{in}) \to (S, \tau')$ is continuous. Now $q : R - - > (S, \tau_{in})$ is continuous by the first paragraph and $id \circ q = q$ in **Ring**. Since q is U-quotient, the identity $id : (S, \tau') \to (S, \tau_{in})$ is continuous.

Conversely, suppose $id : (S, \tau') \to (S, \tau_{in})$ is a homeomorphism. It suffices to show $q : R \to (T, \tau_{in})$ is U-quotient. Let $g : R \to T$ be a continuous ring homomorphism satisfying $g = k \circ q$ in **Ring**. We must show $k : (S, \tau_{in}) \to T$ is continuous. Let τ_k be the initial topology on S with respect to $k : S \to T$. We check that τ_k is in $\mathcal{T}$. By the previous lemma, (S, τ_k) is a topological ring. If U is open in (S, τ_k), then $U = k^{-1}(V)$ for some open V in T. Then $q^{-1}(U) = q^{-1}(k^{-1}(V)) = g^{-1}(V)$ is open since g is continuous. Thus $q : R \to (S, \tau_k)$ is continuous, proving $\tau_k \in \mathcal{T}$. We now conclude that τ_{in} is finer than τ_k (by the universal property of the initial topology). Thus $id : (S, \tau_{in}) \to (S, \tau_k)$ is continuous. By construction, $k : (S, \tau_k) \to T$ is continuous. Thus the composition $k : (S, \tau_{in}) \to T$ is continuous. □

Proposition 4.2.12. *Let U : **TopRng** $\to$ **TopAb** be the forgetful functor. The U-quotient morphisms are precisely the continuous surjective ring homomorphisms.*

Proof. Note the epimorphisms in **TopGrp** are the continuous surjections [Ard69]. Therefore the epimorphisms in **TopAb** are the continuous surjections. Let $f : R \to S$ and $g : R \to T$ be continuous ring homomorphisms with f a surjection. Suppose $k : S \to T$ is a continuous abelian group homomorphism such that $k \circ f = g$. We show that k preserves ring multiplication, making it a morphism in **TopRing**. Let $a, b \in S$. Since f is surjective, we have $f(x) = a$ and $f(y) = b$ for some $x, y \in R$. Then

$$k(ab) = k(f(x)f(y)) = k(f(xy)) = g(xy) = g(x)g(y) = kf(x)kf(y) = k(a)k(b).$$

Here we have used the fact that f and g are *ring* homomorphisms by assumption, and therefore preserve ring multiplication. Thus k is also a ring homomorphism, making f quotient rel. U. Now if $q : R \to S$ is U-quotient, the continuous homomorphism of abelian groups $U(q)$ is an epimorphism, hence a surjection. Therefore, q is a surjection. $\square$

Our results are summarized in the following table:

Forgetful Functor U	Morphisms Quotient rel. U
Grp $\to$ **Set**	Surjective group homomorphisms
Ring $\to$ **Set**	Surjective ring homomorphisms
Top $\to$ **Set**	Topological quotient maps
TopGrp $\to$ **Set**	Surjective open cont. group homomorphisms
TopRng $\to$ **Set**	Surjective open cont. ring homomorphisms
Ring $\to$ **Grp**	Surjective ring homomorphisms
TopGrp $\to$ **Grp**	Surjective open cont. group homomorphisms
TopGrp $\to$ **Top**	Surjective continuous group homomorphisms
TopRng $\to$ **Ab**	Surjective open cont. ring homomorphisms
TopRng $\to$ **Ring**	Codomain has the initial topology τ_{in}
TopRng $\to$ **Top**	Surjective continuous ring homomorphisms
TopRng $\to$ **TopAb**	Surjective continuous ring homomorphisms

Table 4.1: Quotient Morphisms rel. Forgetful Functors

4.3 Galois Theory

We now show how unique lifting morphisms relative to a functor can be used to partially characterize a Galois correspondence.

Definition 4.3.1. [nLa22f] A *partial order* on a set S is a binary relation $\leq$ satisfying the following conditions:

- Reflexivity: $x \leq x$ for all $x \in S$
- Transitivity: $x \leq y \leq z \implies x \leq z$ for all $x, y, z \in S$
- Antisymmetry: $x \leq y \leq x \implies x = y$ for all $x, y \in S$.

A *poset* is a set equipped with a partial order.

Remark 4.3.1. A poset may be viewed as a category with at most one morphism between objects. In this context reflexivity amounts to the existence of identity morphisms and transitivity to composability of morphisms. Antisymmetry states that the only isomorphisms in the category are the identities.

Definition 4.3.2. [nLa22d] Let P be a poset and let $x, y \in P$. The *meet*, *infimum*, or *greatest lower bound* of x and y, written $x \wedge y$, is an element of P satisfing $x \wedge y \leq x$ and $x \wedge y \leq y$ and for any $z \in P$ also satisfying $z \leq x$ and $z \leq y$ we have $z \leq x \wedge y$.

Definition 4.3.3. [nLa22e] Let P be a poset and let $x, y \in P$. The *join*, *supremum*, or *least upper bound* of x and y, written $x \vee y$, is an element of P satisfing $x \leq x \vee y$ and $y \leq x \vee y$ and for any $z \in P$ also satisfying $x \leq z$ and $y \leq z$ we have $x \vee y \leq z$.

Definition 4.3.4. A *lattice* is a partially ordered set P such that for each $x, y \in P$ there exists $x \wedge y$ and $x \vee y$ in P.

Definition 4.3.5. A *field extension* of a field F is a field E such that $F \subseteq E$.

Definition 4.3.6. [nLa22b] For posets A and B a *Galois connection* between A and B is a pair of order reversing functions $f : A \to B$ and $g : B \to A$ such that $a \leq g(f(a))$ and $b \leq f(g(b))$ for all $a \in A$ and $b \in B$. A *Galois correspondence* is a Galois connection such that $a = g(f(a))$ and $b = f(g(b))$ for all $a \in A$ and $b \in B$.

Let E be an arbitrary field extension of the base field k. Let $\mathbf{Sub}(E, k)$ be the subcategory of the category of fields having as objects the subextensions of E (that is, fields F satisfying $k \subseteq F \subseteq E$) and as morphisms the inclusion field homomorphisms. Let $\mathbf{Gal}(E, k)$ be the subcategory of the category of groups having as objects the subgroups of the group $G(E/k)$ of field automorphisms $\sigma : E \to E$ that fix the base field k and as morphisms the inclusion homomorphisms. Let $f : E_1 \to E_2$ be a morphism in $\mathbf{Sub}(E, k)$. Let $A(F)$ denote the group of field automorphisms $\sigma : E \to E$ that fix the subfield F satisfying $k \subseteq F \subseteq E$. Define the contravariant functor $A : \mathbf{Sub}(E, K) \to \mathbf{Gal}(E, k)$ by the assignment

$$E_1 \hookrightarrow E_2$$

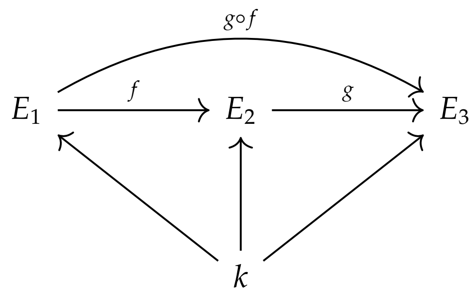
$$A(E_1) \hookleftarrow A(E_2)$$

where the inclusion $f : E_1 \to E_2$ is taken to the inclusion $g : A(E_2) \to A(E_1)$.

Proposition 4.3.2. *As defined above, $A : \mathbf{Sub}(E, k) \to \mathbf{Gal}(E, k)$ is indeed a functor.*

Proof. Let E_1 and E_2 be subextensions of the field extension E/k. If $E_1 \subseteq E_2$ then an automorphism $\tau : E \to E$ fixing E_2 must also fix E_1, so $\tau \in A(E_2) \implies \tau \in A(E_1)$ and we may therefore associate the inclusion field homomorphisms $f : E_1 \to E_2$ to the inclusion group homomorhism $A(f) = g : A(E_2) \to A(E_1)$.

If we are given composable inclusions f and g as in the followind diagram



then a composition of inclusions is an inclusion and inclusions are uniquely defined by the ordered pair (D, C) of their domain and codomain. Therefore, $A(g \circ f) = A(g) \circ A(f)$, and A respects morphism composition.

If we consider the identity homomorphism $1_F : F \to F$, then $A(1_F)$ is the inclusion $A(F) \to A(F)$, or $1_{A(F)}$, and A respects identity morphisms. Thus A defines a functor. □

Definition 4.3.7. Consider a field extensions E_2/E_1 so that $E_1 \subseteq E_2 \subseteq E$. Then we have a morphism $E_1 \to E_2$ in $\mathbf{Sub}(E, k)$. The extension E_2/E_1 is said to be *downward complete* if for any field extension E_2/E_3 with $A(E_1) \subseteq A(E_3)$ we have $E_3 \subseteq E_1$.

Remark 4.3.3. Note that for an arbitrary field extension E/k the objects of $\mathbf{Gal}(E/k)$ need not bijectively correspond to subextensions of E. There may be objects H_1 and H_2 in $\mathbf{Gal}(E/k)$ with the same associated fixed field (those elements of E fixed by all automorphisms in H_i). Likewise, there may be distinct fields E_1 and E_2 in $\mathbf{Sub}(E, k)$ such that $A(E_1) = A(E_2)$. Take for example the field of real numbers, $\mathbb{R}$. For any subfield $F \subseteq \mathbb{R}$, we have $A(\mathbb{R}) = \{id_{\mathbb{R}}\} = A(F)$, since the identity automorphism is the only automorphism of $\mathbb{R}$.

Definition 4.3.8. A field extension E/k is called *Galois* if there exists a Galois correspondence between the lattice of subfields of E above k and the subgroup lattice of $\mathbf{Gal}(E, k)$.

