On the entire functions from the Laguerre–Pólya class having monotonic second quotients of Taylor coefficients

Von der Fakultät für Mathematik und Informatik

der Universität Leipzig

angenommene

DISSERTATION

zur Erlangung des akademischen Grades

DOCTOR RERUM NATURALIUM
(Dr. rer. nat.)

im Fachgebiet

Mathematik

Vorgelegt

von M.Sc. Thu Hien Nguyen

geboren am 11.06.1995 in Berlin, Deutschland

Die Annahme der Dissertation wurde empfohlen von:

1. Prof. Anna Vishnyakova, V.N. Karazin Kharkiv National University (Betreuerin)
2. Prof. Tatjana Eisner, Leipzig University
3. Prof. Vladimir Kostov, Université de Nice

Die Verleihung des akademischen Grades erfolgt mit Bestehen

der Verteidigung am 20.10.2022 mit dem Gesamtprädikat Magna Cum Laude

Dedicated to my beloved mother and my father.

To my dearest advisor and mentor, Anna Vishnyakova, Ukraine and V. N. Karazin Kharkiv National University.

Contents

List of symbols

The following symbols have been frequently used throughout the thesis.

$\mathcal{CZDS}$: the set of complex zero decreasing sequences, i.e. the set of real sequences $(\gamma_k)_{k=0}^{\infty}$, such that for any real polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$ the number of non-real zeros of P does not exceed the number of non-real zeros of the polynomial $\sum_{k=0}^{n} \gamma_k a_k x^k$.

$g_a(x) = \sum_{k=0}^{\infty} \frac{x^k}{a^{k^2}}, a > 1$: the partial theta function.

$\mathcal{HP}$: the set of univariate hyperbolic polynomials, i.e. the set of real univariate polynomials having only real zeros.

$\mathcal{HP}_+$: the set of univariate hyperbolic polynomials with all positive coefficients.

$\mathcal{L} - \mathcal{P}$: the Laguerre–Pólya class of entire functions, i.e. the closure in the topology of the uniform convergence on compacts of the set of hyperbolic polynomials.

$\mathcal{L} - \mathcal{PI}$: the Laguerre–Pólya class of type I of entire functions, i.e. the closure in the topology of the uniform convergence on compacts of the set of hyperbolic polynomials with all positive coefficients.

$\mathcal{MS}$: the set of multiplier sequences, i.e. the set of real sequences $(\gamma)_{k=0}^{\infty}$, such that for any hyperbolic polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$ the polynomial $\sum_{k=0}^{n} \gamma_k a_k x^k$ is also hyperbolic.

$p_n(f) = \frac{a_{n-1}}{a_n}, n \geq 1$: the first quotients of Taylor coefficients of an entire function $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

$q_n(f) = \frac{a_{n-1}^2}{a_{n-2} a_n}, n \geq 2$: the second quotients of Taylor coefficients of an entire function $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

$q_\infty \approx 3.23363666$: the absolute constant which was found by O. Katkova, T. Lobova and A. Vishnyakova. For more details, see the corresponding theorems.

$S_n(x, f) = \sum_{k=0}^{n} a_k x^k$: the n-th partial sum of an entire function $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

$R_n(x, f) = \sum_{k=n}^{\infty} a_k x^k$: the n-th remainder of an entire function $f = \sum_{k=0}^{\infty} a_k x^k$.

$Z_{\mathbb{C}}(P)$: the number of non-real zeros of a real polynomial P counting multiplicities.

$Z_{\mathbb{R}}(P)$: the number of real zeros of a real polynomial P counting multiplicities.

Introduction

We investigate the famous Laguerre–Pólya class of entire functions and its subclass, the Laguerre–Pólya class of type I. The functions from these classes can be expressed in terms of the Hadamard Canonical Factorization (see Chapter 1, Definition 1.2 and 1.3). The prominent theorem by E. Laguerre and G. Pólya gives a complete description of the Laguerre–Pólya class and the Laguerre–Pólya class of type I, showing that these classes are the respective closures in the topology of uniform convergence on compact sets of the set of real polynomials having only real zeros (that is, the set of so-called *hyperbolic polynomials*) and the set of real polynomials having only real negative zeros. Both the Laguerre–Pólya class and the Laguerre–Pólya class of type I play an essential role in complex analysis. For the properties and characterizations of these classes, see, for example, [31] by A. Eremenko, [40] by I.I. Hirschman and D.V. Widder, [43] by S. Karlin, [57] by B.Ja. Levin, [66, Chapter 2] by N. Obreschkov, and [74] by G. Pólya and G. Szegó.

To highlight the importance of the Laguerre–Pólya class we would like to mention the connection between this class and one of the most famous open problems in modern mathematics, the Riemann hypothesis, which was posed by Bernhard Riemann in 1859 (see [76]). The conjecture states that a special meromorphic function known as the Riemann zeta function has (apart from trivial zeros at the negative even integers) only zeros with real parts equal to 1/2. The relation between the Riemann zeta function and the Laguerre–Pólya class can be found in works [24] by G. Csordas, T.S. Norfolk and R.S. Varga, [25] and [26] by G. Csordas and R.S. Varga, [22] by G. Csordas, [28] by G. Csordas and C.-C. Yang, and [29] by D.K. Dimitrov. In particular, the Riemann hypothesis can be reformulated as a statement that a special entire function, the so-called Ξ-function, belongs to the Laguerre–Pólya class. This approach of proving the Riemann Hypothesis is known as the Hilbert–Pólya Conjecture. Although very little is known about its origin, a sketch of the idea can be found, for example, in the paper by O. Katkova, [46].

There are many works devoted to functions from the Laguerre–Pólya class. We mention here only a few of them. In the works [21] by T. Craven,

G. Csordas and W. Smith, and [42] by H. Ki and Y. Kim, Pólya's conjecture is proved which states that for a real entire function of order less than two with a finite number of nonreal zeros the derivative of a sufficiently high order belongs to the Laguerre–Pólya class. In the papers [7] by W. Bergweiler and A. Eremenko and [8] by W. Bergweiler, A. Eremenko and J. Langley the Wiman conjecture is proved concerning the number of nonreal zeros of derivatives of a real entire function of order greater than two.

Among a plenty of recent works devoted to the functions from the Laguerre–Pólya class, we only mention [5] by A. Baricz and S. Singh, [10] by A. Bohdanov, [11] by A. Bohdanov and A. Vishnyakova, [12] by P. Brändén, [14] and [13] by D. Cardon, [16] and [23] by T. Craven and G. Csordas, [23] by G. Csordas and T. Forgácz, [27] by G. Csordas and A. Vishnyakova, [38] by B. He, [56] by M. Lamprecht and [80] by A.D. Sokal.

The question of whether an entire function belongs to the Laguerre–Pólya class (or the Laguerre–Pólya class of type I) can be very difficult. In this thesis, we have found some necessary and sufficient conditions, in terms of its Taylor coefficients, for an entire function to belong to the Laguerre–Pólya class (and Laguerre–Pólya class of type I). We restrict our investigations to the set of entire functions from the Laguerre–Pólya class having positive Taylor coefficients and an increasing sequence of their second quotients. The main purpose of this thesis is to study the properties of the coefficients and the location of the zeros of these functions.

Structure of the thesis. The thesis is organized as follows. Chapter 1 is devoted to the brief historical overview of the existing results. The definitions of the Laguerre–Pólya class and the Laguerre–Pólya class of type I, the multiplier sequence and the complex zero decreasing sequence, the partial theta function and other important objects are provided. A short historical overview of the study of the partial theta function is given. Some previously known results on the functions from the Laguerre–Pólya class are indicated.

In Chapter 2 we describe a necessary condition for an entire function with positive coefficients and with the increasing sequence of second quotients of Taylor coefficients to belong to the Laguerre–Pólya class (see Theorem 2.1).

In Section 2.2, we investigate a special function F_a defined by

$$F_a(z) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{(a^k+1)(a^{k-1}+1)\cdot \ldots \cdot (a+1)}$$

which is related to the partial theta function and and is also known as the q-Kummer function ${}_1\phi_1(q; -q; q, -z)$, where $q = 1/a$. The question is, for which $a > 1$, this function belongs to the Laguerre–Pólya class, and the answer is given by Theorem 2.10 and Theorem 2.11.

In Chapter 3 we present some necessary conditions for entire functions to belong to the Laguerre–Pólya class in terms of their roots with the smallest modulus. For entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with positive coefficients we prove that, if f belongs to the Laguerre–Pólya class and the quotients $q_k(f)$ satisfy the condition $q_2(f) \leq q_3(f)$, then f has at least one zero in the segment $[-\frac{a_1}{a_2}, 0]$. Some necessary and sufficient conditions of the existence of such a zero in terms of the quotients $q_k(f)$ for $k = 2, 3, 4$ are obtained (see Theorem 3.2, Theorem 3.4, and Theorem 3.6).

In Chapter 4, it is proved that if $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0$, is an entire function such that the sequence of its second quotients of Taylor coefficients is non-decreasing and $q_2(f) \geq 2\sqrt[3]{2}$, then all but a finite number of zeros of f are real and simple (see Theorem 4.1 and Theorem 4.3).

We further provide a criterion for entire functions with the non-decreasing sequences of their second quotients of Taylor coefficients for belonging to the Laguerre–Pólya class of type I in terms of the closest to zero roots under additional assumptions on the regularity of increasing of the sequences of the second quotients of Taylor coefficients.

In Chapter 5 we present some necessary conditions for entire functions with the non-decreasing sequences of the second quotients of Taylor coefficients for belonging to the Laguerre–Pólya class of type I (see Theorem 5.1 and Theorem 5.4).

Finally, Chapter 6 includes the remaining problems for future research.

Chapter 1

Background of research

This chapter begins with a short literature overview concerning hyperbolic polynomials of one variable, the Laguerre–Pólya class, linear operators, preserving real-rootedness, properties of the partial theta-function, apolar polynomials, and other topics connected to our investigations. We define necessary notions and formulate some important theorems to which we refer in this thesis.

1.1 Hyperbolic polynomials and the Laguerre–Pólya class

The study of zero distribution of entire functions, their sections and tails has always been one of the central questions in complex analysis, see, for example, detailed reviews on this topic in works by A. A. Goldberg and I.V. Ostrovskii [33], I.I. Hirschman and D.V. Widder [40], B.Ja. Levin [57] and I.V. Ostrovskii [68].

We start with the definition of a hyperbolic polynomial.

Definition 1.1. A real univariate polynomial is said to be *hyperbolic* if all its zeros are real. The set of hyperbolic polynomials is denoted by $\mathcal{HP}$. The set of hyperbolic polynomials with only negative zeros is denoted by $\mathcal{HP}_+$.

Entire functions which can be uniformly approximated in a neighbourhood of zero by hyperbolic polynomials found important applications in other fields, for example, in the theory of integral transforms [40] by I.I. Hirschman and D.V. Widder, approximation theory [78], the theory of total positivity and probability theory [43] and [44] by S. Karlin.

One of the important classes of entire functions is the Laguerre–Pólya class. We give the definitions of the Laguerre–Pólya class and the Laguerre–Pólya class of type I.

Definition 1.2. A real entire function f is said to be in the *Laguerre–Pólya class*, written $f \in \mathcal{L} - \mathcal{P}$, if it can be expressed in the form

$$f(z) = cz^n e^{-\alpha z^2+\beta z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{x_k}\right) e^{zx_k^{-1}}, \tag{1.1}$$

where $c, \alpha, \beta, x_k \in \mathbb{R}$,$x_k \neq 0$, $\alpha \geq 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} x_k^{-2} < \infty$.

Definition 1.3. A real entire function f is said to be in the *Laguerre– Pólya class of type I*, written $f \in \mathcal{L} - \mathcal{P}I$, if it can be expressed in the following form

$$f(z) = cz^n e^{\beta z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{x_k}\right), \tag{1.2}$$

where $c \in \mathbb{R}, \beta \geq 0, x_k > 0$, n is a nonnegative integer, and $\sum_{k=1}^{\infty} x_k^{-1} < \infty$.

The product on the right-hand sides in both definitions can be finite or empty (in the latter case, the product equals 1).

Various important properties and characterizations of the Laguerre–Pólya class and the Laguerre–Pólya class of type I can be found in works by I.I. Hirshman and D.V. Widder [40], B.Ja Levin [57], G. Pólya and G. Szegö [74], G. Pólya and J. Schur [73], monograph by N. Obreschkov [66, Chapter II] and many other works. These classes are essential in the theory of entire functions since it appears that the polynomials with only real zeros (or only real and nonpositive zeros) converge locally uniformly to these and only these functions. The following prominent theorem provides an even stronger result.

Theorem A (E. Laguerre and G. Pólya, see, for example, [40, p. 42–46] and [57, chapter VIII, §3])**.**

(i) *Let* $(P_n)_{n=1}^{\infty}$, $P_n(0) = 1$, *be a sequence of hyperbolic polynomials which converges uniformly on the disc* $|z| \leq A, A > 0$. *Then this sequence converges locally uniformly in* $\mathbb{C}$ *to an entire function from the* $\mathcal{L} - \mathcal{P}$ *class.*

(ii) *For any* $f \in \mathcal{L} - \mathcal{P}$ *there exists a sequence of hyperbolic polynomials, which converges locally uniformly to* f*.*

(iii) *Let* $(P_n)_{n=1}^{\infty}$, $P_n(0) = 1$, *be a sequence of hyperbolic polynomials having only negative zeros which converges uniformly on the disc* $|z| \leq A, A > 0$. *Then this sequence converges locally uniformly in* $\mathbb{C}$ *to an entire function from the class* $\mathcal{L} - \mathcal{P}I$*.*

(iv) *For any $f \in \mathcal{L}-\mathcal{P}I$ there is a sequence of hyperbolic polynomials with only negative zeros which converges locally uniformly to f.*

For a real entire function (not identically zero) of the order less than 2 the property of having only real zeros is equivalent to belonging to the Laguerre–Pólya class. Similarly, for a real entire function with positive coefficients of the order less than 1 having only real negative zeros is equivalent to belonging to the Laguerre–Pólya class of type I. Strikingly, the situation changes for the functions of order 2. For instance, the entire function $f(x) = e^{-x^2}$ belongs to the $\mathcal{L}-\mathcal{P}$ class while the entire function $g(x) = e^{x^2}$ does not.

On the interesting properties and various characterizations of the Laguerre–Pólya class and the Laguerre–Pólya class of type I, see, for example, [31] by A. Eremenko, [40] by I.I. Hirschman and D.V. Widder, [43] by S. Karlin, [57] by B.Ja. Levin, [66, Chapter 2] by N. Obreschkov, and [74] by G. Pólya and G. Szegó. Among plenty of recent works devoted to the functions from the Laguerre–Pólya class, we only mention here [5] by A. Baricz and S. Singh, [10] by A. Bohdanov, [11] by A. Bohdanov and A. Vishnyakova, [12] by P. Brändén, [14] and [13] by D. Cardon, [16] and [23] by T. Craven and G. Csordas, [23] by G. Csordas and T. Forgácz, [27] by G. Csordas and A. Vishnyakova, [38] by B. He, [56] by M. Lamprecht and [80] by A.D. Sokal.

1.2 The first and second quotients of Taylor coefficients

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with real nonzero coefficients. We define the *first quotients of Taylor coefficients* p_n and the *second quotients of Taylor coefficients* q_n as follows.

$$p_n = p_n(f) := \frac{a_{n-1}}{a_n}, \quad n \geq 1,$$
$$q_n = q_n(f) := \frac{p_n}{p_{n-1}} = \frac{a_{n-1}^2}{a_{n-2}a_n}, \quad n \geq 2.$$

From these definitions it follows straightforwardly that one can express coefficients of f in terms of p_n or q_n.

$$a_n = \frac{a_0}{p_1 p_2 \cdot \ldots \cdot p_n}, \quad n \geq 1,$$
$$a_n = a_1 \left(\frac{a_1}{a_0}\right)^{n-1} \frac{1}{q_2^{n-1} q_3^{n-2} \cdot \ldots \cdot q_{n-1}^2 q_n}, \quad n \geq 2.$$

One can see that the second quotients of Taylor coefficients are independent parameters that define a function up to multiplication by a constant and changing $z \to \lambda z$.

1.3 A simple sufficient condition for an entire function to have only real zeros

The problem of understanding whether a given entire function (or polynomial) has only real zeros is considered subtle and complicated. A simple verified description of this class, in terms of the coefficients of a series, is impossible since it is determined by an infinite number of discriminant inequalities. In 1926, J.I. Hutchinson found quite a simple sufficient condition for an entire function with positive coefficients to have only real zeros, which was a generalization of the results by M. Petrovitch [71] and G. Hardy [36], or [37, pp. 95 - 100].

Theorem B (J.I. Hutchinson, [41])**.** *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$,$a_k > 0$ for all k, be an entire function. Then $q_n(f) \geq 4$, for all $n \geq 2$, if and only if the following two conditions are fulfilled:*

(i) The zeros of f are all real, simple and negative, and

(ii) The zeros of any polynomial $\sum_{k=m}^{n} a_k z^k$,$m < n$, formed by taking any number of consecutive terms of f, are all real and non-positive.

For some extensions of Hutchinson's results see, for example, [15, §4].

1.4 Multiplier sequences and complex zero decreasing sequence

Since it is very difficult problem to define whether or not a given real polynomial (or a real entire function) belongs to the class $\mathcal{HP}$ (to the class $\mathcal{L}-\mathcal{P}$) the important role play linear operators that preserve the class $\mathcal{HP}$ (the class $\mathcal{L}-\mathcal{P}$). Now, we need the definition of multiplier sequences.

Definition 1.4. A sequence $(\gamma_k)_{k=0}^{\infty}$ of real numbers is called a *multiplier sequence* if, whenever the real polynomial $P(x) = \sum_{k=0}^{n} a_k z^k$ has only real zeros, the polynomial $\sum_{k=0}^{n} \gamma_k a_k z^k$ has only real zeros. The class of multiplier sequences is denoted by $\mathcal{MS}$.

The following theorem by G. Pólya and I. Schur (1914) fully describes multiplier sequences.

Theorem C (G. Pólya and J.Schur, see [73], and [66, pp. 29–47])**.** *Let* $(\gamma_k)_{k=0}^{\infty}$ *be a given real sequence. The following three statements are equivalent.*

(i) $(\gamma_k)_{k=0}^{\infty}$ *is a multiplier sequence.*

(ii) For every $n \in \mathbb{N}$ *the polynomial* $P_n(z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k$ *has only real zeros of the same sign.*

(iii) The power series $\Phi(z) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ *converges absolutely in the whole complex plane and the entire function* $\Phi(z)$ *or the entire function* $\Phi(-z)$ *admits the representation*

$$Ce^{\sigma z} z^m \prod_{k=1}^{\infty} (1 + \frac{z}{x_k}), \tag{1.3}$$

where $C \in \mathbb{R}, \sigma \geq 0, m \in \mathbb{N} \cup \{0\}, 0 < x_k \leq \infty,$ *and* $\sum_{k=1}^{\infty} \frac{1}{x_k} < \infty.$

In other words, a sequence $(\gamma_k)_{k=0}^{\infty}, \gamma_0 > 0$, is a multiplier sequence if and only if one of the following two entire functions $\Phi(z) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ or $\Phi(-z)$ belongs to the Laguerre–Pólya class of type I.

In 1977, T. Craven and G. Csordas extended the definition of multiplier sequences investigating and characterizing them for more general fields (see [17]).

For further details about multiplier sequences see the original paper on the subject by G. Pólya and I. Schur [73], B.Ja. Levin [57, pp. 340-347], N. Obreshkov [66, Chapter 2], T. Craven and G. Csordas [19], [16], [20], and G. Csordas and T. Forgács [23].

For a real polynomial P we denote by $Z_{\mathbb{R}}(P)$ the number of real zeros of P counting multiplicities and by $Z_{\mathbb{C}}(P)$ the number of nonreal zeros of P counting multiplicities. Now, we define complex zero decreasing sequences.

Definition 1.5. A sequence $(\gamma_k)_{k=0}^{\infty}$ of real numbers is said to be a *complex zero decreasing sequence* (we write $(\gamma_k)_{k=0}^{\infty} \in \mathcal{CZDS}$), if

$$Z_{\mathbb{C}} \left(\sum_{k=0}^{n} \gamma_k a_k z^k \right) \leq Z_{\mathbb{C}} \left(\sum_{k=0}^{n} a_k z^k \right), \tag{1.4}$$

for any real polynomial $\sum_{k=0}^{n} a_k z^k$.

Obviously, we have

$$\mathcal{CZDS} \subset \mathcal{MS}.$$

In addition, the sequence $\gamma_k \equiv 1,\ k = 0, 1, 2, \ldots$, or the sequence $\gamma_k = 2^k,\ k = 0, 1, 2, \ldots$ are (trivial) complex zero decreasing sequences. The existence of nontrivial complex zero decreasing sequences is a consequence of the following remarkable theorem proved by E. Laguerre and extended by G. Pólya.

Theorem D (E. Laguerre, see [66, p. 3.2] and [72])**.**

- *Let* $P(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* $n \in \mathbb{N}$ *and let* $Q(x)$ *be a polynomial with only real zeros, none of which lie in the interval* $(0, n)$*. Then* $Z_{\mathbb{C}}(\sum_{k=0}^{n} Q(k) a_k x^k) \leq Z_{\mathbb{C}}(f)$*.*
- *Let* $P(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* $n \in \mathbb{N}$*, let* $\varphi \in \mathcal{L} - \mathcal{P}$ *and suppose that none of the zeros of* φ *lie in the interval* $(0, n)$*. Then the inequality* $Z_{\mathbb{C}}(\sum_{k=0}^{n} \varphi(k) a_k x^k) \leq Z_{\mathbb{C}}(P)$ *holds.*
- *Let* φ *be an entire function from the Laguerre–Pólya class having only negative zeros. Then* $(\varphi(k))_{k=0}^{\infty} \in \mathcal{CZDS}$*.*

As it follows from the theorem above,

$$\left(a^{-k^2}\right)_{k=0}^{\infty} \in \mathcal{CZDS},\ a \geq 1, \quad \left(\frac{1}{k!}\right)_{k=0}^{\infty} \in \mathcal{CZDS}. \tag{1.5}$$

In the 19th century, E. Laguerre (see [55]) suggested a problem to characterize complex zero decreasing sequences. This problem is also known as the *Karlin-Laguerre problem* (see [43]). This famous problem is still open. Some important results connected with this problem were obtained by A. Bakan, T. Craven, G. Csordas, and A. Golub in [4], A. Bakan and A. Golub in [3], by T. Craven and G. Csordas in [15] and [18]. In particular, the following theorem is valid.

Theorem E (A. Bakan, T. Craven, G. Csordas and A. Golub, [4, Theorem 2])**.** *Let* $(\gamma_k)_{k=0}^{\infty}, \gamma_k > 0$*, be a complex zero decreasing sequence. If*

$$\limsup_{k\to\infty} \gamma_k^{1/k} > 0,$$

then there exists a function $\varphi \in \mathcal{L} - \mathcal{P}$ *of the form*

$$\varphi(z) = \beta e^{\alpha z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{x_n}\right),$$

where $\alpha, \beta \in \mathbb{R}, x_n > 0$ *and* $\sum_{n=1}^{\infty} \frac{1}{x_n} < \infty$*, such that* φ *interpolates the sequence* $(\gamma)_{k=0}^{\infty}$*, that is,* $\gamma_k = \varphi(k)$ *for* $k = 0, 1, 2, \ldots$*.*

During the research connected to the zero distribution of entire functions, T. Craven and G. Csordas faced some interesting new open problems (see, for example, Problem 1.1. and Problem 1.2. in [18]), which involve a set of multiplier sequences. We state here one of these problems.

Problem 1.6 (Problem 1.1 of [18])**.** Characterize the meromorphic functions F which interpolate the multiplier sequences, namely such that the polynomial $\sum_{k=0}^{n} F(k)a_k x^k$ has only real zeros whenever the polynomial $\sum_{k=0}^{n} a_k x^k$ has only real zeros.

The specific entire functions from the class of the generalized Fox–Wright functions, which have a significant role in different mathematical areas, was studied by many authors. The *Fox–Wright* function is defined as follows

$$_p\psi_q(a_1, \ldots, a_p, b_1, \ldots, b_q; x) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdot \ldots \cdot (a_p)_k}{(b_1)_k \cdot \ldots \cdot (b_q)_k} x^k k!,$$

where $p, q \in \mathbb{N} \cup \{0\}$, $a_1, \ldots, a_p, b_1, \ldots, b_q \in \mathbb{C}$, the *Pochhammer* or ascending factorial symbol for $a \in \mathbb{C}$ is defined as $(a)_0 = 1, (a)_k := a(a+1)(a+2) \cdot \ldots \cdot (a+k-1) = \Gamma(a+k)/\Gamma(a), k = 1, 2, 3 \ldots.$

For instance, in case when $p = 1$ and $q = 1$, we get the entire functions defined as

$$_1\psi_1(x) := \sum_{k=0}^{\infty} \frac{\Gamma(ak+1)}{\Gamma(ck+1)} \frac{x^k}{k!}, c \geq a \geq 0,$$

and with the parameter values $a = 1$ and $c = \alpha > 0$, we get that $_1\psi_1(x)$ is the classical *Mittag-Leffler function* (see, for example, [6, vol. 3, Chapter XVIII] by A. Erdélyi, et al.)

$$E_\alpha := \sum_{k=0}^{\infty} \frac{x^k}{\gamma(\alpha k+1)}, \alpha > 0.$$

The function E_α is known as a generalization of the exponential functions of finite order. Moreover, if $\alpha \geq 2$, then it is known that this entire function (of order $1/\alpha$) has only real negative zeros, i.e. $E_\alpha \in \mathcal{L} - \mathcal{PI}$ (see [69, Corollary 3] by I.V. Ostrovskii, I. Peresyolkova).

Remark 1.7. The generalized Mittag-Leffler function $E_{\alpha,\beta}(x)$ plays an important role in analysis where it is used in the theory of integral transforms, fractional calculus, and other areas.

1.5 Apolar polynomials and the Grace Apolarity Theorem

An essential technique in our investigations is related to the Schur composition of polynomials, apolar polynomials, and the Grace Apolarity Theorem. We provide a brief description of Laguerre's Separation Theorem and Grace's Apolarity Theorem, which are often implemented in the proofs of composition theorem for polynomials. For additional citations see P. Borwein and T. Erdélyi [70], M. Marden [58] and N. Obreschkov [66], and the references contained therein.

We define a circular domain as an open or closed disk, an open or closed exterior of a disk, or an open or closed half-plane. The set of circular domains is invariant under Möbius transformations. The following theorem is the invariant form of the Gauss-Lucas Theorem (see also [70, p. 20] by P. Borwein and T. Erdélyi, [58] by M. Marden, [66] by N. Obreschkov).

Theorem F (Laguerre's Separation Theorem, see [58, §13] or [66, §14]). *Let $P(z) = \sum_{k=0}^{n} a_k x^k$ be a complex polynomial of degree $n \geq 2$.*

1. *Suppose that all zeros of P lie in a circular domain D. For $\zeta \notin D$, all of the zeros of the polar derivative $P_\zeta(z) := nP(z) + (\zeta - z)P'(z)$ lie in D.*

2. *Let α be any complex number such that $P(\alpha)P'(\alpha) \neq 0$. Then any circle C passing through the points α and $\alpha - \frac{nP(\alpha)}{P'(\alpha)}$ either passes through all the zeros of P or separates the zeros of P in the sense that there is at least one zero of P in the interior of C and at least one zero of P in the exterior of C.*

Some extensions of Laguerre's theorem and some of its more recent applications in terms of the notion of a generalized center of mass is given by E. Grosswald [35] (see also G. Pólya and G. Szegö [74, Vol. II, Problems 101-120]).

Further, we give the definition of apolar polynomials in order to state Grace's Apolarity Theorem (see [74, Chapter 2, $ 3, Problem 145] by G. Pólya and G. Szegö, [58, p.61] by M. Marden, [66, p. 23] by N. Obreschkov, [70] by P. Borwein and T. Erdélyi, or [34] by A. Goodman and I.J. Schoenberg).

Definition 1.8 (see, for example [74, Chapter 2, §3, p. 59]). Two complex polynomials $P(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k$ and $Q(x) = \sum_{k=0}^{n} \binom{n}{k} b_k x^k$ of degree n are called *apolar* if their coefficients satisfy the relation

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0. \tag{1.6}$$

The following famous theorem due to J.H. Grace states that the complex zeros of two apolar polynomials cannot be separated by a straight line or by a circumference.

Theorem G (J.H. Grace, see for example [74, Chapter 2, $ 3, Problem 145], [58, p.61], [66, p. 23], [70], or [34])**.** *Suppose P and Q are two apolar polynomials of degree $n \geq 1$. If all the zeros of P lie in a circular domain K, then Q has at least one zero in K. (A circular region is a closed or open half-plane, disk or exterior of a disk).*

The Grace Apolarity Theorem can be derived by repeated applications of Laguerre's Separation Theorem (see [58, p. 61] by M. Marden). It is a fundamental result that gives information about the relative location of the zeros of two apolar polynomials and has far-reaching consequences. An example of such a consequence is the following composition theorem.

Theorem H (The Malo-Schur-Szegö Theorem, see [58], [66, §7])**.** *Let $A(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k$ and $B(x) = \sum_{k=0}^{n} \binom{n}{k} b_k x^k$ are complex polynomials and set*

$$C(x) := \sum_{k=0}^{n} \binom{n}{k} a_k b_k x^k.$$

1. *If all the zeros of A lie in a circular domain K, and if $\beta_1, \beta_2, \ldots, \beta_n$ are all the zeros of B, then every zero of C is of the form $\zeta = -w\beta_j$, where $1 \leq j \leq n$, and $w \in K$ (G. Szegö, [81]).*

2. *If all the zeros of A lie in a convex region K containing the origin and if the zeros of B lie in the interval $(-1, 0)$, then all the zeros of C also lie in K (I. Schur, [79]).*

3. *If all the zeros of A lie in the interval $(-a, a)$ and all the zeros of B lie in the interval $(-b, 0)$ (or in $(0, b)$), where $a, b > 0$, then all the zeros of C lie in $(-ab, ab)$.*

4. *If all the zeros of a polynomial $P(x) = \sum_{k=0}^{\mu} a_k x^k$ are real and all the zeros of $Q(x) = \sum_{k=0}^{\nu} b_k x^k$ are real and of the same sign, then all the zeros of the polynomials $H(x) = \sum_{k=0}^{m} k! a_k b_k x^k$ and $L(x) = \sum_{k=0}^{m} a_k b_k x^k$ are also all real, where $m = \min(\mu, \nu)$ (E. Malo [66, p. 29], I. Shur [79]).*

For variations and generalizations of The Malo-Schur-Szegö Theorem, see [58] by M. Marden, [66] by N. Obreschkov and the references included in these monographs, and the more recent works of A. Aziz [1], [2] and Z. Rubinstein [77].

Note that from the point (4) of Theorem H it follows that if a real polynomial $\sum_{k=0}^{n} b_k x^k$ has only real negative zeros, then the sequence $(b_k)_{k=0}^{\infty}$ is a multiplier sequence, where we put $b_k = 0$ if $k > n$.

1.6 The partial theta function and its history overview

A special entire function

$$g_a(z) = \sum_{k=0}^{\infty} a^{-k^2} z^k, \quad a > 1,$$

known as the *partial theta function* (the classical Jacobi theta function is defined by the series $\theta(z) := \sum_{k=-\infty}^{\infty} a^{-k^2} z^k, |a| > 1$), was investigated by many mathematicians (see, for example, the work by E.T. Whittaker [88]), it has many interesting properties and plays an important role in Complex Analysis. Note that for the second quotients of Taylor coefficients for this function we have $q_n(g_a) = a^2$ for all $n \geq 2$.

The partial theta function has a long and fascinating history of its study. We give a brief overview of an interesting survey which was made by S.O. Warnaar in [86]. Initially, S. Ramanujan (see [75]) contributed extensively to the theory of theta functions. G.E. Andrews discovered Ramajuan's lost notebook in 1976. In The Lost Notebook S. Ramanujan stated numerous identities for functions that closely resemble ordinary theta functions. Unfortunately, his notes did not contain any proofs of the partial theta function formulae, making it impossible to determine how S. Ramanujan actually discovered them. Proofs of many of Ramanujan's partial theta function identities were found by G.E. Andrews, whose proofs were based on some identities for basic hypergeometric series. G.E. Andrews was the first to name the function as the "partial theta function", and his student, S.O. Warnaar, made a research [86], containing the history of investigation of the partial theta function and some of its main properties.

The first occurrence of the partial theta function can be found in 1844 in the papers of G. Eisenstein (see [30]). He gave a continues fraction expansion for g_a. The previous result was generalized by E. Heine in 1846 ([39]). Later on, G. Eisenstein's result for g_a was sharpened by F. Bernstein and O. Szász ([9]). L. Tschakaloff (see [82] and [83]) established linear independence results for values of partial theta function.

Besides, the history of investigating the zeros of the partial theta function has arisen our interest. The main question for us is, for which $a > 1$, the partial theta function belongs to the Laguerre–Pólya class. In 1904, G.H. Hardy [36] studied the zeros of entire functions and showed that the roots of g_a for $a^2 \geq 9$ are real and negative with exactly one root in the interval $(-a^{2^n}, -a^{2^{n-1}})$. M. Petrovitch [71] considered real rooted entire functions f all of whose finite sections are real rooted. J.I. Hutchinson has received a sufficient condition (see Theorem B). He improved Hardy's lower bound on a^2 for the roots of the partial theta function to be real and negative from 9 to 4. The paper [47] of O.M. Katkova, T. Lobova, A.M. Vishnyakova gives the exhaustive answer to the question: for which $a > 1$ the entire function g_a belongs to the Laguerre–Pólya class. Moreover, the paper [48] deals with the stability of Taylor sections of the partial theta function. Note that, since $\left(a^{-k^2}\right)_{k=0}^{\infty} \in \mathcal{CZDS}$ for $a \geq 1$, in the paper [47] it was explained that for every $n \geq 2$, there exists a constant $c_n > 1$ such that for each $n \in \mathbb{N}$, $S_n(z, g_a) := \sum_{j=0}^{n} a^{-j^2} z^j \in \mathcal{L} - \mathcal{P}$ if and only if $a^2 \geq c_n$. The notation of the constants c_n having this property will be further used.

Theorem I (O. Katkova, T. Lobova, A. Vishnyakova, [47]). *There exists a constant* q_∞ ($q_\infty \approx 3.23363666$) *such that:*

1. $g_a \in \mathcal{L} - \mathcal{P} \Leftrightarrow a^2 \geq q_\infty$;
2. $g_a \in \mathcal{L} - \mathcal{P} \Leftrightarrow$ *there exists* $z_0 \in (-a^3, -a)$ *such that* $g_a(z_0) \leq 0$
3. *if there exists* $z_0 \in (-a^3, -a)$ *such that* $g_a(z_0) < 0$, *then* $a^2 > q_\infty$;
4. *for a given* $n \geq 2$ *we have* $S_n(z, g_a) \in \mathcal{L} - \mathcal{P} \Leftrightarrow$ *there exists* $z_n \in (-a^3, -a)$ *such that* $S_n(z_n, g_a) \leq 0$;
5. *if there exists* $z_n \in (-a^3, -a)$ *such that* $S_n(z_n, g_a) < 0$, *then* $a^2 > c_n$;
6. $4 = c_2 > c_4 > c_6 > \cdots$ *and* $\lim_{n\to\infty} c_{2n} = q_\infty$;
7. $3 = c_3 < c_5 < c_7 < \cdots$ *and* $\lim_{n\to\infty} c_{2n+1} = q_\infty$.

Calculations show that $c_4 = 1 + \sqrt{5} \approx 3.23607$, $c_6 \approx 3.23364$ and $c_5 \approx 3.23362$, $c_7 \approx 3.23364$.

The partial theta function is of interest to many areas such as statistical physics and combinatorics, see [80] by A. Sokal, Ramanujan type q-series, see [87] by S.O. Warnaar, asymptotic analysis and the theory of (mock) modular forms and problems related to hyperbolic polynomials, see, for example, [36] by G.H. Hardy, [41] by J.I. Hutchinson, [47] by O. Katkova, T. Lobova and A. Vishnyakova, [52] by V.P. Kostov, [54] by V.P. Kostov and B. Shapiro, [68] by I.V. Ostrovskii, [71] by M. Petrovitch, [85] by J.L. Walsh, [45] by

I. Karpenko and A. Vishnyakova, etc. There is a series of works by V.P. Kostov dedicated to various properties of zeros of the partial theta function and its derivative (see [51, 53] and the references therein). For example, in [52] V.P. Kostov studied the so called spectrum of the partial theta function, i.e. the set of values of $a > 1$ for which the function g_a has a multiple real zero.

Theorem J (V.P. Kostov, [52]).

