PROJECTIONS OF LAWLESS SEQUENCES

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UNIVERSITEIT VAN AMSTERDAM

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ACADEMISCH PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN
DOCTOR IN DE WISKUNDE EN NATUURWETENSCHAPPEN
AAN DE UNIVERSITEIT VAN AMSTERDAM,
OP GEZAG VAN DE RECTOR MAGNIFICUS,
DR D.W. BRESTERS,
HOOGLERAAR IN DE FACULTEIT DER
WISKUNDE EN NATUURWETENSCHAPPEN,
IN HET OPENBAAR TE VERDEDIGEN IN DE
AULA DER UNIVERSITEIT
(TIJDELIJK IN DE LUTHERSE KERK,
INGANG SINGEL 411, HOEK SPUI)
OP WOENSDAG 17 FEBRUARI 1982
DES NAMIDDAGS OM 4.00 UUR

DOOR

GERRIT FRANS VAN DER HOEVEN

GEBOREN TE ROTTERDAM

1981
MATHEMATISCH CENTRUM, AMSTERDAM

PROMOTOR: PROF.DR. A.S. TROELSTRA
COREFERENT: PROF.DR. D. VAN DALEN

Aan mijn ouders
Aan Lottie

ACKNOWLEDGEMENTS

I am deeply indebted to my thesis-supervisor A.S. TROELSTRA for guiding me into research in the foundations of intuitionistic mathematics. The many stimulating conversations we had on the subject have been an invaluable support to me during the writing of this thesis.

I am grateful to D. VAN DALEN for his work as coreferee. His suggestions led to a number of improvements in the presentation of my results.

I thank G. RENARDEL for his assistence in proof-reading.
Without his help quite a lot of errors might have remained unnoticed.

My gratitude also concerns the Mathematical Institute of the University of Amsterdam for providing me the opportunity to do research for this thesis, and the Mathematical Centre for the willingness to publish it.

Finally I wish to thank the people involved in the technical realisation of the publication: Mr. D. Zwarst for organizing the production process, and Mr. J. Schipper and his colleagues for the printing and the binding, Mr. T. Baanders for the drawing of the pictures and (last but not least) R. Riechelmann for her careful typing of the almost endless manuscript.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1. INTRODUCTION

In this monograph we investigate (a specific question concerning) intuitionistic Baire-space N, i.e. the universe of sequences of natural numbers, or, as Brouwer calls them, 'choice sequences'.

Our approach to the subject is the analytic one, as described by TROELSTRA in [T81]. That is to say, we do not accept the universe of choice sequences as a single primitive entity, quantification over which is intuitively clear. We look upon N rather as a collection of individual objects, each of them generated by a process of assigning to each argument $n \in \mathbb{N}$ a value $m \in \mathbb{N}$, in which we can distinguish subdomains, according to the type of data that are available to us on a sequence ε at any moment of its generation. The meaning of quantification over choice sequences of a specific type is explained in terms of the sort of data that can become available to us for individual sequences of this type at some stage of the generation process.

Two extreme types of choice sequences to be distinguished are the *lawlike* sequences and the *lawless* sequences.

Lawlike sequences are given to us by a law, i.e. a set of computation rules. In generating a lawlike sequence a, we simply apply these rules to the arguments 0,1,..., in order to find the values a0,a1,... . The data that are available to us on such a sequence do not change during the generation process, they consist of the set of computation rules. One may accept Church's thesis (CT), and identify lawlike with recursive. We shall not do so (though we do not reject CT either).

The lawless sequences are the extreme opposite of the lawlike ones. Here the generation process is divided into countably many stages 0,1,... . At stage 0 we can fix an initial segment of the sequence to be generated according to our needs, after that, we generate values as if we were throwing an infinite-sided die: at each stage we choose a completely arbitrary value, to be assigned to the next argument.

A lawless sequence for which we do *not* specify an initial segment (or in other words an *empty* segment) we call *proto-lawless*.

Lawless sequences were introduced by KREISEL in [K68]. A discussion of lawless sequences of zero's and ones (i.e. sequences comparable to the tossing of a coin) is given already in [K58] (there called 'absolutely free').

Before we discuss the lawless sequences here, two remarks are in order: Firstly, we do not discuss lawless sequences from a probabilistic point of view. The truth of a statement about a lawless sequence is not identified with 'having probability 1'; such a statement is true iff it is intuitionistically provable as certainly true.
Secondly, it is to be noted that we can consider *any* choice sequence at two levels: the extensional and the intensional. (This remark applies to the lawlike sequences as well.) At the extensional level we take into account only the information that is contained in the graph of the sequence (the outcome of the generation process), at the intensional level we consider also the way in which this graph is constructed (the generation process itself). E.g. we can distinguish between intensional and extensional equality of sequences. These do not always coincide: two sequences may result from different generation processes (in the case of lawlike sequences: from different computation laws) but still take the same values. (It turns out that for lawless sequences the difference between intensional and extensional equality disappears.)

The data that are available to us on the graph of a lawless sequence at any stage of its generation process, consist of an initial segment of that sequence only. Of course we do have more information on the sequence we can also tell e.g. which initial segment has been specified in advance and what values have been generated at later stages, but such facts are irrelevant at the extensional level.

On the basis of this insight in the possibly available data on the graph of a lawless sequence, one can justify informally, but rigorously, the axioms for the theory of lawless sequences $\underset{\sim}{\mathrm{LS}}$, as introduced by KREISEL ([K68]), and corrected by TROELSTRA in [T70A].

Some notation:

a and b are variables for lawlike sequences, α,β etc. for lawless ones.

n, v and x are variables for natural numbers, also used as codes for finite sequences of natural numbers.
If ϕ is an element of N then $\bar{\phi}x$ is the finite sequence $\langle\phi 0,\ldots,\phi(x-1)\rangle$, $\langle\ \rangle = \bar{\phi}0$ is the empty sequence.
If ϕ is an element of N and v is (the code of) a finite sequence then $\phi \in v$ expresses 'ϕ has initial segment v'.
If $A(\alpha,\beta_1,\ldots,\beta_p)$ is a formula which contains no lawless parameters besides $\alpha,\beta_1,\ldots,\beta_p$, then $\underline{\forall}\alpha A(\alpha,\beta_1,\ldots,\beta_p)$ denotes: 'for all lawless α distinct from $\beta_1,\ldots,\beta_p$, $A(\alpha,\beta_1,\ldots,\beta_p)$ holds'. e is a variable ranging over a set of neighbourhood-functions for continuous functionals. (This set is discussed in more detail below.) The members of this set are lawlike elements of N which satisfy:
- for all $\phi \in N$ there is an x such that $e(\bar{\phi}x) \neq 0$ and
- $e(\bar{\phi}x) = m+1 \rightarrow e(\bar{\phi}(x+y)) = m+1$.
e is a neighbourhood-function for the continuous $\Psi_e : N \rightarrow \mathbb{N}$ defined by

$$\Psi_e(\phi) = m \ \text{ iff } \exists x(e(\bar{\phi}x) = m+1).$$

We write $e(\phi)$ for $\Psi_e(\phi)$, and $e(\phi_1,\ldots,\phi_p)$ for $e(\nu_p(\phi_1,\ldots,\phi_p))$ where ν_p is some homeomorphism from N^p into N. j is a bijective 'pairing' function, $j : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$.
If $\phi \in N$ then ϕ can be seen as the code of a countable sequence of elements of N, $(\phi)_n$ is the n-th element of this sequence, defined by $(\phi)_n \equiv \lambda z.\phi j(n,z)$.
= between elements of N is used for extensional equality, i.e. $\phi = \psi$ abbreviates $\forall x(\phi x=\psi x)$.
LS finally is the universe of lawless sequences.
We adopt the convention that the choice parameters of a formula are explicitly shown. I.e. $A(\alpha_1,\ldots,\alpha_p)$ is a formula which contains no choice parameters besides (maybe) $\alpha_1,\ldots,\alpha_p$.

The LS-axioms are:

(LS1) $\forall v \exists \alpha(\alpha \in v)$,

i.e. LS lies dense in Baire-space.

(LS2) $\alpha = \beta \vee \alpha \neq \beta$,

i.e. extensional equality between lawless sequences is decidable.

(LS3) $\underline{\forall}\alpha(A(\alpha,\beta_1,\ldots,\beta_p) \rightarrow \exists v(\alpha\in v \wedge \underline{\forall}\gamma\in v\, A(\gamma,\beta_1,\ldots,\beta_p)))$,

the *axiom of open data*, where $A(\alpha,\beta_1,\ldots,\beta_p)$ is a formula expressing an extensional property of $\alpha,\beta_1,\ldots,\beta_p$. This axiom expresses that if A holds for a p+1-tuple $\alpha,\beta_1,\ldots,\beta_p$, α distinct from $\beta_1,\ldots,\beta_p$, then A holds for all lawless γ distinct from $\beta_1,\ldots,\beta_p$ in an open neighbourhood of α.
α.

(LS4) $\underline{\forall}\alpha_1\ldots\underline{\forall}\alpha_p\exists a\, A(\alpha_1,\ldots,\alpha_p,a) \rightarrow$

$$\exists e\exists b\underline{\forall}\alpha_1\ldots\underline{\forall}\alpha_p\, A(\alpha_1,\ldots,\alpha_p,(b)_{e(\alpha_1,\ldots,\alpha_p)}),$$

which expresses that if we can find with each p-tuple of distinct lawless sequences $\alpha_1,\ldots,\alpha_p$ a lawlike a such that $A(\alpha_1,\ldots,\alpha_p,a)$, then there is a countable sequence of lawlike sequences $(b)_0,(b)_1$, etc. coded in the single sequence b and a continuous Ψ_e with neighbourhood-function e such that for all distinct $\alpha_1,\ldots,\alpha_p$ $A(\alpha_1,\ldots,\alpha_p,(b)_n)$ holds, where $n = \Psi_e(\alpha_1,\ldots,\alpha_p)$. Here also A is an extensional property of $\alpha_1,\ldots,\alpha_p$.

The axioms and their motivation are discussed at length in [T77]. The justification of (LS4), which is the most complex of the four axioms, is refined in [T81].

We can distinguish two variant of $\underset{\sim}{\mathrm{LS}}$, according to our definition of the range of e in (LS4). In the strong version (as intended by Kreisel, in keeping with Brouwer's views) e ranges over the inductively defined set K. (A detailed treatment of this set is to be found in [KT70], we give a concise description in 1.3.7-27 below.)
In this version, the schema of bar induction is derivable from (LS4).

In the weaker variant we define the range of e in (LS4) as

$$K_{LS} \equiv \{e: \forall vw(ev\neq 0 \rightarrow ev=e(v*w)) \wedge \forall\alpha\exists x(e(\bar{\alpha}x)\neq 0)\}$$

(where * denotes concatenation of finite sequences), and we adopt the 'extension principle'

EP $e\in K_{LS} \wedge \phi\in N \rightarrow \exists x(e(\bar{\phi}x)\neq 0)$,

which expresses that any continuous Ψ from LS to $\mathbb{N}$ can be extended to a continuous operation on the whole of N.

Our proofs below can be formalized in the weaker system.

Note that the $\underset{\sim}{LS}$-axioms give a contextual definition of quantification over LS:
from density (LS1) and open data (LS3) we find that

$$\exists\alpha A(\alpha,\beta_1,\ldots,\beta_p) \leftrightarrow \exists v \underline{\forall}\alpha\epsilon v\ A(\alpha,\beta_1,\ldots,\beta_p)$$

which explains existential quantification in terms of universal quantification,
(LS4) explains universal quantification over LS in the context of a quantifier $\exists a$ (and hence in combination with $\exists x$ and $\vee$),
and from open data we can derive

$$\underline{\forall}\alpha_1\ldots\underline{\forall}\alpha_p(A(\alpha_1,\ldots,\alpha_p) \rightarrow B(\alpha_1,\ldots,\alpha_p)) \leftrightarrow$$

$$\forall v_1\ldots v_p(\underline{\forall}\alpha_1\epsilon v_1\ldots\underline{\forall}\alpha_p\epsilon v_p\ A(\alpha_1,\ldots,\alpha_p) \rightarrow \underline{\forall}\alpha_1\epsilon v_1\ldots\underline{\forall}\alpha_p\epsilon v_p\ B(\alpha_1,\ldots,\alpha_p))$$

which explains universal quantification in the context of an implication.

This observation is formally reflected in the elimination theorem (formulated by KREISEL in [K58], [K68], for a detailed treatment see [T77]): there is a translation τ from $\underset{\sim}{LS}$-sentences into sentences which do not contain $\underset{\sim}{LS}$-quantifiers, such that:

(i) each $\underset{\sim}{LS}$ sentence A is equivalent to τA (provable in $\underset{\sim}{LS}$)

(ii) if A is a theorem of $\underset{\sim}{LS}$, then τA is derivable in the lawlike part of $\underset{\sim}{LS}$ (i.e. without using (LS1)-(LS4)).

The lawless sequences are a simple type of choice sequence, in the sense that it is easy to see what kind of information we can have on a lawless α at the various stages of its generation process. This simplicity is of great advantage in rigorously justifying axioms for lawlessness, but it is a drawback if one tries to use LS as a basis for e.g. intuitionistic analysis.

To give an example: if one associates with each lawless α a real number generator (i.e. a Cauchy-sequence of rationals) $\langle r_n^\alpha\rangle_n$, in a non-trivial manner, i.e. in such a way that for all finite sequences v there are α and β with the same initial segment v which yield non-equivalent $\langle r_n^\alpha\rangle_n$ and $\langle r_n^\beta\rangle_n$, then the resulting notion of real number does not contain any rationals (to be able to state that $\langle r_n^\alpha\rangle_n$ converges to the rational q we need more information than just an initial segment of α, but initial segments are all we can ever get), and is for instance not closed under addition (for a similar reason).

To put this quite generally: LS has the serious defect that it is not closed under any non-trivial lawlike continuous operation.

Formal systems which, unlike LS, can be used for the foundation of intuitionistic analysis have been proposed by KLEENE and VESLEY [KV65] and by KREISEL and TROELSTRA [KT70]. From the analytical viewpoint the second one is the most interesting one.

The system of [KT70] is called CS (for 'choice sequences'). It is a corrected version of an earlier proposal by KREISEL (in [K63]). Before we formulate and discuss the CS-axioms, we need some more notation.

Let e be a neighbourhood-function for a continuous mapping from $N \to \mathbb{N}$. We can think of e as a countable sequence $e_0, e_1, \ldots$ of such neighbourhood-functions by putting

$$e_n v = e(\langle n\rangle * v)$$

where $\langle n\rangle$ is the finite sequence consisting only of the element n. With the sequence $e_0, e_1, \ldots$, and hence with e, we can associate a continuous mapping Γ_e from N into N by putting

$$\Gamma_e(\phi)(n) = m \quad \text{iff} \quad e_n(\phi) = m.$$

We write $e|\phi$ for $\Gamma_e(\phi)$, we call e a neighbourhood-function for Γ_e. $e|(\phi,\psi)$ abbreviates $e|\nu_2(\phi,\psi)$ where ν_2 is a homeomorphism from N^2 onto N.

The CS-axioms are:

(CS1) $\quad \forall\varepsilon\eta\forall e\exists\zeta(\zeta = e|(\varepsilon,\eta))$,

which expresses closure under pairing and continuous function application.

(CS2) $\quad \forall\varepsilon(A(\varepsilon) \to \exists e(\varepsilon\epsilon e \wedge \forall\eta A(e|\eta)))$,

where A is an extensional property of ε, and $\varepsilon \in e$ abbreviates 'ε lies in the range of Γ_e'. This axiom is called the *axiom of analytic data*, it expresses that if ε has the property A, then we can find a continuous Γ_e: $N \to N$ such that all sequences in its range (among which is ε) have the property A.

(CS3) $\forall\varepsilon\exists aA(\varepsilon,a) \rightarrow \exists e\exists b\forall\varepsilon A(\varepsilon,(b)_{e(\varepsilon)})$,

where A is an extensional property of ε independent of other choice parameters (cf. LS4),
and finally

(CS4) $\forall\varepsilon\exists\eta A(\varepsilon,\eta) \rightarrow \exists e\forall\varepsilon A(\varepsilon,e|\varepsilon)$,

where A is an extensional relation between ε and η, independent of other choice parameters. This axiom expresses that if all sequences lie in the domain of A, then A contains a continuous mapping. This continuous choice principle is sometimes called 'Brouwer's principle for functions'.

In the original formulation of CS, (CS1) is not an axiom but a theorem. We have put it among the axioms here to stress its importance. As a corollary of (CS1) we find e.g. that there exist choice sequences ε and lawlike sequences a which coincide (since for each a there is an e such that for any ϕ $e|\phi=a$), which is refutable for LS.

Note that this system also gives a contextual definition of the quantifiers $\forall\varepsilon,\exists\varepsilon$:
from analytic data and the existence of lawlike η we find

$$\exists\varepsilon A\varepsilon \leftrightarrow \exists aAa,$$

which explains existential choice quantification in the absence of choice parameters as lawlike existential quantification, (CS3) and (CS4) explain universal choice quantification in the context of existential quantification and disjunction, and from analytic data one derives

$$\forall\varepsilon(A\varepsilon \rightarrow B\varepsilon) \leftrightarrow \forall e(\forall\varepsilon A(e|\varepsilon) \rightarrow \forall\varepsilon B(e|\varepsilon))$$

which explains $\forall\varepsilon$ in the context of an implication.

We can formulate and prove an elimination theorem for CS analogous to the one for LS (see [KT70]).

CS has all the properties we would like a formal system for intuitionistic analysis to have: it expresses closure under continuous operations, it has strong continuity axioms and it fully explains choice-quantification. The problem is, that we do not have a fully analyzed notion (subdomain) of choice sequence for which the CS-axioms can be justified.

There are two approaches to the problem of finding interesting universes of choice sequences other than the lawlike and the lawless sequences: the informal approach and the study of universes of projections of lawless sequences.

A general framework for the informal approach has been set up by TROELSTRA [T69]. This was inspired by MYHILL, who developed in [My67] an approach to choice sequences which seemed to be implicit in some of Brouwer's writings. The idea is, that one can think of the generation process of a choice sequence as being a process of generating pairs $\langle x_0,R_0\rangle$, $\langle x_1,R_1\rangle$, etc., where $x_0,x_1,\ldots$ are to be the values of the generated sequence, and $R_0,R_1,\ldots$ are 'restrictions' taken from some fixed universe $\mathcal{R}$, equipped with a partial ordering $\le$ (weaker than). The values $x_n,x_{n+1},\ldots$ must meet the restriction R_n, the restriction R_n must be weaker than the next restriction R_{n+1}, otherwise we are completely free in choosing pairs for the sequence, with the stipulation that an initial segment may be fixed in advance. Subdomains are now distinguished according to the universe $\mathcal{R}$ from which the restrictions are taken.

