

Resource theoretic efficacy of the single copy of a two-qubit entangled state in a sequential network

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Abstract How best one can recycle a given quantum resource, mitigating the various difficulties involved in its preparation and preservation, is of considerable importance for ensuring efficient applications in quantum technology. Here we demonstrate quantitatively the resource theoretic advantage of reusing a single copy of a two-qubit entangled state towards information processing. To this end, we consider a scenario of sequential entanglement detection of a given two-qubit state by multiple independent observers on each of the two spatially separated wings. In particular, we consider equal numbers of sequential ob-

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servers on the two wings. We first determine the upper bound on the number of observers who can detect entanglement employing suitable entanglement witness operators. In terms of the parameters characterizing the entanglement consumed and the robustness of measurements, we then compare the above scenario with the corresponding scenario involving multiple pairs of entangled qubits shared among the two wings. This reveals a clear resource theoretic advantage of recycling a single copy of a two-qubit entangled state in the sequential network.

1 Introduction

One of the most counterintuitive features of quantum mechanics is quantum entanglement [1–3], which leads to nonclassical phenomena like Bell nonlocality [4, 5] and Einstein-Podolsky-Rosen steering [6–8]. Apart from the foundational significance of probing incompatibility between quantum mechanical predictions and the local realist descriptions of nature, quantum entanglement serves as a resource for various information processing and communication tasks. To name a few, some such well-established tasks include quantum teleportation [9], quantum dense coding [10], quantum key distribution [11], certification of genuine randomness [12], and quantum random access codes [13].

In a real laboratory set-up, preparation of any quantum resource always faces different types of complications [14] and such difficulties are quantified by the “preparation cost” associated with the preparation dynamics [15]. Moreover, it is an extremely difficult task to prepare quantum resources having a high degree of isolation from environmental interactions [16, 17]. In fact, quantum correlations in independent environments have been shown to decay asymptotically or even to disappear at a finite time under the action of noise [18–20]. Therefore, the efficient use of quantum resources is one of the primary challenges in the backdrop of current endeavour of building quantum technology. To this end, the possibility of recycling the same resource several times is of great advantage. A particular network scenario suitable for this purpose comprises a single copy of a bipartite entangled state with multiple pairs of independent sequential observers in the two spatially separated wings, where each of these observers performs unsharp measurement and delivers the accessed particle to the next observer [21].

Specifically, in [21], the authors showed that at most two independent observers at one wing can violate the Bell-CHSH (Clauser-Horne-Shimony-Holt) inequality [22] with an observer on the other wing by sharing a pair of entangled spin-1/2 particles. Note that all except the last one in a sequence of multiple observers cannot perform sharp or projective measurements, since it is desired that some amount of entanglement must survive in the post-measured state in order to be utilized by the subsequent observer. It was shown in [23], that the upper bound of two observers on one wing who can share nonlocality of a two-qubit state in the scenario of unbiased measurement settings, is based on the optimality of the unsharp measurement framework [24, 25] with respect

to the trade-off between information gain and disturbance in a quantum measurement [26, 27]. It is thus important to employ such measurements in order to obtain optimal performance in the above sequential network scenario.

The issue of sequential detection of different quantum correlations by multiple observers has been investigated both theoretically and experimentally in several subsequent works using the unsharp measurement formalism. Such works include, for example, steering a single system multiple times [28–34], exploring Bell-type nonlocality in various settings [35–45], witnessing entanglement of bipartite and tripartite states [46–48], sharing of nonlocal advantage of quantum coherence [49], and quantum contextuality [50]. Applications of sequential detection of quantum correlations in different information processing tasks have also been reported, e.g., in the context of randomness certification [51], dimension witness [52], quantum random access codes [53–55], quantum teleportation [56], remote state preparation [57], distinguishing quantum predictions from classical simulations with finite memory [58]. Recently, the technique of choosing different sharpness parameters for the different measurement settings of each observer has been proposed for obtaining the possibility of unbounded number of observers sharing Bell-nonlocality [44, 59, 60], and this type of result has also been probed towards random access code generation [61].

The above works have stimulated wide interest in recycling various types of quantum correlations for their use in multi-observer networks. A natural question emerges in this context as to if any quantitative advantage can be gained from the resource theoretic perspective by the reuse of correlations in a single copy of a quantum state. In the present work we answer this question in the affirmative. In particular, we consider a single copy of a bipartite two-qubit entangled state that is shared between multiple observers on both the wings, who sequentially and independently perform measurements on the state. We first determine the essential figure of merit in this scenario, that is given by the maximum number of observers who can successfully detect entanglement contained in the bipartite two-qubit state. We next focus towards addressing resource theoretic comparison of performance of sequential network schemes based on single copy entangled state, with that of schemes based on multi-copy entangled states shared by the same number of observers. In order to compare the sequential and the non-sequential scenarios we utilize the detectability (or visibility) in terms of the expectation value of the entanglement witness operator [62, 63], as well as the information extraction capability of the involved measurements, defined quantitatively via the robustness of measurement [64]. In terms of the above two parameters, we show that the sequential measurement protocol provides advantages in witnessing entanglement by multiple pair of observers in terms of the resources consumed.

We arrange the rest of the paper in the following way. In Sec. 2, we provide a brief overview of some basic tools employed in our analysis, such as entanglement witness operators, unsharp measurement and robustness of measurements. Next, in Sec. 3, we provide details of our symmetric network scenario. The main results regarding the bounds on the number of observers, and the re-

source theoretic comparisons are discussed in Sec. 4. In Sec. 5 we illustrate the resource theoretic advantage of the sequential scenario through an example of quantum teleportation using witness operators for detecting useful entangled states for quantum teleportation. Sec. 6 contains an analysis of the asymmetric extension of the above scenario. Finally, in Sec. 7 we summarize our results with some concluding discussions.

2 Basic tools

In this section, we present the basic ideas of entanglement witness operators, unsharp measurements and robustness of measurements. These concepts will be used later for presenting the main results of this paper.

2.1 Entanglement Witness Operators

A Hermitian operator W is called an entanglement witness operator if there exists at least one entangled state $\rho_e \notin \mathcal{S}$ such that $\text{Tr}(W\rho_e) < 0$ and $\text{Tr}(W\rho) \geq 0$ for all $\rho \in \mathcal{S}$ with \mathcal{S} being the set of all separable states [62]. One can find out an entanglement witness operator for each entangled state. However, finding out the optimal entanglement witness operator for a given entangled state is not always easy [63]. For detecting entanglement, one is usually interested in decomposing an entanglement witness operator in terms of local quantum measurements. This enables performing the detection process using local quantum measurements [62].

Consider the state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The optimal entanglement witness operator for this state is given by [62],

$$W = \frac{1}{4} \left(\mathbb{I} \otimes \mathbb{I} + \sigma_z \otimes \sigma_z - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y \right). \quad (1)$$

The advantage of this entanglement witness operator is that it can be implemented in the laboratory by performing a three correlated local quantum measurements in the bases associated with the Pauli operators $\{\sigma_x, \sigma_y, \sigma_z\}$.

If in the preparation process of the state $|\psi^+\rangle$ some random noise acts, then the resultant state may turn out to be the Werner state of the form,

$$\rho = p|\psi^+\rangle\langle\psi^+| + (1-p)\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}, \quad (2)$$

where $0 < p \leq 1$; $\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}$ denotes white noise and $(1-p)$ is the strength of the noisy process. The entanglement witness operator W given by Eq.(1) remains optimal for the state ρ as well [62].

2.2 Optimality of Unsharp Measurements

A fundamental feature of quantum theory is that no information about a system can be obtained without perturbing its state [65]. Projective measurements, also known as strong measurements, are the most informative at the cost of maximally disturbing the initial state. On the other hand, weak measurements are characterised by broad pointer states, and are less informative, but affect lesser the initial state [21], thereby reflecting a nontrivial trade off between information gain and disturbance [26, 27]. In our sequential networks, it is important to employ such weak measurements which optimize the information gain-disturbance trade-off to achieve best performance, rather than some random choice of measurement. In [23], it was shown that unsharp measurement which is an one-parameter Positive Operator-Valued Measure (POVM) satisfies the optimality criteria.

Generalized quantum measurement or POVM [24, 25] is defined by a set of positive operators that add to identity, i.e., $E \equiv \{E_i | \sum E_i = \mathbb{I}, 0 \leq E_i \leq \mathbb{I}\}$. Consider the dichotomic observable $\vec{\sigma} \cdot \hat{n}$, which is the spin component observable for qubits along the direction \hat{n} . Here $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector composed of three Pauli operators and \hat{n} is a unit vector in \mathbb{R}^3 . Given the observable $\vec{\sigma} \cdot \hat{n}$, one can define the dichotomic unsharp observable $E_{\hat{n}}^\lambda = E_{+|\hat{n}}^\lambda - E_{-|\hat{n}}^\lambda$ [66] associated with the sharpness parameter $\lambda \in (0, 1]$, where

$$E_{\pm|\hat{n}}^\lambda = \lambda P_{\pm|\hat{n}} + \frac{1-\lambda}{2} \mathbb{I} \quad (3)$$

are the effect operators that satisfy $E_{+|\hat{n}}^\lambda + E_{-|\hat{n}}^\lambda = \mathbb{I}$, $0 \leq E_{\pm|\hat{n}}^\lambda \leq \mathbb{I}$. Here, $P_{\pm|\hat{n}}$ are the projectors given by, $P_{\pm|\hat{n}} = (\mathbb{I} \pm \vec{\sigma} \cdot \hat{n})/2$.