1. *The spectrum* Γ *of the partial theta-function consists of countably many values of* a *denoted by* $\widetilde{a}_1 > \widetilde{a}_2 > \ldots > \widetilde{a}_k > \ldots > 1$, $\lim_{j\to\infty} \widetilde{a}_j = 1$.

2. *For* $\widetilde{a}_k \in \Gamma$ *the function* $g_{\widetilde{a}_k}$ *has exactly one multiple real zero which is of multiplicity 2 and is the rightmost of its real zeros.*

3. *For* $a \in (\widetilde{a}_{k+1}, \widetilde{a}_k)$ *the function* g_a *has exactly* k *complex conjugate pairs of zeros (counted with multiplicities).*

In [49], V.P. Kostov shown that for any fixed value of the parameter a, the partial theta function g_a has only finite number of multiple zeros. For $a \in (1, +\infty)$, there exists a sequence of values of this parameter which tends to 1 such that g_a has double negative zeros which tend to $-e^{\pi}$ (see [50]).

The paper [54] by V.P. Kostov and B. Shapiro among the other results explains the role of the constant q_∞ in the study of the set of entire functions with positive coefficients having all Taylor sections with only real zeros.

Theorem K (V.P. Kostov and B. Shapiro, [54]). *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be an entire function with positive coefficients and* $S_n(z) = \sum_{j=0}^{n} a_j z^j$ *be its sections. Suppose that there exists* $N \in \mathbb{N}$, *such that for all* $n \geq N$ *the sections* S_n *belong to the Laguerre–Pólya class. Then* $\liminf_{n\to\infty} q_n(f) \geq q_\infty$.

A.D. Sokal in [80] studies the leading roots of the partial theta-function. A formal power series

$$f(x, y) = \sum_{n=0}^{\infty} \alpha_n x^n y^{n(n-2)/2},$$

is considered as a formal power series in y whose coefficients are polynomials in x. A.D. Sokal defines the "leading root" of f as a unique formal power series $x_0(y)$ which satisfies the equation $f(x_0(y), y) = 0$. The coefficientwise positivity of $-x_0(y)$ was proved. Moreover, all the coefficients of $1/x_0(y)$ and $1/x_0(y)^2$ after the constant term 1 are strictly negative, except for the vanishing coefficient of y^3 for the latter case.

In [67] by O. Katkova, T. Lobova and A. Vishnyakova, some entire functions with a convergent sequence of second quotients of coefficients are investigated. The main question of [67] is whether the Taylor sections of the function $\prod\limits_{k=1}^{\infty}\left(1+\frac{z}{a^k}\right)$,$a>1$, and $\sum_{k=0}^{\infty}\frac{z^k}{k!a^{k^2}}$, $a\geq 1$, belong to the Laguerre–Pólya class of type I. In [11] by A. Bohdanov and A. Vishnyakova and [10] by A. Bohdanov, some important special functions with non-decreasing sequence of the second quotients of Taylor coefficients are studied.

B. He in [38] considers the entire function as follows

$$A_q^{(\alpha)}(a;x)=\sum_{k=0}^{\infty}\frac{(a;q)_k q^{\alpha k^2}x^k}{(q;q)_k},$$

where $\alpha>0, 0<q<1$, and

$$(a;q)_n=\begin{cases}1, \ n=0\\ \prod\limits_{j=1}^{n-1}(1-aq^j), \ n\geq 1,\end{cases}$$

is the q-shifted factorial. The entire function $A_q^{(\alpha)}(a;x)$ defined as above is the generalization of Ramanujan entire function and the Stieltjes-Wigert polynomial which have only real positive zeros. The paper [38] gives a partial answer to Zhang's question: under what conditions the zeros of the entire function $A_q^{(\alpha)}(a;x)$ are all real.

A. Baricz and S. Singh in [5] investigated the Bessel functions. The Hurwitz theorem on the exact number of nonreal zeros was extended for the Bessel functions of the first kind. In addition, the results on zeros of derivatives of Bessel functions and the cross-product of Bessel functions were obtained.

1.7 The entire functions with the decreasing second quotients of Taylor coefficients

In the paper [62] by T.H. Nguyen and A. Vishnyakova, sufficient conditions for some special entire functions to have only real zeros were found.

Theorem L (T.H. Nguyen, A. Vishnyakova, [62]). *Let $f(z)=\sum_{k=0}^{\infty}a_kz^k$,$a_k>0$ for all k, be an entire function. Suppose that the second quotients of Taylor coefficients $q_n(f)$ are decreasing in n, i.e. $q_2(f)\geq q_3(f)\geq q_4(f)\geq\ldots$, and $\lim\limits_{n\to\infty}q_n(f)=b\geq q_\infty$. Then all the zeros of f are real and negative, in other words $f\in\mathcal{L}-\mathcal{P}$.*

It is easy to see that, if only the estimation of $q_n(f)$ from below is given and the assumption of monotonicity is omitted, then the constant 4 in $q_n(f) \geq 4$ is the smallest possible to conclude that $f \in \mathcal{L} - \mathcal{P}$.

The following function is a generalization of the classical Mittag-Leffler function and was studied by I.V. Ostrovskii and I. Peresyolkova in 1997 in [69]

$$E_\rho(z,\mu) = \sum_{k=0}^{\infty} \frac{z_k}{\Gamma(\mu + k/\rho)}, \rho > 0, \mu \in \mathbb{C}.$$

It plays a fundamental role in the theory of integral transforms and representations created by Dzhrbashyan and other topics of Analysis. Note that $q_k(E_\rho)$ are decreasing in k. I.V. Ostrovskii and I. Peresyolkova found non-asymptotic results on the distribution of zeros of $E_\rho(z,\mu)$. In particular, they obtained that this function belongs to the $\mathcal{L} - \mathcal{P}$ class for $\rho \in (0, 1/2]$.

1.8 The study of the entire functions with the increasing second quotients

The function $\varphi_a(z,m) = \sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}}(k!)^m$, $a > 1, m > 0$ has the increasing second quotients of Taylor coefficients. Indeed, we can observe that

$$q_k(\varphi_a) = \frac{((k-1)!)^{2m} a^{(k-2)^2} a^{k^2}}{a^{2(k-1)^2}((k-2)!)^m (k!)^m} = \left(\frac{k-1}{k}\right)^m a^2$$

is increasing in k. A. Bohdanov in [10] studied the function above and found the estimations on $a > 1$ and $m \in (0,1)$ for which $\varphi_a(z,m)$ and its Taylor sections belong to the Laguerre-Pólya class. In addition, for the case $m = 0$, it is the partial theta function which described in details in [47] and [67] by O. Katkova, T. Lobova and A. Vishnyakova. The case $m > 1$ was investigated in [11] by A. Bohdanov and A. Vishnyakova, where necessary and sufficient conditions were found for $\varphi_a(z,m)$ to belong to the Laguerre–Pólya class. The question is remained open for the case of negative m.

Chapter 2

A necessary condition for an entire function with the increasing second quotients of Taylor coefficients to belong to the Laguerre–Pólya class

In this chapter, we study the entire functions with the increasing second quotients of Taylor coefficients. For an entire function $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0$, we show that f does not belong to the Laguerre–Pólya class if the quotients $q_n(f) = \frac{a_{n-1}^2}{a_{n-2}a_n}$ are increasing in n, and $c := \lim\limits_{n\to\infty} \frac{a_{n-1}^2}{a_{n-2}a_n}$ is smaller than an absolute constant q_∞ ($q_\infty \approx 3.23363666$). In [62] by T.H. Nguyen and A. Vishnyakova, it was shown that if $q_n(f)$ are decreasing in n and they tend to a constant which is greater than or equal to the constant q_∞, then the function f has only real and negative zeros, or, in other words, f belongs to the Laguerre–Pólya class. As it was proved in [47] by O. Katkova, T. Lobova and A. Vishnyakova, if only the estimation of $q_n(f)$ from below is given, and the assumption of monotonicity is omitted, then the constant 4 in the sufficient condition $q_n(f) \geq 4$ is the smallest possible to conclude that $f \in \mathcal{L} - \mathcal{P}$. The main result of this chapter is the following theorem.

Theorem 2.1 (T.H. Nguyen, A. Vishnyakova, [60]). *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, where $a_k > 0$ for all k, be an entire function. Suppose that the quotients $q_n(f)$ are increasing in n, and $\lim\limits_{n\to\infty} q_n(f) = c < q_\infty$. Then the function f does not belong to the Laguerre–Pólya class.*

The theorem above provides the following necessary condition for an entire function with positive coefficients and with the increasing second quotients of Taylor coefficients to belong to the Laguerre–Pólya class.

Corollary 2.2 (T.H. Nguyen, A. Vishnyakova, [60])**.** *Let* $f(x) = \sum_{k=0}^{\infty} a_k x^k$*, where* $a_k > 0$ *for all* k*, be an entire function such that the quotients* $q_n(f)$ *are increasing in* n*. If* f *belongs to the Laguerre–Pólya class, then* $\lim_{n\to\infty} q_n(f) \geq q_\infty$*.*

Note that, by Theorem C (see Chapter 1), every entire function from the $\mathcal{L} - \mathcal{P}$ class generates a new multiplier sequence. So, we obtain the following direct corollary of Theorem 2.1.

Corollary 2.3. *Suppose that a real positive sequence* $(a_k)_{k=0}^{\infty}$ *has the following property: the sequence of its second quotients* $\left(\frac{a_{k-1}^2}{a_{k-2}a_k}\right)_{k=2}^{\infty}$ *is increasing in* k*. Then if* $(k!a_k)_{k=0}^{\infty} \in \mathcal{MS}$*, then* $\lim_{n\to\infty} \frac{a_{k-1}^2}{a_{k-2}a_k} \geq q_\infty$*.*

2.1 Proof of Theorem 2.1

Without loss of generality, we can assume that $a_0 = a_1 = 1$, since we can consider a function $g(x) = a_0^{-1} f(a_0 a_1^{-1} x)$ instead of f, due to the fact that such rescaling of f preserves its property of having real zeros and preserves the second quotients of Taylor coefficients: $q_n(g) = q_n(f)$ for all n. During the proof we use notation p_n and q_n instead of $p_n(f)$ and $q_n(f)$. So, we can write

$$f(x) = 1 + x + \sum_{k=2}^{\infty} \frac{x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}.$$

For convenience in solving inequalities, we further consider the function

$$\varphi(x) = f(-x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$$

instead of f.

Since the quotients q_n are increasing in n, and, under our assumptions, $\lim_{n\to\infty} q_n = c < q_\infty$, we conclude that $q_2 < q_\infty < 4$. The following lemma shows that for $q_2 < 3$, the functions φ does not belong to the class $\mathcal{L} - \mathcal{P}$.

Lemma 2.4 (T.H. Nguyen, A. Vishnyakova, [60])**.** *Let* $\varphi(x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$ *be an entire function, and* $q_k = q_k(\varphi)$ *are increasing in* k*, i.e.* $0 < q_2 \leq q_3 \leq q_4 \leq \ldots$ *If* $\varphi \in \mathcal{L} - \mathcal{P}$*, then* $q_2(\varphi) \geq 3$*.*

Proof. Suppose that $\varphi \in \mathcal{L} - \mathcal{P}$, and denote by $0 < x_1 \leq x_2 \leq x_3 \leq \ldots$ the real roots of φ. We use the Vieta's polynomials

$$\sigma_r = \sum_{1 \leq i_1 < i_2 < \cdots < i_r} x_{i_1}^{-1} x_{i_2}^{-1} \cdot \ldots \cdot x_{i_r}^{-1}, \ r \in \mathbb{N},$$

and the general Newton power sums

$$s_r = \sum_{i=1}^{\infty} x_i^{-r}, \ r \in \mathbb{N}.$$

Therefore, since $s_2 = \sigma_1^2 - 2\sigma_2$, we can observe that

$$0 \leq \sum_{k=1}^{\infty} \frac{1}{x_k^2} = \left(\sum_{k=1}^{\infty} \frac{1}{x_k} \right)^2 - 2 \sum_{1 \leq i < j < \infty} \frac{1}{x_i x_j} = \left(\frac{a_1}{a_0} \right)^2 - 2\frac{a_2}{a_0},$$

whence $a_1^2 \geq 2a_0 a_2$, or $q_2 \geq 2$. According to the Cauchy-Schwarz inequality, we obtain

$$(\frac{1}{x_1} + \frac{1}{x_2} + \ldots)(\frac{1}{x_1^3} + \frac{1}{x_2^3} + \ldots) \geq (\frac{1}{x_1^2} + \frac{1}{x_2^2} + \ldots)^2,$$

or $s_1 s_3 \geq s_2^2$. Since $\sigma_1 = -\frac{a_1}{a_0}, \sigma_2 = \frac{a_2}{a_0}, \sigma_3 = \frac{a_3}{a_0}$, and using $s_3 = \sigma_3^3 - 3\sigma_1\sigma_2 + 3\sigma_3$, we get

$$\sigma_1(\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3) \geq (\sigma_1^2 - 2\sigma_2)^2,$$

or

$$\frac{a_1^2 a_2}{a_0^3} + 3\frac{a_1 a_3}{a_0^2} - 4\frac{a_2^2}{a_0^2} \geq 0.$$

Since $a_0 = a_1 = 1$ and $a_2 = \frac{1}{q_2}, a_3 = \frac{1}{q_2^2 q_3}$, we obtain

$$q_3(q_2 - 4) + 3 \geq 0.$$

Under our assumptions, since $q_2 \leq q_3$, supposing $q_2 < 4$, we conclude that

$$q_2(q_2 - 4) + 3 \geq 0.$$

Therefore, $q_2 \geq 3$. □

Further, we assume that $3 \leq q_2 < q_\infty$. In order to prove Theorem 2.1, we need the following lemma.

Lemma 2.5 (T.H. Nguyen, A. Vishnyakova, [60]). *Let* $\varphi(x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$ *be an entire function. Suppose that* $q_2 \geq 2$, *and* $q_k = q_k(\varphi)$ *are increasing in* k, *i.e.* $q_2 \leq q_3 \leq q_4 \ldots$, $\lim_{n\to\infty} q_n = c < q_\infty$. *Then for any* $x \in [0, q_2]$ *we have* $\varphi(x) > 0$, *i.e. there are no real roots of* φ *in the segment* $[0, q_2]$.

Proof. For $x \in [0, 1]$ we have

$$1 \geq x > \frac{x^2}{q_2} > \frac{x^3}{q_2^2 q_3} > \frac{x^4}{q_2^3 q_3^2 q_4} > \frac{x^5}{q_2^4 q_3^3 q_4^2 q_5} > \cdots,$$

whence

$$\varphi(x) = (1 - x) + \left(\tfrac{x^2}{q_2} - \tfrac{x^3}{q_2^2 q_3}\right) + \left(\tfrac{x^4}{q_2^3 q_3^2 q_4} - \tfrac{x^5}{q_2^4 q_3^3 q_4^2 q_5}\right) + \ldots > 0 \quad \text{for all} \quad x \in [0, 1]. \tag{2.1}$$

Suppose that $x \in (1, q_2]$. Then we obtain

$$1 < x \geq \frac{x^2}{q_2} > \frac{x^3}{q_2^2 q_3} > \cdots > \frac{x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k} > \cdots \tag{2.2}$$

For an arbitrary $m \in \mathbb{N}$ we have

$$\varphi(x) = S_{2m+1}(x, \varphi) + R_{2m+2}(x, \varphi),$$

where

$$S_{2m+1}(x, \varphi) := 1 - x + \sum_{k=2}^{2m+1} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k},$$

and

$$R_{2m+2}(x, \varphi) := \sum_{k=2m+2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}.$$

By (2.2) and the Leibniz criterion for alternating series, we obtain

$$R_{2m+2}(x, \varphi) = \sum_{k=m+1}^{\infty} \left(\tfrac{x^{2k}}{q_2^{2k-1} q_3^{2k-2} \cdot \ldots \cdot q_{2k-1}^2 q_{2k}} - \tfrac{x^{2k+1}}{q_2^{2k} q_3^{2k-1} \cdot \ldots \cdot q_{2k}^2 q_{2k+1}}\right) > 0 \text{ for all } x \in (1, q_2], \tag{2.3}$$

or

$$\varphi(x) > S_{2m+1}(x, \varphi) \quad \text{for all} \quad x \in (1, q_2],\ m \in \mathbb{N}. \tag{2.4}$$

It remains to prove that there exists $m \in \mathbb{N}$ such that

$$S_{2m+1}(x,\varphi) > 0$$

for all $x \in (1, q_2]$. We have

$$\begin{aligned} S_{2m+1}(x,\varphi) = (1-x) + \left(\frac{x^2}{q_2} - \frac{x^3}{q_2^2 q_3}\right) + \left(\frac{x^4}{q_2^3 q_3^2 q_4} - \frac{x^5}{q_2^4 q_3^3 q_4^2 q_5}\right) \\ + \ldots + \left(\frac{x^{2m}}{q_2^{2m-1} q_3^{2m-2} \cdot \ldots \cdot q_{2m-1}^2 q_{2m}} \right. \\ \left. - \frac{x^{2m+1}}{q_2^{2m} q_3^{2m-1} \cdot \ldots \cdot q_{2m}^2 q_{2m+1}}\right). \end{aligned} \tag{2.5}$$

Under our assumptions, q_k are increasing in k, and $\lim_{n\to\infty} q_n = c < q_\infty$. We prove that for any $m \in \mathbb{N}$, any fixed $k = 1, 2, \ldots, m$, and any $x \in (1, q_2]$ the following inequality holds

$$\begin{aligned} \frac{x^{2k}}{q_2^{2k-1} q_3^{2k-2} \cdot \ldots \cdot q_{2k}} - \frac{x^{2k+1}}{q_2^{2k} q_3^{2k-1} \cdot \ldots \cdot q_{2k}^2 q_{2k+1}} \\ \geq \frac{x^{2k}}{c^{2k-1} \cdot c^{2k-2} \cdot \ldots \cdot c} - \frac{x^{2k+1}}{c^{2k} \cdot c^{2k-1} \cdot \ldots \cdot c^2 \cdot c}. \end{aligned}$$

For $x \in (1, q_2]$ and $k = 1, 2, \ldots, m$, we define the following function

$$F(q_2, q_3, \ldots, q_{2k}, q_{2k+1}) := \frac{x^{2k}}{q_2^{2k-1} q_3^{2k-2} \cdot \ldots \cdot q_{2k}} - \frac{x^{2k+1}}{q_2^{2k} q_3^{2k-1} \cdot \ldots \cdot q_{2k}^2 q_{2k+1}}.$$

We observe that

$$\begin{aligned} \frac{\partial F(q_2, q_3, \ldots, q_{2k}, q_{2k+1})}{\partial q_2} = -\frac{(2k-1)x^{2k}}{q_2^{2k} q_3^{2k-2} \cdot \ldots \cdot q_{2k}} \\ + \frac{2k x^{2k+1}}{q_2^{2k+1} q_3^{2k-1} \cdot \ldots \cdot q_{2k}^2 q_{2k+1}} < 0 \Leftrightarrow x < \left(1 - \frac{1}{2k}\right) q_2 q_3 \cdot \ldots \cdot q_{2k} q_{2k+1}. \end{aligned}$$

The right-hand side of the last inequality is strictly increasing in k, so the inequalities for all $k = 1, 2, \ldots, m$ are valid if the inequality for $k = 1$ is valid, or $x < \left(1 - \frac{1}{2}\right) q_2 q_3 = \frac{1}{2} q_2 q_3$. Thus, under our assumptions that $q_2 \geq 3$ and $x \leq q_2$, the function $F(q_2, q_3, \ldots, q_{2k}, q_{2k+1})$ is decreasing in q_2. Since $q_2 \leq q_3$, we have

$$F(q_2, q_3, q_4, \ldots, q_{2k}, q_{2k+1}) \geq F(q_3, q_3, q_4, \ldots, q_{2k}, q_{2k+1}) =$$

$$\frac{x^{2k}}{q_3^{4k-3} q_4^{2k-3} \cdot \ldots \cdot q_{2k}} - \frac{x^{2k+1}}{q_3^{4k-1} q_4^{2k-2} \cdot \ldots \cdot q_{2k+1}}.$$

In particular, if $k = 1$ we obtain

$$F(q_2, q_3) = \frac{x^2}{q_2} - \frac{x^3}{q_2^2 q_3} \geq \frac{x^2}{q_3} - \frac{x^3}{q_3^3}. \tag{2.6}$$

Further, if $k \geq 2$, we have

$$\frac{\partial F(q_3,q_3,q_4,\ldots,q_{2k},q_{2k+1})}{\partial q_3} = -\frac{(4k-3)x^{2k}}{q_3^{4k-2}q_4^{2k-3}\cdot\ldots\cdot q_{2k}} + \frac{(4k-1)x^{2k+1}}{q_3^{4k}q_4^{2k-2}\cdot\ldots\cdot q_{2k+1}} < 0$$
$$\Leftrightarrow x < \tfrac{4k-3}{4k-1} q_3^2 q_4 \cdot \ldots \cdot q_{2k+1}.$$

The right-hand side of the last inequality is strictly increasing in k, so the inequalities for all $k = 2, 3, \ldots, m$ are valid if the inequality for $k = 2$ is valid, or $x < \frac{5}{7} q_3^2 q_4 q_5$. Hence, under our assumptions, $F(q_3, q_3, q_4, \ldots, q_{2k}, q_{2k+1})$ is decreasing in q_3, and since $q_3 \leq q_4$, we obtain

$$F(q_3, q_3, q_4, q_5, \ldots, q_{2k}, q_{2k+1}) \geq F(q_4, q_4, q_4, q_5, \ldots, q_{2k}, q_{2k+1}).$$

Analogously, since

$$F(q_{2k}, q_{2k}, \ldots, q_{2k}, q_{2k+1}) = \frac{x^{2k}}{q_{2k}^{2k^2-k}} - \frac{x^{2k+1}}{q_{2k}^{2k^2+k-1} q_{2k+1}},$$

its partial derivative with respect to q_{2k} is

$$\frac{\partial F(q_{2k}, q_{2k}, \ldots, q_{2k}, q_{2k+1})}{\partial q_{2k}} = -\frac{(2k^2-k)x^{2k}}{q_{2k}^{2k^2-k+1}} + \frac{(2k^2+k-1)x^{2k+1}}{q_{2k}^{2k^2+k} q_{2k+1}} < 0$$
$$\Leftrightarrow x < \left(\frac{2k^2-k}{2k^2+k-1}\right) q_{2k}^{2k-1} q_{2k+1},$$

or, equivalently,

$$x < \left(1 - \frac{2k-1}{2k^2+k-1}\right) q_{2k}^{2k-1} q_{2k+1}.$$

Therefore, under our assumptions, $F(q_{2k}, q_{2k}, \ldots, q_{2k}, q_{2k+1})$ is decreasing in q_{2k}, and since $q_{2k} \leq q_{2k+1}$, we get

$$F(q_{2k}, q_{2k}, \ldots, q_{2k}, q_{2k+1}) \geq F(q_{2k+1}, q_{2k+1}, \ldots, q_{2k+1}, q_{2k+1}),$$

where

$$F(q_{2k+1}, q_{2k+1}, \ldots, q_{2k+1}, q_{2k+1}) = \frac{x^{2k}}{q_{2k+1}^{2k^2-k}} - \frac{x^{2k+1}}{q_{2k+1}^{2k^2+k}}.$$

Consequently, we obtain the following chain of inequalities

$$F(q_2, q_3, q_4, \dots, q_{2k}, q_{2k+1}) \geq F(q_3, q_3, q_4, \dots, q_{2k}, q_{2k+1})$$
$$\geq F(q_4, q_4, q_4, q_5, \dots, q_{2k}, q_{2k+1}) \geq \dots \geq F(q_{2k+1}, q_{2k+1}, \dots, q_{2k+1}, q_{2k+1}).$$

Next, we consider the partial derivative of $F(q_{2k+1}, q_{2k+1}, \dots, q_{2k+1}, q_{2k+1})$

$$\frac{\partial F(q_{2k+1}, q_{2k+1}, \dots, q_{2k+1}, q_{2k+1})}{\partial q_{2k+1}} = -\frac{(2k^2-k)x^{2k}}{q_{2k+1}^{2k^2-k+1}} + \frac{(2k^2+k)x^{2k+1}}{q_{2k+1}^{2k^2+k+1}} < 0$$
$$\Leftrightarrow x < \frac{2k^2-k}{2k^2+k} q_{2k+1}^{2k},$$

or, equivalently,

$$x < \left(1 - \frac{2k}{2k^2+k}\right) q_{2k+1}^{2k}.$$

Thus, $F(q_{2k+1}, q_{2k+1}, \dots, q_{2k+1}, q_{2k+1})$ is decreasing in q_{2k+1}. Besides, since q_k are increasing in k, and $\lim\limits_{n\to\infty} q_n = c$, we conclude that

$$F(q_{2k+1}, q_{2k+1}, \dots, q_{2k+1}, q_{2k+1}) \geq F(c, c, \dots, c, c)$$
$$= \frac{x^{2k}}{c^{k(2k-1)}} - \frac{x^{2k+1}}{c^{k(2k+1)}}.$$

Substituting the last inequality in (2.5) for every $x \in (1, q_2]$ and $k = 1, 2, \dots, m$, we get

$$S_{2m+1}(x,\varphi) \geq (1-x) + \left(\tfrac{x^2}{c} - \tfrac{x^3}{c^3}\right) + \left(\tfrac{x^4}{c^6} - \tfrac{x^5}{c^{10}}\right) + \dots \qquad (2.7)$$
$$+ \left(\tfrac{x^{2m}}{c^{m(2m-1)}} - \tfrac{x^{2m+1}}{c^{m(2m+1)}}\right) = \sum_{k=0}^{2m+1} \tfrac{(-1)^k x^k}{(\sqrt{c})^{k(k-1)}} = S_{2m+1}(-\sqrt{c}x, g_{\sqrt{c}}),$$

where g_a is the partial theta function and $S_{2m+1}(y, g_a)$ is its $(2m+1)$-th partial sum at the point y. Since, by our assumptions, $q_k(g_{\sqrt{c}}) = (\sqrt{c})^2 = c < q_\infty$, using the statement (7) of Theorem I that $3 = c_3 < c_5 < c_7 < \cdots$ and $\lim_{n\to\infty} c_{2n+1} = q_\infty$ (see Chapter 1), we obtain that there exists $m \in \mathbb{N}$ such that $S_{2m+1}(y, g_{\sqrt{c}}) \notin \mathcal{L}-\mathcal{P}$ (or, equivalently, $S_{2m+1}(-\sqrt{c}x, g_{\sqrt{c}}) \notin \mathcal{L}-\mathcal{P}$). Let us choose and fix such m. By the statement (5) of Theorem I (it states that for a given $n \geq 2, S_n(x, g_a) \in \mathcal{L}-\mathcal{P}$ if and only if there exists $x_n \in (-a^3, -a)$ such that $S_n(x_n, q_a) \leq 0$), we obtain that for every x such that $\sqrt{c} < \sqrt{c}x < (\sqrt{c})^3$, we have $S_{2m+1}(-\sqrt{c}x, g_{\sqrt{c}}) > 0$. It means that for every $x : 1 < x < c$ we have $S_{2m+1}(\sqrt{c}x, g_{\sqrt{c}}) > 0$, and, using (2.7) and (2.4), we obtain

$$\varphi(x) > S_{2m+1}(x,\varphi) > 0 \quad \text{for all} \quad x \in (1, q_2) \subset (1, c).$$

It remains to prove that $\varphi(q_2) > 0$. We have

$$\varphi(q_2) = \left(1 - q_2 + q_2 - \frac{q_2}{q_3}\right) + \left(\frac{q_2}{q_3^2 q_4} - \frac{q_2}{q_3^3 q_4^2 q_5}\right) + \left(\frac{q_2}{q_3^4 q_4^3 q_5^2 q_6} - \frac{q_2}{q_3^5 q_4^4 q_5^3 q_6^2 q_7}\right) + \ldots > 0,$$

by our assumptions on q_j. □

The lemma below has a key value in our research. We further provide its generalization in Chapter 3. As we mentioned before, the function φ can be presented in the following form

$$\varphi(x) = S_4(x, \varphi) + R_5(x, \varphi),$$

where

$$S_4(x, \varphi) = 1 - x + \frac{x^2}{q_2} - \frac{x^3}{q_2^2 q_3} + \frac{x^4}{q_2^3 q_3^2 q_4},$$

and

$$R_5(x, \varphi) := \sum_{k=5}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k}.$$

In the following lemma, we estimate the 4-th partial sum of φ from below and set $a := q_2, b := q_3, c := q_4$.

Lemma 2.6 (T.H. Nguyen, A. Vishnyakova, [60]). *Let $P(x) = 1 - x + \frac{x^2}{a} - \frac{x^3}{a^2 b} + \frac{x^4}{a^3 b^2 c}$ be a polynomial, $3 \le a < 4$, and $a \le b \le c$. Then*

$$\min_{0 \le \theta \le 2\pi} |P(ae^{i\theta})| \ge \frac{a}{b^2 c}.$$

Proof. By direct calculation, we have

$$\begin{aligned} |P(ae^{i\theta})|^2 &= (1 - a\cos\theta + a\cos 2\theta - \frac{a}{b}\cos 3\theta + \frac{a}{b^2 c}\cos 4\theta)^2 \\ &\quad + (-a\sin\theta + a\sin 2\theta - \frac{a}{b}\sin 3\theta + \frac{a}{b^2 c}\sin 4\theta)^2 \\ &= 1 + 2a^2 + \frac{a^2}{b^2} + \frac{a^2}{b^4 c^2} - 2a\cos\theta + 2a\cos 2\theta - 2\frac{a}{b}\cos 3\theta \\ &\quad + 2\frac{a}{b^2 c}\cos 4\theta - 2a^2\cos\theta + 2\frac{a^2}{b}\cos 2\theta - 2\frac{a^2}{b^2 c}\cos 3\theta \\ &\quad - 2\frac{a^2}{b}\cos\theta + 2\frac{a^2}{b^2 c}\cos 2\theta - 2\frac{a^2}{b^3 c}\cos\theta. \end{aligned}$$

Next, set $t := \cos\theta, t \in [-1,1]$. Applying that $\cos 2\theta = 2t^2 - 1$, $\cos 3\theta = 4t^3 - 3t$, and $\cos 4\theta = 8t^4 - 8t^2 + 1$, we get

$$\begin{aligned}|P(ae^{i\theta})|^2 = \frac{16a}{b^2c}t^4 &+ \left(-\frac{8a}{b} - \frac{8a^2}{b^2c}\right)t^3 + \left(4a - \frac{16a}{b^2c} + \frac{4a^2}{b} + \frac{4a^2}{b^2c}\right)t^2 \\ &+ \left(-2a + \frac{6a}{b} - 2a^2 + \frac{6a^2}{b^2c} - \frac{2a^2}{b} - \frac{2a^2}{b^3c}\right)t \\ &+ \left(1 + 2a^2 + \frac{a^2}{b^2} + \frac{a^2}{b^4c^2} - 2a + \frac{2a}{b^2c} - \frac{2a^2}{b} - \frac{2a^2}{b^2c}\right).\end{aligned}$$

In the following step, we want to show that $\min_{0\leq\theta\leq2\pi}|P(ae^{i\theta})|^2 \geq \frac{a^2}{b^4c^2}$, or, equivalently, to prove the inequality $\min_{0\leq\theta\leq2\pi}|P(ae^{i\theta})|^2 - \frac{a^2}{b^4c^2} \geq 0$. Using the last expression, we see that the inequality we want to get is equivalent to the following statement: for all $t \in [-1,1]$ the inequality below holds

$$\frac{16a}{b^2c}t^4 - \frac{8a}{b}\Big(1 + \frac{a}{bc}\Big)t^3 + 4a\Big(1 - \frac{4}{b^2c} + \frac{a}{b} + \frac{a}{b^2c}\Big)t^2 - 2a\Big(1 - \frac{3}{b} + a - \frac{3a}{b^2c} +$$
$$\frac{a}{b} + \frac{a}{b^3c}\Big)t + \Big(1 + 2a^2 + \frac{a^2}{b^2} - 2a + \frac{2a}{b^2c} - \frac{2a^2}{b} - \frac{2a^2}{b^2c}\Big) \geq 0.$$

Set $y := 2t$, where $y \in [-2,2]$. We rewrite the last inequality as the following

$$\begin{aligned}&\frac{a}{b^2c}y^4 - \frac{a}{b}\left(1 + \frac{a}{bc}\right)y^3 + a\left(1 - \frac{4}{b^2c} + \frac{a}{b} + \frac{a}{b^2c}\right)y^2 \\ &- a\left(1 - \frac{3}{b} + a - \frac{3a}{b^2c} + \frac{a}{b} + \frac{a}{b^3c}\right)y \\ &+ \left(1 + 2a^2 + \frac{a^2}{b^2} - 2a + \frac{2a}{b^2c} - \frac{2a^2}{b} - \frac{2a^2}{b^2c}\right) \geq 0.\end{aligned}$$

Let us consider the coefficients of the polynomial on the left hand side: the coefficient of y^4 is $\frac{a}{b^2c} > 0$, and the coefficient of y^3 is $-\frac{a}{b}\left(1 + \frac{a}{bc}\right) < 0$. It is easy to show that the other coefficients are also sign-changing. For y^2, since a, b and c are positive, it follows that

$$b^2c > 4.$$

Then we have

$$1 - \frac{4}{b^2c} > 0.$$

Thus, the coefficient of y^2 is

$$1 + \frac{a}{b} + \frac{a}{b^2c} - \frac{4}{b^2c} = \left(1 - \frac{4}{b^2c}\right) + \frac{a}{b} + \frac{a}{b^2c} > 0.$$

As for the coefficients of y, we have

$$1 + a - \frac{3}{b} > 0 \Leftrightarrow ab + b > 3,$$

and

$$\frac{a}{b} - \frac{3a}{b^2c} > 0 \Leftrightarrow \frac{a}{b} > \frac{3a}{b^2c} \Leftrightarrow abc > 3a,$$

since $3 \leq a \leq b \leq c$. Therefore, it follows from the inequalities above that

$$\begin{aligned} 1 + a + \frac{a}{b} + \frac{a}{b^3c} - \frac{3}{b} - \frac{3a}{b^2c} &= \left(1 + a - \frac{3}{b}\right) \\ &+ \left(\frac{a}{b} - \frac{3a}{b^2c}\right) + \frac{a}{b^3c} > 0, \end{aligned}$$

Finally, we observe that

$$1 - 2a + a^2 = (a+1)^2 \geq 0,$$

which holds for any a, and

$$a^2 - 2\frac{a^2}{b} > 0 \Leftrightarrow a^2b > 2a^2 \Leftrightarrow b > 2,$$

which is true under our assumptions that $3 \leq a \leq b$.