E.g. we obtain the lawless sequences if we let $\mathcal{R}$ consist of a single restriction, the empty one U (for universal), which is met by all natural numbers.

If we take $\mathcal{R}$ to be the set $\{U,Z\}$, where Z (for zero) is the restriction of 'being equal to 0', which is met by 0 only, U being (obviously) weaker than Z, we obtain a notion of 'lawless zero sequence', a sequence which we start generating as if it were lawless, but then, at some moment of the generation process, we can decide to continue choosing only zero's.

The alternative approach is to study subsets of $\mathcal{N}$, the elements of which are constructed from lawless sequences by means of continuous operations from $\mathcal{N}$ to $\mathcal{N}$, so called universes of projections of lawless sequences. This approach was followed by VAN DALEN and TROELSTRA in [DT70] and further investigated in [T69B], [T70] and [T70A].
Examples of such universes are (1)-(4) below.

(1) $\{\pi\alpha : \alpha \in LS\}$ (introduced in [DT70]),

where $\pi : \mathcal{N} \to \mathcal{N}$ is defined by

$$\pi\phi(n) = \begin{cases} j_1(\phi n) & \text{iff } \forall m \le n\ (j_2(\phi m) = 0) \\ 0 & \text{otherwise.} \end{cases}$$

where j_1, j_2 are left-inverses to the pairing operation j, i.e. $j_1\ j(x,y) = x$, $j_2\ j(x,y) = y$, and $z \mapsto (j_1 z, j_2 z)$ is a mapping from $\mathbb{N}$ *onto* $\mathbb{N} \times \mathbb{N}$.
This projected universe can be seen to imitate (with a lot of redundancy in the coding) the behaviour of the lawless zero sequences above: the finite sequences $\langle j_2(\alpha 0)\rangle$, $\langle j_2(\alpha 0), j_2(\alpha 1)\rangle, \ldots$ play the rôle of the restrictions $R_0, R_1, \ldots$, a sequence $\langle j_2(\alpha 0), \ldots, j_2(\alpha n)\rangle$ which consists only of zero's corresponds to the empty restriction, if it contains a value unequal zero we have the restriction Z; the values $j_1(\alpha 0), j_2(\alpha 1), \ldots$ are the freely chosen $x_1, x_2, \ldots$, at least for as long as the restriction Z is not imposed.

(2) $\{e|\alpha : e \in K\}$ (discussed in [T77]),

which consists of all continuous images of a fixed lawless α (i.e. this universe is projected from a single lawless sequence).

(3) $\{e|(\alpha_1, \ldots, \alpha_p) : e \in K,\ \#(\alpha_1, \ldots, \alpha_p),\ \alpha_1, \ldots, \alpha_p \in LS\}$

(introduced in [T69B]),

which consists of all continuous images of all p-tuples of mutually distinct lawless sequences (for all p). We shall say more about (2) and (3) below. Finally we mention

(4) $\{n*(\alpha)_n : n \in \mathbb{N}\}$ (introduced in [T70A]),

a countable universe projected from a single lawless sequence α. As before $(\alpha)_n \equiv \lambda z.\alpha j(n,z)$, $*$ denotes concatenation, i.e. $n*(\alpha)_n$ is the result of prefixing the finite sequence with code n to the sequence $(\alpha)_n$.
The universe (4) is a model for the theory of lawless sequences LS, one can prove this fact inside LS. It is of interest to us because it shows that there are non-trivial universes of projections in which all sequences are identified by a natural number ('have a name' so to speak).

An advantage of the study of projections over the informal approach is that properties of projected universes can be proved from the LS-axioms whereas properties of an informal notion can only be justified informally, albeit sometimes quite rigorously.

Another interesting feature of universes of projections is the correspondence between such universes and Beth-models or equivalently topological models over Baire-space. Validity in a universe projected from a single lawless sequence translates immediately into validity in a Beth- or topological model. Under this translation the universe (2) above corresponds to the Moschovakis model of [M73] (cf. [T77]), and the universe (4) can be reinterpreted as a Beth-model for $\underset{\sim}{LS}$ (see the appendix of [D78]). Via (4), the universe (3) is equivalent to

$$\{e|(n_1*(\alpha)_{n_1},\ldots,n_p*(\alpha)_{n_p}): e \in K, \#(n_1,\ldots,n_p)\}$$

projected from the single lawless α, this universe corresponds to the Krol'-model of [K'78] (cf.[T81]).

These points in favour of the study of projections do not argue against the informal approach of conceptual analysis of new primitive notions. In fact there are good reasons to use both approaches simultaneously: the informal description of a notion of choice sequence may suggest to us a universe of projections in which the behaviour of those sequences is imitated (cf. the example under (1)), further study of this universe may help to improve our analysis of the informal concept. Eventually we can thus obtain a fully analyzed notion of choice sequence, together with a reduction of that notion, via projections, to the concept of lawless sequence, the simplest notion of choice sequence. This reduction will generally not be an isomorphism: one can expect to be able to rigorously justify axioms for the informal notion, which are provable for the projected imitations only under suitable language restrictions, necessary to avoid interference between the projected sequences and the lawless sequences from which they are constructed. (See e.g. [DT70].)

If we now return to the problem of finding a type of choice sequence for which the $\underset{\sim}{CS}$-axioms hold, we find that none of the projected universes of [DT70], [T69B,70,70A] and [T77] is a good candidate: these universes are either not closed under non-trivial continuous operations (as e.g. all examples in [DT70]) or, if they have closure properties, as e.g. (2) and (3) above, then it is impossible to derive strong continuity principles for them, at least in $\underset{\sim}{LS}$.

(The universe (3) of continuous images of p-tuples of independent lawless sequences does provide an acceptable basis for intuitionistic analysis, even if it is not a CS-model, cf. [T69B].)

On the informal side there is a proposal for a notion which might fulfill CS, made by Troelstra, first in a restricted form in [T68]: the GUC-sequences, later generalized in [T69,69A] to the concept of a GC-sequence. (GUC and GC stand for 'Generated by Unary Continuous operations' and 'Generated by Continuous operations' respectively.)

This notion is further analyzed in [T77], the analysis is discussed and somewhat refined by DUMMETT in [Du77]. Troelstra's analysis and Dummett's improvements yield convincing arguments showing that the notion is closed under non-trivial continuous operations and pairing and that it satisfies analytic data and $\forall\varepsilon\exists a$-continuity, (CS3).

The questions we shall deal with here are the following:

(a) to give a precise description of the notion of GC-sequence,

(b) to define universes of projections, projected from a single lawless α, which faithfully imitate the behaviour of the GC-sequences,

(c) to prove in LS that these projected universes are CS-models.

A first step towards answering (a)-(c) is taken in [HT80], where a variant of the GUC-sequences is imitated by projections, yielding a universe which is (provably in LS) closed under a restricted set of unary continuous operations, (but not under pairing), and which satisfies variants of analytic data (CS2) and the continuity axioms (CS3) and (CS4). These results are not a special case of the results we obtain here. This is so for technical reasons. At the cost of some extra technical effort we could give a uniform treatment which covers the results of [HT80] as well. In any case, the method of [HT80] remains of interest because of its direct, easily visualizable character.

Question (a) will be answered in chapter 2, where we also analyze the notion of GC(C)-sequence, for C a subset of K. (GC-sequence = GC(K)-sequence.)

As to question (b), we shall define universes of projections which imitate GC(C)-sequences, where C is subject to the restriction that it can be enumerated, modulo equivalence (cf.1.3.11, 1.3.26), by a mapping $J : \mathbb{N} \to C$ (i.e. we do not model GC(K)-sequences themselves).

In answer to question (c) we shall prove that for sufficiently nice enumerable $C \subset K$, the projection model for GC(C)-sequences satisfies the axiom system CS(C) which consists of

CS(C)1 $\forall\varepsilon\eta\forall e\in C\exists\zeta(\zeta=e|(\varepsilon,\eta))$

CS(C)2 $\forall\varepsilon(A(\varepsilon)\to\exists e\in C(\varepsilon\in e\wedge\forall\eta\, A(e|\eta)))$

CS(C)3 $\forall\varepsilon\exists a A(\varepsilon,a)\to\exists b\exists e\forall\varepsilon A(\varepsilon,(b)_{e(\varepsilon)})$

CS(C)4 $\forall\varepsilon\exists\eta\, A(\varepsilon,\eta)\to\forall\varepsilon\exists f\in C\, A(\varepsilon,f|\varepsilon)$,

i.e. all quantifiers $\forall e,\exists e$ in CS which have something to do with closure of the universe under continuous operations are relativized to C, and the quantifier combination $\exists e\forall\varepsilon$ in the conclusion of CS4 is switched.
In the presence of

AC-NF $\forall x\exists a\, A(x,a)\to\exists b\forall x\, A(x,(b)_x)$

one can show that CS = CS(K) (see 1.3.29).

An important tool in the proof of the validity of CS(C) in the projected universes is an elimination translation introduced by DRAGALIN in [Dr74]. This translation generalizes both the elimination translations for LS and CS, and is formulated as a kind of forcing. We return to it in chapter 8. Our results do not give a reduction of the full concept of GC-sequence to lawlessness, nor do they yield a projection model for the system CS itself.

It is to be expected however that if we extend LS with the schema

ECT_0 $\forall x(A(x)\to\exists y\, B(x,y))\to\exists z\forall x(A(x)\to\,!\{z\}(x)\wedge B(x,\{z\}(x)))$,

where A(x) is almost negative, and add variables for lawless sequences ranging over sets {x:A(x)}, A almost negative (cf. [T80A]), then CS(C) can be modelled for any $C\subset K$ which is enumerated by a mapping $J:\{x:A(x)\}\to K$, A almost negative. Since under assumption of ECT_0, K itself has such an enumeration we would obtain a model for CS + ECT_0. (The details of this claim have not yet been completely verified.)

To obtain a projection model for CS without using ECT_0, it seems necessary to work inside a theory LS^K of lawless sequences of K-functions. It is likely that a CS model can be constructed from such lawless K-sequences, but this needs further consideration, in particular the appropriate axiomatization of LS^K.

1.2. GENERAL OUTLINE

Chapter 2 of this monograph is devoted to the precise description of the notions of GC-sequence and GC(C)-sequence.

The chapters 3, 4 and 5 deal with the construction and investigation of projection models for the notion of GC(C)-sequence. Chapter 3 gives the necessary technical auxiliaries, chapter 4 contains the definition of the models, and in chapter 5 we derive a crucial property for the models, the so-called 'overtake-property'.

In chapter 6 the class of 'domains' is introduced. The projection models are special cases of domains. We shall give the proof of the validity of CS(C) in domains, hence CS(C) will hold in all projection models. By generalizing to domains, we achieve that our proofs are independent of some of the peculiarities of the models.

The treatment in the chapters 2-6 is informal in the sense that we do not derive our results inside a formal LS-like system.

In chapter 7 we introduce suitable extensions (modifications) of IDB_1 and LS in which the formalization of the results can be carried out.

Then, in the chapters 8 and 9, we deal with the problem of showing that domains are CS(C)-models, at least for suitable $C \subset K$.

In chapter 8 we describe and investigate an elimination translation τ, similar to the one introduced by DRAGALIN [Dr74], and we prove an elimination theorem for domains which states that a sentence A is valid in a domain iff its translation τA is derivable in the lawlike IDB_1 extension defined in chapter 7.

In chapter 9 we take the final step by showing that indeed all CS(C)-axioms (for suitable $C \subset K$) are derivable under the translation τ.

But before we turn to chapter 2, we present our notational conventions, basic definitions and their properties in the final section 1.3 of this introductory chapter.

1.3. PRELIMINARIES

This section consists of a long list of notations, definitions and simple facts. The notational conventions are mostly those of [T77] and [KT70]. The same holds for the definitions and facts. 'New' here are only 1.3.3 (on finite sets), 1.3.11(b), 1.3.12, 1.3.16, 1.3.21, 1.3.23 (the definitions of $e|w$, $[v]$, s^n, id, $e \propto f$, $e \wedge f$ and their properties), some of the results of 1.3.24, and 1.3.26 on subsets of K. In 1.3.28 and 1.3.29 we give reformulations of the systems LS and CS(C) which deviate slightly from the ones given in the introduction (1.1).

The reader is advised either to skip this section altogether and to consult it only when necessary, or to glance through its contents, with a special eye for the 'new' facts mentioned above.

1.3.1. Sets and variables

$\mathbb{N}$ is the set of natural numbers, we use i,k,m,n,u,v,w,x,y and z (with sub- or superscripts) as variables ranging over $\mathbb{N}$.

N is the set of all mappings from $\mathbb{N}$ into $\mathbb{N}$ (i.e. Baire-space), ϕ,ψ and χ (with sub- or superscripts) are used as variables for elements of N (see also 1.3.4), a,b and c (with sub- or superscripts) range over the lawlike elements of N.

K is the inductively defined subset of N which contains the lawlike neighbourhood functions for continuous functionals from N into N (cf. 1.3.7-27), we use e,f and g (with sub- or superscripts) as variables ranging over K.

LS is the universe of lawless sequences, we use α,β,γ and δ (with sub- or superscripts) as variables for elements of LS.

ε,η and ζ (with sub- or superscripts) are used to range over subsets $U \subset N$ distinct from LS and the set of lawlike sequences.

We use D,D_1,D',D_2, etc. and S,S_0,S',S_1, etc. as variables for sets.

1.3.2. Formulae and terms

(a) *Metavariables*
A,B,C,D, Φ and Ψ are used as metavariables for formulae, t and s are metavariables for number-terms, ϕ,ψ and χ are metavariables for function-terms (denoting elements of N).
(b) *Formulae and terms with parameters*
We write $A(a_1,\dots,a_p)$, where $a_1,\dots,a_p$ is any string of variables, to indicate that some of the parameters of A are in the list $a_1,\dots,a_p$, similarly we use $t[a_1,\dots,a_p]$ and $\phi[a_1,\dots,a_p]$ for number- and function-terms with parameters in the list $a_1,\dots,a_p$.
In formulae of the form A(a) we sometimes omit the brackets, and write Aa.
(c) *Substitution*
Once $A(a_1,\dots,a_p)$, $t[a_1,\dots,a_p]$ or $\phi[a_1,\dots,a_p]$ has been introduced,

$A(b_1,\ldots,b_p)$, $t[b_1,\ldots,b_p]$, $\phi[b_1,\ldots,b_p]$ denote the result of substituting b_i for a_i ($i = 1,\ldots,p$) in A, t or ϕ respectively. Here b_i is a variable or term of the same type as a_i, for $i = 1,\ldots,p$.
$A(b/a)$, $t[b/a]$, $\phi[b/a]$ denote the result of substituting b for a in A, t and ϕ respectively.
(d) *Restricted quantification*
If R is a relation in infix notation, like e.g. < between elements of $\mathbb{N}$ or $\in$ between elements and sets, then

$$\forall aRb\ A(a) \equiv_{def} \forall a(aRb \rightarrow A(a)),$$

$$\exists aRb\ A(a) \equiv_{def} \exists a(aRb \wedge A(a)),$$

where a is a variable and b a term, both of the right type.
(e) *Terms for sets*
If $b_1,\ldots,b_p$ are terms for elements of a set D, then $\{b_1,\ldots,b_p\}$ denotes the finite set with elements $b_1,\ldots,b_p$.
If a is a variable ranging over D, then $\{a:A(a)\}$ denotes the subset of D of all elements with the property A.

1.3.3. Finite sets

If we speak of a finite set, we mean finite in the strong sense of 'being in 1-1 correspondence with an initial segment of $\mathbb{N}$'. That is to say, we assume a finite subset $S \subset \mathbb{N}$ to be given to us by a mapping $\phi \in \mathcal{N}$ which enumerates its elements without repetitions and a natural number n, such that

$$\forall k<n\ \forall m<n(k \neq m \rightarrow \phi k \neq \phi m)$$

and

$$x \in S \quad \text{iff} \quad \exists m<n(x=\phi m).$$

n is the *cardinality of* S, notation card(S).

$\emptyset$ is the empty set with cardinality 0.
Note that with this interpretation of finite, membership of a finite set $S \subset \mathbb{N}$ is always decidable.

1.3.4. Mappings (domain, codomain, range, composition, restriction)

A mapping ϕ from D_1 into D_2, notation $\phi : D_1 \to D_2$, is a process of assigning to each element of D_1 a value in D_2. D_1 is the *domain* of ϕ, D_2 is the *codomain* of ϕ, the set $\{\phi(d) : d \in D_1\} \subset D_2$ is the *range* of ϕ.

$D_2^{D_1}$ is the set of all mappings from D_1 into D_2.

If the domain or the codomain of ϕ is not the set of natural numbers, then ϕ will be lawlike; that is to say, the only choice sequences considered here are choice sequences of natural numbers.

If the domain D of ϕ is a cartesian product, $D = D_1 \times D_2$, then $\phi(d_1,d_2)$ is the value assigned by ϕ to the ordered pair $\langle d_1,d_2\rangle \in D$.

If $\phi : D_1 \to D_2$ and $\psi : D_2 \to D_3$ then $\psi\circ\phi$ is the *composition* of ψ and ϕ; $\psi\circ\phi : D_1 \to D_3$, $\psi\circ\phi(d_1) = \psi(\phi(d_1))$.

If $\phi : D_1 \to D_2$ and $D \subset D_1$ then $\phi \restriction D$ is the *restriction* of ϕ to the domain D; $\phi\restriction D : D \to D_2$, $\phi\restriction D(d)=\phi(d)$.

If a is a variable ranging over D_1 and $b[a]$ is a term such that $\forall a \in D_1(b[a] \in D_2)$, then $a \mapsto b[a]$ and $\lambda a.b[a]$ denote 'b[a] as a function of a', i.e. a mapping with domain D_1 and codomain D_2 which assigns to $d \in D_1$ the value $b[d] \in D_2$.

If D is a set of mappings then we use ϕ,ψ and χ as variables ranging over D (cf.1.3.1. for $D=\mathcal{N}$).

In terms of the form $\phi(a)$ we sometimes omit the brackets and write ϕa.

