A Fast On-Line Estimator of the Two Main Vibration Modes of Flexible Structures From Biased and Noisy Measurements

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***Abstract*—Vibrations are present in many mechanical structures and machines, and are often associated with their elastic parts. Characterizing these vibrations, i.e., obtaining their frequencies, amplitudes and phases, is of most interest in many applications ranging from the maintenance of civil structures to motion control.**

This article presents a method for the on-line and reliable identification of the defining parameters of two unknown sinusoidal signals through the use of their measured sum in the presence of noise and an offset. It is based on the algebraic derivative approach, defined in the frequency domain, which yields exact calculation formulae for the unknown parameters of the signal, i.e., the amplitudes, phases and frequencies of the two sinusoids and the value of the constant term. The on-line estimation is performed in a time interval which is only a fraction of the first full cycle of the slower component of the measured signal. This feature allows the algorithm to be used to monitor time varying parameters in these vibration signals.

This algorithm has been used in experiments with a flexible beam, which is a representative platform of a vibrating mechatronic system. It estimated all the vibration signal parameters quickly and accurately, proved to be insensitive to high frequency noises, and accurately tracked the time variations of the signal parameters.

***Index Terms*—Algebraic Estimator, Vibration, Flexible.**

I. Introduction

VIBRATIONS are present in many mechanical structures and machines, and are often associated with their elastic parts. Characterizing these vibrations, i.e., obtaining their frequencies, amplitudes and phases, is of the utmost interest in many applications ranging from the maintenance of civil structures to motion control.

One of the main applications of the monitoring of mechanical vibrations over the last 20 years has been that of assessing the integrity of mechanical structures by detecting changes in the dynamic parameters from vibration measurement and processing. Several techniques for structural health monitoring based on vibrations measurements have demonstrated their capacity to detect failures in civil structures [1]. These methods estimate the vibration characteristics offline by first measuring the vibrations and later processing the signals recorded.

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The structural vibration suppression of flexible mechanisms is another topic of increasing concern in engineering. Since these flexible structures are often lightweight, they are used extensively in both the aerospace industry, [2], [3] and other mechatronic advanced applications such as micropositioning, e.g.[4], [5]. As these structures have very little damping, it is necessary to use active vibration control to achieve precise motions and positionings. In this context, knowledge of the vibration modes may significantly improve the performance of these control systems [6]. As the dynamics, and hence the vibration modes, of these mechanisms are often time varying as a consequence of their changing configurations (e.g. deployable space structures [7], large robot manipulators [8]) or changing payloads (robotics [9], micropositioners [10], [11]), algorithms able to estimate these vibrations in real-time are of the utmost interest.

The new field of energy harvesting research has also recently demonstrated the need for frequency estimation. As shown in [12], the performance of the device used to produce energy is higher when its natural vibration frequency is properly tuned. In order to maintain the rate of harvested energy in its optimal value it would therefore be very useful to estimate the frequencies of the source of vibration in real-time and implement an autotuning mechanism that would allow the device to track frequency changes.

Research into algorithms for the real-time estimation of vibration modes in mechanical systems has therefore taken place over the last few years. The faster the estimation is, the better the control system will act on the mechanism, or the energy harvesting system will track the optimal functioning point. Moreover, in the case of the maintenance of civil structures, implementing real-time vibration estimators will also increase monitoring possibilities.

Estimating the frequency, amplitude and phase of sinusoidal signals has become an interesting issue in itself since Fourier developed a transformation method in 1822, and it has been applied in many areas of engineering. In fact, one of the two main approaches for sinusoid parameter estimation, the non-parametric methods, is based on the (fast) Fourier transform [13], [14]. Although these methods are usually considered to be more accurate than those of parametrics, their heavy computational burden and leakage make them fairly useless for on-line computations.

The other main approach used for sinusoid estimation, the parametric methods, has proved to have a higher processing speed and a sufficiently high resolution, despite a short data record [15] in many applications, which makes them suitable for on-line estimations and fast system identification. These methods use predetermined functional structures, and are focused on estimating only the parameters of these functions.

Since on-line estimation requires computational efficiency and fast convergence, the parametric method that is most often preferred is that of adaptive notch filtering [16]. Despite the fact that adaptive notch filters are developed in continuous-time, it should be noted that discrete-time identifiers have also been proposed (see Li and Kedem [17], Rife and Boorstyn [18], and references therein).

Several approaches have also been proposed for the general estimation case of n sinusoidal frequencies based on adaptive observers. Marino and Tomei [19], used an adaptive observer with a dimension of 5n-1. Obregon-Pulido et al. [20] later presented a nice simultaneous frequency and state estimation method, whose resulting globally convergent estimator is 3n dimensional. A different estimator, also with a dimension of 3n, was also proposed as an adaptive identifier by Xia [21].

These n sinusoidal frequency estimators share certain features with adaptive notch filters: all of these algorithms require a set of initial parameter values to be chosen - which strongly influences the efficiency of the method -, they require several cycles to converge on the right frequency estimates, and they do not provide information about the remaining sinusoid parameters (amplitudes and phases). A long standing requirement is, therefore, that of an on-line, robust, identification method, which computes the desired frequencies, amplitudes and phases in less time than a vibration period, and without the need to tune any initialization parameters in the algorithm.

Signals provided by the sensors used to monitor mechanical structure vibrations are often noisy and biased. DC offset should not be underestimated in the analogic voltage signals provided by electronic acquisition systems. The aforementioned methods are not well suited to dealing with these problems. It would therefore be desirable for the vibration mode estimation method to be additionally able to accurately identify sinusoid parameters in measurements with these drawbacks.

A completely different approach with which to estimate vibrations in real-time is based on the algebraic framework originally proposed by Fliess and Sira-Ramirez [22]. This methodology has been used to design algorithms for the real-time estimation of the parameters of sinusoids from noisy signals in the cases of: one sinusoid with a bias [23], one sinusoid with a small damping [24], two sinusoids [25], two sinusoids with offset [26], the two main vibration modes of an elastic beam from measurements without offset [27], and the general case of n frequencies [28]. This last algorithm approached the case of identifying sinusoids with an offset by assuming that the first mode to be estimated had a zero frequency value, thus mimicking the offset.

In this article we propose a new method for the real-time estimation of the frequencies, amplitudes and phases of the two main vibration modes (the two with the lowest frequencies) of a flexible structure from noisy signals which exhibit offset and may have parametric variations over the time. The algorithm developed herein is based on the aforementioned algebraic identification framework, and it provides very fast (it converges in a time which is a fraction of the period of the first vibration mode) and sufficiently accurate estimations. It improves on the method developed in [28] as regards four aspects: 1) it provides frequencies, amplitudes and vibrations phases in real-time (unlike [28] which only provides frequencies); 2) it yields reliable estimations in the presence of an offset, converging faster in this case to the real values than the algorithm in [28]; 3) it provides a real-time estimate of the DC offset; and 4) we have experimentally tested the method using a real-time application created in LabView (only simulations were presented in [28]).

The algorithm developed herein also improves on the method developed in [26] as regards four aspects: 1) it provides a real-time estimate of the DC offset; 2) it yields reliable estimations in the presence of noise in the signal through the use of integral filters; 3) it employs all the available data to estimate the values of the amplitudes, phases and offset, thus yielding more accurate results than the algorithm in [26]; and 4) we have experimentally tested the method (only simulations were presented in [26]).

This article also develops and experiments with an application of the above: the estimation of the two main vibration modes of a flexible cantilever beam. Experiments are carried out in which the vibrations are characterized using two different signals: a) position and velocity measurements provided by a laser doppler vibrometer (LDV) and b) momentum measurements provided by strain gauges.