Also,

$$\frac{a^2}{b^2} - 2\frac{a^2}{b^2c} > 0,$$

follows from

$$\frac{a^2}{b^2} > 2\frac{a^2}{b^2c} \Leftrightarrow a^2c > 2a^2 \Leftrightarrow c > 2,$$

which is true under our assumptions that $3 \leq a \leq b \leq c$. Therefore, we have

$$\begin{aligned} &1 + 2a^2 + \frac{a^2}{b^2} - 2a - 2\frac{a^2}{b} - 2\frac{a^2}{b^2c} + 2\frac{a}{b^2c} \\ &= (1 + a^2 - 2a) + (a^2 - 2\frac{a^2}{b}) + (\frac{a^2}{b^2} - 2\frac{a^2}{b^2c}) + 2\frac{a}{b^2c} > 0. \end{aligned}$$

Consequently, the inequality we need holds for any $y \in [-2, 0]$, so it remains to prove it for $y \in [0, 2]$. Multiplying our inequality by $\frac{b^2c}{a}$, we get

$$y^4 - (bc + a)y^3 + (b^2c + abc + a - 4)y^2 - (b^2c + ab^2c + abc + \frac{a}{b} - 3bc - 3a)y$$

$$+(\frac{b^2c}{a} + 2ab^2c + ac - 2b^2c - 2abc - 2a + 2) =: \psi(y),$$

and we want to prove that $\psi(y) \geq 0$ for all $y \in [0, 2]$.
Let $\chi(y) := \psi(y) - \frac{1}{b}(b-a)y$, then

$$\chi(y) = y^4 - (bc+a)y^3 + (b^2c + abc + a - 4)y^2$$
$$-(b^2c + ab^2c + abc + \frac{a}{b} - 3bc - 3a)y$$
$$+\left(\frac{b^2c}{a} + 2ab^2c + ac - 2b^2c - 2abc - 2a + 2\right) - \frac{1}{b}(b-a)y$$
$$= y^4 - (bc+a)y^3 + (b^2c + abc + a - 4)y^2$$
$$-\left(b^2c + ab^2c + abc + \frac{a}{b} - 3bc - 3a + \frac{1}{b}(b-a)\right)y$$
$$+\left(\frac{b^2c}{a} + 2ab^2c + ac - 2b^2c - 2abc - 2a + 2\right)$$
$$= y^4 - (bc+a)y^3 + (b^2c + abc + a - 4)y^2$$
$$-(b^2c + ab^2c + abc - 3bc - 3a + 1)y$$
$$+\left(\frac{b^2c}{a} + 2ab^2c + ac - 2b^2c - 2abc - 2a + 2\right).$$

Since, under our assumptions, $a \leq b$ and $y \in [0, 2]$, we have $\chi(y) \leq \psi(y)$ for all $y \in [0, 2]$. Therefore, it is sufficient to prove that $\chi(y) \geq 0$ for all $y \in [0, 2]$. To begin with, we have

$$\chi(0) = \psi(0) = \frac{b^2c}{a} + 2ab^2c + ac - 2b^2c - 2abc - 2a + 2 \geq 0,$$

as it was previously shown. Next, we observe that $\chi(2) = \psi(2) - \frac{2}{c}(b-a) \geq 0$, since

$$\psi(2) = -2bc - 2\frac{a}{b} + \frac{b^2c}{a} + ac + 2$$
$$= \frac{1}{b}\left(2(b-a) + \frac{b^2c}{a}(b-a) - bc(b-a)\right)$$
$$= \frac{1}{b}(b-a)\left(2 + \frac{bc}{a}(b-a)\right) \geq \frac{2}{b}(b-a) \geq 0.$$

Now we consider the following function

$$\nu(y) := \frac{\partial^2\chi(y)}{\partial y^2} = \frac{\partial^2\psi(y)}{\partial y^2} = 12y^2 - 6(bc+a)y + 2(b^2c + abc + a - 4).$$

The vertex point of this parabola is

$$y_v = \frac{bc+a}{4} \geq 3.$$

Accordingly, we can observe that $\nu(y)$ decreases for $y \in [0,2]$. We have

$$\nu(0) = 2(b^2c + abc + a - 4) > 0,$$

and

$$\nu(2) = 2abc + 2b^2c - 12bc - 10a + 40.$$

We want to show that $\nu(2)$ is positive, which follows from

$$abc + b^2c - 6bc - 5a + 20 = (20 - 5a) + (b^2c - 3bc) + (abc - 3bc)$$

$$= 5(4-a) + bc(c-3) + bc(a-3) > 0,$$

due to our assumptions. We conclude that $\nu(y)$ is nonnegative for $y \in [0,2]$, and it follows that $\chi'(y)$ increases for $y \in [0,2]$.

Next, we prove that $\chi'(y) \leq 0$ for $y \in [0,2]$, and it is sufficient to show that $\chi'(2) \leq 0$. Under our assumptions that $3 \leq a \leq b \leq c$, it follows

$$\chi'(2) = \psi'(2) - \frac{b-a}{b} = 15 - 9bc - 5a + 3b^2c + 3abc - ab^2c$$

$$= 5(3-a) + bc(-9 + 3b + 3a - ab) = 5(3-a) + bc(a-3)(3-b) \leq 0,$$

since $3 - a < 0$, $3 - b < 0$. Thus, $\chi(y)$ decreases in y and $\chi(2) \geq 0$, we can conclude that it is positive for $y \in [0,2]$. Since $\chi(y) \leq \psi(y)$, it follows that $\psi(y)$ is positive for $y \in [0,2]$. □

By Lemma 2.6, since $a = q_2, b = q_3$, and $c = q_4$, we have

$$\min_{0 \leq \theta \leq 2\pi} |S_4(q_2e^{i\theta}, \varphi)| \geq \frac{q_2}{q_3^2 q_4}. \tag{2.8}$$

Now we need the estimation on $|R_5(q_2e^{i\theta}, \varphi)|$ from above. The following lemma is technical.

Lemma 2.7 (T.H. Nguyen, A. Vishnyakova, [60]). *Let q_n be a sequence increasing in n with $\lim\limits_{n\to\infty} q_n(f) = c < q_\infty$, and let $R_5(x,\varphi) := \sum_{k=5}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k}$. Then*

$$\max_{0 \leq \theta \leq 2\pi} |R_5(q_2e^{i\theta}, \varphi)| \leq \frac{q_2}{q_3^3 q_4^3 - q_3^2}.$$

Proof. We have

$$\begin{aligned}|R_5(q_2e^{i\theta},\varphi)| &\leq \sum_{k=5}^{\infty}\frac{q_2^k}{q_2^{k-1}q_3^{k-2}\cdot\ldots\cdot q_k} = \sum_{k=5}^{\infty}\frac{q_2}{q_3^{k-2}\cdot\ldots\cdot q_k}\\
&= \frac{q_2}{q_3^3q_4^2q_5} + \frac{q_2}{q_3^4q_4^3q_5^2q_6} + \frac{q_2}{q_3^5q_4^4q_5^3q_6^2q_7} + \ldots + \frac{q_2}{q_3^{k-2}q_4^{k-3}q_5^{k-4}\cdot\ldots\cdot q_k} + \cdots\\
&= \frac{q_2}{q_3^3q_4^2q_5}\Big(1 + \frac{1}{q_3q_4q_5q_6} + \frac{1}{q_3^2q_4^2q_5^2q_6^2q_7} + \cdots\\
&\quad + \frac{1}{q_3^{k-5}q_4^{k-5}q_5^{k-5}q_6^{k-5}q_7^{k-6}q_8^{k-7}\cdot\ldots\cdot q_k} + \ldots\Big)\\
&\leq \frac{q_2}{q_3^3q_4^2q_5}\Big(1 + \frac{1}{q_3q_4q_5q_6} + \frac{1}{q_3^2q_4^2q_5^2q_6^2} + \cdots + \frac{1}{q_3^{k-5}q_4^{k-5}q_5^{k-5}q_6^{k-5}} + \ldots\Big)\\
&\leq \frac{q_2}{q_3^3q_4^3}\cdot\frac{1}{1-\frac{1}{q_3q_4q_5q_6}} \leq \frac{q_2}{q_3^3q_4^3}\cdot\frac{1}{1-\frac{1}{q_3q_4^3}} = \frac{q_2}{q_3^3q_4^3 - q_3^2},\end{aligned}$$

where we use the fact that $q_2 \leq q_3 \leq q_4 \leq \ldots$. □

In the next step of our proof, we check that

$$\min_{|x|=q_2} S_4(x,\varphi) > \max_{|x|=q_2} R_5(x,\varphi).$$

It follows from Lemma 2.6 and Lemma 2.7, that it is sufficient to prove that

$$\frac{q_2}{q_3^2q_4} > \frac{q_2}{q_3^3q_4^3 - q_3^2},$$

which is equivalent to

$$q_3q_4^3 - 1 > q_4.$$

The last inequality obviously holds under our assumptions. Therefore, according to Rouché's theorem, the functions $S_4(x,\varphi)$ and $\varphi(x)$ have the same number of zeros inside the disk $\{x : |x| < q_2\}$ counting multiplicities.

It remains to prove that $S_4(x,\varphi)$ has zeros in the disk $\{x : |x| < q_2\}$.

We present the following lemma.

Lemma 2.8 (T.H. Nguyen, A. Vishnyakova, [60])**.** *Let* $S_4(z,\varphi) = 1 - z + \frac{z^2}{q_2} - \frac{z^3}{q_2^2q_3} + \frac{z^4}{q_2^3q_3^2q_4}$ *be a real polynomial and* $3 \leq q_2 \leq 4$*. Then* $S_4(z,\varphi)$ *has at least one root in the disk* $\{z \in \mathbb{C} : |z| \leq q_2\}$.

Proof. Firstly, we rewrite S_4 in the form

$$S_4(z,\varphi) = \binom{4}{0} + \binom{4}{1}(-\frac{z}{4}) + \binom{4}{2}\frac{z^2}{6q_2} + \binom{4}{3}(-\frac{z^3}{4q_2^2q_3}) + \binom{4}{4}\frac{z^4}{q_2^3q_3^2q_4}.$$

Let

$$Q(z) = \binom{4}{2} b_2 z^2 + \binom{4}{3} b_3 z^3 + \binom{4}{4} z^4$$

be a complex polynomial. Then the condition for $S_4(z, \varphi)$ and $Q(z)$ to be apolar is the following

$$\binom{4}{0} - \binom{4}{1}\left(-\frac{1}{4}\right) b_3 + \binom{4}{2}\frac{1}{6q_2} b_2 = 0,$$

or, equivalently,

$$1 + b_3 + \frac{b_2}{q_2} = 0.$$

Further, we choose

$$b_3 = \frac{q_2 - 6}{2},$$

and, by the condition of apolarity,

$$b_2 = -q_2(1 + \frac{q_2 - 6}{2}).$$

Therefore, we have

$$\begin{aligned} Q(z) &= -6q_2\left(1 + \frac{q_2 - 6}{2}\right) z^2 + 4\left(\frac{q_2 - 6}{2}\right) z^3 + z^4 \\ &= z^2\left(-3q_2(q_2 - 4) + 2(q_2 - 6)z + z^2\right). \end{aligned}$$

As we can see, the zeros of Q are

$$z_1 = 0, \quad z_2 = 0, \quad z_3 = q_2, \quad z_4 = -3(q_2 - 4).$$

To show that z_4 lies in the closed disk centered in the origin and of radius q_2, we solve the inequality

$$| - 3(q_2 - 4)| \le q_2 \Leftrightarrow -q_2 \le 3(q_2 - 4) \le q_2 \Leftrightarrow 3 \le q_2 \le 6.$$

Hence, by our assumptions $3 \le q_2 \le 4$, all the zeros of Q are in the disk $\{z : |z| \le q_2\}$. Since all the zeros of Q are in the disk $\{z : |z| \le q_2\}$, by Grace's Apolarity theorem (see Theorem G in Chapter 1), we obtain that $S_4(z, \varphi)$ has at least one zero in the disk $\{z : |z| \le q_2\}$. $\square$

Thus, $S_4(z,\varphi)$ has at least one zero in the disk $\{z : |z| \le q_2\}$, and, by Lemma 2.6 applying to $S_4(z,\varphi)$, we obtain that $S_4(z,\varphi)$ does not have zeros on $\{z : |z| = q_2\}$. So, the polynomial $S_4(z,\varphi)$ has at least one zero in the open disk $\{z : |z| < q_2\}$. By Rouché's theorem, the functions $S_4(z,\varphi)$ and $\varphi(z)$ have the same number of zeros inside the disk $\{z : |z| < q_2\}$, whence φ has at least one zero in the open disk $\{z : |z| < q_2\}$. If φ is in the Laguerre–Pólya class, this zero must be real, and, since coefficients of φ are sign-changing, this zero belongs to the real interval $(0, q_2)$. However, by Lemma 2.5 we have $\varphi(x) > 0$ for all $x \in [0, q_2]$. This contradiction leads to the fact that $\varphi \notin \mathcal{L} - \mathcal{P}$.

Theorem 2.1 is proved.

2.2 On the conditions for a special entire function related to the partial theta-function and the q-Kummer functions to belong to the Laguerre–Pólya class

This chapter is based on [59]. We discuss the conditions for the function $F_a(x) = \sum_{k=0}^{\infty} \frac{x^k}{(a+1)(a^2+1)\cdot\ldots\cdot(a^k+1)}$, $a > 1$, to belong to the Laguerre–Pólya class, or, equivalently, to have only real zeros.

To begin with, the following function is known as the *second q-exponential function $E_q(x)$ with $q = 1/a$* (see [32], formula (1.3.6.)). We write $h_a(x)$ instead of $E_{1/a}(x)$.

$$h_a(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{(a^k-1)(a^{k-1}-1)\cdot\ldots\cdot(a-1)} = \prod_{k=1}^{\infty}\left(1 + \frac{x}{a^k}\right),\ a > 1,$$

Remark 2.9. Note that the function h_a has only real negative zeros, namely, $-a, -a^2, -a^3$, etc., since

$$\prod_{k=1}^{\infty}\left(1 + \frac{x}{a^k}\right) = 0 \Leftrightarrow x = -a^k.$$

We study the following companion of $h_a(x)$, which is known as the q-Kummer function ${}_1\phi_1(q; -q; q, -x)$, where $q = 1/a$ (see [32], formula (1.2.22)). Further, we use the notation $F_a(x)$, where

$$F_a(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{(a^k+1)(a^{k-1}+1)\cdot\ldots\cdot(a+1)}, \tag{2.9}$$

and we address the question, for which $a > 1$ this function belongs to the Laguerre–Pólya class. This problem was posed in the problem list of the workshop "Stability, hyperbolicity, and zero localization of functions" (American Institute of Mathematics, Palo Alto, California, 2011, see [84, Problem 8.2]).

Note that we have

$$q_n(F_a) = \frac{\Big((a^{n-1}+1)(a^{n-1}+1)\cdot\ldots\cdot(a+1)\Big)^2}{(a^{n-2}+1)(a^{n-3}+1)\cdot\ldots\cdot(a+1)} \times \frac{1}{(a^n+1)(a^{n-1}+1)\cdot\ldots\cdot(a+1)} = \frac{a^n+1}{a^{n-1}+1},$$

which is an increasing sequence in n for $a > 1$, with the limit value given by a.

The following two theorems are the main results concerning the function F_a.

Theorem 2.10 (T.H. Nguyen, [59])**.** *The entire function $F_a, a > 1$, belongs to the Laguerre–Pólya class if and only if there exists $x_0 \in (-(a^2+1), -(a+1))$ such that $F_a(x_0) \leq 0$.*

The following result estimates the corresponding values of the parameter a.

Theorem 2.11 (T.H. Nguyen, [59])**.**

(i) If $F_a, a > 1$, belongs to the Laguerre–Pólya class, then $a \geq 3.90155$;

(ii) If $a \geq 3.91719$, then F_a belongs to the Laguerre–Pólya class.

The question, for which $a > 1$ the entire function F_a belongs to the Laguerre–Pólya class, has attracted our interest. Unfortunately, up to now we can not prove the following statement, and we leave it as an open problem.

Conjecture 2.12 (T.H. Nguyen, [59])**.** *There exist a constant*

$$a_0 \in [3.90155, 3.91719]$$

such that the function F_a belongs to the Laguerre–Pólya class if and only if $a \geq a_0$.

As we have mentioned earlier, by Theorem C and Theorem D (see Chapter 1), every new entire function from the $\mathcal{L}-\mathcal{P}$ class generates a new multiplier sequence and a new complex zero decreasing sequence. So, we obtain the following direct corollary of Theorem 2.11.

Corollary 2.13. *For $a \geq 3.91719$, we have*

$$\left(\frac{k!}{(a+1)(a^2+1)\cdot\ldots\cdot(a^k+1)}\right)_{k=0}^{\infty} \in \mathcal{MS}$$

and

$$(F_a(k))_{k=0}^{\infty} \in \mathcal{CZDS}.$$

If

$$\left(\frac{k!}{(a+1)(a^2+1)\cdot\ldots\cdot(a^k+1)}\right)_{k=0}^{\infty} \in \mathcal{MS},$$

then $a \geq 3.90155$.

2.3 Proof of Theorem 2.10

Previously, in Lemma 2.4 from Chapter 2 (also, see [60, Lemma 2.1]), we have shown that for an entire function $f(x) = \sum_{k=0}^{\infty} a_k x^k$, if its second quotients $q_k(f)$ are increasing in k, and f belongs to the Laguerre–Pólya class, then $q_2(f) \geq 3$. Therefore, if $q_2(F_a) < 3$, we can conclude that $F_a \notin \mathcal{L}-\mathcal{P}$.

Let us consider the function $\widetilde{F}_a(x) := F_a((a+1)x)$, where

$$\widetilde{F}_a(x) := \sum_{k=0}^{\infty} \frac{(a+1)^k x^k}{(a^k+1)(a^{k-1}+1)\cdot\ldots\cdot(a+1)},$$

and note that

$$q_n(\widetilde{F}_a) = \frac{(a+1)^{2(n-1)}}{\Big((a^{n-1}+1)(a^{n-2}+1)\cdot\ldots\cdot(a+1)\Big)^2}$$

$$\times \frac{\Big((a^{n-2}+1)(a^{n-1}+1)\cdot\ldots\cdot(a+1)\Big)\cdot\Big((a^n+1)(a^{n-1}+1)\cdot\ldots\cdot(a+1)\Big)}{\Big((a+1)^{n-2}\Big)\cdot\Big((a+1)^n\Big)}$$

$$= \frac{(a+1)^{2n-1}}{(a+1)^{2n-2}}\times$$

$$\frac{\Big((a^{n-2}+1)(a^{n-1}+1)\cdot\ldots\cdot(a+1)\Big)\cdot\Big((a^n+1)(a^{n-1}+1)\cdot\ldots\cdot(a+1)\Big)}{\Big((a^{n-1}+1)(a^{n-2}+1)\cdot\ldots\cdot(a+1)\Big)^2}$$

$$= \frac{(a^{n-1}+1)(a^n+1)}{(a^{n-1}+1)^2} = \frac{a^n+1}{a^{n-1}+1} = q_n(F_a),$$

for all $n \in \mathbb{N}, n \geq 2$. Moreover, the statement that $F_a \in \mathcal{L} - \mathcal{P}$ is equivalent to the statement $\widetilde{F}_a = F_a(-(a+1)x) \in \mathcal{L} - \mathcal{P}$. Therefore, by Lemma 2.4, $q_2(\widetilde{F}_a) = q_2(F_a) \geq 3$. If $q_2(F_a) \geq 4$, then for any $j \geq 2$ we have $q_j(F_a) \geq 4$, thus, according to the Hutchinson's theorem (see Theorem B from Chapter 1), $F_a \in \mathcal{L} - \mathcal{P}$. Therefore, it remains to consider the case $q_2(F_a) \in [3, 4)$. Consequently, if $F_a(x) \in \mathcal{L} - \mathcal{P}$, and $q_2(F_a) = \frac{a^2+1}{a+1}$, then we have the following condition on a

$$3 \leq \frac{a^2+1}{a+1} < 4.$$

On the one hand,

$$\frac{a^2+1}{a+1} \geq 3 \Leftrightarrow a^2 + 1 \geq 3(a+1),$$

which is equivalent to the quadratic inequality

$$a^2 - 3a - 2 \geq 0.$$

Since we consider $a > 1$, the solution is

$$a \geq (3 + \sqrt{17})/2 \approx 3.56155281 \geq q_\infty.$$

Besides, note that $\lim_{n\to\infty} q_k(F_a) = a > q_\infty$. On the other hand, if $q_2(F_a) < 4$, then we have

$$\frac{a^2+1}{a+1} < 4 \Leftrightarrow a^2 + 1 < 4(a+1),$$

or, equivalently,

$$a^2 - 4a - 3 < 0.$$

The solution of the quadratic inequality above in case when $a > 1$ is

$$1 < a < 2 + \sqrt{7} \approx 4.64575131.$$

Consequently, we look at

$$a \in \left(\frac{3+\sqrt{17}}{2}, 2 + \sqrt{7} \right).$$

Further, during the proof we need inequalities related to the roots of the function F_a. So, for convenience of dealing with inequalities, we are going to consider the positive roots. Thus, instead of F_a we study the entire function with sign-changing coefficients

$$f_a(x) = F_a(-x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(a^k+1)(a^{k-1}+1)\cdot \ldots \cdot (a+1)},$$

where $q_k(f_a)$ are increasing in k, and $\lim\limits_{n\to\infty} q_k(f_a) = a > q_\infty$. In addition, we further use the denotation $S_{n,a}$ and $R_{n,a}$ for the nth partial sum and the nth partial remainder of f_a consequently. Applying Rouché's theorem, we prove that the function f_a has the same number of zeros in $\{z : |z| < a^2+1\}$ as $S_{2,a}$. In the following lemma we find the minimum value of $S_{2,a}$ on the disk of radius a^2+1.

Lemma 2.14 (T.H. Nguyen, [59])**.** *We have* $\min_{|z|=a^2+1} |S_{2,a}(z)| = 1$.

Proof. For the sake of brevity, we further write q_2 instead of $q_2(f_a)$. Note that

$$q_2 = \frac{a^2+1}{a+1},$$

hence, we have

$$\begin{aligned}
&\left|S_{2,a}\Big((a^2+1)e^{i\theta}\Big)\right|^2 = \left|1 - q_2 e^{i\theta} + q_2 e^{2i\theta}\right|^2 \\
&= (1 - q_2\cos\theta + q_2\cos 2\theta)^2 + (-q_2\sin\theta + q_2\sin 2\theta)^2 \\
&= 1 + q_2^2\cos^2\theta + q_2^2\cos^2 2\theta - 2q_2\cos\theta + 2q_2\cos 2\theta \\
&- 2q_2^2\cos\theta\cos 2\theta + q_2^2\sin^2\theta + q_2^2\sin^2 2\theta - 2q_2^2\sin\theta\sin 2\theta \\
&= 1 + 2q_2^2 - 2q_2\cos\theta + 2q_2\cos 2\theta - 2q_2^2(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta) \\
&= 1 + 2q_2^2 - 2q_2\cos\theta + 2q_2\cos 2\theta - 2q_2^2\cos\theta \\
&= 1 + 2q_2^2 - 2q_2(1+q_2)\cos\theta + 2q_2\cos 2\theta.
\end{aligned}$$

Set $t := \cos\theta$, and applying that

$$\cos 2\theta = 2\cos^2\theta - 1 = 2t^2 - 1,$$

we get

$$\begin{aligned}
&\left|S_{2,a}\Big((a^2+1)e^{i\theta}\Big)\right|^2 = 1 + 2q_2^2 - 2q_2(1+q_2)\cos\theta + 2q_2\cos 2\theta \\
&= 2q_2(2t^2-1) - 2q_2(1+q_2)t + 1 + 2q_2^2 \\
&= 4q_2t^2 - 2q_2(1+q_2)t + 1 - 2q_2 + 2q_2^2 =: \xi(t).
\end{aligned}$$

Next, we calculate the discriminant and get

$$\frac{D_\xi}{4} = (q_2(1+q_2))^2 - 4q_2(1-2q_2+2q_2^2)$$
$$= q_2^4 - 6q_2^3 + 9q_2^2 - 4q_2 = q_2(q_2-1)^2(q_2-4).$$

Since, under our assumptions, $q_2 < 4$, the discriminant D_ξ is negative. Hence, we conclude that $\xi(t)$ is positive and has no zeros for all $t \in [-1,1]$. The vertex point of the parabola is

$$t_v = \frac{1+q_2}{4} \geq 1,$$

since $q_2 \geq 3$. Consequently,

$$\min_{t\in[-1,1]} \xi(t) = \xi(1) = 4q_2 - 2q_2(1+q_2) + 1 - 2q_2 + 2q_2^2$$
$$= 4q_2 - 2q_2 - 2q_2^2 + 1 - 2q_2 + 2q_2^2 = 1.$$

Thus, $\min_{|x|=a^2+1} |S_{2,a}(x)| = 1$. □

As the next step of the proof of Theorem 2.10, since we want to show that the function f_a has the same number of zeros in $\{z : |z| < a^2+1\}$ as $S_{2,a}$, by Rouché's theorem, we obtain an upper bound for the modulus of $R_{3,a}(x) := f_a(x) - S_{2,a}(x)$.

Lemma 2.15 (T.H. Nguyen, [59])**.** *We have*

$$\max_{|z|=a^2+1} |R_{3,a}(z)| \leq \frac{(a^2+1)^2(a^4+1)}{a^2(a+1)(a^3+1)(a^2-1)}.$$

Proof. We observe that

$$R_{3,a}(z) = \sum_{k=3}^{\infty} \frac{(-1)^k z^k}{(a^k+1)(a^{k-1}+1)\cdot\ldots\cdot(a+1)},$$

then we obtain the following estimation of $|R_{3,a}|$ from above:

$$\begin{aligned}
\max_{|z|=a^2+1} |R_{3,a}(z)| &\leq \sum_{k=3}^{\infty} \frac{(a^2+1)^k}{(a+1)(a^2+1)\cdot \ldots \cdot (a^k+1)} \\
&= \sum_{k=3}^{\infty} \frac{(a^2+1)^{k-1}}{(a+1)(a^3+1)\cdot \ldots \cdot (a^k+1)} \\
&= \frac{(a^2+1)^2}{(a+1)(a^3+1)} + \frac{(a^2+1)^3}{(a+1)(a^3+1)(a^4+1)} \\
&\quad + \frac{(a^2+1)^4}{(a+1)(a^3+1)(a^4+1)(a^5+1)} + \cdots \\
&= \frac{(a^2+1)^2}{(a+1)(a^3+1)} \left(1 + \sum_{k=1}^{\infty} \frac{(a^2+1)^k}{(a^4+1)(a^5+1)\cdot \ldots \cdot (a^{k+3}+1)}\right) \\
&\leq \frac{(a^2+1)^2}{(a+1)(a^3+1)} \left(1 + \sum_{k=1}^{\infty} \frac{(a^2+1)^k}{(a^4+1)^k}\right) \\
&= \frac{(a^2+1)^2}{(a+1)(a^3+1)} \cdot \frac{1}{1-(a^2+1)/(a^4+1)} = \frac{(a^2+1)^2(a^4+1)}{a^2(a+1)(a^3+1)(a^2-1)},
\end{aligned}$$

which establishes the claim. □

We observe that, under the assumption $a \geq (3+\sqrt{17})/2$, we have

$$\frac{(a^2+1)^2(a^4+1)}{a^2(a+1)(a^3+1)(a^2-1)} < 1,$$

or, after simplifying,

$$a^7 - 3a^6 - 2a^4 - a^2 - 1 > 0.$$

The numerical calculations show that this inequality valid for all $a \geq 3.2051$, so it is valid for $a \geq (3+\sqrt{17})/2$. Hence, by Lemmas 2.14 and 2.15 we have

$$\max_{|z|=a^2+1} |R_{3,a}(z)| < \max_{|z|=a^2+1} |S_{2,a}(z)| = 1.$$

Consequently, by Rouché's theorem, the functions f_a and $S_{2,a}$ have the same number of zeros (counted with multiplicities) on the open disk $D_{a^2+1} := \{z \in \mathbb{C} : |z| < a^2+1\}$. Since $q_2(f_a) < 4$, we conclude that $S_{2,a}$ has two conjugate zeros of modulus $\sqrt{(a+1)(a^2+1)} < a^2+1$, so that $S_{2,a}$ has exactly two zeros on D_{a^2+1} for all a such that $q_2(f_a) \in [3,4)$. Thus, if $f_a \in \mathcal{L}-\mathcal{P}$ (or $S_{n,a} \in \mathcal{L}-\mathcal{P}$ for $n \geq 2$), then these two zeros are real, and there exists

$x_0 \in (0, a^2+1)$ such that $f_a(x_0) \leq 0$. Since $f_a(-x)$ (and $S_{n,a}(-x)$) has positive Taylor coefficients, the functions f_a (and $S_{n,a}$) does not have zeros on $[-(a^2+1), 0]$. For $x \in [0, a+1]$ we have

$$\frac{x^k}{(a+1)(a^2+1)\cdot \ldots \cdot (a^k+1)} > \frac{x^{k+1}}{(a+1)(a^2+1)\cdot \ldots \cdot (a^{k+1}+1)}$$

for all nonnegative integers k, whence

$$f_a(x) > 0 \quad \text{for all} \quad x \in [0, a+1], \tag{2.10}$$

and

$$S_{n,a}(x) > 0 \quad \text{for all} \quad x \in [0, a+1],\ n \geq 2. \tag{2.11}$$

We obtained that if $f_a \in \mathcal{L}-\mathcal{P}$ (or $S_{n,a} \in \mathcal{L}-\mathcal{P}$ for $n \geq 2$), then there exists $x_0 \in (a+1, a^2+1)$ such that $f_a(x_0) \leq 0$ (also, there exists $x_n \in (a+1, a^2+1)$ such that $S_{n,a}(x_n) \leq 0$).

It remains to prove the converse statement: if there exists $x_0 \in (a+1, a^2+1)$ such that $f_a(x_0) \leq 0$, then $f_a \in \mathcal{L}-\mathcal{P}$. We need the following lemma.

Lemma 2.16 (T.H. Nguyen, [59])**.** *Define $\rho_j(f_a) := q_2(f_a)q_3(f_a)\cdot \ldots \cdot q_j(f_a)\sqrt{q_{j+1}(f_a)}$ for j being a positive integer. Then, for sufficiently large j, the function $f_a(x)$ has exactly j zeros on the disk $D_{\rho_j(f_a)} = \{z : |z| < \rho_j(f_a)\}$.*

Proof. For brevity, we write p_n and q_n instead of $p_n(f_a)$ and $q_n(f_a)$. Then

$$f_a(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1}q_3^{k-2}\cdot \ldots \cdot q_k},$$

where the sequence $q_2, q_3, \ldots$ is strictly increasing with the limit $a > (3+\sqrt{17})/2$. We now dissect the above sum as

$$\begin{aligned} f_a(x) &= \left(\sum_{k=0}^{j-3} + \sum_{k=j-2}^{j+2} + \sum_{k=j+3}^{\infty}\right) \frac{(-1)^k x^k}{q_2^{k-1}q_3^{k-2}\cdot \ldots \cdot q_k} \\ &=: \Sigma_{1,j}(x) + g_j(x) + \Sigma_{2,j}(x). \end{aligned}$$

We further dissect $g_j(x) = \widetilde{g}_j(x) + \xi_j(x)$, where

$$\widetilde{g}_j(x) := \left(\sum_{k=j-2}^{j+1} \frac{(-1)^k x^k}{q_2^{k-1}q_3^{k-2}\cdot \ldots \cdot q_k} + \frac{(-1)^{j+2}x^{j+2}}{q_2^{j+1}q_3^{j}\cdot \ldots \cdot q_{j-2}^5 q_{j-1}^4 q_j^4 q_{j+1}^2}\right),$$

and

$$\xi_j(x) := \frac{(-1)^{j+2}x^{j+2}}{q_2^{j-3}q_3^{j-4}\cdot\ldots\cdot q_{j-2}}\Bigg(\frac{1}{q_2^4q_3^4\cdot\ldots\cdot q_{j-1}^4q_j^3q_{j+1}^2q_{j+2}} \\ -\frac{1}{q_2^4q_3^4\cdot\ldots\cdot q_{j-1}^4q_j^4q_{j+1}^2}\Bigg).$$

Since $\rho_j(f_a) = q_2q_3\cdot\ldots\cdot q_j\sqrt{q_{j+1}}$, we have

$$q_2q_3\cdot\ldots\cdot q_j < \rho_j < q_2q_3\cdot\ldots\cdot q_jq_{j+1}.$$

We get

$$\begin{aligned}&(-1)^{j-2}g_j(\rho_je^{i\theta}) = e^{i(j-2)\theta}q_2q_3^2\cdot\ldots\cdot q_{j-2}^{j-3}q_{j-1}^{j-2}q_j^{j-2}q_{j+1}^{\frac{j-2}{2}}\times\\ &\Big(1-e^{i\theta}q_j\sqrt{q_{j+1}}+e^{2i\theta}q_jq_{j+1}-e^{3i\theta}q_j\sqrt{q_{j+1}}+e^{4i\theta}q_jq_{j+2}^{-1}\Big)\\ &= e^{i(j-2)\theta}q_2q_3^2\cdot\ldots\cdot q_{j-2}^{j-3}q_{j-1}^{j-2}q_j^{j-2}q_{j+1}^{\frac{j-2}{2}}\times\\ &\Big(1-e^{i\theta}q_j\sqrt{q_{j+1}}+e^{2i\theta}q_jq_{j+1}-e^{3i\theta}q_j\sqrt{q_{j+1}}+e^{4i\theta}\Big)\\ &\quad+e^{i(j+2)\theta}q_2q_3^2\cdot\ldots\cdot q_{j-2}^{j-3}q_{j-1}^{j-2}q_j^{j-2}q_{j+1}^{\frac{j-2}{2}}\Big(q_jq_{j+2}^{-1}-1\Big)\\ &= \widetilde{g}_j(\rho_je^{i\theta})+\xi_j(\rho_je^{i\theta}).\end{aligned}$$

Our aim is to show that for every sufficiently large j the following inequality holds:

$$\min_{0\le\theta\le2\pi}|\widetilde{g}_j(\rho_je^{i\theta})| > \max_{0\le\theta\le2\pi}|f_a(\rho_je^{i\theta})-\widetilde{g}_j(\rho_je^{i\theta})|,$$

so that the number of zeros of f_a on the open disk $\{x : |x| < \rho_j\}$ is equal to the number of zeros of $\widetilde{g}_j$ on the same disk. Later in the proof we also find the number of the zeros. First, we find $\min_{0\le\theta\le2\pi}|\widetilde{g}_j(\rho_je^{i\theta})|$:

$$\begin{aligned}&\widetilde{g}_j(\rho_je^{i\theta}) = e^{ij\theta}q_2q_3^2\cdot\ldots\cdot q_{j-2}^{j-3}q_{j-1}^{j-2}q_j^{j-2}q_{j+1}^{\frac{j-2}{2}}\\ &\times\Big(e^{-2i\theta}-e^{-i\theta}q_j\sqrt{q_{j+1}}+q_jq_{j+1}-e^{i\theta}q_j\sqrt{q_{j+1}}+e^{2i\theta}\Big)\\ &= e^{ij\theta}q_2q_3^2\cdot\ldots\cdot q_{j-2}^{j-3}q_{j-1}^{j-2}q_j^{j-2}q_{j+1}^{\frac{j-2}{2}}\\ &\times\Big(2\cos2\theta-2\cos\theta q_j\sqrt{q_{j+1}}+q_jq_{j+1}\Big)\\ &=: e^{ij\theta}q_2q_3^2\cdot\ldots\cdot q_{j-2}^{j-3}q_{j-1}^{j-2}q_j^{j-2}q_{j+1}^{\frac{j-2}{2}}\cdot\psi_j(\theta).\end{aligned}$$

We consider $\psi_j(\theta)$ defined as follows

$$\psi_j(\theta) = \widetilde{\psi}_j(t) := 4t^2-2q_j\sqrt{q_{j+1}}t+(q_jq_{j+1}-2),$$

where $t := \cos\theta$, and where we have used that $\cos 2\theta = 2t^2 - 1$.

