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Impact of cash flow mapping on VaR calculation of bond portfolios

Lúcia Gonçalves Nunes

Master in Finance

Supervisor:
PhD António Manuel Rodrigues Guerra Barbosa, Assistant Professor,
Iscte-Iul

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Resumo

A estimação exata do Value-at-Risk (VaR) é um desafio constante na gestão de risco. Deste modo, vários autores focam-se no estudo de diferentes modelos de estimação da volatilidade ou métodos do VaR. Contudo, a análise da influência da escolha do mapeamento de cash flows no cálculo do VaR não se mostra expressiva. Assim, esta dissertação visa determinar a perda de precisão no cálculo do VaR associada ao mapeamento dos cash flows em portefólios de obrigações, comparando diferentes abordagens com uma alternativa que não carece desse procedimento. Para isso, o nosso estudo requer os parâmetros de modelização da curva das taxas de juro que nos possibilita a obtenção de dados para qualquer maturidade. Os resultados desta dissertação enfatizam a importância da escolha de um mapeamento adequado, nomeadamente quando as maturidades dos cash flows são predominantemente de curto prazo para se evitar erros muito elevados.

Palavras-Chave: Cash flow mapping; Value-at-Risk; Obrigações

Classificação JEL: G12, G32

Abstract

The exact estimation of Value-at-Risk (VaR) is a constant challenge in risk management. Most authors focus their studies on topics such as volatility estimation models or VaR methods; however, the study of the impact of risk factor mapping on VaR estimation has not received much attention. Thus, this dissertation aims to determine the loss of precision on cash flow mapping on the calculation of the VaR of bond portfolios, comparing different cash flow mapping approaches with an alternative that does not involve cash flow mapping. For this, our study requires yield curve modeling parameters that will allow us to have data for any cash flow maturity. The results highlight the importance of choosing a suitable cash flow mapping, namely, when the cash flows' maturities are in the short term, to avoid vast errors in all the methods.

Keywords: Cash flow mapping; Value-at-Risk; Bonds

JEL Classification: G12, G32

Contents

1 Introduction

One of the goals of finance practitioners is the accurate measurement of market risk. There are different tools to quantify the risk of financial loss arising from unexpected fluctuations in market prices or rates. One of the most widely used tools is Value-at-Risk (VaR), which calculates the maximum potential loss with a given time horizon and confidence level of a specific portfolio (Alexander, 2008b).

On a large bond portfolio, cash flows can occur anytime. Consequently, to estimate the Parametric Normal VaR, we would need to consider the volatilities and correlations of interest rates for all the different cash flow dates, leading to a large covariance matrix of risk factors.

Therefore, to reduce the dimensionality and ensure data availability, it is common to select a set of interest rate maturities, called vertex maturities. As a result, if the maturity of a cash flow does not coincide with a vertex maturity, it is distributed over the available vertices through cash flow mapping. Mapping a cash flow involves splitting the cash flow at the previous and the following vertex. Some procedures based on different conditions dictate how to perform this division.

This dissertation analyzes the impact of cash flow mapping techniques on the VaR estimation of bond portfolios. Understanding the influence of each approach is helpful for market risk analysts seeking to adopt the most accurate. The purpose of this dissertation is to answer the following questions: What loss of precision does cash flow mapping entails when estimating the VaR? Do the differences between mapping methods justify the use of more complicated methods that sometimes do not have a solution?

We innovate by comparing the VaR estimates with different cash flow mappings with an approach that does not involve cash flow mapping. The no-mapping method is our reference and requires the interest rates series for all the cash flow dates, which we obtain through the historical series of coefficients for the Svensson (1994) curve.

First, our analysis focuses on a single cash flow, and then, to consider the interactions between cash flows, we analyze portfolios of bonds whose cash flows we simulate randomly. The main goal is to find the magnitude of errors associated with each mapping and consequently detect if the choice of cash flow mapping strongly influences the VaR estimation.

We contribute to the literature by comparing different cash flow mapping techniques. To apply the most accurate procedures, we investigate the volatility invariant mappings performance by considering the covariance between interest rates on the variance formula.

The results show that the magnitude of errors depends on the maturity of the bonds, being more expressive in the short term, where all the methods obtain large values. Additionally, most methods tend to register higher VaR estimation than the reference in extreme maturity ranges (short-term and above 15 years).

2 Literature Review

The VaR is a standard measure of market risk, which tells us the portfolio's potential losses for a given period and a specific probability of occurrence. This popular measure is the focus of the study of many financial practitioners, which try to continuously make refinements in the VaR calculation, namely using different methods and volatility estimation models.

One of the major differences between the VaR estimation methods is the inference of the distribution of returns. According to Alexander (2008b), the three main models to estimate the VaR are the Parametric VaR, the Historical Simulation, and the Monte Carlo Simulation. The Parametric VaR assumes the portfolio returns follow a specific distribution. An example of a well-known parametric approach is the RiskMetrics methodology (Morgan & Reuters, 1996) that bases the VaR estimation under the Normal distribution. Although this assumption simplifies the VaR estimation, it may lead to an underestimation of the VaR because fat tails are common in asset returns, meaning that extreme outcomes occur more frequently than a normal distribution predicts (Hendricks, 1996). On the other hand, the Historical Simulation does not involve a particular assumption of the return distribution except their stationarity. Pérignon and Smith (2010) show this method's popularity, reporting that 73% of banks use Historical Simulation. Despite being an easy method to implement and defend, it relies on a subjective sample size choice, where a larger sample increases the estimation's precision but decreases its relevance. Alternatively, the Monte Carlo VaR models assume a functional form for the stochastic process that generates risk factor returns. This approach is very flexible and applicable to portfolios with non-linear positions (Ammann & Reich, 2001). In terms of disadvantages, it can be computationally intensive for large portfolios and suffer from an incorrect choice of functional forms (Abad et al., 2014).

Another possible method involves quantile regression models that use explanatory variables to improve the forecastability of VaR. Guo et al. (2013) compare the quantile regression approach pioneered by Koenker and Bassett (1978) with traditional VaR models and state that it is a more robust method. Shifting the attention to the behavior of the quantile, Engle and Manganelli (2004) propose conditional autoregressive VaR, which models the quantile directly instead of the whole distribution.

The primary and most determining aspect of many of the models is how to forecast the volatility. Engle (1982) introduces the Autoregressive Conditional Heteroskedasticity (ARCH) model, which replaces the assumption of constant volatility with a variance modeled as a function of lagged squared prediction errors. Bollerslev (1986) extends this model to the

Generalized ARCH (GARCH) model. Additionally, there are extensions to adequately capture the asymmetric behavior of the volatility, including the exponential (Nelson, 1991), the threshold (Zakoian, 1994), and the Glosten-Jagannathan-Runkle (Glosten et al., 1993) GARCH model. Alternatively, another popular estimator for volatility is the Exponentially Weighted Moving Average (EWMA) due to its use by RiskMetrics (Morgan & Reuters, 1996).

Investigating the performance of VaR methodologies, Abad and Benito (2013) show that parametric methods can obtain successful VaR measures if the conditional variance is appropriately estimated. Furthermore, their results indicate that the best model is a parametric method under Student's t distribution with conditional variance estimated by an asymmetric GARCH model. In contrast, multiple authors (e.g., Bekiros & Georgoutsos, 2005; Gençay & Selçuk, 2004; Kuester et al., 2006) argue that the Extreme Value Theory (EVT) framework is a superior modeling alternative, especially at extremely high tails (greater than 99% confidence level). Moreover, studies such as Giamouridis and Ntoula (2009), Marimoutou et al. (2009), and Žiković and Aktan (2009) find that close to the accuracy of the EVT approach is the Filtered Historical Simulation.

Focusing on large portfolios, an inherent challenge to VaR is accurately estimating the high dimensional conditional covariance matrices due to the fastly increasing number of parameters. Based on the idea that a few components can drive co-movements in the market, factor models emerge to tackle the problem, which, jointly with other studies, results in a diversity of alternatives to deal with the dimensionality problem (Chang et al., 2018; Engle et al., 2019; Fan et al., 2008; Li et al., 2016). Exploring the recent one, Engle et al. 2019 use the composite likelihood method of Pakel et al. (2014) and the nonlinear shrinkage method of Ledoit and Wolf (2012) to estimate the correlation matrix.

Another current topic aiming to improve the forecasts of dynamic covariance matrices is via intraday data (Callot et al., 2017; Fiszeder et al., 2019; Lunde et al., 2016; Nard et al., 2022). With also associated with a generalized autoregressive score model seems to improve the prediction of VaR and Expected Shortfall (ES) (Lazar & Xue, 2020). Moreover, such joint improvement is relevant since the ES reinforces its application after the Basel III Accord (2019).

Whichever the VaR method, it is necessary to simplify the portfolio by replacing the current positions with exposures on risk factors. Jorion (2007) differentiates three methods to map fixed income portfolios: duration mapping, principal mapping, and cash flow mapping. Duration mapping considers as a risk factor a zero-coupon bond with maturity equal to the portfolio's duration, while principal mapping considers the portfolio's average maturity. In cash flow mapping, all cash flows are treated separately and mapped to vertices.

The allocation of the original cash flow to the two adjacent vertices commonly satisfies certain conditions, including preservation of value, sign, and risk (Das, 2006). One approach to ensure risk preservation is to use the present value of a basis point (PV01), where the PV01 of the allocated cash flows matches the original cash flow. A closely related procedure is to use duration. Despite the similarity between duration and PV01, Alexander (2008a) carefully distinguishes these interest rate risk measures.

For RiskMetrics (Morgan & Reuters, 1996) the risk preservation condition is based on the variance of financial returns. Normally we know the variance at the standard vertices, but we do not know the variance of the original position. To obtain this variance some authors (Deutsch, 2009; Esch et al., 2005; Morgan & Reuters, 1996) interpolate linearly the standard deviations of the adjacent vertices and then square it. Alternatively, Alexander (2008a) interpolates variances instead of standard deviations, which leads to different results since the relation between variance and standard deviation is not linear.

Schaller (1996) shows that the RiskMetrics methodology (Morgan & Reuters, 1996) maps more than 65% of the cash flow's value to the vertex with higher volatility, even though the maturity of the cash flow is closer to the vertex with lower volatility. To overcome this, Schaller (1996) proposes a new mapping approach, where the allocation to an adjacent vertex tends towards 100%, as the maturity of the original cash flow is close to it. Besides this, his mapping also preserves the risk, matching the variance with the interpolated standard deviation squared.

Updating the RiskMetrics Technical Document of 1996, Mina and Xiao (2001) suggest an alternative mapping that behaves well under extreme vertex volatility and correlation scenarios. This mapping preserves the sensitivity of the present value to changes in the zero rates for the two adjacent vertices but not the present value of the cash flow.