The probabilities of getting the outcomes $+1$ and -1 , when the above unsharp measurement is performed on the state ρ , are given by $\text{Tr}[\rho E_{+|\hat{n}}^\lambda]$ and $\text{Tr}[\rho E_{-|\hat{n}}^\lambda]$, respectively. The expectation value of $E_{\hat{n}}^\lambda$ for a given ρ is defined as,

$$\langle E_{\hat{n}}^\lambda \rangle = \text{Tr}[\rho E_{+|\hat{n}}^\lambda] - \text{Tr}[\rho E_{-|\hat{n}}^\lambda] = \lambda \langle \vec{\sigma} \cdot \hat{n} \rangle, \quad (4)$$

where $\langle \vec{\sigma} \cdot \hat{n} \rangle = \text{Tr}[\rho(P_{+|\hat{n}} - P_{-|\hat{n}})]$ denotes the expectation value of the observable $\vec{\sigma} \cdot \hat{n}$ under projective measurement. The post-measurement state can be determined using the generalized von Neumann-Lüders transformation rule [24, 25] as follows,

$$\rho \rightarrow \frac{\sqrt{E_{\pm|\hat{n}}^\lambda} \rho \sqrt{E_{\pm|\hat{n}}^\lambda}}{\text{Tr}(\rho E_{\pm|\hat{n}}^\lambda)}. \quad (5)$$

In the weak measurement formalism with broad pointer states, the optimal information gain-disturbance trade-off is characterised by a condition involving two parameters called quality factor (F) and precision (G). An optimal pointer satisfies $F^2 + G^2 = 1$, which implies maximal information gain for a given amount of disturbance [21]. In the unsharp measurement formalism described above, it turns out that $G = \lambda, F = \sqrt{1 - \lambda^2}$ [23], thereby satisfying the optimality condition.

2.3 Robustness of Measurements

The concept of ‘‘Robustness of Measurements’’ (RoM) has been formulated recently to quantify the informativeness of a measurement [64]. Given a particular measurement E , RoM of E , denoted by $R(E)$, quantifies to what extent E is a resourceful measurement. When a measurement returns an arbitrary outcome i with probability $q(i)$ independent of the quantum state measured, the measurement is called a trivial measurement. Such a measurement has POVM elements E_i with $E_i = q(i)\mathbb{I}$ for all i . It is evident that trivial measurements are not informative or resourceful at all.

RoM is defined as the minimal amount of noise that needs to be added to the measurement such that the measurement becomes a trivial one. Suppose, instead of always performing the measurement $E = \{E_i\}$, one performs a different measurement $F = \{F_i\}$ sometimes. The informativeness of the measurement E can be captured by the minimal probability of this other measurement F that makes the overall measurement trivial. Hence, the RoM can be formally defined as [64],

$$\begin{aligned}
 R(E) &= \min_{F,q} r, \\
 &\text{such that} \\
 \frac{E_i + rF_i}{1+r} &= q(i)\mathbb{I} \quad \forall i, \\
 F_i &\geq 0 \quad \forall i \quad \text{and} \quad \sum_i F_i = \mathbb{I}.
 \end{aligned} \tag{6}$$

Here, the minimization is taken over all noise measurements $F = \{F_i\}$ and all probability distributions $q = \{q(i)\}$.

RoM can also be written as [64],

$$R(E) = \sum_i \|E_i\|_\infty - 1, \tag{7}$$

where $\|E_i\|_\infty$ is the operator norm of E_i . Note that $\|E_i\|_\infty$ is equal to the maximum eigen value of $\sqrt{E_i^\dagger E_i}$ [67]. Hence, for the unsharp measurement $E_{\hat{n}}^\lambda \equiv \{E_{+|\hat{n}}^\lambda, E_{-|\hat{n}}^\lambda\}$ defined in Eq.(3), we have

$$R(E_{\hat{n}}^\lambda) = \lambda. \tag{8}$$

Using the resource theory of measurement informativeness [64], any trivial measurement can be considered as a free measurement and any measurement which is not trivial is a resourceful measurement. Hence, RoM characterises the amount of resource in a measurement, i.e., its information extraction capacity.

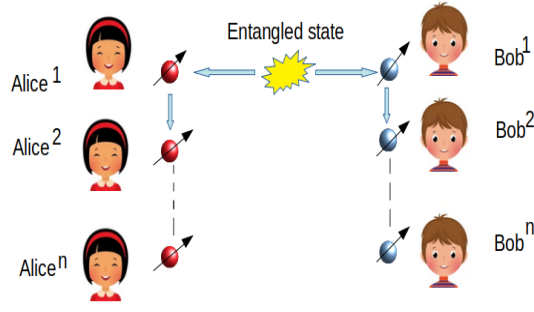


Fig. 1: An entangled state ρ_i of two spin- $\frac{1}{2}$ particles is initially shared between Alice¹ and Bob¹. They perform measurements on the particles in their possession to detect the entanglement, and after the measurements Alice¹ sends her particle to Alice² and Bob¹ sends his particle to Bob². Next, Alice² and Bob² perform measurements to detect entanglement of the shared state, and then sends the particles to the next pair, i.e., Alice³-Bob³ and the process continues. The process is terminated when a sequential pair, say, Alice^m-Bob^m is unable to detect entanglement of the shared state.

3 Setting up the scenario

In the present study, we will consider the particular scenario as described below (see Fig. 1).

Scenario 1 We consider n number of sequential Alices (Alice¹, Alice², Alice³, \dots , Aliceⁿ) and n number of sequential Bobs (Bob¹, Bob², Bob³, \dots , Bobⁿ), where n is a-priori arbitrarily large. At first, the pair Alice¹-Bob¹ detects the entanglement of the initial state ρ_i of two spin- $\frac{1}{2}$ particles using entanglement witness operator. Alice¹ then passes her particle to Alice² and Bob¹ passes his particle to Bob². Alice²-Bob² then detects entanglement of the two-qubit state received after the measurements by Alice¹ and Bob¹. Consequently, Alice² and Bob² pass their particles to Alice³ and Bob³ respectively, and so on. The process is terminated when Alice^m-Bob^m with $m \leq n$ is unable to detect entanglement.

In the above scenario we consider the following assumptions:

- 1) Each Alice (Bob) performs measurements independent of the measurement settings and outcomes of the previous Alices (Bobs).
- 2) All possible measurement settings of each Alice (Bob) are equally probable.
- 3) Each Alice (Bob) employs the same value of the sharpness parameter for all of her (his) measurement settings.

In the case of sharp projective measurement, one obtains the maximum amount of information at the cost of maximum disturbance to the state. In

the scenarios considered by us, Alice^{*i*} (Bob^{*i*}) passes on the respective particle to Alice^{*i+1*} (Bob^{*i+1*}) after performing suitable measurement. In this case, Alice^{*i*} (Bob^{*i*}) needs to perform measurement for detecting entanglement by disturbing the state minimally such that some entanglement remains in the post-measurement state to be detected by Alice^{*i+1*} (Bob^{*i+1*}). This can be achieved in the unsharp measurement formalism as the disturbance is minimized for any fixed amount of information gain in this formalism for qubits [23, 28].

3.1 Modified entanglement witness operator in unsharp measurement formalism

Note that the entanglement witness operator (1) can be implemented in the laboratory by performing projective quantum measurements. Since, in our scenarios, all Alices (Bobs), except the last Alice (Bob) in a sequence, perform unsharp measurements, the entanglement witness operator (1) needs to be modified accordingly. In order to modify the entanglement witness operator (1), we follow the process described in [46, 47].

Suppose, Alice^{*i*} and Bob^{*j*} perform measurement of spin component observables $\vec{\sigma} \cdot \hat{n}_i$ and $\vec{\sigma} \cdot \hat{m}_j$ respectively. The sharpness parameters associated with the measurements by Alice^{*i*} and Bob^{*j*} are denoted by ξ_i and λ_j respectively with $\xi_i, \lambda_j \in (0, 1]$. We will follow this notation throughout the paper.