A flexible cantilever beam that supports several loads on its structure can be used as a basic representative model for a number of advanced lightweight engineering structures [29] such as robot arms, aircraft wings, and civil structures such as buildings and bridges. The identification of its dominant vibrations is a key issue in health monitoring and the vibration control of these structures. This flexible beam has an infinite number of vibration modes. Our algorithm will estimate the two main modes, and higher modes will be treated as noise. The interest in designing an estimator that is robust to noises of relatively low frequencies - although always higher than the second mode frequency - is therefore justified.

There are several applications related to the high accuracy positioning of flexible structures in which LDVs are used as sensors in order to control their vibrations [30], [31]. The use of these devices is often justified because laser measurements avoid the mechanical changes frequently caused by accelerometers or other sensors that must be placed on the structure. Since the displacement measurements provided by the LDVs are simply the integration of the velocity measures, this integrated signal may have a DC offset depending on where the zero displacement was set (note that if the target has a small amount of movement, the zero cannot be set exactly), i.e., the initial position from which the velocity integration process is carried out cannot be exactly determined to yield such an offset.

The most common sensors with which to characterize and remove link vibrations that are used in flexible robotics are, meanwhile, strain gauges[32], [6]. These sensors are used because: a) their placement at the base of the links makes it easier to control the robot; b) strain gauges are usually a cheaper alternative to LDVs; and c) the angular displacements performed by the robot cause great displacements of the tip, which are out of the range of the LDVs. However, strain measures are often plagued by a DC offset, owing to non-idealities in the mechanical or electrical arrangement.

In the proposed application, the use of velocity measurements to characterize vibrations may be troublesome because frequencies of modes higher than two (mainly the third mode) may have an amplitude similar to the amplitudes of the first two modes. These higher than second order modes may then strongly distort the measured signal and impair the estimation of the two main modes (the level of the ”noise” ”seen” by the estimator becomes too high). Displacement measurements do not, however, suffer from this problem as third and higher modes exhibit amplitudes that are significantly smaller than those of the first two modes. These measurements are therefore preferred to those of velocity. Furthermore, as displacement measures and strain measures are affected by offset, it would be desirable if, in addition to estimating the vibration parameters, the algorithm could also estimate the offset value - which may change with time. The proposed application is of utmost interest in this case.

We wish to point out the interest in characterizing the two lowest vibration modes in several applications. For example, in the motion control of flexible robots, at least three vibration modes need to be characterized in order to efficiently damp structural vibrations. Analytical models of beams are available that allow the vibration frequencies of a beam to be calculated if all its mechanical parameters are known. However, an interesting application of the method developed here is the determination of the two lowest vibration frequencies of the beam in the case of tip mass and tip rotational inertia being unknown because they change throughout the robot's operation. In this case, the estimation of these two frequencies allows estimation to take place after the tip mass and rotational inertia by substituting these frequencies in the characteristic equation of the beam (either in the clamped-free of articulated-free cases), and the two equation system yielded to be solved. Once all the beam parameters are known, the analytical model of the beam can yield all the vibration modes of the beam. We again note the interest in obtaining these payload parameters and beam vibration modes as fast as possible for control purposes.

This article is organized as follows. Section II formulates the problem of estimating sinusoid parameters in the ideal case of a sum of two sinusoids plus a constant term without noise. Section III extends the results of the previous section to the case of a noisy signal, and describes some implementation issues. Section IV describes the experimental setup, and Section V presents an actual case of application. Finally, some conclusions are drawn in Section VI.

II. Problem formulation in the noise-free case

Given a signal of the form:

$$x(t) = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2) + K \tag{1}$$

where K is an unknown constant bias perturbation term, it is desired to compute, on-line and as fast as possible, the unknown amplitudes A_1, A_2, the unknown frequencies ω_1, ω_2), the constant phase angles ϕ_1, ϕ_2 and the unknown bias perturbation term K from the available signal.

The Laplace transform of (1) is given by:

$$\begin{aligned} x(s) = & \frac{A_1 \cos(\phi_1)}{s^2+\omega_1^2} + \frac{sA_1 \sin(\phi_1)}{s^2+\omega_1^2} + \frac{A_2 \cos(\phi_2)}{s^2+\omega_2^2} \\ & + \frac{sA_2 \sin(\phi_2)}{s^2+\omega_2^2} + \frac{K}{s} \end{aligned} \tag{2}$$

For the sake of simplicity, we assume that ϕ_1 and ϕ_2 are bounded by the interval $[-\pi/2, \pi/2]$, and the amplitudes can be both positive or negative. The objective of the algebraic manipulations described in the following section is to eliminate the constant term K and obtain expressions for the unknown parameters, involving only integrations or integral convolutions of the signal $x(t)$. The vector components are defined as follows:

$$\begin{aligned} \hat{P}_1 = A_1\omega_1 \cos(\phi_1), \quad Q_1 = A_1 \sin(\phi_1), \quad R_1 = \omega_1^2, \\ \hat{P}_2 = A_2\omega_2 \cos(\phi_2), \quad Q_2 = A_2 \sin(\phi_2), \quad R_2 = \omega_2^2 \end{aligned} \tag{3}$$

Note that parameters A_1, A_2, ω_1, ω_2, ϕ_1, ϕ_2 can be easily obtained from $\hat{P}_1, \hat{P}_2$, Q_1, Q_2, R_1, R_2:

$$\begin{aligned} \omega_1 = \sqrt{R_1}, \quad A_1 = \sqrt{Q_1^2 + \frac{\hat{P}_1^2}{R_1}}, \phi_1 = \arctan\left(\frac{Q_1\sqrt{R_1}}{\hat{P}_1}\right), \\ \omega_2 = \sqrt{R_2}, \quad A_2 = \sqrt{Q_2^2 + \frac{\hat{P}_2^2}{R_2}}, \phi_2 = \arctan\left(\frac{Q_2\sqrt{R_2}}{\hat{P}_2}\right) \end{aligned} \tag{4}$$

Expression (2) may then be rewritten as:

$$x(s) = \frac{\hat{P}_1}{s^2+R_1} + \frac{sQ_1}{s^2+R_1} + \frac{\hat{P}_2}{s^2+R_2} + \frac{sQ_2}{s^2+R_2} + \frac{K}{s} \tag{5}$$

A. On-line computation of the frequency

Upon multiplying out the expression (5) by the polynomial $s(s^2+R_1)(s^2+R_2)$ and rearranging, we obtain:

$$\begin{aligned} & s^5x(s) + s^3(R_1+R_2)x(s) + sR_1R_2x(s) = \\ & s^4(Q_1+Q_2+K) + s^3(\hat{P}_1+\hat{P}_2) \\ & + s^2((Q_1+K)R_2 + (Q_2+K)R_1) \\ & + s(\hat{P}_1R_2 + \hat{P}_2R_1) + KR_1R_2 \end{aligned} \tag{6}$$

If expression (6) were differentiated five times with regard to variable s, unknown constants K, $\hat{P}_1$, $\hat{P}_2$, Q_1, Q_2 would be removed, yielding the expression:

$$\begin{aligned}&\left(s^5+(R_1+R_2)\,s^3+R_1R_2 s\right)\frac{d^5x(s)}{ds^5}\\&+\left(25s^4+15\,(R_1+R_2)\,s^2+5R_1R_2\right)\frac{d^4x(s)}{ds^4}\\&+\left(200s^3+60\,(R_1+R_2)\,s\right)\frac{d^3x(s)}{ds^3}\\&+\left(600s^2+60\,(R_1+R_2)\right)\frac{d^2x(s)}{ds^2}\\&+600s\frac{dx(s)}{ds}+120x(s)=0 \qquad (7)\end{aligned}$$