Henrard (2000) compares six cash flow mappings, where two of them, namely the polar coordinates mapping and the three dimensional mapping, are new mapping systems. For this, he restricts his study to standard maturities and computes for the different mappings the residual risk of a position, which is short 1 million in that maturity and is long by the same million on the two neighboring maturities. His results indicate that globally the best mappings are elementary, rates, polar coordinates, and three dimensional. The author also classifies the rates mapping as the best method due to its calculation speed and accuracy.

3 Methodology

3.1 Bond VaR method

3.1.1 VaR

Our dissertation compares different VaR estimations to assess the performance of the different cash flow mapping approaches. The VaR represents the maximum potential loss that we expect not to exceed if the portfolio is not rebalanced during a given period (h) and under a specific significance level (α). Statistically, the $100\alpha\%$ h-day VaR $(VaR_{h,\alpha})$ corresponds to the symmetric of the α quantile of the h-day discounted return distribution (X_{ht}), that is:

$$P\left(X_h < -VaR_{h,\alpha}\right) = \alpha \tag{1}$$

Throughout our work, we set the α to 1%.

Since this dissertation's main objective is to determine the impact of cash flow mapping in the VaR estimation, we restrict our analysis to the Parametric Normal VaR due to its simplicity. Under the parametric normal VaR, we assume a normal distribution for our portfolios' returns, $X_h \sim N(\mu_h, \sigma_h^2)$, where μ_h and σ_h^2 represent the mean and variance estimates, respectively. Thus, the derivation to the formula for the $100\alpha\%$ h-day Parametric Normal VaR leads to:

$$VaR_{h,\alpha} = \Phi^{-1}(1-\alpha)\sigma_h - \mu_h \tag{2}$$

where Φ^{-1} is the inverse standard normal cumulative distribution function.

According to Alexander (2008b), we can set the drift adjustment (μ_h) to zero since we are estimating 1-day VaR $(h = 1)$. The simplified expression is:

$$VaR_{h,\alpha} = \Phi^{-1}(1-\alpha)\sigma_h \tag{3}$$

3.1.2 Risk Factors of Cash Flows

As the cash flows of the bonds are known, the bond value fluctuates due to changes in the discount rates applicable for the maturity of each cash flow. Therefore, the risk factor is the interest rates, whose exposure is a function of the sensitivity of the present value of cash flows to changes in interest rates, measurable by the PV01.

For a single cash flow with maturity T (c_T), the PV01 measures the change in the present value when the interest rate (r_T) shifts down by one basis point. Assuming continuous compounding, the present value (PV) is:

$$PV(c_T, r_T) = c_T e^{-r_T T} \tag{4}$$

Then, we approximate the PV01 as:

$$\begin{aligned} PV01(c_T, r_T) &\approx \frac{\partial PV(c_T, r_T)}{\partial r_T} \times (-0.01\%) \\ &= T \times PV(c_T, r_T) \times 0.01\% \end{aligned} \tag{5}$$

Similarly, using a first-order Taylor expansion, the change in the present value, defined as the profit & loss (P&L), is:

$$\begin{aligned} \Delta PV(c_T, r_T) &\approx \frac{\partial PV(c_T, r_T)}{\partial r_T} \times \Delta r_T \approx -PV01(c_T, r_T) \times \frac{\Delta r_T}{0.01\%} \\ &= -PV01(c_T, r_T) \times \Delta r_T(b.p.) \end{aligned} \tag{6}$$

Then for a portfolio with several cash flows, the P&L corresponds to the sum of the single cash flows P&Ls:

$$P\&L \equiv \Delta PV(c, r) \approx \sum_{i=1}^{n} -PV01_{T_i} \times \Delta r_{T_i}(b.p.) \tag{7}$$

In vector form, the change in the present value is:

$$\Delta PV \approx \Theta^T \times \Delta r(b.p.) \tag{8}$$

where Θ is the vector with the $-$PV01s for the maturities that we collect data, and Δr represents the vector of changes in the interest rates.

Since the behavior of the risk factors is unknown, we consider that it follows a normal distribution, $\Delta r(b.p.) \sim N(\mu, \Sigma)$, with mean vector μ and covariance matrix Σ (in basis points).

By the properties of a normal distribution, the P&L is a linear transformation of that random variable, implying that:

$$P\&L \equiv \Delta PV \sim N(\Theta^T \mu, \Theta^T \Sigma \Theta) \tag{9}$$

After knowing the variance ($\Theta^T \Sigma \Theta$), we compute the VaR as usual.

3.1.3 Mapping

Generally, we do not have access to historical interest rate series for all the cash flow maturities in a portfolio to have a covariance matrix with all maturities. For this purpose, mapping emerges as a solution and a simplifier since we map the cash flow with maturity T to its neighboring vertices T_1 and T_2 ($T_1 < T < T_2$), for which we have data.

Our dissertation works with two sets of vertices: the first comprises 12 nodes and encompasses the maturities 3 months, 6 months, 1 year, 2 years, 3 years, 4 years, 5 years, 7 years, 10 years, 15 years, 20 years, and 30 years, while the second set of vertices encompasses the maturities from 3 months to 30 years spaced monthly.

We study five different cash flow mappings: PV+PV01; PV+Vol; PV01+Vol; PV+PV01+Vol (Alexander, 2008a) and Mina & Xiao (2001), from now on referred to as Rates (designation attributed by Henrard (2000)).

The PV+PV01 (Alexander, 2008a) notation indicates the simultaneous preservation of the present value and PV01 of the original cash flow.

Without loss of generality, if we consider a cash flow with a present value of 1, the present value invariant condition is:

$$x_{T_1} + x_{T_2} = 1 \tag{10}$$

where x_{T_i} is the percentage of the present value of the original cash flow that you allocate into the vertex maturity T_i, $i = 1{,}2$.

Additionally, the PV01 preservation equation is:

$$x_{T_1} T_1 + x_{T_2} T_2 = T \tag{11}$$

Joining both equations 10 and 11, we have for the PV+PV01 mapping a system whose allocations correspond to:

$$x_{T_1} = \frac{T_2 - T}{T_2 - T_1} \text{ and } x_{T_2} = 1 - x_{T_1} \tag{12}$$

The PV+Vol mapping (Alexander, 2008a) preserves the present value and the cash flow volatility. However, the derivation of the volatility invariant formula made by Alexander (20008a) is not coherent with her previous conditions. Considering equation 6, the variance of the P&L of the original cash flow is:

$$\begin{aligned} \sigma^2_{\Delta PV_o} &= var[-PV01_T \times \Delta r_T(b.p.)] = PV01_T^2 \times \sigma_T^2 \\ &\approx [T \times PV(c_T, r_T) \times 0.01\%]^2 \sigma_T^2 \\ &= (T \times 0.01\%)^2 \sigma_T^2 = T^2 \sigma_T^2 \times (0.01\%)^2 \end{aligned} \tag{13}$$

And the variance of the mapped cash flows is:

$$\begin{aligned} \sigma^2_{\Delta PV} &= var\left[-PV01_{T_1} \times \Delta r_{T_1}(b.p.) - PV01_{T_2} \times \Delta r_{T_2}(b.p.)\right] \\ &\approx \left(T_1 x_{T_1} \times 0.01\%\right)^2 \sigma_{T_1}^2 + \left(T_2 x_{T_2} \times 0.01\%\right)^2 \sigma_{T_2}^2 \\ &+ 2\left(T_1 x_{T_1} \times 0.01\%\right)\left(T_2 x_{T_2} \times 0.01\%\right) \sigma_{T_1,T_2} \\ &= \left[\left(T_1 x_{T_1}\right)^2 \sigma_{T_1}^2 + \left(T_2 x_{T_2}\right)^2 \sigma_{T_2}^2 \right. \\ &+ \left. 2 T_1 x_{T_1} T_2 x_{T_2} \sigma_{T_1,T_2}\right] \times (0.01\%)^2 \end{aligned} \tag{14}$$

Equating the two previous expressions, we obtain a volatility invariant mapping defined as:

$$\left(T_1 x_{T_1}\right)^2 \sigma_{T_1}^2 + \left(T_2 x_{T_2}\right)^2 \sigma_{T_2}^2 + 2 T_1 x_{T_1} T_2 x_{T_2} \sigma_{T_1,T_2} = T^2 \sigma_T^2 \tag{15}$$

where $\sigma_{T_i}^2$ is the variance of absolute basis point changes in interest rates of maturity T_i, $i = 1, 2$, σ_T^2 is the variance of absolute basis point changes of the interest rates for the original cash flow's maturity, and σ_{T_1,T_2} is the covariance between the changes in interest rates of vertex maturities T_1 and T_2.

Thus, by joining equations 10 and 15, the PV+Vol mapping has the following allocations:

$$x_{T_1}^2\left({T_1}^2\sigma_{T_1}^2 + {T_2}^2\sigma_{T_2}^2 - 2T_1T_2\sigma_{T_1,T_2}\right) + 2x_{T_1}\left(T_1T_2\sigma_{T_1,T_2} - {T_2}^2\sigma_{T_2}^2\right) + \left({T_2}^2\sigma_{T_2}^2 - T^2\sigma_T^2\right) = 0 \quad \text{and} \quad x_{T_2} = 1 - x_{T_1} \tag{16}$$

The PV01+Vol mapping preserves simultaneously the PV01 and the volatility. Combining then equations 11 and 15 in a system, we obtain:

$$x_{T_1}^2{T_1}^2\left(\sigma_{T_1}^2 + \sigma_{T_2}^2 - 2\sigma_{T_1,T_2}\right) + 2x_{T_1}T_1T\left(\sigma_{T_1,T_2} - \sigma_{T_2}^2\right) + T^2\left(\sigma_{T_2}^2 - \sigma_T^2\right) = 0 \quad \text{and} \quad x_{T_2} = \frac{T - T_1x_{T_1}}{T_2} \tag{17}$$

The PV+PV01+Vol is a mapping with no algebraic solution, so we solved it numerically using the *fminunc* function in MATLAB. Besides, this method distributes the original cash flow between three vertices, where the third vertex is the closest to the cash flow maturity (disregarding the adjacent vertices that we already consider).

