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# The Dual Expression of Parallel Equidistant Ruled Surfaces in Euclidean 3-Space

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**Abstract:** In this study, we examine the dual expression of Valeontis' concept of parallel $p$-equidistant ruled surfaces well known in Euclidean 3-space, according to the Study mapping. Furthermore, we show that the dual part of the dual angle on the unit dual sphere corresponds to the $p$-distance. We call these ruled surfaces we obtained "dual parallel equidistant ruled surfaces" and we briefly denote them with "DPERS". Furthermore, we find the Blaschke vectors, the Blaschke invariants and the striction curves of these DPERS and we give the relationships between these elements. Moreover, we show the relationships between the Darboux screws, the instantaneous screw axes, the instantaneous dual Pfaff vectors and dual Steiner rotation vectors of these surfaces. Finally, we give an example, which we reinforce this article, and we explain all of these features with the figures on the example. Furthermore, we see that the corresponding dual curves on the dual unit sphere to these DPERS are such that one of them is symmetric with respect to the imaginary symmetry axis of the other.

**Keywords:** dual parallel equidistant ruled surfaces; Study mapping; Blaschke vectors; Blaschke invariants; dual Steiner vector

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## 1. Introduction

E. Study (1901) made the first applications to the geometry of dual numbers [1], which were first conceived by Clifford (1873) [2]. Dual numbers and dual vectors, which have been used by many researchers in various fields since then, have recently become important for studies in the space of lines. As for the concept of the ruled surfaces, they are determined with the movement of any line along any curve. The Study mapping is already based on the relationship between the geometry of the any line in $\mathbb{E}^3$ and the geometry of any point on the unit dual sphere. That is, there exists one-to-one correspondence between the directed lines in line space and dual unit vectors on the dual vector space ($\mathbb{D}$-Modul). According to the correspondence principle, ruled surfaces specify a curve on the unit dual sphere, and this curve is called the dual spherical indicatrix of the ruled surface. The basic concepts related to the ruled surfaces in Euclidean space and on $\mathbb{D}$-Modul exist in many sources. Some of them are from Biran [3], Blaschke [4], Hacisalihoglu [5,6], Hagemann et al. [7], Muller [8], Ozdemir [9], Sabuncuoglu [10] and Senatalar [11]. Moreover, Ali and Abdel Aziz [12], Bilici [13] Oral and Kazaz [14], Saracoglu and Yayli [15,16] and Schaaf [17] have studied the various ruled surfaces. Moreover, several authors have worked on the integral invariants of closed ruled surfaces on the dual vector space [18–21]. As a new concept, Valeontis (1986) has defined parallel $p$-equidistant ruled surfaces (the tangent vectors of two ruled surfaces along the striction curves are parallel and at the same time the distance ($p$-distance) between the corresponding points of the polar planes of these surfaces is constant) and he has given their some features [22]. Based on this definition, Masal and Kuruoglu [23–26] have calculated some characteristic properties of these surfaces in Euclidean space and Minkowski space. Senyurt and As [27] have obtained some features of the parallel $z$-equidistant ruled surfaces obtained by paralleling normal vectors. Furthermore, Fenchel [28] has given the expression of the unit vector in direction of Darboux vector on the curve. In a similar vein, Sarioglugil et al. [29] and Senyurt [30] have studied on the ruled surface determined by the dual centroit curve. On the other hand, the dual vector space concept has been applied to research in kinematics, analysis and synthesis of spatial mechanisms, and many other fields. Some techniques obtained using dual vectors provide advantages in studies on robot kinematics [31]. Saglamer has [32] made robotic calculations in the kinematics of coordination between industrial robots using real and dual quaternion algebra. Sahiner [33] has studied differential properties of motion of a robot end-effector by using the curvature theory of dual spherical curves. In this paper, we comprehensively introduce the dual expression of parallel $p$-equidistant ruled surfaces in the space of lines according to the Study mapping. Furthermore, we show that the dual part of the dual angle on the unit dual sphere corresponds to the $p$-distance. Moreover, we give the relationships between the elements of these surfaces and finally reinforce these surfaces with an example. Finally, we see in this study that the corresponding dual curves on the dual unit sphere to these parallel equidistant ruled surfaces are such that one of them is symmetric with respect to the imaginary symmetry axis of the other. In the future, we are going to combine the result in the paper with singularity theory, sub-manifold theory, etc. from [34–41] to obtain new results and theorems. As a result of this study, further studies can be considered on the following research questions:

- What applications might this study have in the robot kinematics?
- What kind of contributions can this study make to the field of physics if it is studied in the different spaces?

## 2. Preliminaries

### *2.1. Some Preliminaries on $\mathbb{D}$-Modul*

The addition and the multiplication operations on the dual numbers set $\mathbb{D} = \{A = a + \varepsilon a^* \mid a, a^* \in \mathbb{R},\ \varepsilon^2 = 0\}$ for the dual numbers $A$ and $B$ are stated as follows:

$$\begin{aligned} A + B = (a + \varepsilon a^*) + (b + \varepsilon b^*) &= (a + b) + \varepsilon(a^* + b^*)\,, \\ A \cdot B = (a + \varepsilon a^*) \cdot (b + \varepsilon b^*) &= ab + \varepsilon(ab^* + a^* b)\,, \end{aligned}$$

where $a$ and $a^*$ are the real part and the imaginary part of the dual number $A$, respectively [2]. The set $\mathbb{D}$ is a commutative ring with these operations. The division operation for the dual numbers $A$ and $B$, with $b \neq 0$, is stated as the following:

$$\frac{A}{B} = \frac{a + \varepsilon a^*}{b + \varepsilon b^*} = \frac{a}{b} + \varepsilon \frac{a^* b - ab^*}{b^2}.$$

The modulus of a dual number $A = a + \varepsilon a^*$ is just the absolute value of the real part $|a|$.

The set $\mathbb{D} \times \mathbb{D} \times \mathbb{D} = \mathbb{D}^3 = \left\{ \vec{A} \mid \vec{A} = \vec{a} + \varepsilon \vec{a^*},\ \vec{a}, \vec{a^*} \in \mathbb{E}^3 \right\}$, called the dual vector space, is a module with the following addition and scalar product operations on the ring $\mathbb{D}$ and the set denoted by $\mathbb{D} - Modul$:

$$\oplus \; : \; \mathbb{D} - Modul \times \mathbb{D} - Modul \to \mathbb{D} - Modul,\ \vec{A} \oplus \vec{B} = \left(\vec{a} + \vec{b}\right) + \varepsilon\left(\vec{a^*} + \vec{b^*}\right), \quad (1)$$

$$\odot \; : \; \mathbb{D} \times \mathbb{D} - Modul \to \mathbb{D} - Modul,\ \Lambda \odot \vec{A} = \lambda \vec{a} + \varepsilon\left(\lambda \vec{a^*} + \lambda^* \vec{a}\right). \quad (2)$$

The inner and the vectorel product operations for $\vec{A}, \vec{B} \in \mathbb{D} - Modul$ are defined by, respectively:

$$\langle , \rangle \;:\; \mathbb{D}-Modul \times \mathbb{D}-Modul \to \mathbb{D}, \quad \left\langle \vec{A}, \vec{B} \right\rangle = \left\langle \vec{a}, \vec{b} \right\rangle + \varepsilon\left( \left\langle \vec{a}, \vec{b^*} \right\rangle + \left\langle \vec{a^*}, \vec{b} \right\rangle \right), \tag{3}$$

$$\wedge \;:\; \mathbb{D}-Modul \times \mathbb{D}-Modul \to \mathbb{D}-Modul, \; \vec{A} \wedge \vec{B} = \left( \vec{a} \wedge \vec{b} \right) + \varepsilon\left( \vec{a} \wedge \vec{b^*} + \vec{a^*} \wedge \vec{b} \right). \tag{4}$$

The dual angle $\Phi = \phi + \varepsilon\phi^*$ between the unit dual vectors $\vec{A}$ and $\vec{B}$ is stated as follows:

$$\begin{aligned} \left\langle \vec{A}, \vec{B} \right\rangle &= \cos\Phi = \cos\phi - \varepsilon\phi^* \sin\phi, \\ \vec{A} \wedge \vec{B} &= \sin\Phi \vec{N} = (\sin\phi + \varepsilon\phi^* \cos\phi)\vec{N}, \end{aligned} \tag{5}$$

where $\phi$ is the real angle between $\vec{a}$ and $\vec{b}$ lines and $\phi^*$ is the shortest distance between $\vec{a}$ and $\vec{b}$ lines, while $\vec{N} = \vec{n} + \varepsilon\vec{n^*}$ is unit dual vector [4,5]. For $\vec{A} \neq 0$, the norm $\left\| \vec{A} \right\|$ of the dual vector $\vec{A} = \vec{a} + \varepsilon\vec{a^*}$ is defined by:

$$\left\| \vec{A} \right\| = \sqrt{\left\langle \vec{A}, \vec{A} \right\rangle} = \left\| \vec{a} \right\| + \varepsilon \frac{\left\langle \vec{a}, \vec{a^*} \right\rangle}{\left\| \vec{a} \right\|}, \quad \left\| \vec{a} \right\| \neq 0. \tag{6}$$

The set $\mathbb{K} = \{ \vec{A} = \vec{a} + \varepsilon\vec{a^*} \mid \| \vec{A} \| = (1,0), \vec{a}, \vec{a^*} \in \mathbb{E}^3 \}$ is called the unit dual sphere [5].

**Theorem 1** (E. Study Mapping)**.** *The directed lines in $\mathbb{E}^3$ are in one to one correspondence with the dual points of the unit dual sphere [1].*

According to Study theorem, a ruled surface on the space of lines corresponds to a curve on the unit dual sphere. The dual equation of the ruled surface with the base curve $(\alpha)$ and the generating line $\vec{a}(s)$ is defined by:

$$\vec{\varphi}(s,v) = \vec{a}(s) \wedge \vec{a^*}(s) + v\,\vec{a}(s), \qquad \vec{a^*}(s) = \vec{\alpha}(s) \wedge \vec{a}(s) \tag{7}$$

in the space of lines ($\mathbb{E}^3$). The dual spherical curve $\vec{A}(s) = \vec{a}(s) + \varepsilon\vec{a^*}(s)$, with $\vec{A}$ being a unit dual vector, represents by the ruled surface $\vec{\varphi}(s,v)$, where $\vec{a}(s) \wedge \vec{a^*}(s)$ is the base curve of the ruled surface [4,5]. Let $\{\vec{U_1}(s), \vec{U_2}(s), \vec{U_3}(s)\}$ be the dual orthonormal moving system of the ruled surface $\vec{A}(s)$, where $\vec{U_1}(s) = \vec{u_1}(s) + \varepsilon\vec{u_1^*}(s)$, $\vec{U_2}(s) = \vec{u_2}(s) + \varepsilon\vec{u_2^*}(s)$, $\vec{U_3}(s) = \vec{u_3}(s) + \varepsilon\vec{u_3^*}(s)$. These unit dual vectors are defined by the following:

$$\vec{U_1}(s) = \vec{A}(s), \quad \vec{U_2}(s) = \frac{\vec{A'}(s)}{\left\| \vec{A'}(s) \right\|}, \quad \vec{U_3}(s) = \vec{U_1}(s) \wedge \vec{U_2}(s) \tag{8}$$

and are called Blaschke vectors [4]. There are the following relations between the Blaschke vectors and their derivative vectors:

$$\vec{U_1'}(s) = P(s)\vec{U_2}(s), \quad \vec{U_2'}(s) = -P(s)\vec{U_1}(s) + Q(s)\vec{U_3}(s), \quad \vec{U_3'}(s) = -Q(s)\vec{U_2}(s),$$

and here:

$$P(s) = p(s) + \varepsilon p^*(s) = \left\| \vec{A'}(s) \right\|, \quad Q(s) = q(s) + \varepsilon q^*(s) = \frac{\left( \vec{A}(s), \vec{A'}(s), \vec{A''}(s) \right)}{P^2(s)}, \tag{9}$$

are called Blaschke invariants [4]. The real and dual parts of these vectors are:

$$\begin{cases} \overrightarrow{u_1'}(s) = p(s)\overrightarrow{u_2}(s), \\ \overrightarrow{u_2'}(s) = -p(s)\overrightarrow{u_1}(s) + q(s)\overrightarrow{u_3}(s), \\ \overrightarrow{u_3'}(s) = -q(s)\overrightarrow{u_2}(s), \\ \\ \overrightarrow{u_1^{*\prime}}(s) = p^*(s)\overrightarrow{u_2}(s) + p(s)\overrightarrow{u_2^*}(s), \\ \overrightarrow{u_2^{*\prime}}(s) = q^*(s)\overrightarrow{u_3}(s) + q(s)\overrightarrow{u_3^*}(s) - p(s)\overrightarrow{u_1^*}(s) - p^*(s)\overrightarrow{u_1}(s), \\ \overrightarrow{u_3^{*\prime}}(s) = -q^*(s)\overrightarrow{u_2}(s) - q(s)\overrightarrow{u_2^*}(s), \end{cases} \tag{10}$$

where $\{\overrightarrow{u_1}(s), \overrightarrow{u_2}(s), \overrightarrow{u_3}(s)\}$ is the Frenet frame, $p(s) = k_1(s)$ is the curvature and $q(s) = k_2(s)$ is the torsion of curve $\overrightarrow{\alpha}(s)$. According to Expression (7), the equations of the ruled surfaces $\left(\overrightarrow{U_1}\right), \left(\overrightarrow{U_2}\right), \left(\overrightarrow{U_3}\right)$ are as follows:

$$\begin{cases} \left(\overrightarrow{U_1}\right) \equiv \overrightarrow{\Psi_{U_1}}(s,v) = \overrightarrow{u_1}(s) \wedge \overrightarrow{u_1^*}(s) + v\overrightarrow{u_1}(s), & \overrightarrow{u_1^*}(s) = \overrightarrow{\alpha}(s) \wedge \overrightarrow{u_1}(s), \\ \left(\overrightarrow{U_2}\right) \equiv \overrightarrow{\Psi_{U_2}}(s,v) = \overrightarrow{u_2}(s) \wedge \overrightarrow{u_2^*}(s) + v\overrightarrow{u_2}(s), & \overrightarrow{u_2^*}(s) = \overrightarrow{\alpha}(s) \wedge \overrightarrow{u_2}(s), \\ \left(\overrightarrow{U_3}\right) \equiv \overrightarrow{\Psi_{U_3}}(s,v) = \overrightarrow{u_3}(s) \wedge \overrightarrow{u_3^*}(s) + v\overrightarrow{u_3}(s), & \overrightarrow{u_3^*}(s) = \overrightarrow{\alpha}(s) \wedge \overrightarrow{u_3}(s). \end{cases} \tag{11}$$