The vertex of the parabola is $t_j = \frac{q_j\sqrt{q_{j+1}}}{4}$. Under our assumptions, $t_j > 1$. Hence,

$$\min_{t\in[-1,1]} \widetilde{\psi}_j(t) = \widetilde{\psi}_j(1) = 2 - 2q_j\sqrt{q_{j+1}} + q_jq_{j+1} = q_j(\sqrt{q_{j+1}} - 1)^2 - q_j + 2.$$

If $q_j \geq 4$, then $q_{j+1} \geq 4$, and

$$q_j(\sqrt{q_{j+1}} - 1)^2 - q_j + 2 \geq q_j - q_j + 2 > 0.$$

If $q_j < 4$, then, since $q_j \geq q_2 \geq 3$, we have $(\sqrt{q_{j+1}} - 1)^2 \geq (\sqrt{3} - 1)^2 > 0.5$. Therefore, we get

$$q_j(\sqrt{q_{j+1}} - 1)^2 - q_j + 2 > q_j \cdot 0.5 - q_j + 2 = q_j \cdot (-0.5) + 2 > 0.$$

Thus, $\widetilde{\psi}_j(t) > 0$ for all $t \in [-1, 1]$. Consequently, we have obtained the estimate from below

$$\min_{0\leq\theta\leq 2\pi} |\widetilde{g}_j(\rho_j e^{i\theta})| \geq q_2q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3}q_{j-1}^{j-2}q_j^{j-2}q_{j+1}^{\frac{j-2}{2}} \times \left(2 - 2q_j\sqrt{q_{j+1}} + q_jq_{j+1}\right). \tag{2.12}$$

Next, we bound the modulus of Σ_1 from above. We have

$$|\Sigma_1(\rho_j e^{i\theta})| \leq \sum_{k=0}^{j-3} \frac{(q_2 \cdot \ldots \cdot q_j)^k q_{j+1}^{\frac{k}{2}}}{q_2^{k-1}q_3^{k-2} \cdot \ldots \cdot q_k} = \text{(we rewrite the sum from right to left)}$$

$$= q_2q_3^2 \cdot \ldots \cdot q_{j-3}^{j-4}(q_{j-2}q_{j-1}q_j)^{j-3}q_{j+1}^{\frac{j-3}{2}} + q_2q_3^2 \cdot \ldots \cdot q_{j-4}^{j-5}(q_{j-3}q_{j-2}q_{j-1}q_j)^{j-4}q_{j+1}^{\frac{j-4}{2}}$$
$$+ q_2q_3^2 \cdot \ldots \cdot q_{j-5}^{j-6}(q_{j-4}q_{j-3}q_{j-2}q_{j-1}q_j)^{j-5}q_{j+1}^{\frac{j-5}{2}} + \cdots$$
$$= q_2q_3^2 \cdot \ldots \cdot q_{j-3}^{j-4}(q_{j-2}q_{j-1}q_j)^{j-3}q_{j+1}^{\frac{j-3}{2}} \times$$
$$\left(1 + \frac{1}{q_{j-2}q_{j-1}q_j\sqrt{q_{j+1}}} + \frac{1}{q_{j-3}(q_{j-2}q_{j-1}q_j\sqrt{q_{j+1}})^2} + \cdots\right)$$
$$\leq q_2q_3^2 \cdot \ldots \cdot q_{j-3}^{j-4}(q_{j-2}q_{j-1}q_j)^{j-3}q_{j+1}^{\frac{j-3}{2}} \cdot \frac{1}{1 - \frac{1}{q_{j-2}q_{j-1}q_j\sqrt{q_{j+1}}}}$$

(where we have bounded the finite sum from the above by the sum of the infinite geometric progression). Finally, we obtain

$$|\Sigma_1(\rho_j e^{i\theta})| \leq q_2q_3^2 \cdot \ldots \cdot q_{j-3}^{j-4}(q_{j-2}q_{j-1}q_j)^{j-3}q_{j+1}^{\frac{j-3}{2}} \cdot \frac{1}{1 - \frac{1}{q_{j-2}q_{j-1}q_j\sqrt{q_{j+1}}}}.$$

Next, the upper bound of $|\Sigma_2(\rho_j e^{i\theta})|$ can be found by a completely analogous computation

$$|\Sigma_2(\rho_j e^{i\theta})| \leq \sum_{k=j+3}^{\infty} \frac{q_2^k q_3^k \cdot \ldots \cdot q_j^k q_{j+1}^{\frac{k}{2}}}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k} =$$

$$\frac{q_2 q_3^2 \cdot \ldots \cdot q_j^{j-1} q_{j+1}^{\frac{j-3}{2}}}{q_{j+2}^2 q_{j+3}} \cdot \Bigg(1 + \frac{1}{\sqrt{q_{j+1}} q_{j+2} q_{j+3} q_{j+4}} + \frac{1}{(\sqrt{q_{j+1}})^2 q_{j+2}^2 q_{j+3}^2 q_{j+4}^2 q_{j+5}} + \ldots\Bigg).$$

The latter can be estimated from above by the sum of the geometric progression, so, we obtain

$$|\Sigma_2(\rho_j e^{i\theta})| \leq \frac{q_2 q_3^2 \cdot \ldots \cdot q_j^{j-1} q_{j+1}^{\frac{j-3}{2}}}{q_{j+2}^2 q_{j+3}} \cdot \frac{1}{1 - \frac{1}{\sqrt{q_{j+1}} q_{j+2} q_{j+3} q_{j+4}}}.$$

Note that

$$|\xi_j(\rho_j e^{i\theta})| = q_2 q_3^2 \cdots q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \left(q_j q_{j+2}^{-1} - 1\right).$$

Therefore, the desired inequality

$$\min_{0\leq\theta\leq 2\pi} |\widetilde{g}_j(\rho_j e^{i\theta})| > \max_{0\leq\theta\leq 2\pi} |f_a(\rho_j e^{i\theta}) - \widetilde{g}_j(\rho_j e^{i\theta})|$$

follows from

$$q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \cdot \left(2 - 2q_j\sqrt{q_{j+1}} + q_j q_{j+1}\right)$$
$$> q_2 q_3^2 \cdot \ldots \cdot q_{j-3}^{j-4} q_{j-2}^{j-3} q_{j-1}^{j-3} q_j^{j-3} q_{j+1}^{\frac{j-3}{2}} \cdot \frac{1}{1 - \frac{1}{q_{j-2} q_{j-1} q_j \sqrt{q_{j+1}}}}$$
$$+ \frac{q_2 q_3^2 \cdot \ldots \cdot q_j^{j-1} q_{j+1}^{\frac{j-3}{2}}}{q_{j+2}^2 q_{j+3}} \cdot \frac{1}{1 - \frac{1}{\sqrt{q_{j+1}} q_{j+2} q_{j+3} q_{j+4}}}$$
$$+ q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \left(q_j q_{j+2}^{-1} - 1\right).$$

After reducing by $q_2 q_3^2 \cdot \ldots \cdot q_{j-3}^{j-4} q_{j-2}^{j-3} q_{j-1}^{j-3} q_j^{j-3} q_{j+1}^{\frac{j-3}{2}}$ we get

$$q_{j-1} q_j \sqrt{q_{j+1}} \left(2 - 2q_j\sqrt{q_{j+1}} + q_j q_{j+1}\right) > \frac{1}{1 - \frac{1}{q_{j-2} q_{j-1} q_j \sqrt{q_{j+1}}}} \tag{2.13}$$
$$+ \frac{q_{j-1} q_j^2}{q_{j+2}^2 q_{j+3}} \cdot \frac{1}{1 - \frac{1}{\sqrt{q_{j+1}} q_{j+2} q_{j+3} q_{j+4}}} + q_{j-1} q_j \sqrt{q_{j+1}} (q_j q_{j+2}^{-1} - 1).$$

To prove the inequality above for sufficiently large j, we first consider the limiting inequality for $j \to \infty$. Since, under our assumptions, $\lim_{j\to\infty} q_j = a$, we obtain

$$a^2\sqrt{a}(2 - 2a\sqrt{a} + a^2) > \frac{1}{1 - \frac{1}{a^3\sqrt{a}}} + \frac{1}{1 - \frac{1}{a^3\sqrt{a}}} + a^2\sqrt{a} \cdot 0. \qquad (2.14)$$

Equivalently,

$$2 - 2a\sqrt{a} + a^2 > \frac{2a}{a^3\sqrt{a} - 1}.$$

Set $\sqrt{a} =: b$. Then we obtain $(2 - 2b^3 + b^4)(b^7 - 1) > 2b^2$, or

$$b^{11} - 2b^{10} + 2b^7 - b^4 + 2b^3 - 2b^2 - 2 > 0.$$

We have found the roots of the polynomial on the left-hand side of the inequality using numerical methods, and its largest real root is less than 1.47. Thus, the inequality is fulfilled for $b > 1.47$, and, therefore, for $a > 2.17$. Under our assumptions, $a > 3.57$, so the inequality (2.14) is valid under our assumptions on a. Consequently, the inequality (2.13) is valid under our assumptions on a and for j being sufficiently large.

Therefore, we have proved that for all sufficiently large j,

$$\min_{0\le\theta\le2\pi} |\widetilde{g}_j(\rho_j e^{i\theta})| > \max_{0\le\theta\le2\pi} |f_a(\rho_j e^{i\theta}) - \widetilde{g}_j(\rho_j e^{i\theta})|,$$

so the number of zeros of f_a on the open disk $\{z : |z| < \rho_j\}$ is equal to the number of zeros of $\widetilde{g}_j$ on this disk.

In the next stage of the proof, it remains to find the number of zeros of $\widetilde{g}_j$ on the open disk $\{z : |z| < \rho_j\}$. We have

$$\widetilde{g}_j(z) = \sum_{k=j-2}^{j+1} \frac{(-1)^k z^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k} + \frac{(-1)^{j+2} z^{j+2}}{q_2^{j+1} q_3^{j} \cdot \ldots \cdot q_{j-2}^5 q_{j-1}^4 q_j^4 q_{j+1}^2}.$$

Denote $w = z\rho_j^{-1}$, so that $|w| < 1$. This yields

$$\widetilde{g}_j(\rho_j w) = (-1)^{j-2} w^{j-2} q_2 q_3^2 \cdots q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \times (1 - q_j\sqrt{q_{j+1}}w + q_j q_{j+1} w^2 - q_j\sqrt{q_{j+1}}w^3 + w^4).$$

It follows from (2.12) that $\widetilde{g}_j$ does not have zeros on the circle $\{z : |z| = \rho_j\}$, whence $\widetilde{g}_j(\rho_j w)$ does not have zeros on the circle $\{w : |w| = 1\}$. Since $P_j(w) = 1 - q_j\sqrt{q_{j+1}}w + q_j q_{j+1} w^2 - q_j\sqrt{q_{j+1}}w^3 + w^4$ is a self-reciprocal polynomial in w, we conclude that P_j has exactly two zeros on the open disk $\{w : |w| < 1\}$. Hence, $\widetilde{g}_j(x)$ has exactly j zeros on the open disk $\{z : |z| < \rho_j\}$ counting multiplicities, and we have proved the statement of Lemma 2.16. □

Further we need the following lemma.

Lemma 2.17 (T.H. Nguyen, [59]). *Denote by* $\rho_k(f_a) := q_2(f_a)q_3(f_a)\cdot\ldots\cdot q_k(f_a)\sqrt{q_{k+1}(f_a)}$, $k\in\mathbb{N}, k\geq 2$. *If* $a\geq 3$ *then for every* $k\geq 2$ *the following inequality holds:*

$$(-1)^k f_a(\rho_k)\geq 0.$$

Proof. For the sake of brevity, we further write q_n and ρ_n instead of $q_n(f_a)$ and $\rho_n(f_a)$. Then the function takes the following form

$$f_a(x)=\sum_{j=0}^{\infty}\frac{(-1)^j x^j}{q_2^{j-1}q_3^{j-2}\cdot\ldots\cdot q_j},$$

where the sequence $q_2, q_3, \ldots$ is strictly increasing with the limit $a\geq 3$.

Since $\rho_k\in(q_2q_3\cdot\ldots\cdot q_k, q_2q_3\cdot\ldots\cdot q_kq_{k+1})$, we have

$$1<\rho_k<\frac{\rho_k^2}{q_2}<\cdots<\frac{\rho_k^k}{q_2^{k-1}q_3^{k-2}\cdots q_k},$$

and

$$\frac{\rho_k^k}{q_2^{k-1}q_3^{k-2}\cdot\ldots\cdot q_k}>\frac{\rho_k^{k+1}}{q_2^kq_3^{k-1}\cdot\ldots\cdot q_k^2q_{k+1}}>\frac{\rho_k^{k+2}}{q_2^{k+1}q_3^k\cdot\ldots\cdot q_k^3q_{k+1}^2q_{k+2}}>\cdots.$$

Therefore, we get for $k\geq 2$

$$\begin{aligned}(-1)^k f_a(\rho_k) &= \left(\sum_{j=0}^{k-4}+\sum_{j=k-3}^{k+3}+\sum_{j=k+4}^{\infty}\right)\frac{(-1)^{j+k}\rho_k^j}{q_2^{j-1}q_3^{j-2}\cdot\ldots\cdot q_j}\\ &=: m_1(\rho_k)+\mu_k(\rho_k)+m_2(\rho_k).\end{aligned}$$

We note that the terms in $m_1(\rho_k)$ are alternating in sign and increasing in moduli, and the largest term in modulus is positive, whence $m_1(\rho_k)\geq 0$. Analogously, the summands in $m_2(\rho_k)$ are alternating in sign and their moduli are decreasing, and the term of the greatest modulus is positive, thus, $m_2(\rho_k)\geq 0$. Therefore,

$$(-1)^k f_a(\rho_k)\geq \mu_k(\rho_k).$$

Thus, it is sufficient to prove that for every $k\geq 2$ we have $\mu_k(\rho_k)\geq 0$. After factoring out $(\rho_k^{k-3})(q_2^{k-4}q_3^{k-5}\cdots q_{k-3})$, the desired inequality takes the form

$$\begin{aligned}&-1+\frac{\rho_k}{q_2q_3\cdot\ldots\cdot q_{k-3}q_{k-2}}-\frac{\rho_k^2}{q_2^2q_3^2\cdot\ldots\cdot q_{k-2}^2q_{k-1}}+\frac{\rho_k^3}{q_2^3q_3^3\cdot\ldots\cdot q_{k-2}^3q_{k-1}^2q_k}\\ &-\frac{\rho_k^4}{q_2^4q_3^4\cdot\ldots\cdot q_{k-2}^4q_{k-1}^3q_k^2q_{k+1}}+\frac{\rho_k^5}{q_2^5q_3^5\cdot\ldots\cdot q_{k-2}^5q_{k-1}^4q_k^3q_{k+1}^2q_{k+2}}\\ &-\frac{\rho_k^6}{q_2^6q_3^6\cdot\ldots\cdot q_{k-2}^6q_{k-1}^5q_k^4q_{k+1}^3q_{k+2}^2q_{k+3}}\geq 0,\end{aligned}$$

or, using that $\rho_k = q_2 q_3 \cdot \ldots \cdot q_k \sqrt{q_{k+1}}$,

$$\begin{aligned}\nu_k(\rho_k) &:= -1 + q_{k-1}q_k\sqrt{q_{k+1}} - 2q_{k-1}q_k^2 q_{k+1} \\ &+ q_{k-1}q_k^2 q_{k+1}\sqrt{q_{k+1}} + q_{k-1}q_k^2\sqrt{q_{k+1}}q_{k+2}^{-1} - q_{k-1}q_k^2 q_{k+2}^{-2}q_{k+3}^{-1} \geq 0.\end{aligned}$$

After rearranging we get

$$\begin{aligned}\nu_k(\rho_k) &= q_{k-1}q_k^2 q_{k+1}\sqrt{q_{k+1}} - 2q_{k-1}q_k^2 q_{k+1} \\ &+ q_{k-1}q_k\sqrt{q_{k+1}}\left(1 + \frac{q_k}{q_{k+2}}\right) - \left(1 + \frac{q_{k-1}q_k^2}{q_{k+2}^2 q_{k+3}}\right) \geq 0.\end{aligned}$$

It is easy to check that the sequence $\left(\frac{q_k}{q_{k+2}}\right)_{k=2}^{\infty}$ is increasing in k and $\lim_{k\to\infty} \frac{q_k}{q_{k+2}} = 1$, so we have

$$\frac{q_k}{q_{k+2}} \geq \frac{q_2}{q_4} = \frac{(a^2+1)(a^3+1)}{(a+1)(a^4+1)} = \frac{a^5+a^3+a^2+1}{a^5+a^4+a+1} \geq 0.8$$

for $a \geq 0$. In addition,

$$\frac{q_{k-1}q_k^2}{q_{k+2}^2 q_{k+3}} < 1,$$

so it is sufficient to prove the following inequality

$$q_{k-1}q_k^2 q_{k+1}\sqrt{q_{k+1}} - 2q_{k-1}q_k^2 q_{k+1} + q_{k-1}q_k\sqrt{q_{k+1}} \cdot 1.8 - 2 \geq 0.$$

Since

$$2 < \frac{2}{9}q_{k-1}q_k,$$

we have

$$\begin{aligned}&q_{k-1}q_k^2 q_{k+1}\sqrt{q_{k+1}} - 2q_{k-1}q_k^2 q_{k+1} + q_{k-1}q_k\sqrt{q_{k+1}} \cdot 1.8 - 2 \\ &\geq q_{k-1}q_k^2 q_{k+1}\sqrt{q_{k+1}} - 2q_{k-1}q_k^2 q_{k+1} + q_{k-1}q_k\sqrt{q_{k+1}} \cdot 1.8 - \tfrac{2}{9}q_{k-1}q_k.\end{aligned}$$

Next, we need to check that for all $k \geq 2$

$$\begin{aligned}&q_k q_{k+1}\sqrt{q_{k+1}} - 2q_k q_{k+1} + 1.8\sqrt{q_{k+1}} - \frac{2}{9} \\ &= q_k q_{k+1}\left(\sqrt{q_{k+1}} - 2\right) + 1.8\sqrt{q_{k+1}} - \frac{2}{9} \geq 0.\end{aligned}$$

If $q_{k+1} \geq 4$, then $\sqrt{q_{k+1}} - 2 \ \geq 0$ and $1.8\sqrt{q_{k+1}} - \frac{2}{9} \geq 0$, and the last inequality is valid. In case when $q_{k+1} < 4$, since q_k increases in k, if we set $\sqrt{q_{k+1}} = t$, $t > 0$, then we obtain the inequality

$$\begin{aligned} & q_k q_{k+1}\sqrt{q_{k+1}} - 2q_k q_{k+1} + 1.8\sqrt{q_{k+1}} - \tfrac{2}{9} \\ & \geq q_{k+1}^2\sqrt{q_{k+1}} - 2q_{k+1}^2 + 1.8\sqrt{q_k + 1} - \tfrac{2}{9} \\ & \quad = t^5 - 2t^4 + 1.8t - \tfrac{2}{9} \geq 0. \end{aligned}$$

The inequality above holds for $t \geq 1.57685$, since 1.57685 is greater than the largest real root of the polynomial on the left-hand side (we used numerical methods here to find the roots of the polynomial), so it follows that it holds for $q_{k+1} \geq 2.48646$.

Lemma 2.17 is proved. □

Suppose that there exists $x_0 \in (a + 1, a^2 + 1)$, such that $f_a(x_0) \leq 0$. Then, by Lemma 2.17 we have for every $k \geq 2$:

$$f_a(0) > 0, f_a(x_0) \leq 0, f_a(\rho_2) \geq 0, f_a(\rho_3) \leq 0, \ldots, (-1)^k f_a(\rho_k) \geq 0.$$

So, for every $k \geq 2$ the function f_a has at least $k - 1$ real zeros on the open disk $\{x : |x| < \rho_k\}$. By Lemma 2.16 the function f_a has exactly k zeros on the open disk $\{x : |x| < \rho_k\}$ for sufficiently large k . Thus, if there exists $x_0 \in (a + 1, a^2 + 1)$, such that $f_a(x_0) \leq 0$, then all the zeros of f_a are real.

Theorem 2.10 is proved.

2.4 Proof of Theorem 2.11

In order to bound the values of a from below such that f_a belongs to the Laguerre–Pólya class, we consider its section $S_{3,a}$. We have proved that if $f_a \in \mathcal{L} - \mathcal{P}$, then there exists $x_0 \in (a + 1, a^2 + 1)$ such that $f_a(x_0) \leq 0$. Note that, for every $x \in (a + 1, a^2 + 1)$ we have $S_{3,a}(x) < f_a(x)$, which follows that $S_{3,a}(x_0) < 0$.

Lemma 2.18 (T.H. Nguyen, [59])**.** *If there exists $x_0 \in (a + 1, a^2 + 1)$ such that $S_{3,a}(x_0) \leq 0$, then $a \geq 3.90155$.*

Proof. Set $y_0 := x_0/(a + 1)$, where $1 < y_0 < \frac{a^2+1}{a+1} = q_2$. Hence, we get

$$S_{3,a}(x_0) = S_{3,a}((a + 1)y_0) = 1 - y_0 + \frac{y_0^2}{q_2} - \frac{y_0^3}{q_2^2 q_3}, \quad 3 \leq q_2 < q_3.$$

For the sake of brevity, set $b := q_2, c := q_3$. Then we obtain

$$S_{3,a}((a+1)y_0) = 1 - y_0 + \frac{y_0^2}{b} - \frac{y_0^3}{b^2c} =: K(y_0).$$

We would like to find the minimal point of $K(y)$ in the interval $y \in (1, b)$. First, we find the roots of the derivative. The derivative of $K(y)$ is

$$K'(y) = -\frac{1}{b^2c}(3y^2 - 2bcy + b^2c).$$

We consider the discriminant of the quadratic polynomial K'

$$D/4 = b^2c^2 - 3b^2c = b^2c(c-3) > 0,$$

since under our assumptions, $c > 3$. Thus, the roots of derivative are

$$y_\pm = \frac{bc \pm b\sqrt{c(c-3)}}{3}.$$

Now we check if y_- or y_+ lie in $(1, b)$. We consider the following inequality

$$1 < \frac{bc - b\sqrt{c(c-3)}}{3} < b. \tag{2.15}$$

The left-hand side of this inequality is

$$bc - 3 > b\sqrt{c(c-3)},$$

or, equivalently,

$$b^2c^2 - 6bc + 9 > b^2c(c-3).$$

The inequality is fulfilled under our assumptions

$$b^2c - 2bc + 3 = bc(b-2) + 3 > 0.$$

Now we consider the right-hand side of (2.15). It is equivalent to

$$bc - b\sqrt{c(c-3)} < 3b,$$

or

$$c^2 - 6c + 9 < c^2 - 3c.$$

Under our assumptions, $c-3>0$, so the inequality is fulfilled. Therefore, we have verified that $y_- \in (1,b)$. In the next step, we check that $y_+ > b$, or,

$$\frac{bc+b\sqrt{c(c-3)}}{3} > b,$$

or, equivalently,

$$c+\sqrt{c(c-3)} > 3,$$

which is true under our assumptions for c.

Therefore, y_- is the minimal point of $K(y)$ in the interval $1<y<q_2$. Thus, if there exists y_0 such that $1<y_0<q_2$, and $K(y_0)\leq 0$, then $K(y_-)\leq 0$. After substituting y_- into $K(y)$, we obtain the following expression

$$K(y_-) = 1 - \frac{bc-b\sqrt{c(c-3)}}{3} + \frac{(bc-b\sqrt{c(c-3)})^2}{9b} - \frac{(bc-b\sqrt{c(c-3)})^3}{27b^2c}.$$

We require $K(y_-)\leq 0$, or

$$27-9bc+9b\sqrt{c(c-3)}+3bc^2-6bc\sqrt{c(c-3)}+3bc(c-3)-bc^2 + 3bc\sqrt{c(c-3)}-3bc(c-3)+b(c-3)\sqrt{c(c-3)} \leq 0.$$

We rewrite the above and get

$$\sqrt{c(c-3)}(6b-2bc)+(27-9bc+2bc^2)\leq 0. \tag{2.16}$$

We have

$$6b-2bc = 2b(3-c) < 0,$$

since $c>3$. Thus,

$$\sqrt{c(c-3)}(6b-2bc) < 0.$$

Now we show that the following inequality is fulfilled

$$27-9bc+2bc^2 \geq 0.$$

We substitute $b=q_2=(a^2+1)(a+1), c=q_3=(a^3+1)(a^2+1)$. We have

$$27-9\frac{a^2+1}{a+1}\cdot\frac{a^3+1}{a^2+1}+2\frac{a^2+1}{a+1}\cdot\left(\frac{a^3+1}{a^2+1}\right)^2 \geq 0.$$

Equivalently,

$$2a^6 - 9a^5 + 22a^3 + 18a^2 + 27a + 20 \geq 0,$$

or

$$(a+1)^2(2a^4 - 13a^3 + 24a^2 - 13a + 20) \geq 0.$$

It remains to prove that

$$2a^4 - 13a^3 + 24a^2 - 13a + 20 \geq 0.$$

Since $a \geq 3$, set $a = 3 + x, x \geq 0$. We obtain

$$2(3+x)^4 - 13(3+x)^3 + 24(3+x)^2 - 13(3+x) + 20 \geq 0,$$

or

$$2x^4 + 11x^3 + 15x^2 - 4x + 8 \geq 0,$$

which is true for all $x \geq 0$. Consequently, the inequality

$$27 - 9bc + 2bc^2 \geq 0$$

is verified. Thus, we rewrite (2.16) in the form

$$27 - 9bc + 2bc^2 \leq \sqrt{c(c-3)}(2bc - 6b).$$

We can observe that both sides of the inequality are positive. After straightforward calculations we get

$$b^2c^2 - 4b^2c + 18bc - 4bc^2 - 27 \geq 0.$$

We substitute

$$b = q_2 = \frac{a^2+1}{a+1}, \quad c = q_3 = \frac{a^3+1}{a^2+1},$$

and obtain

$$\frac{(a^3+1)^2}{(a+1)^2} - 4\frac{(a^3+1)(a^2+1)}{(a+1)^2} + 18\frac{a^3+1}{a+1} - 4\frac{(a^3+1)^2}{(a+1)(a^2+1)} - 27 \geq 0,$$

or, equivalently,

$$a^8 - 8a^7 + 15a^6 + 12a^5 - 21a^4 - 28a^3 - 43a^2 - 40a - 16 \geq 0.$$

We used numerical methods to find the roots of the polynomial on the left-hand side of the inequality. It is valid when $a \geq 3.90155$, since 3.90155 is greater than the largest real root of the polynomial above. Lemma 2.18 is proved. $\square$

Further, it is known that for any $x_0 \in (a+1, a^2+1)$ and for any $n \in \mathbb{N}$, $S_{2n+1,a}(x_0) \leq f_a(x_0) \leq S_{2n,a}(x_0)$. Thus, if there exists $x_0 \in (a+1, a^2+1)$ such that $S_{6,a}(x_0) \leq 0$, then $f_a(x_0) \leq 0$.

Lemma 2.19 (T.H. Nguyen, [59])**.** *If $a \geq 3.91719$, then there exists $x_0 \in (a+1, a^2+1)$ such that $S_{6,a}(x_0) \leq 0$.*

Proof. We choose

$$x_0 = \frac{2}{3}(a+1)q_2 = \frac{2}{3}(a^2+1) \in (a+1, a^2+1).$$

After substituting x_0 into $S_{6,a}$, we get

$$S_{6,a}\left(\frac{2}{3}(a+1)q_2\right) = 1 - \frac{2}{9}q_2 - \frac{8}{27}\frac{q_2}{q_3} + \frac{16}{81}\frac{q_2}{q_3^2 q_4} - \frac{32}{243}\frac{q_2}{q_3^3 q_4^2 q_5} + \frac{64}{729}\frac{q_2}{q_3^4 q_4^3 q_5^2 q_6}.$$

We need the inequality $S_{6,a}(x_0) \leq 0$ to be fulfilled. Hence, we rewrite the inequality using

$$q_j = \frac{a^j + 1}{a^{j-1} + 1},$$

and after direct calculation we obtain the following inequality

$$\begin{aligned} &729(a+1)(a^3+1)(a^4+1)(a^5+1)(a^6+1) - 162(a^2+1)(a^3+1) \\ &\quad \times (a^4+1)(a^5+1)(a^6+1) - 216(a^2+1)^2(a^4+1)(a^5+1)(a^6+1) \\ &\quad - 144(a^2+1)^3(a^5+1)(a^6+1) - 96(a^2+1)^4(a^5+1) \\ &\quad + 64(a^2+1)^5 \leq 0. \end{aligned}$$

Equivalently,

$$\begin{aligned} &162a^{20} - 513a^{19} - 567a^{18} + 594a^{17} - 567a^{16} - 1134a^{15} - 918a^{14} \\ &\quad - 822a^{13} - 846a^{12} - 228a^{11} - 1927a^{10} - 1125a^9 - 1142a^8 - 750a^7 \\ &\quad - 1030a^6 - 966a^5 - 1360a^4 - 567a^3 + 226a^2 - 729a - 463 \geq 0. \end{aligned}$$

We have used numerical methods to find the roots of the polynomial on the left-hand side of the inequality. The inequality above is valid for $a \geq 3.91719$, which is greater than the value of its largest real root.

Lemma 2.19 is proved. □

Theorem 2.11 is proved.

To conclude this chapter, we studied entire functions with positive Taylor coefficients and with the increasing second quotients of Taylor coefficients.

We have found some necessary conditions for such entire functions to belong to the Laguerre–Pólya class. It occurs that, if the quotients $q_n(f) = \frac{a_{n-1}^2}{a_{n-2}a_n}$ are increasing in n, and $c := \lim\limits_{n\to\infty} \frac{a_{n-1}^2}{a_{n-2}a_n}$ is smaller than the absolute constant q_∞ ($q_\infty \approx 3.2336$), then $f \notin \mathcal{L} - \mathcal{P}$. Moreover, we studied a special entire function $F_a(x) = \sum_{k=0}^{\infty} \frac{x^k}{(a^k+1)(a^{k-1}+1)\cdots(a+1)}$, $a > 1$, which has strictly increasing second quotients of Taylor coefficients, and found the conditions under which F_a belongs to the Laguerre–Pólya class.

Chapter 3

Closest to zero roots and the second quotients of Taylor coefficients of entire functions from the Laguerre–Pólya I class

This chapter deals with the zero location of entire functions with positive Taylor coefficients. For an entire function $f(x) = \sum_{k=0}^{\infty} a_k x^k, a_k > 0$, we show that if f belongs to the Laguerre–Pólya class, and the quotients $q_k(f), k = 2, 3, \dots$ satisfy the condition $q_2(f) \leq q_3(f)$, then f has at least one zero in the segment $\left[-\frac{a_1}{a_2}, 0\right]$. We also give necessary conditions and sufficient conditions of the existence of such a zero in terms of the quotients $q_k(f)$ for $k = 2, 3, 4$.

Let us consider the entire function $f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$, which belongs to the Laguerre–Pólya class of type I. We can observe that its second quotients are $q_k(f) = \frac{a_{k-1}^2}{a_{k-2}a_k} = \frac{k}{k-1}, k \geq 2$.

The following statement is the analogue (for entire functions) of the Newton inequalities which are necessary conditions for real polynomials with positive coefficients to have only real zeros (or, equivalently, to belong to the Laguerre–Pólya class). This fact is well-known to experts, but it is a kind of folklore: it is easier to give the proof than to find an appropriate reference.

Statement 3.1. Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$,$a_k > 0$ for all k, be an entire function from the Laguerre–Pólya class of type I. Then $q_n(f) \geq \frac{n}{n-1}$, for all $n \geq 2$. Moreover, if there exists $m = 2, 3, \dots$, such that $q_m(f) = \frac{m}{m-1}$, then $f(x) = ce^{\alpha x}$, for some $c > 0, \alpha > 0$.

3.1 Proof of Statement 3.1

Proof. We give the proof by induction on k.

Base case: $k = 2$. If f does not have any real roots, then, since the order of f is not greater than 1, we conclude that $f(x) = ce^{\alpha x}$, i.e. the statement is fulfilled (see (1.2) in Definition 1.3). If f has at least one real root, we denote by $\{x_k\}_{k=1}^{\alpha}$, $\alpha \in \mathbb{N} \cup \{\infty\}$ the set of roots of f. Thus,

$$0 < -\sum_{k=1}^{\alpha} \frac{1}{x_k} < \infty,$$

which follows that

$$\sum_{k=1}^{\alpha} \frac{1}{x_k^2} < \infty,$$

whence

$$\sum_{k=1}^{\alpha} \frac{1}{x_k^2} = a_1^2 - 2a_0a_2 > 0.$$

Consequently, $q_2(f) \geq 2$, and if $q_2 = 2$, then $f(x) = ce^{\alpha x}$.

Inductive step: Suppose that the statement is true for $k - 1$, $k = 3, 4, \ldots$. Obviously, if $f \in \mathcal{L} - \mathcal{P}I$, then $f^{(s)} \in \mathcal{L} - \mathcal{P}I$, for any $s \in \mathbb{N}$. Hence,

$$f^{(k-2)}(x) = a_{k-2}(k-2)! + a_{k-1}(k-1)!x + a_k\frac{k!}{2!}x^2 + \cdots \in \mathcal{L} - \mathcal{P}I.$$

Then,

$$q_2(f^{(k-2)}) = \frac{a_{k-1}^2}{a_{k-2}a_k}\frac{2((k-1)!)^2}{(k-2)!k!} = \frac{a_{k-1}^2}{a_{k-2}a_k}\frac{2(k-1)}{k} \geq 2,$$

whence

$$q_k(f) \geq \frac{k}{k-1}.$$

It follows that

$$q_{k-1}(f') = \frac{k-1}{k-2},$$

and, by the induction conjecture,

$$f'(x) = ce^{\alpha x}.$$

Therefore,

$$f(x) = \frac{c}{\alpha}e^{\alpha x} + \lambda,$$

where λ is a constant. Since $f \in \mathcal{L} - \mathcal{P}I$, then $\lambda = 0$, and we are done. □

The following theorem is a necessary condition for an entire function to belong to the Laguerre-Polýa class of type I in terms of the closest to zero roots.

Theorem 3.2 (T.H. Nguyen, A. Vishnyakova, [61])**.** *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, where $a_k > 0$ for all k, be an entire function. Suppose that the quotients $q_n(f)$ satisfy the following condition: $q_2(f) \leq q_3(f)$. If the function f belongs to the Laguerre–Pólya class of type I, then there exists $x_0 \in [-\frac{a_1}{a_2}, 0]$ such that $f(x_0) \leq 0$.*

The following example shows that the entire function $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0$ for all k, from the Laguerre–Pólya class of type I (without the additional condition $q_2(f) \leq q_3(f)$) can be positive in the whole segment $[-\frac{a_1}{a_2}, 0]$.

Example 3.3. For $\alpha > 0$ we consider the entire function

$$f_\alpha(x) = \left(1 + \frac{x}{\alpha+3}\right)^3 e^{\frac{\alpha x}{\alpha+3}} \in \mathcal{L} - \mathcal{P}I.$$

We have

$$f_\alpha(x) = \left(1 + \frac{3x}{\alpha+3} + \frac{3x^2}{(\alpha+3)^2} + \frac{x^3}{(\alpha+3)^3}\right) \cdot \left(1 + \frac{\alpha x}{\alpha+3} + \frac{\alpha^2 x^2}{2(\alpha+3)^2} + \frac{\alpha^3 x^3}{6(\alpha+3)^3} + \cdots\right) = 1 + x + \frac{(\alpha^2+6\alpha+6)}{2(\alpha+3)^2} x^2 + \frac{(\alpha^3+9\alpha^2+18\alpha+6)}{6(\alpha+3)^3} x^3 + \cdots =: \sum_{k=0}^{\infty} a_k(\alpha) x^k.$$

We observe that

$$q_2(f_\alpha) = \frac{2(\alpha+3)^2}{(\alpha^2+6\alpha+6)},$$

$q_2(f_\alpha) < 3$ for all $\alpha > 0$, and $\lim_{\alpha\to 0} q_2(f_\alpha) = 3$. We also observe that

$$q_3(f_\alpha) = \frac{3(\alpha^2+6\alpha+6)^2}{2(\alpha+3)(\alpha^3+9\alpha^2+18\alpha+6)},$$

$q_3(f_\alpha) < 3$ for all $\alpha > 0$, and $\lim_{\alpha\to 0} q_3(f_\alpha) = 3$. The only root (of multiplicity 3) of f_α is $x_0(\alpha) = -(\alpha+3)$, and it is easy to check that

$$|x_0(\alpha)| = \alpha + 3 > \frac{a_1(\alpha)}{a_2(\alpha)} = \frac{2(\alpha+3)^2}{(\alpha^2+6\alpha+6)}.$$

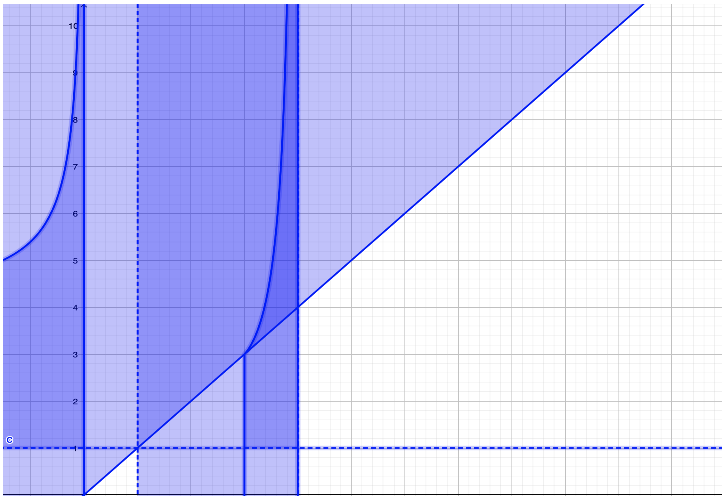

Figure 3.1: The relationship between q_2 and q_3.

The following theorem gives a necessary condition for an entire function to belong to the Laguerre–Polýa class of type I in terms of the second quotients of its Taylor coefficients q_n.

Theorem 3.4 (T.H. Nguyen, A. Vishnyakova, [61]). *If $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0$ for all k, belongs to the Laguerre–Pólya class, $q_2(f) < 4$ and $q_2(f) \leq q_3(f)$, then*

$$q_3(f) \leq \frac{-q_2(f)(2q_2(f)-9) + 2(q_2(f)-3)\sqrt{q_2(f)(q_2(f)-3)}}{q_2(f)(4-q_2(f))}. \tag{3.1}$$

The figure (3.1) illustrates the set of solutions for the inequality (3.1).

The following statement is a simple corollary of Theorem 3.4.

Corollary 3.5 (T.H. Nguyen, A. Vishnyakova, [61]). *Suppose that a real positive sequence $(a_k)_{k=0}^{\infty}$ has the following properties: $\frac{a_1^2}{a_0 a_2} < 4$ and $\frac{a_1^2}{a_0 a_2} \leq \frac{a_2^2}{a_1 a_3}$. Then if $(k! a_k)_{k=0}^{\infty} \in \mathcal{MS}$, then*

$$q_3(f) \leq \frac{-q_2(f)(2q_2(f)-9) + 2(q_2(f)-3)\sqrt{q_2(f)(q_2(f)-3)}}{q_2(f)(4-q_2(f))}.$$

In the proof of Theorem 3.2, using the Hutchinson's idea, we show that if $q_2(f) \geq 4$ (and $q_j > 1, j = 3, 4, \ldots$), then there exists a point $x_0 \in [-\frac{a_1}{a_2}, 0]$ such that $f(x_0) \leq 0$. We present the sufficient condition for the existence of such a point x_0 for the case $q_2(f) < 4$.