The Rates mapping (Mina & Xiao, 2001) performs a volatility invariant mapping for the whole bond portfolio, working with a covariance matrix for the absolute changes in the spot rates for the cash flow maturities and not the vertex maturities. It has built-in that the computation of the cash flow variance at a non-standard maturity considers the covariance between the interest rates, contrary to the typical procedures in the literature of direct linear interpolations of standard deviations or variances. Despite being a more complex method, it has a simple allocation scheme (Henrard, 2000) that encompasses only the consideration of maturities:

$$x_{T_1} = \frac{T}{T_1}\frac{T_2 - T}{T_2 - T_1} \text{ and } x_{T_2} = \frac{T}{T_2}\frac{T - T_1}{T_2 - T_1} \tag{18}$$

3.1.4 Benchmark

To determine the most accurate mapping, we have an approach that does not require cash flow mapping as the reference. For this, we compute the interest rates series for every cash flow maturity by applying the Svensson (1994) model, where the continuously compounded zero-coupon rate for maturity T is:

$$r_c(T) = \beta_0 + \beta_1 \left[\frac{1 - \exp\left(-\frac{T}{\tau_1}\right)}{\frac{T}{\tau_1}} \right] + \beta_2 \left[\frac{1 - \exp\left(-\frac{T}{\tau_1}\right)}{\frac{T}{\tau_1}} - \exp\left(-\frac{T}{\tau_1}\right) \right] + \beta_3 \left[\frac{1 - \exp\left(-\frac{T}{\tau_2}\right)}{\frac{T}{\tau_2}} - \exp\left(-\frac{T}{\tau_2}\right) \right] \tag{19}$$

where $\beta_0, \beta_1, \beta_2$ and β_3 are the parameters that measure respectively, the level, slope and curvatures of the term structure and τ_1 and τ_2 are parameters measuring the rate at which the short-term and medium-term components drop to zero.

It is questionable whether this VaR estimation will be the true VaR. However, the focus of our thesis is the study of the impact of cash flow mapping, and as such, this non-mapping approach is our benchmark to evaluate the different methods.

3.2 Procedures

3.2.1 EWMA

To estimate the covariance matrix, we follow the EWMA model, which gives greater weight to the most recent observations. The decay factor (λ) defines this relationship, and as suggested by Morgan and Reuters (1996), we set its value to 0.94 since our risk horizon is less than one month.

The recursive equation is:

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1 - \lambda) r_{t-1}^2 \tag{20}$$

where the variance for day t depends on the previous day's variance estimate (σ_{t-1}^2) and the previous day's squared returns (r_{t-1}^2).

3.2.2 Variance for a non-standard maturity

To calculate the variance of a cash flow with a non-standard maturity for mapping methods that require this variance as input, we apply the same procedure as the Rates method. Therefore, starting by considering a cash flow with non-standard maturity, we linearly interpolate r_T from the interest rates of the adjacent vertices, r_{T_1} and r_{T_2}:

$$
\begin{aligned}
r_T &= r_{T_1} + \frac{r_{T_2} - r_{T_1}}{T_2 - T_1}(T - T_1) = \frac{T_2 - T}{T_2 - T_1} r_{T_1} + \left(1 - \frac{T_2 - T}{T_2 - T_1}\right) r_{T_2} \\
&= w r_{T_1} + (1 - w) r_{T_2}
\end{aligned} \tag{21}
$$

where $w = \frac{T_2 - T}{T_2 - T_1}$ is the interpolation weight to give to r_{T_1}.

Knowing the covariance matrix:

$$
\begin{bmatrix} var(r_{T_1}) & cov(r_{T_1}, r_{T_2}) \\ cov(r_{T_1}, r_{T_2}) & var(r_{T_2}) \end{bmatrix} = \begin{bmatrix} \sigma_{T_1}^2 & \sigma_{T_1,T_2} \\ \sigma_{T_1,T_2} & \sigma_{T_2}^2 \end{bmatrix}
$$

we compute the variance of r_T as:

$$
\begin{aligned}
var(r_T) &= var(w r_{T_1} + (1 - w) r_{T_2}) \\
&= var(w r_{T_1}) + var((1 - w) r_{T_2}) + 2cov(w r_{T_1}, (1 - w) r_{T_2}) \\
&= w^2 \sigma_{T_1}^2 + (1 - w)^2 \sigma_{T_2}^2 + 2w(1 - w)\sigma_{T_1,T_2}
\end{aligned} \tag{22}
$$

3.2.3 Comparisons

In our work, we pursue several analyses to adequately compare the different cash flows mappings and consequently determine the best approach.

Firstly, we focus on an individual cash flow. Through this first examination, we intend to get a general idea of the behavior of each mapping, namely how accurate it is, if it is similar to other approaches, and investigate the influence of the choice of the vertices. Nevertheless, the single zero-coupon bond analysis is mainly needed to clarify the error evolution between the vertices, allowing us to determine the nodes' cash flow distance that induces a higher relative measurement error.

After that, we consider the interaction of cash flows by doing two studies with bond portfolios. Specifically, we explore the VaR measurements for the same date with 5000 different portfolios, and the VaR calculations once per month over a ten-year period for the 5000 portfolios.

Our portfolios have various maturity ranges. Firstly, the analyses focus on heterogeneous portfolios ranging from 3 months to 30 years. Then we insert restrictions, namely, portfolios with maturities from 3 months to 1 year; 1 year to 5 years; 5 years to 15 years; and 15 years to 30 years.

In the study with bond portfolios, we compute statistical indicators such as the median, standard deviation, minimum, maximum, and first and third quartile. Moreover, our work prioritizes the "mean |error|" indicator, which corresponds to the mean of the absolute value of the relative error, since it does not hide the true magnitude of the errors contrary to the mean where dispersed results with opposite signs cancel out the error amplitude. Another number that we pay special attention to is the percentage of overestimation associated with the mappings, which allows us to explore if a given method tends to have VaR estimations above or below the reference.

4 Data

To answer the initial questions this dissertation requires the historical series of interest rates that are freely available on the European Central Bank (ECB) website. The ECB releases daily euro area yield curves based on the Svensson (1994) model for all triple-A rating issuers. To obtain the historical series of interest rates for every possible maturity, we download the yield curve parameters: Beta 0, Beta 1, Beta 2, Beta 3, Tau 1, and Tau 2.

5 Result Analysis

5.1 Single zero-coupon bond

In this section we investigate the differences in the VaR estimate for a single cash flow corresponding to a zero-coupon bond. The VaR estimation date is 27/05/2021, and the maturity of the zero-coupon bond varies continuously from 3 months (27/08/2021) to 30 years (27/05/2051).

To access the errors in the cash flow mappings, we estimate the VaR for the different mapping methods and our reference method, which does not involve cash flow mapping. Furthermore, we start by analyzing the results with the set of 12 vertices, then with the monthly set, and finally introduce additional studies on the error limitation.

5.1.1 12 vertices

In figure 5.1, we present the relative differences between the results of PV+PV01, PV+Vol, PV01+Vol, PV+PV01+Vol and Rates against our benchmark using the set of 12 vertices.

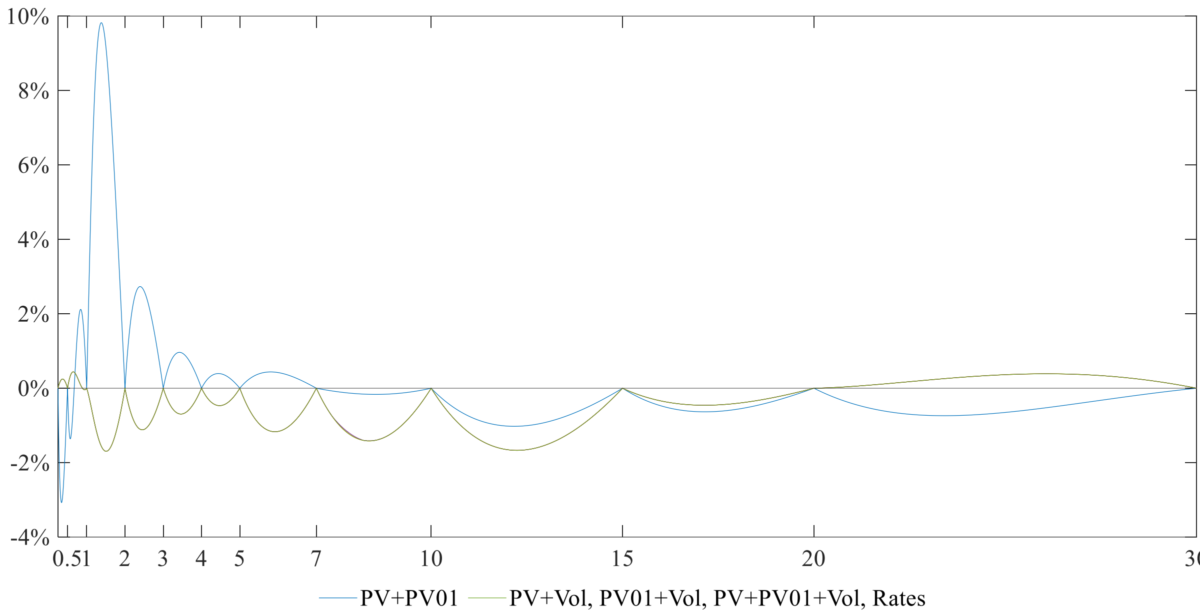

Figure 5.1 - The relative errors in the VaR calculation (1%) associated with PV+PV01, PV+Vol, PV01+Vol, PV+PV01+Vol and Rates mapping. The lines represent the difference between the VaR estimates between the specified cash flow mapping and the reference (no cash flow mapping procedure). The available vertices for allocation are 3 months, 6 months, 1 year, 2 years, 3 years, 4 years, 5 years, 7 years, 10 years, 15 years, 20 years, and 30 years.

Only two lines are distinguishable in figure 5.1 because the four volatility invariant mappings have equivalent errors, resulting in overlapping lines. Nevertheless, the

PV+PV01+Vol has slightly different values since its solution is determined numerically rather than algebraically.

Figure 5.1 shows that none of the methods has the lowest errors throughout all maturities. Besides, it suggests that the position that registers the maximum error discrepancy seems to be near the center of the two adjacent vertices of the cash flow. However, not precisely in the center.

Focusing on studying the position of the peaks (appendix A.1), we obtain an average relative placement of the PV+PV01 mapping of 41.32%, which translates into a left bias, that is, a closer position to the lower vertex. However, this value considers 12 peaks instead of 11 since it registers a local minimum and a local maximum between six-month and one-year maturity. Excluding these two peaks, the peak that presents the most significant distance from the central position occurs in the 10-year distance between the vertices, with a relative position to the lower vertex of 34.03%. However, there is no direct implication between distanced vertices and skewed peaks. An example is the first local, with the smallest distance between the neighboring vertices and one of the most skewed registers. Besides, we also have distinguished values (distancing up to 5.84%) for the same spacing between vertices.

Volatility invariant mappings present an average relative peak distance closer to the PV+PV01 mapping, although higher (49.67% for the PV+Vol, PV01+Vol, Rates, and 49.50% for PV+PV01+Vol). Therefore, it is more reliable to anticipate that maximum errors emerge in the middle of the distance between vertices in volatility invariant mappings.