Proposition 4.3.4. *If the extension E/k is Galois, then every morphism in* $\mathbf{Sub}(E, k)$ *is an A-lifting morphism.*

Proof. Suppose E/k is a Galois extension. Then $A : \mathbf{Sub}(E, k) \to \mathbf{Gal}(E, k)$ is an isomorphism of categories [Bow13]. Thus there is a bijective correspondence between subextensions of E and subgroups of $Aut(E/k)$ and a bijective correspondence between inclusions of subextensions and inclusions of subgroups. Let $f : E_1 \to E_2$ and $g : E_3 \to E_2$ be morphisms in $\mathbf{Sub}(E, k)$. If there exists $k : A(E_1) \to A(E_3)$ such that $k \circ Af = Ag$, then k must have a preimage h under A since A is a full functor. We must have $h : E_3 \to E_1$ since A is bijective on objects. Thus there exists $h : E_3 \to E_1$ such that $A(h) = k$ and f is A-lifting. □

Remark 4.3.5. When the inclusion associated to an extension is a unique A-lifting morphism, we do not necessarily know that the extension is Galois. We do however have some measure of the degree to which a Galois correspondence fails. We know that if $f : E_1 \to E_2$ is a unique A-lifting morphism, then for any subextension E_3 inluded in E_2, if $A(E_1) \subseteq A(E_3)$ then it must be the case that $E_3 \subseteq E_1$. That is, we know the extension E_2/E_1 is downward complete. There may still be a subfield E_3' distinct from E_3 with $A(E_3) = A(E_3')$, but we do know that E_3' must be a subextension of E_1. Thus we preclude *order ignoring* failures of bijectivity on objects, as for example happens in the case of the extension $\mathbb{R}/\mathbb{Q}$.

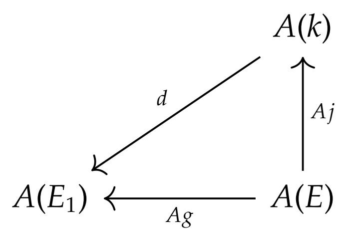
Proposition 4.3.6. *Suppose $f : E_1 \to E_2$ and $g : E_3 \to E_2$ are both unique A-lifting morphisms in $\mathbf{Sub}(E, k)$. Then $E_1 = E_3$ if $A(E_1) = A(E_3)$.*

Proof. Let f, g be A-lifting morphisms and suppose $A(E_1) = A(E_3)$. Then we have the identity $k : A(E_1) \to A(E_3)$. If a triangle can be formed in $\mathbf{Gal}(E/k)$ then it commutes, since the morphisms are inclusions, so we have $k \circ A(f) = A(g)$. Since f is an A-lifting morphism, there must exist a (necessarily unique) morphism $h : E_3 \to E_1$, which will always satisfy $f \circ h = g$ since morphisms in $\mathbf{Sub}(E/k)$ are also inclusions. The same argument using the fact that g is A-lifting shows there exists $h' : E_1 \to E_3$. Thus $E_1 = E_3$ and indeed, $f = g$. □

Lemma 4.3.7. *For any field extension E/k the inclusions $j : k \to E$ and $id_E : E \to E$ are A-lifting.*

Proof. Note that any identity morphism is a unique lifting relative to any functor, since funtors take identities to identities. For $j : k \to E$, suppose we have $g : E_1 \to E$ and

$d : A(E) \to A(E_1)$ such that $d \circ Aj = Ag$.



Since $A(k)$ is the supremum of the subgroup lattice, d must be the identity and $A(E_1) = A(k)$. Thus $Aj = Ag$. Since there is at most one morphism between any two objects in $\mathbf{Sub}(E, k)$ we must have $j = g$, so that $E_1 = k$ and id_k satisfies $f \circ id_k = f$ with $A(id_k) = d$. □

Proposition 4.3.8. *Let* $\mathbf{Al}(E, k)$ *be the sublattice of* $\mathbf{Sub}(E, k)$ *consisting of those objects* E_1 *for which* $f : E_1 \to E$ *is A-lifting and all field inclusions. Then the restriction functor* $A : \mathbf{Al}(E, k) \to \mathbf{Gal}(E, k)$ *is injective on objects.*

Proof. Note that not all morphisms in $\mathbf{Al}(E, k)$ need be A-lifting. Note also that $\mathbf{Al}(E, k)$ does form a lattice, since $id_E : E \to E$ and $i : k \to E$ are both A-lifting, so that $\mathbf{Al}(E, k)$ has a supremum and infimum. If $A(E_1) = A(E_2)$ then consider the inclusions $f : E_1 \to E$ and $g : E_2 \to E$. Since both of these are A-lifting, $E_1 = E_2$ by the above proposition. □

Remark 4.3.9. The above result shows that A-lifting morphisms with the top field extension E as codomain correspond injectively to subgroups H of $\mathbf{Gal}(E/k)$. Thus the A-lifting morphisms pick out those subextensions for which there is an (at least injective) Galois correspondence. This is in analogy with the *lpc*-covering morphisms of Section 4.1.2. For the topological space X, a generalized covering map $p : Y \to X$ is determined uniquely by the associated subgroup of $\pi_1(X, x)$. Thus based generalized covering maps $p : Y \to X$ injectively correspond to the $\pi_1(X, x)$ lattice of subgroups.

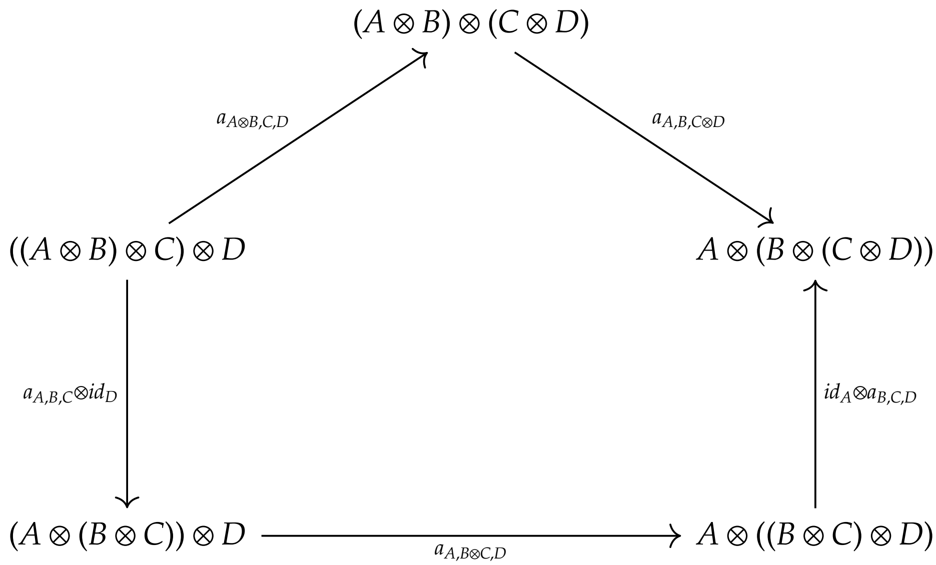
Chapter 5

Monoidal Categories and Tensor Products of F-Quotients

5.1 Tensor Products and F-Quotient Morphisms

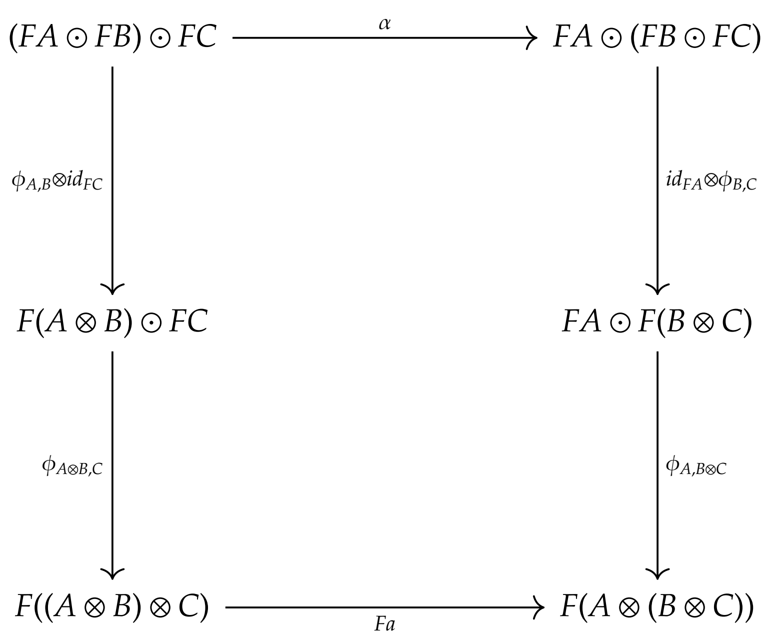
Definition 5.1.1. [JS86] A *monoidal category* $\mathcal{C} = (\mathcal{C}_0, \otimes, I, r, \ell, a)$ is a category $\mathcal{C}_0$ equipped with a bifunctor $\otimes : \mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{C}_0$, called the *tensor product*, a distinguished unit object, or *unitor*, I, having associated natural isomorphisms r and ℓ, and an *associator* natural isomorphism a. These enjoy the following properties: for all objects A, B, C, D in $\mathcal{C}_0$, the following diagrams commute:

$$
\begin{array}{ccc}
(A \otimes I) \otimes C & \xrightarrow{\quad a_{AIC} \quad} & A \otimes (I \otimes C) \\
& {\scriptstyle r \otimes id_I} \searrow \qquad \swarrow {\scriptstyle id_I \otimes \ell} & \\
& A \otimes C &
\end{array}
\tag{5.1}
$$



(5.2)

Definition 5.1.2. [Mac71] Let $\mathcal{C} = (\mathcal{C}_0, \otimes, I_{\mathcal{C}_0}, r, \ell, a)$ and $\mathcal{D} = (\mathcal{D}_0, \odot, I_{\mathcal{D}_0}, \rho, \lambda, \alpha)$ be monoidal categories. A *monoidal functor* is a functor $F : \mathcal{C} \to \mathcal{D}$ equipped with a natural transformation $\Phi : F \circ \odot \Rightarrow \otimes \circ F$ of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$ having components $\phi_{A,B} : FA \odot FB \to F(A \otimes B)$ and morphism $\phi : I_{\mathcal{D}} \to FI_{\mathcal{C}}$ such that the following diagrams to commute:



$$\begin{array}{ccc}
FA \odot I_{\mathcal{D}} & \xrightarrow{id_{FA} \odot \phi} & FA \odot FI_{C} \\
\Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle \phi_{A,I_C}} \\
FA & \xleftarrow[Fr]{} & F(A \otimes I_C)
\end{array}$$

$$\begin{array}{ccc}
I_{\mathcal{D}} \odot FB & \xrightarrow{\phi \odot id_{FB}} & FI_{C} \odot FB \\
\Big\downarrow{\scriptstyle \lambda} & & \Big\downarrow{\scriptstyle \phi_{I_C,B}} \\
FB & \xrightarrow[Fl]{} & F(I_C \otimes B)
\end{array}$$

Definition 5.1.3. A monoidal functor $F : C \to \mathcal{D}$ is said to be a *strong* monoidal functor if the natural transformation $\Phi : F \circ \odot \Rightarrow \otimes \circ F$ is a natural isomorphism, so that the component $\Phi_{A,B} : F(A \odot B) \to F(A) \otimes F(B)$ is an isomorphism for all A, B in C.