= between functions is extensional equality, i.e. $\phi=\psi \equiv_{def} \forall x(\phi x=\psi x)$.

1.3.5. Elementary analysis

(a) The *formal system* EL *for* (lawlike) *elementary analysis* contains variables for natural numbers and (lawlike) sequences of natural numbers, constants: 0 (zero), S (successor), = (equality between natural numbers), λ (abstraction operator), Π (recursor for definition by recursion) and j,j_1,j_2 (a pairing function from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$ with two inverses), and the usual logical constants.

Axioms of EL are:

(1) the successor and equality axioms,

(2) the pairing axioms $j(j_1x,j_2x)=x$, $j_1j(x,y)=x$, $j_2j(x,y)=y$,

(3) the λ-conversion rule $(\lambda x.t[x])(s) = t[s]$,

(4) the axioms for primitive recursion:

$$\Pi(x,a,0)=x, \quad \Pi(x,a,(Sn))=aj(\Pi(x,a,n),n),$$

(5) and a weak choice axiom:

QF-AC $\forall x \exists y\, A(x,y) \to \exists a \forall x\, A(x,ax)$, A quantifier-free.

(b) We use the following symbols for *arithmetical operations and relations*:

$+$ for addition,

$\cdot$ for multiplication,

$\dot{-}$ for 'cut-off subtraction':
if x is larger than y, then $x \dot{-} y$ is the difference between x and y, otherwise $x \dot{-} y$ is zero.

sg for the 'sign-mapping': $sg\ 0=0$, $sg(n+1) = 1$.

$>, \geq, <, \leq$ for 'larger than', larger than or equal to', 'smaller than' and 'smaller than or equal to' respectively.

min for the minimum operator: $\min(x,y)$ is the minimum of x and y, if S is a finite non-empty subset of $\mathbb{N}$, then $\min(S)$ is the smallest element of S, if A is a decidable property of natural numbers and $\exists kAk$, then $\min_k(Ak)$ is the smallest natural number with the property A, $\min_{k<n}(Ak)$ is the smallest k below n with the property A, if such a number does not exist then $\min_{k<n}(Ak) = n$,

max for the maximum operator: $\max(x,y)$ is the maximum of x and y, if S is a finite non-empty subset of $\mathbb{N}$, then $\max(S)$ is the largest element of S, if $\phi \in N$ then $\max_{n\in S}(\phi n)$ is the largest element of $S' \equiv \{m : \exists n \in S(\phi n = m)\}$,

$\sum$ for repeated addition; if S is a finite non-empty subset of $\mathbb{N}$ and $\psi \in N$ then $\sum_{n\in S}(\psi n) \equiv \psi(\phi 0)+\ldots+\psi(\phi(\mathrm{card}(S)\dot{-}1))$, where ϕ is the mapping which enumerates S, $\sum_{n\in\emptyset}(\psi n) = 0$.

(c) *Pairing and p-tuple coding*

In the sequel it is assumed that the pairing j satisfies $j(0,0) = 0$.
For coding of p-tuples we use ν_p with inverses $j_1^p,\ldots,j_p^p$:

$$\nu_p(j_1^p x,\ldots,j_p^p x) = x, \quad j_i^p \nu_p(x_1,\ldots,x_p) = x_i \quad (1 \leq i \leq p).$$

We put

$$\nu_1(x) = x, \quad \nu_{p+1}(x_1,\ldots,x_{p+1}) = j(\nu_p(x_1,\ldots,x_p),x_{p+1}).$$

If $\phi \in N$ then

$$\phi(x_1,\ldots,x_p) \equiv \phi\nu_p(x_1,\ldots,x_p).$$

The use of j, j_1, j_2, ν_p and j_i^p is extended from $\mathbb{N}$ to N by putting (for $\phi, \psi, \phi_1, \ldots, \phi_p \in N$):

$$j(\phi,\psi) \equiv \lambda x.\ j(\phi x, \psi x),$$

$$j_1\phi \equiv \lambda x.\ j_1(\phi x), \quad j_2\phi \equiv \lambda x.\ j_2(\phi x),$$

$$\nu_p(\phi_1,\ldots,\phi_p) \equiv \lambda x.\ \nu_p(\phi_1 x,\ldots,\phi_p x)$$

and

$$j_i^p\phi \equiv \lambda x.\ j_i^p(\phi x).$$

If $\phi \in N$, $n \in \mathbb{N}$ then $(\phi)_n \equiv \lambda z.\ \phi j(n,z)$.

(d) *Finite sequences of natural numbers*

We assume a (primitive recursive) coding of all finite sequences onto the natural numbers to be given. In fact we shall not distinguish between the finite sequence and its code. We shall use (as much as possible) the variables u,v and w for 'a natural number in the rôle of sequence code'.

$\langle x_1,\ldots,x_p\rangle$ is the code-number of the finite sequence $x_1,\ldots,x_p$.

$\langle\ \rangle$ is the empty sequence. In the sequel we assume that $\langle\ \rangle = 0$.

$\hat{x}$ is the finite sequence $\langle x\rangle$.

$*$ is used for concatenation.

lth is the length-function.

tl is the tail-function, i.e. $\mathrm{tl}(\langle\ \rangle) = \langle\ \rangle$, $\mathrm{tl}(\hat{x}*v) = v$.

$(v)_n$ is the n-th element of the sequence v: if $v = \langle x_0,\ldots,x_p\rangle$, and $n<\mathrm{lth}(v)(=p+1)$, then $(v)_n = x_n$, if $n \geq \mathrm{lth}(v)$ then $(v)_n = 0$.

$\preccurlyeq$ is used for 'initial segment of' between finite sequences:
$v \preccurlyeq w \equiv \exists u(v*u=w)$.

$\bar{\phi}n, \bar{\phi}(n)$ is the finite sequence which contains the first n values of $\phi \in N$, i.e. $\bar{\phi}0 = \langle\ \rangle$, $\bar{\phi}(n+1) = \langle\phi 0,\ldots,\phi n\rangle$.

$\phi \in v$ expresses that $\phi \in N$ has initial segment v:
$\phi \in v \equiv_{\mathrm{def}} \forall n<\mathrm{lth}(v)(\phi n=(v)_n)$, i.e. $\phi \in v$ iff $\bar{\phi}(\mathrm{lth}(v)) = v$ iff $\exists n(\bar{\phi}n=v)$.

$k_1, k_2, \ldots, k_i^p$ $(1\leq i\leq p)$ are defined by:
$k_1(\langle\ \rangle) = k_2(\langle\ \rangle) = k_i^p(\langle\ \rangle) = \langle\ \rangle$,
$k_1(v*\hat{x}) = k_1 v*\langle j_1 x\rangle$, $k_2(v*\hat{x}) = k_2 v*\langle j_2 x\rangle$ and $k_i^p(v*\hat{x}) = k_i^p v*\langle j_i^p x\rangle$,
i.e. $k_1(\langle x_1,\ldots,x_p\rangle) = \langle j_1 x_1,\ldots,j_1 x_p\rangle$, $k_2(\langle x_1,\ldots,x_p\rangle) = \langle j_2 x_1,\ldots,j_2 x_p\rangle$ and likewise for k_i^p.
Via these mappings we can treat the finite sequence v as a pair of

finite sequences $k_1 v, k_2 v$ and as a p-tuple $k_1^p v, \ldots, k_p^p v$.

$*$ is also used for concatenation of a finite sequence with an element $\phi \in N$. $v*\phi$ is the sequence satisfying:

$$v*\phi(n) = \begin{cases} (v)_n & \text{if } n < \text{lth}(v) \\ \phi m & \text{if } n = m + \text{lth}(v). \end{cases}$$

1.3.6. <u>FACTS</u>.

(a) $j_1(v*\phi) = k_1 v * j_1 \phi$,

(b) $k_1(v*w) = k_1 v * k_1 w$,

(c) $k_1(\bar{\phi}x) = \overline{j_1\phi}(x)$,

and similarly for k_2 and k_i^p.

<u>The set K</u> (1.3.7-1.3.27)

1.3.7. <u>DEFINITION</u>. K is the set of lawlike sequences of natural numbers, inductively defined by

(K1) $\forall x(\lambda n.Sx) \in K$,

(K2) $a0=0 \wedge \forall x(\lambda v.a(\hat{x}*v) \in K) \rightarrow a \in K$,

(K3) $\forall a(A(a,Q) \rightarrow a \in Q) \rightarrow \forall a(a \in K \rightarrow a \in Q)$,

where $A(a,Q) \equiv \exists x(a=\lambda n.Sx) \wedge \forall x(\lambda v.a(\hat{x}*v) \in Q))$.

(K3) is called *induction over* K, it expresses that K is the smallest set satisfying (K1) and (K2).

1.3.8. $\underset{\sim}{\text{IDB}}_0$ is the formal system which consists of $\underset{\sim}{\text{EL}}$ plus the constant K and the axioms (K1)-(K3).

We use e,f,g etc. to range over K.

1.3.9. <u>FACTS</u>. If $e \in K$ then

(1) $\forall\phi\exists x(e(\bar{\phi}x) \neq 0)$, by induction over K,

(2) $\forall vw(ev \neq 0 \rightarrow ev = e(v*w))$.

1.3.10. <u>COROLLARIES</u> (including the definitions of 'bar', $e(\phi), e|\phi$).

(a) *The set* $\{w : ew\neq 0\}$ *is a bar in the tree of finite sequences: the bar given by* e *or simply the bar* e.

(b) *With each* $\phi \in N$ *there is a unique* y *such that, for some* x, $e(\bar{\phi}x) = y+1$. *For this* y *we write* $e(\phi)$, *we put* $e(\phi_1,\ldots,\phi_p) \equiv e\nu_p(\phi_1,\ldots,\phi_p)$.

(c) *With each* $\phi \in N$ *there is a unique sequence* $\psi \in N$ *such that* $\forall n\exists x(e(\hat{n}*\bar{\phi}x)=1+\psi n)$. *For* ψ *we write* $e|\phi$; $e|(\phi_1,\ldots,\phi_p)\equiv e|\nu_p(\phi_1,\ldots,\phi_p)$.

The mappings $\phi \mapsto e(\phi)$ and $\phi \mapsto e|\phi$ from N to $\mathbb{N}$ and from N to N respectively, are continuous. e is a *neighbourhood-function* for these mappings.

1.3.11. <u>DEFINITION</u> (of $e\simeq f$, $e\restriction w$). (a) Two elements e and f of K are *equivalent*, notation $e\simeq f$, iff $e|\phi = f|\phi$ for all ϕ, i.e. e and f are neighbourhood-functions for the same continuous mapping. Equivalently:

$$e\simeq f \equiv_{def} \forall w(ew\neq 0 \wedge fw\neq 0 \rightarrow ew=fw).$$

(b) $e\restriction w$ is a common initial segment of the sequences $\{e|\phi : \phi\in w\}$. Formally:

$$e\restriction w \equiv \overline{\phi[w]}(t[w])$$

where

$$\phi[w] \equiv \lambda x.e(\hat{x}*w)\dot{-}1$$

and

$$t[w] \equiv \min_{z<\mathrm{lth}(w)}(e(\hat{z}*w)=0).$$

(So $\mathrm{lth}(e\restriction w) \leq \mathrm{lth}(w)$).

1.3.12. <u>FACT</u>. $e\restriction w$ satisfies:

(a) $\forall x\exists y\leq x\ (e\restriction\bar{\phi}x = \overline{e|\phi}(y))$,

(b) $\forall y\exists x\geq y\ (\overline{e|\phi}(y) \preccurlyeq e\restriction\bar{\phi}x)$.

1.3.13. <u>LEMMA</u> (Closure properties of K).

(3) *If* $e \in K$, $\forall v(ev\neq 0 \rightarrow \lambda w.f(v*w)\in K)$, *and* $\forall vw(fv\neq 0 \rightarrow fv=f(v*w))$, *then* $f \in K$, *i.e.* K *is closed under 'unions over* $e \in K$'.

(4) *If* $e \in K$ *then* $\forall v(\lambda w.e(v*w)\in K)$, *i.e.* K *is closed under 'restrictions'*.

(5) *If* $e \in K$ *and* $f \in K$ *then* $\lambda v.e(f\restriction v) \in K$, *i.e.* K *is closed under '* ; *composition'* (cf.1.3.17 below for;).

<u>PROOF</u>. (3) and (4) by induction over K w.r.t. e.

(5) is more complicated, we outline the idea.

First one generalizes $f \upharpoonright w$ to $f \upharpoonright_n w$, putting

$$f \upharpoonright_n w \equiv \overline{\phi[n,w]}\ (t[n,w])$$

where

$$\phi[n,w] \equiv \lambda x.f(\langle n+x\rangle * w) \dot{-} 1$$

and

$$t[n,w] \equiv (\min_{z<\mathrm{lth}(w)}(f(\hat{z}*w)=0)) \dot{-} n,$$

i.e. $f \upharpoonright w = f \upharpoonright_0 w$, and if $n < \mathrm{lth}(w)$, $\forall m \leq n(f(\langle m\rangle * w) \neq 0)$ then

$$f \upharpoonright_n w = \langle f(\langle n\rangle * w) \dot{-} 1\rangle * f \upharpoonright_{n+1} w.$$

Now one proves by induction over K w.r.t. e

$$\forall n(\lambda v.e(f \upharpoonright_n v) \in K).$$

This is trivial for $e = \lambda z.Sx$. Assume $e0 = 0$ and for all x,n $\lambda v.e(\langle x\rangle * f \upharpoonright_n v) \in K$. To prove that $\lambda v.e(f \upharpoonright_m v) \in K$ it suffices by (3) to show that for some $g \in K$ we have:

$$(*) \qquad gw \neq 0 \to \lambda v.e(f \upharpoonright_m (w*v)) \in K.$$

Take g such that $gw \neq 0 \to m < \mathrm{lth}(w) \wedge \forall k \leq n(f(\langle k\rangle * w) \neq 0)$.
(For the existence of such a $g \in K$ we need $f \in K$, (3) and (6) below.)
Note that for this g, $gw \neq 0 \to \exists x(f \upharpoonright_m w = \langle x\rangle * f \upharpoonright_{m+1} w)$, and apply the induction hypothesis, which yields (*). □

1.3.14. <u>LEMMA</u> (a special element of K).
(6) *For all* n, $\lambda v.sg(\mathrm{lth}(v) \dot{-} n) \in K$.

<u>PROOF</u>. By induction w.r.t. n, using (K1), (K2). □

1.3.15. <u>COROLLARY</u>. *If* e *satisfies*

$$e0=0, \qquad e(\hat{x}*v) = sg(\mathrm{lth}(v) \dot{-} t[x]) \cdot (1+s[x,(v)_{t[x]}]),$$

where t[x] *is independent of* v *and* s *depends on no other values of* v *except* $(v)_{t[x]}$, *then* $e \in K$.

PROOF. Immediate from (6), (3), (K1) and (K2). □

1.3.16. FACT. (Including the 'definitions' of $[v], s^n$ and id.) From 1.3.15 it follows that K contains:

- for each v a mapping [v] such that $[v]|a = v*a$,
- for each n a mapping s^n ('shift over n') such that $s^n|a = \lambda z.a(n+z)$,
- for i = 1,2 mappings j_i such that $j_i|a = j_i(a)$.

The precise definitions of these mappings are irrelevant, we leave them to the reader.

We put

- $id \in K$ is the mapping [0], i.e. $id|a = 0*a = a$.

Derived closure conditions and operations on K (1.3.17–1.3.23)

1.3.17. DEFINITION. $e;f \equiv \lambda v.e(f\restriction v)$.

FACTS. If $e,f \in K$ then $e;f \in K$ by (5), $e;f$ satisfies $e;f|a = e(f|a)$.

1.3.18. DEFINITION. e:f is the mapping such that

$$e{:}f(0) = 0, \quad e{:}f(\hat{x}*v) = e(\hat{x}*(f\restriction v)).$$

FACTS. If $e,f \in K$ then $e{:}f \in K$ by (4), (5) and (K2). $e{:}f$ satisfies $e{:}f|a = e|(f|a)$.

1.3.19. DEFINITION (of h(e,u)). $h : K\times\mathbb{N} \to \mathbb{N}$ is the mapping which satisfies

$$h(e,u) = \begin{cases} 0 & \text{if } eu = 0 \\ 1+\phi[e,u] & \text{otherwise,} \end{cases}$$

where $\phi[e,u] \equiv$ the shortest initial segment v of u for which $ev \neq 0$.

FACT. If $e \in K$ then $\lambda u.h(e,u) \in K$ by (3).

1.3.20. DEFINITION. h_c is the mapping from $K \times \mathbb{N}$ into $\mathbb{N}$ which satisfies

$$h_c(e,0)=0, \ h_c(e,v*\hat{x}) = \begin{cases} 0 & \text{if } ev = 0, \\ h_c(e,v)*\hat{x} & \text{otherwise.} \end{cases}$$

FACT. $h_c(e,v)$ satisfies $ev\neq 0 \rightarrow v=(h(e,v)\dot{-}1)*h_c(e,v)$, i.e. $h_c(e,v)$ is the complement of $h(e,v)\dot{-}1$ w.r.t. v, provided $ev\neq 0$.

1.3.21. DEFINITION. $e\times f \equiv \lambda w.sg(ew)\cdot f(\langle h(e,w)\dot{-}1\rangle * h_c(e,w))$.

If $eu \neq 0$ then $sg(e(u*w)) = 1$, $h(e,u*w)\dot{-}1 = u'$ and $h_c(e,u*w) = u''*w$ for some u',u'' such that $u'*u'' = u$ (by 1.3.19,20). Hence $e\times f(u*w) = f(\langle u'\rangle * u''*w)$, so, if $e,f \in K$ then $e\times f \in K$ by (3) and (4).
In the context of $e\times f$, $f \in K$ is to be considered as representing the mapping $\phi : n \mapsto \lambda v.f(\hat{n}*v)$.
$e\times f$ is the 'composition' of the bars ϕn over the bar e, i.e. $e\times f(w) \neq 0$ iff $w = n*u$, n is the shortest initial segment of w such that $en \neq 0$ and $\phi(n)u \neq 0$. $e\times f$ is comparable to e/f in [KT70].

1.3.22. FACT. If $e \in K$ then $\lambda w.e(k_i w) \in K$ for $i = 1,2$, as follows from (5) and 1.3.16 by the observation that we can define j_i in such a way that $j_i|w = k_i w$.