Regarding the amplitude of the deviations, the PV+PV01 method has a maximum peak of 9.82%, which is a large error. In contrast, the volatility invariant mappings have their most significant deviation with a value of -1.70%. Although the maximum value for the various methods appears between the vertices of one and two years, there is significant variability in the results. Indeed, it is doubtful to associate that a period with a high error in PV+PV01 will also have a high error with the volatility invariant mappings. A clear example of this situation is the period from 0.25 to 1, where the methods that preserve volatility have their peaks with the lowest values, in contrast with the PV+PV01 results.

Moreover, it is interesting to detect that the volatility invariant mappings have the peaks with the lowest absolute error until 5 years. After that, such behavior changes, with the errors even becoming the largest.

On the other hand, the behavior of the error is also not uniform. There are several moments in which the PV+PV01 is registering values above the reference value, while for those same periods, the methods that preserve volatility underestimate. Thus, to check the influence of the vertex choice, we proceed to the study with vertices spaced monthly.

5.1.2 Monthly vertices

As expected, the errors decrease remarkably by redoing the calculations with vertices spaced monthly (appendix B.1) since null errors in vertex maturities imply that the more vertices available, the smaller the magnitude of the errors. Figure 5.2 represents the discrepancies up to 5 years of maturity to clarify the curvature behavior.

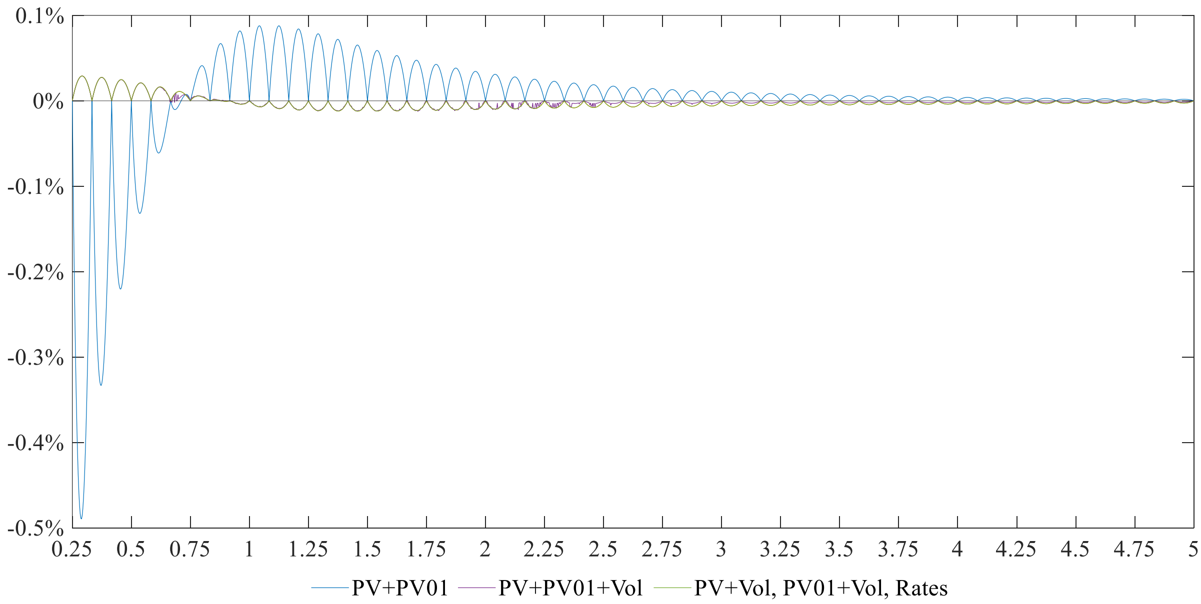

Figure 5.2 - The relative errors in the VaR calculation associated with the mappings PV+PV01, PV+Vol, PV01+Vol, PV+PV01+Vol and Rates from 3 months till 5 years with vertices spaced monthly (reference no mapped).

This figure confirms that the highest error amplitudes appear in the closest maturities. Beyond this, it shows that the first interval is where all the methods register their maximum error. For the PV+PV01, it corresponds to the value of -0.49%, whereas for the volatility invariant mappings are 0.03%. On the other hand, we recheck a disassociation between the error signs of the PV+PV01 and the volatility invariant mappings, resembling the 12-vertex results. Additionally, with monthly vertices, we also do not have a method that continuously obtains the lowest error peaks. Moreover, the winning methods maintain similarities in both vertex sets, with the PV+PV01 registering mainly lower peaks than the volatility invariant mappings between 5 and 15 years.

Overall, with the increase in the cash flow maturity, there are reductions in the amplitudes of the peaks. Nevertheless, there are reverse movements, namely one visually observable between 0.6667 and 1.0833 years in the PV+PV01 method. Therefore, the increase in the number of vertices helps increase the accuracy of the mappings, both in the short and long term, despite being more critical in the short term.

5.1.3 Limit error

As observed in the study with the standard 12 vertices, we verify that the nearest vertex maturities are insufficient to avoid expressive errors in the mappings. Furthermore, that vertex choice is not very suitable in the long term since there is no need for such a higher number of nodes. As a result, it is helpful to determine a set of vertices that leads to a homogeneous error throughout all maturities.

Starting from the set of vertices with monthly availability, we represent in figure 5.3 the discrepancies for the optimal spacing between the vertices that leads to magnitudes in absolute value close to but not exceeding 1% for the PV+PV01 method.

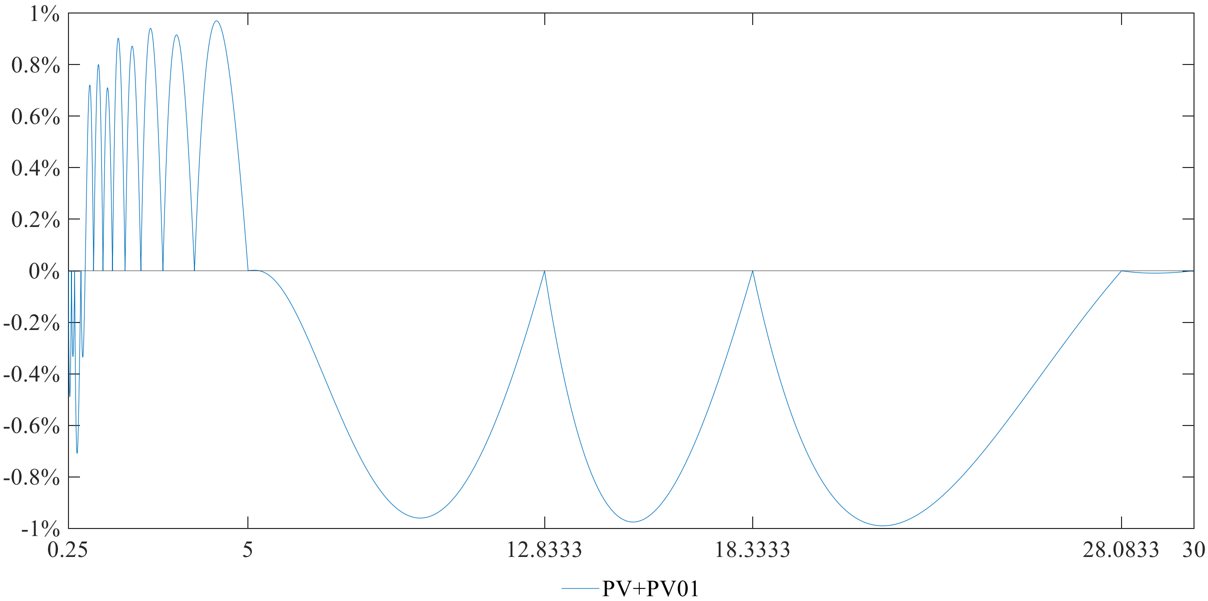

Figure 5.3 - The relative errors in the VaR calculation associated with the mapping PV+PV01 (reference no mapped). The available vertices for allocation are 3, 4, 5, 7, 11 months, 1 year and 2 months, 1 year and 5 months, 1 year and 9 months, 2 years and 2 month, 2 years and 9 months, 3 years and 7 months, 5 years, 12 years and 10 months, 18 years and 4 months, 28 years and 1 month and 30 years.

This representation involves sixteen vertices, demonstrating that we can significantly reduce the errors by using just a few additional vertices than the 12-typical set. Specifically, this set of vertices has ten nodes whose maturities are less than 3 years; namely, five with maturities below 1 year; three between 1 and 2 years and the remaining two vertices between 2 and 3 years. This result underlines the importance of the availability of vertices for the nearest maturities. From then, the distance between the vertices increases, with the vertex at five years setting the beginning of very distant vertices, where even with distances exceeding nine years, the error remains low.

However, this set of vertices is unsuitable for the volatility invariant mappings, namely in the range from 5 years to 12 years and ten months, that registers an error more than nine times higher than the intended limit (appendix B.2). By inference from figures 5.1 and 5.2, this result was already predictable. Since the PV+PV01 has lower errors in the intermediate ranges than the volatility invariant mappings, it can control the error with fewer vertices.

Thus, to ensure no limit exceedance of the maximum 1% error in the volatility invariant mappings, their vertices set has different maturities and requires a greater expression in intermediate maturities than in the short term, achievable by a total of 11 vertices (figure 5.4).

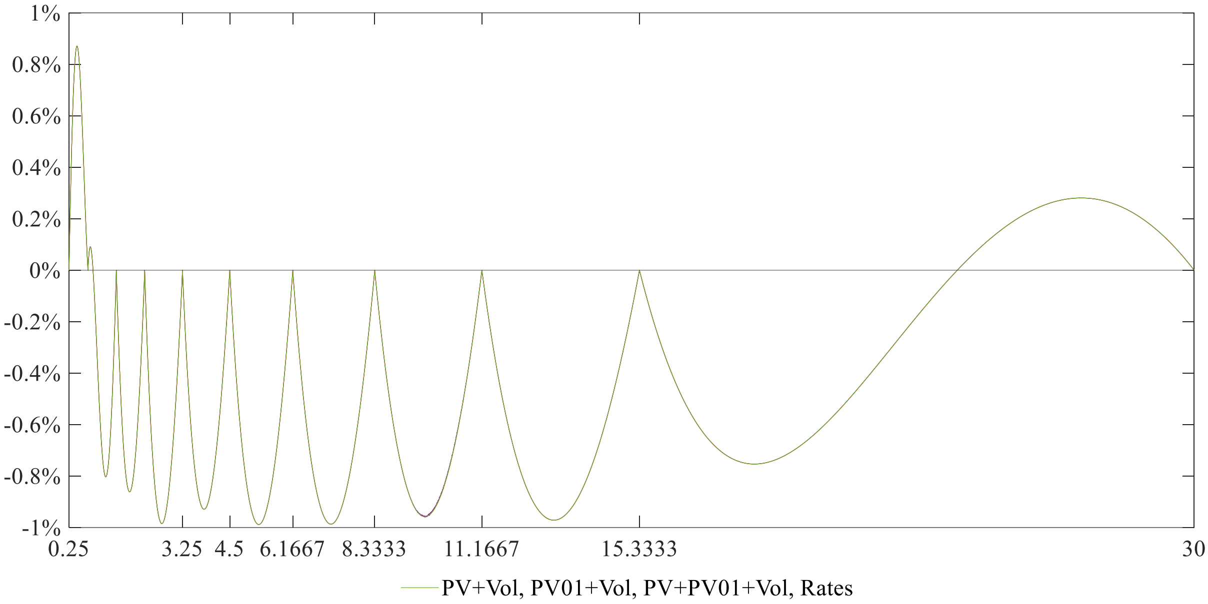

Figure 5.4 - The relative errors in the VaR calculation associated with the mappings PV+Vol, PV01+Vol, PV+PV01+Vol, and Rates (reference no mapped). The available vertices for allocation are 3 months, 9 months, 1 year and 6 months, 2 years and 3 months, 3 years and 3 months, 4 years and 6 months, 6 years and 2 months, 8 years and 4 months, 11 years and 2 months, 15 years and 4 months and 30 years.