If we derive the vectorial moment vectors $\overrightarrow{u_1^*}(s), \overrightarrow{u_2^*}(s), \overrightarrow{u_3^*}(s)$ according to $s$, we get:

$$\begin{cases} \overrightarrow{u_1^{*\prime}}(s) = p(s)\overrightarrow{u_2^*}(s), \\ \overrightarrow{u_2^{*\prime}}(s) = \overrightarrow{u_3}(s) + q(s)\overrightarrow{u_3^*}(s) - p(s)\overrightarrow{u_1^*}(s), \\ \overrightarrow{u_3^{*\prime}}(s) = -\overrightarrow{u_2}(s) - q(s)\overrightarrow{u_2^*}(s). \end{cases} \tag{12}$$

The striction curve of the ruled surface $\overrightarrow{A}(s)$ is given as follows:

$$\overrightarrow{Y}(s) = \overrightarrow{a}(s) \wedge \overrightarrow{a^*}(s) + \frac{\left(\overrightarrow{a}(s), \overrightarrow{a'}(s), \overrightarrow{a^{*\prime}}(s)\right)}{\left\langle \overrightarrow{a'}(s), \overrightarrow{a'}(s)\right\rangle}\overrightarrow{a}(s), \tag{13}$$

where the curvature and the torsion of this striction curve are $\frac{p(s)}{q^*(s)}$ and $\frac{q(s)}{q^*(s)}$, respectively [4]. The Darboux screw are defined as follows [17]:

$$\overrightarrow{W}(s) = \overrightarrow{w}(s) + \varepsilon\overrightarrow{w^*}(s) = Q(s)\overrightarrow{U_1}(s) + P(s)\overrightarrow{U_3}(s), \tag{14}$$

where $\overrightarrow{w}(s)$ is the Darboux vector of the curve $\overrightarrow{\alpha}(s)$ and $\overrightarrow{w}^*(s)$ is the vectorial moment of the vector $\overrightarrow{w}(s)$. The instantaneous screw axis $\overrightarrow{B}(s)$ of the ruled surface $\overrightarrow{A}(s)$ $\overrightarrow{B}(s)$ :

$$\overrightarrow{B}(s) = \overrightarrow{b}(s) + \varepsilon\overrightarrow{b^*}(s) = \frac{Q(s)\overrightarrow{U_1}(s) + P(s)\overrightarrow{U_3}(s)}{\sqrt{P^2(s) + Q^2(s)}}, \tag{15}$$

where $\overrightarrow{b}(s)$ is the instantaneous Pfaff vector of the curve $\overrightarrow{\alpha}(s)$ and $\overrightarrow{b}^*(s)$ is the vectorial moment of the vector $\overrightarrow{b}(s)$ [4]. The dual Steiner rotation vector $\overrightarrow{D}(s) = \overrightarrow{d}(s) + \varepsilon\overrightarrow{d^*}(s)$ of the motion is found with the below equation:

$$\overrightarrow{D}(s) = \oint \overrightarrow{W}(s)ds = \overrightarrow{U_1}(s) \oint Q(s)ds + \overrightarrow{U_3}(s) \oint P(s)ds\,, \tag{16}$$

where $\vec{d}(s)$ is the Steiner vector of the curve $\vec{\alpha}(s)$ and $\vec{d}^*(s)$ is the vectorial moment of the vector $\vec{d}(s)$ [5]:

$$\begin{cases} \vec{d}(s) = \vec{u_1}(s)\oint q(s)ds + \vec{u_3}(s)\oint p(s)ds, \\ \vec{d^*}(s) = \vec{u_1^*}(s)\oint q(s)ds + \vec{u_1}(s)\oint q^*(s)ds + \vec{u_3^*}(s)\oint p(s)ds + \vec{u_3}(s)\oint p^*(s)ds. \end{cases} \tag{17}$$

### 2.2. The Parallel p-Equidistant Ruled Surfaces in $\mathbb{E}^3$

**Definition 1.** *Let $\vec{\varphi}(s,v)$ and $\vec{\overline{\varphi}}(\overline{s},\overline{v})$ be two ruled surfaces in $\mathbb{E}^3$. If the following two conditions are satisfied, these ruled surfaces are called parallel p-equidistant ruled surfaces [22]:*

- *The generator vectors are parallel;*
- *The distance p between the polar planes at the suitable points of the ruled surfaces $\vec{\varphi}(s,v)$ and $\vec{\overline{\varphi}}(\overline{s},\overline{v})$ is constant.*

Let $\{\vec{u_1}(s), \vec{u_2}(s), \vec{u_3}(s)\}$ be Frenet frame at the point $\vec{\alpha}(s)$ of the curve $(\alpha)$. If the generator vector of the ruled surface $\vec{\varphi}(s,v)$ is taken $\vec{u_1}(s)$, the parametric expressions of these ruled surfaces are stated as follows:

$$\vec{\varphi}(s,v) = \vec{\alpha}(s) + v\vec{u_1}(s), \quad \vec{\overline{\varphi}}(\overline{s},\overline{v}) = \vec{\overline{\alpha}}(\overline{s}) + \overline{v}\vec{u_1}(\overline{s}). \tag{18}$$

## 3. The Dual Expression of Parallel Equidistant Ruled Surfaces in Euclidean 3-Space

Let $\alpha : \mathbb{I} \to \mathbb{E}^3$ and $\beta : \mathbb{I} \to \mathbb{E}^3$ be any two curves. Let $\vec{t}(s)$ and $\vec{t^*}(s) = \vec{\alpha}(s) \wedge \vec{t}(s)$ be the tangent vector and its vectorial moment of the curve $\alpha$ and let $\vec{\overline{t}}(\overline{s})$ and $\vec{\overline{t}^*}(\overline{s}) = \vec{\beta}(\overline{s}) \wedge \vec{\overline{t}}(\overline{s})$ be the tangent vector and its vectorial moment of the curve $\beta$ , respectively. Furthermore, let $\Phi(s) = \phi(s) + \varepsilon\phi^*(s)$ be the dual angle between the dual vectors $\vec{T}(s) = \vec{t}(s) + \varepsilon\vec{t^*}(s)$ and $\vec{\overline{T}}(\overline{s}) = \vec{\overline{t}}(\overline{s}) + \varepsilon\vec{\overline{t}^*}(\overline{s})$ on the unit dual sphere (or the dual arc length $\widehat{T\overline{T}} = \phi^*(s)$ of the dual great circle passing through $\vec{T}(s)$ and $\vec{\overline{T}}(\overline{s})$ dual points on the unit dual sphere, Figure 1), where the dual great circle of the dual unit sphere is the intersection of the sphere and a dual plane that passes through the center point of the dual sphere [5]. These dual vectors describe dual spherical curves on the dual unit sphere and the dual curves correspond to the ruled surfaces in $\mathbb{E}^3$.

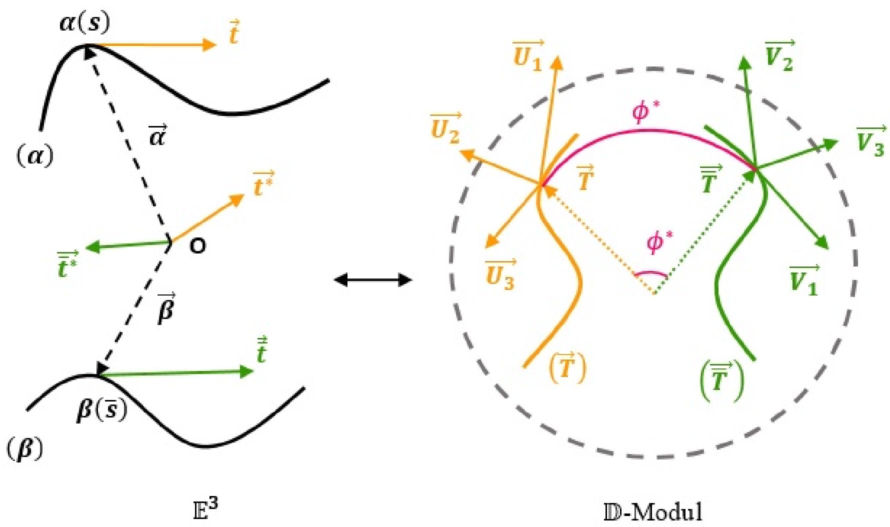


**Figure 1.** The dual spherical curves and their Blaschke vectors (the imaginary figure for $\mathbb{D}$-Modul).

From Expression (8), the Blaschke vectors of these ruled surfaces are given by the following (Figure 1):

$$\begin{cases} \overrightarrow{U_1}(s) = \overrightarrow{T}(s) = \overrightarrow{u_1}(s) + \varepsilon \overrightarrow{u_1^*}(s), \\ \overrightarrow{U_2}(s) = \dfrac{\overrightarrow{T}'(s)}{\left\| \overrightarrow{T}'(s) \right\|} = \overrightarrow{u_2}(s) + \varepsilon \overrightarrow{u_2^*}(s), \\ \overrightarrow{U_3}(s) = \overrightarrow{U_1}(s) \wedge \overrightarrow{U_2}(s) = \overrightarrow{u_3}(s) + \varepsilon \overrightarrow{u_3^*}(s), \end{cases} \quad \begin{cases} \overrightarrow{V_1}(\bar{s}) = \overrightarrow{\overline{T}}(\bar{s}) = \overrightarrow{v_1}(\bar{s}) + \varepsilon \overrightarrow{v_1^*}(\bar{s}), \\ \overrightarrow{V_2}(\bar{s}) = \dfrac{\overrightarrow{\overline{T}}'(\bar{s})}{\left\| \overrightarrow{\overline{T}}'(\bar{s}) \right\|} = \overrightarrow{v_2}(\bar{s}) + \varepsilon \overrightarrow{v_2^*}(\bar{s}), \\ \overrightarrow{V_3}(\bar{s}) = \overrightarrow{V_1}(\bar{s}) \wedge \overrightarrow{V_2}(\bar{s}) = \overrightarrow{v_3}(\bar{s}) + \varepsilon \overrightarrow{v_3^*}(\bar{s}), \end{cases}$$

where the axes $\{\overrightarrow{u_1}(s), \overrightarrow{u_2}(s), \overrightarrow{u_3}(s)\}$ and $\{\overrightarrow{v_1}(\bar{s}), \overrightarrow{v_2}(\bar{s}), \overrightarrow{v_3}(\bar{s})\}$ of these dual vectors intersect on the points of striction of these ruled surfaces and these points are on the axes $\overrightarrow{u_1}(s)$ and $\overrightarrow{v_1}(\bar{s})$, respectively [4].

**Definition 2** (*Dual Parallel Equidistant Ruled Surfaces*)**.** *Let two curves be $(\alpha)$, $(\beta)$ and the unit tangent vectors be $\overrightarrow{t}(s)$, $\overrightarrow{\bar{t}}(s)$ and their vectorial moments be $\overrightarrow{t^*}(s)$, $\overrightarrow{\bar{t}^*}(s)$ of these curves in Euclidean 3-space $\mathbb{E}^3$, respectively. Furthermore, let the dual angle between the unit dual vectors $\overrightarrow{T}(s) = \overrightarrow{t}(s) + \varepsilon \overrightarrow{t^*}(s)$ and $\overrightarrow{\overline{T}}(\bar{s}) = \overrightarrow{\bar{t}}(\bar{s}) + \varepsilon \overrightarrow{\bar{t}^*}(\bar{s})$ generated by these unit vectors (or the arc length between the corresponding dual points of the dual curves $\left(\overrightarrow{T}\right)$, $\left(\overrightarrow{\overline{T}}\right)$ described on the dual unit sphere of these unit dual vectors) be $\Phi(s) = \phi(s) + \varepsilon\phi^*(s)$. If the angle (or the arc length) is non zero, constant and only dual ($\phi(s) = 0$), that is $\Phi(s) = \phi^*(s)$, the ruled surfaces corresponding to these curves according to Study mapping are called dual parallel equidistant ruled surfaces.*

We will briefly denote these surfaces with DPERS from now on. The angle and length in question are explained in Figures 1 and 2. Here, we draw imaginary figures on the unit dual sphere for these surfaces. According to this definition, the following results along the striction curves of these DPERS are obtained:

- The dual generator vectors $\overrightarrow{T}(s) = \overrightarrow{U_1}(s)$ and $\overrightarrow{\overline{T}}(\bar{s}) = \overrightarrow{V_1}(\bar{s})$ are parallel;
- The distance $\phi^*$ between these vectors at the corresponding points of the ruled surfaces is constant.