1.3.23. DEFINITION. $e\wedge f$, the *pairing* of e and f, is defined by:

$$e\wedge f(0) = 0,$$

$$e\wedge f(\hat{x}*v) = sg(\phi_1[x,v])\cdot sg(\phi_2[x,v])\cdot(1+j(\phi_1[x,v]\dot{-}1, \phi_2[x,v]\dot{-}1)),$$

where

$$\phi_1[x,v] \equiv e(\hat{x}*k_1 v), \quad \phi_2[x,v] \equiv f(\hat{x}*k_2 v).$$

FACTS. If e and f belong to K then so does $e\wedge f$, by (4) and (3).
$e\wedge f$ is characterized by the following property:
$e\wedge f|(a,b) = j(e|a, f|b)$, or equivalently
$j_1(e\wedge f|a) = e|j_1 a$ and $j_2(e\wedge f|a) = f|j_2 a$.

1.3.24. LEMMA.
(a) *Composition of neighbourhood-functions is associative modulo equivalence:*

$$(e:f):g \simeq e:(f:g).$$

(*Therefore we omit brackets in the context of an equivalence.*)

(b) $\quad e\simeq e' \wedge f\simeq f' \rightarrow e:f \simeq e':f'$.

(c) $\quad \forall a(e|a\in w) \rightarrow e \simeq [w] : s^n : e$, *where* $n=\mathrm{lth}(w)$.

(d) $\quad f:[v] \simeq [f|v] : s^n : f : [v]$, *where* $n=\mathrm{lth}(f|v)$.

(e) *Pairing* $\wedge$ *is* 1-1 *modulo equivalence:*

$$e\simeq e' \wedge f\simeq f' \leftrightarrow (e\wedge f) \simeq (e'\wedge f').$$

(f) *Composition* : *is distributive over pairing* $\wedge$*:*

$$(e\wedge f):(e'\wedge f') \simeq (e:e')\wedge(f:f').$$

(g) $$[k_1 v] \wedge [k_2 v] \simeq [v], \quad id\wedge id \simeq id, \quad s^m \wedge s^m \simeq s^m.$$

PROOF. Is left to the reader. □

Note that the mapping $\phi \mapsto [w] : s^n|\phi$, $n=lth(w)$, has the effect of replacing the initial segment $\bar{\phi}n$ of ϕ by w. In [KT70], [T77] and [HT80] a separate K element is used as neighbourhood-function for this mapping. They write $w|\phi$ where we have $[w] : s^n|\phi$.

1.3.25. REMARK. The properties of K that are used in the sequel can be derived from (K1), (K2), (1)-(5) above. I.e. we do not use induction over K.

1.3.26. Subsets of K

Below we shall define a concept of choice sequence and projection models for that concept, relative to a subset C of K. We assume such a subset to be closed w.r.t. equivalence, i.e. by $C \subset K$ we mean that $\forall e\in C(e\in K)$ *and* $\forall ef(e\in C \wedge f \simeq e \to f\in C)$.

The reason for this convention is, that we are primarily interested in the continuous mappings $\phi \mapsto e|\phi$, $e \in C$, and not so much in the elements of C themselves. At one point in the definition of the primitive concept of choice sequence w.r.t. C however, it is essentail that C is a set of neighbourhood-functions and not a set of continuous mappings from N into N, namely in the construction of upb (see 2.8.1-3).

1.3.27. $\underset{\sim}{IDB}_1$ is a reformulation of $\underset{\sim}{IDB}_0$ in a richer language, containing K-variables e,f etc., K-terms like e:f, e;f etc., and K-term application $e|\cdot$ and $e(\cdot)$, strengthened with the choice axiom:

(AC-NF) $$\forall n\exists a\, A(n,a) \to \exists b\forall n\, A(n,(b)_n),$$

where $(b)_n \equiv \lambda z.bj(n,z)$, see 1.3.5(c).
We define a variant $\underset{\sim}{IDBF}_1$ of this system, suitable for our purposes, in 7.2.8-11.

The systems $\underset{\sim}{LS}$ and $\underset{\sim}{CS}(C)$ reformulated (1.3.28-29)

1.3.28. $\underset{\sim}{LS}$ is the formal theory of lawless sequences, of which $\underset{\sim}{IDB}_1$ is the lawlike part. We shall use the extension $\underset{\sim}{LSF}^*$ of this system, defined in 7.2.14-15. For the sake of completeness we give the axioms for lawless sequences of $\underset{\sim}{LS}$:

(LS1) $\forall v \exists \alpha (\alpha \in v)$ (density),

(LS2) $\alpha = \beta \vee \alpha \neq \beta$ (decidable equality),

(LS3) $\underline{\forall}\alpha(A(\alpha,\beta_1,\ldots,\beta_p) \rightarrow \exists v(\alpha \in v \wedge \underline{\forall}\gamma \in v\ A(\gamma,\beta_1,\ldots,\beta_p))$ (open data),

where A contains no lawless parameters besides those shown and
$\underline{\forall}\alpha\ \Phi(\alpha,\beta_1,\ldots,\beta_p) \equiv \forall\alpha(\Lambda_{i=1}^{p}\ \alpha \neq \beta_i \rightarrow \Phi(\alpha,\beta_1,\ldots,\beta_p))$,

(LS4) $\underline{\forall}\alpha_1 \ldots \underline{\forall}\alpha_p \exists a\ A(\alpha_1,\ldots,\alpha_p,a) \rightarrow$

$\exists e \forall v[ev \neq 0 \rightarrow \exists a \underline{\forall}\alpha_1,\ldots,\underline{\forall}\alpha_p\ A(\alpha_1,\ldots,\alpha_p,a)]$ (continuity),

where A contains no lawless parameters besides those shown and a is a metavariable for 'any lawlike variable'.

In the context of $\underset{\sim}{LS}$, AC-NF is restricted to predicates without lawless parameters.

Note that the formulation of (LS4) given in 1.1 (which is the usual one) is derivable from the one given here by AC-NF.

Our results can be formalized using a weaker variant of $\underset{\sim}{LS}$ where e in (LS4) ranges over the set

$$K_{LS} \equiv \{e: \forall vw(ev \neq 0 \rightarrow ev = e(v*w)) \wedge \forall\alpha\exists x(e(\bar{\alpha}x) \neq 0)\},$$

but using the extension principle

EP $\quad e \in K_{LS} \wedge \phi \in N \rightarrow \exists x(e(\bar{\phi}x) \neq 0)$.

The conditions (1)-(5) on K above are derivable from EP for K_{LS}.

1.3.29. Finally we reformulate $\underset{\sim}{CS}(C)$:
CS(C)1 (closure) $\forall \varepsilon \eta\ \forall e \in C\ \exists \zeta(\zeta = e|(\varepsilon,\eta))$,
CS(C)2 (analytic data)

$$\forall\varepsilon(A(\varepsilon) \to \exists e{\in}C(\varepsilon{\in}e \wedge \forall\eta\ A(e|\eta))),$$

where ε is the only choice parameter in A and $\varepsilon{\in}e \equiv \exists\eta(\varepsilon{=}e|\eta)$.
CS(C)3 (continuity for lawlike objects)

$$\forall\varepsilon\exists a\ A(\varepsilon,a) \to \exists e\forall v(ev{\neq}0 \to \exists a\forall\varepsilon\ A([v]|\varepsilon,a)),$$

where ε is the only choice parameter in A, a is a meta-variable for 'any lawlike variable' (n,a or e), and [v] is the K-element introduced in 1.3.16.
CS(C)4 ($\forall\varepsilon\exists\eta$-continuity)

$$\forall\varepsilon\exists\eta\ A(\varepsilon,\eta) \to \forall\varepsilon\exists e{\in}C\ A(\varepsilon,e|\varepsilon),$$

where ε and η are the only choice parameters in A.

In the presence of AC-NF, the formulations of CS(C)3 as given here and in the introduction are equivalent.

CS = CS(K), to see this we must show that CS(K)4:

$$\forall\varepsilon\exists\eta\ A(\varepsilon,\eta) \to \forall\varepsilon\exists e\ A(\varepsilon,e|\varepsilon),$$

is equivalent to the usual CS4:

$$\forall\varepsilon\exists\eta\ A(\varepsilon,\eta) \to \exists e\forall\varepsilon\ A(\varepsilon,e|\varepsilon).$$

CS4 implies CS(K)4 trivially, for the converse implication assume that $\forall\varepsilon\exists\eta\ A(\varepsilon,\eta)$ and apply CS(K)4, this yields $\forall\varepsilon\exists e\ A(\varepsilon,e|\varepsilon)$.
To this sentence we can apply CS(K)3, and find an $f \in K$ such that

$$\forall v(fv{\neq}0 \to \exists e\forall\varepsilon\ A([v]|\varepsilon,\ e|([v]|\varepsilon))).$$

Now put f and e together. First we apply AC-NF, yielding an e' such that

$$\forall v(fv{\neq}0 \to \forall\varepsilon\ A([v]|\varepsilon,\ \lambda w.e'(\langle v\rangle * w)|([v]|\varepsilon))).$$

Then we define g by

$$g0 = 0$$

$$g(\hat{x}*w) = \begin{cases} 0 & \text{if} \quad fw = 0, \\ e'(\langle h(f,w)\dot{-}1\rangle * \hat{x} * (h(f,w)\dot{-}1) * w) & \text{otherwise.} \end{cases}$$

One easily shows that $g \in K$, and that $\forall\varepsilon\ A(\varepsilon, g|\varepsilon)$. □

CHAPTER 2

GC-SEQUENCES AND GC-CARRIERS

2.1. The concept of GC-sequence was introduced by TROELSTRA (in [T68], [T69], [T69A]) as a candidate for a model of the CS-axioms. In [T77], appendix C, convincing, but not completely rigorous arguments are given for the validity of the principle of analytic data and $\forall\varepsilon\exists x$-continuity in the universe of GC-sequences. The description of this universe is elaborated and refined by DUMMETT ([Du77], see also [T80]). This chapter will be devoted to an even more rigorous, but still informal description of the primitive notion of GC-sequence (deviating in some respects from the one given by DUMMETT), which is to be used as a basis for the construction of a universe of projections, imitating the behaviour of the primitive concept.

First, we quote the description of the GC-sequence of [T77]:
"We think of a choice sequence α as started by generating values $\alpha 0, \alpha 1, \ldots$ - then, at some stage we decide to make α dependent on another, "fresh" sequence α_0 by means of a continuous operation, i.e. $\alpha = \Gamma_0\alpha_0$ ($\Gamma_0 : N \to N$); from then on, α is determined by choosing values of α_0 - at a later stage we may in turn wish to make α_0 dependent on another sequence α_1, so $\alpha_0 = \Gamma_1\alpha_1$, etc. (...).
So far we have presented a simplified picture, in as much as we omitted to take into account the possibility that a choice sequence is obtained from two or more other choice sequences i.e. (...)

$$\alpha_k = \Gamma_k \nu_{r(k)}(\alpha_{k+1,1}, \ldots, \alpha_{k+1,r(k)})."$$

(In this quotation a misprint in the original text has been corrected (line 4: $\alpha = \Gamma_0\alpha_0$ instead of $\alpha_0 = \Gamma_0\alpha_0$). Note that the variable-conventions in the quotation above, deviate from the ones we have adopted: we should use $\varepsilon, \varepsilon_0, \varepsilon_1, \varepsilon_k$ etc. instead of $\alpha, \alpha_0, \alpha_1, \alpha_k$ etc.)

It will be clear from this description, that the universe GC of GC-sequences is not a collection of individual objects, but rather a network in which gradually more dependencies can be created.

2.2. GC (THE UNIVERSE OF GC-SEQUENCES) IS CONSTRUCTED FROM GCC (THE UNIVERSE OF GC-CARRIERS)

The decision to make a sequence ε dependent on another sequence ε_0, or on a p-tuple $\varepsilon_{0,1},\ldots,\varepsilon_{0,p}$, or rather the description of that decision, presupposes something like the ability to call sequences 'by their name'. The existence of countable models for $\underset{\sim}{\underline{LS}}$ in which all sequences are indexed by a natural number ($\mathcal{U}_\alpha \equiv \{n*(\alpha)_n : n\in\mathbb{N}\}$ is such a universe) shows that it is feasible to consider universes of sequences in which all elements are identified by a natural number.

2.2.1. Hence we assume from now on:
the universe GC of GC-sequences is constructed from the countable universe GCC $\equiv \{\varepsilon_{\underline{n}} : n\in\mathbb{N}\}$ of GC-*carriers*. (*carriers* for short).
$\underline{n}$ is the *name* of the sequence $\varepsilon_{\underline{n}}$. Names are underlined to distinguish them from subscripts.
The construction of GCC is given in 2.3-2.8, the construction of GC from GCC in 2.10. The relation between GC and GCC, will be comparable to the relation between lawless and proto-lawless.

2.3. INTRODUCTION TO THE CONSTRUCTION OF GCC

One may think of the name $\underline{n}$ of a carrier as the name of an unbounded register for storage of natural numbers. The construction of GCC is an infinite (mental) process, divided into stages 1,2,3,..., in which the registers are filled with natural numbers (i.e. all sequences are constructed simultaneously). $\varepsilon_{\underline{n}}x$ is the x-th number in register $\underline{n}$. With each pair $(\underline{n},x)$ there is a stage $\bar{z}$ in the filling process at which sufficiently many data have been provided to determine $\varepsilon_{\underline{n}}x$. $\varepsilon_{\underline{n}}$ is the infinite sequence $\varepsilon_{\underline{n}}0,\varepsilon_{\underline{n}}1,\ldots$. In general we shall not have a finite description of $\varepsilon_{\underline{n}}$. An assertion like '$\varepsilon_{\underline{n}}$ has property P' is made at some stage z of the construction of GCC, on the basis of the data that are available to us on the contents of register $\underline{n}$ at that stage. This is characteristic for choice sequences.
The description of GC quoted above can be rephrased for GCC as:

at each stage of the construction of GCC we can either put some values in register $\underline{n}$, or make the contents of this register dependent on the values in the registers $\underline{n}_1,\ldots,\underline{n}_p$ via some continuous operation.
That is to say, if we decide to the second alternative at stage z, we associate a computation law to the register $\underline{n}$, by which for each x the value $\varepsilon_{\underline{n}}x$ can be determined from initial segments of the sequences $\varepsilon_{\underline{n}_1},\ldots,\varepsilon_{\underline{n}_p}$. These initial segments are to be found in the registers $\underline{n}_1,\ldots,\underline{n}_p$ at a stage z' later than z.

2.4. THE CREATION OF DEPENDENCIES BETWEEN GC-CARRIERS (1)

2.4.1. Initially all carriers are independent.
At each stage of the construction of GCC we can decide to make *at most one* carrier dependent on *at most two* others, or in other words: at each stage we can choose a pair $(\underline{k},\underline{m})$ or a triple $(\underline{k},\underline{m},\underline{n})$, $\underline{m}$ and $\underline{n}$ distinct from $\underline{k}$, and decide that $\varepsilon_{\underline{k}}$ will depend on $\varepsilon_{\underline{m}}$ or $\varepsilon_{\underline{m}}$ and $\varepsilon_{\underline{n}}$.
Not every choice of $\underline{m}$ and $\underline{n}$ is permitted:
the carriers $\varepsilon_{\underline{k}}$ is made dependent upon at stage z, must be *fresh* at stage z, where

2.4.2. <u>DEFINITION</u> (of a fresh carrier)
A GC-carrier $\varepsilon_{\underline{n}}$ is *fresh* at stage z, if it has not been made dependent on other carriers at any stage $z' \leq z$.

2.4.3. If we make $\varepsilon_{\underline{k}}$ dependent on $\varepsilon_{\underline{m}}$ or on $\varepsilon_{\underline{m}}$ and $\varepsilon_{\underline{n}}$ at stage z, we say that $\varepsilon_{\underline{k}}$ *jumps to* $\varepsilon_{\underline{m}}$ *at stage* z or *jumps to* $\varepsilon_{\underline{m}}$ *and* $\varepsilon_{\underline{n}}$ *at stage* z. If we are not especially interested in the sequence or sequences on which $\varepsilon_{\underline{k}}$ comes to depend, we simply say that $\varepsilon_{\underline{k}}$ *jumps at stage* z.

2.4.4. Note that there are two restrictions in this description of the creation of dependencies among GC-carriers, not to be found in the original description of GC-sequences, namely

- at each stage *at most one* carrier can be made dependent on others (*the single jump property*),
- a carrier can be made dependent on *at most two* others at the time (*at most binary jumps*).

As we shall see later, these restrictions are not essential, "at most one" and "at most two" can both be weakened to "finitely many". They are introduced to make it technically easier to imitate the concept by means of projections.

2.4.5. If we follow a particular carrier, say $\varepsilon_{\underline{3}}$, through the various stages, we can picture its history of dependencies (its history of jumps) by means of a sequence of labelled finite binary trees as in fig. 1.
Note: stage 0 is the stage preceding the actual construction of GCC, the other stages are stages in the construction process.

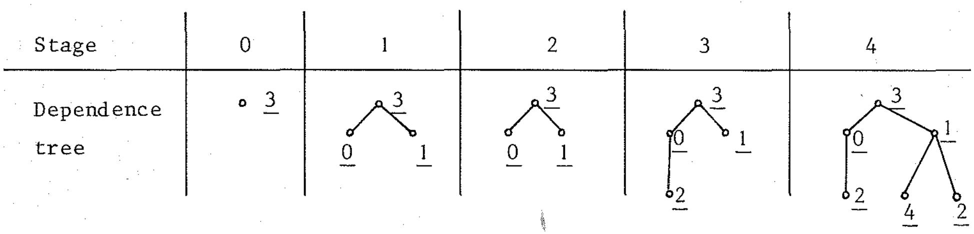


fig. 1

At stage 1 $\varepsilon_{\underline{3}}$ jumps to (is made dependent on) $\varepsilon_{\underline{0}}$ and $\varepsilon_{\underline{1}}$.
At stage 2 no dependencies affecting $\varepsilon_{\underline{3}}$ are made.
At stage 3 $\varepsilon_{\underline{0}}$ jumps to $\varepsilon_{\underline{2}}$, whence $\varepsilon_{\underline{3}}$ now depends on $\varepsilon_{\underline{2}}$ and $\varepsilon_{\underline{1}}$.
At stage 4 $\varepsilon_{\underline{1}}$ jumps to $\varepsilon_{\underline{4}}$ and $\varepsilon_{\underline{2}}$ whence $\varepsilon_{\underline{3}}$ now depends on $\varepsilon_{\underline{4}}$ and (two occurrences of) $\varepsilon_{\underline{2}}$.