To present a set of vertices that meets the limit over time for all methods and using the estimation of parameters to obtain the curve for any day (and not for monthly nodes), we have the following result:

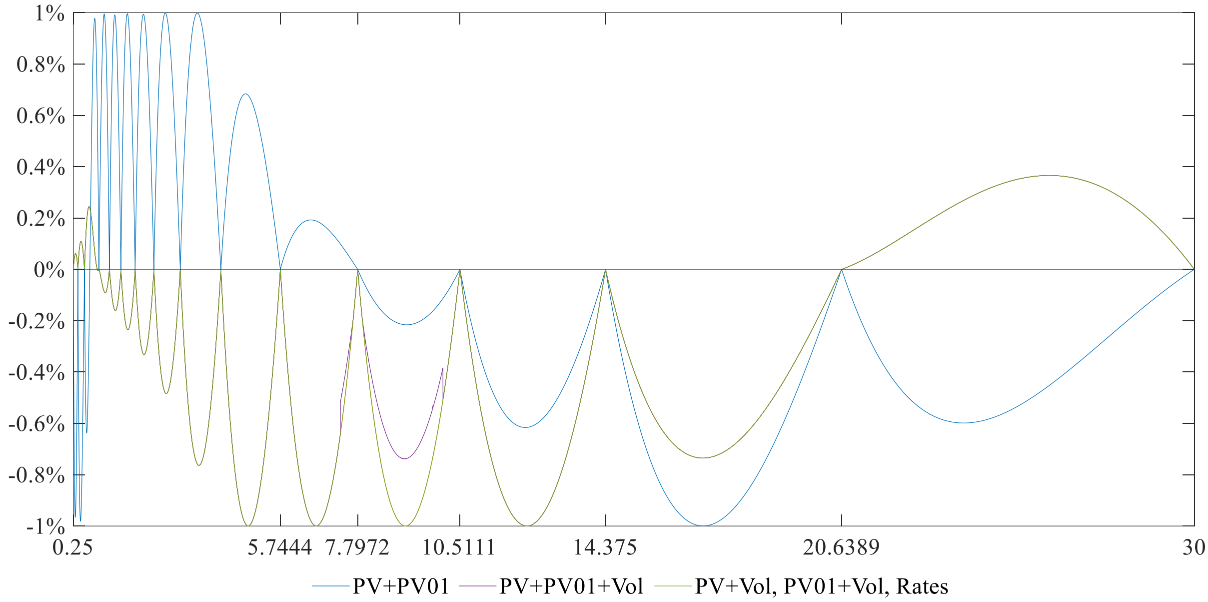

Figure 5.5 - The relative errors in the VaR calculation associated with the mappings PV+PV01, PV+Vol, PV01+Vol, PV+PV01+Vol and Rates (no mapped reference). The available vertices for allocation are 3 months, 4 months and 14 days, 6 months and 17 days, 11 months and 5 days, 1 year 2 months and 15 days, 1 year 6 months and 5 days, 1 year 10 months and 21 days, 2 years 4 months and 20 days, 3 years 1 month and 2 days, 4 years and 2 months, 5 years 8 months and 28 days, 7 years 9 months and 17 days, 10 years 6 months and 4 days, 14 years 4 months and 16 days, 20 years 7 months and 20 days, and 30 years.

Simplifying the spacing between the vertices, it starts with one month-distance, followed by two months, five months, three months, four months, six months, six months, nine months, 13 months, 18 months, 25 months, 33 months, 46 months, 75 months, and finally ends up with two nodes 113 months apart.

This composition of 16 vertices is the consequence of a combination of previously explored results. To control the shorter maturities, the excessive number of vertices is due to the error behavior of PV+PV01. In contrast, in the intermediate ranges, the volatility invariant mappings are the ones that dictate the need for more vertices.

5.2 Bonds Portfolios

To consider the interactions between cash flows, we analyze bond portfolios.

In this section, each table contains two different studies. The first corresponds to the estimate of the VaR for the date 27/05/2021 for 5000 portfolios, each composed of 200 randomly simulated zero-coupon bonds. Since an analysis based exclusively on one day implies dependence on the observation of a single covariance matrix, in our second study, we measure the VaR over ten years (daily VaR estimated once per month) for the same 5000 now fixed portfolios. The last-mentioned study does not include the PV+PV01+Vol mapping due to its execution time delay.

We start with the set of 12 vertices and then with the monthly spaced nodes.

5.2.1 12 vertices

Table 5.1 presents indicators for the relative difference between each mapping and the reference method for portfolios with heterogeneous maturities (3 months to 30 years).

Overall, the mean errors are less than 0.5%, which is quite good and conveys almost an indifference in the mapping choice. However, scrutinizing the results, the PV+PV01 has more pronounced errors suggesting that in a heterogeneous portfolio with a common set of standard vertices, the volatility invariant mappings necessarily denote more precision.

Ordering the mappings for the two studies based on their mean of the absolute value of the relative errors clarifies this finding. For the first study, the best result is for PV01+Vol with 0.1479%. Following it are Rates, PV+Vol, PV+PV01+Vol, and finally PV+PV01. For the second study, the Rates is the method that positively stands out, with a percentage of 0.3146%. Then, it is PV01+Vol, PV+Vol, and PV+PV01.

Globally, the second study presents more expressive and differentiable errors, with all the methods registering maximums above 2%, which in the case of PV+PV01 even reaches 4.9622%. Nevertheless, the quartile column maintains absolute values below 0.5%, reinforcing that the magnitude of errors is mainly in this range of values for any mapping. However, the possibility of much higher errors makes the mapping selection critical.

Interestingly, in the first study, all the methods almost always underestimate the VaR (excluding a few outliers). In contrast, in the second study, the PV+PV01 has a massive value of overestimation while the volatility invariant mappings register a low percentage.

Mappings	Mean |error|	Median	Std dev	Min	Max	Q1	Q3	> 0	Outlier
First Study - 1 Date - 5000 Portfolios with 200 bonds (0.25-30)									
PV+PV01	0.3876%	-0.3882%	2.6146e-4	-0.4749%	-0.2821%	-0.4054%	-0.3707%	0%	0.50%
PV+Vol	0.1739%	-0.1728%	4.8037e-4	-0.3798%	-0.0046%	-0.2052%	-0.1406%	0%	0.40%
PV01+Vol	0.1479%	-0.1468%	4.8501e-4	-0.3559%	0.0185%	-0.1796%	-0.1147%	0.02%	0.44%
PV+PV01+Vol	0.2555%	-0.2535%	6.0901e-4	-0.4926%	-0.0659%	-0.2962%	-0.2132%	0%	0.38%
Rates	0.1481%	-0.1471%	4.8521e-4	-0.3559%	0.0185%	-0.1798%	-0.1150%	0.02%	0.44%
Second Study - 10 Years - 5000 Portfolios with 200 bonds (0.25-30)									
PV+PV01	0.4770%	0.2941%	0.0064	-2.2224%	4.9622%	0.1438%	0.3984%	84.0586%	21.3677%
PV+Vol	0.3290%	-0.1179%	0.0048	-2.1661%	2.2347%	-0.3015%	0.0548%	30.1631%	10.0301%
PV01+Vol*	0.3161%	-0.1008%	0.0046	-1.4278%	2.3356%	-0.2862%	0.0973%	33.2826%	8.1961%
Rates	0.3146%	-0.1013%	0.0046	-1.2585%	2.3356%	-0.2864%	0.0958%	33.1403%	8.0520%

Table 5.1 - Summary of statistics. An outlier corresponds to a value that is more than three scaled median absolute deviations away from the median.
*1 observation without solution.

However, the individual analysis of the bonds showed us that the cash-flow maturity dramatically influences the results. Therefore, we restrict the maturities of the portfolios to assess this effect. Table 5.2 presents the mean |error| indicators for the portfolios with maturity restrictions confirming dependence on the bonds' maturities.

Mappings	0.25-1	1-5	5-15	15-30
First Study - 1 Date - 5000 Portfolios with 200 bonds				
PV+PV01	0.1517%	1.0439%	0.4141%	0.4386%
PV+Vol	0.3350%	0.4105%	1.0092%	0.0750%
PV01+Vol	0.1619%	0.4840%	1.0026%	0.0969%
PV+PV01+Vol	0.0604%	0.4938%	1.2695%	0.0931%
Rates	0.2426%	0.4840%	1.0026%	0.0969%
Second Study - 10 Years - 5000 Portfolios with 200 bonds				
PV+PV01	2.0428%	0.6326%	0.3627%	0.6202%
PV+Vol	0.6647%	0.4234%	0.7096%	0.3666%
PV01+Vol	0.6459%*	0.5487%	0.6961%**	0.3519%
Rates	0.6951%	0.5648%	0.6959%	0.3500%

Table 5.2 - Summary of the mean of the absolute value of the relative error in portfolios with maturities restrictions. *1 observation without solution. **3 observations without solution.

With this set of 12 standard vertices, the approaches that preserve the volatility obtain remarkably worse mean errors than the PV+PV01 in the range of 5-15, whose finding is in line with the results of the single zero-coupon bond. On the other hand, in the ranges 1-5 and 15-30, we have a superiority of the volatility invariant mappings. Particularly between 1 to 5 years, the winning method is the PV+Vol. The most ambiguous range is 0.25-1; however, since the second study is more representative as it is an analysis over time, we also get lower mean |errors| with the volatility invariant mappings.