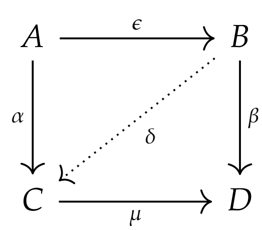
Definition 5.1.4. [Mac71] A monoidal category C is said to be *closed* if for every object A in C the functor $- \otimes A : C \to C$ has a right adjoint written $C(A, -) : C \to C$. We call $C(A, -)$ the *internal hom functor* associated to A. This is in analogy with the situation in **Set** where the usual hom functor is right adjoint to the cartesian product. The object $C(A, B)$ is called an *exponential object*. The counit of the adjunction $- \otimes A \dashv C(A, -)$ at the object B is known as the *evaluation* morphism and written $\mathrm{eval} : C(A, B) \otimes A \to B$. In the case that $C = \mathbf{Set}$, for example, this will be the evaluation function for $a \in A$ and $f : A \to B$: $\mathrm{eval}(f, a) = f(a)$.

Remark 5.1.1. In a closed monoidal category C, when we have a morphism $f : X \otimes Y \to Z$ in $C(X \otimes Y, Z)$, we will write its adjoint in $C(Y, C(X, Z))$ as $\overline{f} : Y \to C(X, Z)$. Symmetrically, if we have already defined $g : Y \to C(X, Z)$, we will also write $\overline{g} : X \otimes Y \to Z$ for its adjoint. Thus $\overline{\overline{f}} = f$ in all cases.

Definition 5.1.5. [AHG09] A *terminal object* in category C is an object A in C such that for all objects B in C, there exists exactly one morphism $f : B \to A$.

Definition 5.1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be closed monoidal categories. Let $\epsilon_Z : \mathcal{C}(W, Z) \otimes W \to Z$ be the evaluation morphism at Z, that is, the component at Z of the counit natural transformation (i.e. $\overline{id_{\mathcal{C}(W,Z)}}$) relative to the tensor-internal hom adjunction in $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor with structure isomorphisms $\Psi_{W,Z} : FW \otimes FZ \to F(W \otimes Z)$. Then a *monoidal hom transform* is the unique morphism $\phi_{W,Z} : F(\mathcal{C}(W, Z)) \to \mathcal{D}(FW, FZ)$ which is the adjoint of $F(\epsilon) \circ \Psi_{\mathcal{C}(W,Z),W}$ for W, Z in $\mathcal{C}$, that is, $\phi_{W,Z} = \overline{F(\epsilon) \circ \Psi_{\mathcal{C}(W,Z),W}}$.

Definition 5.1.7. [AHG09][253] [21d] Dual to the notion of strong epimorphism (Definition 1.3.2) is the notion of *strong monomorphism*. Thus, $\mu : C \to D$ is a strong monomorphism if given an epimorphism $\epsilon : A \to B$ and morphisms $\alpha : A \to C$ and $\beta : B \to D$ with $\beta \circ \epsilon = \mu \circ \alpha$ there exists $\delta : B \to C$ satisfying $\delta \circ \epsilon = \alpha$ and $\mu \circ \delta = \beta$.



Lemma 5.1.2. *All monomorphisms in* **Set** *are strong monomorphisms.*

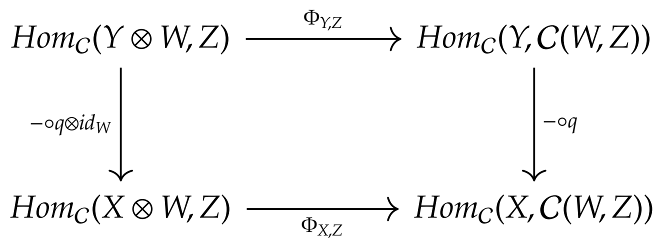
Proof. Let $\alpha, \beta, \mu, \epsilon$ be as above in **Set** and define $\delta : B \to C$ by $\delta(b) = \alpha(\epsilon^{-1}(b))$, which is well-defined since ϵ is an epi in **Set** and thus a surjection. Then $\delta(\epsilon(a)) = \alpha(\epsilon^{-1}(\epsilon(a))) = \alpha(a)$ and $\mu \circ \delta(b) = \mu(\alpha(\epsilon^{-1}(b)) = (\mu \circ \alpha)(\epsilon^{-1}(b)) = \beta \circ \epsilon(\epsilon^{-1}(b)) = \beta(b)$. □

We will need the following result for certain commuting triangles in closed monoidal categories.

Lemma 5.1.3. *Let W be an object in the closed monoidal category $(\mathcal{C}, \otimes)$ and $q : X \to Y$ a morphism in $\mathcal{C}$. Then for any morphism $g : Y \otimes W \to Z$ we have $\overline{g} \circ q = \overline{g \circ (q \otimes id_W)}$.*

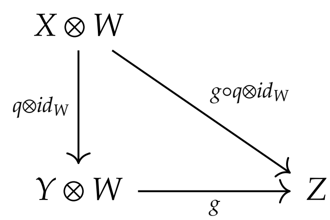
Proof. Let $\Phi_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A \otimes W, B) \to \mathrm{Hom}_{\mathcal{C}}(A, \mathcal{C}(W, B)$ be the hom set bijection natural in A and B associated with the adjunction $(- \otimes W) \dashv \mathcal{C}(W, -)$, where we write $\Phi_{A,B}(g) = \overline{g}$.

Then naturality in the first variable implies that for each fixed Z in $\mathcal{C}$ the following triangle commutes for all $q : X \to Y$ in $\mathcal{C}$:

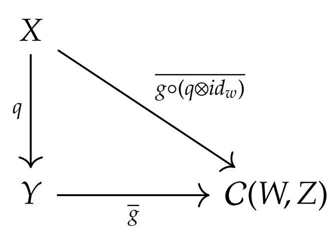


Therefore, if $g : Y \otimes W \to Z$, then $\overline{g} \circ q = \overline{g \circ (q \otimes id_W)}$. □

We interpret the previous lemma as follows: given the following triangle commutes



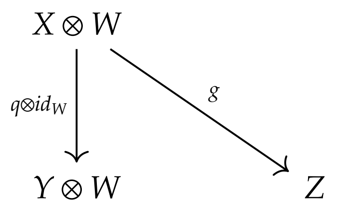
the naturality of Φ implies the following triangle also commutes.

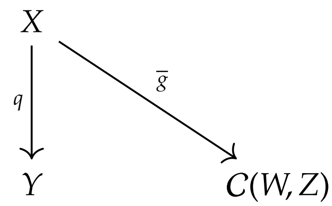


Therefore, for each fixed W and $q : X \to Y$ in $\mathcal{C}$, whenever we pre-compose a morphism g with $q \otimes id_W$ we have a commuting triangle $\overline{g} \circ q = \overline{g \circ (q \otimes id_w)}$ and vice versa. We refer to the last two diagrams as representing *adjoint triangles*. We refer to passing back and forth between triangles of this form as *taking adjoint triangles*.

Proposition 5.1.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor between closed monoidal categories. Let $q : X \to Y$ be F-quotient in $\mathcal{C}$ and W an object in $\mathcal{C}$. If for all Z in $\mathcal{C}$, the monoidal hom transform $\phi_{W,Z}$ is a strong monomorphism, then $q \otimes id_W$ is F-quotient.*

Proof. Let $q : X \to Y$ be quotient rel. $F : \mathcal{C} \to \mathcal{D}$. Fix W in $\mathcal{C}$. Consider the following morphisms in $\mathcal{C}$:

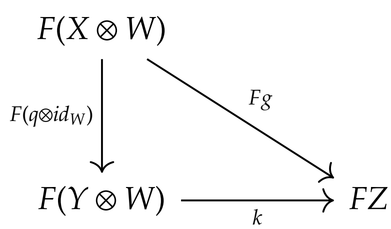




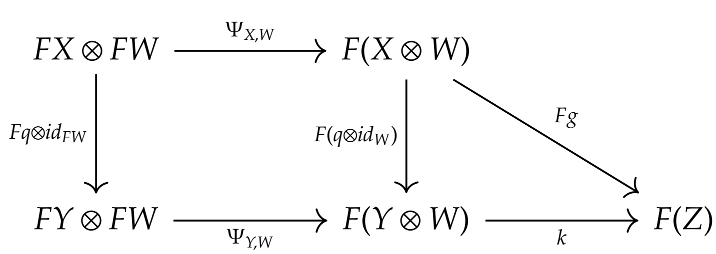
where $\overline{g}$ denotes the adjoint of g with respect to the hom set adjunction

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes W, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, \mathcal{C}(W, Z)).$$

Suppose the following triangle commutes in $\mathcal{D}$:



To show $q \otimes id_W$ is cocartesian it suffices to find a unique morphism $h : Y \otimes W \to Z$ such that $F(h) = k$ and $h \circ (q \otimes id_W) = g$. Since F is strong monoidal, we have an isomorphism $\Psi_{W,Z} : FW \otimes FZ \to F(W \otimes Z)$ natural in W and Z for all $W, Z \in \mathcal{C}$, so that for all $X, Y, W \in \mathcal{C}$, the square in the following diagram commutes. In fact, we have a natural isomorphism $\Psi : (- \otimes -) \circ (F \times F) \Rightarrow F \circ (- \otimes -)$ of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$.