2.5. THE CREATION OF DEPENDENCIES BETWEEN GC-CARRIERS (2)

The dependencies among carriers are made *via* continuous operations. If, at some stage, we decide to make $\varepsilon_{\underline{n}}$ dependent on other carriers, we also choose an $e \in K$, a neighbourhoodfunction for a $\Gamma : N \xrightarrow{cts} N$. We call e the *jumpfunction*.
The effect of the decision to make $\varepsilon_{\underline{k}}$ jump to $\varepsilon_{\underline{m}}$ with jumpfunction e is, that $\varepsilon_{\underline{k}}$ is completely (lawlike) determined relative to $\varepsilon_{\underline{m}}$. The equation which expresses the relation between $\varepsilon_{\underline{k}}$ and $\varepsilon_{\underline{m}}$ after the first one has jumped to the second one with jumpfunction e will be given in 2.7. As a first approximation to that equation, think of

(1) $\varepsilon_{\underline{k}} = e|\varepsilon_{\underline{m}}$.

Likewise

(2) $\varepsilon_{\underline{k}} = e|(\varepsilon_{\underline{m}}, \varepsilon_{\underline{n}})$

can be used as a first approximation to the relation between $\varepsilon_{\underline{k}}$ and $(\varepsilon_{\underline{m}},\varepsilon_{\underline{n}})$ if $\varepsilon_{\underline{k}}$ has jumped to $\varepsilon_{\underline{m}}$ and $\varepsilon_{\underline{n}}$ with jumpfunction e.
The jumpfunctions can be added to the dependence trees for $\varepsilon_{\underline{3}}$ of fig. 1.
This results in fig. 2

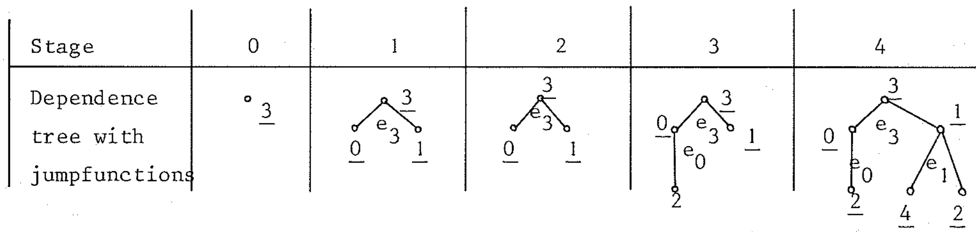


fig. 2

At stage 1 $\varepsilon_{\underline{3}}$ jumps to $\varepsilon_{\underline{0}}$ and $\varepsilon_{\underline{1}}$ with jumpfunction e_3.
At stage 3 $\varepsilon_{\underline{0}}$ jumps to $\varepsilon_{\underline{2}}$ with jumpfunction e_0.
At stage 4 $\varepsilon_{\underline{1}}$ jumps to $\varepsilon_{\underline{2}}$ and $\varepsilon_{\underline{4}}$ with jumpfunction e_1.

2.6. THE GENERATION OF VALUES FOR GC-CARRIERS (1)

2.6.1. Initially, all carriers (or rather: all registers $\underline{n}$) are empty. *At stage* 1 we can choose an initial segment of values for a finite number of carriers. We make this choice after we have decided whether any carrier will jump, and if so, which one. We only choose values for carriers that are still fresh. E.g. in the example of fig. 2, we could choose the initial segment m_0 for $\varepsilon_{\underline{0}}$ and m_1 for $\varepsilon_{\underline{1}}$.

2.6.2. DEFINITION. A carrier is *empty* at stage z, iff at no stage $z' < z$ we have decided to make it dependent on other carriers, or have chosen values for it.

2.6.3. *At stage* $z > 1$ we choose a segment of values for all carriers that are non-empty at stage z, but still fresh, and possibly for a finite number of empty ones as well. Again, we choose values after having chosen the jump (if any). In the example of fig. 2 we could choose
at stage 2: the segments m_0', m_1' for $\varepsilon_{\underline{0}}$, $\varepsilon_{\underline{1}}$ respectively,

then $\qquad \varepsilon_{\underline{0}} \in m_0 * m_0', \qquad \varepsilon_{\underline{1}} \in m_1 * m_1',$

at stage 3: the segment m_1'' for $\varepsilon_{\underline{1}}$ and the initial segment m_2 for $\varepsilon_{\underline{2}}$,

then $\quad \varepsilon_{\underline{1}} \in m_1 * m_1' * m_1''$, $\quad \varepsilon_{\underline{2}} \in m_2$,

(for $\varepsilon_{\underline{0}}$ see section 2.7 below),
at stage 4: the segment m_2' for $\varepsilon_{\underline{2}}$ and the initial segment m_4 for $\varepsilon_{\underline{4}}$,

then $\quad \varepsilon_{\underline{2}} \in m_2 * m_2'$, $\quad \varepsilon_{\underline{4}} \in m_4$,

(for $\varepsilon_{\underline{1}}$ see section 2.7 below).
The pictures of fig. 2 can be adapted to show also the generated values. Thus we obtain fig. 3.

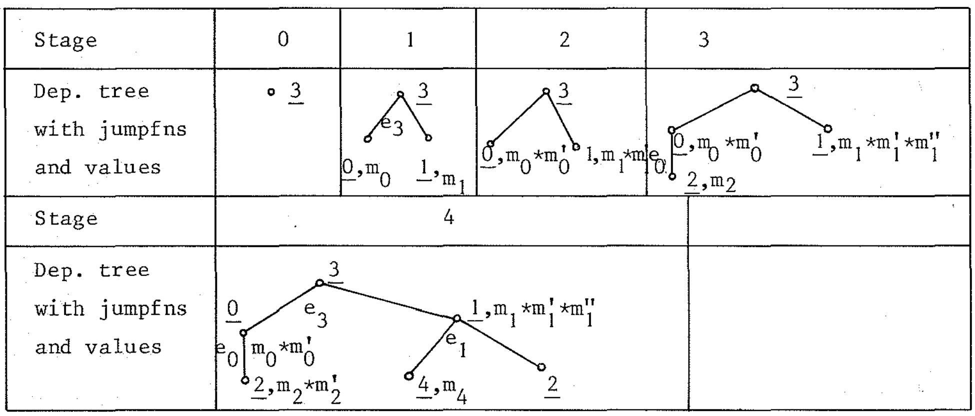


fig. 3

2.6.4. For each $\underline{n}$ and y the initial segment $\overline{\varepsilon_{\underline{n}} y}$ must be available to us at some stage of the construction of GCC. Hence certainly no carrier must remain empty. If carrier $\underline{n}$ is still empty and fresh at stage n + 1, then we generate an initial segment for it at this stage.
So, in our example above, we were forced to choose an initial segment for $\varepsilon_{\underline{0}}$ at stage 1, but we might have left $\varepsilon_{\underline{1}}$ empty. However, in that case we would have been forced to choose values for $\varepsilon_{\underline{1}}$ at stage 2.

2.7. THE CREATION OF DEPENDENCIES BETWEE GC-CARRIERS (3)

In the example of figure 3, the initial segment m_0 is generated for $\varepsilon_{\underline{0}}$ at stage 1, and the segment m_0' at stage 2, i.e. then

(1) $\varepsilon_{\underline{0}} \in m_0 * m_0'$.

At stage 3, $\varepsilon_{\underline{0}}$ jumps to $\varepsilon_{\underline{2}}$ with jumpfunction e_0. If we keep to our first approximation to the relation that now exists between $\varepsilon_{\underline{0}}$ and $\varepsilon_{\underline{2}}$, (see 2.5(1)) we find

(2) $\varepsilon_{\underline{0}} = e_0 | \varepsilon_{\underline{2}}$.

(1) and (2) may be in conflict. Hence we replace (2) by

(3) $\lambda z.\varepsilon_{\underline{0}}(k+z) = e_0 | \varepsilon_{\underline{2}}$,

where $k = \mathrm{lth}(m_0 * m_0')$. (1) and (3) together yield

(4) $\varepsilon_{\underline{0}} = m_0 * m_0' * (e_0 | \varepsilon_{\underline{2}})$.

In general: if $\varepsilon_{\underline{k}}$ is made dependent on other carriers at stage 1 then this dependency applies only to the values of $\varepsilon_{\underline{k}}$ that are not yet determined. That is to say, as a second approximation to the relation which exists between $\varepsilon_{\underline{k}}$ and the sequence(s) $\varepsilon_{\underline{m}}$ (and $\varepsilon_{\underline{n}}$) to which it jumps at stage z with jumpfunction e, we put

(5)
$$\varepsilon_{\underline{k}} = m_k * (e | \varepsilon_{\underline{m}}),$$
$$\varepsilon_{\underline{k}} = m_k * (e | (\varepsilon_{\underline{m}}, \varepsilon_{\underline{n}})) \text{ respectively,}$$

where m_k is the segment of values generated for $\varepsilon_{\underline{k}}$ at the stages before z.

At stage 4 in fig. 3 we have: $\varepsilon_{\underline{1}}$ jumps to $\varepsilon_{\underline{4}}$ and $\varepsilon_{\underline{2}}$ with jumpfunction e_1. At stage 3 we know already that

(6) $\varepsilon_1 \in m_1 * m_1' * m_1''$,

hence (5) would yield

(7) $\underline{\varepsilon_1} = m_1 * m_1' * m_1'' * (e_1 | (\underline{\varepsilon_4}, \underline{\varepsilon_2}))$.

We start to generate values for $\underline{\varepsilon_4}$ at stage 4, but $\underline{\varepsilon_2}$ is nonempty at this stage, at stage 3 we have already chosen the initial segment m_2. So, $\underline{\varepsilon_1}$ is made dependent at stage 4 on values that have been generated at stage 3. This is inconvenient for technical reasons. Therefore, we replace (7) by

(8) $\underline{\varepsilon_1} = m_1 * m_1' * m_1'' * (e_1 | (\underline{\varepsilon_4}, \lambda z.\underline{\varepsilon_2}(k+z)))$,

where $k = \text{lth}(m_2)$.
In general: if we make a carrier $\underline{\varepsilon_k}$ dependent on one or two others at stage z, then it will depend only on those values of the carrier(s) it jumps to, that become available at the stages $z' \geq z$. That is to say,

2.7.1. if $\underline{\varepsilon_k}$ jumps at stage z, with jumpfunction f, then the relation between $\underline{\varepsilon_k}$ and the carrier(s) $\underline{\varepsilon_m}$ (and $\underline{\varepsilon_n}$) it jumps to, is given by

$$\underline{\varepsilon_k} = m_k * f | \lambda z.\underline{\varepsilon_m}(y_m + z), \text{ or}$$

(9)

$$\underline{\varepsilon_k} = m_k * f | (\lambda z.\underline{\varepsilon_m}(y_m + z), \lambda z.\underline{\varepsilon_n}(y_n + z)),$$

where m_k is the initial segment of $\underline{\varepsilon_k}$ available to us after stage $z - 1$, and y_m, y_n are the lengths of the corresponding initial segments for $\underline{\varepsilon_m}$ and and $\underline{\varepsilon_n}$ respectively.
This formulation if final.

2.7.2. Note that for the range of all possible relations after a jump, it makes no difference whether we adopt (5) or (9). If we keep to (5) and $\underline{\varepsilon_k}$ jumps to $\underline{\varepsilon_m}$ with jumpfunction $f : s^{y_m}$, then we have the same relation between $\underline{\varepsilon_k}$ and $\underline{\varepsilon_m}$ as when we keep to (9) and $\underline{\varepsilon_k}$ jumps to $\underline{\varepsilon_m}$ with jumpfunction f. For a jump to two carriers $\underline{\varepsilon_m}$ and $\underline{\varepsilon_n}$, the choice of the jumpfunction $f : (s^{y_m} \wedge s^{y_n})$ with (5), gives the same result as the choice of f with (9).
Conversely, if we keep to (9) and $\underline{\varepsilon_k}$ jumps to $\underline{\varepsilon_n}$ with jumpfunction $e : [u_m]$, where u_m is the initial segment of $\underline{\varepsilon_m}$ available to us after stage $z - 1$ (i.e. $\text{lth}(u_m) = y_m$, $\underline{\varepsilon_m} = u_m * \lambda z.\underline{\varepsilon_m}(y_m + z)$) then this gives the same result as when $\underline{\varepsilon_k}$ jumps to $\underline{\varepsilon_m}$ with jumpfunction e, if we keep to (5). For a jump to two

carriers $\varepsilon_{\underline{m}}$ and $\varepsilon_{\underline{n}}$, $e:([u_m]\wedge[u_n])$ in (9) gives the same relation as e in (5) where $\varepsilon_{\underline{n}} \in u_n$ after stage z-1). For [u] and s^y see 1.3.16.

In the tree at stage 4 in fig. 3, the right most occurrence of $\underline{2}$ is not labelled with a sequence of generated values. The values generated for $\varepsilon_{\underline{2}}$ at that stage are $m_2 * m_2'$, as is shown by the label for the leftmost occurrence of $\underline{2}$. The rightmost occurrence of $\underline{2}$ results from a dependency between $\varepsilon_{\underline{1}}$ and $(\varepsilon_{\underline{4}}, \varepsilon_{\underline{2}})$, that is created at stage 4. In the foregoing we have stated that the initial segment m_2 of $\varepsilon_{\underline{2}}$ is not involved in this dependency. Hence we should label the rightmost $\underline{2}$ with m_2' only. This gives us fig. 4.

At stage 4:

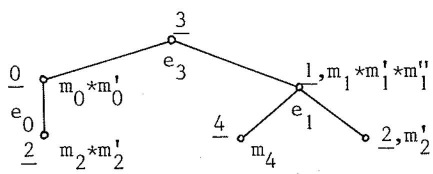


fig. 4

We have the following equations for $\varepsilon_{\underline{3}}, \varepsilon_{\underline{0}}, \varepsilon_{\underline{1}}$ at stage 4:

$\varepsilon_{\underline{3}} = e_3 | (\varepsilon_{\underline{0}}, \varepsilon_{\underline{1}})$,

$\varepsilon_{\underline{0}} = m_0 * m_0' * e_0 | \varepsilon_{\underline{2}}$,

$\varepsilon_{\underline{1}} = m_1 * m_1' * m_1'' * e_1 | (\varepsilon_{\underline{4}}, \lambda z.\varepsilon_{\underline{2}}(k+z))$, where $k = \mathrm{lth}(m_2)$,

while for $\varepsilon_{\underline{2}}$, $\varepsilon_{\underline{4}}$ we have

$\varepsilon_{\underline{2}} \in m_2 * m_2'$, $\lambda z.\varepsilon_{\underline{2}}(k+z) \in m_2'$ and $\varepsilon_{\underline{4}} \in m_4$.

2.8. THE GENERATION OF VALUES FOR GC-CARRIERS (2)

Consider the possible sequence of dependence trees with jumpfunctions for $\varepsilon_{\underline{0}}$ in fig. 5

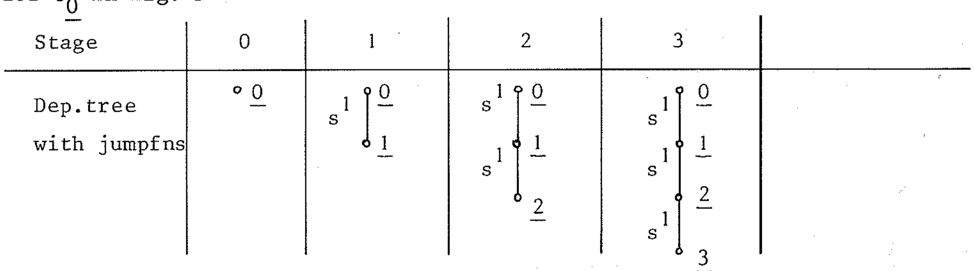


fig. 5

s^1 is a neighbourhoodfunction for the shift mapping $\Delta:\phi \mapsto \lambda x.\phi(x+1)$. Assume that at stage 1 we generate the initial segment $\langle 0\rangle$ for $\varepsilon_{\underline{1}}$, then we have:

(1) $$\varepsilon_{\underline{1}}0 = 0, \quad \varepsilon_{\underline{0}} = s^1|\varepsilon_{\underline{1}} = \lambda x.\varepsilon_{\underline{1}}(x+1).$$

At stage 2 we might generate the initial segment $\langle 1\rangle$ for $\varepsilon_{\underline{2}}$, then

(2) $$\varepsilon_{\underline{2}}0 = 1, \quad \varepsilon_{\underline{1}} = \langle 0\rangle * s^1|\varepsilon_{\underline{2}} = \langle 0\rangle * \lambda x.\varepsilon_{\underline{2}}(x+1),$$
$$\varepsilon_{\underline{0}} = \lambda x.\varepsilon_{\underline{1}}(x+1) = \lambda x.\varepsilon_{\underline{2}}(x+1).$$

If at stage 3 we generate the initial segment $\langle 2\rangle$ for $\varepsilon_{\underline{3}}$ then

(3) $$\varepsilon_{\underline{3}}0 = 2, \quad \varepsilon_{\underline{2}} = \langle 1\rangle * \lambda x.\varepsilon_{\underline{3}}(x+1),$$
$$\varepsilon_{\underline{1}} = \langle 0\rangle * \lambda x.\varepsilon_{\underline{2}}(x+1) = \langle 0\rangle * \lambda x.\varepsilon_{\underline{3}}(x+1),$$
$$\varepsilon_{\underline{0}} = \lambda x.\varepsilon_{\underline{2}}(x+1) = \lambda x.\varepsilon_{\underline{3}}(x+1).$$

None of the sets of equations (1), (2) and (3) determines $\varepsilon_{\underline{0}}0$, and there is no guarantee that it will be determined at a stage $z > 3$. The process of generating values as described in 2.6, must be adapted so as to provide this guarantee. It is possible to refine the process in such a way, that at stage $n + 1$ the initial segments $\overline{\varepsilon_{\underline{m}}}(n+1)$ are available to us for all $m \leq n$. We make a more radical change in the method of generating values, to the effect that at stage $n + 1$ the initial segments $\overline{\varepsilon_{\underline{m}}}(n+1)$ are determined for *all* m. We motivate our approach at the end of this section.

To generate values for carriers we proceed as follows.

2.8.1. *At stage* 1 we first deal with the creation of dependencies. So we start generating values e.g. in a situation as pictured in fig. 6. (Carriers not shown are all empty.)