The joint probability of obtaining the outcomes a_i, b_j (with $a_i, b_j \in \{+1, -1\}$), when Alice^{*i*} and Bob^{*j*} perform the above unsharp measurements, can be evaluated using the expression,

$$\text{Tr} \left[\rho \left(E_{a_i|\hat{n}_i}^{\xi_i} \otimes E_{b_j|\hat{m}_j}^{\lambda_j} \right) \right],$$

where ρ is the state shared by Alice^{*i*} and Bob^{*j*}; and the expressions of $E_{a_i|\hat{n}_i}^{\xi_i}$ and $E_{b_j|\hat{m}_j}^{\lambda_j}$ are defined following Eq.(3). The expectation value of the above joint measurement in the state ρ is given by,

$$\begin{aligned} \langle E_{\hat{n}_i}^{\xi_i} \otimes E_{\hat{m}_j}^{\lambda_j} \rangle &= \text{Tr} \left[\left\{ \left(E_{+|\hat{n}_i}^{\xi_i} - E_{-|\hat{n}_i}^{\xi_i} \right) \otimes \left(E_{+|\hat{m}_j}^{\lambda_j} - E_{-|\hat{m}_j}^{\lambda_j} \right) \right\} \rho \right] \\ &= \xi_i \lambda_j \text{Tr} \left[\left\{ \left(P_{+|\hat{n}_i} - P_{-|\hat{n}_i} \right) \otimes \left(P_{+|\hat{m}_j} - P_{-|\hat{m}_j} \right) \right\} \rho \right] \\ &= \xi_i \lambda_j \langle \vec{\sigma} \cdot \hat{n}_i \otimes \vec{\sigma} \cdot \hat{m}_j \rangle, \end{aligned} \quad (9)$$

where $\langle \vec{\sigma} \cdot \hat{n}_i \otimes \vec{\sigma} \cdot \hat{m}_j \rangle$ is the expectation value under projective measurements. We can hence use the substitution $\langle \vec{\sigma} \cdot \hat{n}_i \otimes \vec{\sigma} \cdot \hat{m}_j \rangle \rightarrow \xi_i \lambda_j \langle \vec{\sigma} \cdot \hat{n}_i \otimes \vec{\sigma} \cdot \hat{m}_j \rangle$ in order to obtain the modified entanglement witness operator for the case of unsharp measurements. For any $\xi_i, \lambda_j \in (0, 1]$, the modified entanglement witness operator for the state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ has the following form,

$$W^{(\xi_i, \lambda_j)} = \frac{1}{4} \left(\mathbb{I} \otimes \mathbb{I} + \xi_i \sigma_z \otimes \lambda_j \sigma_z - \xi_i \sigma_x \otimes \lambda_j \sigma_x - \xi_i \sigma_y \otimes \lambda_j \sigma_y \right)$$

$$= \frac{1}{4} \left[\mathbb{I} \otimes \mathbb{I} + \xi_i \lambda_j (\sigma_z \otimes \sigma_z - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y) \right]. \quad (10)$$

Now, for any separable state $\rho_s \in \mathcal{S}$, we have

$$\begin{aligned} \text{Tr} \left(W^{(\xi_i, \lambda_j)} \rho_s \right) &= \text{Tr} \left[(\xi_i \lambda_j W + \frac{1}{4} (1 - \xi_i \lambda_j) \mathbb{I} \otimes \mathbb{I}) \rho_s \right] \\ &= \xi_i \lambda_j \text{Tr} (W \rho_s) + \frac{1}{4} (1 - \xi_i \lambda_j). \end{aligned} \quad (11)$$

This implies that $\text{Tr} (W^{(\xi_i, \lambda_j)} \rho_s) \geq 0$ for all $\rho_s \in \mathcal{S}$ as $0 < \xi_i, \lambda_i \leq 1$. Hence, $W^{(\xi_i, \lambda_j)}$ is a valid entanglement witness operator.

4 Witnessing entanglement by sequential observers

We now focus on the task of entanglement detection in our sequential network scenario. We take three different types of the initially shared states ρ_i .

4.1 Initially shared maximally entangled two-qubit state

Let us consider that Alice¹ and Bob¹ initially share the Bell state given by, $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Different pairs of Alice and Bob (i.e., Alice¹-Bob¹, Alice²-Bob², Alice³-Bob³, ..., Aliceⁿ-Bobⁿ) try to detect entanglement sequentially. At first, we will address the following question: how many such pairs of Alice and Bob can sequentially detect entanglement.

Let Aliceⁱ and Bobⁱ ($i \in \{1, 2, \dots, n\}$) perform unsharp measurements with sharpness parameters ξ_i and λ_i respectively. The pair Alice¹-Bob¹ can detect entanglement if the following condition is satisfied:

$$\text{Tr} \left[W^{(\xi_1, \lambda_1)} |\psi^+\rangle \langle \psi^+| \right] < 0, \quad (12)$$

where the operator $W^{(\xi_1, \lambda_1)}$ is given by Eq.(10). After simplification, we get the following condition from (12),

$$\xi_1 \lambda_1 > \frac{1}{3} \quad (13)$$

Next, let us find out the post measurement state received by the pair Alice²-Bob² from Alice¹-Bob¹. As Alice² (Bob²) acts independent of the measurement setting and outcome of Alice¹ (Bob¹) in each experimental run, we take average over the measurement settings and outcomes by Alice¹ and Bob¹. Hence, the state received, on average, by Alice²-Bob² from Alice¹-Bob¹ is given by,

$$\rho_{A_2 B_2} = \frac{1}{9} \sum_{n_1, m_1, a_1, b_1} \left(\sqrt{E_{a_1|n_1}^{\xi_1}} \otimes \sqrt{E_{b_1|m_1}^{\lambda_1}} \right) |\psi^+\rangle \langle \psi^+| \left(\sqrt{E_{a_1|n_1}^{\xi_1}} \otimes \sqrt{E_{b_1|m_1}^{\lambda_1}} \right), \quad (14)$$

with $\hat{n}_1, \hat{m}_1 \in \{\hat{x}, \hat{y}, \hat{z}\}$ and $a_1, b_1 \in \{+1, -1\}$. Here, we have used the fact that each of Alice¹ and Bob¹ performs any of the three local unsharp measurements associated with the observables $\sigma_x, \sigma_y, \sigma_z$ in each experimental run in order to implement the entanglement witness operator (10). We have also used here the assumption that all possible measurement settings of Alice¹ and that of Bob¹ are equally probable. After simplification, we get from Eq.(14),

$$\rho_{A_2 B_2} = p|\psi^+\rangle\langle\psi^+| + (1-p)\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}$$

$$\text{with } p = \frac{1}{9} \left(1 + 2\sqrt{1 - \xi_1^2}\right) \left(1 + 2\sqrt{1 - \lambda_1^2}\right). \quad (15)$$

Since, the state (15) has the form given by Eq.(2), the Alice²-Bob² pair again uses the same entanglement witness operator given by Eq.(10) to detect entanglement. Hence, Alice²-Bob² can detect entanglement if the following condition is satisfied,

$$\text{Tr}\left[W^{(\xi_2, \lambda_2)} \rho_{A_2 B_2}\right] < 0, \quad (16)$$

which implies the condition,

$$\xi_2 \lambda_2 > \frac{3}{\left(1 + 2\sqrt{1 - \xi_1^2}\right) \left(1 + 2\sqrt{1 - \lambda_1^2}\right)}. \quad (17)$$

Proceeding in a similar way, it can be shown that the state $\rho_{A_3 B_3}$ received, on average, by Alice³-Bob³ from Alice²-Bob² has the similar form of Werner state (2) and the pair Alice³-Bob³ can detect entanglement if

$$\text{Tr}\left[W^{(\xi_3, \lambda_3)} \rho_{A_3 B_3}\right] < 0, \quad (18)$$

i.e., when

$$\xi_3 \lambda_3 > \frac{27}{\prod_{i=1}^2 \left[\left(1 + 2\sqrt{1 - \xi_i^2}\right) \left(1 + 2\sqrt{1 - \lambda_i^2}\right)\right]}. \quad (19)$$

Repeating the above steps, it can be shown that Alice⁴-Bob⁴ can detect entanglement if the following condition is satisfied,

$$\xi_4 \lambda_4 > \frac{243}{\prod_{i=1}^3 \left[\left(1 + 2\sqrt{1 - \xi_i^2}\right) \left(1 + 2\sqrt{1 - \lambda_i^2}\right)\right]}. \quad (20)$$

Similar conditions can be found out that ensure entanglement detection by the other pairs, i.e., Aliceⁿ-Bobⁿ.

Now, our purpose is to investigate what is the maximum number of the sequential pairs succeed in witnessing the entanglement of the shared two-qubit state. Combining Eqs.(13), (17), (19) and (20) and performing some analytical

calculations (see Appendix A for details), we get that Alice¹-Bob¹, Alice²-Bob² and Alice³-Bob³ can detect entanglement if the following conditions are satisfied simultaneously,

$$\xi_1 = \lambda_1 = 0.58 + \delta_1 \quad \text{with } 0 \leq \delta_1 \ll 1, \quad (21)$$

$$\xi_2 = \lambda_2 = 0.66 + \delta_2 \quad \text{with } 0 \leq \delta_2 \ll 1, \quad (22)$$

$$\xi_3 = \lambda_3 = 0.79 + \delta_3 \quad \text{with } 0 \leq \delta_3 \ll 1. \quad (23)$$

Here the numerical digits appearing in the the above conditions are rounded to two decimal places.

If Alice¹-Bob¹, Alice²-Bob² and Alice³-Bob³ perform measurements with sharpness parameters satisfying Eqs.(21) (22), (23), then it can be shown that Alice⁴-Bob⁴ cannot witness the entanglement even if they perform projective measurements, i.e., with $\xi_4 = \lambda_4 = 1$ (see Appendix A for details). Thus, at most three sequential pairs of Alice and Bob can witness the entanglement in this case.