Therefore the outer triangle commutes:

$$\begin{array}{ccc} FX \otimes FW & & \\ {\scriptstyle Fq\otimes id_{FW}}\big\downarrow & \searrow^{(Fg)'} & \\ FY \otimes FW & \xrightarrow[k']{} & FZ \end{array}$$

where $(Fg)' = Fg \circ \Psi_{X,W}$ and $k' = k \circ \Psi_{Y,W}$. Then using adjoint triangles again gives:

$$\begin{array}{ccc} FX & & \\ {\scriptstyle Fq}\big\downarrow & \searrow^{\overline{(Fg)'}} & \\ FY & \xrightarrow[\overline{k'}]{} & \mathcal{D}(FW, FZ) \end{array}$$

We have the commuting triangle

$$\begin{array}{ccc} X & & \\ {\scriptstyle \overline{g}}\big\downarrow & \searrow^{\overline{g}} & \\ \mathcal{C}(W, Z) & \xrightarrow[id_{\mathcal{C}(W,Z)}]{} & \mathcal{C}(W, Z) \end{array}$$

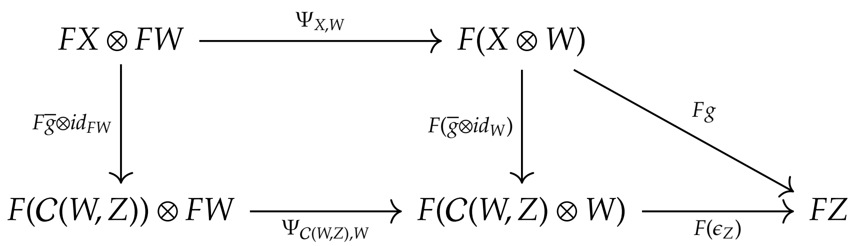
Taking adjoint triangles, we get

$$\begin{array}{ccc} X \otimes W & & \\ {\scriptstyle \overline{g}\otimes id_W}\big\downarrow & \searrow^{g} & \\ \mathcal{C}(W, Z) \otimes W & \xrightarrow[\epsilon_Z]{} & Z \end{array}$$

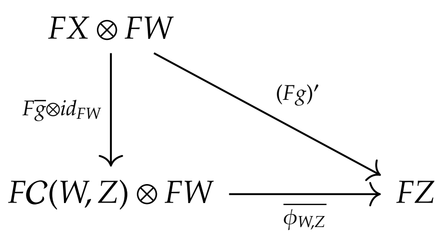
where ϵ_Z is the component at Z of the counit natural transformation

$$(- \otimes W) \circ \mathcal{C}(W, -) \xRightarrow{\epsilon} 1_{\mathcal{C}}.$$

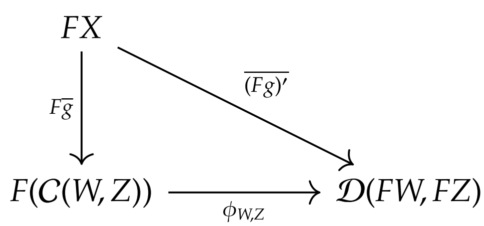
Applying the monoidal functor F to the above, we then have the following commuting diagram in $\mathcal{D}$



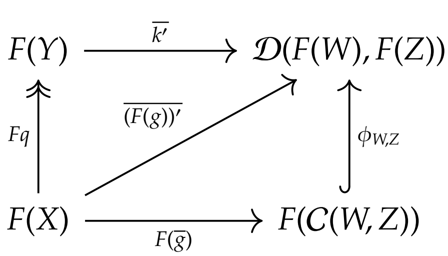
Condition for hom transform: we require by definition for all $W, Z \in C$, $\overline{\phi_{W,Z}} = F(\epsilon_Z) \circ \Psi_{C(W,Z),W}$. Therefore we get the commuting triangle



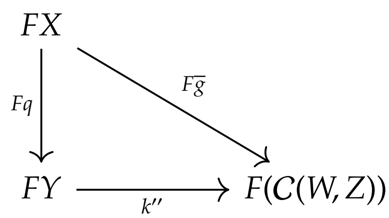
Taking adjoint triangles with reference to the left adjoint functor $(- \otimes FW)$, the following diagram commutes in $\mathcal{D}$:



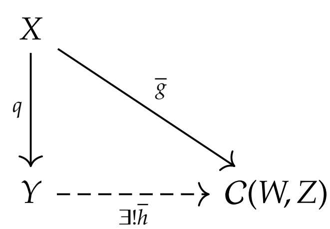
Then the internal triangles of the following diagram commute:



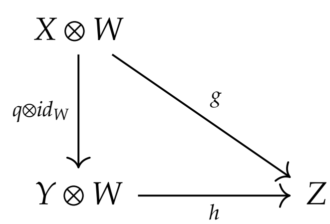
Therefore the outer square commutes. Since Fq is an epimorphism, the hypothesis that $\phi_{W,Z}$ is a strong monomorphism implies the existence of $k'' : FY \to F(\mathcal{C}(W, Z))$ such that the following triangle commutes in $\mathcal{D}$



Since q is F-quotient, there exists a unique $\bar{h} : Y \to \mathcal{C}(W, Z)$ making the following triangle commutes in $\mathcal{C}$ such that $F(\bar{h}) = k$:



Taking adjoint triangles once again gives



Since Fq is an epimorphism and epimorphisms are preserved by left adjoints, $Fq \otimes id_{FW}$ is an epimorphism. Therefore, since

$$k' \circ Fq \otimes id_{FW} = (Fg)' = Fg \circ \Psi_{X,W} = (Fh \circ \Psi_{Y,W}) \circ Fq \otimes id_{FW}$$

we have $F(h) \circ \Psi_{Y,W} = k' = k \circ \Psi_{Y,W}$. Since $\Psi_{Y,W}$ is an epimorphism it follows that $F(h) = k$.

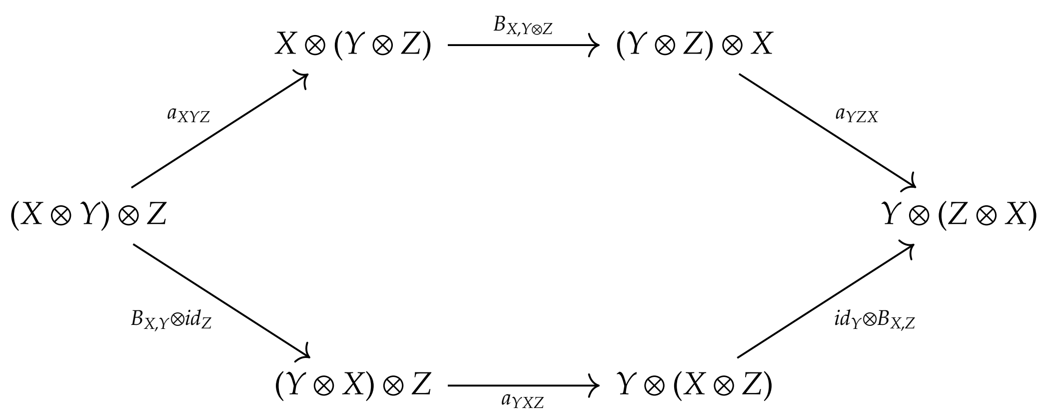
Since $q \otimes id_W$ is an epimorphism, h uniquely satisfies $h \circ q \otimes id_W = g$. Therefore $q \otimes id_W$

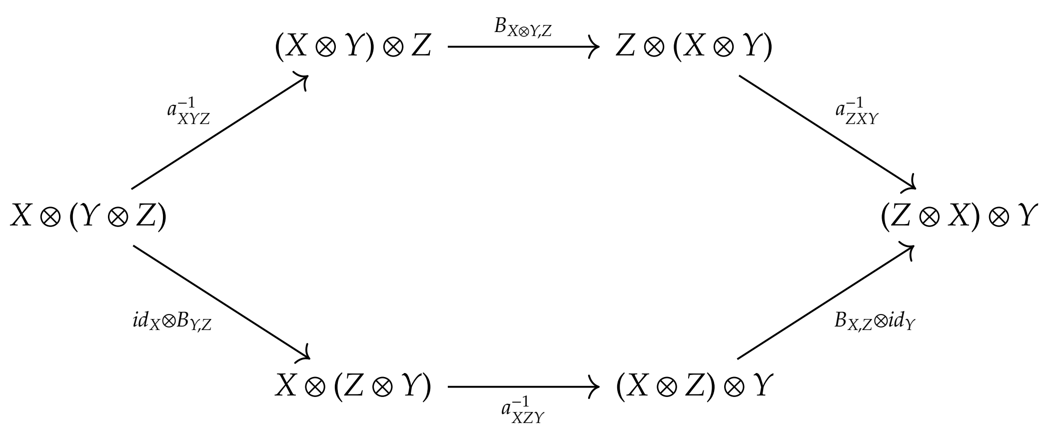
is cocartesian.

As q is F-quotient, Fq is an epimorphism. Since $\Psi_{C(W,Z),W}$ and $Fq \otimes id_{FW}$ are epimorphisms, and $\Psi_{C(W,Z),W} \circ (Fq \otimes id_{FW}) = F(q \otimes id_W) \circ \Psi_{X,W}$, $F(q \otimes id_W)$ is also an epimorphism. Therefore, $q \otimes id_W$ is an F-quotient morphism. $\square$

5.2 Braidings and F-Quotient Morphisms

Definition 5.2.1. [JS86] A *braided monoidal category* is a a monoidal category C equipped with a *braiding*: a natural isomorphism $B_{X,Y} : X \otimes Y \to Y \otimes X$ for each pair of objects $X, Y \in C$ compatible with the monoidal associator a, so that the following diagrams commute for all $X, Y, Z \in C$





A braided monoidal category C such that for all $X, Y \in C$ we have $B_{Y,X} \circ B_{X,Y} = id_{X\otimes Y}$ is called *symmetric* [nLa21d].