Upon denoting $X = R_1 + R_2$ and $Z = R_1R_2$, and rearranging terms we obtain:

$$\begin{aligned}&\left[s^3\frac{d^5x(s)}{ds^5}+15s^2\frac{d^4x(s)}{ds^4}+60s\frac{d^3x(s)}{ds^3}+60\frac{d^2x(s)}{ds^2})\right]X\\&+\left[s\frac{d^5x/s)}{ds^5}+5\frac{d^4x(s)}{ds^4}\right]Z=\\&-s^5\frac{d^5x(s)}{ds^5}-25s^4\frac{d^4x(s)}{ds^4}-200s^3\frac{d^3x(s)}{ds^3}\\&-600s^2\frac{d^2x(s)}{ds^2}-600s\frac{dx(s)}{ds}-120x(s) \qquad (8)\end{aligned}$$

If expression (8) is multiplied out by the factor s^{-5} (which represents five iterated integrations in the time domain), then an expression free of terms containing positive powers of the complex variable s is obtained (positive powers of s must be avoided because they represent undesired repeated time differentiations of the signals involved). An expression involving only time convolutions of the signal $x(t)$ is therefore yielded:

$$\begin{aligned}&\left[s^{-2}\frac{d^5x(s)}{ds^5}+15s^{-3}\frac{d^4x(s)}{ds^4}+60s^{-4}\frac{d^3x(s)}{ds^3}\right.\\&\left.+60s^{-5}\frac{d^2x(s)}{ds^2}\right]X+\left[s^{-4}\frac{d^5x(s)}{ds^5}+5s^{-5}\frac{d^4x(s)}{ds^4}\right]Z=\\&-\frac{d^5x(s)}{ds^5}-25s^{-1}\frac{d^4x(s)}{ds^4}-200s^{-2}\frac{d^3x(s)}{ds^3}\\&-600s^{-3}\frac{d^2x(s)}{ds^2}-600s^{-4}\frac{dx(s)}{ds}-120x(s) \qquad (9)\end{aligned}$$

Let $\mathcal{L}$ denote the usual operational calculus transform acting on exponentially bounded signals with bounded left support. Recall that $\mathcal{L}^{-1}s(\cdot) = d/dt(\cdot)$, $\mathcal{L}^{-1}1/s(\cdot) = \int_0^t(\cdot)(\sigma)d\sigma$, and $\mathcal{L}^{-1}d^v/ds^v(\cdot) = (-1)^v t^v(\cdot)$. Expression (9) can thus be written in the time domain as follows:

$$\begin{aligned}&\left[-\int^{(2)}t^5x(t)+15\int^{(3)}t^4x(t)-60\int^{(4)}t^3x(t)\right.\\&\left.+60\int^{(5)}t^2x(t)\right]X+\left[-\int^{(4)}t^5x(t)+5\int^{(5)}t^4x(t)\right]Z=\\&t^5x(t)-25\int t^4x(t)+200\int^{(2)}t^3x(t)\\&-600\int^{(3)}t^2x(t)+600\int^{(4)}tx(t)-120\int^{(5)}x(t) \qquad (10)\end{aligned}$$

Expression (10) can be written in a compact form as:

$$\eta(t)X+\xi(t)Z=q(t) \qquad (11)$$

where $\eta(t)$, $\xi(t)$, and $q(t)$ can be calculated in real time because they are the outputs of the following time-varying linear unstable filters:

$$\begin{array}{lll}q(t)=t^5x(t)-z_1 & \xi=z_6 & \eta=z_{11}\\ \dot{z_1}=z_2+25t^4x(t) & \dot{z_6}=z_7 & \dot{z_{11}}=z_{12}\\ \dot{z_2}=z_3-200t^3x(t) & \dot{z_7}=z_8 & \dot{z_{12}}=z_{13}-t^5x(t)\\ \dot{z_3}=z_4+600t^2x(t) & \dot{z_8}=z_9 & \dot{z_{13}}=z_{14}+15t^4x(t)\\ \dot{z_4}=z_5-600tx(t) & \dot{z_9}=z_{10}-t^5x(t) & \dot{z_{14}}=z_{15}-60t^3x(t)\\ \dot{z_5}=120x(t) & \dot{z_{10}}=5t^4x(t) & \dot{z_{15}}=60t^2x(t)\end{array} \qquad (12)$$

whose initial states are set to zero.

The linear equation (11) has two unknowns, X and Z, which can be obtained from a least squares error fitting in a time interval $[t_i, t_f]$ (where the interval $[t_i, t_f]$ is equal to the interval of time between the first and the last available sample). Upon defining a cost:

$$\varepsilon=\int_{t_i}^{t_f}\left\{\left[\begin{array}{cc}\eta(\tau) & \xi(\tau)\end{array}\right]\cdot\left[\begin{array}{c}X\\Z\end{array}\right]-q(\tau)\right\}^2 d\tau, \qquad (13)$$

its minimization leads to:

$$\begin{aligned}\left[\begin{array}{c}X\\Z\end{array}\right]&=\left[\int_{t_i}^{t_f}\left[\begin{array}{c}\eta(\tau)\\\xi(\tau)\end{array}\right]\cdot\left[\begin{array}{cc}\eta(\tau) & \xi(\tau)\end{array}\right]d\tau\right]^{-1}\\&\cdot\int_{t_i}^{t_f}\left[\begin{array}{c}\eta(\tau)\\\xi(\tau)\end{array}\right]q(\tau)d\tau \qquad (14)\end{aligned}$$

The batch formula has been expressed in (14) for the sake of clarity, and it was first used in connection with algebraic parameter estimation techniques in [33]. However, in order to increase the computational efficiency of the algorithm, we have used the standard recursive formula for the least squares algorithm [34]. We should mention that a recursive formula of the least squares algorithm was used in [35] to estimate the parameters of a DC motor using an algebraic estimator.

The parameters ω_1 and ω_2 are only weakly linearly identifiable. This signifies that once X and Z have been identified, parameters R_1 and R_2 can be obtained, and ω_1 and ω_2 can be easily determined from them by using the non-linear relations:

$$\omega_1=\sqrt{\frac{1}{2}\left[X\pm\sqrt{X^2-4Z}\right]},\omega_2=\sqrt{\frac{2Z}{X\pm\sqrt{X^2-4Z}}} \qquad (15)$$

Note that only positive solutions for ω_1 and ω_2 have physical sense. The square root term: $\sqrt{X^2-4Z}$ must have the same sign in both expressions, since otherwise $Z = \omega_1^2\omega_2^2$ is not verified. Finally note that the solutions found for ω_1 and ω_2 can be interchanged.

B. Exact calculation of the amplitudes, phases and offset

When the frequencies ω_1 and ω_2 have been computed after n samples, the set of data available is the n samples used for the computation and the two estimated frequencies. All of these data will be used to compute the remaining unknowns.

The equality:

$$\begin{aligned} x(t) &= A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2) + K \\ &= A_1 \cos(\phi_1) \sin(\omega_1 t) + A_1 \sin(\phi_1) \cos(\omega_1 t) \\ &+ A_2 \cos(\phi_2) \sin(\omega_2 t) + A_2 \sin(\phi_2) \cos(\omega_2 t) + K \end{aligned} \quad (16)$$

can be used in a compact form:

$$x(t) = \mathbf{\Phi}(t)^T \cdot \mathbf{\Theta} \quad (17)$$

where:

$$\mathbf{\Phi}(t) = \left[\begin{array}{ccccc} \sin(\omega_1 t) & \cos(\omega_1 t) & \sin(\omega_2 t) & \cos(\omega_2 t) & 1 \end{array} \right]^T$$

and

$$\mathbf{\Theta} = \left[A_1 \cos(\phi_1), A_1 \sin(\phi_1), A_2 \cos(\phi_2), A_2 \sin(\phi_2), K \right]^T,$$

to estimate the values of amplitudes, phases, and offset that minimize the error determined by:

$$\varepsilon = \int_{t_i}^{t_f} \left[\mathbf{\Phi}^T(\tau) \cdot \mathbf{\Theta} - x(\tau) \right]^2 d\tau, \quad (18)$$