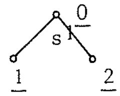
Carrier	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$
Dep.tree with jumpfns		∘ $\underline{1}$	∘ $\underline{2}$	∘ $\underline{3}$	∘ $\underline{4}$

fig. 6

(a) We choose finite segments of values for a finite number of fresh carriers, or equivalently: we generate a natural number x, and associate with each fresh carrier $\underline{n}$ the finite sequence $(x)_n$, which is empty for all but finitely many n. We call $(x)_n$ the *preliminary choice of values* for $\varepsilon_{\underline{n}}$. E.g. in fig. 6 we could choose x = <<1>, <2,3>, <4>>, this yields <2,3> and <4> as preliminary choice of values for $\varepsilon_{\underline{1}}$, $\varepsilon_{\underline{2}}$ respectively, and < > for all others.

(b) The preliminary choice may be insufficient to determine values for non-fresh carriers. In our example e.g., we need at least two values for $\varepsilon_{\underline{1}}$ and $\varepsilon_{\underline{2}}$ to determine $\varepsilon_{\underline{0}}0$, whereas the preliminary choice for $\varepsilon_{\underline{2}}$ consists of the single value 4. We now extend our preliminary choice to an infinite supply of values for each fresh carrier, by putting:

the *guiding sequence* for a fresh carrier $\underline{n}$, is the sequence $gs_n = (x)_n * \lambda z.0$.

In our example, the guiding sequences for $\varepsilon_{\underline{1}}$ and $\varepsilon_{\underline{2}}$ are <2,3> $*$ $\lambda z.0$ and <4> $*$ $\lambda z.0$ respectively, all other fresh carriers have $\lambda z.0$ for their guiding sequence.

(c) The final choice of values for each carrier $\underline{n}$ is to be an initial segment of its guiding sequence. In finding suitable (i.e. sufficiently long) initial segments, we distinguish two cases:

- if no carrier has jumped, then $\langle gs_n 0\rangle$ is the initial segment generated for $\varepsilon_{\underline{n}}$, i.e. we choose $\varepsilon_{\underline{n}}0 = gs_n 0$.
- if a carrier $\varepsilon_{\underline{k}}$ has been made dependent on one or two others, then we have an equation for $\varepsilon_{\underline{k}}$, either

$$\varepsilon_{\underline{k}} = e|\varepsilon_{\underline{m}},$$

or

$$\varepsilon_{\underline{k}} = e|(\varepsilon_{\underline{m}},\varepsilon_{\underline{n}}),$$

where $\underline{\varepsilon}_m$ and $\underline{\varepsilon}_n$ are fresh carriers. In our example we have $\underline{\varepsilon}_0 = s^1|(\underline{\varepsilon}_1,\underline{\varepsilon}_2)$. Now we substitute $\overline{gs}_m$, $\overline{gs}_n$ for $\underline{\varepsilon}_m$ and $\underline{\varepsilon}_n$ respectively in this equation, which yields $\underline{\varepsilon}_0 = s^1|(\overline{<2,3>*\lambda z.0},\overline{<4>*\lambda z.0})$. In general, we find either

$$\underline{\varepsilon}_k = e|gs_m,$$

or

$$\underline{\varepsilon}_k = e|(gs_m,gs_n).$$

From this equation we can determine $\underline{\varepsilon}_k 0$; the computation of that value requires only an initial segment of either gs_m or (gs_m,gs_n). We put:

the *upperbound* for the relevant values of the guiding sequences at stage 1 is

$upb_1 \equiv$ the minimal $z \in \mathbb{N}$ such that $\underline{\varepsilon}_k 0$ is determined by $\overline{gs_m}(z)$ or $\overline{(gs_m,gs_n)}(z)$ respectively, i.e. such that $e(<0>*\overline{gs_m}(z)) \neq 0$ or $e(<0>*\overline{(gs_m,gs_n)}(z)) \neq 0$ respectively.

In the example $upb_1 = 2$ (i.e. assuming that s^1 has the optimal modulus of continuity).
Once we have computed upb_1, we put

$\overline{gs_n}(1+upb_1)$ is the sequence of values generated for the fresh carrier $\underline{\varepsilon}_n$ at stage 1.

We use $1+upb_1$ instead of just upb_1 here, to provide for the case that $upb_1 = 0$. In our example we would end up with

<2,3,0> as the initial segment of $\underline{\varepsilon}_1$,
<4,0,0> as the initial segment of $\underline{\varepsilon}_2$, and
<0,0,0> as the initial segment of all other fresh carriers.

From the equation $\underline{\varepsilon}_0 = s^1|(\underline{\varepsilon}_1,\underline{\varepsilon}_2)$ we find

$$\underline{\varepsilon}_0 0 = j(3,0) \qquad \underline{\varepsilon}_0 1 = j(0,0).$$

2.8.2. *At the next stages* we essentially repeat this procedure. To continue our example, let $\varepsilon_{\underline{3}}$ be made dependent on $\varepsilon_{\underline{1}}$ via e at stage 2, see fig. 7.

Carrier	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$
Dep.tree with jumpfns and gen. values	$\underline{0}$ (via s^1) with children $\underline{1}$, <2,3,0> and $\underline{2}$, <4,0,0>	∘ 1,<2,3,0>	∘ $\underline{2}$,<4,0,0>	∘ $\underline{3}$,<0,0,0> (via e) ∘ $\underline{1}$	∘ $\underline{4}$,<0,0,0>

fig. 7

First we generate a y, e.g. y = <0,<0,1>,0,<2>,<3,4,5>>, i.e. as preliminary choice of values we have

$(y)_1$ = <0,1> for $\varepsilon_{\underline{1}}$

$(y)_4$ = <3,4,5> for $\varepsilon_{\underline{4}}$

$(y)_n$ = < > for all fresh n, $n \notin \{1,4\}$,

and as guiding sequences

$gs_1 = \langle 0,1\rangle * \lambda z.0, \quad gs_4 = \langle 3,4,5\rangle * \lambda z.0,$

and

$gs_n = \lambda z.0$ for $\varepsilon_{\underline{n}}$ fresh, $n \notin \{1,4\}$.

At this stage we have to provide for the determination of $\varepsilon_{\underline{n}}1$, for all carriers. There are two dependencies now, which yield two equations to be considered:

(1) $\varepsilon_{\underline{0}} = s^1|(\varepsilon_{\underline{1}},\varepsilon_{\underline{2}})$,

(2) $\varepsilon_{\underline{3}} = \langle 0,0,0\rangle * e|\lambda z.\ \varepsilon_{\underline{1}}(3+z)$.

(Cf. 2.7.1, 3 is the length of the initial segment generated for $\varepsilon_{\underline{1}}$ at stage 1.) Now we substitute the guiding sequences for *the parts of* $\varepsilon_{\underline{1}}$ *and* $\varepsilon_{\underline{2}}$ *that are not yet available*, i.e. gs_1 replaces $\lambda z.\varepsilon_{\underline{1}}(3+z)$ and gs_2 replaces $\lambda z.\varepsilon_{\underline{2}}(3+z)$, which yields

$$\varepsilon_{\underline{0}} = s^1|(<2,3,0>*gs_1,<4,0,0>*gs_2),$$

$$\varepsilon_{\underline{3}} = <0,0,0> * e|gs_1.$$

Obviously, we do not need any values of gs_1, gs_2 to determine $\varepsilon_{\underline{0}}1$ and $\varepsilon_{\underline{3}}1$ from these equations, that is, we find $upb_2 = 0$. The generated values for $\varepsilon_{\underline{n}}$, n fresh at stage 2, are $\overline{gs_n}(1+upb_2)$, i.e. $<gs_n0>$. So now we have

$$\varepsilon_{\underline{1}} \in <2,3,0> * <0> \text{ since } gs_1 0 = 0,$$
$$\varepsilon_{\underline{2}} \in <4,0,0> * <0> \text{ since } gs_2 0 = 0,$$
$$\varepsilon_{\underline{4}} \in <0,0,0> * <3> \text{ since } gs_4 0 = 3,$$
$$\varepsilon_{\underline{0}} = s^1|(\varepsilon_{\underline{1}},\varepsilon_{\underline{2}}), \text{ whence } \varepsilon_{\underline{0}} \in <j(3,0),j(0,0),j(0,0)>, \text{ and}$$
$$\varepsilon_{\underline{3}} = <0,0,0> * e|\lambda z.\varepsilon_{\underline{1}}(3+z), \text{ whence } \varepsilon_{\underline{3}} \in <0,0,0>.$$

All other carriers have initial segment <0,0,0,0>.

Fig. 8 shows the situation after stage 2.

Carrier	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$
Dep.tree with jumpfns and gen.values	$\underline{0}$; s 1; $\underline{1},m_1$; $\underline{2},m_2$	$\underline{1},m_1$	$\underline{2},m_2$	$\underline{3},m_3$; e; $\underline{1}$,<0>	$\underline{4},m_4$

$m_1 = <2,3,0>*<0>$, $m_2 = <4,0,0>*<0>$, $m_3 = <0,0,0>$, $m_4 = <0,0,0>*<3>$

fig. 8

Carrier	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$
Dep.tree with jumpfunctions and gen.values	s 1; $\underline{1},m_1$; $\underline{2},m_2$; f; $\underline{4}$	$\underline{1},m_1$; f; $\underline{4}$	2,m_2	$\underline{3},m_3$; e; $\underline{1}$,<0>; f; $\underline{4}$	4,m_4

fig. 9

Figure 9 shows the situation which occurs if we decide to make $\varepsilon_{\underline{1}}$ dependent on $\varepsilon_{\underline{4}}$ at stage 3.
At this stage $\varepsilon_{\underline{n}}2$ must become available for all n. In fact, these values are already available at stage 2. I.e. the upb_3 computation will yield 0, and there will be one value generated for each fresh $\varepsilon_{\underline{n}}:gs_n0$.
Assuming that $gs_2 0 = 1$, $gs_4 0 = 2$, we reach the situation of fig. 10.

Carrier	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$
Dep.tree with jumpfns and gen.values	$\underline{0}$; s, 1: $\underline{1}, m_1$ — f — $\underline{4}, \langle 2 \rangle$; $\underline{2}, m_2 * \langle 1 \rangle$	e: $\underline{1}, m_1$ — $\underline{4}, \langle 2 \rangle$	$\underline{2}, m_2 * \langle 1 \rangle$	$\underline{3}, m_3$ — e — $\underline{1}, \langle 0 \rangle$ — f — $\underline{4}, \langle 2 \rangle$	$\underline{4}, m_4 * \langle 2 \rangle$

fig. 10

At stage 4 we do not create new dependencies, i.e. we start generating values in the situation of fig. 10.
First we determine the guiding sequences, then we make a list of all carriers that depend on other ones. This list consists of $\underline{0}$, $\underline{1}$, and $\underline{3}$ in our example. The equations relating these non-fresh carriers to the fresh ones are:

(3) $\varepsilon_{\underline{0}} = s^1 | (\varepsilon_{\underline{1}}, \varepsilon_{\underline{2}})$,

(4) $\varepsilon_{\underline{1}} = m_1 * f | \lambda z.\varepsilon_{\underline{4}}(4+z)$, where $4 = \mathrm{lth}(m_4)$ is the number of values that were available for $\varepsilon_{\underline{4}}$ when $\varepsilon_{\underline{1}}$ came to depend upon it at stage 3.

(5) $\varepsilon_{\underline{3}} = m_3 * e | \lambda z.\varepsilon_{\underline{1}}(3+z)$, see (2) above.

If we substitute (4) in (3) we find

(6) $\varepsilon_{\underline{0}} = s^1 | (m_1 * f | \lambda z.\varepsilon_{\underline{4}}(4+z), \varepsilon_{\underline{2}})$,

substituting (4) in (5) yields

(7) $\varepsilon_{\underline{3}} = m_3 * e|(\langle 0\rangle * f|\lambda z.\varepsilon_{\underline{4}}(4+z))$, where $\langle 0\rangle = \langle (m_1)_3\rangle$, the first value of $\lambda z.\varepsilon_{\underline{1}}(3+z)$.

We do already have initial segments $m_2 * \langle 1\rangle$ and $m_4 * \langle 2\rangle$, both with length 5 for $\varepsilon_{\underline{2}}$ and $\varepsilon_{\underline{4}}$ respectively, so if we substitute gs_2 and gs_4 for the parts of $\varepsilon_{\underline{2}}$ and $\varepsilon_{\underline{4}}$ that are not yet available, we find

(8) $\varepsilon_{\underline{0}} = s^1|(m_1 * f|(\langle 2\rangle * gs_4),\ m_2 * \langle 1\rangle * gs_2)$,

(9) $\varepsilon_{\underline{3}} = m_3 * e|(\langle 0\rangle * f|(\langle 2\rangle * gs_4))$,

(10) $\varepsilon_{\underline{1}} = m_1 * f|(\langle 2\rangle * gs_4)$.

From these equations we can compute $\varepsilon_{\underline{0}}3$, $\varepsilon_{\underline{1}}3$ and $\varepsilon_{\underline{3}}3$, the values that must become available to us at this stage. We determine $upb_4 \equiv$ minimal z, such that $\overline{gs_2}(z)$ and $\overline{gs_4}(z)$ suffice to perform these computations. (upb_4 will probably be unequal to 0, depending on e and f). As before, $\overline{gs_n}(1+upb_4)$ is the sequence of generated values for each fresh n at this stage.

2.8.3. Summarizing: in generating values for fresh carriers at stage z + 1 one takes the following steps:

- Determine a preliminary choice of values (completely arbitrary).
- Determine guiding sequences.
- List all 'depency equations', either of the form $\varepsilon_{\underline{k}} = \phi(\varepsilon_{\underline{m}},\varepsilon_{\underline{n}})$ or of the form $\varepsilon_{\underline{k}} = \phi(\varepsilon_{\underline{m}})$.
- If chains of dependencies exist, make substitutions in this list, to obtain only equations of the form $\varepsilon_{\underline{k}} = \phi(\varepsilon_{\underline{n}_1},\ldots,\varepsilon_{\underline{n}_p})$, where $\underline{n}_1,\ldots,\underline{n}_p$ are fresh at stage z + 1.
- Make a list $\underline{m}_1,\ldots,\underline{m}_q$ of all fresh carriers that occur in the right hand side of an equation in the list, and substitute gs_{m_i} for the part of $\varepsilon_{\underline{m_i}}$ not yet available at stage z + 1 in all equations of the list, for $i = 1,\ldots,q$.
- Determine the minimal y such that $\overline{gs_{m_1}}(y),\ldots,\overline{gs_{m_q}}(y)$ suffice to compute $\varepsilon_{\underline{k}}(z)$ from the equation for $\varepsilon_{\underline{k}}$ in the list, for all non-fresh k. This y we call upb_{z+1}.
- The generated values for $\varepsilon_{\underline{n}}$ at stage z + 1, $\varepsilon_{\underline{n}}$ fresh, are $\overline{gs_n}(1+upb_{z+1})$.

Note that in order to compute upb it is essential that jumpfunctions are neighbourhoodfunctions for continuous mappings, and not the continuous mappings themselves.

2.8.4. This method of generating values does not leave us full freedom in the choice of values for $\varepsilon_{\underline{n}}$ at stage z ($\underline{n}$ fresh), nevertheless, we do have freedom of continuation for carriers locally, in the following sense: if $\underline{n}_1,\ldots,\underline{n}_p$ are fresh at stage z, $\overline{\varepsilon_{\underline{n}_i}}(y)$ is the segment of values available to us for $\varepsilon_{\underline{n}_i}$, $i = 1,\ldots,p$ at this stage (note that all these segments will indeed have the same length $y = \sum_{1\le z'<z} (upb_{z'}+1))$, and $\overline{\varepsilon_{\underline{n}_i}}(y) = \overline{\phi_i}(y)$ for $i = 1,\ldots,p$, $\phi_i \in N$ arbitrary, then we can arrange by a suitable preliminary choice of values, that after this stage we have $\overline{\varepsilon_{\underline{n}_i}}(y') = \overline{\phi_i}(y')$, $i = 1,\ldots,p$. $y' > y$, where $\overline{\varepsilon_{\underline{n}_i}}(y')$ is the segment of values now available for $\varepsilon_{\underline{n}_i}$.

2.8.5. It may seem unnatural to use an infinite supply of zero's, in order to achieve that for *all* carriers $\underline{n}$ at stage z + 1 the value $\varepsilon_{\underline{n}}(z)$ is available. This gives the number zero a special status in the universe of GC-carriers GCC: GCC satisfies $\forall x\exists n(\overline{\varepsilon_{\underline{n}}}(x) = \overline{\lambda z.0}(x))$, but not e.g. $\forall x\exists n(\overline{\varepsilon_{\underline{n}}}(x) = \overline{\lambda z.y+1}(x))$.
However, in the construction of GC, the universe of GC-sequences, this special rôle of the zero is made invisible (see 2.10.6), that is to say: for the construction of GC it makes no difference whether we define GCC as we do here, or use a (non-equivalent) variant, in which it is guaranteed only that for the carriers $\varepsilon_{\underline{n}}$, $n \le z$, an initial segment $\overline{\varepsilon_{\underline{n}}}(z+1)$ is determined at stage z + 1.

Our choice of definition is motivated by a technical reason: if we choose a more liberal approach, which requires the specification of sufficiently many values at each stage only for a finite set of carriers, and leaves us full freedom w.r.t. the others, then we have to take additional steps in the generation of values, distinguishing between carriers for which the choice of a sufficiently long segment is forced upon us, and others, where we are (still) free to choose any segment we want. This would further complicate a faithful imitation of GCC and GC by means of projections. (We feel that the projectionmodel is already complicated enough.)
Moreover (and maybe even more important) it is technically most convenient that at each stage z the segments of values generated for the fresh carriers have the same length $1 + upb_z$.

2.8.6. With this section we conclude the description of GCC. We have defined this universe more narrowly than seems natural, in order to prepare for the possibility of "coding" the construction of carriers by means of projections. The artificial character of those restrictions is on reflection seen to be inessential: the freedom of continuing and creating dependencies in a finite set of GC-carriers is not affected by them.