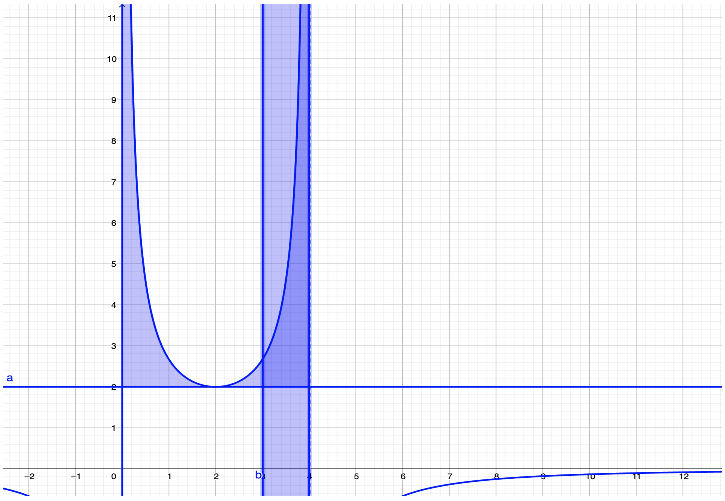

Figure 3.2: The relationship between q_2, q_3 and q_4.

Theorem 3.6 (T.H. Nguyen, A. Vishnyakova, [61]). *Let* $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0$ *for all* k, *be an entire function and* $3 \le q_2(f) < 4, q_3(f) \ge 2$, *and* $q_4(f) \ge 3$. *If*

$$q_3(f) \le \frac{8}{d(4-d)}, \tag{3.2}$$

where $d = \min(q_2(f), q_4(f))$, *then there exists* $x_0 \in [-\frac{a_1}{a_2}, 0]$ *such that* $f(x_0) \le 0$.

The figure (3.2) illustrates the set of solutions for the inequality (3.2).

3.2 Proof of Theorem 3.2

Without loss of generality, we can assume that $a_0 = a_1 = 1$, since we can consider a function $g(x) = a_0^{-1} f(a_0 a_1^{-1} x)$ instead of $f(x)$, due to the fact that such rescaling of f preserves its property of having real zeros and preserves the second quotients: $q_n(g) = q_n(f)$ for all n. During the proof we use notation p_n and q_n instead of $p_n(f)$ and $q_n(f)$.

We consider a function

$$\varphi(x) = f(-x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$$

instead of f.

To begin with, we consider the simple case when $q_2 \geq 4$ (and $q_j > 1$, where $j = 3, 4, \ldots$, by Statement 3.1, since φ belongs to the Laguerre–Pólya class of type I). We use the idea of J. I. Hutchinson, see [41].

Suppose that $x \in (1, q_2)$. Then we obtain

$$1 < x > \frac{x^2}{q_2} > \frac{x^3}{q_2^2 q_3} > \cdots > \frac{x^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k} > \cdots .$$

Thus, for $x \in (1, q_2)$ we have

$$\begin{aligned}\varphi(x) &= \left(1 - x + \frac{x^2}{q_2}\right) - \left(\frac{x^3}{q_2^2 q_3} - \frac{x^4}{q_2^3 q_3^2 q_4}\right) \\ &\quad - \left(\frac{x^5}{q_2^4 q_3^3 q_4^2 q_5} - \frac{x^6}{q_2^5 q_3^4 q_4^3 q_5^2 q_6}\right) - \cdots \\ &< \left(1 - x + \frac{x^2}{q_2}\right).\end{aligned}$$

Note that, $\varphi(0) = 1 > 0$, and for $x_0 = \sqrt{q_2}$ we obtain

$$\varphi(x_0) < 1 - \sqrt{q_2} + 1 = 2 - \sqrt{q_2} \leq 0,$$

under our assumptions that $q_2 \geq 4$. Consequently, φ has a zero $x_0 \in (0; q_2)$. Thus, for $q_2 \geq 4$, Theorem 3.2 is proved.

In the next step of our proof, we consider the case when $q_2 < 4$ and $q_2 \leq q_3$. By Lemma 2.4 from Chapter 2 (also, see [60, Lemma 2.1]), if $f(x) = \sum_{k=0}^{\infty} a_k x^k, a_k > 0$, belongs to $\mathcal{L} - \mathcal{P}I$, then $q_3(q_2 - 4) + 3 \geq 0$. In particular, if $q_3 \geq q_2$, then $q_2 \geq 3$. Therefore, we suppose that $q_2 \in [3, 4), q_3 \geq q_2$ and, applying Statement 3.1, $q_4 \geq \frac{4}{3}$ (since φ belongs to the Laguerre–Pólya class of type I).

The following lemma (from [60, Lemma 2.3]) plays a key role in the proof of Theorem 3.2. It is a generalisation of Lemma 2.6 from Chapter 2. Besides, the first version of this lemma for the simplest case $a = b = c \geq 3$ was proved in [47] by O. Katkova, T. Lobova and A. Vishnyakova (see [47, Lemma 1]).

Lemma 3.7 (T.H. Nguyen, A. Vishnyakova, [61]). *Let*

$$P(x) = 1 - x + \frac{x^2}{a} - \frac{x^3}{a^2 b} + \frac{x^4}{a^3 b^2 c}$$

be a polynomial, where $3 \leq a < 4$, $b \geq a$, *and* $c \geq 4/3$. *Then*

$$\min_{0 \leq \theta \leq 2\pi} |P(ae^{i\theta})| \geq \frac{a}{b^2 c}.$$

Proof. By direct calculation, we have

$$\begin{aligned}
|P(ae^{i\theta})|^2 &= (1 - a\cos\theta + a\cos 2\theta - \frac{a}{b}\cos 3\theta + \frac{a}{b^2c}\cos 4\theta)^2 \\
&\quad + (-a\sin\theta + a\sin 2\theta - \frac{a}{b}\sin 3\theta + \frac{a}{b^2c}\sin 4\theta)^2 \\
&= 1 + 2a^2 + \frac{a^2}{b^2} + \frac{a^2}{b^4c^2} - 2a\cos\theta + 2a\cos 2\theta - 2\frac{a}{b}\cos 3\theta \\
&\quad + 2\frac{a}{b^2c}\cos 4\theta - 2a^2\cos\theta + 2\frac{a^2}{b}\cos 2\theta - 2\frac{a^2}{b^2c}\cos 3\theta \\
&\quad - 2\frac{a^2}{b}\cos\theta + 2\frac{a^2}{b^2c}\cos 2\theta - 2\frac{a^2}{b^3c}\cos\theta.
\end{aligned}$$

Set $t := \cos\theta, t \in [-1, 1]$. Using that

$$\begin{aligned}
\cos 2\theta &= 2t^2 - 1, \\
\cos 3\theta &= 4t^3 - 3t, \\
\cos 4\theta &= 8t^4 - 8t^2 + 1,
\end{aligned}$$

we get

$$\begin{aligned}
|P(ae^{i\theta})|^2 &= \frac{16a}{b^2c}t^4 + \left(-\frac{8a}{b} - \frac{8a^2}{b^2c}\right)t^3 + \left(4a - \frac{16a}{b^2c} + \frac{4a^2}{b} + \frac{4a^2}{b^2c}\right)t^2 \\
&\quad + \left(-2a + \frac{6a}{b} - 2a^2 + \frac{6a^2}{b^2c} - \frac{2a^2}{b} - \frac{2a^2}{b^3c}\right)t \\
&\quad + \left(1 + 2a^2 + \frac{a^2}{b^2} + \frac{a^2}{b^4c^2} - 2a + \frac{2a}{b^2c} - \frac{2a^2}{b} - \frac{2a^2}{b^2c}\right).
\end{aligned}$$

Further, we want to show that

$$\min_{0\le\theta\le 2\pi} |P(ae^{i\theta})|^2 \ge \frac{a^2}{b^4c^2},$$

or to prove the inequality

$$\min_{0\le\theta\le 2\pi} |P(ae^{i\theta})|^2 - \frac{a^2}{b^4c^2} \ge 0.$$

Using the last expression, we see that the inequality we want to get is equivalent to the following: for all $t \in [-1, 1]$ the next inequality holds:

$$\begin{aligned}
&\frac{16a}{b^2c}t^4 - \frac{8a}{b}\Big(1 + \frac{a}{bc}\Big)t^3 + 4a\Big(1 - \frac{4}{b^2c} + \frac{a}{b} + \frac{a}{b^2c}\Big)t^2 \\
&-2a\Big(1 - \frac{3}{b} + a - \frac{3a}{b^2c} + \frac{a}{b} + \frac{a}{b^3c}\Big)t + \Big(1 + 2a^2 + \frac{a^2}{b^2} - 2a + \frac{2a}{b^2c} - \frac{2a^2}{b} - \frac{2a^2}{b^2c}\Big) \ge 0.
\end{aligned}$$

Set $y := 2t$, $y \in [-2, 2]$. We rewrite the last inequality in the form

$$\frac{a}{b^2c}y^4 - \frac{a}{b}\left(1 + \frac{a}{bc}\right)y^3 + a\left(1 - \frac{4}{b^2c} + \frac{a}{b} + \frac{a}{b^2c}\right)y^2 - a\left(1 - \frac{3}{b}\right.$$
$$\left.+a - \frac{3a}{b^2c} + \frac{a}{b} + \frac{a}{b^3c}\right)y + \left(1 + 2a^2 + \frac{a^2}{b^2} - 2a + \frac{2a}{b^2c} - \frac{2a^2}{b} - \frac{2a^2}{b^2c}\right) \geq 0.$$

Let us observe the coefficients of the polynomial on the left hand side: the coefficient of y^4 is $\frac{a}{b^2c} > 0$. Since

$$\frac{a}{b}\left(1 + \frac{a}{bc}\right) > 0,$$

the coefficient of y^3 is negative. It is easy to show that the other coefficients are also sign-changing. For y^2, since a, b and c are positive, it follows that $b^2c > 4$. Then we have

$$1 - \frac{4}{b^2c} > 0.$$

Thus, the coefficient of y^2 is

$$1 + \frac{a}{b} + \frac{a}{b^2c} - \frac{4}{b^2c} = \left(1 - \frac{4}{b^2c}\right) + \frac{a}{b} + \frac{a}{b^2c} > 0.$$

As for the coefficients of y, we have

$$1 + a - \frac{3}{b} > 0 \Leftrightarrow ab + b > 3,$$

and

$$\frac{a}{b} - \frac{3a}{b^2c} > 0 \Leftrightarrow \frac{a}{b} > \frac{3a}{b^2c} \Leftrightarrow abc > 3a \Leftrightarrow bc > 3,$$

since $b \geq 3$ and $c \geq 4/3 > 1$. Therefore, it follows from the inequalities above that

$$1 + a + \frac{a}{b} + \frac{a}{b^3c} - \frac{3}{b} - \frac{3a}{b^2c} = \left(1 + a - \frac{3}{b}\right)$$
$$+ \left(\frac{a}{b} - \frac{3a}{b^2c}\right) + \frac{a}{b^3c} > 0.$$

Finally, we have

$$1 + 2a^2 + \frac{a^2}{b^2} - 2a - 2\frac{a^2}{b} - 2\frac{a^2}{b^2c} + 2\frac{a}{b^2c}$$
$$= (1 + a^2 - 2a) + (a^2 - 3\frac{a^2}{b}) + (\frac{a^2}{b} - 2\frac{a^2}{b^2c}) + \frac{a^2}{b^2} + 2\frac{a}{b^2c} > 0,$$

since

$$1 - 2a + a^2 = (a+1)^2 \geq 0,$$

which, obviously, holds for any a,

$$a^2 - 3\frac{a^2}{b} \geq 0 \Leftrightarrow a^2 b \geq 3a^2 \Leftrightarrow b \geq 3,$$

which is true under our assumptions that $3 \leq a \leq b$, and

$$\frac{a^2}{b} - 2\frac{a^2}{b^2 c} > 0 \Leftrightarrow 1 > \frac{2}{bc},$$

which holds by our assumptions.

Consequently, the inequality we need holds for any $y \in [-2, 0]$, and it is sufficient to prove it for $y \in [0, 2]$. Multiplying our inequality by $\frac{b^2 c}{a}$, we get

$$y^4 - (bc + a)y^3 + (b^2 c + abc + a - 4)y^2 - (b^2 c + ab^2 c + abc + \frac{a}{b} - 3bc - 3a)y$$

$$+(\frac{b^2 c}{a} + 2ab^2 c + ac - 2b^2 c - 2abc - 2a + 2) =: \psi(y),$$

and we want to prove that $\psi(y) \geq 0$ for all $y \in [0, 2]$.

At first, we consider the case $b \geq a \in [3, 4), c \geq \frac{5}{3}$. Let

$$\chi(y) := \psi(y) - \frac{1}{b}(b-a)y,$$

whence $\chi(y) \leq \psi(y)$ for all $y \in [0, 2]$. It is sufficient to prove that $\chi(y) \geq 0$ for all $y \in [0, 2]$. We have

$$\chi(0) = \psi(0) = \frac{b^2 c}{a} + 2ab^2 c + ac - 2b^2 c - 2abc - 2a + 2 \geq 0,$$

as it was previously shown. Moreover, note that

$$\chi(2) = \psi(2) - \frac{2}{b}(b-a) \geq 0,$$

since

$$\begin{aligned}
\psi(2) &= -2bc - 2\frac{a}{b} + \frac{b^2 c}{a} + ac + 2 \\
&= \frac{1}{b}\left(2(b-a) + \frac{b^2 c}{a}(b-a) - bc(b-a)\right) \\
&= \frac{1}{b}(b-a)\left(2 + \frac{bc}{a}(b-a)\right) \geq \frac{2}{b}(b-a) \geq 0.
\end{aligned}$$

Now we consider the following function

$$\nu(y) := \chi''(y) = \psi''(y) = 12y^2 - 6(bc+a)y + 2(b^2c + abc + a - 4).$$

The vertex point of this parabola is

$$y_v = \frac{bc+a}{4} \geq 2$$

for $b \geq a \geq 3, c \geq \frac{5}{3}$. Therefore,

$$\nu(y) \geq \nu(2) \text{ for all } y \in [0,2].$$

We have

$$\begin{aligned}\nu(2) &= 48 - 12(bc+a) + 2b^2c + 2abc + 2a - 8\\ &= 40 - 12bc - 10a + 2b^2c + 2abc\\ &= 2bc(a-3) + 2bc(b-3) + 10(4-a) \geq 0,\end{aligned}$$

under our assumption. Thus, $\chi'(y)$ is increasing for $y \in [0,2]$. Besides,

$$\begin{aligned}\chi'(y) = {} & 4y^3 - 3(bc+a)y^2 + 2(b^2c + abc + a - 4)y\\ & - (b^2c + ab^2c + abc + \frac{a}{b} - 3bc - 3a) - \frac{1}{b}(b-a).\end{aligned}$$

We want to prove that $\chi'(y) \leq 0$ for all $y \in [0,2]$. Since χ' is an increasing function, it is sufficient to show that $\chi'(2) \leq 0$. We have

$$\begin{aligned}\chi'(2) &= 32 - 12(bc+a) + 4(b^2c + abc + a - 4)\\ &\quad - (b^2c + ab^2c + abc + \frac{a}{b} - 3bc - 3a) - 1 + \frac{a}{b}\\ &= 15 - 9bc - 5a + 3b^2c + 3abc - ab^2c\\ &= (15 - 5a) - bc(a-3)(b-3)\\ &= 5(3-a) - bc(a-3)(b-3) \leq 0,\end{aligned}$$

under our assumption that $b \geq a \geq 3$.

Thus, we have proved that χ is decreasing for $y \in [0,2]$. Hence, the fact that $\chi(y) \geq 0$ for all $y \in [0,2]$, is equivalent to $\chi(2) \geq 0$, that was proved above. We obtain $\psi(y) \geq \chi(y) \geq 0$ for all $y \in [0,2]$. So, for the case $b \geq a \in [3,4), c \geq \frac{5}{3}$ Lemma 3.7 is proved.

It remains to consider the case $3 \leq a < 4, b \geq a, \frac{4}{3} \leq c < \frac{5}{3}$. We want to prove that

$$\psi''(y) = \chi''(y) = 12y^2 - 6(bc+a)y + 2(b^2c + abc + a - 4) \geq 0$$

for all $y \in [0, 2]$. If the vertex point of this parabola $y_v = \frac{bc+a}{4}$ is not less than 2, then we have proved that

$$\psi''(y) \geq \psi''(2) \geq 0.$$

Suppose that

$$0 < y_v = \frac{bc+a}{4} < 2.$$

Then we want to show that $\psi''(y_v) \geq 0$. We have

$$\psi''(y_v) = 12\left(\frac{bc+a}{4}\right)^2 - 6(bc+a)\left(\frac{bc+a}{4}\right) + 2(b^2c + abc + a - 4) \quad (3.3)$$
$$= -\frac{1}{4}\Big(3b^2c^2 - (2ab+8b^2)c + (3a^2 - 8a + 32)\Big) =: -\frac{1}{4}h_{a,b}(c).$$

We want to show that, under our assumption, $h_{a,b}(c) \leq 0$. We can observe that

$$3a^2 - 8a + 32 > 0$$

for all a, and the expression $3a^2 - 8a + 32$ is increasing for $a \in [3, 4)$, which follows that

$$3a^2 - 8a + 32 \leq 3 \cdot 16 - 8 \cdot 4 + 32 = 48.$$

Thus, to prove that the function $h_{a,b}(c)$ from (3.3) is negative, it is sufficient to show that

$$3b^2c^2 - (2ab + 8b^2)c + 48 \leq 0.$$

Since $a \geq 3$, it is sufficient to prove that

$$\eta_b(c) := 3b^2c^2 - (6b + 8b^2)c + 48 \leq 0.$$

The vertex point of this parabola is

$$c_v = \frac{6b + 8b^2}{6b^2} = \frac{1}{b} + \frac{4}{3} \in \left(\frac{4}{3}, \frac{5}{3}\right)$$

for $b \geq 3$. Thus, to prove that $\eta_b(c) \leq 0$ for all $c \in \left[\frac{4}{3}, \frac{5}{3}\right)$, we need to show that $\eta_b(\frac{4}{3}) \leq 0$ and $\eta_b(\frac{5}{3}) \leq 0$. We have

$$\eta_b(\frac{4}{3}) = -\frac{16}{3}b^2 - 8b + 48 \leq -\frac{16}{3} \cdot 9 - 8 \cdot 3 + 48 = -24 < 0,$$

and

$$\eta_b(\frac{5}{3}) = -5b^2 - 10b + 48 \leq -45 - 30 + 48 = -27 < 0.$$

Consequently, we have proved that $\psi''(y) \geq 0$ for all $y \in [0, 2]$. The rest of the proof is the same as in the previous case. Lemma 3.7 is proved. □

Further, we need the following technical lemma to obtain the estimation of $R_5(x, \varphi)$ from above.

Lemma 3.8 (T.H. Nguyen, A. Vishnyakova, [61])**.** *If $q_j > 1$ for all $j \geq 2$, let $R_5(x, \varphi) := \sum_{k=5}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k}$. Then*

$$\max_{0 \leq \theta \leq 2\pi} |R_5(q_2 e^{i\theta}, \varphi)| \leq \frac{q_2 q_6}{q_3^3 q_4^2 q_5 q_6 - q_3^2 q_4}.$$

Proof. We have

$$\begin{aligned}
|R_5(q_2 e^{i\theta}, \varphi)| &\leq \sum_{k=5}^{\infty} \frac{q_2^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k} = \sum_{k=5}^{\infty} \frac{q_2}{q_3^{k-2} q_4^{k-3} \cdot \ldots \cdot q_k} \\
&= \frac{q_2}{q_3^3 q_4^2 q_5} + \frac{q_2}{q_3^4 q_4^3 q_5^2 q_6} + \cdots + \frac{q_2}{q_3^{k-2} \cdot \ldots \cdot q_k} + \cdots \\
&= \frac{q_2}{q_3^3 q_4^2 q_5}(1 + \frac{1}{q_3 q_4 q_5 q_6} + \frac{1}{q_3^2 q_4^2 q_5^2 q_6^2 q_7} + \cdots) \\
&\leq \frac{q_2}{q_3^3 q_4^2 q_5} \cdot \frac{1}{1 - \frac{1}{q_3 q_4 q_5 q_6}} = \frac{q_2 q_6}{q_3^3 q_4^2 q_5 q_6 - q_3^2 q_4}.
\end{aligned}$$

□

Let us check that

$$\frac{q_2}{q_3^2 q_4} > \frac{q_2 q_6}{q_3^3 q_4^2 q_5 q_6 - q_3^2 q_4},$$

which is equivalent to

$$q_3 q_4 q_5 q_6 > q_6 + 1.$$

The last inequality obviously holds under our assumption. Therefore, according to the Rouché's theorem, the functions $S_4(x, \varphi)$ and $\varphi(x)$ have the same number of zeros inside the disk $\{x : |x| < q_2\}$ counting multiplicities.

It remains to prove that $S_4(x, \varphi)$ has zeros in the disk $\{x : |x| < q_2\}$. We apply Grace's theorem about the complex zeros of apolar polynomials (see Theorem G in Chapter 1).

We consider

$$S_4(x,\varphi) = \binom{4}{0} + \binom{4}{1}(-\frac{1}{4})x + \binom{4}{2}\frac{1}{6q_2}x^2$$
$$+\binom{4}{3}(-\frac{1}{4q_2^2q_3})x^3 + \binom{4}{4}\frac{1}{q_2^3q_3^2q_4}x^4.$$

Let

$$Q(x) = \binom{4}{2}b_2x^2 + \binom{4}{3}b_3x^3 + \binom{4}{4}x^4.$$

Then the condition for $S_4(x,\varphi)$ and $Q(x)$ to be apolar is the following:

$$\binom{4}{0} - \binom{4}{1}\left(-\frac{1}{4}\right)b_3 + \binom{4}{2}\frac{1}{6q_2}b_2 = 0.$$

If $q_2 \geq 3$, then all the zeros of Q are in the disk $\{x : |x| \leq q_2\}$. Therefore, by Grace's theorem, we obtain that $S_4(x,\varphi)$ has at least one zero in the disk $\{x : |x| \leq q_2\}$.

Thus, $S_4(x,\varphi)$ has at least one zero in the disk $\{x : |x| \leq q_2\}$, and, by Lemma 3.8 applying to $S_4(x,\varphi)$,$S_4(x,\varphi)$ does not have zeros in $\{x : |x| = q_2\}$. Hence, the polynomial $S_4(x,\varphi)$ has at least one zero in the open disk $\{x : |x| < q_2\}$. By Rouché's theorem, the functions $S_4(x,\varphi)$ and $\varphi(x)$ have the same number of zeros inside the disk $\{x : |x| < q_2\}$, whence φ has at least one zero in the open disk $\{x : |x| < q_2\}$. If φ is in the Laguerre–Pólya class, this zero must be real, and, since the coefficients of φ are sign-changing, this zero belongs to $(0, q_2)$.

Theorem 3.2 is proved.

3.3 Proof of Theorem 3.4

In the proof of Theorem 3.2, using the Hutchinson's idea (see [41]), we show that if $q_2(f) \geq 4$ (and $q_j > 1, j = 3, 4, \ldots$), then there exists a point $x_0 \in [-\frac{a_1}{a_2}, 0]$ such that $f(x_0) \leq 0$. In Theorem 3.4, we prove the sufficient condition for the existence of the point x_0 for the case when $q_2(f) < 4$.

Proof. According to theorem 3.2, if $\varphi(x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1}q_3^{k-2}\cdot\ldots\cdot q_{k-1}^2 q_k} \in \mathcal{L}-\mathcal{PI}$, and $q_3 \geq q_2 \geq 3$, then there exists $x_0 \in (0, q_2)$ such that $\varphi(x_0) \leq 0$.

For $x \in [0,1]$ we have

$$1 \geq x > \frac{x^2}{q_2} > \frac{x^3}{q_2^2q_3} > \frac{x^4}{q_2^3q_3^2q_4} > \cdots,$$

whence

$$\varphi(x) > 0 \quad \text{for all} \quad x \in [0,1]. \tag{3.4}$$

Suppose that $x \in (1, q_2]$. Then we obtain

$$1 < x \geq \frac{x^2}{q_2} > \frac{x^3}{q_2^2 q_3} > \cdots > \frac{x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k} > \cdots \tag{3.5}$$

For an arbitrary $m \in \mathbb{N}$ we have

$$\varphi(x) = S_{2m+1}(x, \varphi) + R_{2m+2}(x, \varphi).$$

By (3.5) and the Leibniz criterion for alternating series, we obtain that $R_{2m+2}(x, \varphi) > 0$ for all $x \in (1, q_2]$, or

$$\varphi(x) > S_{2m+1}(x, \varphi) \quad \text{for all} \quad x \in (1, q_2], \; m \in \mathbb{N}. \tag{3.6}$$

Analogously,

$$\varphi(x) \leq S_{2m}(x, \varphi) \quad \text{for all} \quad x \in (1, q_2], \; m \in \mathbb{N}. \tag{3.7}$$

As the next step of our proof, we consider

$$S_3(x, \varphi) = 1 - x + \frac{x^2}{q_2} - \frac{x^3}{q_2^2 q_3}.$$

For the sake of brevity, set $a := q_2, b := q_3$. Then we obtain

$$S_3(x) := 1 - x + \frac{x^2}{a} - \frac{x^3}{a^2 b}, \; b \geq a \geq 3.$$

Therefore, if there exists $x_0 \in (0, a)$ such that $\varphi(x_0) \leq 0$, then $S_3(x_0) \leq 0$.

First, we find the roots of the derivative. We have

$$S_3'(x) = -\frac{1}{a^2 b}(3x^2 - 2abx + a^2 b).$$

We consider the discriminant of the quadratic polynomial S_3' :

$$\frac{D_{S_3'}}{4} = a^2 b^2 - 3a^2 b = a^2 b(b-3).$$

Under our assumption, $b \geq 3$, so $\frac{D_{S_3'}}{4} \geq 0$. Thus, the roots of S_3' are

$$x_1 = \frac{ab - a\sqrt{b(b-3)}}{3}$$

and

$$x_2 = \frac{ab + a\sqrt{b(b-3)}}{3}.$$

It is easy to check that if $b \geq a \geq 3$, then the following conditions hold: $x_1 \in (0, a]$ and $x_2 \geq a$. Therefore, we can conclude that x_1 is the minimal point of $S_3(x)$ in the interval $(0, a)$. Now we check if $S_3(x_1) \leq 0$.

After substituting x_1 into $S_3(x)$, we obtain the following expression

$$S_3(x_1) = 1 - \frac{ab - a\sqrt{b(b-3)}}{3} + \frac{(ab - a\sqrt{b(b-3)})^2}{9a} - \frac{(ab - a\sqrt{b(b-3)})^3}{27a^2b}.$$

We want $S_3(x_1) \leq 0$, or, equivalently,

$$27 - 9ab + 9a\sqrt{b(b-3)} + 3ab^2 - 6ab\sqrt{b(b-3)} + 3ab(b-3) - ab^2$$

$$+3ab\sqrt{b(b-3)} - 3ab(b-3) + a(b-3)\sqrt{b(b-3)} \leq 0.$$

We rewrite and get

$$\sqrt{b(b-3)}(6a - 2ab) + (27 - 9ab + 2ab^2) \leq 0. \tag{3.8}$$

We observe that

$$6a - 2ab = 2a(3 - b) \leq 0,$$

since $b \geq 3$, under our assumption. Thus,

$$\sqrt{b(b-3)}(6a - 2ab) \leq 0.$$

Now we consider the expression

$$27 - 9ab + 2ab^2 =: y(b)$$

as a quadratic function of b. Its discriminant

$$D = 81a^2 - 216a = 27a(3a - 8)$$

is positive under our assumption that $a \geq 3$. The roots of $y(b)$ are

$$b_1 = \frac{9a - 3\sqrt{3a(3a-8)}}{4a} \text{ and } b_2 = \frac{9a + 3\sqrt{3a(3a-8)}}{4a}.$$

It is easy to check that

$$b_1 < \frac{9}{4} < 3 \leq b_2.$$

For

$$b \in [3, \frac{9a + 3\sqrt{3a(3a-8)}}{4a}],$$

we have $y(b) \leq 0$, and (3.8) is fulfilled. Further, we consider the case when

$$b > \frac{9a + 3\sqrt{3a(3a-8)}}{4a}.$$

Then, (3.8) is equivalent to

$$27 - 9ab + 2ab^2 \leq \sqrt{b(b-3)}(2ab - 6a),$$

or

$$a^2b^2 - 4a^2b - 4ab^2 + 18ab - 27 \geq 0.$$

We rewrite the inequality above in the following way:

$$b^2a(4-a) + 2a(2a-9)b + 27 \leq 0. \tag{3.9}$$

We note that the coefficient of b^2 is positive, since, under our assumption, $a < 4$. Then, its discriminant

$$\begin{aligned} D/4 &= a^2(2a-9)^2 - 27a(4-a) \\ &= 4a^4 - 36a^3 + 108a^2 - 108a \\ &= 4a(a-3)^3 \geq 0. \end{aligned}$$

We obtain the roots of the left-hand side of (3.9):

$$\beta_1 = \frac{-a(2a-9) - 2(a-3)\sqrt{a(a-3)}}{a(4-a)},$$

$$\beta_2 = \frac{-a(2a-9) + 2(a-3)\sqrt{a(a-3)}}{a(4-a)}.$$

Therefore, b should be in the interval (β_1, β_2) for (3.9) to be fulfilled, which is equivalent to the inequality below:

$$\frac{-a(2a-9) - 2(a-3)\sqrt{a(a-3)}}{a(4-a)} < b < \frac{-a(2a-9) + 2(a-3)\sqrt{a(a-3)}}{a(4-a)}. \tag{3.10}$$

Next, we show that the following inequality is fulfilled under our assumption that $b \in [3, \frac{9a+3\sqrt{3a(3a-8)}}{4a}]$:

$$\frac{-a(2a-9)-2(a-3)\sqrt{a(a-3)}}{a(4-a)} \leq \frac{9a+3\sqrt{3a(3a-8)}}{4a}.$$

It is equivalent to

$$a^2 < 8(a-3)\sqrt{a(a-3)} + 3(4-a)\sqrt{3a(3a-8)},$$

or

$$a^3 < 64(a-3)^3 + 27(4-a)^2(3a-8) + 48(a-3)(4-a)\sqrt{3(a-3)(3a-8)}.$$

After straightforward calculation we get

$$3(a^3 - 10a^2 + 33a - 36) + (a-3)(4-a)\sqrt{3(a-3)(3a-8)} \geq 0,$$

or

$$3(a-3)^2(a-4) + (a-3)(4-a)\sqrt{3(a-3)(3a-8)} \geq 0.$$

Divided by $(a-3)(4-a)$, the inequality takes the form:

$$\sqrt{3(a-3)(3a-8)} \geq 3(a-3).$$

It is simple to verify that the equation above is fulfilled for any $a \geq 3$. Consequently, we obtain the following condition for b :

$$3 \leq b \leq \frac{-a(2a-9)+2(a-3)\sqrt{a(a-3)}}{a(4-a)}. \tag{3.11}$$

Theorem 3.4 is proved. □

Remark 3.9. From Lemma 2.6, we have $b(a-4)+3 \geq 0$, or $b \leq \frac{3}{4-a}$. It is easy to check that the following inequality holds:

$$b \leq \frac{-a(2a-9)+2(a-3)\sqrt{a(a-3)}}{a(4-a)} \leq \frac{3}{4-a}. \tag{3.12}$$

Let us consider the right hand side of the inequality (3.12). Since under our assumptions $3 \leq a < 4$, we multiply it by $a(4-a)$ and obtain

$$-a(2a-9)+2(a-3)\sqrt{a(a-3)} \leq 3a,$$

or, equivalently,

$$2a(a-3) \geq 2(a-3)\sqrt{a(a-3)}.$$

As $a > 3$, we can divide the inequality above by $2(a-3)$ and get

$$a \geq \sqrt{a(a-3)}.$$

After squaring both sides of the inequality, we have

$$a^2 \geq a^2 - 3a,$$

which is valid under our assumptions on a.

3.4 Proof of Theorem 3.6

Proof. Note that, for every $x_0 \in (0, q_2)$ and for any $n \in \mathbb{N} : \varphi(x_0) \leq S_{2n}(x_0, \varphi)$ (see (3.7)). Hence, if there exists $x_0 \in (0, q_2)$ such that $S_4(x_0, \varphi) \leq 0$, then $\varphi(x_0) \leq 0$.

Let

$$P(x) := 1 - x + \frac{x^2}{a} - \frac{x^3}{a^2 b} + \frac{x^4}{a^3 b^2 c}, \quad a \geq 3, b \geq 2, c \geq 3.$$

First, if $a \leq c$, then

$$P(x) \leq 1 - x + \frac{x^2}{a} - \frac{x^3}{a^2 b} + \frac{x^4}{a^4 b^2} =: \widetilde{P}(x).$$

Thus, if there exists $x_0 \in (0, a)$ such that $\widetilde{P}(x_0) \leq 0$, then $P(x_0) \leq 0$.

Next, if $a \geq c$, then we consider

$$P(x) = (1 - x) + \left(\frac{x^2}{a} - \frac{x^3}{a^2 b}\right) + \frac{x^4}{a^3 b^2 c}.$$

Since

$$\left(\frac{x^2}{a} - \frac{x^3}{a^2 b}\right)'_a = -\frac{x^2}{a^2} + \frac{2x^3}{a^3 b}$$

is negative for any $x \in (0, a)$, it follows that $P(x)$ decreases monotonically for any $x \in (0, a)$. Therefore,

$$P(x) \leq 1 - x + \frac{x^2}{c} - \frac{x^3}{c^2 b} + \frac{x^4}{c^4 b^2} =: \widetilde{\widetilde{P}}(x).$$

Analogously to the previous case, if there exists $x_0 \in (0, a)$ such that $\widetilde{\widetilde{P}}(x_0) \leq 0$, then $P(x_0) \leq 0$.

Thus, we can set $d := \min(a, c)$, and consider the following polynomial

$$T(x) = 1 - x + \frac{x^2}{d} - \frac{x^3}{d^2 b} + \frac{x^4}{d^4 b^2}.$$

We substitute $x = d\sqrt{b}y$, then $y = \frac{x}{d\sqrt{b}}, y \in (0, \frac{1}{\sqrt{b}})$. We have

$$Q(y) := T(d\sqrt{b}y) = y^2 \left(\left(\frac{1}{y^2} + y^2 \right) - d\sqrt{b} \left(\frac{1}{y} + y \right) + db \right).$$

Let us set $w(y) = y + y^{-1}$, then

$$\left(\left(\frac{1}{y^2} + y^2 \right) - d\sqrt{b} \left(\frac{1}{y} + y \right) + db \right) = w^2 - d\sqrt{b}w + db - 2.$$

If we find a point $w_0 \in \left(\sqrt{b} + \frac{1}{\sqrt{b}}, \infty \right)$, such that

$$w_0^2 - d\sqrt{b}w_0 + db - 2 \leq 0,$$

then we find $y_0, 0 < y_0 < \frac{1}{\sqrt{b}} < 1$, such that $Q(y_0) \leq 0$.

The vertex of the quadratic function $w^2 - d\sqrt{b}w + db - 2$ is in the point $w_v = \frac{d\sqrt{b}}{2}$, and, by our assumption,

$$\frac{d\sqrt{b}}{2} \geq \sqrt{b} + \frac{1}{\sqrt{b}}.$$

Therefore, the existence of a point $w_0 \in (\sqrt{b} + \frac{1}{\sqrt{b}}, \infty)$, such that $w_0^2 - d\sqrt{b}w_0 + db - 2 \leq 0$, is equivalent to the condition that its discriminant is non-negative:

$$D = d^2 b - 4db + 8 \geq 0.$$

Then, $(4 - d)db \leq 8$, or (since $3 \leq d < 4$)

$$b \leq \frac{8}{d(4 - d)}.$$

Theorem 3.6 is proved. □

To summarise this chapter, we studied the zero location of entire functions $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with positive coefficients, such that their second quotients of Taylor coefficients $q_k(f) = \frac{a_{k-1}^2}{a_{k-2} a_k}, k = 2, 3, \ldots$ satisfy the condition $q_2(f) \leq q_3(f)$. We have shown that if f belongs to the Laguerre–Pólya class of type I, then it has at least one zero in the segment $\left[-\frac{a_1}{a_2}, 0 \right]$. Moreover, we have obtained necessary and sufficient conditions for existence of such a zero in terms of the second quotients q_k, for $k = 2, 3, 4$.