The minimization of this cost function leads to:

$$\mathbf{\Theta} = \left[\int_{t_i}^{t_f} \mathbf{\Phi}(\tau) \mathbf{\Phi}(\tau)^T d\tau \right]^{-1} \cdot \int_{t_i}^{t_f} \mathbf{\Phi}(\tau) x(\tau) d\tau. \quad (19)$$

Since it is necessary to know the estimates of the frequencies prior to computing (19), and we wish to obtain estimates for the amplitudes, phases and bias as soon as the estimates of the frequencies are available, the batch formula has been used. The batch formula (19) uses the estimated frequencies together with the data recorded until the frequencies have been estimated, in order to compute the estimates of the remaining parameters immediately after the frequencies have been obtained. Note also that recording the samples from t_i to t_f, i.e., from the first sample to the sample at which the estimation of the frequencies is available, only requires a limited amount of memory because the proposed algorithm has been endowed with a reinitiation step which is triggered by the completion of the estimation process (as will be seen in the following section). This prevents any possible overflow. By knowing $\mathbf{\Theta}$, i.e., the values of P_1, Q_1, P_2 and Q_2, (where $P_1 = A_1 \cos(\theta_1)$ and $P_2 = A_2 \cos(\theta_2)$) it is quite straightforward to obtain the values of the amplitudes and phases with the following expressions:

$$\begin{aligned} \phi_1 &= \arctan\left(\frac{Q_1}{P_1}\right), \; A_1 = P_1 \sqrt{\frac{Q_1^2}{P_1^2} + 1}, \\ \phi_2 &= \arctan\left(\frac{Q_2}{P_2}\right), \; A_2 = P_2 \sqrt{\frac{Q_2^2}{P_2^2} + 1} \end{aligned} \quad (20)$$

where, for the sake of simplicity, it is assumed that ϕ_1 and ϕ_2 are bounded by the interval $[-\pi/2, \pi/2]$, and the amplitudes can be both positive or negative.

III. Parameter estimation of the noisy case

In this case, the measured signal is of the form:

$$y(t) = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2) + K + \xi(t) \quad (21)$$

A. On-line computation of the frequency in the noisy case

Given an uncertain noisy signal of the form: $y(t) = x(t) + \xi(t)$ where $\xi(t)$ is a zero mean stochastic process (otherwise, the process could be considered as a noise of zero mean whose mean value is included in the offset to be identified), but is otherwise of completely unknown statistics, noisy signal $y(t)$ is therefore the only available signal with direct information about $x(t)$.

Since $x(t)$ cannot be exactly measured, the formulae developed for the exact computation of X and Z will be used, but it must be stressed that $y(t)$ represents the noisy measured signal, and $x(t)$ must therefore be substituted for the measured signal $y(t)$.

In order to filter the noise present in $y(t)$, it is necessary to apply an integral filter in (14), and the numerators and the denominators of the quotients resulting from operating that expression will thus be filtered independently with the same integral filter.

This invariant filtering operation finds its full justification when the expressions for the numerator and the denominator are considered in the frequency domain. It is clear that using identical integral filter transfer functions for the numerator and the denominator does not affect the quotient in the noise free case, and it certainly enhances the signal to noise ratio of both the numerator and denominator in the high frequency noise-perturbed case. In order to implement this filter, expression (14) is modified to:

$$\begin{aligned} \left[\begin{array}{c} \hat{X} \\ \hat{Z} \end{array} \right] &= \left[\int_{t_i}^{t_f} \left[\int_{t_i}^{t_f} \left[\begin{array}{c} \eta(\tau) \\ \xi(\tau) \end{array} \right] \cdot \left[\begin{array}{cc} \eta(\tau) & \xi(\tau) \end{array} \right] d\tau \right] d\tau \right]^{-1} \\ &\cdot \int_{t_i}^{t_f} \left[\int_{t_i}^{t_f} \left[\begin{array}{c} \eta(\tau) \\ \xi(\tau) \end{array} \right] q(\tau) d\tau \right] d\tau. \end{aligned} \quad (22)$$

Once $\hat{X}$ and $\hat{Z}$ have been calculated, frequencies ω_1 and ω_2 can be computed from (15).

B. Exact calculation of the amplitudes, phases and offset

In the case of dealing with a noisy signal it is necessary to use the estimation of the frequencies ω_1 and ω_2 computed from the noisy signal. The whole calculation process is similar to the noisy-free case. The only difference is that $y(t)$ is used instead of $x(t)$ in (19), because noisy signal $y(t)$ is the only one available.

The simplistic treatment of signals with noise proposed here -which is based on modeling noise by using a high frequency component of the signal instead of using its statistical properties, which is removed with a low pass filter- has proved to be quite effective in practice: it produces accurate estimations without increasing the complexity of the algorithm, as has been shown in previous experiments with a carbon fiber flexible link [24] and an electric wave generator [25] (both of which had perceptible noise), and will be shown in the more complex experiments carried out in Section V. We refer to the study of the properties of our algebraic estimators in the presence of noise carried out in [23], in which the robustness of these algorithms was checked with parameters such as the sampling time, signal-to-noise ratio (SNR) and high frequency sinusoidal noise.

C. Reinitiation algorithm

In order to carry out the on-line estimation of the signal parameters, it is necessary to reset and reinitiate the frequency estimation process described in Section II because: 1) Linear time varying filters are unstable, and the values of the variables involved in the estimation may therefore become very high as time increases; 2) the computer storage capacity of the data needed to calculate (19) is limited; and 3) signal parameters are often time varying. Signal parameters may change for several reasons: the damping of structure vibrations produces variations in the amplitude of the signal (and therefore in the amplitude of the vibration modes), and variations in the mass or the rotational inertia of the beam (for example, in the load to be supported at its tip) produce changes in the frequencies of the vibration modes.

The estimation algorithm must therefore be reset and reinitiated from time to time. We reinitiate the algorithm at the time when the estimations of the two frequencies have converged to steady state values. This condition is expressed by using the following algorithm.

The method used to determine the time of reinitiation was originally proposed in [27] and is based on the moving average and the moving standard deviation of the estimated frequencies. The frequency estimator of subsection II-A requires a discrete implementation. Its output is therefore two sequences, $\{\omega_1(n)\}$ and $\{\omega_2(n)\}$, where n is the ordinal that refers to the estimation provided by the algorithm at time $n \cdot T_s$, T_s being the sampling time. Let us define a window of length M_i, $i = 1, 2$ for each frequency estimation, which includes samples $n - M_i + 1 \leq k \leq n$. The moving average:

$$\bar{\omega}_i(n) = \frac{1}{M_i} \sum_{k=0}^{M_i-1} \omega_i(n-k) \tag{23}$$

and the moving standard deviation:

$$\sigma_i(n) = \sqrt{\frac{\sum_{k=0}^{M_i-1} (\omega_i(n-k) - \bar{\omega}_i(n))^2}{M_i}} \tag{24}$$

are then associated with these windows, and the proposed criterium is: *the algorithm is reinitiated when the condition:*

$$\frac{\sigma_i(n)}{|\bar{\omega}_i(n)|} \leq \delta_i \tag{25}$$

is simultaneously verified for $i = 1, 2$*, where* δ_i *is the tolerance parameter that determines the accuracy of the estimate of each frequency*.

The sample at which condition (25) is simultaneously verified by the estimation of the two frequencies is denoted as $\hat{n}$. The frequency estimations provided by the algorithm are therefore $\omega_1^e = \bar{\omega}_1(\hat{n})$ and $\omega_2^e = \bar{\omega}_2(\hat{n})$. The time $\hat{n} \cdot T_s$ is the *operation time interval*, and it corresponds to the interval of time from the beginning of the estimation process (it is assumed that $n = 0$ when the estimation starts) until the frequency estimates have converged and the algorithm yields its result. If condition (25) has not been verified after a time T_r, the algorithm is also reinitiated, but the estimates yielded are not considered valid. The tuning of the value of T_r is based on experience: a good compromise is often that of choosing a value of the order of the period of the lower frequency that has to be estimated.