4.2 Initially shared two-qubit Werner state

Let Alice¹ and Bob¹ initially share the two-qubit Werner state mentioned in Eq.(2). For this state, the optimal entanglement witness operator remains the same as before, i.e., it is W given by Eq.(1) [62].

In this case at most three sequential pairs of Alice and Bob (for example, Alice¹-Bob¹, Alice²-Bob² and Alice³-Bob³) can witness the entanglement using the entanglement witness operator given by Eq.(10). Furthermore, we get the following results:

(1) When $0.80 < p \leq 1$, each of the pairs Alice¹-Bob¹, Alice²-Bob² and Alice³-Bob³ can detect the entanglement. Other pairs Alice^{*i*}-Bob^{*i*} with $i \in \{4, 5, 6, \dots\}$ cannot detect entanglement.

(2) When $0.57 < p \leq 0.80$, each of the two pairs Alice¹-Bob¹ and Alice²-Bob² can detect the entanglement. Other pairs Alice^{*i*}-Bob^{*i*} with $i \in \{3, 4, 5, \dots\}$ cannot detect entanglement.

(3) When $0.33 < p \leq 0.57$, only the pair Alice¹-Bob¹ can detect the entanglement. Other pairs Alice^{*i*}-Bob^{*i*} with $i \in \{2, 3, 4, \dots\}$ cannot detect entanglement.

4.3 Initially shared non-maximally entangled two-qubit pure state

Suppose that Alice¹ and Bob¹ initially share a non-maximally entangled two-qubit pure state given by,

$$|\Psi\rangle = \cos \theta |01\rangle + \sin \theta |10\rangle, \quad (24)$$

with $0 < \theta < \frac{\pi}{4}$. For this state also, the optimal entanglement witness operator is given by Eq.(1) [62].

In this case too, at most three pairs of Alice and Bob (e.g., Alice¹-Bob¹, Alice²-Bob², Alice³-Bob³) can witness entanglement sequentially through the entanglement witness operator (10). Further, the following results are obtained.

(1) When $\frac{\pi}{8} \leq \theta < \frac{\pi}{4}$, each of the pairs Alice¹-Bob¹, Alice²-Bob² and Alice³-Bob³ can detect the entanglement. Other pairs Alice^{*i*}-Bob^{*i*} with $i \in \{4, 5, 6, \dots\}$ cannot detect entanglement.

(2) When $\frac{\pi}{17} \leq \theta < \frac{\pi}{8}$, each of the two pairs Alice¹-Bob¹ and Alice²-Bob² can detect the entanglement. Other pairs Alice^{*i*}-Bob^{*i*} with $i \in \{3, 4, 5, \dots\}$ cannot detect entanglement.

(3) When $0 < \theta < \frac{\pi}{17}$, only the pair Alice¹-Bob¹ can detect the entanglement. Other pairs Alice^{*i*}-Bob^{*i*} with $i \in \{2, 3, 4, \dots\}$ cannot detect entanglement.

4.4 Advantage of the sequential measurement scenario

For sequential detection of entanglement by multiple pairs of observers, it is important to ensure that the expectation values of the witness operator for different pairs become as much negative as possible, for feasibility of practical detection of entanglement. In our case, as we discussed earlier, maximum three pairs of Alice and Bob can detect entanglement. Hence, for our purpose, we define ‘Detectability’ (D) as the minus one times the sum of the expectation values of the entanglement witness operators for all the three pairs.

Mathematically, detectability is defined as

$$D = (-1) \sum_{i=1}^3 D_{ii} = (-1) \sum_i \text{Tr} \left[W^{(\xi_i, \lambda_i)} \rho_{A_i B_i} \right] \\ \text{with } \text{Tr} \left[W^{(\xi_i, \lambda_i)} \rho_{A_i B_i} \right] < 0 \forall i, \quad (25)$$

where $\rho_{A_i B_i}$ is the state shared by the pair Alice^{*i*}-Bob^{*i*}. Since, negative expectation value of the witness operator implies detection of entanglement, we have taken minus sign in the above definition to make D positive when each pair detects entanglement.

Now it is of practical demand to look for a measurement strategy that would yield optimum witness of entanglement in the sequential measurement scenario. For that we need to define the maximum detectability, D_{\max} which is obtained by maximizing D over all possible sharpness parameters (ξ_i, λ_i) of all the three pairs of observers under the constraint that each of the three pairs can detect entanglement, i.e., $\text{Tr} \left[W^{(\xi_i, \lambda_i)} \rho_{A_i B_i} \right] < 0 \forall i \in \{1, 2, 3\}$.

Mathematically, maximum detectability D_{\max} is defined as,

$$D_{\max} = \max_{\xi_1, \lambda_1, \xi_2, \lambda_2, \xi_3, \lambda_3} D$$

such that

$$D_{ii} = \text{Tr} \left[W^{(\xi_i, \lambda_i)} \rho_{A_i B_i} \right] < 0 \quad \forall i \in \{1, 2, 3\}. \quad (26)$$

D_{\max} serves as a tool to quantify the overall ability of the three pairs of observers with their best measurement strategy to detect entanglement in an experiment in a sequential measurement scenario. Here, the best measurement strategy implies the one that makes the expectation values of the witness operators as much negative as possible for all the three pairs simultaneously. Note that when D in the above definition is maximized, individual $\text{Tr} \left[W^{(\xi_i, \lambda_i)} \rho_{A_i B_i} \right]$ may not be optimized. This is because the expectation values of the witness operators of all the three pairs are not optimized simultaneously. Also, for example, when the expectation value of the witness operator for the first pair of Alice and Bob is optimized, other subsequent pairs may not detect any entanglement. Since our objective in the present paper is to optimize the expectation values of the witness operators for all the three pairs simultaneously, we have taken the above definition of maximum detectability.

Also note here that although the above definition is expressed for three pairs of Alice and Bob (relevant for the present study), it can be generalized to any number of pairs of observers depending on the specific context under consideration.

We now demonstrate the advantage of the sequential scenario when Alice¹ and Bob¹ initially share the Bell state $\rho_{A_1 B_1} = |\psi^+\rangle\langle\psi^+|$ with $|\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. In this case, at most three pairs can detect entanglement. Hence, D in this case is given by,

$$\begin{aligned} D &= -D_{11} - D_{22} - D_{33} \\ &= (-1) \sum_{i=1}^3 \text{Tr} \left[W^{(\xi_i, \lambda_i)} \rho_{A_i B_i} \right], \end{aligned} \quad (27)$$

where $\rho_{A_i B_i}$ is the state shared, on average, by the pair Alice^{*i*}-Bob^{*i*}. Based on the analysis described in Sec. 4.1, it can be shown that

$$\begin{aligned} D &= -\frac{1}{4}(1 - 3\xi_1 \lambda_1) - \frac{1}{12} \left[3 - \left(1 + 2\sqrt{1 - \xi_1^2} \right) \left(1 + 2\sqrt{1 - \lambda_1^2} \right) \xi_2 \lambda_2 \right] \\ &\quad - \frac{1}{108} \left[27 - \left(1 + 2\sqrt{1 - \xi_1^2} \right) \left(1 + 2\sqrt{1 - \lambda_1^2} \right) \right. \\ &\quad \left. \left(1 + 2\sqrt{1 - \xi_2^2} \right) \left(1 + 2\sqrt{1 - \lambda_2^2} \right) \xi_3 \lambda_3 \right]. \end{aligned} \quad (28)$$

Now, we get $D_{\max} = 0.20$, which is obtained for $\xi_1 = \lambda_1 = 0.73$, $\xi_2 = \lambda_2 = 0.80$, and $\xi_3 = \lambda_3 = 1$.

Next, let us evaluate the total RoM that is needed to achieve the above-mentioned D_{\max} . As mentioned earlier, for the unsharp measurement $E_{\hat{n}}^\lambda \equiv \{E_{+|\hat{n}}^\lambda, E_{-|\hat{n}}^\lambda\}$ defined by Eq.(3), we have $R(M) = \lambda$. Hence, the total RoM, denoted by $R_{\text{total}}(M)$, needed to achieve the above-mentioned D_{\max} is given by,

$$R_{\text{total}}(M) = \sum_{i=1}^3 (\xi_i + \lambda_i)$$

such that

$$D = D_{\max} = 0.20. \quad (29)$$

Consequently, we have that

$$\begin{aligned} R_{\text{total}}(M) &= 2(0.73 + 0.80 + 1) \\ &= 5.06 \end{aligned} \quad (30)$$

$R_{\text{total}}(M)$ quantifies the total amount of resource consumed while performing the unsharp measurements necessary for achieving the maximum detectability.

Next, we will compare the resource requirement in terms of the entanglement consumed and total RoM between the sequential measurement scenario (1) and the corresponding non-sequential measurement scenario. The non-sequential measurement scenario involves three pairs of observers- Alice¹-Bob¹, Alice²-Bob² and Alice³-Bob³, where each of these three pairs share one copy of a two-qubit entangled state. Hence, this non-sequential measurement scenario involves total three pairs of entangled qubits. Each pair detects entanglement of their shared state using the witness operator (10). For the purpose of meaningful comparison of the sequential and the non-sequential scenarios, we need to ensure that the detectability D remains the same for both the scenarios. This will confirm that the overall ability of all the three pairs of observers to detect entanglement is the same in both the scenarios. Next, we will perform the aforementioned comparison when different types of entangled states are shared between the three pairs of Alice and Bob in the non-sequential scenario.