2.9. DRESSINGS, FRAMES AND RESTRICTIONS

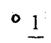
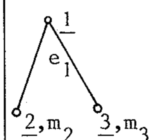
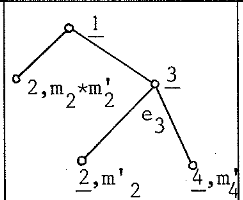
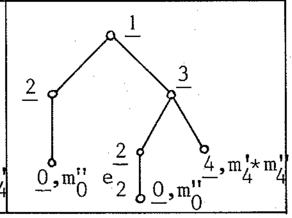
Stage	0	1	2	3
Dep. tree with jumpfns and gen. values				

fig. 11

Fig. 11 shows the possible history of carrier $\underline{1}$ through the stages 1,2,3. (The labelling with jumpfunctions and generated values is restricted to the changes w.r.t. the situation at the previous stage.)

2.9.1. <u>DEFINITION</u>. For each z, E_z is a mapping from the set $\{n : \underline{n}$ fresh at stage $z\}$ into N, defined by

$E_z(n) \equiv$ the part of $\varepsilon_{\underline{n}}$ which becomes available only after stage z, i.e. if we write UPB_z for the common length of the initial segments of the fresh carriers that have become available through the stages $1,\ldots,z$ ($UPB_0 = 0$), then

$$E_z(n) \equiv \lambda x.\varepsilon_{\underline{n}}(UPB_z+x)$$

E stands for 'empty', we call $E_z(n)$ the *empty part of* $\varepsilon_{\underline{n}}$ *at stage* z. Note that $E_0(n)$ is defined for all n and equal to $\varepsilon_{\underline{n}}$.

From fig. 11 we can read for each $z \in \{1,2,3\}$ a list of equations relating $\varepsilon_{\underline{1}}$ to empty parts of fresh carriers at stage z. At stage 1 we find:

(1) $\quad \varepsilon_{\underline{1}} = e_1|(\varepsilon_{\underline{2}},\varepsilon_{\underline{3}}),\quad \varepsilon_{\underline{2}} = m_2*E_1(2),\quad \varepsilon_{\underline{3}} = m_3*E_1(3)$, or equivalently

(2) $\quad \varepsilon_{\underline{2}} = [m_2]|E_1(2), \quad \varepsilon_{\underline{3}} = [m_3]|E_1(3),$ and substituting (2) in (1) yields:

(3) $\quad \varepsilon_{\underline{1}} = e_1|([m_2]|E_1(2), [m_3]|E_1(3)).$

At stage 2 we find additional equations for $E_1(2)$ and $E_1(3)$. First $\varepsilon_{\underline{3}}$ jumps to $\varepsilon_{\underline{2}}$ and $\varepsilon_{\underline{4}}$ at this stage, with jumpfunction e_3:

(4) $\quad E_1(3) = e_3|(E_1(2), E_1(4)).$

Recall from 2.7 that if $\underline{k}$ jumps to $\underline{n}$ and $\underline{m}$ at stage z+1, then the values of $\varepsilon_{\underline{k}}$ not yet available (i.e. $E_z(k)$) are determined from the values of $\varepsilon_{\underline{n}}$ and $\varepsilon_{\underline{n}}$ that are not yet available (i.e. $E_z(n)$ and $E_z(m)$) via the jumpfunction.) Moreover for $\varepsilon_{\underline{2}}$ and $\varepsilon_{\underline{4}}$ we generate the values m_2' and m_4' respectively at this stage:

(5) $\quad E_1(2) = [m_2']|E_2(2), \quad E_1(4) = [m_4']|E_2(4).$

We can substitute (5) in (4), and the resulting equation and (5) in (3), to find

(6) $\quad \varepsilon_{\underline{1}} = e_1|([m_2]|([m_2']|E_2(2)), [m_3]|(e_3|([m_2']|E_2(2),[m_4']|E_2(4)))).$

At stage 3 we find the following additional equations for $E_2(2)$ and $E_2(4)$:

(7) $\quad E_2(2) = e_2|E_2(0),$

(8) $\quad E_2(0) = [m_0'']|E_3(0),$

(9) $\quad E_2(4) = [m_4'']|E_3(4),$

which yield together with (6) an even more unreadable equation for $\varepsilon_{\underline{1}}$.

2.9.2. It will be clear that for each carrier $\underline{n}$ at each stage z we have an equation

$$\varepsilon_{\underline{n}} = \Gamma_z(\mathrm{src}(n,z)),$$

where Γ_z is a continuous mapping from N into N and src(n,z), the *source* *for* $\varepsilon_{\underline{n}}$ *at stage* z, is an element of N constructed from empty parts of fresh carriers at stage z, i.e. src(n,z) is a sequence of which no values are known to us at stage z.

2.9.3. The *dressing for* $\varepsilon_{\underline{n}}$ *at stage* z, is a standard neighbourhood function for Γ_z, the *frame for* $\varepsilon_{\underline{n}}$ *at stage* z is a structure which tells us how the source src(n,z) is constructed and from which empty parts.
The mappings d_n: z $\mapsto$ the dressing for $\varepsilon_{\underline{n}}$ at stage z, and
f_n: z $\mapsto$ the frame for $\varepsilon_{\underline{n}}$ at stage z
will play a key rôle in the imitation of GC-carriers by means of projections. We shall not give the formal definition of d_n and f_n here, but we shall explain their construction, using the example of fig. 11. For that explanation we need some tools.

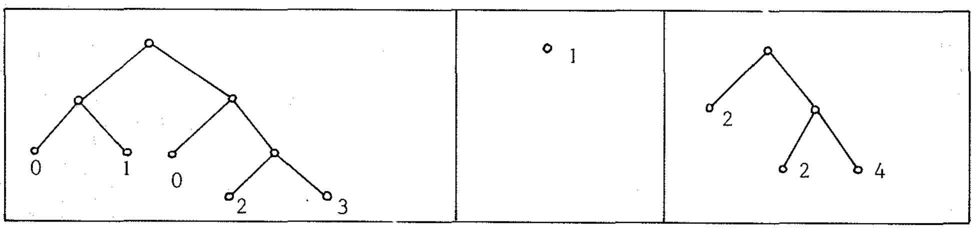


fig. 12

Fig. 12 shows three pictures of frames.

2.9.4. <u>DEFINITION</u>. A *frame* is a finite strictly binary tree, i.e. a finite tree in which each node has either two immediate descendants or none at all, the terminal nodes of which are labelled by natural numbers.

(A detailed formal treatment of frames is given in chapter 3.)

Let D be either K or N. Let p: D × D → D, the pairing on D, be ∧ or j respectively. (For ∧ see 1.3.23.)
Fig. 13a shows a finite strictly binary tree T, with a mapping ϕ from its terminal nodes into D.

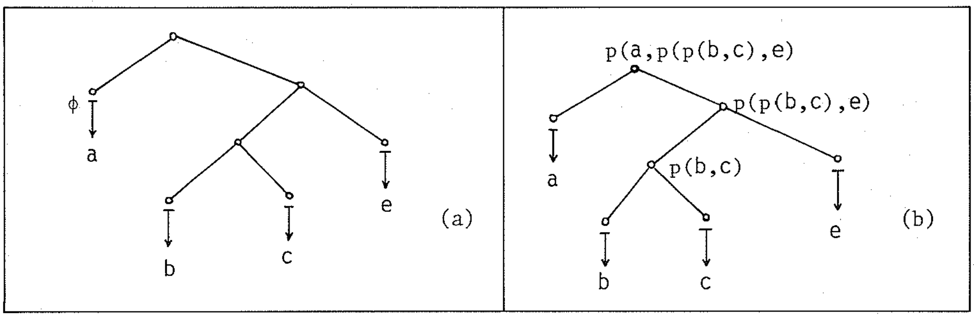


fig. 13

Fig. 13b shows how this mapping can be extended to one with domain all nodes of T.

2.9.5. DEFINITION. (i) The *extension* of a mapping ϕ: terminal nodes of T $\to$ D is the mapping ψ: nodes of T $\to$ D which satisfies:

$\psi(n) = \phi(n)$ if n is a terminal node of T,

$\psi(n) = p(a,b)$ if n is non-terminal in T, and a and b are the values of ψ on the left hand and the right hand immediate descendant of n respectively.

(ii) The T-*nesting* of ϕ: terminal nodes of T $\to$ D is the image of the top-node of T under the extension of ϕ.
(For a formal treatment of nestings see chapter 3.)
If $a \in D$ is the T-nesting of ϕ, then we say that ϕ *represents* a *in* T.

2.9.6. CLAIM. Application .|. is distributive over nesting, i.e. if $\phi \in N$ is represented by ϕ' in T as in fig. 14a, and $\psi \in K$ is represented by ψ' in T as in fig. 14b, then $\psi|\phi$ is represented as in fig. 14c.

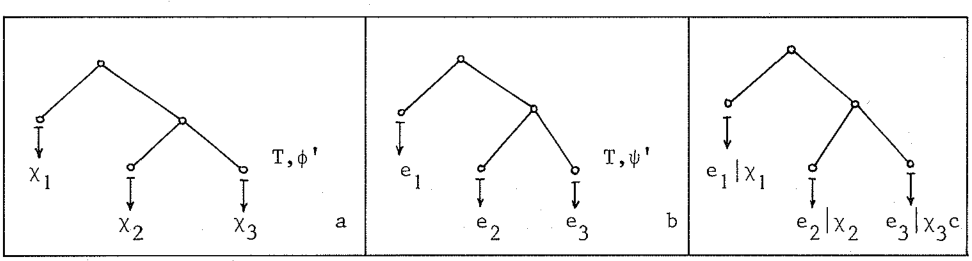


fig. 14

PROOF. See 3.2.16(c) . □

2.9.7. Now we show $d_n z$ and $f_n z$ are constructed for $z = 0,1,2,3$, $n = 1$ where the history of carrier $\underline{1}$ through the stages 0,1,2,3 is pictured in fig. 11.

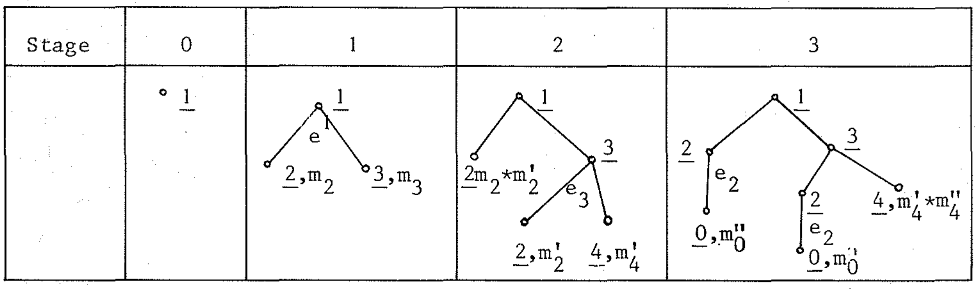


fig. 11 (repeated)

At stage 0 the source for $\varepsilon_{\underline{1}}$ is just $\varepsilon_{\underline{1}} = E_0(1)$. The values of $\varepsilon_{\underline{1}}$ are computed from those of the source via the identity mapping.
We put $d_1(0) = id$, $f_1(0) = {}^\circ 1$, the frame with a single node, labelled 1.
At stage 1 first $E_0(1)$ is made dependent on $E_0(2)$ and $E_0(3)$ via e_1, i.e. we have an equation

$$E_0(1) = e_1 | \chi_1,$$

where χ_1 can be represented as in fig. 15a.

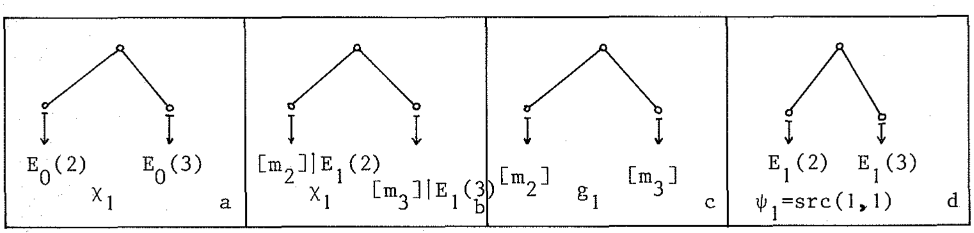


fig. 15

Next we generate values, m_1 for $\varepsilon_{\underline{2}}$ and m_3 for $\varepsilon_{\underline{3}}$. We can now refine the representation of χ_1 to the one given in fig. 15b. We use distributivity of application over nesting, and find that

$$\chi_1 = g_1|\psi_1,$$

where g_1 is represented as in fig. 15c, and ψ_1 as in 15d. We put $d_1(1) = e_1:g_1$, the source for ε_1 at stage 1, src(1,1) is ψ_1, and $f_1(1)$ is the structure obtained from fig. 15d by replacing $E_1(2)$ and $E_1(3)$ by their names 2 and 3 respectively.

At stage 2 we first decide that

$$E_1(3) = e_3|(E_1(2),E_1(4)),$$

i.e. the representation of the source src(1,1) as given in fig. 15d. is refined to the one of fig. 16a.

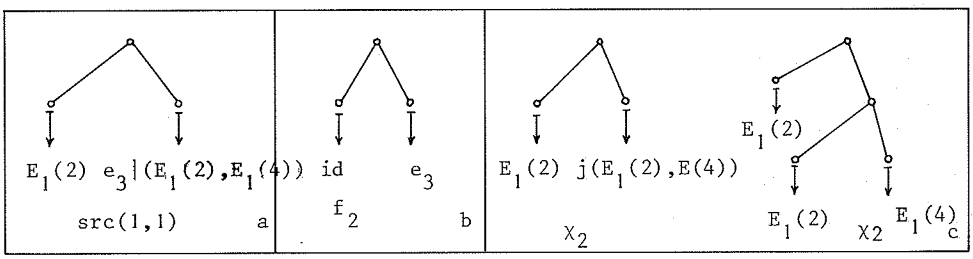


fig. 16

Using distributivity we find that

$$src(1,1) = f_2|\chi_2,$$

f_2 represented as in fig. 16b, χ_2 represented as in fig. 16c.
After generating values the representation of χ_2 can be refined to the one in fig. 17a, application of distributivity yields

$$\chi_2 = g_2|\psi_2,$$

g_2 as in 17b, ψ_2 as in 17c.

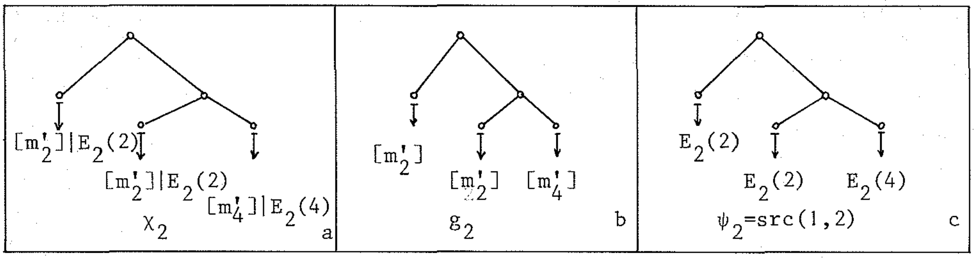


fig. 17

ψ_2 is the source for ε_1 at stage 2, src(1,2). The dressing for ε_1 at stage 2, $d_1(2) \equiv d_1(1)$: $f_2 : \overline{g}_2$, the frame for ε_1 at stage 2, $f_1(2)$ is obtained by replacing the empty parts of carriers in 17c by their names. (i.e. 2 for $E_2(2)$, 4 for $E_2(4)$).

At stage 3 we decide that

$$E_2(2) = e_2|E_2(0)$$

i.e. 17c is replaced by 18a. Using distributivity we find that we now have

$$\text{src}(1,2) = f_3|\chi_3,$$

f_3 and χ_3 represented as in 18b and c.

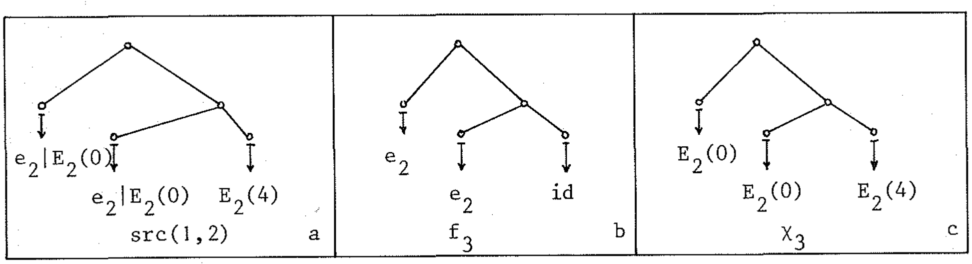


fig. 18

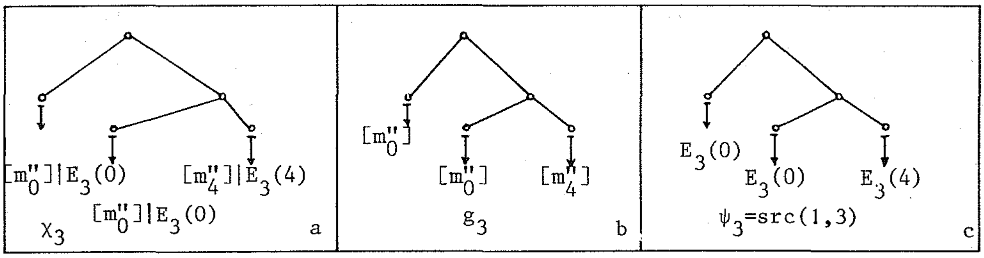


fig. 19

After generating values we can replace 18c by 19a; using distributivity we find that

$$\chi_3 = g_3 | \psi_3,$$

g_3 and ψ_3 represented by 19b and c respectively. As before ψ_3 is src(1,3), the source for $\underline{\varepsilon_1}$ at stage 3, $d_1(3) \equiv d_1(2)$: $f_3 : g_3$, $f_1(3)$, the frame for $\underline{\varepsilon_1}$ at stage 3, is obtained from 19c by replacing empty parts by their names.

2.9.8. The example is characteristic for the construction of d_n and f_n in general. Summarizing:

- The frame for $\underline{\varepsilon_n}$ at stage 0 is °n.

  We obtain $f_n(z+1)$ from $f_n(z)$ as follows:

(i) if none of the labels of $f_n(z)$ refers to a carrier which is made dependent on one or two others at stage z+1, then $f_n(z+1) = f_n(z)$,

(ii) if k is a label of $f_n(z)$, and $\underline{\varepsilon_k}$ jumps to $\underline{\varepsilon_m}$ at stage z+1 (i.e. $E_z(k) = e|E_z(m)$, e the jumpfunction) then k is replaced by m to obtain $f_n(z+1)$,

(iii) if $\underline{\varepsilon_k}$ jumps to $\underline{\varepsilon_{m_1}}, \underline{\varepsilon_{m_2}}$, then the label k is replaced by the pair m_1, m_2 to obtain $f_n(z+1)$, that is to say, we extend the tree of $f_n(z)$ by adding two immediate descendants for each terminal node with label k, label these new terminal nodes with m_1 and m_2, m_1 to the left, m_2 to the right, and erase the original label.