In summary, the proposed real-time estimation algorithm consists of of the following steps:

1) In each sampling period, the input of the algorithm is the sample $y(n)$ of the signal, and new estimates of the frequencies $\omega_1(n)$ and $\omega_2(n)$ are calculated from expressions (22) and (15).
2) Estimations $\omega_i(n-M+1) \ldots \omega_i(n)$ are used to compute (23), (24).
3) Check the reinitiation criterion (25). If this criterion is verified, the last value of each moving average will be considered to be the correct estimation of the frequencies ω_1^e and ω_2^e. If T_r is reached and (25) is not fulfilled, return to step 1) without considering ω_1^e and ω_2^e as valid.
4) These estimations of frequencies are used to compute amplitudes, phases and offset from (19) and (20)
5) Finally, the algorithm is reinitiated.

IV. Experimental setup

The platform used in the experiments is shown in Fig. 1. It is a flexible aluminium cantilevered beam which is 1.26 m long, with a transversal area of 50×3 mm. The flexible beam has one end free (the tip), while other end is clamped.

The sensors used are: on the one hand, a Polytec LDV (model OFV-5000) with a sensor head model OFV-505 in order to take displacement and velocity measurements, and on the other, strain gauges placed at the base of the link (at the clamped end). LDVs have several advantages: they have high resolution, bandwidth and accuracy, and since they do not require any physical contact with the object being measured, they do not perturb its dynamic behaviour. Strain gauges are useful to measure the vibrations of the link when the angular movement of the base of the link causes very high displacements of the tip of the beam, which cannot be measured with the LDV.

The data obtained from the experimental platform were computed in real time using a PC equipped with LabView and a PCI-6221 for data acquisition. The sampling time was 1 ms.

In order to characterize the dynamics of the beam, a blow was applied at 42 cm from the tip. The displacement and velocity of the tip were then simultaneously measured using the LDV (see Fig. 2). The fast Fourier transforms of these two signals are shown in Fig. 3, which illustrates four peaks

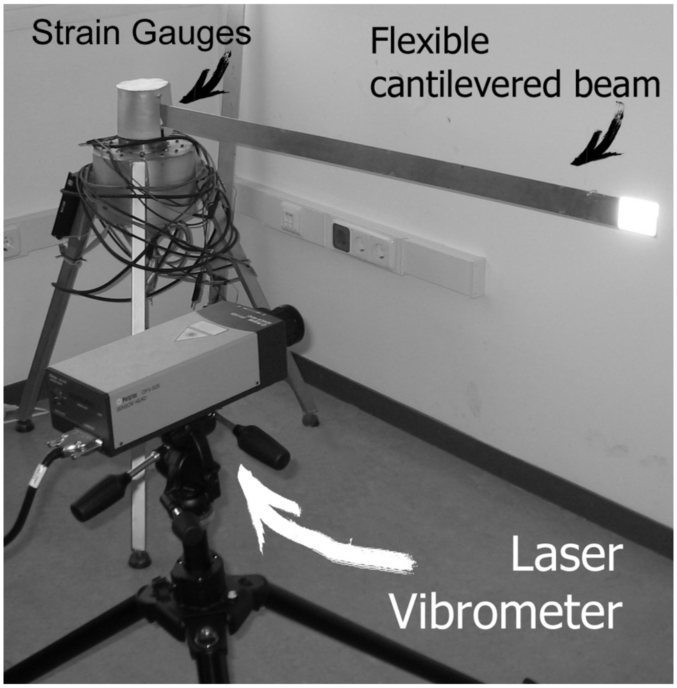

Fig. 1. Experimental Platform

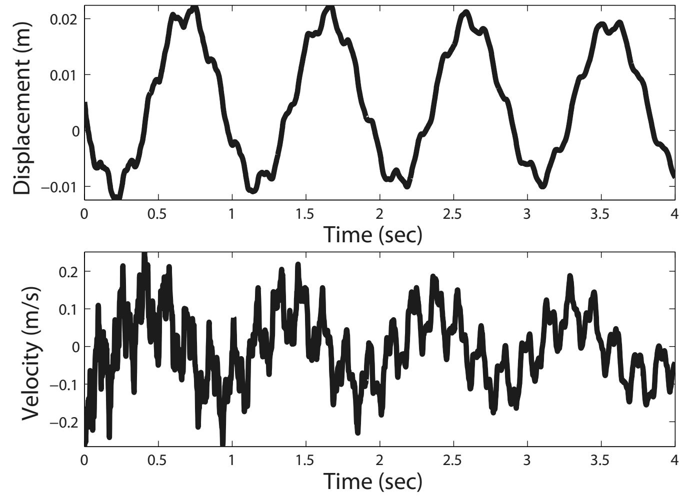

Fig. 2. Experimental displacement (a) and velocity (b) measures

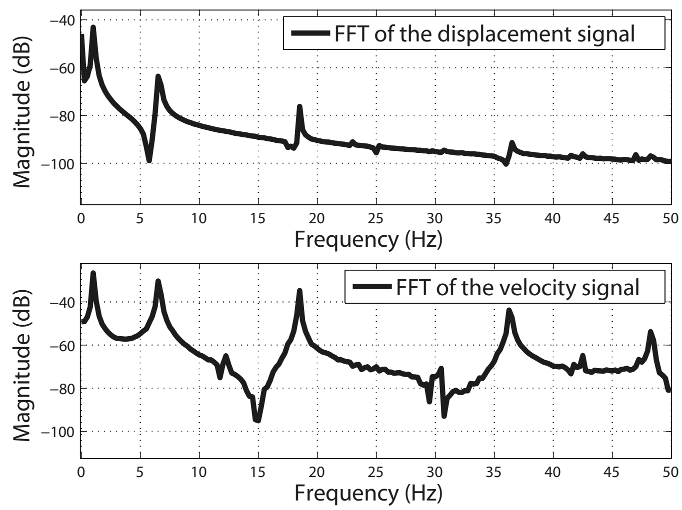

Fig. 3. FFT of the experimental displacement (a) and velocity (b) measures

corresponding to the resonant frequencies $f_1 = 1.068$ Hz, $f_2 = 6.61$ Hz, $f_3 = 18.43$ Hz, and $f_4 = 36.32$ Hz,. This figure shows that the magnitudes of the peaks diminish in both spectra as the values of the resonant frequencies increase. It also shows that the magnitudes of these peaks are relatively larger in the velocity than in the displacement spectrum and, more significantly, the amplitudes of the first three modes are similar in the velocity spectrum, while the amplitudes of the third mode is smaller than the amplitude of modes one and two in the displacement spectrum. In fact, this figure shows that the magnitude of the peak of the third vibration mode is comparable to the amplitude of the first and second harmonic when the velocity signal is considered. The significant amplitude exhibited by the third resonant frequency in the velocity spectrum may therefore preclude the accurate functioning of any estimator of two frequencies applied to velocity data.

The aforementioned flexible cantilever beam with no payload at its free end has been modelled and simulated. Its dynamic behaviour has been modelled as an Euler-Bernouilli beam with the following parameters: linear mass density $\rho = 0.268\ Kg/m$, uniform bending stiffness $EI = 2.4\ N \cdot m$, length $L = 1.26\ m$, beam mass $M_B = 0.338\ Kg$, and rotational inertia $J_B = 0.179\ Kg \cdot m^2$. The frequencies of the vibration modes and the complete dynamic model were obtained by using the method described in [36]. The equations provided by this method are not reproduced here because they are well known and have been extensively used in scientific literature. Table I shows the lower four resonant frequencies obtained from the spectra of the signals recorded in the experiment, in addition to those obtained from the theoretical model based on the Euler-Bernouilli beam. The results obtained from these two approaches are quite similar: the table shows differences of under 2%.