4.4.1 Comparison with non-sequential scenario involving three pairs of entangled qubits in Werner states

Let us first consider that the following Werner state,

$$\rho_{\text{W}}(p_i) = p_i |\psi^+\rangle\langle\psi^+| + (1 - p_i) \frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2} \quad (31)$$

with $0 < p_i \leq 1$ is shared between the pair Alice ^{i} -Bob ^{i} in the aforementioned non-sequential scenario with $i \in \{1, 2, 3\}$. Here, p_1 , p_2 , and p_3 are different in general. Let $\tilde{\xi}_i$ and $\tilde{\lambda}_i$ denote the sharpness parameters for the measurements by Alice ^{i} and Bob ^{i} respectively in the non-sequential scenario. On the other hand, as mentioned earlier, ξ_i and λ_i are the sharpness parameters for the measurements by Alice ^{i} and Bob ^{i} respectively in the sequential measurement case. Here $\tilde{\xi}_i$ may or may not be equal to ξ_i and $\tilde{\lambda}_i$ may or may not be equal to λ_i for all $i \in \{1, 2, 3\}$ in general.

Now, we would like to evaluate the minimum amount of the total entanglement (denoted by η) that is necessary in the non-sequential measurement scenario for the detectability in the non-sequential measurement scenario given by, $D^{\text{NS}} = (-1) \sum_{i=1}^3 \text{Tr} \left[W^{(\tilde{\xi}_i, \tilde{\lambda}_i)} \rho_{\text{W}}(p_i) \right] = (-1) \sum_{i=1}^3 \frac{1}{4} (1 - 3p_i \tilde{\xi}_i \tilde{\lambda}_i)$ being equal to the maximum detectability in the sequential measurement scenario

		Sequential measurement scenario	Non-sequential measurement scenario
Detectability D		0.20	0.20
Total robustness of measurement $R_{\text{total}}(M)$		5.06	5.06
Total entanglement consumed η	with $\rho_W(p_i)$ in non-sequential case	1 ebit	≥ 1.11 ebits
	with ρ_{p_i} in non-sequential case	1 ebit	≥ 1.11 ebits
	with $ \Psi(\theta_i)\rangle$ in non-sequential case	1 ebit	≥ 1.11 ebits

Table 1: The total amount of required entanglement η when each party performs equally resourceful measurements in both scenarios. Here, the non-sequential measurement scenario involves either three pairs of entangled qubits in the Werner states $\rho_W(p_i)$, or three pairs of entangled qubits in the mixed states with colored noise ρ_{p_i} , or three pairs of qubits in non-maximally entangled pure states $|\Psi(\theta_i)\rangle$. All numerical values presented in this table are rounded to two decimal places.

denoted by $D_{\text{max}}^S = 0.2$ and the total RoM in the non-sequential measurement case given by $\sum_{i=1}^3 (\tilde{\xi}_i + \tilde{\lambda}_i)$ being equal to that in the sequential measurement case, i.e., equal to 5.06. Also, we must ensure that each of the pairs of Alice and Bob can detect entanglement in the non-sequential scenario while performing the above minimization problem. This last constraint is a natural demand for a meaningful comparison.

The total concurrence of the three copies of the Werner states (31) is, $\eta = \sum_{i=1}^3 C(\rho_W(p_i)) = \sum_{i=1}^3 \frac{1}{2}(3p_i - 1)$. Thus, now the task is to minimize η with the following constraints: $\sum_{i=1}^3 (\tilde{\xi}_i + \tilde{\lambda}_i) = 5.06$, $\sum_{i=1}^3 \frac{1}{4}(1 - 3p_i \tilde{\xi}_i \tilde{\lambda}_i) = -0.2$ and $\frac{1}{4}(1 - 3p_i \tilde{\xi}_i \tilde{\lambda}_i) < 0 \ \forall i \in \{1, 2, 3\}$. Now it turns out that the minimum of η under the above constraints is given by, $\eta_{\text{min}} = 1.11$. This is achieved when $p_1 = 0.54$, $p_2 = 0.54$, $p_3 = 0.65$, $\tilde{\xi}_1 = \tilde{\lambda}_1 = 0.79$, $\tilde{\xi}_2 = \tilde{\lambda}_2 = 0.79$, $\tilde{\xi}_3 = \tilde{\lambda}_3 = 0.95$. Now, as mentioned earlier, D_{max}^S in the sequential scenario is achieved for $\xi_1 = \lambda_1 = 0.73$, $\xi_2 = \lambda_2 = 0.80$, $\xi_3 = \lambda_3 = 1$. Hence, we don't need to take equal sharpness parameter in both the scenarios for each of the observers in order to satisfy equal amount of the total RoM in both scenarios and $D^{\text{NS}} = D_{\text{max}}^S$.

On the other hand, only 1 ebit is sufficient in the sequential measurement scenario for achieving D_{max}^S as only one copy of the maximally entangled state is involved. Hence, for achieving the same amount of detectability in the sequential and the non-sequential measurement scenarios with the total resourcefulness of the measurements being equal in both the scenarios, the non-sequential measurement scenario needs greater amount of entanglement compared to the sequential measurement scenario. This result is summarized in Table 1. This demonstrates a resource theoretic advantage of the sequential measurement scenario over the non-sequential measurement scenario.

It is possible to demonstrate further the advantage of the sequential measurement scenario from another perspective as well. Let us now take the total

		Sequential measurement scenario	Non-sequential measurement scenario
Detectability D		0.20	0.20
Total entanglement consumed η		1 ebit	1 ebit
Total robustness of measurement	with $\rho_W(p_i)$ in non-sequential case	5.06	≥ 5.16
	with ρ_{p_i} in non-sequential case	5.06	≥ 5.17
	with $ \Psi(\theta_i)\rangle$ in non-sequential case	5.06	≥ 5.17

Table 2: The required total robustness of measurement $R_{\text{total}}(M)$ when the same amount of total entanglement is consumed in the both scenarios. The non-sequential measurement scenario involves either three pairs of entangled qubits in the Werner states $\rho_W(p_i)$, or three pairs of entangled qubits in the mixed states with colored noise ρ_{p_i} , or three pairs of qubits in non-maximally entangled pure states $|\Psi(\theta_i)\rangle$. All numerical values presented in the table are rounded to two decimal places.

entanglement consumed η in the two scenarios to be equal. The detectability D^{NS} in the non-sequential measurement scenario is also taken to be equal to the maximum detectability $D_{\text{max}}^{\text{S}}$ in the sequential measurement scenario. We also ensure that each of the pairs of Alice and Bob can detect entanglement in the non-sequential scenario. Under these conditions, we compare the total RoM, i.e., $R_{\text{total}}(M)$ in the two scenarios.

Total entanglement consumption in the sequential measurement scenario starting with a maximally entangled initial state is $\eta = 1$ ebit. Hence, for the non-sequential measurement scenario we use three different Werner states, $\rho_W(p_i)$, $i = 1, 2, 3$ with the total amount of entanglement being equal to that in the sequential measurement scenario, i.e., $\sum_{i=1}^3 C(\rho_W(p_i)) = 1$, i.e., $p_1 + p_2 + p_3 = 1.67$. The other constraints are $D^{\text{NS}} = (-1) \sum_{i=1}^3 \text{Tr} [W^{(\tilde{\xi}_i, \tilde{\lambda}_i)} \rho_W(p_i)] = D_{\text{max}}^{\text{S}} = 0.20$ and $\text{Tr} [W^{(\tilde{\xi}_i, \tilde{\lambda}_i)} \rho_W(p_i)] < 0 \forall i \in \{1, 2, 3\}$. Now, the task is to minimize the total RoM, $R_{\text{total}}(M) = \sum_{i=1}^3 (\tilde{\xi}_i + \tilde{\lambda}_i)$ in the non-sequential scenario with the aforementioned constraints. It turns out that the minimum total RoM in the non-sequential scenario is 5.16.

On the other hand, the total RoM in the sequential measurement scenario is given by 5.06, as mentioned earlier. Hence, for achieving the same amount of detectability using the same amount of total initial entanglement in the sequential and the non-sequential measurement scenarios, the total resourcefulness of the measurements (in terms of total RoM) necessary in the non-sequential measurement scenario turns out to be greater than that in the sequential measurement scenario. Thus a resource theoretic advantage of the sequential measurement scenario over the non-sequential one is clearly implied. These results are depicted in Table 2.

4.4.2 Comparison with non-sequential scenario involving three pairs of entangled qubits in mixed states with colored noise

Now, we consider that the pair Aliceⁱ-Bobⁱ (with $i \in \{1, 2, 3\}$) in the non-sequential measurement scenario shares the following mixed state with colored noise,

$$\rho_{p_i} = p_i |\phi^+\rangle\langle\phi^+| + \frac{1-p_i}{2} (|01\rangle\langle 01| + |10\rangle\langle 10|), \quad (32)$$

with $0 < p_i \leq 1$.