Proposition 5.2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be braided, closed, monoidal categories and $F : \mathcal{C} \to \mathcal{D}$ a strong monoidal functor. Let $q : X \to Y$ be F-quotient in $\mathcal{C}$ and W an object of $\mathcal{C}$. If for all Z in $\mathcal{C}$, the monoidal hom transform $\phi_{W,Z} : F(\mathcal{C}(W,Z)) \to \mathcal{D}(F(W), F(Z))$ is a strong monomorphism, then $id_W \otimes q$ is F-quotient.*

Proof. Since $\mathcal{C}$ is a braided category, for all objects $X, Y \in \mathcal{C}$ there exist isomorphisms $B_{X,Y} : X \otimes Y \to Y \otimes X$ natural in both X and Y. So for a fixed object W, we have a natural isomorphism $B_{-,W} : (- \otimes W) \to (W \otimes -)$ and for $q : X \to Y$ the following square commutes

$$\begin{array}{ccc} X \otimes W & \xrightarrow{B_{X,W}} & W \otimes X \\ {\scriptstyle q \otimes id_W}\downarrow & & \downarrow{\scriptstyle id_W \otimes q} \\ Y \otimes W & \xrightarrow[B_{Y,W}]{} & W \otimes Y \end{array}$$

by Proposition 5.1.4, if $q : X \to Y$ is F-quotient, then so is $q \otimes id_W$.

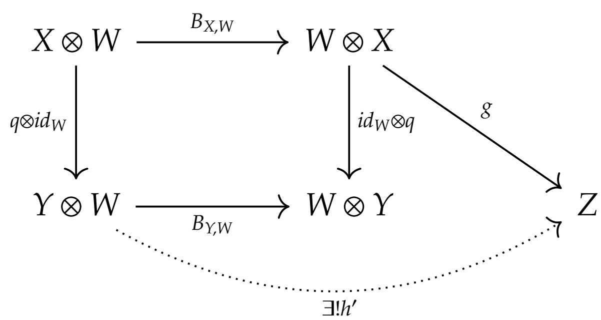
Suppose we are given $g : W \otimes X \to Z$ and $k : F(W \otimes Y) \to FZ$ such that the following triangle commutes:

$$\begin{array}{ccc} F(W \otimes X) & & \\ {\scriptstyle F(id_W \otimes q)}\downarrow & \searrow^{F(g)} & \\ F(W \otimes Y) & \xrightarrow[k]{} & FZ \end{array}$$

Applying F to the commuting square above then gives the following commuting diagram in $\mathcal{D}$:

$$\begin{array}{ccccc} F(X \otimes W) & \xrightarrow{F(B_{X,W})} & F(W \otimes X) & & \\ {\scriptstyle F(q \otimes id_W)}\downarrow & & {\scriptstyle F(id_W \otimes q)}\downarrow & \searrow^{F(g)} & \\ F(Y \otimes W) & \xrightarrow[F(B_{Y,W})]{} & F(W \otimes Y) & \xrightarrow[k]{} & FZ \end{array}$$

Note the outer triangle commutes. Since $q \otimes id_W$ is cocartesian, there exists unique h' such that the outer square in the following diagram commutes



Let $h = h' \circ B^{-1}_{Y,W} : W \otimes Y \to Z$.

We then have

$$\begin{aligned} h \circ id \otimes q &= (h' \circ B^{-1}_{Y,W}) \circ id_W \otimes q = h' \circ (B^{-1}_{Y,W} \circ id_W \otimes q) \\ &= h' \circ (q \otimes id_W \circ B^{-1}_{X,W}) = (h' \circ q \otimes id_W) \circ B^{-1}_{X,W} = (g \circ B_{X,W}) \circ B^{-1}_{X,W} \\ &= g \circ (B_{X,W} \circ B^{-1}_{X,W}) \\ &= g \end{aligned}$$

and

$$\begin{aligned} F(h) &= F(h' \circ B^{-1}_{Y,W}) = F(h') \circ F(B^{-1}_{Y,W}) \\ &= (k \circ F(B_{Y,W})) \circ F(B^{-1}_{Y,W}) = k \circ F(B_{Y,W} \circ B^{-1}_{Y,W}) = k \circ id_{F(W \otimes Y)} \\ &= k. \end{aligned}$$

The morphism h is unique in satisfying $h \circ id_W \otimes q = g$ since h' uniquely satisfies $h' \circ q \otimes id_W = g \circ B_{X,W}$. Then, since $F(h) = k$, $id_W \otimes q$ is cocartesian with respect to F.

As $q \otimes id_W$ is F-quotient, $F(q \otimes id_W)$ is an epimorphism. To show $F(id_W \otimes q)$ is an epimorphism we consider the following diagram in $\mathcal{D}$:

$$
\begin{array}{ccccccc}
F(X \otimes W) & \xrightarrow{\Psi^{-1}_{X,W}} & FX \otimes FW & \xrightarrow{B_{FX,FW}} & FW \otimes FX & \xrightarrow{\Psi_{W,X}} & F(W \otimes X) \\
\downarrow{\scriptstyle F(q \otimes id_W)} & & \downarrow{\scriptstyle Fq \otimes id_{FW}} & & \downarrow{\scriptstyle id_{FW} \otimes Fq} & & \downarrow{\scriptstyle F(id_W \otimes q)} \\
F(Y \otimes W) & \xrightarrow[\Psi^{-1}_{Y,W}]{} & FY \otimes FW & \xrightarrow[B_{FY,FW}]{} & FW \otimes FY & \xrightarrow[\Psi_{W,X}]{} & F(W \otimes Y)
\end{array}
$$

This rectangle commutes by the naturality of the braiding B and the natural transformation Ψ associated with the strong monoidal functor F. Since $(\Psi_{W,X} \circ B_{FY,FW} \circ \Psi^{-1}_{Y,W}) \circ F(q \otimes id_W)$ is a composition of epimorphisms, and $\Psi_{W,X} \circ B_{FX,FW} \circ \Psi^{-1}_{X,W}$ is an epimorphism, $F(id_W \otimes q)$ is an epimorphism, and $id_W \otimes q$ is F-quotient. □

Theorem 5.2.2. *Let $\mathcal{C}, \mathcal{D}$ be braided closed monoidal categories and $F : \mathcal{C} \to \mathcal{D}$ a strong monoidal functor. Let $q_1 : X_1 \to Y_1$ and $q_2 : X_2 \to Y_2$ be F-quotient morphisms in $\mathcal{C}$. If for all W in $\mathcal{C}$, the monoidal hom transforms $\phi_{X_2,Z}$ and $\phi_{Y_1,Z}$ are strong monomorphisms, then $q_1 \otimes q_2$ is F-quotient.*

Proof. Let $\mathcal{C}$ be a category and let $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ be morphisms in $\mathcal{C}$. By definition of the morphisms and morphism composition in the product category $\mathcal{C} \times \mathcal{C}$, the following triangle commutes for any bifunctor $F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$:

$$
F(X_1 \times X_2) \xrightarrow{F(f \times id_{X_2})} F(Y_1 \times X_2) \xrightarrow{F(id_{Y_1} \times g)} (Y_1 \times Y_2), \qquad F(X_1 \times X_2) \xrightarrow{F(f \times g)} (Y_1 \times Y_2)
$$

Let $q_1 : X_1 \to Y_1$ and $q_2 : X_2 \to Y_2$ be F-quotient morphisms with respect to the strong monoidal functor $F : \mathcal{C} \to \mathcal{D}$. Then applying the tensor product functor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ gives the following commutative diagram

$$
X_1 \otimes X_2 \xrightarrow{q_1 \otimes id_{X_2}} Y_1 \otimes X_2 \xrightarrow{id_{Y_1} \otimes q_2} Y_1 \otimes Y_2, \qquad X_1 \otimes X_2 \xrightarrow{q_1 \otimes q_2} Y_1 \otimes Y_2
$$

By Proposition 5.1.4 we have $q_1 \otimes id_{X_2}$ is F-quotient and by Proposition 5.2.1 we have $id_{Y_1} \otimes q_2$ is F-quotient. Then as a composition of F-quotient morphisms is F-quotient by Proposition 2.1.12, the product $q_1 \otimes q_2$ is an F-quotient morphism. □

Corollary 5.2.3. *Let $C, \mathcal{D}$ be braided closed monoidal categories and $F : C \to \mathcal{D}$ be a strong monoidal functor such that all monoidal hom-transforms $\phi_{A,B} : F(C(A,C)) \to \mathcal{D}(F(A), F(B))$ in $\mathcal{D}$ are strong monomorphisms. Then F-quotient morphisms in C are closed under taking tensor products.*

5.3 The Category of Compactly Genererated Spaces

Definition 5.3.1. [Bor94a] A category C is said to be *Cartesian closed* if it has a terminal object, all products, and all exponential objects.

The category **Top** of topological spaces and continuous maps is not Cartesian closed [Wyl73]. Consider the space $\mathbb{R}^\infty$. A countably infinite product of non-compact spaces is not locally compact and local compactness is equivalent to exponentiability for Hausdorff spaces. Even the category of locally compact spaces is not Cartesian closed, since for arbitrary locally compact spaces X, Y the hom set $[X, Y]$ of maps from X to Y equipped with the compact-open topology is not locally compact [Bor94b][360].

Definition 5.3.2. [Bro06][182] A topological space X is said to be *compactly generated* if it is equipped with the final topology with respect to all continuous functions $f : Y \to X$ with Y compact Hausdorff. That is to say, $g : X \to Z$ is continuous if and only if $g \circ f$ is continuous for all continuous $f : Y \to X$ with Y compact Hausdorff. We denote by **CG** the category of compactly generated spaces and continuous maps between them.

Remark 5.3.1. The category **CG** is the coreflective hull (definition 4.1.3) of the category of all compact Hausdorff spaces [Bro06][182]. Thus **CG** is a coreflective subcategory of **Top** and is equipped with coreflection functor k : **Top** $\to$ **CG** which is right adjoint to the inclusion functor i : **CG** $\to$ **Top**.

Definition 5.3.3. Let $C(A, B)$ denote the hom-set of continuous functions between topological spaces A and B. The *compact-open topology* on $C(A, B)$ is given by the subbasis of open sets $\langle U, V\rangle$, where U is compact in A and V is open in B, consisting of maps $f : A \to B$ such that $f(U) \subseteq B$.