TABLE I
COMPARISON OF NATURAL FREQUENCIES OBTAINED EXPERIMENTALLY AND FREQUENCIES OBTAINED BY SIMULATION

Harmonic	Experimental frequencies (Hz)	Simulated frequencies (Hz)	Error (%)
First harmonic	1.068	1.055	1.22
Second harmonic	6.61	6.61	0.02
Third harmonic	18.43	18.5	0.38
Fourth harmonic	36.32	36.27	0.14

Finally, we should mention that in the case of an articulated-free beam - which corresponds to the case of a robot with flexible links which has to follow a desired trajectory - an alternative to the proposed algorithm is that of obtaining the vibration modes by identifying an input-output model, i.e., a transfer function. The proposed algebraic estimator framework could also be used in this case. Some examples of this are [37], [38], [39], in which these estimators have been applied and experimented on flexible robots in order to carry out adaptive control.

V. Real time estimation for signals with time-varying parameters

This section studies the capability of the proposed estimator to identify time varying parameters of harmonic signals in real time. This algorithm is therefore used to estimate not only time varying frequencies of the measured signal, but also the time-varying amplitudes of the different harmonics and the time varying amplitude of the bias term of the signal.

In the tests carried out in this section, the estimation process must be reinitiated each time the convergence criterion (25) is achieved, or each time the elapsed time of the algorithm reaches T_r without having achieved (25), in order to provide the time varying estimates of each parameter, and thus be able to track their changing values. The period of time between two consecutive reinitiations is called the *operation time interval* of the estimator, and signifies the time that the estimator takes to yield the estimates in a step. The time chosen for T_r was $1\ s$ throughout the experiments.

The experimental results are presented in this section, and four cases are studied: the tracking of time-varying mode amplitudes, the tracking of time-varying mode frequencies, the tracking of time-varying offset, and the identification of the modes of vibration under an external forced vibration. Some of the issues related to the data from the following experiments are:

1) The values used as references to compute the precision attained in the estimation of the mode frequencies are those obtained by applying the FFT to the experimental data recorded from the vibrating beam. The errors shown in the subsequent tables therefore concern these values.
2) The data obtained in the experiment on time-varying amplitudes was processed using both the algebraic method proposed in this article and the the modified Prony's method. The difference in the amplitudes produced by both estimators is shown in Table II.
3) The error data shown in the subsequent tables are the absolute values of the errors obtained in the estimations.

A. Time-varying amplitudes

The amplitudes of all the vibration signals obtained on the experimental platform decrease with time owing to the damping of the flexible beam. The data analyzed in this subsection was gathered by hitting the tip of the flexible beam when it was in steady state and not carrying a payload, and then recording the displacement of the tip when the beam oscillated.

In order to assess the performance of the proposed estimator as regards identifying the amplitudes of an harmonic signal with time-varying amplitudes, its results were compared with those provided by a well known method: the modified Prony's method. Section IV showed that our beam has four perceptible modes. The Prony's method was therefore applied by fitting a model with four damped harmonics plus an offset to the recorded data, yielding the following result:

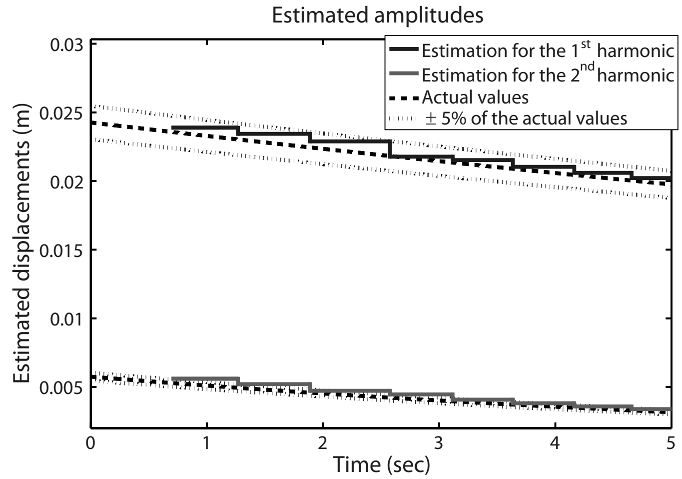

Fig. 4. Estimation of the amplitudes when the amplitudes are time-varying (simulated case)

$$\begin{aligned} y(t)=&0.0242e^{-0.0411t}sin(1.068\cdot 2\pi t)\\ &+0.0057e^{-0.1197t}sin(6.61\cdot 2\pi t)\\ &+0.0007e^{-0.296t}sin(18.43\cdot 2\pi t)\\ &+0.00006e^{-1.5096t}sin(36.32\cdot 2\pi t)-0.001. \quad (26)\end{aligned}$$

The amplitude estimates yielded by the proposed estimator for the first two harmonics when considering the instantaneous amplitudes of (26) as a reference are shown in Fig. 4. This figure shows that the amplitude estimates are accurate and that the estimator can track the changes of the amplitude over time, since the estimates are produced in a short period of time (in less than one cycle of the lower frequency harmonic). Note that continuously varying parameters - as in this case - is the worst scenario for our estimator. The best scenario would be stepwise variations in the parameters (they remain constant during most of the operation time intervals of the identification process).

Table II shows the errors in the frequency estimates of the proposed estimator in comparison to the real values obtained by applying the FFT. It also shows the difference in the amplitude estimates between the proposed estimator and the modified Prony's method. The *operation time interval* of the estimator alters slightly from one estimation step to another. The last row in Table II shows the most representative characteristics of this time: its mean value and its standard deviation.

It should be noted that Prony's method is not suited to yielding fast real-time parameter estimates, while our method is.

B. Time-varying frequency

In order to produce a time-varying frequency signal, the flexible beam was hit at $42\ cm$ from its end with no load mounted at the tip. After $2.7\ s$, a load of $0.065\ Kg$ was added to the tip. This load change produced a stepwise variation in the vibration frequencies of the beam. The vibration

TABLE II
CHARACTERISTICS OF THE ERRORS PRODUCED IN THE ESTIMATIONS AND OPERATION TIME INTERVAL, WHEN A SIGNAL WITH TIME-VARYING MODE AMPLITUDES ARE USED

	Min. value	Mean value	Max. value	Standard deviation
1^{st} harmonic frequency error	0.7475 (%)	2.1169 (%)	2.7308(%)	0.76919 (%)
2^{nd} harmonic frequency error	0.0714 (%)	0.2944 (%)	0.5815 (%)	0.1876 (%)
1^{st} harmonic amplitude diff.	0.2362 (%)	1.0505 (%)	1.9535 (%)	0.5840 (%)
2^{nd} harmonic amplitude diff.	2.5399 (%)	5.9327 (%)	3.8210 (%)	1.5054 (%)
Bias term error	$1.036 \cdot 10^{-3}$	$1.526 \cdot 10^{-3}$	$1.906 \cdot 10^{-3}$	$3.085 \cdot 10^{-4}$
Operation time interval (s)	0.4930	0.5820	0.6950	0.0773

TABLE III
CHARACTERISTICS OF THE ERRORS PRODUCED IN THE ESTIMATIONS AND OPERATION TIME INTERVAL, WHEN A SIGNAL WITH TIME-VARYING MODE FREQUENCIES ARE USED

	Min. value	Mean value	Max. value	Standard deviation
1^{st} harmonic frequency error	0.0583 (%)	2.6224 (%)	6.3715 (%)	2.0470(%)
2^{nd} harmonic frequency error	0.0321 (%)	0.1930 (%)	0.5515 (%)	0.1626 (%)
Bias term error	$5.975 \cdot 10^{-4}$	$2.952 \cdot 10^{-3}$	$5.438 \cdot 10^{-3}$	$1.707 \cdot 10^{-3}$ deviation
Operation time interval (s)	0.5490	0.6577	1.0000	0.1319

frequencies of the experimental platform were, in this case, estimated using measurements obtained from strain gauges placed at the base of the link. The use of strain gauges allows us to show that our algorithm could be applied to signals provided by sensors of different technologies, and that it could be used for the motion control of flexible robots, which often use these sensors. The frequencies estimated from this data, again have the problems of the existence of third and fourth vibration modes, high frequency noise, non negligible offset, and damping in the amplitudes of the harmonics.