Table 5.3 presents the percentage of overestimation for each correspondent in table 5.2.

Mappings	0.25-1	1-5	5-15	15-30
First Study - 1 Date - 5000 Portfolios with 200 bonds				
PV+PV01	60.24%	100%	0%	0%
PV+Vol	100%	0%	0%	99.9%
PV01+Vol	100%	0%	0%	100%
PV+PV01+Vol	90.42%	0%	0%	100%
Rates	100%	0%	0%	100%
Second Study - 10 Years - 5000 Portfolios with 200 bonds				
PV+PV01	50.0034%	93.1427%	40.8511%	83.9557%
PV+Vol	51.8414%	9.2674%	6.0627%	44.9944%
PV01+Vol	42.2796%*	1.7462%	6.1115%**	49.1116%
Rates	45.0137%	1.7085%	6.1099%	49.0474%

Table 5.3 - Summary of the percentage of overestimation in portfolios with maturities restrictions.
*1 observation without solution. **3 observations without solution.

In this table, we can see that the overestimation percentages are also very dependent on the maturities of the cash flows. Indeed, there are ranges where the methods register total overestimation while others entirely underestimate the VaR. The volatility invariant mappings tend to register higher percentages in extreme ranges (0.25-1 and 15-30), whereas they underestimate the VaR estimates in the intermediate ranges. The PV+PV01 shows a disassociation between the two studies. However, the 1-5 range for PV+PV01 is clear, tending to overestimate the results.

Appendix C details additional statistics. It clarifies that among the volatility invariant methods, the PV01+Vol and Rates are the two with the most similar results, namely for portfolios with maturities higher than 1 year. Additionally, the second study's 0.25-1 indicators contain the maximum possible errors for all the methods. PV+PV01 goes from -10.2057% to 7.7583%, whereas volatility invariant mapping errors are roughly between -5% and 6%. Nevertheless, the volatility invariant mappings overcome those last-mentioned percentages, particularly the PV+Vol, which registers a high error of -8.9649%.

5.2.2 Monthly vertices

Table 5.4 shows the statistics for the heterogeneous portfolios considering the monthly availability of vertices.

Mappings	Mean |error|	Median	Std dev	Min	Max	Q1	Q3	> 0	Outlier
First Study - 1 Date - 5000 Portfolios with 200 bonds (0.25-30)									
PV+PV01	1.1935e-4%	1.1634e-4%	4.2605e-7	3.6457e-6%	3.2815e-4%	8.8908e-5%	1.4621e-4%	100%	0.54%
PV+Vol	8.8851e-5%	-9.0418e-5%	3.0543e-7	-1.9939e-4%	4.0498e-5%	-1.0990e-4%	-6.9487e-5%	0.58%	0.72%
PV01+Vol	1.5586e-4%	-1.5476e-4%	2.1655e-7	-2.4525e-4%	-8.4409e-5%	-1.6975e-4%	-1.4089e-4%	0%	0.56%
PV+PV01+Vol	1.1390e-4%	-1.1369e-4%	1.4736e-7	-1.8047e-4%	-6.2899e-5%	-1.2333e-4%	-1.0393e-4%	0%	0.56%
Rates	1.5598e-4%	-1.5492e-4%	2.1622e-7	-2.4525e-4%	-8.4409e-5%	-1.6989e-4%	-1.4103e-4%	0%	0.54%
Second Study - 10 Years - 5000 Portfolios with 200 bonds (0.25-30)									
PV+PV01	1.7883e-4%	1.7665e-4%	6.6349e-7	-1.9715e-4%	6.6439e-4%	1.3773e-4%	2.1562e-4%	99.2979%	2.1446%
PV+Vol	6.1673e-5%	-1.5623e-5%	1.2982e-6	-0.0049%	0.0038%	-4.9255e-5%	2.7338e-5%	39.2489%	4.3713%
PV01+Vol*	1.1269e-4%	-1.0920e-4%	4.5604e-7	-4.6045e-4%	2.2308e-4%	-1.3519e-4%	-8.7015e-5%	1.6925%	3.4766%
Rates	1.1280e-4%	-1.0932e-4%	4.5624e-7	-4.6045e-4%	2.2308e-4%	-1.3530e-4%	-8.7112e-5%	1.6923%	3.4832%

Table 5.4 - Summary of statistics with vertices spaced monthly.
*1 observation without solution.

This massive availability of vertices makes the errors practically null, demonstrating that using this set of nodes remarkably positively impacts all these mappings. Moreover, the maximum and minimum columns have most of their values at/higher than the sixth decimal, which indicates that monthly vertices lead to errors around that magnitude considering a portfolio with heterogeneous maturities.

Curiously, the best and worst results of each study undergo several changes compared to the 12-vertex studies. In the first study of table 5.4, the mean error indicators order from the best to the worst is PV+Vol, PV+PV01+Vol, PV+PV01, PV01+Vol, and Rates. In the second study, the order is PV+Vol, PV01+Vol, Rates, and PV+PV01. Thus, although the PV+Vol emerges as the appropriate choice, the magnitudes of the errors are so small by using an expressive availability of vertices, as this monthly set in a portfolio of heterogeneous maturities, that the choice of the mapping method is highly flexible, achieving good results with any of the procedures.

On the other hand, the error sign differs between the methods. These two studies clarify that the PV+PV01 will overestimate, while the volatility invariant mappings will mainly show VaR estimates below the reference value.

Table 5.5 contains the mean |error| indicators for the restricted portfolios.

Mappings	0.25-1	1-5	5-15	15-30
First Study - 1 Date - 5000 Portfolios with 200 bonds				
PV+PV01	0.0110%	0.0073%	5.3069e-5%	5.4284e-5%
PV+Vol	0.0593%	0.0027%	4.7705e-4%	1.1550e-5%
PV01+Vol	0.0079%	0.0034%	4.9306e-4%	1.2041e-5%
PV+PV01+Vol	0.0005%	0.0020%	3.4302e-4%	3.9924e-5%
Rates	0.0079%	0.0034%	4.9306e-4%	1.2041e-5%
Second Study - 10 Years - 5000 Portfolios with 200 bonds				
PV+PV01	0.0808%	0.0044%	1.8379e-4%	5.6516e-5%
PV+Vol	0.0537%	0.0030%	3.5141e-4%	4.5530e-5%
PV01+Vol	0.0264%	0.0039%	3.8215e-4%	3.7918e-5%
Rates	0.0268%	0.0039%	3.8215e-4%	3.7918e-5%

Table 5.5 - Summary of the mean of the absolute value of the relative error in portfolios with maturities restrictions with vertices spaced monthly.

In this table, we persistently check that the intervals of the shortest maturities are those in which, in global terms, the error is higher, with the worst results being in PV+PV01. In addition, we observe that the mean errors tend to decrease as the cash flow maturities increase. Such a relation reinforces that the high errors registered by the volatility invariant mappings in the range of 5-15 in table 5.2 of 12 vertices are clearly due to the lower availability of vertices.

Although there are discrepancies among the indicators in table 5.5, the methods that register the lowest errors in each range are practically the same as the set of 12 vertices. Therefore, we can generally expect the most promising results in the 5-15 interval if we use the PV+PV01 method, either with the set of 12 or monthly spaced vertices. For the remaining ranges, the methods that stand out positively are those that preserve volatility. However, it is difficult to determine which is preferable to use. In the first study with the monthly availability of vertices, the most accurate methods among the volatility invariants are PV+PV01+Vol (0.25-15) and PV+Vol (15-30). In comparison, the second study has the lowest mean |errors| with PV01+Vol (0.25-1), PV+Vol (1-15), and Rates (15-30). Moreover, there is a remarkable similarity between PV01+Vol and Rates indicators from the 1-5 range, whereas oppositely, the PV+Vol differs from the other volatility invariant methods in the 0.25-1 range.

Table 5.6 indicates the percentages of overestimation.

Mappings	0.25-1	1-5	5-15	15-30
First Study - 1 Date - 5000 Portfolios with 200 bonds				
PV+PV01	8.24%	100%	0.56%	0%
PV+Vol	100%	0%	0%	0.06%
PV01+Vol	100%	0%	0%	0.06%
PV+PV01+Vol	94.04%	0%	0%	0%
Rates	100%	0%	0%	0.06%
Second Study - 10 Years - 5000 Portfolios with 200 bonds				
PV+PV01	48.4462%	93.1891%	47.3234%	80.9296%
PV+Vol	88.8368%	11.0022%	7.4701%	42.4489%
PV01+Vol	41.6113%	1.7012%	4.2130%	38.3915%
Rates	41.6024%	1.7012%	4.2130%	38.3915%

Table 5.6 - Summary of the percentage of overestimation in portfolios with maturities restrictions with vertices spaced monthly.

This table resembles table 5.3 (study with the 12 vertices and maturity restrictions), except in the range 15-30, where the volatility invariant mappings go from a total overestimation to underestimation. Thus, the general similarity makes it easier to anticipate that those are the expectable behavior of the methods.

Therefore, despite exceptions, regardless of the vertex set, the volatility invariant mappings tend to underestimate from 1-5 and 5-15, registering higher values at the edges. Additionally, while the PV+PV01 has an evident overestimation in the range 1-5, it is more ambiguous in the remaining intervals.

Appendix D presents additional statistics for the studies with monthly spaced vertices. It is worth emphasizing the second study's short-term errors, which reach the highest values between -0.3990% (PV+PV01) and 0.5788% (PV+Vol). Additionally, in line with the decrease in the magnitude of errors, the standard deviation indicators also decrease as bonds' maturities increase. Hence, a similarity that persists regardless of the vertex set is that the PV+PV01 is the method whose errors are most heterogeneous up to 5 years.

6 Conclusion

This thesis provides an empirical study of the impact of cash flow mapping on the VaR on bond portfolios. The results show that the loss of precision in each method depends on three main factors: its inherent characteristics, the set of available vertices, and the maturities of the bonds constituting the portfolio.

The analysis of the individual zero-coupon bond demonstrates that the maximum error peak, despite deviations, essentially appears in the maturities in the middle of the adjacent vertices, especially in volatility invariant mappings. Moreover, it highlights that the typical set of vertices is not the most suitable to keep the error within a stipulated limit since it has insufficient nodes in the short term and an excessive number in the long term.

In bond portfolios, the mean of the absolute value of the relative errors reinforces the relation between higher errors and shorter maturities, independent of the vertex selection. The PV+PV01 is the method that registers the worse and most heterogeneous results in the short term. Nevertheless, in the range from 5 to 15 years, it beats the volatility invariant mapping with both set of vertices. Therefore, we do not have a cash flow mapping that is the best in all scenarios.

Among the volatility invariant mappings, there is a remarkable similarity between PV01+Vol and Rates, whereas by the contrary, the PV+Vol differs from them in the nearest maturities. Regarding the PV+PV01+Vol, it is not a functional method as it is time-consuming.