Before proceeding, we will construct the optimal entanglement witness operator of the above state using the method described in [62, 68, 69]. First, we compute the eigenvector corresponding to the negative eigenvalue of $\rho_{p_i}^{TB}$, where $\rho_{p_i}^{TB}$ denotes the partial transposition of ρ_{p_i} . Then the entanglement witness operator is given by the partially transposed projector onto that eigenvector. Following this approach, we obtain the following optimal entanglement witness operator for the state (32),

$$\widetilde{W} = \frac{1}{4} (\mathbb{I} \otimes \mathbb{I} - \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z). \quad (33)$$

Now, suppose in the non-sequential measurement scenario Aliceⁱ performs unsharp measurements with sharpness parameter $\tilde{\xi}_i \in (0, 1]$ and Bobⁱ performs unsharp measurements with sharpness parameter $\tilde{\lambda}_i \in (0, 1]$. Following the calculations mentioned in Sec. 3.1, it can be shown that the modified entanglement witness operator of the state (32) for Aliceⁱ and Bobⁱ is given by,

$$\widetilde{W}^{(\tilde{\xi}_i, \tilde{\lambda}_i)} = \frac{1}{4} \left[\mathbb{I} \otimes \mathbb{I} - \tilde{\xi}_i \tilde{\lambda}_i (\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) \right]. \quad (34)$$

Following the method mentioned in Sec. 3.1, it can be easily shown that for any separable state $\rho_s \in \mathcal{S}$, $\text{Tr}(\widetilde{W}^{(\tilde{\xi}_i, \tilde{\lambda}_i)} \rho_s) \geq 0$ for all $\tilde{\xi}_i, \tilde{\lambda}_i \in (0, 1]$.

As mentioned earlier, in the sequential measurement scenario, D_{\max}^S is achieved for $\xi_1 = \lambda_1 = 0.73$, $\xi_2 = \lambda_2 = 0.80$, and $\xi_3 = \lambda_3 = 1$, i.e., total RoM=5.06. Now, we would like to evaluate what is the minimum amount of the total entanglement η that is necessary in the non-sequential measurement scenario for the the total RoM in the non-sequential measurement case being equal to that in the sequential measurement case and for the detectability in the non-sequential measurement scenario given by, $D^{\text{NS}} = (-1) \sum_{i=1}^3 \text{Tr}[\widetilde{W}^{(\tilde{\xi}_i, \tilde{\lambda}_i)} \rho_{p_i}] = (-1) \sum_{i=1}^3 \frac{1}{4} [1 + \tilde{\xi}_i \tilde{\lambda}_i (1 - 4p_i)]$ being equal to $D_{\max}^S = 0.20$. Here also we must ensure that each pair can detect entanglement in the non-sequential scenario. The total concurrence of the three states is, $\eta = \sum_{i=1}^3 C(\rho_{p_i}) = \sum_{i=1}^3 (2p_i - 1)$. Thus, now the task is to minimize η with the following constraints: $\sum_{i=1}^3 (\tilde{\xi}_i + \tilde{\lambda}_i) = 5.06$, $\sum_{i=1}^3 \frac{1}{4} [1 + \tilde{\xi}_i \tilde{\lambda}_i (1 - 4p_i)] = -0.2$, and $\frac{1}{4} [1 + \tilde{\xi}_i \tilde{\lambda}_i (1 - 4p_i)] < 0$ for all $i \in \{1, 2, 3\}$. Now it turns out that $\eta_{\min} = 1.11$. On the other hand, only 1 ebit is sufficient in the sequential measurement scenario for achieving D_{\max}^S as only one copy of the maximally

entangled state is involved. So here also, the total entanglement consumption in the non-sequential scenario is greater than the sequential scenario for achieving the same detectability using equally resourceful measurements in the two scenarios. This result is summarized in Table 1. This again demonstrates a resource theoretic advantage of the sequential measurement scenario over the non-sequential measurement scenario.

Next, we take the total entanglement consumed η in the non-sequential scenario to be equal to that in the sequential scenario ($= 1$ ebit) which gives rise the following constraint, $p_1 + p_2 + p_3 = 2$. Also, the detectability D^{NS} in the non-sequential measurement scenario has to be equal to the maximum detectability $D_{\text{max}}^{\text{S}}$ in the sequential measurement scenario, i.e., $\sum_{i=1}^3 \frac{1}{4} [1 + \tilde{\xi}_i \tilde{\lambda}_i (1 - 4p_i)] = -0.2$. Finally, we must ensure that each pair in the non-sequential scenario detects entanglement- $\frac{1}{4} [1 + \tilde{\xi}_i \tilde{\lambda}_i (1 - 4p_i)] < 0$, $\forall i \in \{1, 2, 3\}$. Under these conditions, we compare the total RoM, i.e., $R_{\text{total}}(M)$ in the two scenarios. Here too, we observe that the minimum amount of total RoM required in the non-sequential measurement scenario is given by, $R_{\text{total}}^{\text{min}}(M) = 5.17$. These results are depicted in Table 2. This again signifies a resource theoretic advantage of the sequential measurement scenario over the non-sequential one.

4.4.3 Comparison with non-sequential scenario involving three pairs of qubits in non-maximally entangled pure states

Next, we perform a similar comparison between the sequential and the non-sequential measurement scenario, where the pair Alice^{*i*}-Bob^{*i*} (with $i \in \{1, 2, 3\}$) in the non-sequential measurement scenario shares the following non-maximally entangled pure state,

$$|\Psi(\theta_i)\rangle = \cos \theta_i |01\rangle + \sin \theta_i |10\rangle, \quad (35)$$

with $0 < \theta_i < \frac{\pi}{4}$. In the non-sequential measurement scenario, each pair of Alice and Bob detects entanglement of the above pure state through the entanglement witness operator given by Eq.(10).

Similar to the earlier cases, at first, we calculate the total amount of entanglement η that is necessary in the non-sequential measurement scenario involving three different non-maximally entangled pure states $|\Psi(\theta_i)\rangle$ for the detectability in the non-sequential measurement scenario: $D^{\text{NS}} = \sum_{i=1}^3 \text{Tr} \left[W^{(\tilde{\xi}_i, \tilde{\lambda}_i)} |\Psi(\theta_i)\rangle \langle \Psi(\theta_i)| \right]$ to be equal to the maximum detectability in the sequential measurement scenario and for the total RoM in the non-sequential measurement scenario to be equal to that of the sequential scenario. Also, we ensure that each pair in the non-sequential case detects entanglement. The results in this case can be obtained following the similar steps as adopted in the previous two cases. These results are summarized in Table 1. Here too, it is evident that the sequential measurement scenario is advantageous over the non sequential scenario.

Now, let us consider that the total entanglement consumed η in the non-sequential measurement scenario is equal to that in the sequential measurement scenario ($= 1$ ebit). Also, consider that the detectability in the non-sequential measurement scenario is equal to the maximum detectability in the sequential measurement scenario and each of the three pairs in the non-sequential case detects entanglement. Under these conditions we have again found that the minimum amount of total measurement resource required in the non-sequential scenario is greater than that in the sequential one (see Table 2). Thus, the advantage of the sequential measurement scenario over the non-sequential one is reinforced.

5 Quantum teleportation

In this section, we illustrate the resource theoretic efficacy of the sequential measurement scheme by presenting an example of witnessing entangled states useful for quantum teleportation. There exist a class of Hermitian witness operators [70] that can detect entangled states useful for performing quantum teleportation [9]. We will show below that a single copy of an entangled state can be recycled such that multiple pairs of observers can detect entangled states suitable for quantum teleportation. This thus demonstrates that if a pair detects entanglement using a quantum state, then the residual entanglement after this entanglement detection can again be used to perform quantum teleportation by another pair of observers. However, in the conventional approach employing non-sequential measurement strategies, more than one copies of entangled states are required for sequentially performing the two tasks- entanglement detection and quantum teleportation.

Here, we consider that the following maximally entangled mixed state [71] is initially shared in the sequential measurement scenario,

$$\rho_{\text{MEMS}} = \begin{pmatrix} h(c) & 0 & 0 & \frac{c}{2} \\ 0 & 1 - 2h(c) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{c}{2} & 0 & 0 & h(c) \end{pmatrix},$$

$$\text{with } h(c) = \begin{cases} \frac{c}{2}, & \text{if } c \geq \frac{2}{3}, \\ \frac{1}{3}, & \text{if } c < \frac{2}{3}. \end{cases} \quad (36)$$

with c being the concurrence of ρ_{MEMS} . The teleportation witness operator for this state is given by [70],

$$W_{\text{tel}} = \frac{1}{4} \left(\mathbb{I} \otimes \mathbb{I} - \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z \right), \quad (37)$$

which satisfies the following properties: $\text{Tr}(W_{\text{tel}} \sigma) \geq 0$ for all states σ that are not useful for teleportation and $\text{Tr}(W_{\text{tel}} \rho) < 0$ for at least one state ρ that is useful for teleportation.