The results obtained for the estimation of the frequencies are shown in Fig. 5. This figure shows that the frequency estimates are accurate and that they track the frequency changes over time (note that the real values of the first and second harmonic are those obtained using the FFT with the data before and after the load addition).

The characteristics of the errors produced in the estimations and *operation time intervals* are shown in Table III. The statistical characteristics of the estimation of each parameter have been obtained from the set of all the values yielded by the estimator.

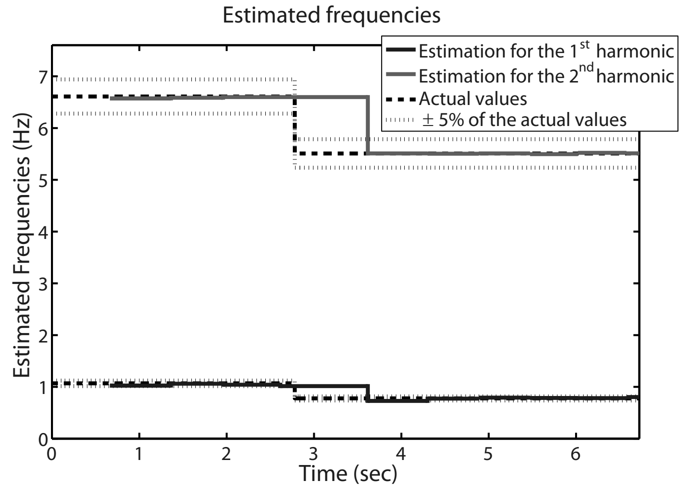

Fig. 5. Estimation of the frequencies when the frequencies are time-varying (experimental case)

C. *Time-varying offset*

The experiment carried out in this study consisted of hitting the beam, without a load at its tip, at $42\ cm$ from its end. The tip displacement was measured with the LDV. $3\ s$ after the blow, the LDV was displaced a small distance backwards to produce a change in the bias term from a value of 0.0085 to a value of 0.1085.

The results obtained in the estimation of the offset are shown in Fig. 6. This figure shows that the estimates are again very accurate and track the changes over time. It also shows that the algorithm may yield a wrong estimate in the operation time interval at which the abrupt change in the offset value is produced, but it yields an accurate estimate in the following operating time interval.

The parameters of the errors produced in the estimation of the offset and the *operation time intervals* are shown in Table IV. Since the signal was obtained by measuring the experimental platform, the only parameters known with sufficient accuracy to be able to study the performance of the estimator are the frequencies of the signal harmonics and the bias term. The performance of the algorithm when estimating these frequencies is also shown in this table. The statistical characteristics of the estimation of each parameter have been obtained from the set of all the values yielded by the estimator, but not that provided in the step at which the sudden change in the offset occurred. This value has been excluded because it is not representative of the estimator's performance, since it did not work correctly in this step.

D. *Forced vibration between the 1^{st} and the 2^{nd} harmonics*

This subsection studies the performance of the proposed estimator when it processes measurements of the vibrations originated by a forced oscillation applied to the flexible beam. In this case, the vibration was produced by varying the angular position of the base of the flexible link in such a way that it followed a sinusoidal signal with a frequency of 3 Hz (between the first and the second modes) and an amplitude of 0.02 rad.

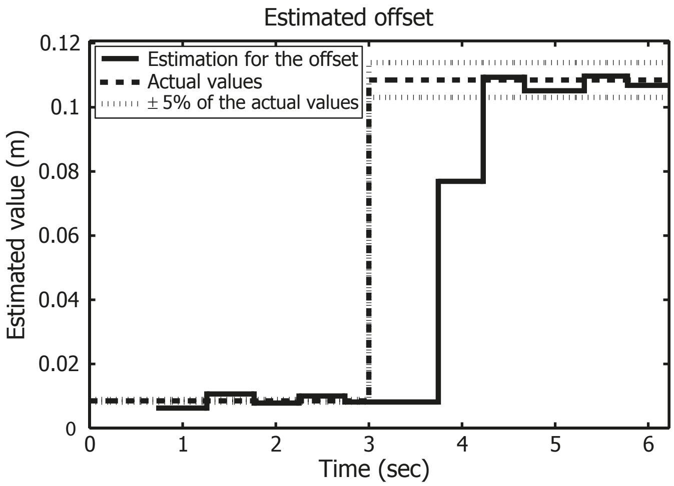

Fig. 6. Estimation of the offset when the offset is time-varying (experimental case)

TABLE IV
CHARACTERISTICS OF THE ERRORS PRODUCED IN THE ESTIMATIONS AND OPERATION TIME INTERVAL, WHEN A SIGNAL WITH TIME-VARYING OFFSET IS USED

	Min. value	Mean value	Max. value	Standard deviation
1^{st} harmonic frequency error	0.013805 (%)	1.0707 (%)	2.5807 (%)	0.99934(%)
2^{nd} harmonic frequency error	0.31386 (%)	2.0211 (%)	6.4055 (%)	2.0882 (%)
Bias term error	0.00036183	0.0015622	0.0033455	0.00092295
Operation time interval (s)	0.446	0.53067	0.715	0.089425

The experimental data recorded in this case was the velocity of the tip measured with the LDV.

The results obtained during the estimation of the harmonics are shown in Fig. 7 and in Fig. 8. These figures show that, when the proposed method (shown in Fig. 7) is used, only the first harmonic can be estimated. Since the frequency of the excitation signal is situated between the first two natural modes of vibration, the proposed estimator only "sees" the first natural mode of vibration and the excitation signal. Note that, even considering this limitation, the results obtained are quite accurate (see Table V).

The velocity measures do not have any offset. The algebraic framework allows estimators to be developed for signals with more than two modes without any offset (e.g. [28]). The estimator of three modes without any offset shown in the Appendix has been applied to the data recorded in this experiment. This estimator identifies the first two modes of vibration and the excitation signal very accurately, as Fig. 8 shows.

We should mention that, in this experiment, the last estimator performed well, since the amplitudes of the third and fourth vibration modes of the beam were significantly smaller than the amplitudes of the two lowest vibration modes and the forced oscillation. Note that the algorithm proposed throughout

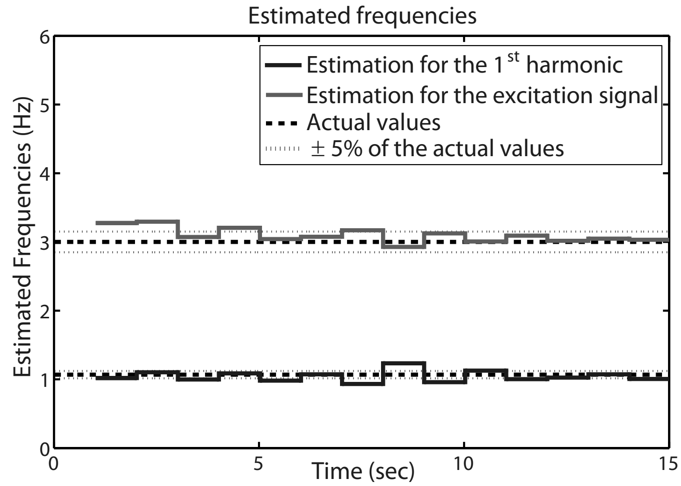

Fig. 7. Estimation of the frequencies under forced vibration. Proposed method

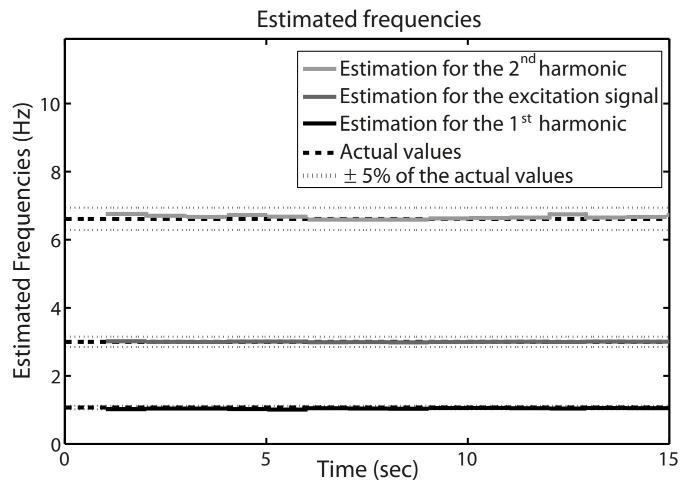

Fig. 8. Estimation of the frequencies under forced vibration. Three harmonics estimator