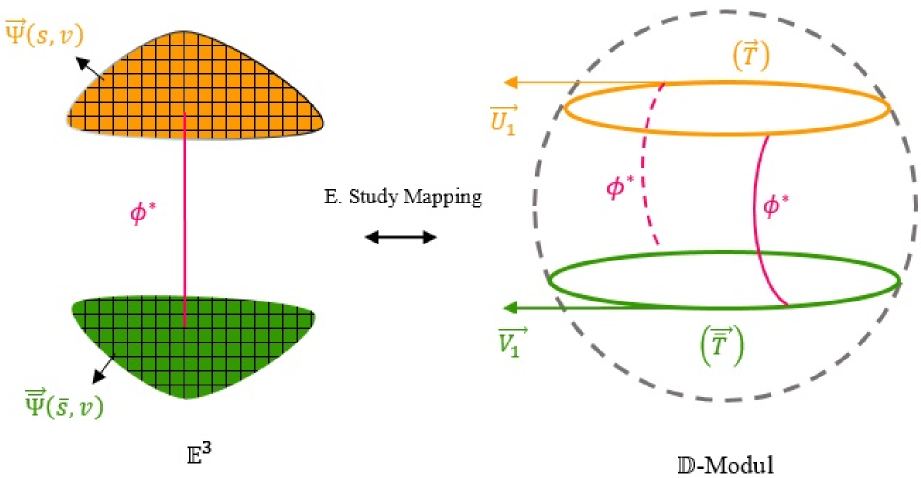


**Figure 2.** The dual parallel equidistant ruled surfaces.

These results are equivalent to Valeontis' expression in Definition 2. From Expression (7), the parametric expressions of the ruled surfaces corresponding in the space of lines to the dual curves $\left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{\overline{T}}\right)$ generated by the vectors $\overrightarrow{U_1}(s)$ and $\overrightarrow{V_1}(\bar{s})$ on $\mathbb{D}$-Modul are, respectively, written as follows:

$$\begin{cases} \left(\overrightarrow{T}\right) \equiv \overrightarrow{\Psi}(s,v) = \overrightarrow{u_1}(s) \wedge \overrightarrow{u_1^*}(s) + v\overrightarrow{u_1}(s), & \overrightarrow{u_1^*}(s) = \overrightarrow{\alpha}(s) \wedge \overrightarrow{u_1}(s), \\ \left(\overrightarrow{\overline{T}}\right) \equiv \overrightarrow{\overline{\Psi}}(\bar{s},\bar{v}) = \overrightarrow{v_1}(\bar{s}) \wedge \overrightarrow{v_1^*}(\bar{s}) + \bar{v}\overrightarrow{v_1}(\bar{s}), & \overrightarrow{v_1^*}(\bar{s}) = \overrightarrow{\beta}(\bar{s}) \wedge \overrightarrow{v_1}(\bar{s}), \end{cases} \tag{19}$$

where:

$$\overrightarrow{\beta}(\bar{s}) = \overrightarrow{\alpha}(s) + \phi^*(s)\overrightarrow{u_1}(s) + z(s)\overrightarrow{u_2}(s) + r(s)\overrightarrow{u_3}(s). \tag{20}$$

Furthermore, $\phi^*(s), z(s), r(s)$ are the perpendicular projection distances on the unit vectors $\overrightarrow{u_1}(s)$, $\overrightarrow{u_2}(s)$, $\overrightarrow{u_3}(s)$ of the vector $\overrightarrow{\beta}(\bar{s}) - \overrightarrow{\alpha}(s)$, respectively [22].

*3.1. The Relationships between the Blaschke Vectors and the Blaschke Invariants and of DPERS*

**Theorem 2.** *Let* $\{\overrightarrow{U_1}(s), \overrightarrow{U_2}(s), \overrightarrow{U_3}(s)\}$ *and* $\{\overrightarrow{V_1}(\bar{s}), \overrightarrow{V_2}(\bar{s}), \overrightarrow{V_3}(\bar{s})\}$ *be the Blaschke frames of DPERS* $\overrightarrow{\Psi}(s,v)$ *and* $\overrightarrow{\overline{\Psi}}(\bar{s},\bar{v})$ *corresponding in the space of lines to the dual curves* $\left(\overrightarrow{T}\right)$ *and* $\left(\overrightarrow{\overline{T}}\right)$ *on* $\mathbb{D}$*-Modul, respectively. There is the following relationship between the real and imaginary parts of these vectors:*

$$\begin{bmatrix} \overrightarrow{v_1}(\bar{s}) \\ \overrightarrow{v_2}(\bar{s}) \\ \overrightarrow{v_3}(\bar{s}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overrightarrow{u_1}(s) \\ \overrightarrow{u_2}(s) \\ \overrightarrow{u_3}(s) \end{bmatrix},$$
$$\begin{bmatrix} \overrightarrow{v_1^*}(\bar{s}) \\ \overrightarrow{v_2^*}(\bar{s}) \\ \overrightarrow{v_3^*}(\bar{s}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overrightarrow{u_1^*}(s) \\ \overrightarrow{u_2^*}(s) \\ \overrightarrow{u_3^*}(s) \end{bmatrix} + \begin{bmatrix} 0 & r(s) & -z(s) \\ -r(s) & 0 & \phi^*(s) \\ z(s) & -\phi^*(s) & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{u_1}(s) \\ \overrightarrow{u_2}(s) \\ \overrightarrow{u_3}(s) \end{bmatrix}. \tag{21}$$

**Proof.** First, since the vectors $\overrightarrow{U_1}(s)$ and $\overrightarrow{V_1}(\bar{s})$ are parallel, $\left\langle \overrightarrow{U_1}(s)\overrightarrow{V_1}(\bar{s}) \right\rangle = 1$. Here, $\left\langle \overrightarrow{u_1}(s), \overrightarrow{v_1}(\bar{s}) \right\rangle = 1$ and so $\overrightarrow{u_1}(s)$ and $\overrightarrow{v_1}(\bar{s})$ are parallel. We can take $\overrightarrow{v_1}(\bar{s}) = \overrightarrow{u_1}(s)$. Furthermore, since $\overrightarrow{v_1^*}(\bar{s}) = \overrightarrow{\beta}(\bar{s}) \wedge \overrightarrow{v_1}(\bar{s})$ and $\overrightarrow{\beta}(\bar{s}) = \overrightarrow{\alpha}(s) + \phi^*(s)\overrightarrow{u_1}(s) + z(s)\overrightarrow{u_2}(s) + r(s)\overrightarrow{u_3}(s)$, we get:

$$\overrightarrow{v_1^*}(\bar{s}) = \overrightarrow{u_1^*}(s) + r(s)\overrightarrow{u_2}(s) - z(s)\overrightarrow{u_3}(s)$$

Second, from $\overrightarrow{V_2}(\bar{s}) = \frac{\overrightarrow{V_1}'(\bar{s})}{\left\| \overrightarrow{V_1}'(\bar{s}) \right\|}$, we get $\overrightarrow{v_2}(\bar{s}) = \overrightarrow{u_2}(s)$ and:

$$\overrightarrow{v_2^*}(\bar{s}) = \overrightarrow{u_2^*}(s) - r(s)\overrightarrow{u_1}(s) + \phi^*(s)\overrightarrow{u_3}(s)$$

Finally, from $\overrightarrow{V_3}(\bar{s}) = \overrightarrow{V_1}(\bar{s}) \wedge \overrightarrow{V_2}(\bar{s})$, we get $\overrightarrow{v_3}(\bar{s}) = \overrightarrow{u_3}(s)$ and:

$$\overrightarrow{v_3^*}(\bar{s}) = \overrightarrow{u_3^*}(s) + z(s)\overrightarrow{u_1}(s) - \phi^*(s)\overrightarrow{u_2}(s).$$

□

**Theorem 3.** *Let* $P(s) = p(s) + \varepsilon p^*(s)$, $Q(s) = q(s) + \varepsilon q^*(s)$ *and* $\overline{P}(\bar{s}) = \overline{p}(\bar{s}) + \varepsilon \overline{p^*}(\bar{s})$, $\overline{Q}(\bar{s}) = \overline{q}(\bar{s}) + \varepsilon \overline{q^*}(\bar{s})$ *be the Blaschke invariants of DPERS* $\overrightarrow{\Psi}(s, v)$ *and* $\overrightarrow{\overline{\Psi}}(\bar{s}, \bar{v})$ *corresponding to the space of the lines to the dual curves* $\left(\overrightarrow{T}\right)$ *and* $\left(\overrightarrow{\overline{T}}\right)$ *on* $\mathbb{D}$*-Modul, respectively. There are the following relationships between these invariants:*

$$\begin{cases} \overline{P}(\bar{s}) = P(s)\dfrac{ds}{d\bar{s}} + \varepsilon(r'(s) + z(s)q(s))\dfrac{ds}{d\bar{s}}, \\ \overline{Q}(\bar{s}) = Q(s)\dfrac{ds}{d\bar{s}} - \varepsilon(z(s)p(s))\dfrac{ds}{d\bar{s}}. \end{cases} \tag{22}$$

**Proof.** If we derive the vector $\overrightarrow{v_1^*}(\bar{s}) = \overrightarrow{u_1^*}(s) + r(s)\overrightarrow{u_2}(s) - z(s)\overrightarrow{u_3}(s)$ with respect to $s$, we get:

$$\frac{d\overrightarrow{v_1^*}(\bar{s})}{d\bar{s}}\frac{d\bar{s}}{ds} = \overrightarrow{u_1^{*\prime}}(s) + r'(s)\overrightarrow{u_2}(s) + r(s)\overrightarrow{u_2'}(s) - z'(s)\overrightarrow{u_3}(s) - z(s)\overrightarrow{u_3'}(s).$$

Furthermore, using Expressions (10) and (21), we obtain:

$$\left(-r(s)\overline{p}(\bar{s})\overrightarrow{u_1}(s) + \overline{p^*}(\bar{s})\overrightarrow{u_2}(s) + \phi^*(s)\overline{p}(\bar{s})\overrightarrow{u_3}(s) + \overline{p}(\bar{s})\overrightarrow{u_2^*}(s)\right)\frac{d\bar{s}}{ds} = -r(s)p(s)\overrightarrow{u_1}(s)$$
$$+\left(p^*(s) + r'(s) + z(s)q(s)\right)\overrightarrow{u_2}(s) + \left(r(s)q(s) - z'(s)\right)\overrightarrow{u_3}(s) + p(s)\overrightarrow{u_2^*}(s).$$

From the last equation, we have the following equations:

$$\overline{p}(\bar{s}) = p(s)\frac{ds}{d\bar{s}} = k_1(s)\frac{ds}{d\bar{s}} = \frac{r(s)q(s) - z'(s)}{\phi^*(s)}\frac{ds}{d\bar{s}}, \tag{23}$$

$$\overline{p^*}(\bar{s}) = \left(p^*(s) + r'(s) + z(s)q(s)\right)\frac{ds}{d\bar{s}}. \tag{24}$$

Likewise, if we derive the vector $\overrightarrow{v_3^*}(\bar{s}) = \overrightarrow{u_3^*}(s) + r(s)\overrightarrow{u_2}(s) - z(s)\overrightarrow{u_3}(s)$ with respect to $s$, we get:

$$\frac{d\overrightarrow{v_3^*}(\bar{s})}{d\bar{s}}\frac{d\bar{s}}{ds} = \overrightarrow{u_3^{*\prime}}(s) + z'(s)\overrightarrow{u_1}(s) + z(s)\overrightarrow{u_1'}(s) - \phi^*(s)\overrightarrow{u_2'}(s).$$

Furthermore, if we use Expressions (10) and (21), we obtain:

$$\left(r(s)\overline{q}(\bar{s})\overrightarrow{u_1}(s) - \overline{q^*}(\bar{s})\overrightarrow{u_2}(s) - \phi^*(s)\overline{q}(\bar{s})\overrightarrow{u_3}(s) - \overline{q}(\bar{s})\overrightarrow{u_2^*}(s)\right)\frac{d\bar{s}}{ds} = (\phi^*(s) + z'(s))\overrightarrow{u_1}(s)$$
$$+(z(s)p(s) - q^*(s))\overrightarrow{u_2}(s) - \phi^*(s)q(s)\overrightarrow{u_3}(s) - q(s)\overrightarrow{u_2^*}(s).$$

From the last equation, we have following equations:

$$\overline{q}(\bar{s}) = q(s)\frac{ds}{d\bar{s}} = k_2(s)\frac{ds}{d\bar{s}} = \frac{\phi^*(s) + z'(s)}{r(s)}\frac{ds}{d\bar{s}}, \tag{25}$$

$$\overline{q^*}(\bar{s}) = (q^*(s) - z(s)p(s))\frac{ds}{d\bar{s}}. \tag{26}$$

□

**Corollary 1.** *The dual arc length $\phi^*(s)$ between the corresponding dual points of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overline{\Psi}}(\bar{s},\bar{v})$ corresponding in the space of lines to the dual curves $\left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{\overline{T}}\right)$ on $\mathbb{D}$-Modul is found as follows:*

$$\phi^*(s) = \frac{r(s)q(s) - z'(s)}{p(s)}. \tag{27}$$

**Proof.** It is clear from Expressions (23) or (25). □

This is the main conclusion of the article. This angle corresponds to the distance $p$ in the dual expression of parallel $p$-equidistant surfaces; it is constant and nonzero.

If $z(s) = r(s) = 0$, from Expressions (21) and (27), the Blaschke frames $\{\overrightarrow{U_1}(s), \overrightarrow{U_2}(s), \overrightarrow{U_3}(s)\}$ and $\{\overrightarrow{V_1}(\bar{s}), \overrightarrow{V_2}(\bar{s}), \overrightarrow{V_3}(\bar{s})\}$ of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overline{\Psi}}(\bar{s},\bar{v})$ corresponding in the space of lines to the dual curves $\left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{\overline{T}}\right)$ on $\mathbb{D}$-Modul become equivalent.

In the corollaries obtained later in the paper (up to Section 3.3), we specifically chose $\overrightarrow{U_1}(s)$ as the generating line and thus considered the specific results in the Expressions (11) and (12).