Proposition 5.3.2. *The sets* $\{\langle U, V\rangle \mid U$ *compact in* A, V *open in* $B\}$ *form a subbasis for a topology* τ *on* $C(A, B)$*.*

Proof. Since $\emptyset$ is open in B, for any open nonempty compact set U in A we have $\langle U, \emptyset\rangle = \emptyset$. Thus $\emptyset \in \tau$. Since B is open in B,for any compact set U in A and continuous map $f : A \to B$ we have $f(U) \subseteq B$, the set $\angle U, B\rangle = C(A, B)$, so that $C(A, B) \in \tau$. □

It can be shown that for compactly generated spaces X, Y, the set of continuous functions $f : X \to Y$ equipped with the compact-open topology is also compactly generated. Indeed, all compactly generated spaces are exponentiable [nLa21a]. Since **CG** is coreflective in **Top**, it is complete and cocomplete [nLa21a]. That is, all limits and colimits exist in **CG**. Thus we may form the product limit $X \times_{\mathbf{CG}} Y$ in **CG**. This has the same underlying set as $X \times Y$ in **Top**, namely the cartesian product $X \times Y$, but not equipped with the standard product topology. Rather, it is the standard product space under the image of the coreflection functor: $X \times_{\mathbf{CG}} Y = c(X \times Y)$ [nLa21a]. Thus **CG** forms a cartesian closed category. Since **CG** has all limits it has all finite limits and is therefore monoidal with respect to the categorical product. The unit of the monoidal structure is given by the empty product, which is the terminal object of the category. Since $X \times_{\mathbf{CG}} Y$ is homeomorphic to $Y \times_{\mathbf{CG}} X$, the category **CG** is also symmetric monoidal and thus braided monoidal.

Proposition 5.3.3. *The forgetful functor* $U : \boldsymbol{CG} \to \boldsymbol{Set}$ *is monoidal.*

Proof. Let W, Z be compactly generated spaces. Let $\mathbf{Set}_{\mathbf{CG}}(W, Z)$ denote the set $U(\mathbf{CG}(W, Z))$ of underlying set functions of continuous functions between W and Z. Note that for compactly generated spaces W, Z if we forget the topologies on W and Z and take the cartesian product $W \times Z$ in **Set** we arrive at the same set as we do by forming the product space and then forgetting the product topology. Thus $U(W) \times U(Z) = U(W \times Z)$. Let $\Psi_{W,Z} : U(W) \times U(Z) \to U(W \times Z)$ be the identity function $id_{W \times Z} : W \times Z \to W \times Z$. This is a natural isomorphism, making U a monoidal functor. ◻

Proposition 5.3.4. *The forgetful functor U : $\boldsymbol{CG} \to \boldsymbol{Set}$ is equipped with a monoidal hom transform $\phi_{W,Z}$: $U(\boldsymbol{CG}(W,Z)) \to \boldsymbol{Set}(UW, UZ)$ given by the inclusion of the underlying set functions of continuous functions in* **CG** *into the set of all set functions between underlying sets $\boldsymbol{Set}(UW, UZ)$.*

Proof. We first verify that $\phi_{W,Z}$ is natural in W and Z. Note this is a consequence rather than a requirement of definition 5.1.6. Let $f : W \to W'$ be a continuous function between compactly generated spaces. We want to show that $\psi_{-,Z}$ defines a natural transformation between composite contravariant functors $\phi_{-,Z} : U \circ \mathbf{CG}(-, Z) \Rightarrow \mathbf{Set}(-, UZ) \circ U$. Thus we need the following diagram to commute in **Set**

$$\begin{array}{ccc} U(\mathbf{CG}(W,Z)) & \xrightarrow{\phi_{W,Z}} & \mathbf{Set}(UW, UZ) \\ {\scriptstyle U(-\circ f)}\uparrow & & \uparrow{\scriptstyle -\circ Uf} \\ U(\mathbf{CG}(W',Z)) & \xrightarrow[\phi_{W',Z}]{} & \mathbf{Set}(UW', UZ) \end{array}$$

Note that $U(\mathbf{CG}(f,Z) = U(- \circ f)$ and $\mathbf{Set}(f, UZ) = - \circ Uf$. Now consider a morphism $g : W' \to Z$ in $\mathbf{CG}(W', Z)$. Let $\underline{g}$ denote the underlying set function of the continuous function g. Then

$$U(\mathbf{CG}(f,Z)(g) = U(g \circ f) = \underline{g \circ f}$$

and $\phi_{W,Z}(\underline{g \circ f}) = \underline{g \circ f} \in \mathbf{Set}(UW, UZ)$. We also have

$$\mathbf{Set}(f, UZ) \circ \phi_{W',Z}(g) = \underline{g} \circ Uf = \underline{g} \circ \underline{f} = \underline{g \circ f}$$

making the square commmute.

We now show that $\phi_{W,-}$ defines a natural transformation between composite covariant functors $\phi_{W,-} : U \circ \mathbf{CG}(W,-) \Rightarrow \mathbf{Set}(UW,-) \circ U$. Letting $h : Z \to Z'$ be a morphism in $\mathbf{CG}$, we need the following square to commute in $\mathbf{Set}$:

$$\begin{array}{ccc} U(\mathbf{CG}(W,Z)) & \xrightarrow{\phi_{W,Z}} & \mathbf{Set}(UW,UZ) \\ \downarrow{\scriptstyle U(h\circ -)} & & \downarrow{\scriptstyle Uh\circ -} \\ U(\mathbf{CG}(W,Z')) & \xrightarrow[\phi_{W,Z'}]{} & \mathbf{Set}(UW,UZ') \end{array}$$

Given a continuous $k : W \to Z$ in $\mathbf{CG}$ we have

$$\phi_{W,Z'} \circ U(\mathbf{CG}(W,h))(k) = \phi_{W,Z'} \circ U(h \circ k) = \phi_{W,Z'}(\underline{h \circ k}) = \underline{h \circ k} \in \mathbf{Set}(UW,UZ')$$

and

$$\mathbf{Set}(UW,Uh) \circ \phi_{W,Z}(k) = \mathbf{Set}(UW,Uh)(\underline{k}) = Uh \circ \underline{k} = \underline{h} \circ \underline{k} = \underline{h \circ k} \in \mathbf{Set}(UW,UZ').$$

Since $\mathbf{CG}$ is cartesian closed it is equipped for each fixed compactly generated space W with a counit natural transformation ϵ having at the compactly generated space Z the component $\epsilon_Z : \mathbf{CG}(W,Z) \times_{\mathbf{CG}} W \to Z$, where $\mathbf{CG}(W,Z)$ denotes the set of maps between compactly generated spaces W and Z equipped with the compact-open topology. Here ϵ_Z is the continuous evaluation map $\epsilon_Z(f,w) = f(w)$. For $\phi_{W,Z}$ to satisfy the definition of a monoidal hom transform we must also have for all $\mathbf{CG}$ spaces W,Z that $\overline{\phi_{W,Z}} = U(\epsilon_Z) \circ \Psi_{\mathbf{CG}(W,Z),W}$. Since our monoidal functor is $U : \mathbf{CG} \to \mathbf{Set}$ we have $\Psi_{\mathbf{CG}(W,Z),W} = id_{\mathbf{CG}(W,Z)\times W}$. Note that $U(\epsilon_Z)$ is just the set-theoretic evalutation $eval : \mathbf{Set}(W,Z) \times W \to Z$ restricted to the underying set functions of continuous functions between W and Z where both spaces are equipped with compactly generated topologies. Note $\phi_{W,Z}$ is the inclusion of the underlying set functions of such continuous maps into the hom set of all set functions between the underlying sets of W and Z. So, for $f : W \to Z$ in $\mathbf{CG}$ we have $\phi_{W,Z} : U(\mathbf{CG}(W,Z)) \to \mathbf{Set}(UW,UZ)$ given by $\phi_{W,Z}(f) = \underline{f} : UW \to UZ$ and

$\overline{\phi_{W,Z}} : UW \times U(\mathbf{CG}(W, Z)) \to UZ$ is given by

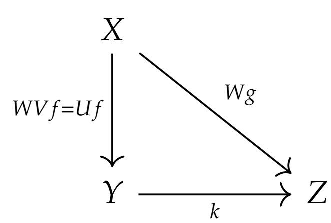
$$\overline{\phi_{W,Z}}(w, \underline{f}) = \underline{f}(w) = U(\epsilon_Z(f, w)) = U(\epsilon_Z) \circ id_{\mathbf{CG}(W,Z)\times W}(\underline{f}, w) = U(\epsilon_Z) \circ \Psi_{\mathbf{CG}(W,Z),W}(\underline{f}, w).$$

□

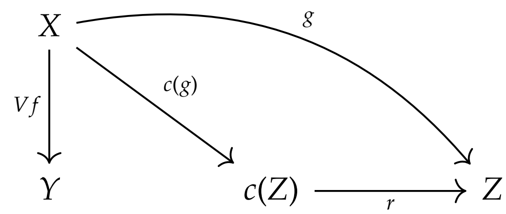
The monoidal hom transform $\phi_{W,Z}$ is clearly an injection, hence a strong monomorphism in **Set**. Therefore, by Proposition 5.1.4, if $q : X \to Y$ is quotient with respect to U, then for all compactly generated spaces W, $q \times_{\mathbf{CG}} id_W$ is quotient relative to U. And since **CG** is symmetric monoidal, if $q_1 : X_1 \to Y_1$ and $q_2 : X_2 \to Y_2$ are both quotient relative to U, then by Proposition 5.2.2, $q_1 \times_{\mathbf{CG}} q_2$ is quotient relative to U.