Now using the same prescription described earlier it can be shown that at most three pairs of Alice and Bob in the aforementioned sequential measurement scenario can get negative expectation values of the witness operator (37) when a single copy of the state (36) is initially shared. Further, three pairs can have negative expectation values of the witness operator (37) when ρ_{MEMS} with concurrence $c \geq 0.93$ is recycled.

Next, in the present context, let us compare the sequential scenario and the corresponding non-sequential scenario. Let in the sequential scenario, Alice¹ and Bob¹ initially share the state ρ_{MEMS} with the minimum amount of necessary concurrence such that three pairs can get negative expectation values of the witness operator (37), i.e., with concurrence $c = 0.93$. Following the similar calculations as described earlier it can be shown that the maximum detectability in the sequential case is given by, $D_{\text{max}}^{\text{S}} = 0.11$. This occurs for total RoM being equal to 4.96.

Next, in the non-sequential scenario, we consider that each pair Alice^{*i*}-Bob^{*i*} with $i \in \{1, 2, 3\}$ shares the state ρ_{MEMS} given by (36) with concurrence c_i . Now, we compute the total amount of entanglement $\eta = c_1 + c_2 + c_3$ that is necessary in the non-sequential measurement scenario for the detectability in the non-sequential measurement scenario being equal to the maximum detectability in the sequential measurement scenario and for the total RoM in the non-sequential measurement scenario to be equal to that in the sequential scenario. Also, we ensure that each pair in the non-sequential case detects useful state for teleportation. It turns out that minimum total entanglement consumed in the non-sequential scenario is 1.98 ebits, thus clearly implying an advantage of the sequential scenario.

Therefore, this analysis implies that the sequential scenario requires less resource compared to the corresponding non-sequential scenario for executing sequential detection of entangled states or useful states for quantum teleportation.

6 More number of observers in an asymmetric scenario

In the scenario considered so far in this work, we assume that the number of sequential Alices on one side is equal to the number of sequential Bobs on the other. We call this scenario as a ‘‘Symmetric Scenario’’. At this stage it may be pertinent to ask the question as to what happens if we relax this condition of symmetry. To this end, let us now consider sequential detection of entanglement in the case when the number of Alices is not equal in general to the number of Bobs. We consider that the Bell state given by, $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is initially shared.

It may be noted here that an asymmetric scenario involving a single Alice and multiple Bobs (Bob¹, Bob², Bob³, \dots , Bob^{*m*}) was discussed in [46]. In this case, it was shown that at most twelve Bobs can detect entanglement with the single Alice [46].

Next, consider the scenario involving two Alices (Alice¹ and Alice²) and multiple Bobs (Bob¹, Bob², Bob³, \dots , Bob ^{m}). At first, the pair Alice¹-Bob¹ detects entanglement using the entanglement witness operator (10). As mentioned in Sec. 4.1, this pair can detect entanglement if the condition (13) is satisfied. The state $\rho_{A_2B_2}$ received, on average, by Alice²-Bob² from Alice¹-Bob¹ is given by Eq.(15). Since, there are only two Alices in the sequence, Alice² performs projective measurements. On the other hand, all subsequent Bobs (i.e., Bob², Bob³, \dots , Bob ^{m}) perform unsharp measurements. The sharpness parameter associated with the measurement by Bob ^{i} (with $i \in \{2, 3, \dots, m\}$) is denoted by λ_i . Hence, the modified entanglement witness operator used by the pair Alice²-Bob ^{i} is given by,

$$W^{(\lambda_i)} = \frac{1}{4} \left[\mathbb{I} \otimes \mathbb{I} + \lambda_i (\sigma_z \otimes \sigma_z - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y) \right]. \quad (38)$$

This can be obtained from the entanglement witness operator given by Eq.(1) and following the analysis mentioned in Sec. 3.1. It can be shown that for any separable state $\rho_s \in \mathcal{S}$, we have $\text{Tr} \left(W^{(\lambda_j)} \rho_s \right) \geq 0$.

Alice² and Bob² can witness entanglement if

$$\text{Tr} \left[W^{(\lambda_2)} \rho_{A_2B_2} \right] < 0, \quad (39)$$

which implies the following condition,

$$\lambda_2 > \frac{3}{\left(1 + 2\sqrt{1 - \xi_1^2}\right) \left(1 + 2\sqrt{1 - \lambda_1^2}\right)}. \quad (40)$$

The average state shared between Alice² and Bob³ is given by,

$$\rho_{A_2B_3} = \frac{1}{3} \sum_{m_2, b_2} \left(\mathbb{I} \otimes \sqrt{E_{b_2|\hat{m}_2}^{\lambda_2}} \right) \rho_{A_2B_2} \left(\mathbb{I} \otimes \sqrt{E_{b_2|\hat{m}_2}^{\lambda_2}} \right), \quad (41)$$

with $\hat{m}_2 \in \{\hat{x}, \hat{y}, \hat{z}\}$ and $b_2 \in \{+1, -1\}$. Here also, we use the assumption that all possible measurement settings of Bob² are equally probable.

Similarly, Alice² and Bob³ can witness entanglement if the following condition is satisfied,

$$\text{Tr} \left[W^{(\lambda_3)} \rho_{A_2B_3} \right] < 0. \quad (42)$$

The above condition implies the following,

$$\lambda_3 > \frac{9}{\left(1 + 2\sqrt{1 - \xi_1^2}\right) \left(1 + 2\sqrt{1 - \lambda_1^2}\right) \left(1 + 2\sqrt{1 - \lambda_2^2}\right)}. \quad (43)$$

Repeating the above calculations for other subsequent Bobs (i.e., Bob⁴, Bob⁵, \dots , Bob ^{m}), we can derive the conditions on $\lambda_4, \lambda_5, \dots$.

Next, we determine what is the maximum number of Bobs succeed in witnessing the entanglement of the shared state with Alice². Combining the conditions (13), (40), (43), other similar conditions on $\lambda_4, \lambda_5, \dots$, and performing

analytical calculations as described in Appendix A, we get that Bob², Bob³, \dots , Bob⁸ can detect entanglement with Alice², if the following conditions are satisfied simultaneously,

$$\begin{aligned}
\xi_1 &= \lambda_1 = 0.58 + \tilde{\delta}_1 \text{ with } 0 \leq \tilde{\delta}_1 \ll 1, \\
\lambda_2 &= 0.44 + \tilde{\delta}_2 \text{ with } 0 \leq \tilde{\delta}_2 \ll 1, \\
\lambda_3 &= 0.47 + \tilde{\delta}_3 \text{ with } 0 \leq \tilde{\delta}_3 \ll 1, \\
\lambda_4 &= 0.51 + \tilde{\delta}_4 \text{ with } 0 \leq \tilde{\delta}_4 \ll 1, \\
\lambda_5 &= 0.56 + \tilde{\delta}_5 \text{ with } 0 \leq \tilde{\delta}_5 \ll 1, \\
\lambda_6 &= 0.63 + \tilde{\delta}_6 \text{ with } 0 \leq \tilde{\delta}_6 \ll 1, \\
\lambda_7 &= 0.74 + \tilde{\delta}_7 \text{ with } 0 \leq \tilde{\delta}_7 \ll 1, \\
\lambda_8 &= 0.95 + \tilde{\delta}_8 \text{ with } 0 \leq \tilde{\delta}_8 \ll 1.
\end{aligned} \tag{44}$$

Here the numerical values appearing in the above conditions are rounded to two decimal places.

If Alice¹, Alice², Bob¹, Bob², Bob³, \dots , Bob⁸ perform particular measurements with sharpness parameters satisfying the conditions mentioned in (44), then it can be shown that Bob⁹ cannot witness entanglement with Alice² even if Bob⁹ performs projective measurements, i.e., with $\lambda_9 = 1$. Hence, at most eight sequential Bobs can detect entanglement in this scenario with two Alices.

Next, let us consider the scenario involving three Alices (Alice¹, Alice² and Alice³) and multiple Bobs (Bob¹, Bob², Bob³, \dots , Bob ^{m}). At first, the pair Alice¹-Bob¹ detects entanglement using the entanglement witness operator (10). Then Alice¹ and Bob¹ pass their particles to Alice² and Bob², respectively. The pair Alice²-Bob² detects entanglement using the same entanglement witness operator (10) and passes the respective particles to the pair Alice³-Bob³ who performs measurements to detect entanglement. Now, Bob³ passes his particle to Bob⁴ so that the pair Alice³-Bob⁴ can detect entanglement. Subsequently, Bob⁴ passes the particle to Bob⁵, and so on. In this scenario, following the aforementioned calculations, it can be shown that Bob³, Bob⁴ and Bob⁵ can detect entanglement with Alice³. No additional Bob can detect entanglement with Alice³. Hence, at most five Bobs can detect entanglement.

Finally, if we consider that there are four Alices (Alice¹, Alice², Alice³ and Alice⁴) and multiple Bobs (Bob¹, Bob², Bob³, \dots , Bob ^{m}), then the result derived in Sec. 4.1 for the symmetric scenario tells us that it is not possible for four pairs (i.e., Alice¹-Bob¹, Alice²-Bob², Alice³-Bob³ and Alice⁴-Bob⁴) to detect entanglement sequentially. Entanglement detection is possible only up to the third pair.