**Corollary 2.** *For DPERS $\overrightarrow{\Psi_{U_1}}(s,v)$ and $\overrightarrow{\overline{\Psi}_{V_1}}(\bar{s},\bar{v})$ corresponding in the space of lines to the dual curves $\left(\overrightarrow{U_1}\right) = \left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{V_1}\right) = \left(\overrightarrow{\overline{T}}\right)$ on $\mathbb{D}$-Modul, Expression (22) becomes as following:*

$$\begin{cases} \overline{P}(\bar{s}) = P(s)\dfrac{ds}{d\bar{s}} = p(s)\dfrac{ds}{d\bar{s}}, \\ \overline{Q}(\bar{s}) = Q(s)\dfrac{ds}{d\bar{s}} + \varepsilon\left(1 - \dfrac{ds}{d\bar{s}}\right) = q(s)\dfrac{ds}{d\bar{s}} + \varepsilon. \end{cases} \tag{28}$$

**Proof.** For the generator vectors $\overrightarrow{U_1}(s)$ and $\overrightarrow{V_1}(\bar{s})$, from Expressions (10) and (12), we obtain [16]:

$$\bar{p}^*(\bar{s}) = p^*(s) = 0, \qquad \bar{q}^*(\bar{s}) = q^*(s) = 1, \tag{29}$$

Thus, we have

$$P(s) = k_1(s) = p(s), \qquad Q(s) = k_2(s) + \varepsilon = q(s) + \varepsilon \tag{30}$$

and:

$$\overline{P}(\bar{s}) = \overline{k_1}(\bar{s}) = \bar{p}(\bar{s}), \qquad \overline{Q}(\bar{s}) = \overline{k_2}(\bar{s}) + \varepsilon = \bar{q}(\bar{s}) + \varepsilon, \tag{31}$$

where $k_1(s)$ and $\overline{k_1}(\bar{s})$ are the curvatures, $k_2(s)$ and $\overline{k_2}(\bar{s})$ are the torsions of the curves $\overrightarrow{\alpha}(s)$ and $\overrightarrow{\beta}(s)$, respectively. If these results are used in Expressions (24) and (26), we get the following equations:

$$\left(r'(s) + z(s)q(s)\right)\frac{ds}{d\bar{s}} = 0 \tag{32}$$

and:

$$\left(1 - z(s)p(s)\right)\frac{ds}{d\bar{s}} = 1. \tag{33}$$

If Expressions (32) and (33) are substituted in Expression (22), Expression (28) is obtained. □

**Corollary 3.** *There is the following relationship between the arc-length parameters of the base curves of DPERS* $\overrightarrow{\Psi_{U_1}}(s,v)$ *and* $\overrightarrow{\overrightarrow{\Psi_{V_1}}}(\bar{s},\bar{v})$ *corresponding in the space of lines to the dual curves* $\left(\overrightarrow{U_1}\right)=\left(\overrightarrow{T}\right)$ *and* $\left(\overrightarrow{V_1}\right)=\left(\overrightarrow{\overrightarrow{T}}\right)$ *on* $\mathbb{D}$*-Modul:*

$$\frac{d\bar{s}}{ds}=1-z(s)p(s). \tag{34}$$

**Proof.** It is clear from Expression (26). □

In this corollary, if $z=0$, from Expression (34), we get $d\bar{s}=ds$. That is, the arc-length parameters of the base curves of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overrightarrow{\Psi}}(\bar{s},\bar{v})$ corresponding in the space of lines to the dual curves $\left(\overrightarrow{U_1}\right)=\left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{V_1}\right)=\left(\overrightarrow{\overrightarrow{T}}\right)$ on $\mathbb{D}$-Modul are equivalent.

### *3.2. The Relationship Between the Striction Curves of DPERS*

**Theorem 4.** *Let* $\overrightarrow{\Upsilon}(s)$ *and* $\overrightarrow{\overrightarrow{\Upsilon}}(\bar{s})$ *be the striction curves of DPERS* $\overrightarrow{\Psi}(s,v)$ *and* $\overrightarrow{\overrightarrow{\Psi}}(\bar{s},\bar{v})$ *corresponding to the space of lines to the dual curves* $\left(\overrightarrow{T}\right)$ *and* $\left(\overrightarrow{\overrightarrow{T}}\right)$ *on* $\mathbb{D}$*-Modul, respectively. There is the following relationship between these curves:*

$$\overrightarrow{\overrightarrow{\Upsilon}}(\bar{s})=\overrightarrow{\Upsilon}(s)+\left(\frac{r(s)q(s)-z'(s)}{p(s)}\right)\overrightarrow{u_1}(s)+z(s)\overrightarrow{u_2}(s)+r(s)\overrightarrow{u_3}(s). \tag{35}$$

**Proof.** From Expression (13), the striction curves's equations of these ruled surfaces are obtained as follows:

$$\begin{cases}\overrightarrow{\Upsilon}(s)=\overrightarrow{u_1}(s)\wedge\overrightarrow{u_1^*}(s)+\dfrac{\left(\overrightarrow{u_1}(s),\overrightarrow{u_1'}(s),\overrightarrow{u_1^{*\prime}}(s)\right)}{\left\langle\overrightarrow{u_1'}(s),\overrightarrow{u_1'}(s)\right\rangle}\overrightarrow{u_1}(s),\\[2ex] \overrightarrow{\overrightarrow{\Upsilon}}(s)=\overrightarrow{v_1}(s)\wedge\overrightarrow{v_1^*}(s)+\dfrac{\left(\overrightarrow{v_1}(s),\overrightarrow{v_1'}(s),\overrightarrow{v_1^{*\prime}}(s)\right)}{\left\langle\overrightarrow{v_1'}(s),\overrightarrow{v_1'}(s)\right\rangle}\overrightarrow{v_1}(s).\end{cases} \tag{36}$$

If we use Expression (10), we obtain:

$$\begin{cases}\overrightarrow{\Upsilon}(s)=\overrightarrow{u_1}(s)\wedge\overrightarrow{u_1^*}(s)+\langle\overrightarrow{u_3}(s),\overrightarrow{u_2^*}(s)\rangle\overrightarrow{u_1}(s),\\[1ex] \overrightarrow{\overrightarrow{\Upsilon}}(s)=\overrightarrow{v_1}(s)\wedge\overrightarrow{v_1^*}(s)+\langle\overrightarrow{v_3}(s),\overrightarrow{v_2^*}(s)\rangle\overrightarrow{v_1}(s).\end{cases} \tag{37}$$

From Expression (21), we have:

$$\overrightarrow{\overrightarrow{\Upsilon}}(s)=\overrightarrow{u_1}(s)\wedge\overrightarrow{u_1^*}(s)+\Big(\langle\overrightarrow{u_3}(s),\overrightarrow{u_2^*}(s)\rangle+\phi^*(s)\Big)\overrightarrow{u_1}(s)+z(s)\overrightarrow{u_2}(s)+r(s)\overrightarrow{u_3}(s). \tag{38}$$

From Expressions (37) and (38), we get:

$$\overrightarrow{\overrightarrow{\Upsilon}}(\bar{s})=\overrightarrow{\Upsilon}(s)+\phi^*(s)\overrightarrow{u_1}(s)+z(s)\overrightarrow{u_2}(s)+r(s)\overrightarrow{u_3}(s).$$

If Expression (27) is used in the last expression, the proof is completed. □

**Corollary 4.** *The striction curves of DPERS* $\overrightarrow{\Psi_{U_1}}(s,v)$ *and* $\overrightarrow{\overline{\Psi}_{V_1}}(\bar{s},\bar{v})$ *corresponding in the space of lines to the dual curves* $\left(\overrightarrow{U_1}\right) = \left(\overrightarrow{T}\right)$ *and* $\left(\overrightarrow{V_1}\right) = \left(\overrightarrow{\overline{T}}\right)$ *on* $\mathbb{D}$*-Modul become their base curves.*

**Proof.** For these surfaces, if the vectors $\overrightarrow{u_1^*}(s) = \overrightarrow{\alpha}(s) \wedge \overrightarrow{u_1}(s)$, $\overrightarrow{u_2^*}(s) = \overrightarrow{\alpha}(s) \wedge \overrightarrow{u_2}(s)$, $\overrightarrow{v_1^*}(s) = \overrightarrow{\beta}(s) \wedge \overrightarrow{v_1}(s)$, $\overrightarrow{v_2^*}(s) = \overrightarrow{\beta}(s) \wedge \overrightarrow{v_2}(s)$ are substituted in Expression (37), we get:

$$\overrightarrow{\Upsilon}(s) = \alpha(s), \qquad \overrightarrow{\overline{\Upsilon}}(s) = \beta(s). \tag{39}$$

□

**Corollary 5.** *The Frenet frame, the curvature and the torsion of the striction curves of DPERS* $\overrightarrow{\Psi_{U_1}}(s,v)$ *and* $\overrightarrow{\overline{\Psi}_{V_1}}(\bar{s},\bar{v})$ *corresponding in the space of lines to the dual curves* $\left(\overrightarrow{U_1}\right) = \left(\overrightarrow{T}\right)$ *and* $\left(\overrightarrow{V_1}\right) = \left(\overrightarrow{\overline{T}}\right)$ *on* $\mathbb{D}$*-Modul are* $\{\overrightarrow{u_1}(s), \overrightarrow{u_2}(s), \overrightarrow{u_3}(s)\}$, $k_1(s) = p(s)$, $k_2(s) = q(s)$ *and* $\{\overrightarrow{v_1}(\bar{s}), \overrightarrow{v_2}(\bar{s}), \overrightarrow{v_3}(\bar{s})\}$, $\overline{k_1}(\bar{s}) = \overline{p}(\bar{s})$, $\overline{k_2}(\bar{s}) = \overline{q}(\bar{s})$*, respectively.*

### *3.3. The Relationships Between the Darboux Screws and The Instantaneous Screw Axes of DPERS*

In all theorems obtained in the following parts of the article, Expression (29) has been taken into account, as it will not change the general situation and for ease of operations.

**Theorem 5.** *Let* $\overrightarrow{W}(s) = \overrightarrow{w}(s) + \varepsilon\overrightarrow{w^*}(s)$ *and* $\overrightarrow{\overline{W}}(\bar{s}) = \overrightarrow{\overline{w}}(\bar{s}) + \varepsilon\overrightarrow{\overline{w}^*}(\bar{s})$ *be the Darboux screws of DPERS* $\overrightarrow{\Psi}(s,v)$ *and* $\overrightarrow{\overline{\Psi}}(\bar{s},\bar{v})$ *corresponding in the space of lines to the dual curves* $\left(\overrightarrow{U_1}\right) = \left(\overrightarrow{T}\right)$ *and* $\left(\overrightarrow{V_1}\right) = \left(\overrightarrow{\overline{T}}\right)$ *on* $\mathbb{D}$*-Modul, respectively. There is the following relationship between these vectors:*

$$\overrightarrow{\overline{W}}(\bar{s}) = \frac{\overrightarrow{W}(s)}{1 - z(s)p(s)} + \varepsilon\frac{z'(s)\overrightarrow{u_2}(s) - z(s)q(s)\overrightarrow{u_3}(s)}{1 - z(s)p(s)}. \tag{40}$$

**Proof.** From Expressions (14), (30) and (31), respectively, we write:

$$\begin{cases} \overrightarrow{W}(s) = \overrightarrow{w}(s) + \varepsilon\overrightarrow{w^*}(s) = (q(s) + \varepsilon)\overrightarrow{U_1}(s) + p(s)\overrightarrow{U_3}(s), \\ \overrightarrow{\overline{W}}(\bar{s}) = \overrightarrow{\overline{w}}(\bar{s}) + \varepsilon\overrightarrow{\overline{w}^*}(\bar{s}) = (\overline{q}(\bar{s}) + \varepsilon)\overrightarrow{V_1}(\bar{s}) + \overline{p}(\bar{s})\overrightarrow{V_3}(\bar{s}), \end{cases} \tag{41}$$

If Expressions (21), (23), (25) and (27) are substituted in Expression (41), the real and imaginary parts of these vectors are written as follows:

$$\begin{cases} \overrightarrow{w}(s) = q(s)\overrightarrow{u_1}(s) + p(s)\overrightarrow{u_3}(s), \\ \overrightarrow{w^*}(s) = \overrightarrow{u_1}(s) + q(s)\overrightarrow{u_1^*}(s) + p(s)\overrightarrow{u_3^*}(s), \\ \overrightarrow{\overline{w}}(\bar{s}) = \dfrac{\overrightarrow{w}(s)}{1 - z(s)p(s)} = \dfrac{q(s)\overrightarrow{u_1}(s) + p(s)\overrightarrow{u_3}(s)}{1 - z(s)p(s)}, \\ \overrightarrow{\overline{w}^*}(\bar{s}) = \dfrac{\overrightarrow{w^*}(s)}{1 - z(s)p(s)} + \dfrac{z'(s)\overrightarrow{u_2}(s) - z(s)q(s)\overrightarrow{u_3}(s)}{1 - z(s)p(s)}. \end{cases} \tag{42}$$

Thus, from Expressions (41) and (42), Expression (40) is obtained. □

In this theorem, if $z = 0$, the Darboux screws of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overline{\Psi}}(\bar{s},\bar{v})$ corresponding in the space of lines to the dual curves $\left(\overrightarrow{U_1}\right) = \left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{V_1}\right) = \left(\overrightarrow{\overline{T}}\right)$ on $\mathbb{D}$-Modul are equivalent.