Proposition 5.3.5. *In the category **CG** of compactly generated spaces, morphisms quotient relative to the forgetful functor $U : \mathbf{CG} \to \mathbf{Set}$ are true topological quotient maps.*

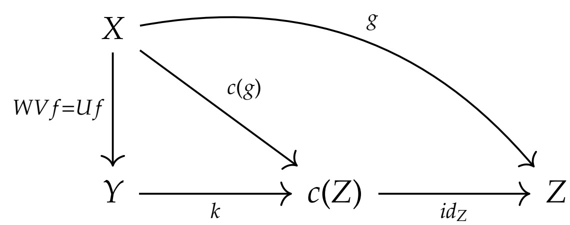
Proof. Suppose $f : X \to Y$ is quotient rel. $U : \mathbf{CG} \to \mathbf{Set}$. If $V : \mathbf{CG} \to \mathbf{Top}$ is the forgetful functor, then $Vf : X \to Y$ is quotient rel. the forgetful functor $W : \mathbf{Top} \to \mathbf{Set}$, and is therefore a true topological quotient map. To see this, consider $Vf : X \to Y$ and $g : X \to Z$ in **Top**. Suppose we have $k : Y \to Z$ in **Set** such that the following triangle commutes:



Let $c : \mathbf{Top} \to \mathbf{CG}$ be the coreflection (right adjoint to inclusion $V : \mathbf{CG} \to \mathbf{Top}$.) Note V does not so much forget that a space X is compactly generated as it forgets the assumption that all spaces mapping to and from X are compactly generated. Applying V, in **Top** we have



where r denotes the component at Z of the counit natural transformation. Recall that $c(Z)$ and Z have the same underlying set, so that $Wr : c(Z) \to Z$ is the identity id_Z, and we have in **Set** the diagram



Now since $f : X \to Y$ is quotient rel. U, there exists a unique $h : Y \to c(Z)$ in **CG** such that $h \circ f = c(g)$ and $Uh = k$. Then we have $r \circ Vh : Y \to Z$ such that $Vh \circ Vf = V(c(f))$ and $W(r \circ Vh) = id_Z \circ WV(h) = U(h) = k$. Now $h' = r \circ Vh : Y \to Z$ uniquely satisfies $W(h') = k$ and $(h') \circ Vf = g$, making Vf quotient with respect to W. □

Proposition 5.3.6. *In the category **CG** the product of two quotient maps is a quotient map.*

Proof. Since morphisms quotient relative to U : **CG** → **Set** are topological quotient maps and products of U-quotient morphisms are U-quotient, it follows that the product of two morphisms quotient relative to U is a topological quotient map. □

Proposition 5.3.7. *Let C be a braided monoidal cartesian closed coreflective subcategory of **Top**. Then quotient morphisms in C are true topological quotient maps and the tensor/product of two quotient maps is a quotient map.*

Proof. Since any exponential object in a subcategory of **Top** will have the same underlying set function and the conditions required for the internal hom functor are satisfied in Set, the above result holds for any braided monoidal cartesian closed coreflective subcategory of **Top**. □

Appendices

Appendix A

Basic Category Theory

Definition A.0.1. We loosely follow [Bor94a]. A *category* C is a collection of *objects* and *morphisms* between objects. If X is an object in C we write $X \in \mathrm{Ob}(C)$, or $X \in C$. We write $f : X \to Y$ to indicate f is a morphism from X to Y, and refer to X and Y as the *domain* and *codomain* of f, respectively. The collection of morphisms from X to Y in C is denoted by $\mathrm{Hom}_C(X, Y)$ or sometimes simply $C(X, Y)$ or $C[X, Y]$ and $\mathrm{Hom}(C)$ denotes the collection of all morphisms in C. If both $\mathrm{Ob}(C)$ and $\mathrm{Hom}(C)$ are sets, C is called a *small* category. If $\mathrm{Hom}_C(X, Y)$ is a set for all pairs of objects X, Y in $\mathrm{Ob}(C)$, then C is called a *locally small* category. If a category is not small, it is *large*.

If f and g are morphisms in C such that cod f = dom g then we can form the composition $g \circ f$: dom f $\to$ cod g. Composition of morphisms is defined locally as a function on hom sets:

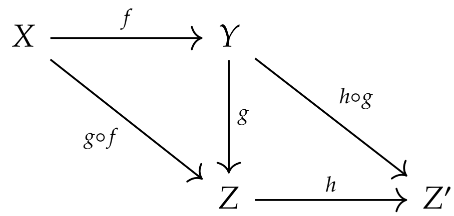
$$\circ : \mathrm{Hom}_C(X, Y) \times \mathrm{Hom}_C(Y, Z) \to \mathrm{Hom}_C(X, Z)$$

$$(f, g) \mapsto g \circ f$$

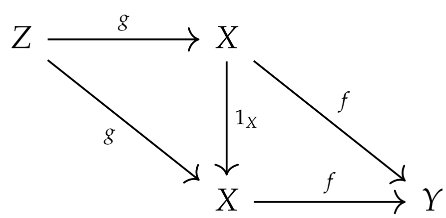
Morphisms in a category adhere to the following axioms:

1. Morphism composition is associative: $h \circ (g \circ f) = (h \circ g) \circ f)$. This fact is expressed

by commutativity of the diagram



2. To each object X in C we associate the *identity* morphism, written id_X or sometimes 1_X, such that the following diagram commutes for all compatible morphisms f and g



Note the axioms of a category dictate when morphisms must be present and proscribe aspects of their behavior, but say essentially nothing about the nature of the objects in a category.

Definition A.0.2. Two morphisms $f : X \to Y$ and $g : Y \to X$ in the category C such that $g \circ f = 1_X$ and $f \circ g = 1_Y$ are said to be inverses of each other, and we may write in this case $f = g^{-1}$ and $g = f^{-1}$. Morphisms with full inverses are called *isomorphisms*.

Example A.0.1. Sets and set functions form the category **Set**. Russell's paradox implies **Set** is not a small category. Note the composition of set functions is associative and the identity function is defined for all sets (including the empty set, where $id_{\varnothing} : \varnothing \to \varnothing$ is just the empty morphism.)

Example A.0.2. The category **Top** has topological spaces for objects and continuous functions as morphisms. Note the composition of continuous functions is continuous, as is the identity function.

The objects of a category are often sets equipped with additional structure, with the morphisms being functions that preserve that structure, such as the topology on a set, or the operation on a group.) A category whose objects have underlying sets is called a *concrete* category. All the above categories are concrete. An example of a non-concrete category is the category **Ho(Top)** with objectes topological spaces and morphisms homotopy classes of continuous maps [nLa22c].

A.1 Functors and Natural Transformations

Definition A.1.1. Let C and $\mathcal{D}$ be categories. A *functor* $F : C \to \mathcal{D}$ is a morphism of categories: to each object $X \in \mathrm{Ob}(C)$ is associated an object $F(X) \in \mathrm{Ob}(\mathcal{D})$ and to each morphism $f : X \to Y$ is associated a morphism $F(f) : F(X) \to F(Y)$. Functors respect morphism composition: $F(g) \circ F(f) = F(g \circ f)$. Functors also take identity morphisms to identity morphisms: $F(id_X) = id_{F(X)}$ for all $X \in \mathrm{Ob}(C)$.

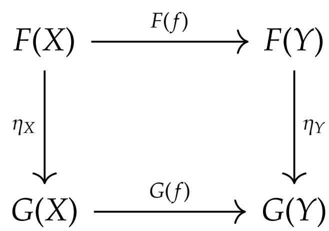
Functors $F : C \to \mathcal{D}$ can preserve domain and codoman in the sense that if $f : X \to Y$ in C then $Ff : FX \to FY$ in $\mathcal{D}$, and the co(domain) of Ff is the image of the co(domain) of f. Such functors are called *covariant*. Functors in which the image domain and codomain have been reversed, so that if $f : X \to Y$ in C, then $Ff : FY \to FX$ in $\mathcal{D}$, are called *contravariant*.

Example A.1.1. Let U : **Top** $\to$ **Set** be the functor that takes a topological space X to its set of underlying points and a continuous map to the underlying set function. Then U defines the so-called *forgetful* functor from **Top** to **Set**.

Definition A.1.2. Let $F : C \to \mathcal{D}$ be a covariant functor between locally small categories. Then for each pair of objects X, Y in C there is an induced set function $F_{X,Y} : \mathrm{Hom}_C[X, Y] \to \mathrm{Hom}_{\mathcal{D}}[FX, FY]$ given by $F_{X,Y}(f) = Ff$. F is said to be *full* if for each pair X, Y the set function $F_{X,Y}$ is surjective, *faithful* if it is injective, and *fully faithful* if it is bijective.

Definition A.1.3. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *essentially surjective on objects* or simply *essentially surjective* if for each object $Y \in \mathcal{D}$ there exists some $X \in \mathcal{C}$ such that $F(X) \cong Y$.

Definition A.1.4. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* η from F to G, sometimes written $\eta : F \Rightarrow G$ is defined by morphisms in $\mathcal{D}$ indexed by the objects of X, such that if $f : X \to Y$ in $\mathcal{C}$, then the following square commutes in $\mathcal{D}$:

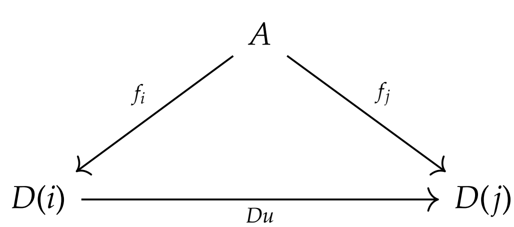


The morphisms η_X and η_Y are referred to as the *components* of η at X and Y respectively. The morphism η_X is said to be *natural in X* if it is the component at X of a natural transformation η.

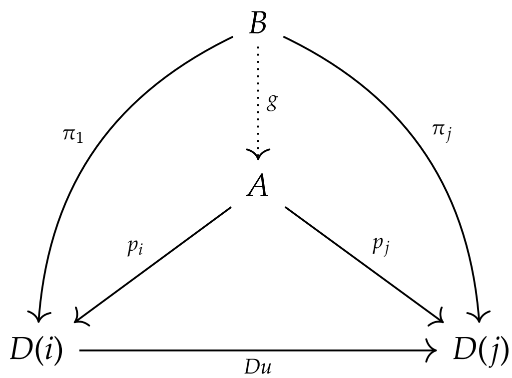
A.2 Limits and Colimits

Definition A.2.1. Let $\mathcal{A}$ be a category and let $\mathbf{I}$ be a small category (its collection of objects defines a set.) A functor $I \to \mathcal{A}$ is a *diagram* in $\mathcal{A}$ of shape $\mathbf{I}$.