Before concluding, it may be noted that a resource theoretic comparison of the above asymmetric scenarios can be performed with the corresponding non-sequential scenarios involving multiple copies of mixed or non-maximally entangled pure initial states. As expected, similar to the case of the symmetric

scenario, here too it is possible to observe advantages of the sequential scenario in terms of the entanglement consumed and robustness of measurement.

7 Concluding Discussions

To summarize, our analysis presented in this paper clearly shows a hitherto unexplored resource theoretic advantage of recycling a single copy of a two-qubit quantum state towards detecting entanglement by multiple observers. Specifically, we have analyzed in detail a scenario involving multiple independent observers acting sequentially on each of the two spatially separated wings that initially share the resource of a single copy of a two-qubit entangled state. The number of observers on the two wings may be equal or unequal, depending upon the type of network considered. We have estimated the maximum number of observers that can detect entanglement sequentially using only one pair of qubits. Our results indicate that one can reduce the number of physical qubits needed in the context of performing different entanglement-assisted quantum tasks multiple times in network scenarios.

Furthermore, we have performed a quantitative analysis of the advantage of the sequential measurement scenario involving multiple sequential observers on both wings from the perspective of resource requirements. In particular, we have shown that the above scenario can help in reducing the necessary requirement of the total initial entanglement as well as the resource cost of the measurements necessary for detecting entanglement by multiple sequential observers. These advantages demonstrate the benefits of recycling single-shot entanglement in various entanglement-assisted quantum information processing and communication tasks in practical contexts, in comparison with various standard schemes employing either multiple copies of two-qubit entangled states, or multipartite entangled states for performing such tasks.

The analysis of this paper may be extended to certain interesting directions. The scheme of unsharp measurement that we have adopted in this work employs the same value of the sharpness parameter associated with measurements in all directions by an individual observer. Considering different sharpness parameters for different measurement input settings by an observer can lead to a significant increase in the number of allowed observers [44, 59, 60]. The resource theoretic efficacy of such a scheme would then be worthwhile to study. A fertile direction of study could also be to generalize the scheme formulated in the present paper towards studying resource theoretic efficacy of sequential sharing of single-shot entanglement in the context of multi-qubit and two-qudit higher dimensional entangled states. Finally, categorizing various information processing tasks involving sequential measurements [51–58, 61] in terms of their resource theoretic advantages is another potentially attractive direction of future study.

Before concluding, it may be noted that several entanglement witnessing based tasks (for example, detecting eavesdropping in quantum key distribution [72], estimating localizable entanglement [73]) may be required to be

performed sequentially in order to execute some communication/computation protocol in practical situations. This can be achieved in one of the following two ways: (i) by using different copies of quantum states for different rounds of tasks, or (ii) by recycling the same copy of a quantum states. Obviously, the second approach requires less number of physical systems. However, from a resource theoretic point of view, it had been unclear before the analysis of our present work whether the second method really consumes less resource (either in terms of total entanglement, or in terms of total resourcefulness of all the measurements performed). This is because it may so happen that the total amount of entanglement of all quantum states necessary in the first approach is smaller than the necessary entanglement of the single copy of the quantum state needed in the second approach. It is this void in the literature that the present paper seeks to fill in by establishing a resource centric advantage of the second method. However, the present study does not capture the advantage of the sequential scenario from all perspectives. For example, sharing one copy of an entangled state and sharing multiple copies of entangled states between spatially separated observers require different experimental efforts. Therefore, probing a more general resource based comparison between sequential and non-sequential scenario incorporating all such factors related to practical implementations is worth for future research and our present analysis indeed motivates future studies along this direction.

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Data availability statement

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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A Appendix

The condition for Alice¹ and Bob¹ to witness entanglement is given by,

$$\xi_1 \lambda_1 > \frac{1}{3} \quad (45)$$

Similarly, the condition for Alice² and Bob² to witness entanglement is given by,

$$\xi_2 \lambda_2 > \frac{3}{\left(1 + 2\sqrt{1 - \xi_1^2}\right)\left(1 + 2\sqrt{1 - \lambda_1^2}\right)}. \quad (46)$$

Now, in order to ensure maximum number of sequential observers witnessing entanglement, Alice¹ and Bob¹ should choose the sharpness parameters of their measurements in such a way that these can detect entanglement causing minimal disturbance to the state. In any unsharp measurement of the form (3) on qubits, the disturbance can be reduced by reducing the associated sharpness parameter [23, 28]. Hence, Alice¹ and Bob¹ should choose sharpness parameters satisfying the following relation,

$$\xi_1 \lambda_1 = \frac{1}{3} + \epsilon_1 \quad \text{with } 0 < \epsilon_1 \ll 1. \quad (47)$$

Next, the sharpness parameters of the measurements by Alice² and Bob² also should ensure detection of entanglement causing minimal disturbance to the state. For this, we have to minimize the right hand side of (46) under the constraint given by Eq.(47). In other words, we have to perform the following optimization problem,

$$\begin{aligned} & \max_{\xi_1, \lambda_1} f_1(\xi_1, \lambda_1) \\ & \text{such that} \\ & \xi_1 \lambda_1 = \frac{1}{3} + \epsilon_1 \\ & 0 < \epsilon_1 \ll 1, \\ & 0 < \xi_1, \lambda_1 \leq 1, \end{aligned} \quad (48)$$

where

$$f_1(\xi_1, \lambda_1) = \left(1 + 2\sqrt{1 - \xi_1^2}\right)\left(1 + 2\sqrt{1 - \lambda_1^2}\right). \quad (49)$$

Now, taking $\epsilon_1 = 10^{-2}$, we obtain $\xi_1 = \lambda_1 = 0.58$. Hence, the condition (46) becomes

$$\xi_2 \lambda_2 > 0.43. \quad (50)$$

(Note that all numerical values appearing in this appendix are rounded to two decimal places.)

Next, we find out the conditions under which Alice¹-Bob¹, Alice²-Bob² and Alice³-Bob³ can witness entanglement in such a way that the measurement by each observer causes minimum possible disturbance to the state. Alice³-Bob³ can detect entanglement if

$$\xi_3 \lambda_3 > \frac{27}{\prod_{i=1}^2 \left[\left(1 + 2\sqrt{1 - \xi_i^2}\right)\left(1 + 2\sqrt{1 - \lambda_i^2}\right) \right]}. \quad (51)$$

In order to ensure minimum possible disturbance by Alice³-Bob³ while witnessing entanglement, we minimize the right hand side of (51) performing the following optimization problem,

$$\max_{\xi_2, \lambda_2} f_2(\xi_2, \lambda_2)$$

$$\begin{aligned}
& \text{such that} \\
& \xi_2 \lambda_2 = 0.43 + \epsilon_2, \\
& 0 < \epsilon_2 \ll 1, \\
& 0 < \xi_2, \lambda_2 \leq 1,
\end{aligned} \tag{52}$$

where

$$f_2(\xi_2, \lambda_2) = \prod_{i=1}^2 \left[\left(1 + 2\sqrt{1 - \xi_i^2}\right) \left(1 + 2\sqrt{1 - \lambda_i^2}\right) \right] \quad \text{with} \quad \xi_1 = \lambda_1 = 0.58. \tag{53}$$

Taking $\epsilon_2 = 10^{-2}$, we obtain $\xi_2 = \lambda_2 = 0.66$. With these values, the condition (51) becomes

$$\xi_3 \lambda_3 > 0.62. \tag{54}$$

Proceeding in a similar way, we check the conditions on the parameters under which Alice¹-Bob¹, Alice²-Bob², Alice³-Bob³ and Alice⁴-Bob⁴ can witness entanglement. Alice⁴-Bob⁴ can detect entanglement if

$$\xi_4 \lambda_4 > \frac{243}{\prod_{i=1}^3 \left[\left(1 + 2\sqrt{1 - \xi_i^2}\right) \left(1 + 2\sqrt{1 - \lambda_i^2}\right) \right]}. \tag{55}$$

Here the corresponding optimization problem is,

$$\begin{aligned}
& \max_{\xi_3, \lambda_3} f_3(\xi_3, \lambda_3) \\
& \text{such that} \\
& \xi_3 \lambda_3 = 0.62 + \epsilon_3, \\
& 0 < \epsilon_3 \ll 1, \\
& 0 < \xi_3, \lambda_3 \leq 1,
\end{aligned} \tag{56}$$

where

$$f_3(\xi_3, \lambda_3) = \prod_{i=1}^3 \left[\left(1 + 2\sqrt{1 - \xi_i^2}\right) \left(1 + 2\sqrt{1 - \lambda_i^2}\right) \right] \quad \text{with} \quad \xi_1 = \lambda_1 = 0.58, \xi_2 = \lambda_2 = 0.66. \tag{57}$$

Taking $\epsilon_3 = 10^{-2}$, we get $\xi_3 = \lambda_3 = 0.79$. However, with these values, Eq.(55) becomes

$$\xi_4 \lambda_4 > 1.13. \tag{58}$$

Since $\xi_4, \lambda_4 \in (0, 1]$, the above condition cannot be satisfied. Therefore, at most three pairs (Alice¹-Bob¹, Alice²-Bob², Alice³-Bob³) can detect entanglement.