Chapter 4

Entire functions from the Laguerre–Pólya I class having the increasing second quotients of Taylor coefficients

We show that if $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0$, is an entire function such that the sequence of its second quotients of Taylor coefficients is non-decreasing in k and $q_2(f) \geq 2\sqrt[3]{2}$, then all but a finite number of zeros of f are real and simple. We also present a criterion in terms of the closest to zero roots for such a function to have only real zeros (in other words, for belonging to the Laguerre–Pólya class of type I) under additional assumption on the regularity of increasing for the sequence $q_k(f)$.

Theorem 4.1 (T.H. Nguyen, A. Vishnyakova, [64]). *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0, k = 0, 1, 2, \ldots$, be an entire function such that*

$$2\sqrt[3]{2} \leq q_2(f) \leq q_3(f) \leq q_4(f) \leq \ldots$$

($2\sqrt[3]{2} \approx 2.51984$). Then all but a finite number of zeros of f are real and simple.

In connection with the theorem above, we formulate the following conjecture.

Conjecture 4.2 (T.H. Nguyen, A. Vishnyakova, [64]). *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0, k = 0, 1, 2, \ldots$, be an entire function such that $1 < q_2(f) \leq q_3(f) \leq q_4(f) \leq \ldots$ Then all but a finite number of zeros of f are real and simple.*

As the second result of this chapter, we present the following criterion for belonging to the Laguerre–Pólya class of type I for real entire functions with the regularly non-decreasing sequence of second quotients of Taylor coefficients.

In order to clarify the statement of the next theorem, we apply Lemma 2.4 from Chapter 2 (also, see Lemma 1.2 from [61], cf. Lemma 2.1 from [60]). Thus, if $f(x) = \sum_{k=0}^{\infty} a_k x^k, a_k > 0$, belongs to $\mathcal{L} - \mathcal{P}I$, then $q_3(f)(q_2(f) - 4) + 3 \geq 0$. In particular, if $q_3(f) \geq q_2(f)$, then $q_2(f) \geq 3$. Hence, if we investigate whether a real entire function with the non-decreasing sequence of second quotients of Taylor coefficients belongs to the Laguerre–Pólya class of type I, then the necessary condition is $q_2(f) \geq 3$. Our main result of this chapter is the following theorem.

Theorem 4.3 (T.H. Nguyen, A. Vishnyakova, [64]). *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, where $a_k > 0$ for all k, be an entire function. Suppose that:*

1. $3 \leq q_2(f) \leq q_3(f) \leq q_4(f) \leq \ldots$;
2. *In the case when there is an integer $j_0 \geq 2$ such that $q_{j_0}(f) < 4$, and $q_{j_0+1}(f) \geq 4$, one of the following conditions holds true:*

   (i) $q_{j_0-1}(f)/q_{j_0+1}(f) \geq 0.525$;

   (ii) $q_{j_0}(f) \geq 3.4303$.

Then $f \in \mathcal{L} - \mathcal{P}I$ if and only if there exists $x_0 \in [-a_1/a_2, 0]$ such that $f(x_0) \leq 0$.

Remark 4.4. Unfortunately, at the moment we do not know whether the additional assumptions in the theorem above are essential.

4.1 Proof of Theorem 4.1

Proof. To prove Theorem 4.1 we need the following lemma.

Lemma 4.5 (T.H. Nguyen, A. Vishnyakova, [64]). *Let $f(x) = \sum_{k=0}^{\infty} (-1)^k a_k x^k$, where $a_k > 0$ for all k, be an entire function such that $2\sqrt[3]{2} \leq q_2(f) \leq q_3(f) \leq q_3(f) \leq \ldots$ For an arbitrary integer $j \geq 2$ we define*

$$\rho_j(f) := q_2(f) q_3(f) \cdot \ldots \cdot q_j(f) \sqrt{q_{j+1}(f)}.$$

Then, for all sufficiently large j, the function f has exactly j zeros on the disk $\{z \ : \ |z| < \rho_j(f)\}$ counting multiplicities.

Proof. For simplicity, we will write q_j instead of $q_j(f)$ and ρ_j instead of $\rho_j(f)$. We have

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k},$$

where the sequence $q_2, q_3, \ldots$ is non-decreasing. We now dissect the above sum as

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k} = \left(\sum_{k=0}^{j-3} + \sum_{k=j-2}^{j+2} + \sum_{k=j+3}^{\infty} \right) =: \Sigma_{1,j}(x) + g_j(x) + \Sigma_{2,j}(x).$$

We represent the term $\frac{(-1)^{j+2} x^{j+2}}{q_2^{j+1} q_3^{j} \cdot \ldots \cdot q_{j+1}^2 q_{j+2}}$ in the form

$$\frac{(-1)^{j+2} x^{j+2}}{q_2^{j+1} q_3^{j} \cdot \ldots \cdot q_{j-1}^4 q_j^3 q_{j+1}^2 q_{j+2}} = \frac{(-1)^{j+2} x^{j+2}}{q_2^{j+1} q_3^{j} \cdot \ldots \cdot q_{j-2}^5 q_{j-1}^4 q_j^4 q_{j+1}^2}$$
$$+ \frac{(-1)^{j+2} x^{j+2}}{q_2^{j+1} q_3^{j} \cdot \ldots \cdot q_{j-1}^4 q_j^3 q_{j+1}^2 q_{j+2}} - \frac{(-1)^{j+2} x^{j+2}}{q_2^{j+1} q_3^{j} \cdot \ldots \cdot q_{j-2}^5 q_{j-1}^4 q_j^4 q_{j+1}^2}.$$

Hence,

$$g_j(x) = \left(\sum_{k=j-2}^{j+1} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k} + \frac{(-1)^{j+2} x^{j+2}}{q_2^{j+1} q_3^{j} \cdot \ldots \cdot q_{j-2}^5 q_{j-1}^4 q_j^4 q_{j+1}^2} \right) \tag{4.1}$$
$$+ \frac{(-1)^{j+2} x^{j+2}}{q_2^{j-3} q_3^{j-4} \cdot \ldots \cdot q_{j-2}} \left(\frac{1}{q_2^4 q_3^4 \cdot \ldots \cdot q_{j-1}^4 q_j^3 q_{j+1}^2 q_{j+2}} - \frac{1}{q_2^4 q_3^4 \cdot \ldots \cdot q_{j-1}^4 q_j^4 q_{j+1}^2} \right)$$
$$=: \widetilde{g}_j(x) + \xi_j(x).$$

By the definition of ρ_j, we have $q_2 q_3 \cdot \ldots \cdot q_j < \rho_j < q_2 q_3 \cdot \ldots \cdot q_j q_{j+1}$. We get

$$(-1)^{j-2} g_j(\rho_j e^{i\theta}) = (-1)^{j-2} e^{i(j-2)\theta} q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \tag{4.2}$$
$$\times \left(1 - e^{i\theta} q_j \sqrt{q_{j+1}} + e^{2i\theta} q_j q_{j+1} - e^{3i\theta} q_j \sqrt{q_{j+1}} + e^{4i\theta} q_j q_{j+2}^{-1} \right)$$
$$= (-1)^{j-2} \Bigg(e^{i(j-2)\theta} q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}}$$
$$\times \left(1 - e^{i\theta} q_j \sqrt{q_{j+1}} + e^{2i\theta} q_j q_{j+1} - e^{3i\theta} q_j \sqrt{q_{j+1}} + e^{4i\theta} \right)$$
$$+ e^{i(j+2)\theta} q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \left(q_j q_{j+2}^{-1} - 1 \right) \Bigg)$$
$$= (-1)^{j-2} \left(\widetilde{g}_j(\rho_j e^{i\theta}) + \xi_j(\rho_j e^{i\theta}) \right).$$

Our aim is to show that for every sufficiently large j the following inequality holds:

$$\min_{0\le\theta\le 2\pi} |\widetilde{g}_j(\rho_j e^{i\theta})| > \max_{0\le\theta\le 2\pi} |f(\rho_j e^{i\theta}) - \widetilde{g}_j(\rho_j e^{i\theta})|,$$

so that the number of zeros of f in the disk $\{z : |z| < \rho_j\}$ is equal to the number of zeros of $\widetilde{g}_j$ in the same disk. Later in the proof, we also find the number of zeros of $\widetilde{g}_j$ in this disk. First, we find $\min_{0\le\theta\le 2\pi} |\widetilde{g}_j(\rho_j e^{i\theta})|$. We have

$$\begin{aligned}
\widetilde{g}_j(\rho_j e^{i\theta}) &= e^{ij\theta} q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \qquad (4.3)\\
&\quad \times \left(e^{-2i\theta} - e^{-i\theta} q_j \sqrt{q_{j+1}} + q_j q_{j+1} - e^{i\theta} q_j \sqrt{q_{j+1}} + e^{2i\theta}\right)\\
&= e^{ij\theta} q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}}\\
&\quad \times \left(2\cos 2\theta - 2\cos\theta q_j \sqrt{q_{j+1}} + q_j q_{j+1}\right)\\
&=: e^{ij\theta} q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \cdot \psi_j(\theta).
\end{aligned}$$

We consider $\psi_j(\theta)$ as follows

$$\psi_j(\theta) = \widetilde{\psi}_j(t) := 4t^2 - 2q_j\sqrt{q_{j+1}}t + (q_j q_{j+1} - 2),$$

where $t := \cos\theta$, and where we have used that $\cos 2\theta = 2t^2 - 1$.

The vertex of the parabola is $t_j = q_j\sqrt{q_{j+1}}/4$. Under our assumptions,

$$2\sqrt[3]{2} \le q_2 \le q_3 \le \ldots,$$

so that

$$q_j\sqrt{q_{j+1}}/4 \ge q_2\sqrt{q_2}/4 \ge 2\sqrt[3]{2} \times \sqrt{2\sqrt[3]{2}}/4 = 1,$$

therefore, we have $t_j \ge 1$. Hence,

$$\min_{t\in[-1,1]} \widetilde{\psi}_j(t) = \widetilde{\psi}_j(1) = 2 - 2q_j\sqrt{q_{j+1}} + q_j q_{j+1} = q_j\sqrt{q_{j+1}}(\sqrt{q_{j+1}} - 2) + 2.$$

If $q_{j+1} \ge 4$, then

$$q_j\sqrt{q_{j+1}}(\sqrt{q_{j+1}} - 2) + 2 > 0.$$

If $q_{j+1} < 4$, then

$$\begin{aligned}
q_j\sqrt{q_{j+1}}(\sqrt{q_{j+1}} - 2) + 2 &\ge q_{j+1}\sqrt{q_{j+1}}(\sqrt{q_{j+1}} - 2) + 2\\
&= q_{j+1}^2 - 2q_{j+1}\sqrt{q_{j+1}} + 2.
\end{aligned}$$

We set $y = \sqrt{q_{j+1}} \geq 0$, and obtain

$$g(y) = y^4 - 2y^3 + 2.$$

It is easy to calculate that

$$\min_{y \geq 0} g(y) = g(\frac{3}{2}) = \frac{5}{16} > 0.$$

Therefore, we get

$$2 - 2q_j\sqrt{q_{j+1}} + q_j q_{j+1} > 0.$$

Thus, $\widetilde{\psi}_j(t) > 0$ for all $t \in [-1, 1]$. Consequently, we have obtained the estimation of $\widetilde{g}_j(\rho_j e^{i\theta})$ from below

$$\min_{0 \leq \theta \leq 2\pi} |\widetilde{g}_j(\rho_j e^{i\theta})| \geq q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \times \left(2 - 2q_j\sqrt{q_{j+1}} + q_j q_{j+1}\right). \tag{4.4}$$

Further, we estimate the modulus of Σ_1 from above. We have

$$|\Sigma_1(\rho_j e^{i\theta})| \leq \sum_{k=0}^{j-3} \frac{q_2^k q_3^k \cdot \ldots \cdot q_j^k q_{j+1}^{\frac{k}{2}}}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k} = \tag{4.5}$$

(we rewrite the sum from right to left)

$$= \Big(q_2 q_3^2 \cdot \ldots \cdot q_{j-3}^{j-4} q_{j-2}^{j-3} q_{j-1}^{j-3} q_j^{j-3} q_{j+1}^{\frac{j-3}{2}} + q_2 q_3^2 \cdot \ldots \cdot q_{j-4}^{j-5} q_{j-3}^{j-4} q_{j-2}^{j-4} q_{j-1}^{j-4} q_j^{j-4} q_{j+1}^{\frac{j-4}{2}}$$

$$+ q_2 q_3^2 \cdot \ldots \cdot q_{j-5}^{j-6} q_{j-4}^{j-5} q_{j-3}^{j-5} q_{j-2}^{j-5} q_{j-1}^{j-5} q_j^{j-5} q_{j+1}^{\frac{j-5}{2}} + \cdots \Big)$$

$$= q_2 q_3^2 \cdot \ldots \cdot q_{j-3}^{j-4} q_{j-2}^{j-3} q_{j-1}^{j-3} q_j^{j-3} q_{j+1}^{\frac{j-3}{2}}$$

$$\times \left(1 + \frac{1}{q_{j-2} q_{j-1} q_j \sqrt{q_{j+1}}} + \frac{1}{q_{j-3} q_{j-2}^2 q_{j-1}^2 q_j^2 (\sqrt{q_{j+1}})^2} + \cdots \right)$$

$$\leq q_2 q_3^2 \cdots q_{j-3}^{j-4} q_{j-2}^{j-3} q_{j-1}^{j-3} q_j^{j-3} q_{j+1}^{\frac{j-3}{2}} \cdot \frac{1}{1 - \frac{1}{q_{j-2} q_{j-1} q_j \sqrt{q_{j+1}}}}$$

(we estimate the finite sum from above by the sum of the infinite geometric progression). Finally, we obtain

$$|\Sigma_1(\rho_j e^{i\theta})| \leq q_2 q_3^2 \cdot \ldots \cdot q_{j-3}^{j-4} q_{j-2}^{j-3} q_{j-1}^{j-3} q_j^{j-3} q_{j+1}^{\frac{j-3}{2}} \times \frac{1}{1 - \frac{1}{q_{j-2} q_{j-1} q_j \sqrt{q_{j+1}}}}. \tag{4.6}$$

Next, the estimation of $|\Sigma_2(\rho_j e^{i\theta})|$ from above can be made analogously.

$$|\Sigma_2(\rho_j e^{i\theta})| \leq \sum_{k=j+3}^{\infty} \frac{q_2^k q_3^k \cdot \ldots \cdot q_j^k q_{j+1}^{\frac{k}{2}}}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k} = \frac{q_2 q_3^2 \cdot \ldots \cdot q_j^{j-1} q_{j+1}^{\frac{j-3}{2}}}{q_{j+2}^2 q_{j+3}}$$
$$\times \left(1 + \frac{1}{\sqrt{q_{j+1}} q_{j+2} q_{j+3} q_{j+4}} + \frac{1}{(\sqrt{q_{j+1}})^2 q_{j+2}^2 q_{j+3}^2 q_{j+4}^2 q_{j+5}} + \cdots\right).$$

The latter can be estimated from above by the sum of the geometric progression, so, we obtain

$$|\Sigma_2(\rho_j e^{i\theta})| \leq \frac{q_2 q_3^2 \cdot \ldots \cdot q_j^{j-1} q_{j+1}^{\frac{j-3}{2}}}{q_{j+2}^2 q_{j+3}} \times \frac{1}{1 - \frac{1}{\sqrt{q_{j+1}} q_{j+2} q_{j+3} q_{j+4}}}. \tag{4.7}$$

Note that

$$|\xi_j(\rho_j e^{i\theta})| = q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \left(1 - q_j q_{j+2}^{-1}\right).$$

Therefore, the desired inequality

$$\min_{0 \leq \theta \leq 2\pi} |\widetilde{g}_j(\rho_j e^{i\theta})| > \max_{0 \leq \theta \leq 2\pi} |f(\rho_j e^{i\theta}) - \widetilde{g}_j(\rho_j e^{i\theta})|$$

follows from

$$q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \cdot \left(2 - 2q_j\sqrt{q_{j+1}} + q_j q_{j+1}\right)$$
$$> q_2 q_3^2 \cdot \ldots \cdot q_{j-3}^{j-4} q_{j-2}^{j-3} q_{j-1}^{j-3} q_j^{j-3} q_{j+1}^{\frac{j-3}{2}} \times \frac{1}{1 - \frac{1}{q_{j-2} q_{j-1} q_j \sqrt{q_{j+1}}}}$$
$$+ \frac{q_2 q_3^2 \cdot \ldots \cdot q_j^{j-1} q_{j+1}^{\frac{j-3}{2}}}{q_{j+2}^2 q_{j+3}} \times \frac{1}{1 - \frac{1}{\sqrt{q_{j+1}} q_{j+2} q_{j+3} q_{j+4}}}$$
$$+ q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \left(1 - q_j q_{j+2}^{-1}\right).$$

Or, equivalently,

$$q_{j-1} q_j \sqrt{q_{j+1}} \left(2 - 2q_j\sqrt{q_{j+1}} + q_j q_{j+1}\right) \tag{4.8}$$
$$> \frac{1}{1 - \frac{1}{q_{j-2} q_{j-1} q_j \sqrt{q_{j+1}}}} + \frac{q_{j-1} q_j^2}{q_{j+2}^2 q_{j+3}} \frac{1}{1 - \frac{1}{\sqrt{q_{j+1}} q_{j+2} q_{j+3} q_{j+4}}}$$
$$+ q_{j-1} q_j \sqrt{q_{j+1}} (1 - q_j q_{j+2}^{-1}).$$

Since, under our assumptions $q_2 \leq q_3 \leq q_4 \leq \ldots$, the sequence $(q_j)_{j=2}^{\infty}$ has the limit, that is finite or infinite. At first, we consider the case when this limit is finite and put $\lim_{j\to\infty} q_j =: a$, $a \geq 2\sqrt[3]{2}$. We firstly investigate the limiting inequality

$$a^2\sqrt{a}(2 - 2a\sqrt{a} + a^2) > \frac{1}{1 - \frac{1}{a^3\sqrt{a}}} + \frac{1}{1 - \frac{1}{a^3\sqrt{a}}} + a^2\sqrt{a} \cdot 0. \tag{4.9}$$

Equivalently,

$$2 - 2a\sqrt{a} + a^2 > \frac{2a}{a^3\sqrt{a} - 1}.$$

Set $\sqrt{a} =: b$, then we obtain $(2 - 2b^3 + b^4)(b^7 - 1) > 2b^2$, or

$$b^{11} - 2b^{10} + 2b^7 - b^4 + 2b^3 - 2b^2 - 2 > 0.$$

We have found the roots of the polynomial on the left-hand side of the inequality using the computer, and its greatest real root is less than 1.47. Thus, the inequality is fulfilled for $b > 1.47$, and, therefore, for $a > 2.17$. Under our assumptions, $a \geq 2\sqrt[3]{2} > 2.51$, so the inequality (4.9) is valid. Whence, for the case when the sequence $(q_j)_{j=2}^{\infty}$ has the finite limit, the inequality (4.8) is valid for all j being large enough.

Now we consider the case when $\lim_{j\to\infty} q_j = +\infty$. The inequality (4.8) follows from

$$q_{j-1}q_j\sqrt{q_{j+1}}\left(2 - 2q_j\sqrt{q_{j+1}} + q_jq_{j+1}\right) \tag{4.10}$$
$$> \frac{1}{1 - \frac{1}{q_{j-2}q_{j-1}q_j\sqrt{q_{j+1}}}} + \frac{1}{1 - \frac{1}{\sqrt{q_{j+1}}q_{j+2}q_{j+3}q_{j+4}}} + q_{j-1}q_j\sqrt{q_{j+1}},$$

or

$$2 - 2q_j\sqrt{q_{j+1}} + q_jq_{j+1} > \frac{1}{q_{j-1}q_j\sqrt{q_{j+1}}} \times \frac{1}{1 - \frac{1}{q_{j-2}q_{j-1}q_j\sqrt{q_{j+1}}}}$$
$$+ \frac{1}{q_{j-1}q_j\sqrt{q_{j+1}}} \times \frac{1}{1 - \frac{1}{\sqrt{q_{j+1}}q_{j+2}q_{j+3}q_{j+4}}} + 1.$$

The left-hand side of the inequality above tends to infinity, and the right-hand side tends to 1. So, the last inequality is valid for all j being large enough. Whence, for the case when the sequence $(q_j)_{j=2}^{\infty}$ has the infinite limit, the inequality (4.8) is valid for all j being large enough. Consequently, we have proved that for all j being large enough

$$\min_{0\leq\theta\leq 2\pi} |\tilde{g}_j(\rho_j e^{i\theta})| > \max_{0\leq\theta\leq 2\pi} |f(\rho_j e^{i\theta}) - \tilde{g}_j(\rho_j e^{i\theta})|,$$

so the number of zeros of f in the disk $\{z : |z| < \rho_j\}$ is equal to the number of zeros of $\widetilde{g}_j$ in this disk.

In the next stage of the proof, it remains to find the number of zeros of $\widetilde{g}_j$ in the disk $\{z : |z| < \rho_j\}$. We have

$$\widetilde{g}_j(x) = \sum_{k=j-2}^{j+1} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k} + \frac{(-1)^{j+2} x^{j+2}}{q_2^{j+1} q_3^{j} \cdot \ldots \cdot q_{j-2}^5 q_{j-1}^4 q_j^4 q_{j+1}^2}.$$

Let us set $w = x\rho_j^{-1}$, so that $|w| < 1$. This yields

$$\begin{aligned}\widetilde{g}_j(\rho_j w) = (-1)^{j-2} w^{j-2} q_2 q_3^2 \cdot \ldots \cdot q_{j-2}^{j-3} q_{j-1}^{j-2} q_j^{j-2} q_{j+1}^{\frac{j-2}{2}} \\ \times (1 - q_j\sqrt{q_{j+1}} w + q_j q_{j+1} w^2 - q_j\sqrt{q_{j+1}} w^3 + w^4).\end{aligned}$$

It follows from (4.4) that $\widetilde{g}_j$ does not have zeros on the circumference $\{z : |z| = \rho_j\}$, while $\widetilde{g}_j(\rho_j w)$ does not have zeros on the circumference $\{w : |w| = 1\}$. Since

$$P_j(w) = 1 - q_j\sqrt{q_{j+1}} w + q_j q_{j+1} w^2 - q_j\sqrt{q_{j+1}} w^3 + w^4$$

is a self-reciprocal polynomial in w, we can conclude that P_j has exactly two zeros in the disk $\{w : |w| < 1\}$. Hence, $\widetilde{g}_j(x)$ has exactly j zeros in the disk $\{x : |x| < \rho_j\}$, and we have proved the statement of Lemma 4.5. □

Theorem 4.1 is a simple corollary of Lemma 4.5. □

4.2 Proof of Theorem 4.3

Without loss of generality, we can assume that $a_0 = a_1 = 1$, since we can consider the function

$$\psi(x) = a_0^{-1} f(a_0 a_1^{-1} x)$$

instead of f, due to the fact that such rescaling of f preserves its property of having real zeros and preserves the second quotients of Taylor coefficients: $q_n(\psi) = q_n(f)$ for all n. For brevity, during the proof we write p_n and q_n instead of $p_n(f)$ and $q_n(f)$. Then, we have

$$f(x) = 1 + x + \sum_{k=2}^{\infty} \frac{x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}.$$

Further, during the proof, we need the inequalities related to the roots of the function f. So, for the convenience of dealing with inequalities, we are going to consider the positive roots. Thus, we consider the entire function

$$\varphi(x) = f(-x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$$

instead of f.

In the previous section, it was proved that if $\varphi \in \mathcal{L} - \mathcal{P}I$ and $q_2(f) \leq q_3(f)$, then there exists $x_0 \in (0; a_1/a_2] = (0, q_2]$ such that $\varphi(x_0) \leq 0$ (see Theorem 3.2). We want to prove the inverse statement. In order to do this, we need the following lemma.

Lemma 4.6 (T.H. Nguyen, A. Vishnyakova, [64]). *According to Lemma 4.5, we denote by*

$$\rho_k = \rho_k(\varphi) := q_2(\varphi) q_3(\varphi) \cdot \ldots \cdot q_k(\varphi) \sqrt{q_{k+1}(\varphi)},$$

for $k \in \mathbb{N}$*. Under the assumptions of Theorem 4.3, for every* $k \geq 2$ *the following inequality holds:*

$$(-1)^k \varphi(\rho_k) \geq 0.$$

Proof. Since $\rho_k \in (q_2 q_3 \cdot \ldots \cdot q_k, q_2 q_3 \cdot \ldots \cdot q_k q_{k+1})$, we have

$$1 < \rho_k < \frac{\rho_k^2}{q_2} < \cdots < \frac{\rho_k^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k},$$

and

$$\frac{\rho_k^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k} > \frac{\rho_k^{k+1}}{q_2^k q_3^{k-1} \cdot \ldots \cdot q_k^2 q_{k+1}} > \frac{\rho_k^{k+2}}{q_2^{k+1} q_3^k \cdot \ldots \cdot q_k^3 q_{k+1}^2 q_{k+2}} > \cdots .$$

Therefore, we get for $k \geq 2$

$$(-1)^k \varphi(\rho_k) \geq \sum_{j=k-3}^{k+3} \frac{(-1)^{j+k} \rho_k^j}{q_2^{j-1} q_3^{j-2} \cdot \ldots \cdot q_j} =: \mu_k(\rho_k),$$

and it is sufficient to prove that for every $k \geq 2$ we have $\mu_k(\rho_k) \geq 0$. After factoring out $\frac{\rho_k^{k-3}}{q_2^{k-4} q_3^{k-5} \cdot \ldots \cdot q_{k-3}}$ the desired inequality is expressed in the following form

$$-1+\frac{\rho_k}{q_2q_3\cdot\ldots\cdot q_{k-3}q_{k-2}}-\frac{\rho_k^2}{q_2^2q_3^2\cdot\ldots\cdot q_{k-2}^2q_{k-1}}+\frac{\rho_k^3}{q_2^3q_3^3\cdot\ldots\cdot q_{k-2}^3q_{k-1}^2q_k}$$
$$-\frac{\rho_k^4}{q_2^4q_3^4\cdot\ldots\cdot q_{k-2}^4q_{k-1}^3q_k^2q_{k+1}}+\frac{\rho_k^5}{q_2^5q_3^5\cdot\ldots\cdot q_{k-2}^5q_{k-1}^4q_k^3q_{k+1}^2q_{k+2}}$$
$$-\frac{\rho_k^6}{q_2^6q_3^6\cdot\ldots\cdot q_{k-2}^6q_{k-1}^5q_k^4q_{k+1}^3q_{k+2}^2q_{k+3}}\geq 0,$$

or, using that $\rho_k=q_2q_3\cdot\ldots\cdot q_k\sqrt{q_{k+1}}$,

$$\nu_k(\rho_k):=-1+q_{k-1}q_k\sqrt{q_{k+1}}-2q_{k-1}q_k^2q_{k+1}$$
$$+q_{k-1}q_k^2q_{k+1}\sqrt{q_{k+1}}+\frac{q_{k-1}q_k^2\sqrt{q_{k+1}}}{q_{k+2}}-\frac{q_{k-1}q_k^2}{q_{k+2}^2q_{k+3}}\geq 0.$$

We observe that

$$\nu_k(\rho_k)=(q_{k-1}q_k\sqrt{q_{k+1}}-1)+q_{k-1}q_k^2q_{k+1}(\sqrt{q_{k+1}}-2)$$
$$+\frac{q_{k-1}q_k^2\sqrt{q_{k+1}}}{q_{k+2}}\left(1-\frac{1}{\sqrt{q_{k+1}}q_{k+2}q_{k+3}}\right).$$

Firstly, we consider the case when $q_{k+1}\geq 4$. Thus, we have

$$q_{k-1}q_k\sqrt{q_{k+1}}-1>0,$$
$$q_{k-1}q_k^2q_{k+1}(\sqrt{q_{k+1}}-2)\geq 0,$$

and

$$\frac{q_{k-1}q_k^2\sqrt{q_{k+1}}}{q_{k+2}}\left(1-\frac{1}{\sqrt{q_{k+1}}q_{k+2}q_{k+3}}\right)>0.$$

Therefore, in the case $q_{k+1}\geq 4$ the desired inequality $\nu_k(\rho_k)\geq 0$ is proved.

Next, we consider the case when $q_{k+1}<4$ and either $q_{k+2}<4$ (so that $\frac{q_k}{q_{k+2}}\geq\frac{3}{4}\geq 0.525$), or $q_{k+2}\geq 4$ and $\frac{q_k}{q_{k+2}}\geq 0.525$.

After rearranging we get

$$\nu_k(\rho_k)=q_{k-1}q_k^2q_{k+1}\sqrt{q_{k+1}}-2q_{k-1}q_k^2q_{k+1}$$
$$+q_{k-1}q_k\sqrt{q_{k+1}}\left(1+\frac{q_k}{q_{k+2}}\right)-\left(1+\frac{q_{k-1}q_k^2}{q_{k+2}^2q_{k+3}}\right)\geq 0.$$

Since q_k is non-decreasing in k, it is easy to see that

$$\frac{q_{k-1}q_k^2}{q_{k+2}^2q_{k+3}}\leq 1,$$

hence, it is sufficient to prove the following inequality

$$q_{k-1}q_k^2q_{k+1}\sqrt{q_{k+1}} - 2q_{k-1}q_k^2q_{k+1} + q_{k-1}q_k\sqrt{q_{k+1}}\left(1 + \frac{q_k}{q_{k+2}}\right) - 2 \geq 0.$$

Under our assumptions that q_k are non-decreasing in k and $q_2 \geq 3$, we have $2 < \frac{2}{9}q_{k-1}q_k$, and we can observe that

$$\begin{aligned} &q_{k-1}q_k^2q_{k+1}\sqrt{q_{k+1}} - 2q_{k-1}q_k^2q_{k+1} + q_{k-1}q_k\sqrt{q_{k+1}} \cdot 1.525 - 2 \\ \geq\ &q_{k-1}q_k^2q_{k+1}\sqrt{q_{k+1}} - 2q_{k-1}q_k^2q_{k+1} + q_{k-1}q_k\sqrt{q_{k+1}} \cdot 1.525 - \frac{2}{9}q_{k-1}q_k. \end{aligned}$$

So, we need to check that for all $k \geq 2$

$$\begin{aligned} &q_kq_{k+1}\sqrt{q_{k+1}} - 2q_kq_{k+1} + 1.525\sqrt{q_{k+1}} - \frac{2}{9} \\ &= q_kq_{k+1}\left(\sqrt{q_{k+1}} - 2\right) + 1.525\sqrt{q_{k+1}} - \frac{2}{9} \geq 0. \end{aligned}$$

Since q_k is non-decreasing in k, we get

$$\begin{aligned} &q_kq_{k+1}\sqrt{q_{k+1}} - 2q_kq_{k+1} + 1.525\sqrt{q_{k+1}} - \frac{2}{9} \\ &\geq q_{k+1}^2\sqrt{q_{k+1}} - 2q_{k+1}^2 + 1.525\sqrt{q_{k+1}} - \frac{2}{9}. \end{aligned}$$

Set $\sqrt{q_{k+1}} = t, t \geq 0$, then we obtain the following inequality

$$t^5 - 2t^4 + 1.525t - \frac{2}{9} \geq 0.$$

This inequality holds for $t \geq 1.73051$ (we used numerical methods to find that the greatest real root of the polynomial on the left-hand side is less than 1.73051), so it follows that it holds for $q_{k+1} \geq 2.99466$. Thus, in the case $q_{k+1} < 4$ and either $q_{k+2} < 4$, or $q_{k+2} \geq 4$ and $q_k/q_{k+2} \geq 0.525$ the desired inequality $\nu_k(\rho_k) \geq 0$ is proved.

It remains to consider the case when $q_{k+1} < 4$, $q_{k+2} \geq 4$, and $q_{k+1} \geq 3.4303$. We have

$$\begin{aligned} \nu_k(\rho_k) &:= (q_{k-1}q_k\sqrt{q_{k+1}} - 1) + q_{k-1}q_k^2q_{k+1}(\sqrt{q_{k+1}} - 2) \\ &+ \frac{q_{k-1}q_k^2\sqrt{q_{k+1}}}{q_{k+2}}\left(1 - \frac{1}{\sqrt{q_{k+1}}q_{k+2}q_{k+3}}\right) \geq (q_{k-1}q_k\sqrt{q_{k+1}} - 1) \\ &+ q_{k-1}q_k^2q_{k+1}(\sqrt{q_{k+1}} - 2) \geq \left(q_{k-1}q_k\sqrt{q_{k+1}} - \frac{q_{k-1}q_k}{9}\right) \\ &+ q_{k-1}q_k^2q_{k+1}(\sqrt{q_{k+1}} - 2) \\ &= q_{k-1}q_k\left(\left(\sqrt{q_{k+1}} - \frac{1}{9}\right) + q_kq_{k+1}(\sqrt{q_{k+1}} - 2)\right). \end{aligned}$$

We want to show that

$$\left(\sqrt{q_{k+1}} - \frac{1}{9}\right) + q_k q_{k+1}(\sqrt{q_{k+1}} - 2) \geq 0.$$

Since $\sqrt{q_{k+1}} - 2 < 0$, and $q_k \leq q_{k+1}$, the last inequality follows from

$$\left(\sqrt{q_{k+1}} - \frac{1}{9}\right) + q_{k+1}^2(\sqrt{q_{k+1}} - 2) \geq 0.$$

Set $t = \sqrt{q_{k+1}}$, then we get the inequality

$$t^5 - 2t^4 + t - \frac{1}{9} \geq 0.$$

We have found the roots of the polynomial on the left-hand side of the inequality using the computer, and its greatest real root is less than 1.8521. Thus, this inequality is valid for $q_{k+1} \geq 3.4303$. So, in the case when $q_{k+1} < 4$, $q_{k+2} \geq 4$, and $q_{k+1} \geq 3.4303$ the desired inequality $\nu_k(\rho_k) \geq 0$ is also proved. Thus, Lemma 4.6 is proved. □

Suppose that there exists $x_0 \in (1, q_2)$, such that $\varphi(x_0) \leq 0$. Then, by Lemma 4.6, we have for every $k \geq 2$

$$\varphi(0) > 0, \varphi(x_0) \leq 0, \varphi(\rho_2) \geq 0, \varphi(\rho_3) \leq 0, \ldots, (-1)^k \varphi(\rho_k) \geq 0.$$

So, for every $k \geq 2$ the function φ has at least $k-1$ real positive zeros on the disk $\{z \in \mathbb{C} : |z| < \rho_k\}$. By Lemma 4.5, the function φ has exactly k zeros on the disk $\{z \in \mathbb{C} : |z| < \rho_k\}$ for k being large enough. So, for all k being large enough all the zeros of φ on the disk $\{z \in \mathbb{C} : |z| < \rho_k\}$ are real. Thus, if there exists $x_0 \in (1, q_2)$, such that $\varphi(x_0) \leq 0$, then all the zeros of φ are real and positive. Therefore, $\varphi \in \mathcal{L} - \mathcal{P}I$.

Theorem 4.3 is proved.

To summarise this chapter, we have proved that if an entire function $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with positive coefficients such that its second quotients of Taylor coefficients $q_k(f), k \geq 2$ is a non-decreasing in k sequence, and $q_2 \geq 2\sqrt[3]{2}$, then all but a finite number of zeros of f are real and simple. Moreover, we obtained a criterion in terms of the closest to zero roots for such a function to have only real zeros, or, in other words, to belong to the Laguerre–Pólya class of type I under some additional assumptions on $q_k(f)$.

Chapter 5

Number of real zeros of real entire functions with a non-decreasing sequence of the second quotients of Taylor coefficients

In this chapter, we find a necessary conditions for an entire function $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0$, with a non-decreasing sequence of the second quotients of Taylor coefficients to belong to the Laguerre–Pólya class of type I. In addition, we estimate the possible number of non-real zeros for such functions. Moreover, we prove that the following conditions on the second quotients of Taylor coefficients are necessary for the function to belong to the Laguerre–Pólya I class.

Theorem 5.1 (T.H. Nguyen, A. Vishnyakova, [65]). *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0, k = 0, 1, 2, \ldots$, be an entire function such that $q_2(f) \leq q_3(f) \leq q_4(f) \leq \cdots$. If $f \in \mathcal{L} - \mathcal{P}I$, then for any $k = 1, 2, 3, \ldots$, the following inequality holds: $q_{2n+1}(f) > c_{2k+1}$ (the constants c_{2k+1} defined as in Theorem I concerning the partial theta-function, see Chapter 1).*

We prove the following interesting corollary of Theorem 5.1.

Corollary 5.2 (T.H. Nguyen, A. Vishnyakova, [65]). *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0, k = 0, 1, 2, \ldots$, be an entire function such that $q_2(f) \leq q_3(f) \leq q_4(f) \leq \cdots$. If $f \in \mathcal{L} - \mathcal{P}$, then $q_2(f) > 3$.*

The following statement is a simple corollary of Theorem 5.1 and Corollary 5.2.