**Theorem 6.** *Let $\overrightarrow{B}(s) = \overrightarrow{b}(s) + \varepsilon\overrightarrow{b^*}(s)$ and $\overrightarrow{\overline{B}}(\bar{s}) = \overrightarrow{\overline{b}}(\bar{s}) + \varepsilon\overrightarrow{\overline{b}^*}(\bar{s})$ be the instantaneous screws axes of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overline{\Psi}}(\bar{s},\bar{v})$ corresponding in the space of lines to the dual curves $\left(\overrightarrow{U_1}\right) = \left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{V_1}\right) = \left(\overrightarrow{\overline{T}}\right)$ on $\mathbb{D}$-Modul, respectively. There is the following relationship between these vectors:*

$$\overrightarrow{\overline{B}}(\bar{s}) = \overrightarrow{B}(s) + \varepsilon\frac{z(s)p(s)q^2(s)\overrightarrow{u_1}(s) + z'(s)(p^2(s)+q^2(s))\overrightarrow{u_2}(s) - z(s)q^3(s)\overrightarrow{u_3}(s)}{(p^2(s)+q^2(s))^{3/2}}. \quad (43)$$

**Proof.** According to Expression (15), from Expressions (30) and (31), we have:

$$\begin{cases} \overrightarrow{B}(s) = \overrightarrow{b}(s) + \varepsilon\overrightarrow{b^*}(s) = \dfrac{(q(s)+\varepsilon)\overrightarrow{U_1}(s) + p(s)\overrightarrow{U_3}(s)}{\sqrt{p^2(s)+q^2(s)+2\varepsilon q(s)}}, \\ \overrightarrow{\overline{B}}(\bar{s}) = \overrightarrow{\overline{b}}(\bar{s}) + \varepsilon\overrightarrow{\overline{b}^*}(\bar{s}) = \dfrac{(\overline{q}(\bar{s})+\varepsilon)\overrightarrow{V_1}(\bar{s}) + \overline{p}(\bar{s})\overrightarrow{V_3}(\bar{s})}{\sqrt{\overline{p}^2(\bar{s})+\overline{q}^2(\bar{s})+2\varepsilon\overline{q}(\bar{s})}}. \end{cases} \quad (44)$$

By using Expressions (2), (21), (23), (25) and (29), the vectors $\overrightarrow{b}(s)$, $\overrightarrow{b^*}(s)$, $\overrightarrow{\overline{b}}(\bar{s})$, $\overrightarrow{\overline{b}^*}(\bar{s})$ are as follows:

$$\begin{cases} \overrightarrow{b}(s) = \overrightarrow{\overline{b}}(\bar{s}) = \dfrac{q(s)\overrightarrow{u_1}(s) + p(s)\overrightarrow{u_3}(s)}{\sqrt{p^2(s)+q^2(s)}}, \\ \overrightarrow{b^*}(s) = \dfrac{q(s)\overrightarrow{u_1^*}(s) + p(s)\overrightarrow{u_3^*}(s)}{\sqrt{p^2(s)+q^2(s)}} + \dfrac{p(s)\left(p(s)\overrightarrow{u_1}(s) - q(s)\overrightarrow{u_3}(s)\right)}{\left(p^2(s)+q^2(s)\right)^{3/2}}, \\ \overrightarrow{\overline{b}^*}(\bar{s}) = \overrightarrow{b^*}(s) + \dfrac{z(s)p(s)q^2(s)\overrightarrow{u_1}(s) + z'(s)\left(p^2(s)+q^2(s)\right)\overrightarrow{u_2}(s) - z(s)q^3(s)\overrightarrow{u_3}(s)}{\left(p^2(s)+q^2(s)\right)^{3/2}}. \end{cases} \quad (45)$$

Thus, from Expressions (44) and (45), Expression (43) is obtained. □

In this theorem, if $z = 0$, the instantaneous screw axes of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overline{\Psi}}(\bar{s},\bar{v})$ corresponding in the space of lines to the dual curves $\left(\overrightarrow{U_1}\right) = \left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{V_1}\right) = \left(\overrightarrow{\overline{T}}\right)$ on $\mathbb{D}$-Modul are equivalent.

**Theorem 7.** *Let $\Theta(s) = \theta(s) + \varepsilon\theta^*(s)$ be the dual angle between the vectors $\overrightarrow{U_3}(s)$ and $\overrightarrow{W}(s)$, $\overline{\Theta}(\bar{s}) = \overline{\theta}(\bar{s}) + \varepsilon\overline{\theta^*}(\bar{s})$ be the dual angle between the vectors $\overrightarrow{V_3}(\bar{s})$ and $\overrightarrow{\overline{W}}(\bar{s})$ of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overline{\Psi}}(\bar{s},\bar{v})$ corresponding in the space of lines to the dual curves $\left(\overrightarrow{U_1}\right) = \left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{V_1}\right) = \left(\overrightarrow{\overline{T}}\right)$ on $\mathbb{D}$-Modul, respectively. There is the following relationship between these angles:*

$$\overline{\Theta}(\bar{s}) = \pm\Theta(s) + 2k\pi \mp \varepsilon z(s)p(s)\theta^*(s), \qquad k \in \mathbb{Z}. \quad (46)$$

**Proof.** From Expressions (6), (41) and (42), we have:

$$\left\|\overrightarrow{W}(s)\right\| = \|\overrightarrow{w}(s)\| + \varepsilon\frac{\left\langle \overrightarrow{w}(s), \overrightarrow{w^*}(s)\right\rangle}{\|\overrightarrow{w}(s)\|} = \sqrt{p^2(s)+q^2(s)} + \varepsilon\frac{q(s)}{\sqrt{p^2(s)+q^2(s)}},$$
$$\left\|\overrightarrow{\overline{W}}(\bar{s})\right\| = \left\|\overrightarrow{\overline{w}}(\bar{s})\right\| + \varepsilon\frac{\left\langle \overrightarrow{\overline{w}}(\bar{s}), \overrightarrow{\overline{w}^*}(\bar{s})\right\rangle}{\left\|\overrightarrow{\overline{w}}(\bar{s})\right\|} = \frac{\sqrt{p^2(s)+q^2(s)}}{1-z(s)p(s)} + \varepsilon\frac{q(s)}{\sqrt{p^2(s)+q^2(s)}}. \tag{47}$$

For the dual angles $\Theta(s) = \theta(s) + \varepsilon\theta^*(s)$ and $\overline{\Theta}(\bar{s}) = \overline{\theta}(\bar{s}) + \varepsilon\overline{\theta^*}(\bar{s})$, from Expressions (5) and (47), we write $\left\langle \overrightarrow{W}(s), \overrightarrow{U_3}(s)\right\rangle = \cos\Theta\left\|\overrightarrow{W}(s)\right\|$ and $\left\langle \overrightarrow{\overline{W}}(\bar{s}), \overrightarrow{V_3}(\bar{s})\right\rangle = \cos\overline{\Theta}\left\|\overrightarrow{\overline{W}}(\bar{s})\right\|$, then we get:

$$\left\langle \overrightarrow{W}(s), \overrightarrow{U_3}(s)\right\rangle = \cos\theta\sqrt{p^2(s)+q^2(s)} + \varepsilon\left(\frac{q(s)\cos\theta}{\sqrt{p^2(s)+q^2(s)}} - \theta^*\sin\theta\sqrt{p^2(s)+q^2(s)}\right),$$
$$\left\langle \overrightarrow{\overline{W}}(\bar{s}), \overrightarrow{V_3}(\bar{s})\right\rangle = \frac{\cos\overline{\theta}\sqrt{p^2(s)+q^2(s)}}{1-z(s)p(s)} + \varepsilon\left(\frac{q(s)\cos\overline{\theta}}{\sqrt{p^2(s)+q^2(s)}} - \frac{\overline{\theta}^*\sin\overline{\theta}\sqrt{p^2(s)+q^2(s)}}{1-z(s)p(s)}\right). \tag{48}$$

On the other hand, from Expressions (3), (21) and (42), we get:

$$\left\langle \overrightarrow{W}(s), \overrightarrow{U_3}(s)\right\rangle = \langle \overrightarrow{w}(s), \overrightarrow{u_3}(s)\rangle + \varepsilon\left(\left\langle \overrightarrow{w^*}(s), \overrightarrow{u_3}(s)\right\rangle + \left\langle \overrightarrow{w}(s), \overrightarrow{u_3^*}(s)\right\rangle\right) = p(s),$$
$$\left\langle \overrightarrow{\overline{W}}(\bar{s}), \overrightarrow{V_3}(\bar{s})\right\rangle = \left\langle \overrightarrow{\overline{w}}(\bar{s}), \overrightarrow{v_3}(\bar{s})\right\rangle + \varepsilon\left(\left\langle \overrightarrow{\overline{w}^*}(\bar{s}), \overrightarrow{v_3}(\bar{s})\right\rangle + \left\langle \overrightarrow{\overline{w}}(\bar{s}), \overrightarrow{v_3^*}(\bar{s})\right\rangle\right) = \frac{p(s)}{1-z(s)p(s)}. \tag{49}$$

From the equality of Expressions (48) and (49), we obtain:

$$\cos\overline{\theta} = \cos\theta = \frac{p(s)}{\sqrt{p^2(s)+q^2(s)}}, \quad \sin\overline{\theta} = \pm\sin\theta = \pm\frac{q(s)}{\sqrt{p^2(s)+q^2(s)}}, \tag{50}$$

$$\frac{q(s)\cos\theta}{\sqrt{p^2(s)+q^2(s)}} = \theta^*\sin\theta\sqrt{p^2(s)+q^2(s)}, \quad \frac{q(s)\cos\overline{\theta}}{\sqrt{p^2(s)+q^2(s)}} = \frac{\overline{\theta}^*\sin\overline{\theta}\sqrt{p^2(s)+q^2(s)}}{1-z(s)p(s)}. \tag{51}$$

From the solution of the equalities in Expression (50), we get:

$$\overline{\theta}(\bar{s}) = 2k\pi \pm \theta(s), \quad k \in \mathbb{Z}. \tag{52}$$

Furthermore, Expression (50) is substituted in Expression (51); thus, we have:

$$\theta^*(s) = \frac{p(s)}{p^2(s)+q^2(s)}, \quad \overline{\theta}^*(s) = \pm\frac{p(s)(1-z(s)p(s))}{p^2(s)+q^2(s)}. \tag{53}$$

Thus, from Expression (53), we arrive at the following relationship:

$$\overline{\theta}^*(s) = \pm\theta^*(s) \mp \frac{z(s)p^2(s)}{p^2(s)+q^2(s)}. \tag{54}$$

Thus, from Expressions (52) and (53), Expression (46) is obtained. In the next steps, we will get $\sin\overline{\theta} = \sin\theta$. □

In this theorem, if $z = 0$, the dual angle between the vectors $\overrightarrow{U_3}(s)$ and $\overrightarrow{W}(s)$, and the dual angle between the vectors $\overrightarrow{V_3}(\bar{s})$ and $\overrightarrow{\overline{W}}(\bar{s})$ of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overline{\Psi}}(\bar{s},\bar{v})$ corresponding in the space of lines to the dual curves $\left(\overrightarrow{U_1}\right) = \left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{V_1}\right) = \left(\overrightarrow{\overline{T}}\right)$ on $\mathbb{D}$-Modul are equivalent.

**Theorem 8.** *Let $\Theta(s) = \theta(s) + \varepsilon\theta^*(s)$ be the dual angle between the vectors $\overrightarrow{U_3}(s)$ and $\overrightarrow{W}(s)$, $\overline{\Theta}(\bar{s}) = \overline{\theta}(\bar{s}) + \varepsilon\overline{\theta^*}(\bar{s})$ be the dual angle between the vectors $\overrightarrow{V_3}(\bar{s})$ and $\overrightarrow{\overline{W}}(\bar{s})$ of DPERS $\overrightarrow{\Psi_{U_1}}(s, v)$ and $\overrightarrow{\Psi_{V_1}}(\bar{s}, \bar{v})$ corresponding in the space of lines to the dual curves $\left(\overrightarrow{U_1}\right) = \left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{V_1}\right) = \left(\overrightarrow{\overline{T}}\right)$ on $\mathbb{D}$-Modul, respectively. There is the following relationship between the instantaneous screw axes $\overrightarrow{B}(s) = \overrightarrow{b}(s) + \varepsilon\overrightarrow{b^*}(s)$ and $\overrightarrow{\overline{B}}(\bar{s}) = \overrightarrow{\overline{b}}(\bar{s}) + \varepsilon\overrightarrow{\overline{b}^*}(\bar{s})$ of these DPERS:*

$$\overrightarrow{\overline{B}}(\bar{s}) = \overrightarrow{B}(s) + \varepsilon(z(s)\cos\theta(1 - p(s)\theta^*(s))\overrightarrow{u_1}(s) + (r(s)\sin\theta - \phi^*(s)\cos\theta)\overrightarrow{u_2}(s)$$
$$-z(s)\sin\theta(1 - p(s)\theta^*(s))\overrightarrow{u_3}(s)). \tag{55}$$

**Proof.** From Expressions (44), (50) and (53), we get:

$$\begin{aligned}
\frac{q(s) + \varepsilon}{\sqrt{p^2(s) + q^2(s) + 2\varepsilon q(s)}} &= \frac{q(s)}{\sqrt{p^2(s) + q^2(s)}} + \varepsilon\frac{p^2(s)}{(p^2(s) + q^2(s))^{3/2}} \\
&= \sin\theta + \varepsilon\theta^*\cos\theta = \sin\Theta, \\
\frac{p(s)}{\sqrt{p^2(s) + q^2(s) + 2\varepsilon q(s)}} &= \frac{p(s)}{\sqrt{p^2(s) + q^2(s)}} - \varepsilon\frac{p(s)q(s)}{(p^2(s) + q^2(s))^{3/2}} \\
&= \cos\theta - \varepsilon\theta^*\sin\theta = \cos\Theta,
\end{aligned} \tag{56}$$

and:

$$\begin{aligned}
\frac{\bar{q}(\bar{s}) + \varepsilon}{\sqrt{\bar{p}^2(\bar{s}) + \bar{q}^2(\bar{s}) + 2\varepsilon\bar{q}(\bar{s})}} &= \frac{q(s)}{\sqrt{p^2(s) + q^2(s)}} + \varepsilon\frac{p^2(s)}{(p^2(s) + q^2(s))^{3/2}}\frac{d\bar{s}}{ds} \\
&= \sin\overline{\theta} + \varepsilon\overline{\theta}^*\cos\overline{\theta} = \sin\overline{\Theta}, \\
\frac{\bar{p}(s)}{\sqrt{\bar{p}^2(\bar{s}) + \bar{q}^2(\bar{s}) + 2\varepsilon\bar{q}(\bar{s})}} &= \frac{p(s)}{\sqrt{p^2(s) + q^2(s)}} - \varepsilon\frac{p(s)q(s)}{(p^2(s) + q^2(s))^{3/2}}\frac{d\bar{s}}{ds} \\
&= \cos\overline{\theta} - \varepsilon\overline{\theta}^*\sin\overline{\theta} = \cos\overline{\Theta},
\end{aligned} \tag{57}$$