Definition A.2.2. Let $\mathcal{A}$ be a category. Let $\mathbf{I}$ be a small category, and $D : \mathbf{I} \to \mathcal{A}$ a diagram in $\mathcal{A}$. A *cone* on D consists of an object $A \in \mathcal{A}$ (called the *vertex* of the cone), and family of $\mathcal{A}$-morphisms $\left(f_i : A \to D(i)\right)_{i \in \mathbf{I}}$ indexed by objects i in $\mathbf{I}$ and satisfying for each $u : i \to j$ in $\mathbf{I}$, the following diagram commutes in $\mathcal{A}$:

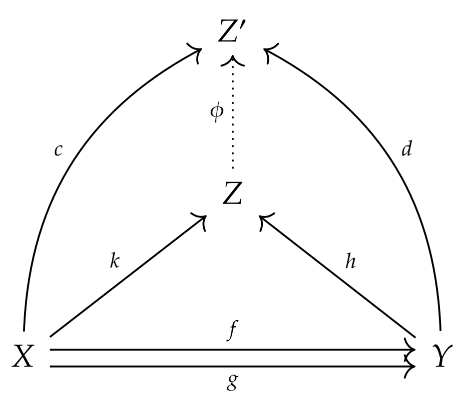


Definition A.2.3. a limit of a diagram is a universal cone. That is, a cone $(A, (p_i)_{i \in \mathbf{I}})$ on D such that for any other cone $(B, (\pi_i)_{i \in \mathbf{I}})$ on D there exists a unique morphism $g : B \to A$ such that for all $i \in \mathbf{I}$ we have $\pi_i = p_i \circ g$.



Thus an arbitrary cone on D uniquely factors through the limiting cone. By duality we can define cocones and colimits in an analogous manner.

Example A.2.1. Let $D : J \to C$ be a diagram from the category with two objects and exactly two non-identity morphisms $a, b : A \to B$. Then a diagram of shape D in C is a pair of morphisms $f, g : X \to Y$ where $f = D(a)$ and $g = D(b)$. We call the colimit of this diagram the *coequalizer* of f and g. We have the following cone over D



If ϕ uniquely satisfies $\phi \circ h = d$ then since $k = h \circ f$ we have $\phi \circ k = \phi \circ h \circ f = d \circ f = c$. We deduced this without needing the fact that $\phi \circ k = c$ because k is determined by h.

Therefore we leave out the data of k and c and write the coequalizer diagram as follows

$$
\begin{array}{ccccc}
X & \overset{f}{\underset{g}{\rightrightarrows}} & Y & \xrightarrow{h} & Z \\
 & & & \searrow^{d} & \uparrow{\scriptstyle \phi} \\
 & & & & Z'
\end{array}
$$

Dual to coequalizer is the notion of equalizer, which is the limit (rather than colimit) of a diagram consisting of a parallel pair $f, g : X \to Y$.

A.3 Adjoint Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ functors. Then F is *left adjoint* to G and G is *right adjoint* to F, written $F \dashv G$, if for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ there is a hom-set bijection $\Phi_{X,Y} : \mathcal{D}(FX, Y) \to \mathcal{C}(X, GY)$ that is natural in the variables X and Y. That is, if we fix $Y \in \mathcal{D}$ and let $f : X' \to X$ be any morphism in $\mathcal{C}$, the following square commutes in **Set**

$$
\begin{array}{ccc}
\mathcal{D}(FX, Y) & \xrightarrow{\Phi_{X,Y}} & \mathcal{C}(X, GY) \\
\downarrow{\scriptstyle (Ff)^\#} & & \downarrow{\scriptstyle f^\#} \\
\mathcal{D}(FX', Y) & \xrightarrow[\Phi_{X',Y}]{} & \mathcal{C}(X', GY)
\end{array}
$$

and if we fix $X \in \mathcal{C}$ and let $g : Y \to Y'$ in $\mathcal{D}$ the following square commutes

$$
\begin{array}{ccc}
\mathcal{D}(FX, Y) & \xrightarrow{\Phi_{X,Y}} & \mathcal{C}(X, GY) \\
\downarrow{\scriptstyle g^\#} & & \downarrow{\scriptstyle (Gg)^\#} \\
\mathcal{D}(FX, Y') & \xrightarrow[\Phi_{X,Y'}]{} & \mathcal{C}(X, GY')
\end{array}
$$

These bijections are natural in that they define components of natural isomorphisms

$$\Phi_{-,Y} : \mathrm{Hom}_{\mathcal{D}}(-, Y) \circ F \Rightarrow \mathrm{Hom}_{\mathcal{C}}(-, GY)$$

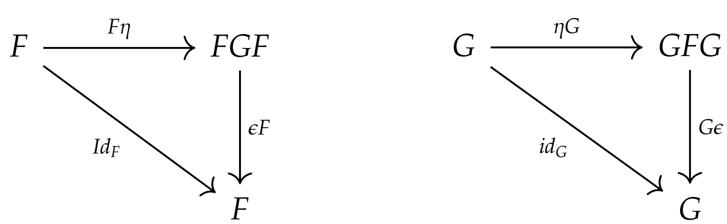
and

$$\Phi_{X,-} : \mathrm{Hom}_{\mathcal{D}}(FX, -) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(X, -) \circ G$$

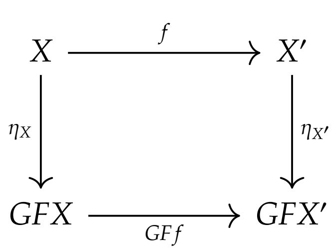
respectively.

We refer to the above as a *hom set adjunction.*

Definition A.3.1. A *unit-counit adjunction* consists of two natural transformations, the *unit* $\eta : 1_{\mathcal{D}} \Rightarrow G \circ F$ and *counit* $\epsilon : F \circ G \Rightarrow 1_{\mathcal{C}}$ satisfying the following "triangle identities."



A unit-counit adjunction may be derived from the hom set adjunction as follows. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors such that $F \dashv G$. If we set $Y = FX$ in the above hom adjunction square then $id_{FX} : FX \to FX$ has adjoint $\widetilde{id_{FX}} = \eta_X : X \to GFX$. We show the morphisms $\eta_X : X \to GFX$ define components of the unit natural transformation $\eta : id_{\mathcal{C}} \Rightarrow G \circ F$. Given a morphism $f : X \to X'$ in $\mathcal{C}$ we must show the following diagram commutes



so that we require

$$\eta_{X'} \circ f = GFf \circ \eta_X.$$

Consider the following diagram in **Set**

$$
\begin{array}{ccc}
\mathcal{D}[FX, FX] & \xrightarrow{\Phi_{X,FX}} & \mathcal{C}[X, GFX] \\
\Big\downarrow{\scriptstyle Ff\circ -} & & \Big\downarrow{\scriptstyle GFf\circ -} \\
\mathcal{D}[FX, FX'] & \xrightarrow[\Phi_{X,FX'}]{} & \mathcal{C}[X, GFX'] \\
\Big\uparrow{\scriptstyle -\circ Ff} & & \Big\uparrow{\scriptstyle -\circ f} \\
\mathcal{D}[FX', FX'] & \xrightarrow[\Phi_{X',FX'}]{} & \mathcal{C}[X', GFX']
\end{array}
$$

Here Φ denotes the isomorphism natural in $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ associated to the hom set bijection $\mathcal{D}[FX, Y] \cong \mathcal{C}[X, GY]$. The top square commutes by naturality of $\Phi_{X,-}$ applied to $Ff : FX \to FX'$. The bottom square commutes by the naturality of $\Phi_{-,FX'}$ applied to $f : X \to X'$. Chasing id_{FX} around the top square gives

$$GFf \circ \eta_X = \widetilde{Ff}$$

and chasing $id_{FX'}$ around the bottom square we have

$$\widetilde{Ff} = \eta_{X'} \circ f$$

thus $\eta_{X'} \circ f = GFf \circ \eta_X$.

Now let $g : Y \to Y'$ be a morphism in $\mathcal{D}$. Analogously to the above we require the following square to commute in Set

$$
\begin{array}{ccc}
FGY & \xrightarrow{FGg} & FGY' \\
\Big\downarrow{\scriptstyle \epsilon_Y} & & \Big\downarrow{\scriptstyle \epsilon_{Y'}} \\
Y & \xrightarrow[g]{} & Y'
\end{array}
$$

Thus we need $g \circ \epsilon_Y = \epsilon_{Y'} \circ FGg$. Let $X = GY$ and $g : Y \to Y'$. Then we have

$$
\begin{array}{ccc}
\mathcal{D}[FGY, Y] & \xrightarrow{\Phi_{GY,Y}} & \mathcal{C}[GY, GY] \\
\downarrow{\scriptstyle g\circ -} & & \downarrow{\scriptstyle Gg\circ -} \\
\mathcal{D}[FGY, Y'] & \xrightarrow[\Phi_{GY,Y'}]{} & \mathcal{C}[GY, GY'] \\
\uparrow{\scriptstyle -\circ FGg} & & \uparrow{\scriptstyle Gg\circ -} \\
\mathcal{D}[FGY', Y'] & \xrightarrow[\Phi_{GY',Y'}]{} & \mathcal{C}[GY', GY']
\end{array}
$$

Since ϵ_Y is the adjoint of 1_{GY} it follows that $\widetilde{g \circ \epsilon_Y} = Gg$. Since $\widetilde{\epsilon_{Y'}} = id_{GY'}$ it follows that $\widetilde{\epsilon_{Y'} \circ FGg} = Gg$ so we have

$$\widetilde{g \circ \epsilon_Y} = Gg = \widetilde{\epsilon_{Y'} \circ FGg}$$

and therefore $g \circ \epsilon_Y = \epsilon_{Y'} \circ FGg$.

A proof of that the triangle identities can be derived from a hom set adjunction can be found in [Lei14][52].

A hom-set adjunction can also be derived from a unit-counit adjunction, so that both describe the same underlying relationship between F and G, which we call an *adjunction*.

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