Corollary 5.3. *Suppose that a real positive sequence $(a_k)_{k=0}^{\infty}$ has the following property: the sequence of its second quotients $\left(\frac{a_{k-1}^2}{a_{k-2}a_k}\right)_{k=2}^{\infty}$ is increasing in k. Then if $(k!a_k)_{k=0}^{\infty} \in \mathcal{MS}$, then for any $k = 1, 2, 3, \ldots$, the following inequality holds: $\frac{a_{2n}^2}{a_{2n-1}a_{2n+1}} > c_{2k+1}$ (the constants c_{2k+1} defined as in Theorem I concerning the partial theta-function, see Chapter 1). Moreover, the inequality $\frac{a_1^2}{a_0 a_2} > 3$ is valid.*

Our next theorem estimates the possible number of nonreal zeros for such entire functions.

Theorem 5.4 (T.H. Nguyen, A. Vishnyakova, [65])**.** *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0, k = 0, 1, 2, \ldots$, be an entire function such that $2\sqrt[3]{2} \approx 2.51984 \leq q_2(f) \leq q_3(f) \leq q_4(f) \leq \cdots$. If there exist $j_0 = 2, 3, 4, \ldots$ and $m_0 \in \mathbb{N}$, such that $q_{j_0} \geq c_{2m_0}$, then the number of nonreal zeros of f does not exceed $j_0 + 2m_0 - 2$ (the constants c_{2k} defined as in defined as in Theorem I concerning the partial theta-function, see Chapter 1).*

We recall that

$$4 = c_2 > c_4 > c_6 > \cdots, \quad \lim_{n\to\infty} c_{2n} = q_\infty;$$

$$3 = c_3 < c_5 < c_7 < \cdots, \quad \lim_{n\to\infty} c_{2n+1} = q_\infty.$$

Calculations show that $c_4 = 1 + \sqrt{5} \approx 3.23607$, $c_6 \approx 3.23364$ and $c_5 \approx 3.23362$, $c_7 \approx 3.23364$.

5.1 Proof of Theorem 5.1

Without loss of generality, we can assume that $a_0 = a_1 = 1$, since we can consider a function $g(x) = a_0^{-1} f(a_0 a_1^{-1} x)$ instead of $f(x)$, due to the fact that such rescaling of f preserves its property of having real zeros as well as the second quotients: $q_n(g) = q_n(f)$ for all $n \in \mathbb{N}$. During the proof instead of $p_n(f)$ and $q_n(f)$ we use notation p_n and q_n. It is more convenient to consider a function

$$\varphi(x) = f(-x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$$

instead of f.

As we proved in Theorem 3.2, if φ belongs to the Laguerre–Pólya class then there exists a point $x_0 \in [0, \frac{a_1}{a_2}] = [0, q_2]$ such that $\varphi(x_0) \leq 0$. Let us introduce some more notation.

First, we need the following lemma.

Lemma 5.5 (T.H. Nguyen, A. Vishnyakova, [65]). *Let* $\varphi(x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$ *be an entire function. Suppose that* q_k *are non-decreasing in* $k : 1 < q_2 \leq q_3 \leq q_4 \leq \cdots$. *If there exists* $x_0 \in [0, q_2]$ *such that* $\varphi(x_0) \leq 0$, *then* $x_0 \in (1, q_2]$.

Proof. For $x \in [0, 1]$ we have:

$$1 \geq x > \frac{x^2}{q_2} > \frac{x^3}{q_2^2 q_3} > \frac{x^4}{q_2^3 q_3^2 q_4} > \cdots ,$$

whence

$$\varphi(x) > 0 \quad \text{for all} \quad x \in [0, 1]. \tag{5.1}$$

□

Lemma 5.6 (T.H. Nguyen, A. Vishnyakova, [65]). *Let* $\varphi(x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$ *be an entire function. Suppose that* q_k *are non-decreasing in* $k : 1 < q_2 \leq q_3 \leq q_4 \leq \cdots$. *If there exists* $x_0 \in (1, q_2]$ *such that* $\varphi(x_0) \leq 0$, *then for any* $n \in \mathbb{N}$, $S_{2n+1}(x_0) < 0$.

Proof. Suppose that $x \in (1, q_2]$. Then we obtain

$$1 < x \geq \frac{x^2}{q_2} > \frac{x^3}{q_2^2 q_3} > \cdots > \frac{x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k} > \cdots \tag{5.2}$$

For an arbitrary $n \in \mathbb{N}$ we have:

$$\varphi(x) = S_{2n+1}(x, \varphi) + R_{2n+2}(x, \varphi).$$

By (5.2) and the Leibniz criterion for alternating series, we conclude that $R_{2n+2}(x, \varphi) > 0$ for all $x \in (1, q_2]$, or

$$\varphi(x) > S_{2n+1}(x, \varphi) \quad \text{for all} \quad x \in (1, q_2], n \in \mathbb{N}. \tag{5.3}$$

Consequently, if there exists a point $x_0 \in (1, q_2]$ such that $\varphi(x_0) \leq 0$, then for any $n \in \mathbb{N}$ we have $S_{2n+1}(x_0) < 0$. □

Thus, we proved that if $\varphi \in \mathcal{L} - \mathcal{P}$, then there exists $x_0 \in (1, q_2]$ such that the inequalities $S_{2n+1}(x_0) < 0$ hold for any $n \in \mathbb{N}$.

In Lemma 2.4 it was proved that if an entire function

$$\varphi(x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$$

belongs to the Laguerre–Pólya class, where $0 < q_2 \leq q_3 \leq q_4 \leq \cdots$, then $q_2 \geq 3$. So we assume that $q_2 \geq 3$.

Lemma 5.7 (T.H. Nguyen, A. Vishnyakova, [65]). *Let* $\varphi(x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$ *be an entire function. Suppose that* $3 \le q_2 \le q_3 \le q_4 \cdots$. *Then the inequality* $S_{2n+1}(x, \varphi) \ge S_{2n+1}(\sqrt{q_{2n+1}}x, g_{\sqrt{q_{2n+1}}})$ *holds for any* $n \in \mathbb{N}$ *and any* $x \in (1, q_2]$ *(here* g_a *is the partial theta function and* $S_{2n+1}(y, g_a)$ *is its* $(2n+1)$*-th partial sum at the point* y*).*

Proof. We have

$$S_{2n+1}(x, \varphi) = (1 - x) + \left(\frac{x^2}{q_2} - \frac{x^3}{q_2^2 q_3}\right) + \left(\frac{x^4}{q_2^3 q_3^2 q_4} - \frac{x^5}{q_2^4 q_3^3 q_4^2 q_5}\right) + \cdots \tag{5.4}$$
$$+ \left(\frac{x^{2n}}{q_2^{2n-1} q_3^{2n-2} \cdot \ldots \cdot q_{2n-1}^2 q_{2n}} - \frac{x^{2n+1}}{q_2^{2n} q_3^{2n-1} \cdot \ldots \cdot q_{2n}^2 q_{2n+1}}\right).$$

Under our assumptions, q_k are non-decreasing in k. We prove that for any fixed $k = 1, 2, \ldots, n$ and $x \in (1, q_2]$, the following inequality holds

$$\frac{x^{2k}}{q_2^{2k-1} q_3^{2k-2} \cdot \ldots \cdot q_{2k-1}^2 q_{2k}} - \frac{x^{2k+1}}{q_2^{2k} q_3^{2k-1} \cdot \ldots \cdot q_{2k-1}^3 q_{2k}^2 q_{2k+1}}$$
$$\ge \frac{x^{2k}}{q_{2k+1}^{2k-1} q_{2k+1}^{2k-2} \cdot \ldots \cdot q_{2k+1}^2 q_{2k+1}} - \frac{x^{2k+1}}{q_{2k+1}^{2k} q_{2k+1}^{2k-1} \cdot \ldots \cdot q_{2k+1}^2 q_{2k+1}}$$
$$= \frac{x^{2k}}{q_{2k+1}^{k(2k-1)}} - \frac{x^{2k+1}}{q_{2k+1}^{k(2k+1)}} = \frac{x^{2k}}{q_{2k+1}^{k(2k-1)}} \cdot \left(1 - \frac{x}{q_{2k+1}^{2k}}\right).$$

For $x \in (1, q_2]$ and any fixed $k = 1, 2, \ldots, n$, we define the following function

$$F(q_2, q_3, \ldots, q_{2k}, q_{2k+1}) := \frac{x^{2k}}{q_2^{2k-1} q_3^{2k-2} \cdot \ldots \cdot q_{2k-1}^2 q_{2k}}$$
$$- \frac{x^{2k+1}}{q_2^{2k} q_3^{2k-1} \cdot \ldots \cdot q_{2k-1}^3 q_{2k}^2 q_{2k+1}}.$$

We observe that

$$\frac{\partial F(q_2, q_3, \ldots, q_{2k}, q_{2k+1})}{\partial q_2} = -\frac{(2k-1) \cdot x^{2k}}{q_2^{2k} q_3^{2k-2} \cdot \ldots \cdot q_{2k-1}^2 q_{2k}}$$
$$+ \frac{2k \cdot x^{2k+1}}{q_2^{2k+1} q_3^{2k-1} \cdot \ldots \cdot q_{2k-1}^3 q_{2k}^2 q_{2k+1}} < 0 \Leftrightarrow x < \left(1 - \frac{1}{2k}\right) \cdot q_2 q_3 \cdot \ldots \cdot q_{2k} q_{2k+1}.$$

Therefore, since

$$\left(1 - \frac{1}{2k}\right) q_2 q_3 \cdot \ldots \cdot q_{2k} q_{2k+1} \ge \frac{1}{2} q_2 q_3 \cdot \ldots \cdot q_{2k} q_{2k+1} \ge \frac{1}{2} q_2 q_3 > q_2$$

(under our assumptions $q_3 \geq q_2 \geq 3$), we conclude that $F(q_2, q_3, \ldots, q_{2k}, q_{2k+1})$ is decreasing in q_2 for each fixed $x \in (1, q_2]$. Since $q_2 \leq q_3$, for $k = 1$ we get:

$$F(q_2, q_3) = \frac{x^2}{q_2} - \frac{x^3}{q_2^2 q_3} \geq \frac{x^2}{q_3} - \frac{x^3}{q_3^2 q_3} = \frac{x^2}{q_3} - \frac{x^3}{q_3^3},$$

and the desired inequality is proved for $k = 1$. For $k \geq 2$ we have

$$\begin{aligned} &F(q_2, q_3, q_4, \ldots, q_{2k}, q_{2k+1}) \geq F(q_3, q_3, q_4, \ldots, q_{2k}, q_{2k+1}) \\ &= \frac{x^{2k}}{q_3^{4k-3} q_4^{2k-3} \cdot \ldots \cdot q_{2k-1}^2 q_{2k}} - \frac{x^{2k+1}}{q_3^{4k-1} q_4^{2k-2} \cdot \ldots \cdot q_{2k-1}^3 q_{2k}^2 q_{2k+1}}. \end{aligned}$$

Further, we consider its derivative with respect to q_3:

$$\begin{aligned} &\frac{\partial F(q_3, q_3, q_4, \ldots, q_{2k}, q_{2k+1})}{\partial q_3} = -\frac{(4k-3) \cdot x^{2k}}{q_3^{4k-2} q_4^{2k-3} \cdot \ldots \cdot q_{2k-1}^2 q_{2k}} \\ &\quad + \frac{(4k-1) \cdot x^{2k+1}}{q_3^{4k} q_4^{2k-2} \cdot \ldots \cdot q_{2k+1}} < 0 \\ &\quad \Leftrightarrow x < \frac{4k-3}{4k-1} q_3^2 q_4 \cdot \ldots \cdot q_{2k-1}^3 q_{2k}^2 q_{2k+1}. \end{aligned}$$

Under our assumptions,

$$\frac{4k-3}{4k-1} \cdot q_3^2 q_4 \cdot \ldots \cdot q_{2k+1} \geq \frac{5}{7} \cdot q_3^2 q_4 q_5 > q_2,$$

and we obtain that $F(q_3, q_3, q_4, \ldots, q_{2k}, q_{2k+1})$ is decreasing in q_3 for each fixed $x \in (1, q_2]$ and, since $q_3 \leq q_4$, we receive

$$F(q_3, q_3, q_4, \ldots, q_{2k}, q_{2k+1}) \geq F(q_4, q_4, q_4, q_5, \ldots, q_{2k}, q_{2k+1}).$$

Thus, for the lth step we have

$$\begin{aligned} &F(q_{l-1}, q_{l-1}, \ldots, q_{l-1}, q_l, q_{l+1}, \ldots, q_{2k}, q_{2k+1}) \\ &= \frac{x^{2k}}{q_{l-1}^{(4k-l+1)(l-2)/2} q_l^{2k-l+1} q_{l+1}^{2k-l} \cdot \ldots \cdot q_{2k-1}^2 q_{2k}} \\ &- \frac{x^{2k+1}}{q_{l-1}^{(4k-l+3)(l-2)/2} q_l^{2k-l+2} q_{l+1}^{2k-l+1} \cdot \ldots \cdot q_{2k-1}^3 q_{2k}^2 q_{2k+1}}. \end{aligned}$$

We consider its partial derivative with respect to q_{l-1} :

$$\begin{aligned}&\frac{\partial F(q_{l-1},q_{l-1},\dots,q_{l-1},q_l,q_{l+1},\dots,q_{2k},q_{2k+1})}{\partial q_{l-1}}\\&=-\frac{\frac{1}{2}(4k-l+1)(l-2)\cdot x^{2k}}{q_{l-1}^{1+(4k-l+1)(l-2)/2}q_l^{2k-l+1}q_{l+1}^{2k-l}\cdot\ldots\cdot q_{2k-1}^2q_{2k}}\\&+\frac{\frac{1}{2}(4k-l+3)(l-2)\cdot x^{2k+1}}{q_{l-1}^{1+(4k-l+3)(l-2)/2}q_l^{2k-l+2}q_{l+1}^{2k-l+1}\cdot\ldots\cdot q_{2k-1}^3q_{2k}^2q_{2k+1}}<0,\end{aligned}$$

which is equivalent to the inequality

$$x<\frac{4k-l+1}{4k-l+3}\cdot q_{l-1}^{l-2}q_lq_{l+1}\cdot\ldots\cdot q_{2k-1}q_{2k}q_{2k+1}.$$

The inequality above is valid, since

$$\begin{aligned}&\frac{4k-l+1}{4k-l+3}\cdot q_{l-1}^{l-2}q_lq_{l+1}\cdot\ldots\cdot q_{2k-1}q_{2k}q_{2k+1}\\&\quad\geq\frac{9-l}{11-l}\cdot q_{l-1}^{l-2}q_lq_{l+1}\cdot\ldots\cdot q_{2k-1}q_{2k}q_{2k+1}>q_2.\end{aligned}$$

Hence, the function $F(q_{l-1},q_{l-1},\dots,q_{l-1},q_l,q_{l+1},\dots,q_{2k},q_{2k+1})$ is decreasing in q_{l-1}. Since, under our assumptions, $q_{l-1}\leq q_l$, we obtain

$$\begin{aligned}&F(q_{l-1},q_{l-1},\dots,q_{l-1},q_l,q_{l+1},\dots,q_{2k},q_{2k+1})\\&\quad\geq F(q_l,q_l,\dots,q_l,q_{l+1},\dots,q_{2k},q_{2k+1}).\end{aligned}$$

Analogously, by the same computation, at the $(2k+1)$-th step we get

$$F(q_{2k},q_{2k}\dots,q_{2k},q_{2k+1})=\frac{x^{2k}}{q_{2k}^{k(2k-1)}}-\frac{x^{2k+1}}{q_{2k}^{(k+1)(2k-1)}\cdot q_{2k+1}}.$$

Its derivative with respect to q_{2k} is

$$\begin{aligned}&\frac{\partial F(q_{2k},q_{2k}\dots,q_{2k},q_{2k+1})}{\partial q_{2k}}=-\frac{k(2k-1)\cdot x^{2k}}{q_{2k}^{2k^2-k+1}}\\&+\frac{(2k^2+k-1)\cdot x^{2k+1}}{q_{2k}^{2k^2+k}q_{2k+1}}<0\Leftrightarrow x<\frac{2k^2-k}{2k^2+k-1}\cdot q_{2k}^{2k-1}q_{2k+1}.\end{aligned}$$

Since we assume that

$$\frac{2k^2-k}{2k^2+k-1}\cdot q_{2k}^{2k-1}q_{2k+1}\geq\frac{2}{3}\cdot q_{2k}^{2k-1}q_{2k+1}>q_2,$$

we conclude that the function $F(q_{2k}, q_{2k} \ldots, q_{2k}, q_{2k+1})$ is decreasing in q_{2k}. While $q_{2k} \leq q_{2k+1}$, we get

$$F(q_{2k}, q_{2k} \ldots, q_{2k}, q_{2k+1}) \geq F(q_{2k+1}, q_{2k+1}, \ldots, q_{2k+1}, q_{2k+1}).$$

Thus, we obtain the following chain of inequalities

$$\begin{aligned} F(q_2, q_3, q_4, \ldots, q_{2k}, q_{2k+1}) &\geq F(q_3, q_3, q_4, \ldots, q_{2k}, q_{2k+1}) \\ &\geq F(q_4, q_4, q_4, q_5, \ldots, q_{2k}, q_{2k+1}) \geq \cdots \geq F(q_{2k}, q_{2k}, \ldots, q_{2k}, q_{2k+1}) \\ &\geq F(q_{2k+1}, q_{2k+1}, \ldots, q_{2k+1}, q_{2k+1}). \end{aligned}$$

Consequently,

$$\begin{aligned} F(q_2, q_3, q_4, \ldots, q_{2k}, q_{2k+1}) &\geq F(q_{2k+1}, q_{2k+1}, \ldots, q_{2k+1}, q_{2k+1}) \\ &= \frac{x^{2k}}{q_{2k+1}^{k(2k-1)}} - \frac{x^{2k+1}}{q_{2k+1}^{k(2k+1)}}. \end{aligned}$$

Finally, we note that under our assumptions, the expression

$$\frac{x^{2k}}{q_{2k+1}^{k(2k-1)}} - \frac{x^{2k+1}}{q_{2k+1}^{k(2k+1)}}$$

is decreasing in q_{2k+1} for each fixed $x \in (1, q_2]$, so we obtain

$$F(q_2, q_3, q_4, \ldots, q_{2k}, q_{2k+1}) \geq \frac{x^{2k}}{q_{2k+1}^{k(2k-1)}} - \frac{x^{2k+1}}{q_{2k+1}^{k(2k+1)}} \geq \frac{x^{2k}}{q_{2n+1}^{k(2k-1)}} - \frac{x^{2k+1}}{q_{2n+1}^{k(2k+1)}}.$$

Substituting the last inequality in (5.4) for every $x \in (1, q_2]$ and $k = 1, 2, \ldots, n$, we get

$$\begin{aligned} S_{2n+1}(x, \varphi) &\geq (1 - x) + \left(\frac{x^2}{q_{2n+1}} - \frac{x^3}{q_{2n+1}^3}\right) + \left(\frac{x^4}{q_{2n+1}^6} - \frac{x^5}{q_{2n+1}^{10}}\right) + \\ &\cdots + \left(\frac{x^{2n}}{q_{2n+1}^{n(2n-1)}} - \frac{x^{2n+1}}{q_{2n+1}^{n(2n+1)}}\right) = \sum_{k=0}^{2n+1} \frac{(-1)^k x^k}{\sqrt{q_{2n+1}}^{k(k-1)}} \\ &= S_{2n+1}(-\sqrt{q_{2n+1}}x, g_{\sqrt{q_{2n+1}}}), \end{aligned} \tag{5.5}$$

where g_a is the partial theta function and $S_{2n+1}(y, g_a)$ is its $(2n+1)$-th partial sum at the point y. $\square$

Since we have $S_{2n+1}(x, \varphi) \geq S_{2n+1}(-\sqrt{q_{2n+1}}x, g_{\sqrt{q_{2n+1}}})$ for any $n \in \mathbb{N}$, if there exists a point $x_0 \in (1, q_2]$ such that $S_{2n+1}(x_0, \varphi) \leq 0$, then $S_{2n+1}(-\sqrt{q_{2n+1}}x_0, g_{\sqrt{q_{2n+1}}}) < 0$. Therefore for $y_0 = \sqrt{q_{2n+1}}x_0$, we have $\sqrt{q_{2n+1}} \leq y_0 \leq \sqrt{q_{2n+1}}q_2 \leq (\sqrt{q_{2n+1}})^3$. Using the statement (5) of Theorem I, we obtain that $q_{2n+1} > c_{2n+1}$, which completes the proof of Theorem 5.1.

5.2 Proof of Corollary 5.2

As we have proved in the previous theorem, if $f \in \mathcal{L} - \mathcal{P}$, then $q_3(f) > 3$. In Theorem 3.4 it is proved that, under the assumptions of the Corollary, if $q_2(f) < 4$, then

$$q_3(f) \leq \frac{-q_2(f)(2q_2(f) - 9) + 2(q_2(f) - 3)\sqrt{q_2(f)(q_2(f) - 3)}}{q_2(f)(4 - q_2(f))}.$$

We have mentioned that if $f \in \mathcal{L} - \mathcal{P}$, then $q_2(f) \geq 3$. If $q_2(f) = 3$, then the inequality above states $q_3(f) \leq 3$. This contradiction proves the Corollary 5.2. $\square$

5.3 Proof of Theorem 5.4

As in the proof of Theorem 5.1 we assume that $a_0 = a_1 = 1$, and we consider the function $\varphi(x) = f(-x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k}$ instead of f. We need the following lemma.

Lemma 5.8 (T.H. Nguyen, A. Vishnyakova, [65]). *Let $\varphi(x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$ be an entire function. Suppose that $1 < q_2 \leq q_3 \leq q_4 \leq \cdots$. If there exist $j_0 = 3, 4, \ldots$ and $m_0 \in \mathbb{N}$, such that $q_{j_0} \geq c_{2m_0}$, then for all $j \geq j_0 + 2m_0 - 3$, there exists $x_j \in (q_2 q_3 \cdot \ldots \cdot q_j, q_2 q_3 \cdot \ldots \cdot q_j q_{j+1})$ such that the following inequality holds:*

$$(-1)^j \varphi(x_j) \geq 0.$$

Proof. Choose an arbitrary $j \geq j_0 + 2m_0 - 3$ and fix this j. For every $x \in (q_2 q_3 \cdots q_j,\, q_2 q_3 \cdots q_j q_{j+1})$ we have

$$1 < x < \frac{x^2}{q_2} < \frac{x^3}{q_2^2 q_3} < \cdots < \frac{x^j}{q_2^{j-1} q_3^{j-2} \cdot \ldots \cdot q_{j-1}^2 q_j},$$

and

$$\frac{x^j}{q_2^{j-1} q_3^{j-2} \cdot \ldots \cdot q_{j-1}^2 q_j} > \frac{x^{j+1}}{q_2^j q_3^{j-1} \cdot \ldots \cdot q_{j-1}^3 q_j^2 q_{j+1}}$$
$$> \frac{x^{j+2}}{q_2^{j+1} q_3^j \cdot \ldots \cdot q_{j-1}^4 q_j^3 q_{j+1}^2 q_{j+2}} > \cdots.$$

We observe that

$$(-1)^j\varphi(x) = \sum_{k=0}^{j-2m_0} \frac{(-1)^{k+j}x^k}{q_2^{k-1}q_3^{k-2}\cdot\ldots\cdot q_{k-1}^2 q_k} + \sum_{k=j-2m_0+1}^{j+1} \frac{(-1)^{k+j}x^k}{q_2^{k-1}q_3^{k-2}\cdot\ldots\cdot q_{k-1}^2 q_k}$$
$$+ \sum_{k=j+2}^{\infty} \frac{(-1)^{k+j}x^k}{q_2^{k-1}q_3^{k-2}\cdot\ldots\cdot q_{k-1}^2 q_k} =: \Sigma_1(x) + h(x) + \Sigma_2(x).$$

Summands in $\Sigma_1(x)$ are increasing in modulus and the sign of the last (biggest) summand is positive. So, for all $x \in (q_2q_3\cdot\ldots\cdot q_j, q_2q_3\cdot\ldots\cdot q_jq_{j+1})$, we have $\Sigma_1(x) > 0$. Summands in $\Sigma_2(x)$ are decreasing in modulus and the sign of the first (biggest) summand is positive. Consequently, for all $x \in (q_2q_3\cdot\ldots\cdot q_j, q_2q_3\cdot\ldots\cdot q_jq_{j+1})$, we get $\Sigma_2(x) > 0$. Thus, we obtain

$$\begin{aligned}(-1)^j\varphi(x) > h(x) &= \sum_{k=j-2m_0+1}^{j+1} \frac{(-1)^{k+j}x^k}{q_2^{k-1}q_3^{k-2}\cdot\ldots\cdot q_{k-1}^2 q_k} \qquad (5.6)\\
&= -\frac{x^{j+1}}{q_2^{j}q_3^{j-1}\cdot\ldots\cdot q_j^2 q_{j+1}} + \frac{x^j}{q_2^{j-1}q_3^{j-2}\cdot\ldots\cdot q_{j-1}^2 q_j}\\
&\quad - \frac{x^{j-1}}{q_2^{j-2}q_3^{j-3}\cdot\ldots\cdot q_{j-2}^2 q_{j-1}} + \ldots\\
&\quad + \frac{x^{j-2m_0+2}}{q_2^{j-2m_0+1}q_3^{j-2m_0}\cdot\ldots\cdot q_{j-2m_0+1}^2 q_{j-2m_0+2}}\\
&\quad - \frac{x^{j-2m_0+1}}{q_2^{j-2m_0}q_3^{j-2m_0-1}\cdot\ldots\cdot q_{j-2m_0}^2 q_{j-2m_0+1}}\end{aligned}$$

(we rewrite the sum from the end to the beginning). After factoring out the term

$$\frac{x^{j+1}}{q_2^{j}q_3^{j-1}\cdot\ldots\cdot q_j^2 q_{j+1}},$$

we get

$$(-1)^j\varphi(x) > h(x) = \frac{x^{j+1}}{q_2^j q_3^{j-1}\cdot\ldots\cdot q_j^2 q_{j+1}}\cdot\Big(-1+\frac{q_2q_3\cdot\ldots\cdot q_jq_{j+1}}{x} \tag{5.7}$$
$$-\frac{(q_2q_3\cdot\ldots\cdot q_jq_{j+1})^2}{x^2q_{j+1}}+\frac{(q_2q_3\cdot\ldots\cdot q_jq_{j+1})^3}{x^3q_{j+1}^2q_j}-\ldots$$
$$+\frac{(q_2q_3\cdot\ldots\cdot q_jq_{j+1})^{2m_0-1}}{x^{2m_0-1}q_{j+1}^{2m_0-2}q_j^{2m_0-3}\cdots q_{j-2m_0+5}^2q_{j-2m_0+4}}$$
$$-\frac{(q_2q_3\cdot\ldots\cdot q_jq_{j+1})^{2m_0}}{x^{2m_0}q_{j+1}^{2m_0-1}q_j^{2m_0-2}\cdot\ldots\cdot q_{j-2m_0+5}^3q_{j-2m_0+4}^2q_{j-2m_0+3}}\Big)$$
$$=: \frac{x^{j+1}}{q_2^j q_3^{j-1}\cdot\ldots\cdot q_j^2 q_{j+1}}\cdot\psi(x).$$

Now we introduce some more notation. Set

$$y := \frac{q_2q_3\cdot\ldots\cdot q_jq_{j+1}}{x},$$

and observe that $x\in(q_2q_3\cdot\ldots\cdot q_j, q_2q_3\cdot\ldots\cdot q_jq_{j+1}) \Leftrightarrow y\in(1,q_{j+1})$. Further we change the numeration of the second quotients of Taylor coefficients as follows

$$s_2 := q_{j+1},\ s_3 := q_j,\ s_4 := q_{j-1},\ \ldots, s_{2m_0-1} := q_{j-2m_0+4},\ s_{2m_0} := q_{j-2m_0+3}.$$

By our assumptions, $q_2\le q_3\le q_4\le\cdots$, thus, we get $s_2\ge s_3\ge s_4\ge\cdots\ge s_{2m_0}>1$, and $y\in(1,s_2)$. In the new notation we have

$$\psi(y) = -1+y-\sum_{k=2}^{2m_0}\frac{(-1)^k y^k}{s_2^{k-1}s_3^{k-2}\cdot\ldots\cdot s_{k-1}^2 s_k}. \tag{5.8}$$

We want to prove that there exists a point $y_j\in(1,q_{j+1})=(1,s_2)$ such that $h(y_j)\ge 0$. Hence, we compare the expression in brackets with the corresponding partial sum of the partial theta function. We have

$$\psi(y) = (-1+y)+\left(-\frac{y^2}{s_2}+\frac{y^3}{s_2^2s_3}\right)+\left(-\frac{y^4}{s_2^3s_3^2s_4}+\frac{y^5}{s_2^4s_3^3s_4^2s_5}\right)+\cdots+ \tag{5.9}$$
$$\left(-\frac{y^{2m_0-2}}{s_2^{2m_0-3}s_3^{2m_0-4}\cdot\ldots\cdot s_{2m_0-3}^2s_{2m_0-2}}+\frac{y^{2m_0-1}}{s_2^{2m_0-2}s_3^{2m_0-3}\cdot\ldots\cdot s_{2m_0-2}^2s_{2m_0-1}}\right)$$
$$-\frac{y^{2m_0}}{s_2^{2m_0-1}s_3^{2m_0-2}\cdot\ldots\cdot s_{2m_0-2}^3s_{2m_0-1}^2s_{2m_0}}.$$

Firstly, under our assumptions, one can see that

$$\begin{aligned}
&-\frac{y^{2m_0}}{s_2^{2m_0-1}s_3^{2m_0-2}\cdot\ldots\cdot s_{2m_0-1}^2 s_{2m_0}} \\
&\geq -\frac{y^{2m_0}}{s_{2m_0}^{2m_0-1}s_{2m_0}^{2m_0-2}\cdot\ldots\cdot s_{2m_0}^2 s_{2m_0}} = -\frac{y^{2m_0}}{s_{2m_0}^{m_0(2m_0-1)}}.
\end{aligned} \tag{5.10}$$

We prove that for any fixed $k = 1, 2, \ldots, m_0 - 1$, the following inequality holds

$$\begin{aligned}
&-\frac{y^{2k}}{s_2^{2k-1}s_3^{2k-2}\cdot\ldots\cdot s_{2k}} + \frac{y^{2k+1}}{s_2^{2k}s_3^{2k-1}\cdot\ldots\cdot s_{2k}^2 s_{2k+1}} \\
&\geq -\frac{y^{2k}}{s_{2m_0}^{2k-1}s_{2m_0}^{2k-2}\cdot\ldots\cdot s_{2m_0}} + \frac{y^{2k+1}}{s_{2m_0}^{2k}s_{2m_0}^{2k-1}\cdot\ldots\cdot s_{2m_0}^2 s_{2m_0}} \\
&= -\frac{y^{2k}}{s_{2m_0}^{k(2k-1)}} + \frac{y^{2k+1}}{s_{2m_0}^{k(2k+1)}}.
\end{aligned} \tag{5.11}$$

Firstly, we consider (5.11) for $k = 1$. Since $s_2 \geq s_3$, we have

$$-\frac{y^2}{s_2} + \frac{y^3}{s_2^2 s_3} \geq -\frac{y^2}{s_2} + \frac{y^3}{s_2^3}.$$

We observe that

$$\frac{\partial}{\partial s_2}\left(-\frac{y^2}{s_2} + \frac{y^3}{s_2^3}\right) = \frac{y^2}{s_2^2} - \frac{3y^3}{s_2^4} > 0 \Leftrightarrow y < \frac{s_2^2}{3}.$$

The inequality above is valid if $y < q_{j+1} = s_2$ and $s_2 > 3$. We suppose that there exist $j_0 = 2, 3, 4, \ldots$ and $m_0 \in \mathbb{N}$, such that $q_{j_0} \geq c_{2m_0}$, then we fix an arbitrary $j \geq j_0 + 2m_0 - 3$ and get $s_2 \geq s_{2m_0} = q_{j-2m_0+3} \geq q_{j_0} \geq c_{2m_0} > 3$. Therefore, the function $\left(-\frac{y^2}{s_2} + \frac{y^3}{s_2^3}\right)$ is increasing in s_2, whence

$$-\frac{y^2}{s_2} + \frac{y^3}{s_2^2 s_3} \geq -\frac{y^2}{s_2} + \frac{y^3}{s_2^3} \geq -\frac{y^2}{s_{2m_0}} + \frac{y^3}{s_{2m_0}^3}. \tag{5.12}$$

We apply analogous reasoning to prove (5.11) for every $k = 1, 2, \ldots, m_0 - 1$. Let us define the following function

$$\begin{aligned}
H(s_2, s_3, \ldots, s_{2k}, s_{2k+1}) &:= -\frac{y^{2k}}{s_2^{2k-1}s_3^{2k-2}\cdot\ldots\cdot s_{2k-1}^2 s_{2k}} \\
&+ \frac{y^{2k+1}}{s_2^{2k}s_3^{2k-1}\cdot\ldots\cdot s_{2k-1}^3 s_{2k}^2 s_{2k+1}}
\end{aligned}$$

for $s_2 \geq s_3 \geq \cdots \geq s_{2k+1}$. Obviously,

$$\begin{aligned} H(s_2, s_3, \ldots, s_{2k}, s_{2k+1}) &\geq H(s_2, s_3, \ldots, s_{2k}, s_{2k}) \\ &= -\frac{y^{2k}}{s_2^{2k-1} s_3^{2k-2} \cdot \ldots \cdot s_{2k-1}^2 s_{2k}} + \frac{y^{2k+1}}{s_2^{2k} s_3^{2k-1} \cdot \ldots \cdot s_{2k-1}^3 s_{2k}^3}. \end{aligned}$$

We have

$$\frac{\partial H(s_2, s_3, \ldots, s_{2k}, s_{2k})}{\partial s_{2k}} = \frac{y^{2k}}{s_2^{2k-1} s_3^{2k-2} \cdot \ldots \cdot s_{2k-1}^2 s_{2k}^2} - \frac{3y^{2k+1}}{s_2^{2k} s_3^{2k-1} \cdot \ldots \cdot s_{2k-1}^3 s_{2k}^4}.$$

Thus,

$$\frac{\partial H(s_2, s_3, \ldots, s_{2k}, s_{2k})}{\partial s_{2k}} > 0 \Leftrightarrow y < \frac{s_2 s_3 \cdot \ldots \cdot s_{2k-1} s_{2k}^2}{3}.$$

Since $y \in (1, s_2) \Rightarrow y < s_2$, we obtain that the function $H(s_2, s_3, \ldots, s_{2k}, s_{2k})$ is increasing in s_{2k}, whence

$$\begin{aligned} &H(s_2, s_3, \ldots, s_{2k-1}, s_{2k}, s_{2k+1}) \geq H(s_2, s_3, \ldots, s_{2k-1}, s_{2k}, s_{2k}) \\ &\geq H(s_2, s_3, \ldots, s_{2k-1}, s_{2m_0}, s_{2m_0}) = -\frac{y^{2k}}{s_2^{2k-1} s_3^{2k-2} \cdot \ldots \cdot s_{2k-1}^2 s_{2m_0}} \\ &+ \frac{y^{2k+1}}{s_2^{2k} s_3^{2k-1} \cdot \ldots \cdot s_{2k-1}^3 s_{2m_0}^3}. \end{aligned}$$

Now we consider the derivative of the latter function

$$\begin{aligned} &\frac{\partial H(s_2, s_3, \ldots, s_{2k-1}, s_{2m_0}, s_{2m_0})}{\partial s_{2k-1}} \\ &= \frac{2y^{2k}}{s_2^{2k-1} s_3^{2k-2} \cdot \ldots \cdot s_{2k-1}^3 s_{2m_0}} - \frac{3y^{2k+1}}{s_2^{2k} s_3^{2k-1} \cdot \ldots \cdot s_{2k-1}^4 s_{2m_0}^3}. \end{aligned}$$

Hence,

$$\frac{\partial H(s_2, s_3, \ldots, s_{2k-1}, s_{2m_0}, s_{2m_0})}{\partial s_{2k-1}} > 0 \Leftrightarrow y < \frac{2 s_2 s_3 \cdot \ldots \cdot s_{2k-1} s_{2k-1} s_{2m_0}^2}{3}.$$

The inequality above is valid since $y < s_2$ and $s_{2m_0} > 3$, therefore, we obtain that the function $H(s_2, s_3, \ldots, s_{2k-1}, s_{2m_0}, s_{2m_0})$ is increasing in s_{2k-1}, whence

$$\begin{aligned} &H(s_2, s_3, \ldots, s_{2k-2}, s_{2k-1}, s_{2m_0}, s_{2m_0}) \geq H(s_2, s_3, \ldots, s_{2k-2}, s_{2m_0}, s_{2m_0}, s_{2m_0}) \\ &= -\frac{y^{2k}}{s_2^{2k-1} s_3^{2k-2} \cdot \ldots \cdot s_{2k-2}^3 s_{2m_0}^3} + \frac{y^{2k+1}}{s_2^{2k} s_3^{2k-1} \cdot \ldots \cdot s_{2k-2}^4 s_{2m_0}^6}. \end{aligned}$$

Applying similar arguments we get the following chain of inequalities.

$$H(s_2, s_3, \dots, s_{2k}, s_{2k+1}) \geq H(s_2, s_3, \dots, s_{2k-1}, s_{2m_0}, s_{2m_0})$$
$$\geq H(s_2, s_3, \dots, s_{2k-2}, s_{2m_0}, s_{2m_0}, s_{2m_0}) \geq \dots \geq H(s_{2m_0}, s_{2m_0}, \dots, s_{2m_0}, s_{2m_0}).$$

Thus, we have proved (5.11).