Furthermore, the instantaneous screw axes in Expression (44) are obtained as:

$$\begin{cases}
\overrightarrow{B}(s) = \sin\Theta\overrightarrow{U_1}(s) + \cos\Theta\overrightarrow{U_3}(s), \\
\overrightarrow{\overline{B}}(\bar{s}) = \sin\overline{\Theta}\overrightarrow{V_1}(\bar{s}) + \cos\overline{\Theta}\overrightarrow{V_3}(\bar{s}).
\end{cases} \tag{58}$$

Whether we divide these vectors into real and imaginary parts, or from the Expression (45), we get the vectors $\overrightarrow{b}(s), \overrightarrow{b^*}(s), \overrightarrow{\overline{b}}(\bar{s}), \overrightarrow{\overline{b}^*}(\bar{s})$ as follows:

$$\begin{cases}
\overrightarrow{b}(s) = \overrightarrow{\overline{b}}(\bar{s}) = \sin\theta\overrightarrow{u_1}(s) + \cos\theta\overrightarrow{u_3}(s), \\
\overrightarrow{b^*}(s) = \sin\theta\overrightarrow{u_1^*}(s) + \cos\theta\overrightarrow{u_3^*}(s) + \theta^*(s)\left(\cos\theta\overrightarrow{u_1}(s) - \sin\theta\overrightarrow{u_3}(s)\right), \\
\overrightarrow{\overline{b}^*}(\bar{s}) = \overrightarrow{b^*}(s) + z(s)\cos\theta\sin^2\theta\overrightarrow{u_1}(s) + (r(s)\sin\theta - \phi^*(s)\cos\theta)\overrightarrow{u_2}(s) - z(s)\sin^3\theta\overrightarrow{u_3}(s),
\end{cases} \tag{59}$$

where $\sin^2\theta = 1 - p(s)\theta^*(s)$. Thus, from Expressions (58) and (59), Expression (55) is obtained. Here, the vectors $\vec{b}(s)$ and $\vec{\bar{b}}(\bar{s})$ are the instantaneous Pfaff vectors of the curves $\vec{\alpha}(s)$ and $\vec{\beta}(\bar{s})$, respectively. □

**Theorem 9.** *Let $\vec{C}(s) = \vec{c}(s) + \varepsilon\vec{c^*}(s)$ and $\vec{\bar{C}}(\bar{s}) = \vec{\bar{c}}(\bar{s}) + \varepsilon\vec{\bar{c}^*}(\bar{s})$ be the instantaneous dual Pfaff vectors of DPERS $\vec{\Psi}(s,v)$ and $\vec{\bar{\Psi}}(\bar{s},\bar{v})$ corresponding in the space of lines to the dual curves $\left(\vec{U_1}\right) = \left(\vec{T}\right)$ and $\left(\vec{V_1}\right) = \left(\vec{\bar{T}}\right)$ on $\mathbb{D}$-Modul, respectively. There is the following relationship between these vectors:*

$$\vec{\bar{C}}(\bar{s}) = \vec{C}(s) + \varepsilon(z(s)\cos\theta\vec{u_1}(s) + (r(s)\sin\theta - \phi^*(s)\cos\theta)\vec{u_2}(s) - z(s)\sin\theta\vec{u_3}(s)). \quad (60)$$

**Proof.** From Expression (59), the instantaneous Pfaff vectors of the curves $\vec{\alpha}(s)$ and $\vec{\beta}(\bar{s})$ are:

$$\vec{c}(s) = \vec{\bar{c}}(\bar{s}) = \sin\theta\vec{u_1}(s) + \cos\theta\vec{u_3}(s), \quad (61)$$

The vector moments $\vec{\alpha}(s) \wedge \vec{c}(s)$ and $\vec{\beta}(\bar{s}) \wedge \vec{\bar{c}}(\bar{s})$ of these vectors are:

$$\begin{cases} \vec{c^*}(s) = \sin\theta\vec{u_1^*}(s) + \cos\theta\vec{u_3^*}(s), \\ \vec{\bar{c}^*}(\bar{s}) = \vec{c^*}(s) + z(s)\cos\theta\vec{u_1}(s) + (r(s)\sin\theta - \phi^*(s)\cos\theta)\vec{u_2}(s) - z(s)\sin\theta\vec{u_3}(s). \end{cases} \quad (62)$$

Furthermore, from Expressions (61) and (62), the dual instantaneous Pfaff vectors of DPERS $\vec{\Psi}(s,v)$ and $\vec{\bar{\Psi}}(\bar{s},\bar{v})$ are, respectively:

$$\begin{cases} \vec{C}(s) = \sin\theta\vec{U_1}(s) + \cos\theta\vec{U_3}(s), \\ \vec{\bar{C}}(\bar{s}) = \sin\bar{\theta}\vec{V_1}(\bar{s}) + \cos\bar{\theta}\vec{V_3}(\bar{s}), \end{cases} \quad (63)$$

Thus, from Expressions (62) and (63), Expression (60) is obtained. □

In this theorem, if $z = 0$, from Expressions (27) and (60), we get:

$$r(s)\sin\theta - \phi^*(s)\cos\theta = r(s)\frac{p(s)}{\sqrt{p^2(s)+q^2(s)}} - \phi^*(s)\frac{q(s)}{\sqrt{p^2(s)+q^2(s)}} = \frac{z'(s)}{\sqrt{p^2(s)+q^2(s)}} = 0.$$

That is, the instantaneous Pfaff vectors of DPERS $\vec{\Psi}(s,v)$ and $\vec{\bar{\Psi}}(\bar{s},\bar{v})$ corresponding in the space of lines to the dual curves $\left(\vec{U_1}\right) = \left(\vec{T}\right)$ and $\left(\vec{V_1}\right) = \left(\vec{\bar{T}}\right)$ on $\mathbb{D}$-Modul are equivalent.

Furthermore, if $\theta^*(s) = 0$ (that is, the dual angle $\Theta(s)$ between the vectors $\vec{U_3}(s)$ and $\vec{W}(s)$ of ruled surface $\vec{\Psi}(s,v)$ is only real), the vectors $\vec{b^*}(s)$ and $\vec{\bar{b}^*}(\bar{s})$ in Expression (59) are equivalent to the vectors $\vec{c^*}(s)$ and $\vec{\bar{c}^*}(\bar{s})$ in Expression (62). Thus, from the Expressions (58) and (63), $\vec{B}(s) = \vec{C}(s)$ and $\vec{\bar{B}}(\bar{s}) = \vec{\bar{C}}(\bar{s})$. That is, the Darboux screw axes and the instantaneous Pfaff vectors of DPERS $\vec{\Psi}(s,v)$ and $\vec{\bar{\Psi}}(\bar{s},\bar{v})$ corresponding in the space of lines to the dual curves $\left(\vec{U_1}\right) = \left(\vec{T}\right)$ and $\left(\vec{V_1}\right) = \left(\vec{\bar{T}}\right)$ on $\mathbb{D}$-Modul are equivalent, respectively.

**Theorem 10.** *Let $\overrightarrow{D}(s) = \overrightarrow{d}(s) + \varepsilon\overrightarrow{d^*}(s)$ and $\overrightarrow{\overline{D}}(\overline{s}) = \overrightarrow{\overline{d}}(\overline{s}) + \varepsilon\overrightarrow{\overline{d}^*}(\overline{s})$ be the dual Steiner vectors of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overline{\Psi}}(\overline{s},\overline{v})$ corresponding in the space of lines to the dual curves $\left(\overrightarrow{U_1}\right) = \left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{V_1}\right) = \left(\overrightarrow{\overline{T}}\right)$ on $\mathbb{D}$-Modul, respectively. There is the following relationship between these vectors:*

$$\begin{aligned}\overrightarrow{\overline{D}}(\overline{s}) = \overrightarrow{D}(s) + \varepsilon\Bigg(&\left(-\oint z(s)p(s)ds + z(s)\oint p(s)ds\right)\overrightarrow{u_1}(s)\\ &+\left(r(s)\oint q(s)ds - \phi^*(s)\oint p(s)ds\right)\overrightarrow{u_2}(s) - z(s)\overrightarrow{u_3}\oint q(s)ds\Bigg).\end{aligned} \tag{64}$$

**Proof.** According to Expression (16), the dual Steiner rotation vectors of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overline{\Psi}}(\overline{s},\overline{v})$ are, respectively:

$$\begin{cases}\overrightarrow{D}(s) = \oint \overrightarrow{W}(s)ds = \overrightarrow{U_1}(s)\oint (q+\varepsilon)(s)ds + \overrightarrow{U_3}(s)\oint p(s)ds,\\ \overrightarrow{\overline{D}}(\overline{s}) = \oint \overrightarrow{\overline{W}}(\overline{s})d\overline{s} = \overrightarrow{V_1}(\overline{s})\oint (\overline{q}+\varepsilon)(\overline{s})d\overline{s} + \overrightarrow{V_3}(\overline{s})\oint \overline{p}(\overline{s})d\overline{s},\end{cases} \tag{65}$$

Expressions (65) can be written respect to the real part and the imaginary part as follows:

$$\begin{cases}\overrightarrow{d}(s) = \overrightarrow{\overline{d}}(\overline{s}) = \overrightarrow{u_1}(s)\oint q(s)ds + \overrightarrow{u_3}(s)\oint p(s)ds,\\ \overrightarrow{d^*}(s) = \overrightarrow{u_1^*}(s)\oint q(s)ds + \overrightarrow{u_1}(s)\oint ds + \overrightarrow{u_3^*}(s)\oint p(s)ds,\\ \overrightarrow{\overline{d}^*}(\overline{s}) = \overrightarrow{d^*}(s) + \left(-\oint z(s)p(s)ds + z(s)\oint p(s)ds\right)\overrightarrow{u_1}(s)\\ \qquad + \left(r(s)\oint q(s)ds - \phi^*(s)\oint p(s)ds\right)\overrightarrow{u_2}(s) - z(s)\overrightarrow{u_3}\oint q(s)ds.\end{cases} \tag{66}$$

Thus, from Expressions (65) and (66), Expression (64) is obtained. □

In this theorem, if $z(s) = r(s) = 0$, from Expression (25), $\phi^*(s) = 0$. Thus, it is clear from Expression (66) that the dual Steiner vectors of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overline{\Psi}}(\overline{s},\overline{v})$ corresponding in the space of lines to the dual curves $\left(\overrightarrow{U_1}\right) = \left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{V_1}\right) = \left(\overrightarrow{\overline{T}}\right)$ on $\mathbb{D}$-Modul are equivalent.

### *3.4. An Example For DPERS*

Now, let us show all these results for the DPERS on an example to reinforce the study and make it easier for the reader to understand the subject. For this, first let us take two curves $(\alpha)$ and $(\beta)$ in Euclidean 3-space. Furthermore, let us get the unit dual vectors $\overrightarrow{T}(s) = \overrightarrow{t}(s) + \epsilon\overrightarrow{t^*}(s)$ and $\overrightarrow{\overline{T}}(s) = \overrightarrow{\overline{t}}(s) + \epsilon\overrightarrow{\overline{t}^*}(s)$ obtained with the tangent vectors $\overrightarrow{t}(s)$, $\overrightarrow{\overline{t}}(s)$ of these curves and the vector moments $\overrightarrow{t^*}(s)$, $\overrightarrow{\overline{t}^*}(s)$ of these vectors. Next, let us write the equations of the DPERS with base curves $\overrightarrow{t}(s) \wedge \overrightarrow{t^*}(s)$ , $\overrightarrow{\overline{t}}(s) \wedge \overrightarrow{\overline{t}^*}(s)$ and generating lines $\overrightarrow{t}(s)$, $\overrightarrow{\overline{t}}(s)$. Furthermore, let us find Blaschke vectors, Blaschke invariants, Darboux screws and instantaneous Pfaff vectors of these ruled surfaces. Then, let us show the figures of these surfaces by drawing the surface equations we have obtained.