the article is designed for biased signals, but it can also be applied to unbiased signals, as Fig. 7 shows.

The parameters of the errors produced in the estimation of the offset and the *operation time intervals* are shown in Table V.

VI. CONCLUSIONS

A flexible beam has been considered as a representative model for a wide number of lightweight engineering structures, and a mechatronic system composed of a flexible beam instrumented with an LDV and strain gauges has been used as an experimental platform. The measures gathered using the LDV at the tip of the beam or the measures from strain gauges at the base of the beam have been used as analysis signals. The time-varying parameters of these signals (composed of two harmonics and a bias term) have been quickly and accurately estimated by means of an algebraic approach.

Although the proposed algorithm has been designed to work with signals with constant parameters (frequencies, amplitudes, initial phases and offset), the feature of the fast convergence of the estimator that has been achieved - estimation

TABLE V
CHARACTERISTICS OF THE ERRORS PRODUCED IN THE ESTIMATIONS AND OPERATION TIME INTERVAL. WHEN THE PROPOSED ALGORITHM IS USED (UPPER TABLE), AND WHEN THE THREE HARMONICS EXTRAPOLATION IS USED (LOWER TABLE)

	Min. value	Mean value	Max. value	Standard deviation
1^{st} harmonic frequency error	0.4016 (%)	6.0794 (%)	15.4199 (%)	4.3955(%)
Excitation signal error	0.2205 (%)	3.6256 (%)	9.8176 (%)	3.1030(%)
Operation time interval (s)	1.000	1.000	1.000	0.000

	Min. value	Mean value	Max. value	Standard deviation
1^{st} harmonic frequency error	0.9344 (%)	2.5379 (%)	5.6893 (%)	1.3154(%)
2^{nd} harmonic frequency error	0.1905 (%)	0.9718 (%)	2.1289 (%)	0.6666 (%)
Excitation signal error	0.0038 (%)	0.2088 (%)	0.6418 (%)	0.2004 (%)
Operation time interval (s)	0.9060	0.9958	1.000	0.0253

takes less than one period of the first harmonic of the signal - allows this algorithm to be used in several applications that involve time-varying parameters. These applications involved the real time implementation of our estimator, and were studied in Section V. It should be noted that these applications cannot be dealt with by the other algorithms presented in the Introduction, which were also developed for the real time estimation of harmonics, since only yield frequency estimates and are very slow as regards tracking the varying parameters, and are not therefore practical. This claim was developed and justified in a previous article [24]. These estimators have not therefore been considered in the study presented herein.

The first real time application consisted of estimating constant frequencies and offset, and time-varying amplitudes of the first and second modes. Comparison between our real time estimator and the modified Prony's method, which is a batch method, showed a good tracking of these amplitudes despite the fact that the algorithm had to estimate harmonic parameters from a signal whose mode amplitudes continuously varied throughout each operation time interval.

The estimator was then applied to signals that undergo sudden changes in their frequencies and in their offset. Experimental results also showed a good tracking of the varying parameters in both cases, but the estimation provided in the operation time interval in which the sudden change occurred was erroneous. However, the algorithm yielded an accurate estimation in the following step.

In all the tests, the operation time interval of the algorithm proved to be quite stable: with a mean value close to but of less than $0.7\ s$ and a standard deviation lower than 0.14 (from Tables II, III and IV).

The proposed algorithm can consequently be used to estimate the amplitudes and frequencies of the first and second vibration modes of a flexible beam, in addition to the offset value, and is robust to: 1) unmodelled higher vibration modes, 2) high frequency noise, 3) offset, and 4) time-varying parameters. This estimator is therefore particularly suitable for the real time tracking of time-varying parameters from data obtained by sensors based on different technologies.

APPENDIX
ALGEBRAIC ESTIMATOR FOR A THREE FREQUENCY SIGNAL

The methodology proposed in this article can be easily generalized to signals composed of three frequencies without offset:

$$x(t) = A_1 sin(\omega_1 t+\phi_1)+A_2 sin(\omega_2 t+\phi_2)+A_3 sin(\omega_3 t+\phi_3) \quad (27)$$

The equation used to estimate the unknown frequencies is therefore:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \left[\int_{t_i}^{t_f} \begin{bmatrix} \eta(\tau) \\ \beta(\tau) \\ \xi(\tau) \end{bmatrix} \cdot \begin{bmatrix} \eta(\tau) & \beta(\tau) & \xi(\tau) \end{bmatrix} d\tau \right]^{-1} \cdot \int_{t_i}^{t_f} \begin{bmatrix} \eta(\tau) \\ \beta(\tau) \\ \xi(\tau) \end{bmatrix} q(\tau) d\tau \quad (28)$$

where X, Y, and Z are:

$$\begin{aligned} X &= \omega_1^2 + \omega_2^2 + \omega_3^2, \\ Y &= \omega_1^2\omega_2^2 + \omega_1^2\omega_3^2 + \omega_2^2\omega_3^2, \\ Z &= \omega_1^2\omega_2^2\omega_3^2 \end{aligned} \quad (29)$$

and where $\eta(t)$, $\beta(t)$, $\xi(t)$, and $q(t)$ can be calculated using the following time-varying linear unstable filters:

$$\begin{array}{ll} q(t) = -t^6 x(t) - z_1 & \beta = z_{13} \\ \dot{z}_1 = z_2 - 36t^5 x(t) & \dot{z}_{13} = z_{14} \\ \dot{z}_2 = z_3 + 450t^4 x(t) & \dot{z}_{14} = z_{15} \\ \dot{z}_3 = z_4 - 2400t^3 x(t) & \dot{z}_{15} = z_{16} \\ \dot{z}_4 = z_5 + 5400t^2 x(t) & \dot{z}_{16} = z_{17} + t^6 x(t) \\ \dot{z}_5 = z_6 - 4320t x(t) & \dot{z}_{17} = z_{18} - 12t^5 x(t) \\ \dot{z}_6 = 720x(t) & \dot{z}_{18} = 30t^4 x(t) \\ \eta = z_7 & \xi = z_{19} \\ \dot{z}_7 = z_8 & \dot{z}_{19} = z_{20} \\ \dot{z}_8 = z_9 & \dot{z}_{20} = z_{21} + t^6 x(t) \\ \dot{z}_9 = z_{10} & \dot{z}_{21} = z_{22} - 24t^5 x(t) \\ \dot{z}_{10} = z_{11} & \dot{z}_{22} = z_{23} + 180t^4 x(t) \\ \dot{z}_{11} = z_{12} & \dot{z}_{23} = z_{24} - 480t^3 x(t) \\ \dot{z}_{12} = t^6 x(t) & \dot{z}_{24} = 360t^2 x(t) \end{array} \quad (30)$$

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