We substitute the inequality (5.10) and (5.11) into (5.9) to get the following

$$\psi(y) \geq -\sum_{k=0}^{2m_0} \frac{(-1)^k y^k}{s_{2m_0}^{\frac{k(k-1)}{2}}} = -S_{2m_0}(-\sqrt{s_{2m_0}}y, g_{\sqrt{s_{2m_0}}}), \tag{5.13}$$

where g_a is a partial theta function and $S_n(x, g_a) := \sum_{j=0}^n x^j a^{-j^2}$ is its partial sum. By our assumption $(\sqrt{s_{2m_0}})^2 = s_{2m_0} = q_{j-2m_0+3}$ and $j \geq j_0 + 2m_0 - 3$, so $s_{2m_0} = q_{j-2m_0+3} \geq q_{j_0} \geq c_{2m_0}$, and we conclude that $S_{2m_0}(x, g_{s_{2m_0}}) \in \mathcal{L} - \mathcal{P}$ (see Theorem I). Whence, by part (4) of Theorem I, there exists $x_0 \in (-(\sqrt{s_{2m_0}})^3, -\sqrt{s_{2m_0}})$ such that $S_{2m_0}(x_0, g_{s_{2m_0}}) \leq 0$. We put $-\sqrt{s_{2m_0}}y_0 := x_0$, i.e. $y_0 := -\frac{x_0}{\sqrt{s_{2m_0}}} \in (1, s_{2m_0}) \subset (1, s_2)$, and we have

$$S_{2m_0}(-\sqrt{s_{2m_0}}y_0, g_{\sqrt{s_{2m_0}}}) \leq 0.$$

Substituting the last inequality in (5.13) we obtain

$$\psi(y_0) \geq -S_{2m_0}(-\sqrt{s_{2m_0}}y_0, g_{\sqrt{s_{2m_0}}}) \geq 0. \tag{5.14}$$

Using (5.14) and substituting (5.13) into (5.7), we get

$$(-1)^j \psi(x) > h(x) = \frac{x^{j+1}}{q_2^j q_3^{j-1} \cdot \ldots \cdot q_j^2 q_{j+1}} \cdot \psi(y_0) \geq 0,$$

which is the desired inequality. It remains to recall that $x_j := \frac{q_2 q_3 \cdots q_j q_{j+1}}{y_0}$, and, since $y_0 \in (1, s_2) = (1, q_{j+1})$, we have $x_j \in (q_2 q_3 \cdot \ldots \cdot q_j, q_2 q_3 \cdot \ldots \cdot q_j q_{j+1})$. □

Now we apply Lemma 4.5, which states that if $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0, k = 0, 1, 2, \dots$, is an entire function such that $2\sqrt[3]{2} \leq q_2(f) \leq q_3(f) \leq q_4(f) \leq \cdots$, then, for all sufficiently large k, the function f has exactly k zeros on the disk $\{z : |z| < q_2(f)q_3(f) \cdot \ldots \cdot q_k(f)\sqrt{q_{k+1}(f)}\}$ counting multiplicities.

Let us choose an arbitrary $k \geq 2$, being large enough to get the statement of Lemma 4.5, and $k \geq j_0 + 2m_0 - 2$. Then the number of zeros of φ (counting multiplicities) in the disk $\{z : |z| < q_2 q_3 \cdot \ldots \cdot q_k \sqrt{q_{k+1}}\}$ is equal to k. By

Lemma 5.8 we have

$$\begin{aligned}
&\operatorname{sgn}\varphi(x_{j_0+2m_0-3}) = -\operatorname{sgn}\varphi(x_{j_0+2m_0-2}),\\
&\operatorname{sgn}\varphi(x_{j_0+2m_0-2}) = -\operatorname{sgn}\varphi(x_{j_0+2m_0-1}),\\
&\dots\\
&\operatorname{sgn}\varphi(x_{k-2}) = -\operatorname{sgn}\varphi(x_{k-1}),
\end{aligned}$$

and

$$0 < x_{j_0+2m_0-3} < x_{j_0+2m_0-2} < \cdots < x_{k-1} < q_2q_3 \cdot \ldots \cdot q_k < q_2q_3 \cdot \ldots \cdot q_k\sqrt{q_{k+1}}.$$

Hence, the function φ has $k - j_0 - 2m_0 + 3$ sign changes in the interval $(0, q_2q_3 \cdot \ldots \cdot q_k\sqrt{q_{k+1}})$, whence the number of real zeros of φ in the disk $\{z : |z| < q_2q_3 \cdots q_k\sqrt{q_{k+1}}\}$ is at least $k - j_0 - 2m_0 + 2$. Therefore, the number of nonreal zeros of φ in this disk is less than or equal to $j_0 + 2m_0 - 2$. Since k is an arbitrary large enough integer, we get that φ has not more than $j_0 + 2m_0 - 2$ nonreal zeros.

Therefore, Theorem 5.4 is proved.

In this chapter, we proved necessary conditions for an entire function $f(x) = \sum_{k=0}^{\infty} a_kx^k, a_k > 0$, with a non-decreasing sequence of its second quotients of Taylor coefficients to belong to the Laguerre–Pólya class of type I. Besides, we obtained an estimation of the possible number of non-real zeros for such functions.

Chapter 6

Further questions

We would like to formulate some open questions for further research.

Question 1. What are the conditions for an entire function with positive coefficients to belong to the Laguerre–Pólya class in case when $\{q_k\}_{k=0}^{\infty}$ is not a monotonic sequence?

As a first attempt, one could consider the case when the second quotients have only 2 values, for instance, $q_k \in \{a, b\}$. For instance, one can look at the following entire function:

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k},$$

where

$$q_2 = q_4 = q_6 = \ldots = a > 1,$$
$$q_3 = q_5 = q_7 = \ldots = b > 1,$$

or

$$f(x) = 1 + x + \frac{x^2}{a} + \frac{x^3}{a^2 b} + \frac{x^4}{a^3 b^2 a} + \frac{x^5}{a^4 b^3 a^2 b} + \ldots$$

and identify conditions for which $\{a, b\}$ the entire function f belongs to the Laguerre-Pólya I class. Partial results for the case when $a < b$ are obtained by T.H. Nguyen and A. Vishnyakova (see [63]), however, the question still remains open for the case $a > b$.

We recall that as a consequence of Hutchinson's theorem (see Chapter 1, Theorem B), it is known that if $q_k(f) \in [4, +\infty)$, then f belongs to the Laguerre–Pólya class.

Question 2. Is it possible to formulate an analogue of Hutchinson's Theorem and to find an interval such that if $q_k(f) \in [a, b]$, $a < 4$ it follows that f belongs to the Laguerre–Pólya class?

Question 3. Is it possible to generalize Hutchinson's constant in Theorem B for the cases of many variables? (The question was posed by Petter Brändén).

Acknowledgements

The research of this thesis was conducted at the Department of Fundamental Mathematics, School of Mathematics and Computer Sciences, V.N. Karazin Kharkiv National University. The research was supported by the Akhiezer Foundation (2019), Ministry of Education and Science of Ukraine research project 0118U002036 “Operators in Banach, Hilbert and functional spaces and Fourier’s quasicrystals” in 2018-2020, and the National Research Foundation of Ukraine funded by Ukrainian State budget in frames of the project 2020.02/0096 “Operators in infinite-dimensional spaces: the interplay between geometry, algebra and topology”, 2020-2022.

First of all, I would like to express my gratitude to my scientific advisor and mentor, Prof. Dr. Anna Vishnyakova, who helped me to go through a dark time in my life and inspired me to pursue my mathematical interest, overcoming the life crisis. I would not complete any part of this research without Anna’s patience, kindness, support, and guidance. She has drawn my attention to the interesting topic of zero distribution of entire functions and, in particular, the beautiful Laguerre–Pólya class which turned into this work and further research. I still remember meeting Anna as the very first lecturer at the Mathematical Department of Kharkiv University when I was a freshman, and she impressed me with very clear, thorough, and detailed explanations. Thanks to her training and the knowledge that Anna has passed to me, I can continue the current research as well as broaden my research area. I am especially deeply grateful for the past 6 years of direct communication with Anna as my advisor since I became a Master student, for always teaching me in her gentle manner and rooting for me in every single problem I have been facing. Thanks to Anna I was able to become the person I am, have the opportunity to meet other wonderful mathematicians and visit different interesting mathematical events and places. No words are enough to express all my feelings towards Anna.

I wish to thank Prof. Dr. Tatjana Eisner who supported me a lot when I had to flee my home city Kharkiv to Germany due to the Russian invasion of Ukraine. Not only did Tatjana help me to continue doing my Ph.D. studies but also helped me with various big and small problems I encountered being a refugee in an unfamiliar country and not knowing the language. Moreover, I would not be able to submit my thesis without Tanja and her patient guidance with all the necessary formal documents.

I would like to highlight that I only managed to finish my thesis and prepare it for submission thanks to the support of the Mathematisches Forschungsinstitut Oberwolfach within the program Oberwolfach Leibniz Fellows, where I could have a shelter, a work place, access to a wonderful library and, more importantly, meet a lot of interesting mathematicians from different places and fields of expertise. I would like to especially thank Prof. Dr. Gerhard Huisken, for considering my special situation and accepting me to the Institute, Prof. Dr. Matthias Hieber, for connecting me with other mathematicians, Ms. Susanne Riester, Ms. Petra Lein, Ms. Silke Okon and the other administrators, for always supporting and helping me in every problems that occured during my stay at the MFO.

I also want to express my gratitude to Prof. Dr. Bernd Sturmfels and Dr. Jörg Lehnert, for inviting me to the seminar at the Max Planck Institute for Mathematics in the Sciences to communicate with the young mathematicians from the research group in Nonlinear Algebra. I would like to thank Mrs. Heike Rackwitz and Ms. Saskia Gutzschebauch, for supporting me with administrative questions and giving me valuable advice to resolve them.

I would like to thank Prof. Dr. Petter Brändén and the organizers of the Göran Gustafsson Symposium, for having me at the Mittag-Lefler Institute and the KTH and inspiring me for further research.

Furthermore, I would like to show my appreciation for all of the teachers in my life. My high school teacher, Tatiana Kukarekina, who was my Maths teacher from secondary school to high school, gave me a solid background in school mathematics. Dr. Mayya Golitsyna has been my mentor and showed me the beauty of Math when I was a teenager and encouraged me to look into the mathematical problems beyond the school program. I was inspired by Mayya to make up my mind to become a mathematician. I would like to express special appreciation to Prof. Dr. Vladimir Kadets who is an excellent lecturer and a patient teacher, for always having my back. I would also like to thank all of my university professors and teachers Prof. Dr. Serhii Favorov, Prof. Dr. Alexander Yamplosky, Dr. Eugene Karolinsky, Prof. Dr. Vyacheslav Gordevskyy, Dr. Natalia Girya, and others.

I should also mention the support by the Dean of our School of Mathematics and Computer Sciences, Prof. Dr. Grygoriy Zholtkevych and the administration of the Department of Fundamental Mathematics. I would like to thank the experts in Complex Analysis from the other parts of Ukraine, Dr. Evgeny Sevost'yanov and Prof. Dr. Volodymyr Dilnyi, and all the referees of our papers, for valuable feedback and comments.

I wish to thank all of my friends who showed their patience, kindness and cheered me up during this long journey.

Last, but not least, I would like to thank my mother who unfortunately passed away when I just finished my Bachelor's Degree, who always supported me in every decision and who I could always rely on, and my father who was very patient and understanding when I could not complete all of my responsibilities as a daughter due to work overload. I am forever grateful to my parents who brought me up to become who I am and worked hard for me to get an education and choose my own career path.

Thu Hien Nguyen

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Dissertation selbständig und ohne unzulässige fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die angeführten Quellen und Hilfsmittel benutzt und sämtliche Textstellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen Personen bereitgestellten Materialen oder erbrachten Dienstleistungen als solche gekennzeichnet.

....
(Ort, Datum)

....
(Unterschrift)

Curriculum Vitae

Education

- *September 2012 – June 2016*: V. N. Karazin Kharkiv National University, Ukraine, School of Mathematics and Computer Sciences, major: Pure Mathematics, Bachelor of Science.

- *September 2016 – June 2018*: V. N. Karazin Kharkiv National University, Ukraine, School of Mathematics and Computer Sciences, major: Pure Mathematics, Master of Science (with honours), Advisor: Prof. Anna Vishnyakova.

- *October 2018 – February 2022*: V. N. Karazin Kharkiv National University, Ukraine, Department of Fundamental Mathematics, Ph.D. studies (interrupted due to Russian invasion of Ukraine). Advisor: Prof. Anna Vishnyakova.

- *May 2022 – present*: University of Leipzig, Germany, Faculty of Mathematics and Computer Science, Ph.D. student.

Positions

- *September 2018 – December 2021*: V. N. Karazin National University, School of Mathematics and Computer Sciences, Department of Fundamental Mathematics, Teaching & Scientific Assistant. Undergraduate courses: Introduction to Algebra and Number Theory, Linear Algebra.

- *1 April 2022 – 30 June 2022*: Mathematical Research Institute Oberwolfach, Oberwolfach Leibniz Fellow.

- *from 1 July 2022*: University of Leipzig, Institute of Mathematics, Scientific Assistant.

Publications

1. On the entire functions from the Laguerre–Pólya class having the decreasing second quotients of Taylor coefficients, Journal of Mathematical Analysis and Applications, 465, 348 – 358 (2018). (with Anna Vishnyakova)

2. On a necessary condition for an entire function with the increasing second quotients of Taylor coefficients to belong to the Laguerre–Pólya class, Journal of Mathematical Analysis and Applications, 480, 123433 (2019). (with Anna Vishnyakova)

3. On the closest to zero roots and the second quotients of Taylor coefficients of entire functions from the Laguerre–Pólya I class, Results in Mathematics, 75, 115 (2020). (with Anna Vishnyakova)

4. On the entire functions from the Laguerre–Pólya I class having the increasing second quotients of Taylor coefficients, Journal of Mathematical Analysis and Applications, 498, 124955 (2021). (with Anna Vishnyakova)

5. On entire functions from the Laguerre–Pólya I class with non-monotonic second quotients of Taylor coefficients, Matematychni Studii, 56, 149 – 161 (2021). (with Anna Vishnyakova)

6. On the number of real zeros of real entire functions with a non-decreasing sequence of the second quotients of Taylor coefficients, Mathematical Inequalities and Applications, 25, 73 – 89, (2022). (with Anna Vishnyakova)

7. On the conditions for a special entire function related to the partial theta-function and the q-Kummer functions to belong to the Laguerre–Pólya class, Computational Methods and Function Theory, 22, 7 – 25 (2022).

Scholarships and Grants

1. Akhiezer Foundation Scholarship, March 12, 2019 (one-time payment).

2. Ukrainian Ministry of Science and Education Research Program 0118U002036, September 1, 2018 – December 31, 2019.

3. Support by the National Research Foundation of Ukraine funded by the Ukrainian State budget in frames of the project 2020.02/0096 “Operators

in infinite-dimensional spaces: the interplay between geometry, algebra, and topology", October 1, 2020 – December 31, 2021 (contract should have been renewed from March 1, 2022, but was terminated due to Russian invasion of Ukraine).

4. Oberwolfach Leibniz Fellowship, Mathematical Research Institute Oberwolfach within the program Oberwolfach Leibniz Fellow, April 1 – June 30, 2022.

Bibliography

[1] Abdul Aziz. "On composite polynomials whose zeros are in a half-plane". In: *Bulletin of the Australian Mathematical Society* 36.3 (1987), pp. 449–460 (cit. on p. 9).

[2] Abdul Aziz. "On the location of the zeros of certain composite polynomials". In: *Pacific Journal of Mathematics* 118.1 (1985), pp. 17–26 (cit. on p. 9).

[3] Andrew G. Bakan and Anatoly P. Golub. "On sequences that do not increase the number of real roots of polynomials". In: *Ukrainian Mathematical Journal* 45.10 (1993), pp. 1481–1489 (cit. on p. 6).

[4] Andrew G. Bakan et al. "Weakly increasing zero-diminishing sequences". In: *Serdica Mathematical Journal* 22.4 (1996), pp. 547–570 (cit. on p. 6).

[5] Árpád Baricz and Sanjeev Singh. "Zeros of some special entire functions". In: *Proceedings of the American Mathematical Society* 146.5 (2018), pp. 2207–2216 (cit. on pp. ix, 3, 13).

[6] Harry Bateman et al. *Higher transcendental functions.* 3 vols. New York: McGraw–Hill Book Company, 1953 (cit. on p. 7).

[7] Walter Bergweiler and Alexandre Eremenko. "Proof of a conjecture of Pólya on the zeros of successive derivatives of real entire functions". In: *Acta mathematica* 197.2 (2006), pp. 145–166 (cit. on p. ix).

[8] Walter Bergweiler, Alexandre Eremenko, and James Langley. "Real entire functions of infinite order and a conjecture of Wiman". In: *Geometric and Funcional Analysis* 13.5 (2003), pp. 975–991 (cit. on p. ix).

[9] Felix Bernstein and Otto Szász. "Über Irrationalität unendlicher Kettenbrüche mit einer Anwendung auf die Reihe $\sum_{\nu=0}^{\infty} q^{\nu^2} x^{\nu}$". In: *Mathematische Annalen* 76.2-3 (1915), pp. 295–300 (cit. on p. 10).

[10] Anton Bohdanov. "Determining bounds on the values of parameters for a function $\varphi_a(z,m) = \sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}} (k!)^m$, $m \in (0,1)$, to belong to the Laguerre–Pólya class". In: *Computational Methods and Function Theory* 18.1 (2018), pp. 35–51 (cit. on pp. ix, 3, 13, 14).

[11] Anton Bohdanov and Anna Vishnyakova. "On the conditions for entire functions related to the partial theta-function to belong to the Laguerre–Pólya class". In: *Journal of Mathematical Analysis and Applications* 434.2 (2016), pp. 1740–1752 (cit. on pp. ix, 3, 13, 14).

[12] Petter Brändén. "On operators on polynomials preserving real-rootedness and the Neggers-Stanley conjecture". In: *Journal of Algebraic Combinatorics* 20.2 (2004), pp. 119–130 (cit. on pp. ix, 3).

[13] David A. Cardon. "Complex zero strip decreasing operators". In: *Journal of Mathematical Analysis and Applications* 426.1 (2015), pp. 406–422 (cit. on pp. ix, 3).

[14] David A. Cardon. "Convolution operators and zeros of entire functions". In: *Proceedings of the American Mathematical Society* 130.6 (2002), pp. 1725–1734 (cit. on pp. ix, 3).

[15] Thomas Craven and George Csordas. "Complex zero decreasing sequences". In: *Methods and Applications of Analysis* 2.4 (1995), pp. 420–441 (cit. on pp. 4, 6).

[16] Thomas Craven and George Csordas. "Composition theorems, multiplier sequences and complex zero decreasing sequences". In: *Value distribution theory and related topics*. Advances in Complex Analysis and its Applications 3. Boston, MA: Kluwer Academic Publishers, 2004, pp. 131–166 (cit. on pp. ix, 3, 5).

[17] Thomas Craven and George Csordas. "Multiplier sequences for fields". In: *Illinois Journal of Mathematics* 21.4 (1977), pp. 801–817 (cit. on p. 5).

[18] Thomas Craven and George Csordas. "On a converse of Laguerre's theorem". In: *Electronic Transactions on Numerical Analysis* 5 (Mar. 1997), pp. 7–17 (cit. on pp. 6, 7).

[19] Thomas Craven and George Csordas. "Problems and theorems in the theory of multiplier sequences". In: *Serdica Mathematical Journal* 22.4 (1996), pp. 515–524 (cit. on p. 5).

[20] Thomas Craven and George Csordas. "The Fox–Wright functions and Laguerre multiplier sequences". In: *Journal of Mathematical Analysis and Applications* 314.1 (2006), pp. 109–125 (cit. on p. 5).

[21] Thomas Craven, George Csordas, and Wayne Smith. "Zeros of derivatives of entire functions". In: *Proceedings of the American Mathematical Society* 101.2 (1987), pp. 323–326 (cit. on p. viii).

[22] George Csordas. "Complex zero decreasing sequences and the Riemann hypothesis II". In: 3rd Congress of the International Society for Analysis, its Applications and Computation (Aug. 2001). Ed. by Heinrich G. W. Begehr, Robert P. Gilbert, and Man Wah Wong. International Society for Analysis, Applications and Computation 10. Berlin, 2003, pp. 121–134 (cit. on p. viii).

[23] George Csordas and Tamás Forgácss. "Multiplier sequences, classes of generalized Bessel functions and open problems". In: *Journal of Mathematical Analysis and Applications* 433.2 (2016), pp. 1368–1389 (cit. on pp. ix, 3, 5).

[24] George Csordas, Timothy S. Norfolk, and Richard S. Varga. "The Riemann hypothesis and the Turán inequalities". In: *Transactions of the American Mathematical Society* 296.2 (1986), pp. 521–541 (cit. on p. viii).

[25] George Csordas and Richard S. Varga. "Moment inequalities and the Riemann hypothesis". In: *Constructive Approximation* 4.2 (1988), pp. 175–198 (cit. on p. viii).

[26] George Csordas and Richard S. Varga. "Necessary and sufficient conditions and the Riemann hypothesis". In: *Advances in Applied Mathematics* 11.3 (1990), pp. 328–357 (cit. on p. viii).

[27] George Csordas and Anna Vishnyakova. "The generalized Laguerre inequalities and functions in the Laguerre–Pólya class". In: *Central European Journal of Mathematics* 11.9 (2013), pp. 1643–1650 (cit. on pp. ix, 3).

[28] George Csordas and Chung-Chun Yang. "On the zeros of the Riemann ζ-function". In: *Southwest Journal of Pure and Applied Mathematics* 1 (2003), pp. 33–42 (cit. on p. viii).

[29] Dimitar K. Dimitrov. "Higher order Turán inequalities". In: *Transactions of the American Mathematical Society* 126.7 (1998), pp. 2033–2037 (cit. on p. viii).

[30] Gotthold Eisenstein. "Transformations remarquables de quelques séries". In: *Journal für die Reine und Angewandte Mathematik* 27 (1844), pp. 193–197 (cit. on p. 10).

[31] Alexandre Eremenko. *Topics in entire functions. Lectures in Kent University.* Mar. 2015. URL: `https://www.math.purdue.edu/~eremenko/dvi/kent.pdf` (cit. on pp. viii, 3).

[32] George Gasper and Mizan Rahman. *Basic hypergeometric series*. 2nd ed. Encyclopedia of Mathematics and Its Applications 96. Cambridge: Cambridge University Press, 2004 (cit. on p. 29).

[33] Anatoly A. Goldberg and Iosif V. Ostrovskiĭ. *Value distribution of meromorphic functions*. Trans. by Mikhail Ostrovskii. Translations of Mathematical Monographs 236. Providence, RI: American Mathematical Society, 2008 (cit. on p. 1).

[34] Adolph W. Goodman and Isaac J. Schoenberg. "A proof of Grace's theorem by induction". In: *Honam Mathematical Journal* 9.1 (1987), pp. 1–6 (cit. on pp. 8, 9).

[35] Emil Grosswald. "Recent applications of some old work of Laguerre". In: *American Mathematical Monthly* 86.8 (1979), pp. 648–658 (cit. on p. 8).

[36] G. H. Hardy. "On the zeros of a class of integral functions". In: *The Messenger of Mathematics* 34 (1904), pp. 97–101 (cit. on pp. 4, 11).

[37] G. H. Hardy. "On the zeros of a class of integral functions". In: *Collected papers*. Vol. 4. Ed. by A committee appointed by the London Mathematical Society. Oxford: Clarendon Press, 1969, pp. 95–99 (cit. on p. 4).

[38] Bing He. "On zeros of some entire functions". In: *Results in Mathematics* 74.1, 52 (2019) (cit. on pp. ix, 3, 13).

[39] Eduarde Heine. "Über die Reihe $1 + \frac{(q^\alpha-1)(q^\beta-1)}{(q-1)(q^\gamma-1)}x + \frac{(q^\alpha-1)(q^{\alpha+1}-1)(q^\beta-1)(q^{\beta+1}-1)}{(q-1)(q^2-1)(q^\gamma-1)(q^{\gamma+1}-1)}x^2 + \cdots$". In: *Journal für die Reine und Angewandte Mathematik* 32 (1846), pp. 210–212 (cit. on p. 10).

[40] Isidore I. Hirschman and David V. Widder. *The convolution transform*. Princeton, NJ: Princeton University Press, 1955 (cit. on pp. viii, 1–3).

[41] J. I. Hutchinson. "On a remarkable class of entire functions". In: *Transactions of the American Mathematical Society* 25.3 (1923), pp. 325–332 (cit. on pp. 4, 11, 54, 61).

[42] Haseo Ji and Young-One Kim. "On the number of nonreal zeros of real entire functions and the Fourier—Pólya conjecture". In: *Duke Mathematical Journal* 104.1 (2000), pp. 45–73 (cit. on p. ix).

[43] Samuel Karlin. *Total positivity*. Vol. 1. Stanford, CA: Stanford University Press, 1968 (cit. on pp. viii, 1, 3, 6).

[44] Samuel Karlin and James L. McGregor. "The differential equations of birth and death processes, and the Stieltjes moment problem". In: *Transactions of the American Mathematical Society* 85 (1957), pp. 489–546 (cit. on p. 1).

[45] Irina Karpenko and Anna Vishnyakova. "On sufficient conditions for a polynomial to be sign-independently hyperbolic or to have real separated zeros". In: *Mathematical Inequalities and Applications* 20.1 (2017), pp. 237–245 (cit. on p. 11).

[46] Olga M. Katkova. *Multiple positivity and the Riemann zeta-function.* 2005. arXiv: `math/0505174 [math.CV]` (cit. on p. viii).

[47] Olga M. Katkova, Tetyana Lobova, and Anna Vishnyakova. "On power series having sections with only real zeros". In: *Computational Methods and Function Theory* 3.2 (2003), pp. 425–441 (cit. on pp. 11, 14, 15, 54).

[48] Olga M. Katkova and Anna Vishnyakova. "On the stability of Taylor sections of a function $\sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}}$, $a > 1$". In: *Computational Methods and Function Theory* 9.1 (2009), pp. 305–322 (cit. on p. 11).

[49] Vladimir Petrov Kostov. "A property of a partial theta function". In: *Dokladi na Bolgarskata Akademiya na Naukite–Comptes Rendus de l'Académie Bulgare des Sciences* 67.10 (2014), pp. 1319–1326 (cit. on p. 12).

[50] Vladimir Petrov Kostov. "Asymptotics of the spectrum of partial theta function". In: *Revista Matemática Complutense* 27.2 (2014), pp. 677–684 (cit. on p. 12).

[51] Vladimir Petrov Kostov. "On a partial theta function and its spectrum". In: *Proceedings of the Royal Society of Edinburgh—Section A: Mathematics* 146.3 (2016), pp. 609–623 (cit. on p. 12).

[52] Vladimir Petrov Kostov. "On the zeros of a partial theta function". In: *Bulletin des Sciences Mathématiques* 137.8 (2013), pp. 1018–1030 (cit. on pp. 11, 12).

[53] Vladimir Petrov Kostov. "The closest to 0 spectral number of the partial theta function". In: *Dokladi na Bolgarskata Akademiya na Naukite–Comptes Rendus de l'Académie Bulgare des Sciences* 69.9 (2016), pp. 105–1112 (cit. on p. 12).

[54] Vladimir Petrov Kostov and Boris Shapiro. "Hardy-Petrovitch-Hutchinson's problem and partial theta function". In: *Duke Mathematical Journal* 162.5 (2013), pp. 825–861 (cit. on pp. 11, 12).

[55] Edmond Laguerre. “Sur quelques points de la théorie des équations numériques”. In: *Acta mathematica* 4.1 (1884), pp. 97–120 (cit. on p. 6).

[56] Martin Lamprecht. “Suffridge’s convolution theorem for polynomials and entire functions having only real zeros”. In: *Advances in Mathematics* 288 (2016), pp. 426–463 (cit. on pp. ix, 3).

[57] Boris Yakovlevich Levin. *Distribution of zeros of entire functions*. Trans. by R. P. Boas et al. Revised edition. Translations of Mathematical Monographs 5. Providence, RI: American Mathematical Society, 1980 (cit. on pp. viii, 1–3, 5).

[58] Morris Marden. *Geometry of polynomials*. 2nd ed. Mathematical Surveys 3. Providence, RI: American Mathematical Society, 1966 (cit. on pp. 8, 9).

[59] Thu Hien Nguyen. “On the conditions for a special entire function related to the partial theta-function and the q-Kummer functions to belong to the Laguerre-Pólya class”. In: *Computational Methods and Function Theory* 22.1 (2022), pp. 7–25 (cit. on pp. 29, 30, 33, 34, 36, 41, 43, 47).

[60] Thu Hien Nguyen and Anna Vishnyakova. “On a necessary condition for an entire function with the increasing second quotients of Taylor coefficients to belong to the Laguerre–Pólya class”. In: *Journal of Mathematical Analysis and Applications* 480.2, 123433 (2019) (cit. on pp. 15, 16, 18, 22, 26, 27, 31, 54, 70).

[61] Thu Hien Nguyen and Anna Vishnyakova. “On the closest to zero roots and the second quotients of Taylor coefficients of entire functions from the Laguerre-Pólya I class”. In: *Results in Mathematics* 75.3, 115 (2020) (cit. on pp. 51–54, 60, 70).

[62] Thu Hien Nguyen and Anna Vishnyakova. “On the entire functions from the Laguerre–Pólya class having the decreasing second quotients of Taylor coefficients”. In: *Journal of Mathematical Analysis and Applications* 465.1 (2018), pp. 348–358 (cit. on pp. 13, 15).

[63] Thu Hien Nguyen and Anna Vishnyakova. “On the entire functions from the Laguerre–Pólya I class with non-monotonic second quotients of Taylor coefficients”. In: *Matematychni Studii* 56.2 (2022), pp. 149–161. arXiv: `2107.13061 [math.CV]` (cit. on p. 95).

[64] Thu Hien Nguyen and Anna Vishnyakova. “On the entire functions from the Laguerre-Pólya I class having the increasing second quotients of Taylor coefficients”. In: *Journal of Mathematical Analysis and Applications* 498.1, 124955 (2021) (cit. on pp. 69, 70, 77).

[65] Thu Hien Nguyen and Anna Vishnyakova. "On the number of real zeros of real entire functions with a non-decreasing sequence of the second quotients of Taylor coefficients". In: *Mathematical Inequalities and Applications* 25.1 (2022), pp. 73–89 (cit. on pp. 81–84, 88).

[66] Nikola Obreschkov. *Verteilung und Berechnung der Nullstellen reeller Polynome*. Berlin: VEB Deutscher Verlag der Wissenschaften, 1963 (cit. on pp. viii, 2, 3, 5, 6, 8, 9).

[67] Tetyana Lobova Olga Katkova and Anna Vishnyakova. "On entire functions having Taylor sections with only real zeros". In: *Matematicheskaya Fizika, Analiz, Geometriya. Khar'kovskiĭ Matematicheskiĭ Zhurnal* 11.4 (2004), pp. 449–469 (cit. on pp. 12–14).

[68] Iosif Vladimirovich Ostrovskiĭ. "On zero distribution of sections and tails of power series". In: Entire functions in modern analysis (1997). Israel Mathematical Conference Proceedings 15. Tel Aviv, 2001, pp. 297–310 (cit. on pp. 1, 11).

[69] Iosif Vladimirovich Ostrovskiĭ and I. N. Peresyolkova. "Nonasymptotic results on distribution of zeros of the function $E_\rho(z,\mu)$". In: *Analysis Mathematica* 23.4 (1997), pp. 283–296 (cit. on pp. 7, 14).

[70] Tamás Erdélyi Peter Borwein. *Polynomials and polynomial inequalities*. Graduate Texts in Mathematics 161. New York: Springer-Verlag, 1995 (cit. on pp. 8, 9).

[71] M. Petrovich. "Une classe remarquable de séries entières". In: IV Congresso Internazionale dei Mathematicci (Apr. 1908). Vol. 2. ICM Proceedings. Roma, 1909, pp. 36–43 (cit. on pp. 4, 11).

[72] Georg Pólya. "Über einen Satz von Laguerre". In: *Pólya: his collected works*. Vol. 2: *Location of zeros*. Ed. by R. P. Boas. Massachusetts Institute of Mathematics, 1974, pp. 314–321 (cit. on p. 6).

[73] Georg Pólya and Issai Schur. "Über zwei Arten vol Faktorenfolgen in der Theorie der algebraische Gleichungen". In: *Pólya: his collected works*. Vol. 2: *Location of zeros*. Ed. by R. P. Boas. Massachusetts Institute of Mathematics, 1974, pp. 54–70 (cit. on pp. 2, 5).

[74] George Pólya and Gabor Szegö. *Problems and theorems in analysis*. 2 vols. Classics in Mathematics. Reprint of the 1978 English translation. Springer Science+Business Media, 1998 (cit. on pp. viii, 2, 3, 8, 9).

[75] Srinivasa Ramanujan. *The lost notebook and other unpublished papers*. New Delhi: Narosa Publishing House, 1988 (cit. on p. 10).

[76] Bernhard Riemann. “Über die Anzahl der Primzahlen unter einer gegebenen Grosse [On the number of primes less than a given magnitude]”. In: *Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin* (1859), pp. 671–680 (cit. on p. viii).

[77] Zalman Rubinstein. “Remarks on a paper by A. Aziz”. In: *Proceedings of the American Mathematical Society* 94.2 (1985), pp. 236–238 (cit. on p. 9).

[78] Isaac J. Schoenberg and Anne Whitney. “On Pólya frequence functions, III: The positivity of translation determinants with an application to the interpolation problem by spline curves”. In: *Transactions of the American Mathematical Society* 74 (1953), pp. 246–259 (cit. on p. 1).

[79] Issai Schur. “Zwei Sätze über algebraische Gleichungen mit lauter reellen Wurzeln”. In: *Journal für die Reine und Angewandte Mathematik* 144 (1914), pp. 75–88 (cit. on p. 9).

[80] Alan D. Sokal. “The leading root of the partial theta function”. In: *Advances in Mathematics* 229.5 (2012), pp. 2603–2621 (cit. on pp. ix, 3, 11, 12).

[81] Gabor Szegö. “Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen”. In: *Mathematische Zeitschrift* 13.1 (1922), pp. 28–55 (cit. on p. 9).

[82] Ljubomir Tschakaloff. “Arithmetische Eigenschaften der unendlichen Reihe $\sum_{\nu=0}^{\infty} \frac{x^\nu}{a^{\frac{\nu(\nu-1)}{2}}}$”. In: *Mathematische Annalen* 80.1 (1919), pp. 62–74 (cit. on p. 10).

[83] Ljubomir Tschakaloff. “Arithmetische Eigenschaften der unendlichen Reihe $\sum_{\nu=0}^{\infty} \frac{x^\nu}{a^{\frac{\nu(\nu-1)}{2}}}$”. In: *Mathematische Annalen* 84.1-2 (1921), pp. 100–114 (cit. on p. 10).

[84] Anna Vishnyakova. *Problem 8.2*. Ed. by Matthew Chasse and Olga Holtz. URL: `http://aimpl.org/hyperbolicpoly/8/` (cit. on p. 30).

[85] Joseph L. Walsh. “On the location of the roots of certain types of polynomials”. In: *Transactions of the American Mathematical Society* 24.3 (1922), pp. 163–180 (cit. on p. 11).

[86] S. Ole Warnaar. *Partial theta functions*. Sept. 2018. URL: `https://www.researchgate.net/publication/327791878_Partial_theta_functions` (cit. on p. 10).

[87] S. Ole Warnaar. “Partial theta functions. I. Beyond the lost notebook”. In: *Proceedings of the London Mathematical Society*. 3rd ser. 87.2 (2003), pp. 363–395 (cit. on p. 11).

[88] Edmund T. Whittaker and George N. Watson. *A course of modern analysis*. 4th ed. Cambridge: Cambridge University Press, 1952 (cit. on p. 10).