**Example.** Let us consider two helix curves:

$$\overrightarrow{\alpha}(s)=\left(-\sin\frac{s}{\sqrt{2}},\cos\frac{s}{\sqrt{2}},\frac{s}{\sqrt{2}}\right),\quad \overrightarrow{\beta}(\bar{s})=\left(\cos\frac{\bar{s}}{\sqrt{2}},\sin\frac{\bar{s}}{\sqrt{2}},\frac{\bar{s}}{\sqrt{2}}\right).$$

Let us get $\bar{s}=s+\frac{\pi}{\sqrt{2}}$, $(d\bar{s}=ds)$. In this case:

$$\overrightarrow{\beta}(\bar{s})\left(-\sin\frac{s}{\sqrt{2}},\cos\frac{s}{\sqrt{2}},\frac{s}{\sqrt{2}}+\frac{\pi}{2}\right)$$

and the tangent vectors of these curves are obtained as follows:

$$\overrightarrow{t}(s)=\overrightarrow{\bar{t}}(\bar{s})=\left(-\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),$$

Furthermore, the moment vectors of these vectors are computed, respectively, as follows:

$$\overrightarrow{t^*}(s)=\left(\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}+\frac{s}{2}\sin\frac{s}{\sqrt{2}},\ -\frac{s}{2}\cos\frac{s}{\sqrt{2}}+\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

and:

$$\overrightarrow{\bar{t}^*}(\bar{s})=\left(\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}+\left(\frac{s}{2}+\frac{\pi}{2\sqrt{2}}\right)\sin\frac{s}{\sqrt{2}},\ -\left(\frac{s}{2}+\frac{\pi}{2\sqrt{2}}\right)\cos\frac{s}{\sqrt{2}}+\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),$$

From Expression (19), the parametric expressions of the ruled surfaces corresponding in the space of lines to the dual curves $\left(\overrightarrow{T}\right)$ and $\left(\overrightarrow{\bar{T}}\right)$ generated by the vectors $\overrightarrow{T}(s)=\overrightarrow{t}(s)+\epsilon\,\overrightarrow{t^*}(s)$ and $\overrightarrow{\bar{T}}(s)=\overrightarrow{\bar{t}}(s)+\epsilon\,\overrightarrow{\bar{t}^*}(s)$ on $\mathbb{D}$-Modules are computed, respectively, as the following:

$$\begin{cases}\overrightarrow{\Psi}(s,v)=\left(\dfrac{s-2v}{2\sqrt{2}}\cos\dfrac{s}{\sqrt{2}}-\sin\dfrac{s}{\sqrt{2}},\ \cos\dfrac{s}{\sqrt{2}}+\dfrac{s-2v}{2\sqrt{2}}\sin\dfrac{s}{\sqrt{2}},\ \dfrac{s+2v}{2\sqrt{2}}\right),\\[2ex] \overrightarrow{\bar{\Psi}}(\bar{s},v)=\left(\left(\dfrac{s-2v}{2\sqrt{2}}+\dfrac{\pi}{4}\right)\cos\dfrac{s}{\sqrt{2}}-\sin\dfrac{s}{\sqrt{2}},\ \cos\dfrac{s}{\sqrt{2}}+\left(\dfrac{s-2v}{2\sqrt{2}}+\dfrac{\pi}{4}\right)\sin\dfrac{s}{\sqrt{2}},\dfrac{s+2v}{2\sqrt{2}}+\dfrac{\pi}{4}\right),\end{cases}$$

where we got $v=\bar{v}$. We find the Blaschke vectors $\{\overrightarrow{U_1}(s),\overrightarrow{U_2}(s),\overrightarrow{U_3}(s)\}$ and $\{\overrightarrow{V_1}(\bar{s}),\overrightarrow{V_2}(\bar{s}),\overrightarrow{V_3}(\bar{s})\}$ of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\bar{\Psi}}(\bar{s},v)$. The real and imaginary parts of these vectors are:

$$\overrightarrow{v_1}(s)=\overrightarrow{u_1}(s)=\left(-\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),$$

$$\overrightarrow{v_2}(s)=\overrightarrow{u_2}(s)=\left(\sin\frac{s}{\sqrt{2}},-\cos\frac{s}{\sqrt{2}},0\right),$$

$$\overrightarrow{v_3}(s)=\overrightarrow{u_3}(s)=\left(\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),$$

$$\overrightarrow{u_1^*}(s) = \left(\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}} + \frac{s}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, -\frac{s}{\sqrt{2}}\cos\frac{s}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$$\overrightarrow{u_2^*}(s) = \left(\frac{s}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, 0\right),$$

$$\overrightarrow{u_3^*}(s) = \left(\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}} - \frac{s}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\cos\frac{s}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),$$

$$\overrightarrow{v_1^*}(s) = \left(\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}} + \left(\frac{s}{2} + \frac{\pi}{2\sqrt{2}}\right)\sin\frac{s}{\sqrt{2}}, -\left(\frac{s}{2} + \frac{\pi}{2\sqrt{2}}\right)\cos\frac{s}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$$\overrightarrow{v_2^*}(s) = \left(\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right)\cos\frac{s}{\sqrt{2}}, \left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right)\sin\frac{s}{\sqrt{2}}, 0\right),$$

$$\overrightarrow{v_3^*}(s) = \left(\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}} - \left(\frac{s}{2} + \frac{\pi}{2\sqrt{2}}\right)\sin\frac{s}{\sqrt{2}}, \left(\frac{s}{2} + \frac{\pi}{2\sqrt{2}}\right)\cos\frac{s}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

In Expression (20), if the inner product is applied to both sides of the equation:

$$\overrightarrow{\beta}(\bar{s}) = \overrightarrow{\alpha}(s) + \phi^*(s)\overrightarrow{u_1}(s) + z(s)\overrightarrow{u_2}(s) + r(s)\overrightarrow{u_3}(s)$$

with $\overrightarrow{u_1}(s)$, $\overrightarrow{u_2}(s)$ and $\overrightarrow{u_3}(s)$, respectively, we get:

$$\phi^*(s) = \frac{\pi}{2\sqrt{2}}, \quad z(s) = 0 \quad \textit{and} \quad r(s) = \frac{\pi}{2\sqrt{2}}.$$

From Expression (21), the relationships between real and imaginary parts of the Blaschke vectors $\{\overrightarrow{U_1}(s), \overrightarrow{U_2}(s), \overrightarrow{U_3}(s)\}$ and $\{\overrightarrow{V_1}(\bar{s}), \overrightarrow{V_2}(\bar{s}), \overrightarrow{V_3}(\bar{s})\}$ of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overline{\Psi}}(\bar{s},v)$ are obtained as follows:

$$\begin{cases} \overrightarrow{V_1}(\bar{s}) = \overrightarrow{v_1}(\bar{s}) + \varepsilon\overrightarrow{v_1^*}(\bar{s}) = \overrightarrow{u_1}(s) + \varepsilon\left(\overrightarrow{u_1^*}(s) + \frac{\pi}{2\sqrt{2}}\overrightarrow{u_2}(s)\right), \\ \overrightarrow{V_2}(\bar{s}) = \overrightarrow{v_2}(\bar{s}) + \varepsilon\overrightarrow{v_2^*}(\bar{s}) = \overrightarrow{u_2}(s) + \varepsilon\left(\overrightarrow{u_2^*}(s) - \frac{\pi}{2\sqrt{2}}\overrightarrow{u_1}(s) + \frac{\pi}{2\sqrt{2}}\overrightarrow{u_3}(s)\right), \\ \overrightarrow{V_3}(\bar{s}) = \overrightarrow{v_3}(\bar{s}) + \varepsilon\overrightarrow{v_3^*}(\bar{s}) = \overrightarrow{u_3}(s) + \varepsilon\left(\overrightarrow{u_3^*}(s) - \frac{\pi}{2\sqrt{2}}\overrightarrow{u_2}(s)\right). \end{cases}$$

Moreover, from Expression (28), the relations between the Blaschke invariants of these ruled surfaces are obtained as follows:

$$\overline{P}(\bar{s}) = P(s) = p(s) = \frac{1}{2}, \qquad \overline{Q}(\bar{s}) = Q(s) = q(s) + \varepsilon = \frac{1}{2} + \varepsilon.$$

Furthermore, from Expression (37), the striction curves of these ruled surfaces are obtained as follows:

$$\begin{cases} \overrightarrow{\Upsilon}(s) = \left(-\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right), \\ \overrightarrow{\overline{\Upsilon}}(\bar{s}) = \left(-\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} + \frac{\pi}{2}\right). \end{cases}$$

From the last expression, it is understood that the striction curves of these ruled surfaces are the base curves. That is, these ruled surfaces are drawn along the curves of striction.

Furthermore, from Expression (40), the relationship between the Darboux screws is given as follows:

$$\overrightarrow{\overline{W}}(\bar{s}) = \overrightarrow{W}(s) = \frac{1}{2}\left(\overrightarrow{u_1}(s) + \overrightarrow{u_3}(s)\right) + \varepsilon\left(\overrightarrow{u_1}(s) + \frac{1}{2}\left(\overrightarrow{u_1^*}(s) + \overrightarrow{u_3^*}(s)\right)\right),$$

From Expression (42), the real and imaginary parts of these screws are:

$$\overrightarrow{\overline{w}^*}(\bar{s}) = \overrightarrow{\overline{w}}(\bar{s}) = \overrightarrow{w^*}(s) = \overrightarrow{w}(s) = \left(0, 0, \frac{1}{\sqrt{2}}\right).$$

In addition, from Expressions (50) and (53), the relationship between the dual angles between the vectors $\overrightarrow{U_3}(s)$ and $\overrightarrow{W}(s)$ and the vectors $\overrightarrow{V_3}(\bar{s})$ and $\overrightarrow{\overline{W}}(\bar{s})$ is:

$$\Theta(s) = \overline{\Theta}(\bar{s}) = \frac{\pi}{4} + \varepsilon,$$

where:

$$\cos\overline{\theta} = \cos\theta = \sin\overline{\theta} = \sin\theta = \frac{\sqrt{2}}{2}, \quad \theta = \overline{\theta} = \frac{\pi}{4}, \quad \theta^*(s) = \overline{\theta}^*(s) = 1.$$

Furthermore, from Expression (55), the relationship between the instantaneous screw axes of these surfaces is gotten as follows:

$$\overrightarrow{\overline{B}}(\bar{s}) = \overrightarrow{B}(s) = \frac{\sqrt{2}}{2}\left(\overrightarrow{u_1}(s) + \overrightarrow{u_3}(s)\right) + \frac{\sqrt{2}\varepsilon}{2}\left(\overrightarrow{u_1}(s) + \overrightarrow{u_1^*}(s) - \overrightarrow{u_3}(s) + \overrightarrow{u_3^*}(s)\right),$$

where, from Expression (59), the real and imaginary parts of these axes are:

$$\overrightarrow{\overline{b}}(\bar{s}) = \overrightarrow{b}(s) = (0, 0, 1), \qquad \overrightarrow{\overline{b}^*}(\bar{s}) = \overrightarrow{b^*}(s) = (0, 0, 0).$$

Lastly, from Expression (60), the relationship between the instantaneous Pfaff vectors of these surfaces is found as follows:

$$\overrightarrow{\overline{C}}(\bar{s}) = \overrightarrow{C}(s) = \frac{\sqrt{2}}{2}\left(\overrightarrow{u_1}(s) + \overrightarrow{u_3}(s)\right) + \frac{\sqrt{2}\varepsilon}{2}\left(\overrightarrow{u_1^*}(s) + \overrightarrow{u_3^*}(s)\right).$$

where, from Expression (61) and (62), the real and imaginary parts of these vectors are:

$$\overrightarrow{\overline{c}}(\bar{s}) = \overrightarrow{c}(s) = (0, 0, 1), \qquad \overrightarrow{\overline{c}^*}(\bar{s}) = \overrightarrow{c^*}(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, 0\right).$$

Let us now draw the figures of these ruled surfaces. If we get $s = -2\pi : \pi/2 : 2\pi$ (the incremental values of $\pi/2$ in the range $-2\pi$ to $2\pi$) and v= -5 : 1/2 : 5, we can draw the graphics in (Figure 3) for the ruled surface $\overrightarrow{\Psi}(s, v)$ and we can draw the graphics in (Figure 4) for the ruled surface $\overrightarrow{\overline{\Psi}}(\bar{s}, v)$ in $\mathbb{E}^3$.