Thus, the combination of our studies leads us to suggest the global application of the Rates mapping due to its accuracy, simplicity, and speed. Besides, it is a method whose allocations do not require solving a quadratic equation, so it always has a solution. Such a recommendation coincides with Henrard (2000).

Regarding the percentage of higher or lower estimations of the VaR, the volatility invariant mappings tend to underestimate their values; nevertheless, if the bond portfolio is constituted mainly by cash flows with extreme maturities, we observe the opposite. On the other hand, the behavior of the PV+PV01 is ambiguous; however, it is evident that it overestimates the VaR measurements for portfolios with bond maturities ranging from 1 to 5 years.

Our study shows that despite the cash flow mappings resemblances, they behave differently and can induce significant accuracy losses on the VaR calculation on bond portfolios.

This dissertation focuses on European bonds-AAA data and the Parametric Normal VaR. Thus, future research may include data from other markets, different instruments, and additional

VaR methods. Even though the currency and stock risk insertion will probably dissipate the impact of cash-flow mapping, it might be interesting to see if there are parallels.

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8 Appendices

8.1 Appendix A - Peaks Coordinates

***Note*:** We only consider the highest absolute peaks existing between the vertices for each mapping. However, if both a negative and a positive peak exist, both are represented.

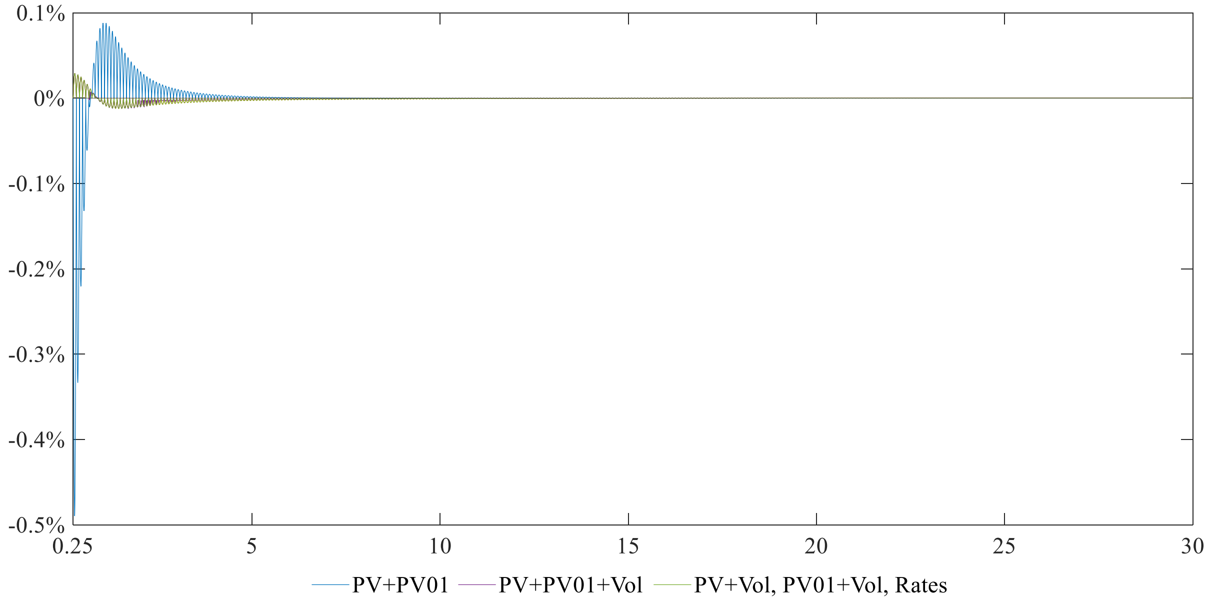
Appendix A.1 - Peaks Coordinates - PV+PV01, PV+Vol, PV01+Vol, Rates and PV+PV01+Vol.

We present coordinates (x,y) for each peak where x denotes the relative distance to the nearest vertices in percentage and y the relative deviation in percentage from the method that does not involve cash flow mapping.

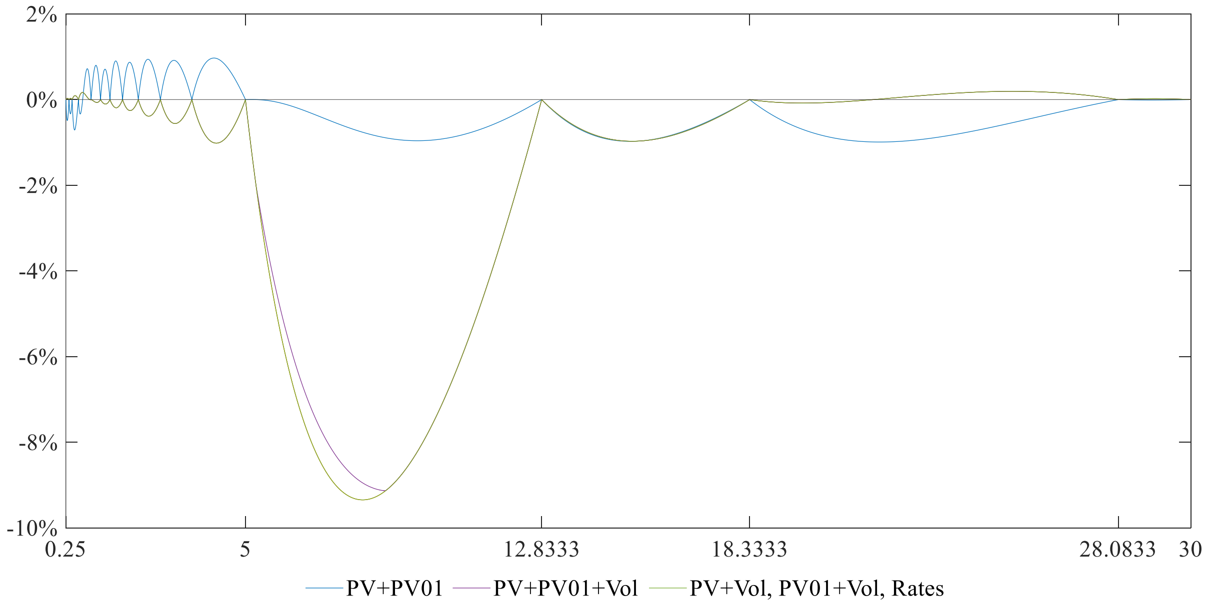
PV+PV01	**PV+Vol, PV01+Vol and Rates**	**PV+PV01+Vol**
(36.67%, -3.07%)	(47.78%, 0.25%)	(47.78%, 0.25%)
(13.89%, -1.36%) (68.33%, 2.12%)	(29.44%, 0.44%) (89.44%, -0.04%)	(29.44%, 0.44%) (89.44%, -0.04%)
(38.33%, 9.82%)	(50.83%, -1.70%)	(50.83%, -1.70%)
(39.44%, 2.73%)	(45.28%, -1.12%)	(45.28%, -1.12%)
(42.50%, 0.96%)	(46.11%, -0.69%)	(46.39%, -0.69%)
(44.17%, 0.39%)	(46.94%, -0.47%)	(46.94%, -0.47%)
(40.28%, 0.44%)	(45.83%, -1.17%)	(45.69%, -1.17%)
(51.76%, -0.17%)	(46.02%, -1.42%)	(43.80%, -1.42%)
(43.39%, -1.02%)	(44.89%, -1.67%)	(44.78%, -1.67%)
(43.00%, -0.63%)	(42.83%, -0.46%)	(42.94%, -0.46%)
(34.03%, -0.74%)	(60.67%, 0.39%)	(60.72%, 0.39%)

8.2 Appendix B - Graph Single Bond

Appendix B.1 - The relative errors in the VaR calculation associated with the mappings PV+PV01, PV+Vol, PV01+Vol, PV+PV01+Vol, and Rates with vertices spaced monthly from 3 months till 30 years (reference no mapped).



Appendix B.2 - The relative errors in the VaR calculation associated with the mappings PV+PV01, PV+Vol, PV01+Vol, PV+PV01+Vol and Rates (reference no mapped). The available vertices for allocation are 3, 4, 5, 7, 11 months, 1 year and 2 months, 1 year and 5 months, 1 year and 9 months, 2 years and 2 month, 2 years and 9 months, 3 years and 7 months, 5 years, 12 years and 10 months, 18 years and 4 months, 28 years and 1 month and 30 years.



8.3 Appendix C - Statistics for bond portfolios with 12 vertices

Appendix C.1 - Summary of statistics for portfolios with maturities between 0.25 and 1 year.

Mappings	Median	Std dev	Min	Max	Q1	Q3	Outlier
First Study - 1 Date - 5000 Portfolios with 200 bonds (0.25-1)							
PV+PV01	0.0502%	0.0018	-0.6457%	0.6820%	-0.0768%	0.1717%	0.24%
PV+Vol	0.3371%	8.7248e-4	0.0331%	0.6674%	0.2781%	0.3953%	0.32%
PV01+Vol	0.1673%	2.9858e-4	0.0053%	0.2491%	0.1461%	0.1836%	2.04%
PV+PV01+Vol	0.0598%	4.0439e-4	-0.1182%	0.2016%	0.0318%	0.0837%	1.30%
Rates	0.2428%	2.1634e-4	0.1665%	0.3129%	0.2282%	0.2570%	0.32%
Second Study - 10 Years - 5000 Portfolios with 200 bonds (0.25-1)							
PV+PV01	1.9962e-4%	0.0276	-10.2057%	7.7583%	-1.4660%	1.5430%	2.7239%
PV+Vol	0.0282%	0.0126	-8.9649%	4.3469%	-0.4199%	0.3970%	6.6793%
PV01+Vol*	-0.1340%	0.0101	-4.8208%	5.5879%	-0.5806%	0.2256%	5.9104%
Rates	-0.1327%	0.0105	-4.9485%	5.6187%	-0.6424%	0.3645%	4.9115%

*1 observation without solution.

Appendix C.2 - Summary of statistics for portfolios with maturities between 1 and 5 years.