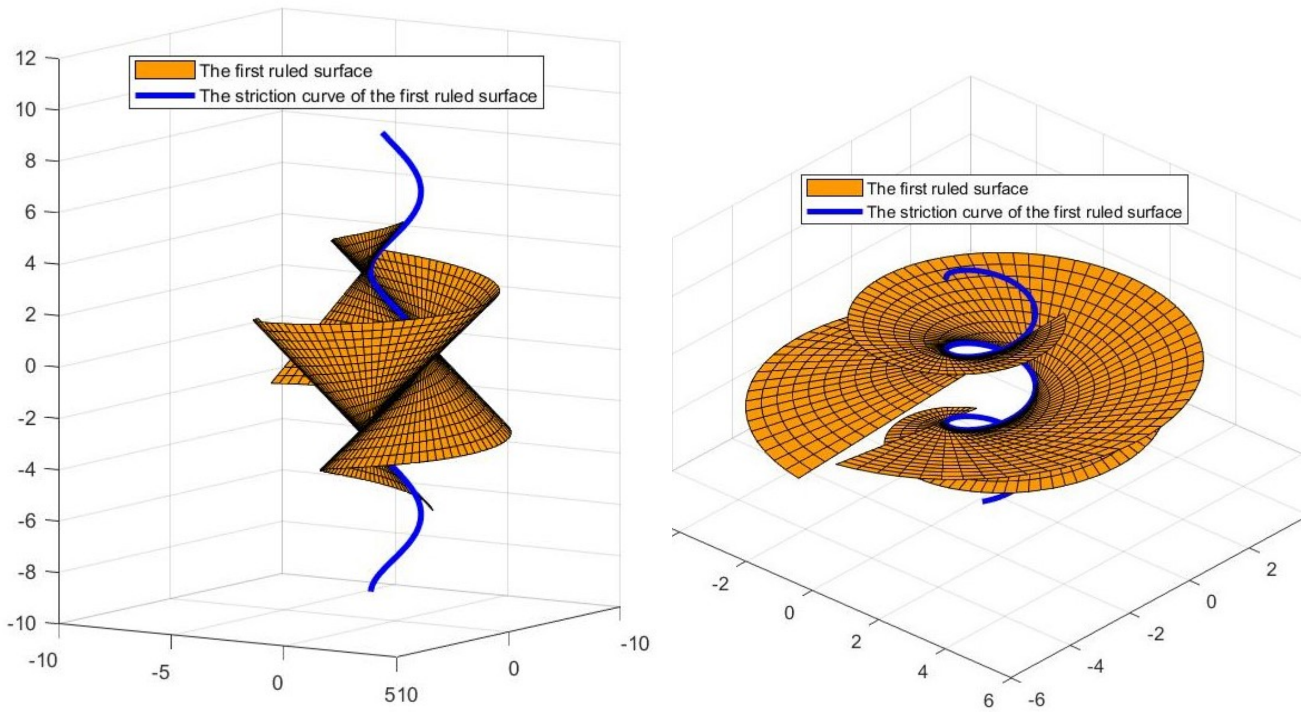


**Figure 3.** The first ruled surface and its striction curve in $\mathbb{E}^3$.

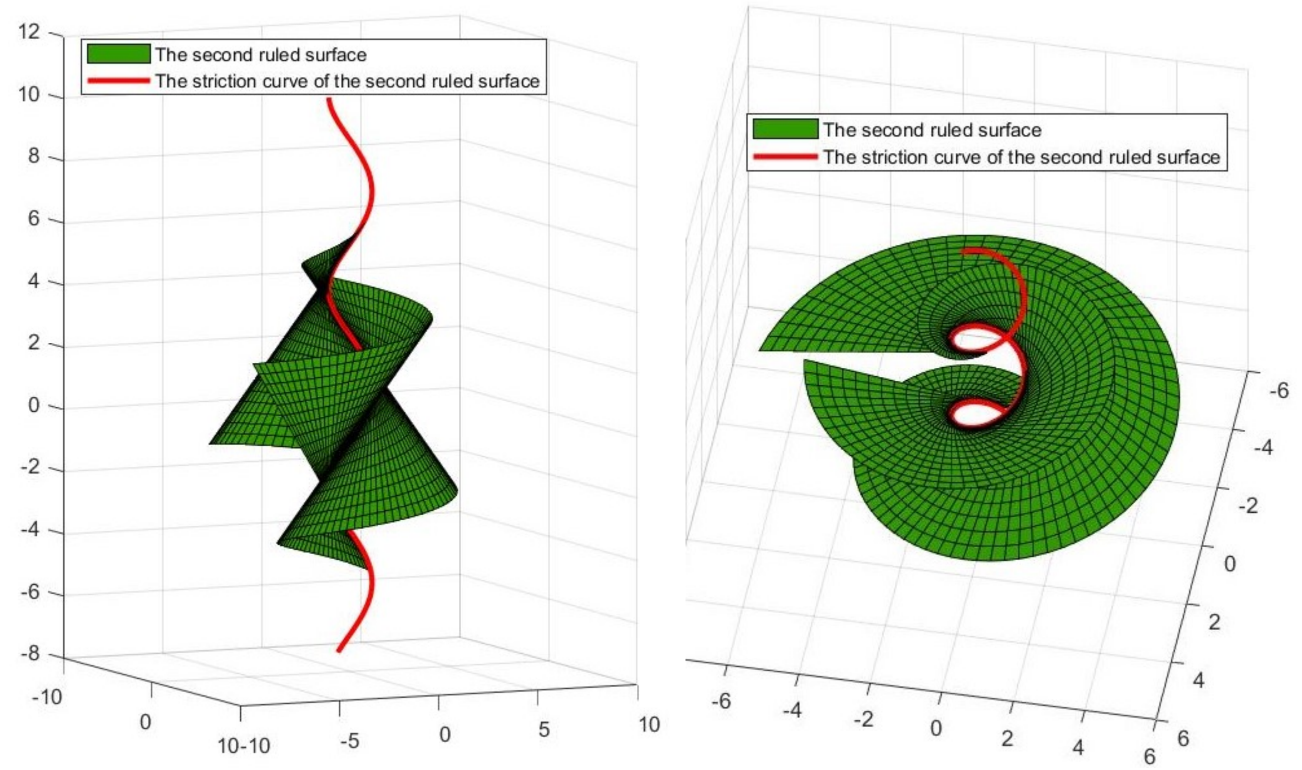


**Figure 4.** The second ruled surface and its striction curve in $\mathbb{E}^3$.

Now, let us draw these ruled surfaces together in $\mathbb{E}^3$, (on the left) in Figure 5. Furthermore, if we imagine that we show these DPERS on the dual unit sphere, they correspond to two dual parallel curves similar to that shown (on the right) in Figure 5. Here, $s = -2\pi : \pi/2 : 2\pi$ and $v = -5 : 1/2 : 5$.

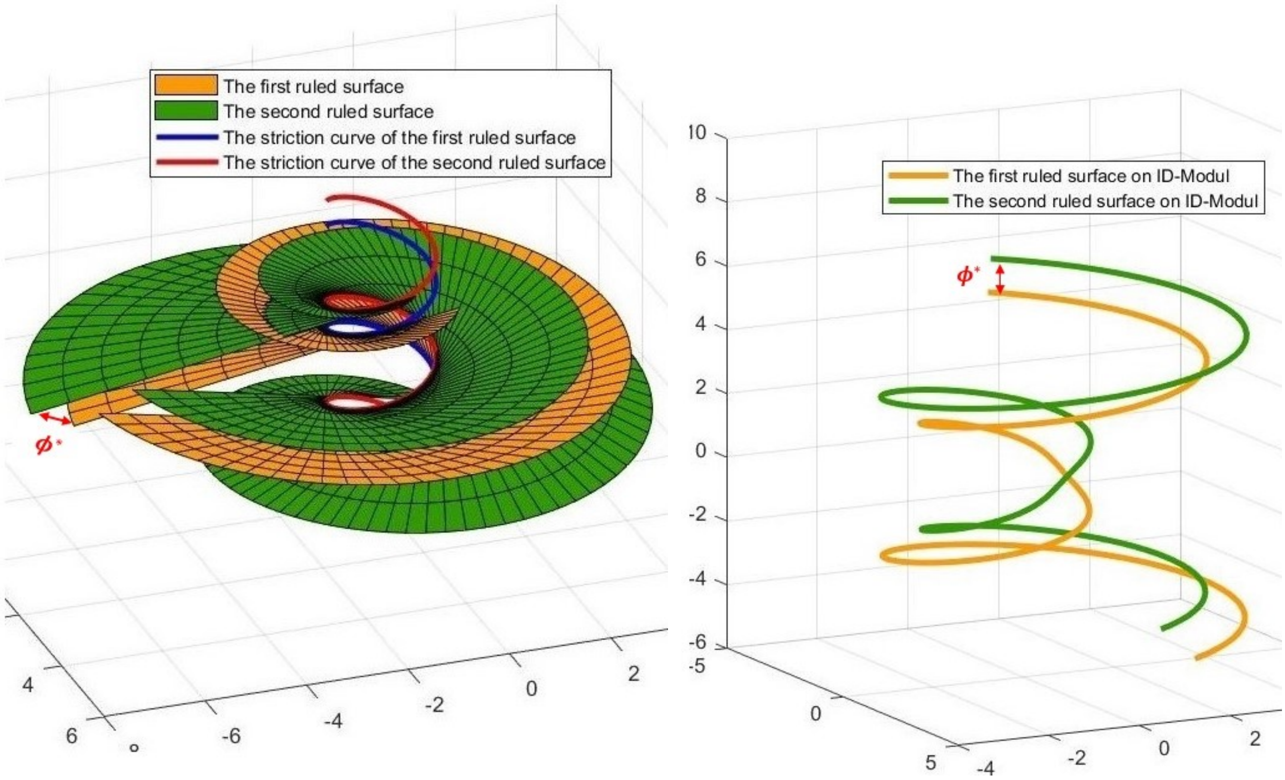


**Figure 5.** The dual parallel equidistant ruled surfaces for $s = -2\pi : \pi/2 : 2\pi$.

If we get $s = -13\pi : \pi/2 : 13\pi$ and $v = -5 : 1/2 : 5$, we can draw the graphics in Figure 6 for the DPERS $\overrightarrow{\Psi}(s, v)$ and $\overrightarrow{\overline{\Psi}}(\bar{s}, v)$ in $\mathbb{E}^3$ and on $\mathbb{D}$-Modul .

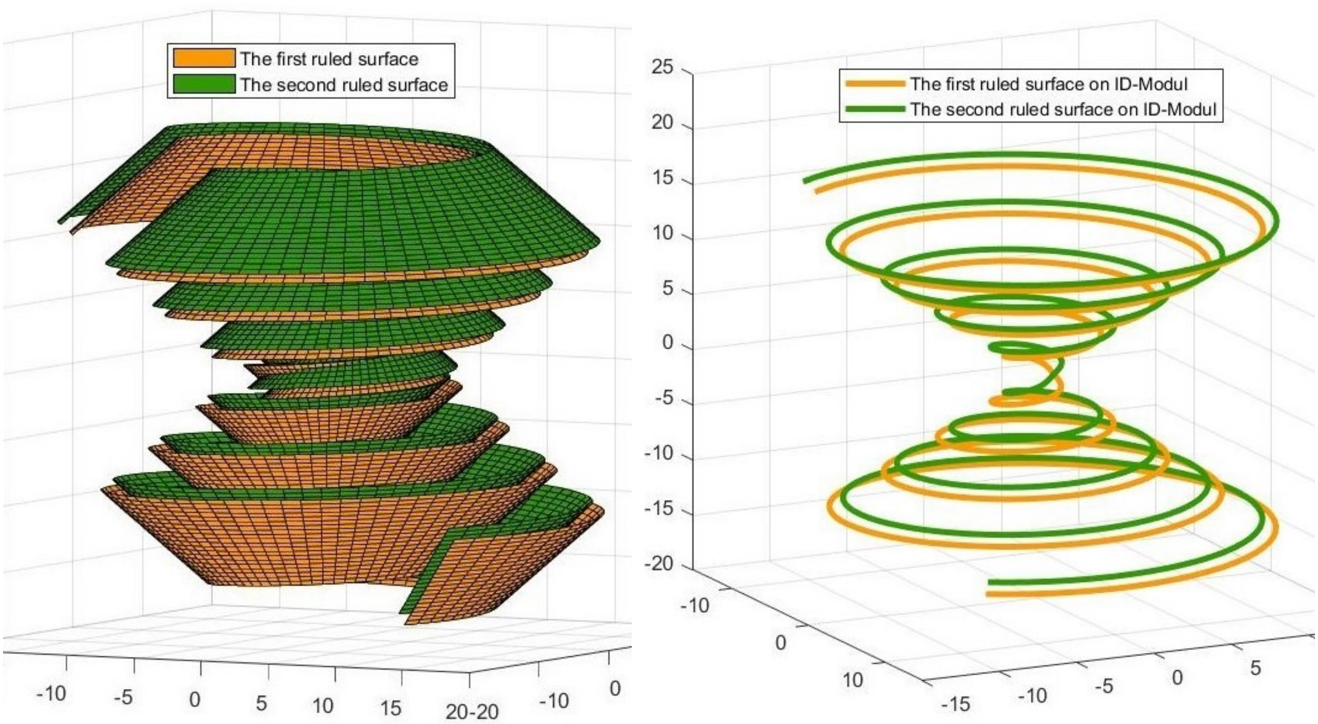


**Figure 6.** The dual parallel equidistant ruled surfaces for $s = -13\pi : \pi/2 : 13\pi$.

Finally, as seen in Figure 7, the dual curves corresponding to the DPERS on the dual unit sphere are such that one of them is symmetric with respect to the imaginary symmetry axis of the other.

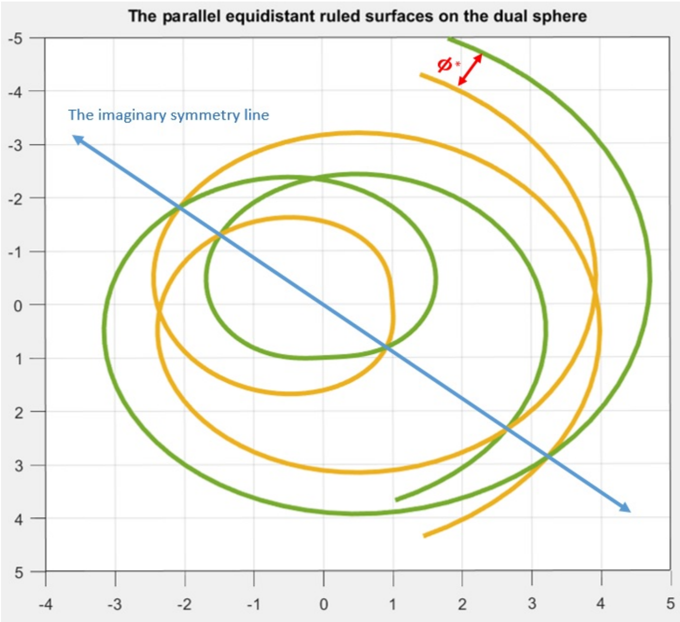


**Figure 7.** The dual parallel equidistant ruled surfaces on dual unit sphere.

## 4. Discussion and Conclusions

If the tangent vectors of two ruled surfaces along the striction curves are parallel and at the same time the distance ($p$-distance) between the corresponding points of the polar planes of these surfaces is constant, these surfaces are called parallel $p$-equidistant ruled surfaces. In this study, firstly, the dual expression of parallel p-equidistant ruled surfaces in the space of lines is given according to the Study mapping, and it is shown that the dual part of the dual angle on the unit dual sphere corresponds to the $p$-distance. Then, the relationships between Blaschke vectors, Blaschke invariants, the striction curves, the Darboux screws, the instantaneous screw axes, the instantaneous dual Pfaff vectors and the dual Steiner rotation vectors of these surfaces are expressed. Finally, an example for these surfaces is given and the drawings of these surfaces are made. The objective of this study is to make a new expansion for the parallel equidistant ruled surfaces on the dual vector space ($\mathbb{D}$-Modul). This expansion can also be studied by considering their principal normal vectors, binormal vectors or Darboux vectors instead of the tangent vectors of any two ruled surfaces, and thus, new parallel equidistant ruled surfaces can be obtained. Similarly, these surfaces can be studied in the Lorentzian space and the dual Lorentzian space, which have an important place in physics. In addition, these dual parallel equidistant ruled surfaces can be applied to studies on the robot kinematics.

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## Abbreviations

The following abbreviations are used in this manuscript:

| | |
|---|---|
| DPERS | Dual Parallel Equidistant Ruled Surfaces |

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