Mappings	Median	Std dev	Min	Max	Q1	Q3	Outlier
First Study - 1 Date - 5000 Portfolios with 200 bonds (1-5)							
PV+PV01	1.0389%	0.0010	0.7352%	1.4417%	0.9732%	1.1079%	0.48%
PV+Vol	-0.4102%	2.2118e-4	-0.4922%	-0.3216%	-0.4254%	-0.3960%	0.34%
PV01+Vol	-0.4837%	2.5617e-4	-0.5820%	-0.3992%	-0.5009%	-0.4664%	0.32%
PV+PV01+Vol	-0.4935%	2.5829e-4	-0.5924%	-0.4086%	-0.5109%	-0.4759%	0.32%
Rates	-0.4837%	2.5617e-4	-0.5820%	-0.3992%	-0.5009%	-0.4664%	0.32%
Second Study - 10 Years - 5000 Portfolios with 200 bonds (1-5)							
PV+PV01	0.5995%	0.0042	-0.5032%	2.5308%	0.3868%	0.8112%	3.6947%
PV+Vol	-0.3770%	0.0031	-2.0446%	1.8263%	-0.5829%	-0.2112%	2.8253%
PV01+Vol	-0.5473%	0.0023	-2.3212%	0.5890%	-0.6875%	-0.4031%	2.7373%
Rates	-0.5536%	0.0025	-2.6596%	0.2571%	-0.7013%	-0.4119%	3.2316%

Appendix C.3 - Summary of statistics for portfolios with maturities between 5 and 15 years.

Mappings	Median	Std dev	Min	Max	Q1	Q3	Outlier
First Study - 1 Date - 5000 Portfolios with 200 bonds (5-15)							
PV+PV01	-0.4144%	3.5951e-4	-0.5345%	-0.2638%	-0.4386%	-0.3895%	0.22%
PV+Vol	-1.0096%	4.2884e-4	-1.1494%	-0.8147%	-1.0380%	-0.9814%	0.36%
PV01+Vol	-1.0033%	4.1877e-4	-1.1420%	-0.8139%	-1.0308%	-0.9758%	0.36%
PV+PV01+Vol	-1.2701%	5.5260e-4	-1.4968%	-1.0313%	-1.3067%	-1.2328%	0.32%
Rates	-1.0033%	4.1877e-4	-1.1420%	-0.8139%	-1.0308%	-0.9758%	0.36%
Second Study - 10 Years - 5000 Portfolios with 200 bonds (5-15)							
PV+PV01	-0.0976%	0.0046	-1.7203%	1.5249%	-0.3750%	0.1813%	0.9891%
PV+Vol	-0.6271%	0.0049	-2.0801%	1.4857%	-0.8347%	-0.4503%	8.1644%
PV01+Vol**	-0.6139%	0.0049	-2.0722%	1.5015%	-0.8272%	-0.4302%	7.6112%
Rates	-0.6134%	0.0049	-2.0722%	1.5015%	-0.8272%	-0.4289%	7.5950%

**3 observations without solution.

Appendix C.4 - Summary of statistics for portfolios with maturities between 15 and 30 years.

Mappings	Median	Std dev	Min	Max	Q1	Q3	Outlier
First Study - 1 Date - 5000 Portfolios with 200 bonds (15-30)							
PV+PV01	-0.4389%	1.9113e-4	-0.5087%	-0.3593%	-0.4515%	-0.4259%	0.34%
PV+Vol	0.0752%	2.2806e-4	-0.0134%	0.1551%	0.0595%	0.0907%	0.26%
PV01+Vol	0.0974%	2.1573e-4	0.0120%	0.1687%	0.0823%	0.1116%	0.34%
PV+PV01+Vol	0.0936%	2.1602e-4	0.0082%	0.1659%	0.0786%	0.1078%	0.32%
Rates	0.0974%	2.1573e-4	0.0120%	0.1687%	0.0823%	0.1116%	0.34%
Second Study - 10 Years - 5000 Portfolios with 200 bonds (15-30)							
PV+PV01	0.3089%	0.0090	-2.1015%	5.9272%	0.2516%	0.3791%	33.4672%
PV+Vol	-0.0269%	0.0060	-1.8093%	3.3339%	-0.2213%	0.1517%	11.6214%
PV01+Vol	-0.0058%	0.0056	-1.2141%	2.8400%	-0.1976%	0.1713%	10.4926%
Rates	-0.0063%	0.0056	-1.2007%	2.8400%	-0.1978%	0.1734%	10.2892%

8.4 Appendix D - Statistics for bond portfolios with monthly spaced vertices

Appendix D.1 - Summary of statistics with vertices spaced monthly for portfolios with maturities between 0.25 and 1 year.

Mappings	Median	Std dev	Min	Max	Q1	Q3	Outlier
First Study - 1 Date - 5000 Portfolios with 200 bonds (0.25-1)							
PV+PV01	-0.0102%	7.8316e-5	-0.0428%	0.0151%	-0.0156%	-0.0051%	0.34%
PV+Vol	0.0591%	5.5330e-5	0.0396%	0.0804%	0.0555%	0.0630%	0.30%
PV01+Vol	0.0078%	7.6892e-6	0.0049%	0.0108%	0.0073%	0.0084%	0.20%
PV+PV01+Vol	4.2806e-4%	3.0545e-6	-0.0014%	0.0027%	2.7301e-4%	5.8217e-4%	3.84%
Rates	0.0079%	7.8240e-6	0.0049%	0.0108%	0.0074%	0.0085%	0.20%
Second Study - 10 Years - 5000 Portfolios with 200 bonds (0.25-1)							
PV+PV01	-0.0030%	0.0011	-0.3990%	0.3015%	-0.0623%	0.0596%	2.4279%
PV+Vol	0.0294%	8.1000e-4	-0.2747%	0.5788%	0.0104%	0.0492%	12.1489%
PV01+Vol	-0.0061%	4.8220e-4	-0.3800%	0.1444%	-0.0220%	0.0110%	5.1250%
Rates	-0.0061%	4.9059e-4	-0.3872%	0.1444%	-0.0228%	0.0111%	5.1210%

Appendix D.2 - Summary of statistics with vertices spaced monthly for portfolios with maturities between 1 and 5 years.

Mappings	Median	Std dev	Min	Max	Q1	Q3	Outlier
First Study - 1 Date - 5000 Portfolios with 200 bonds (1-5)							
PV+PV01	0.0072%	7.0316e-6	0.0051%	0.0101%	0.0068%	0.0077%	0.38%
PV+Vol	-0.0027%	1.6998e-6	-0.0034%	-0.0021%	-0.0028%	-0.0026%	0.36%
PV01+Vol	-0.0034%	1.7713e-6	-0.0041%	-0.0027%	-0.0035%	-0.0032%	0.36%
PV+PV01+Vol	-0.0020%	1.3723e-6	-0.0026%	-0.0016%	-0.0021%	-0.0019%	0.38%
Rates	-0.0034%	1.7713e-6	-0.0041%	-0.0027%	-0.0035%	-0.0032%	0.36%
Second Study - 10 Years - 5000 Portfolios with 200 bonds (1-5)							
PV+PV01	0.0042%	2.8666e-5	-0.0036%	0.0178%	0.0027%	0.0056%	3.7044%
PV+Vol	-0.0025%	2.9803e-5	-0.0474%	0.0682%	-0.0040%	-0.0013%	4.4268%
PV01+Vol	-0.0038%	1.7021e-5	-0.0178%	0.0017%	-0.0048%	-0.0029%	3.1674%
Rates	-0.0038%	1.7025e-5	-0.0178%	0.0017%	-0.0048%	-0.0029%	3.1694%

Appendix D.3 - Summary of statistics with vertices spaced monthly for portfolios with maturities between 5 and 15 years.

Mappings	Median	Std dev	Min	Max	Q1	Q3	Outlier
First Study - 1 Date - 5000 Portfolios with 200 bonds (5-15)							
PV+PV01	-5.3771e-5%	1.9894e-7	-1.1678e-4%	3.0138e-5%	-6.7186e-5%	-3.9929e-5%	0.26%
PV+Vol	-4.7645e-4%	2.9184e-7	-6.0640e-4%	-3.8998e-4%	-4.9579e-4%	-4.5734e-4%	0.28%
PV01+Vol	-4.9239e-4%	3.0445e-7	-6.2805e-4%	-4.0248e-4%	-5.1265e-4%	-4.7266e-4%	0.30%
PV+PV01+Vol	-3.4260e-4%	1.8990e-7	-4.2529e-4%	-2.8438e-4%	-3.5528e-4%	-3.3050e-4%	0.40%
Rates	-4.9239e-4%	3.0445e-7	-6.2805e-4%	-4.0248e-4%	-5.1265e-4%	-4.7266e-4%	0.30%
Second Study - 10 Years - 5000 Portfolios with 200 bonds (5-15)							
PV+PV01	-1.3073e-5%	2.3060e-6	-8.5735e-4%	7.1084e-4%	-1.5518e-4%	1.5422e-4%	0.1480%
PV+Vol	-3.5104e-4%	2.0313e-6	-9.3223e-4%	5.6713e-4%	-4.3076e-4%	-2.5320e-4%	6.0443%
PV01+Vol	-3.8176e-4%	1.8367e-6	-0.0010%	5.2290e-4%	-4.6581e-4%	-2.9876e-4%	5.1132%
Rates	-3.8176e-4%	1.8367e-6	-0.0010%	5.2290e-4%	-4.6581e-4%	-2.9876e-4%	5.1132%

Appendix D.4 - Summary of statistics with vertices spaced monthly for portfolios with maturities between 15 and 30 years.

Mappings	Median	Std dev	Min	Max	Q1	Q3	Outlier
First Study - 1 Date - 5000 Portfolios with 200 bonds (15-30)							
PV+PV01	-5.4181e-5%	3.9352e-8	-7.0872e-5%	-4.0242e-5%	-5.6917e-5%	-5.1587e-5%	0.28%
PV+Vol	-1.1448e-5%	4.1226e-8	-2.6895e-5%	1.6545e-6%	-1.4257e-5%	-8.6444e-6%	0.30%
PV01+Vol	-1.1936e-5%	4.2174e-8	-2.8289e-5%	1.4353e-6%	-1.4795e-5%	-9.0664e-6%	0.34%
PV+PV01+Vol	-3.9805e-5%	3.8903e-8	-5.5484e-5%	-2.6182e-5%	-4.2525e-5%	-3.7212e-5%	0.34%
Rates	-1.1936e-5%	4.2174e-8	-2.8289e-5%	1.4353e-6%	-1.4795e-5%	-9.0664e-6%	0.34%
Second Study - 10 Years - 5000 Portfolios with 200 bonds (15-30)							
PV+PV01	2.8183e-5%	8.1684e-7	-2.6499e-4%	5.3881e-4%	1.7658e-5%	3.5628e-5%	30.1530%
PV+Vol	-4.9656e-6%	1.2539e-6	-0.0044%	0.0041%	-2.6011e-5%	1.9501e-5%	10.6279%
PV01+Vol	-8.1790e-6%	5.8332e-7	-1.5468e-4%	3.1673e-4%	-2.7996e-5%	1.6845e-5%	10.7884%
Rates	-8.1792e-6%	5.8332e-7	-1.5468e-4%	3.1673e-4%	-2.7996e-5%	1.6845e-5%	10.7884%