- The dressing for $\underline{\varepsilon_n}$ at stage 0 is id.

  $d_n(z+1)$ has the form $d_n(z)$: $f_{n,z+1} : g_{n,z+1}$.

$f_{n,z+1}$ is represented by a mapping from the terminal nodes $f_n(z)$ into K, which assigns to a terminal node n with label k the value id if $\varepsilon_{\underline{k}}$ does not jump at stage z+1, and the jumpfunction if it does.
$g_{n,z+1}$ is represented by a mapping from the terminal nodes of $f_n(z+1)$ into K, which assigns to a node n with label k the value $[m_k]$, where m_k are the values generated for $\varepsilon_{\underline{k}}$ at stage z+1.

2.9.9. Recall that in the process of generating values we have to determine at each stage a value upb. The construction of dressings for carriers can be used to reformulate the computation of upb. We illustrate this by means of the example above. (2.9.7.)
At stage 1 we found that

$$\varepsilon_{\underline{1}} = d_1(1)|src(1,1)$$

where src(1,1) is represented as in fig. 20a. (=fig.15d.)
After having decided that at stage 2, $\varepsilon_{\underline{3}}$ jumps to $\varepsilon_{\underline{2}}$ and $\varepsilon_{\underline{4}}$ with jumpfunction e_2 we have

$$\varepsilon_{\underline{1}} = d_1(1)\colon f_2|\chi_2,$$

χ_2 represented as in fig. 20b. (=fig.16c.)
At stage 2, $\varepsilon_{\underline{1}}$ 1 must become available. To achieve this we choose a suitable initial segment of the guiding sequences gs_2 and gs_4 as generated values for $\varepsilon_{\underline{2}}$ and $\varepsilon_{\underline{4}}$ (the carriers on which $\varepsilon_{\underline{1}}$ depends) respectively. To find such suitable initial segments, we substitute gs_n for $E_1(n)$ in fig. 20b, which yields 20c. The sequence represented in fig. 20c is called the guiding sequence for $\varepsilon_{\underline{1}}$ at stage 2 : gs_1.

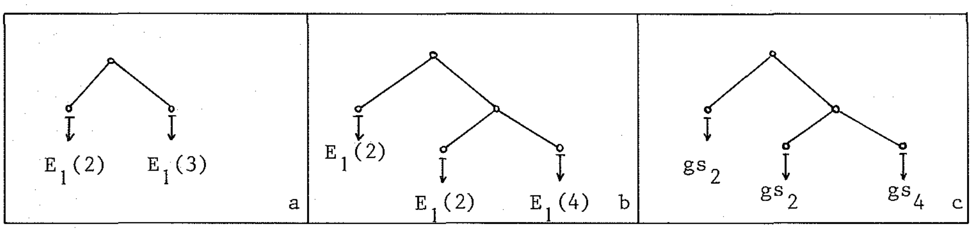


fig. 20

Then we determine the smallest z such that

$$(d_1(1):f_2)(\langle 1\rangle * \overline{gs_1}(z)) \neq 0.$$

If we generate $\overline{gs_2}(z)$ and $\overline{gs_4}(z)$ for $\underline{\varepsilon_2}$ and $\underline{\varepsilon_4}$ respectively, then we shall find that $d_1(2) = d_1(1): f_2 : [\overline{gs_1}(z)]$, whence there is a y such that

(1) $$(d_1(2)|\phi)(1) = y$$

for all ϕ, i.e. in particular we have

(2) $$\underline{\varepsilon_1}(1) = (d_1(2)|src(1,2))(1) = y.$$

We shall not generate $\overline{gs_2}(z)$ and $\overline{gs_4}(z)$ however. Before generating values we repeat the construction of a minimal z as above for all non-fresh carriers, the maximum of all these values we call upb_2, and we generate for each fresh n $\overline{gs_n}(1+upb_2)$. But then (1) and (2) will hold a fortiori, and we have similar equations for all non-fresh carriers at stage 2. Since at least one value is generated for all fresh carriers, we are also sure to have determined $\underline{\varepsilon_n}1$ for $\underline{\varepsilon_n}$ fresh, so we have $\underline{\varepsilon_m}1$ for all m.

In general: we generate values for $\underline{\varepsilon_n}$, n fresh at stage z+1 in such a way that

(3) $$\forall n \exists y \forall \phi [(d_n(z+1)|\phi)(z) = y].$$

Together with the equation

(4) $$\underline{\varepsilon_n} = d_n(z+1)|src(n,z+1)$$

this yields

(5) $$\underline{\varepsilon_n} = \bigcap_z \text{ range } (\lambda\phi . d_n(z+1)|\phi).$$

Finally we put

2.9.10. DEFINITION.

(i) A *restriction* is a pair (e,F), $e \in K$, F a frame

(ii) The *restriction for* $\underline{\varepsilon_n}$ *at stage* z is the pair $(d_n z, f_n z)$

The restriction for $\varepsilon_{\underline{n}}$ at stage z contains all information that is available to us on the values of $\varepsilon_{\underline{n}}$ at stage z. (5) might suggest that this information is already contained in $d_n z$. Note however that the growth of the dressings is regulated by the frames, that is to say, the relation between $d_n(z+1)$ and $d_n z$ depends on $f_n(z)$ and $f_n(z+1)$. Note also that the frame for $\varepsilon_{\underline{n}}$ at stage z contains information on the relation between the values of $\varepsilon_{\underline{n}}$ and the values of other sequences.

2.9.11. <u>REMARKS</u>. (a) It might appear strange that we should find such highly intensional information as the names of the carriers on which $\varepsilon_{\underline{n}}$ depends among the extensional data (as labels of the frame) for $\varepsilon_{\underline{n}}$ at stage z. However, they serve as markers only: if π is some permutation of $\mathbb{N}$ then we can just as well replace all names of carriers m in the frame by the value πm. (The use of the actual names is a matter of convenience.)
(b) Fig. 21 shows the frames and the dependence trees for the carrier $\varepsilon_{\underline{1}}$ of our example in the stages 0-3. There is an obvious resemblance: the frame can be obtained from the depence tree by deleting its non-terminal labels, and contracting pairs of nodes n, n', where n' is the only immediate descendant of n, into a single node.

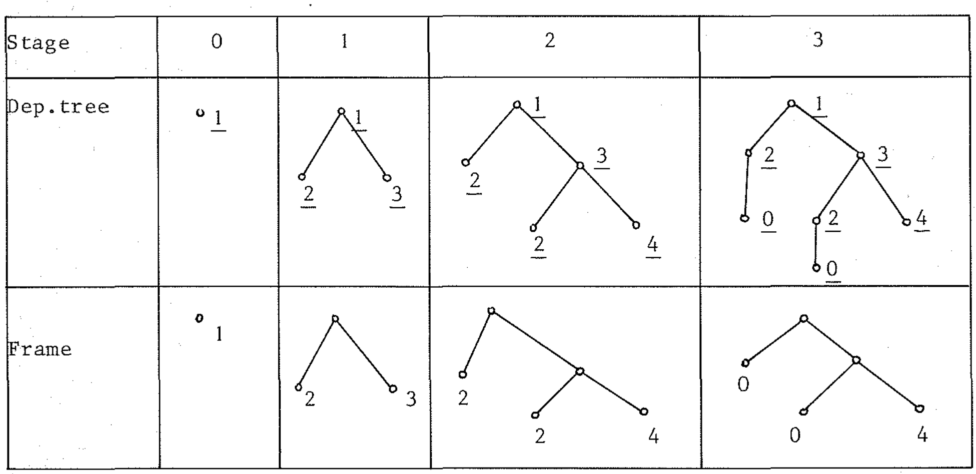


fig. 21

2.10. THE CONSTRUCTION OF GC FROM GCC

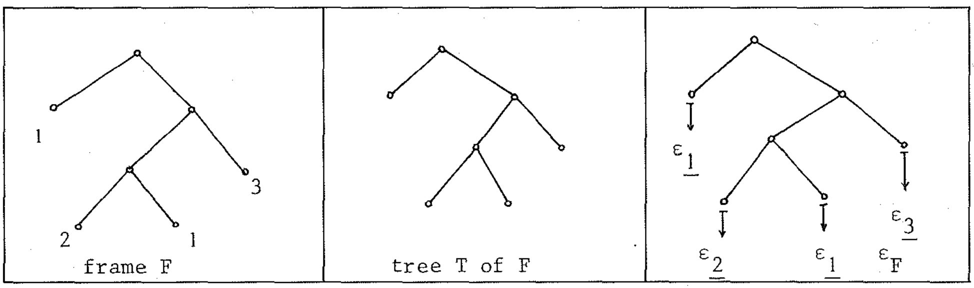


fig. 22

2.10.1. <u>DEFINITION</u>. (of ε_F)
Let F be a frame with tree T. The *nest of* GC-*carriers* ε_F is the T-nesting of the mapping ϕ: terminal nodes of T $\to$ GCC defined by

$$\phi n = \varepsilon_{\underline{k}} \quad \text{iff k is the label of } n \text{ in F.}$$

(See fig. 22, where $\varepsilon_F = j(\varepsilon_{\underline{1}}, j(j(\varepsilon_{\underline{2}}, \varepsilon_{\underline{1}}), \varepsilon_{\underline{3}}))$.)

2.10.2. <u>DEFINITION</u> (of GC, the universe of GC-sequences).

$$GC \equiv \{e|\varepsilon_F : (e,F) \text{ a restriction}\},$$

i.e. each GC-sequence ε is given to us by a restriction (e,F), the *initial restriction* for ε, and conversely, each restriction is the initial restriction of some $\varepsilon \in$ GC. If (e,F) is the initial restriction for $\varepsilon \in$ GC, then e is the *initial dressing* for ε, and F the *initial frame*.

2.10.3. <u>REMARK</u>. One may compare the construction of GC from GCC to the construction of LS from PLS (the universe of proto-lawless sequences). The data available to us on the values of a proto-lawless α at stage z of its construction, consist of:

(i) an initial segment v of α, and

(ii) the name $\underline{\alpha}$ of the source of future values (which plays a rôle in deciding the extensional equality betwee proto-lawless sequences).

The restriction (e,F) for a carrier at stage z, is the analogon of the pair $(v,\underline{\alpha})$ for a proto-lawless sequence. Proto-lawless sequences are, unlike GC-carriers, individualistic. There is a condition on the set R_z of all available pairs $(v,\underline{\alpha})$ at stage z in PLS namely

$$\forall\underline{\alpha}\exists!v((v,\underline{\alpha}) \in R_z).$$

Now LS can be defined as

$$LS \equiv \{v*\alpha : (v,\underline{\alpha}) \in R_0\},$$

where R_0 satisfies

(1) $\forall\underline{\alpha}\exists!v((v,\underline{\alpha}) \in R_0)$

and

(2) $\forall v\exists\underline{\alpha}((v,\underline{\alpha}) \in R_0)$,

i.e. LS is obtained from PLS by 'prefixing' a complete (i.e. satisfying condition (2)) and consistent (i.e. satisfying condition (1)) set of initial pairs $(v,\underline{\alpha})$.
Analogously, GC is obtained from GCC by 'prefixing' a complete set of initial restrictions. (Complete in the sense that all restrictions occur as initial restriction.) In this case there is no consistency condition, at least not modulo extensional equivalence.

2.10.4. <u>LEMMA</u> (Closure of GC under continuous-function-application and pairing).
If $\varepsilon,\eta \in$ GC *and* $e \in K$, *then* $e|\varepsilon \in$ GC *and* $j(\varepsilon,\eta) \in$ GC.

<u>PROOF</u>. If $\varepsilon \in$ GC is given by the initial restriction (f,F), then $e|\varepsilon$ is given by (e:f,F).
If $\varepsilon = f|\varepsilon_F$ and $\eta = g|\varepsilon_G$, then $j(\varepsilon,\eta) = (f\wedge g)|j(\varepsilon_F,\varepsilon_G)$. (For $f \wedge g$ see 1.3.23.) $j(\varepsilon_F,\varepsilon_G) = \varepsilon_{F\wedge G}$, where $F \wedge G$ is obtained by putting F and G below a common topnode, F to the left of G. (See fig. 23, recall the definition of nesting, 2.9.5.)
So $j(\varepsilon,\eta)$ has the initial restriction $(f\wedge g, F\wedge G)$. □

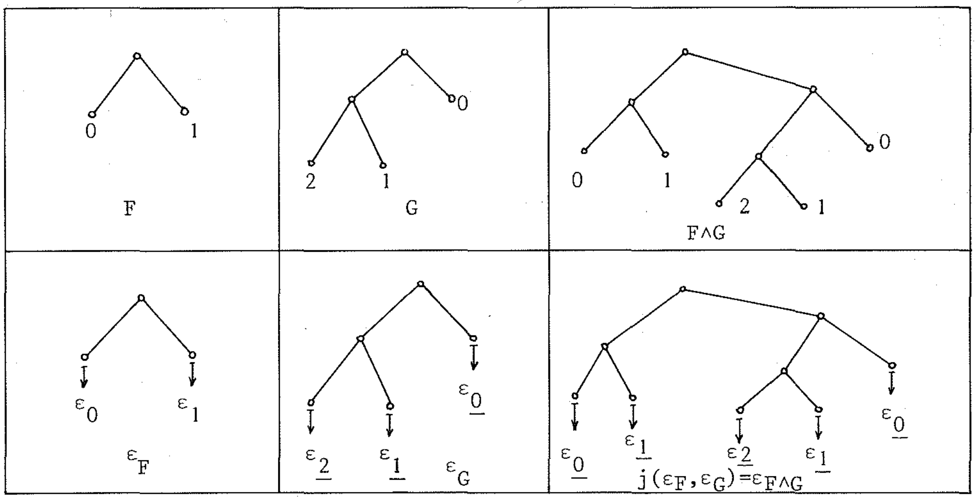


fig. 23

2.10.5. The *restriction for* ε *at stage* z (ε∈GC) is defined as follows (example).
The restriction for ε at stage 0 is the initial restriction for ε. Let this restriction be (e,F), as in fig. 24a, then

(1) $\varepsilon = e|\varepsilon_F$,

ε_F represented as in fig. 24b.
At stage z+1 we have equations

$$\varepsilon_{\underline{n}} = d_n(z+1)|src(n,z+1)$$

for each n, in particular for the n which occur as label in F, so the representation of ε_F can be refined to the one given in fig. 24c.

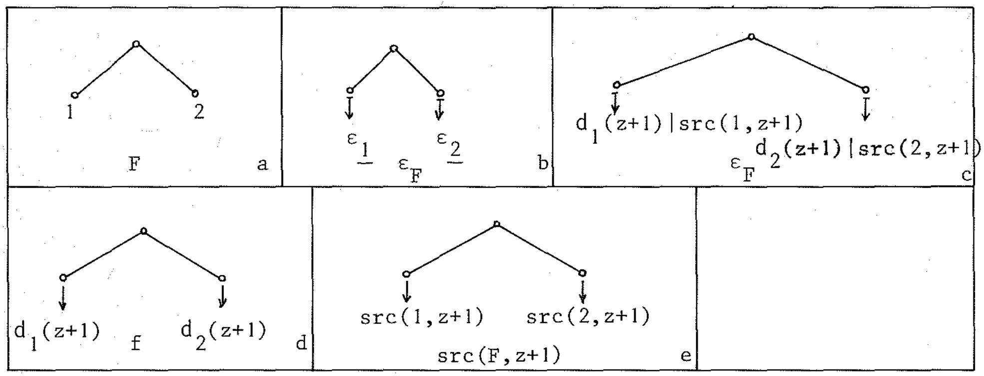


fig. 24

Using distributivity of .|. over nesting we find that

(2) $\varepsilon_F = f|src(F,z+1)$,

f represented as in fig. 24d, src(F,z+1), the *source for* ε_F *at stage* z+1 represented as in 24e.

We write $d_F(z+1)$ for the mapping f of (2), and put:

the *dressing for* ε *at stage* z+1 is $e : d_F(z+1)$, e as in (1), i.e. the initial dressing. $d_F(z+1)$ is the *dressing for* ε_F *at stage* z+1.

For each n we have a frame $f_n(z+1)$ at stage z+1 and a corresponding representation of src(n,z+1), the source for $\varepsilon_{\underline{n}}$ at stage z+1 (see fig. 25).

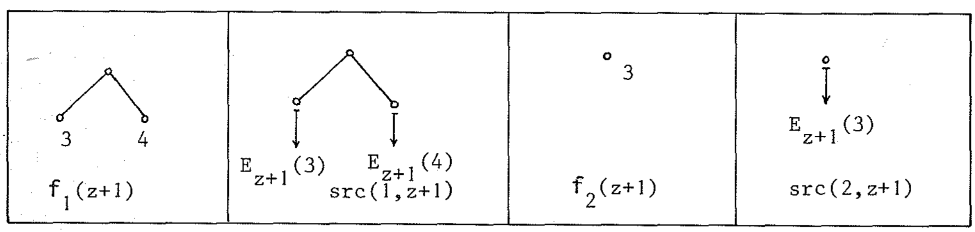


fig. 25

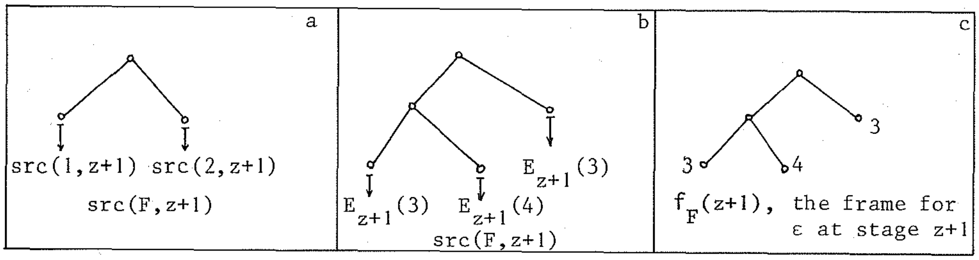


fig. 26

So the representation of src(F,z+1) of fig. 24e (=fig.26a) can be refined to the one of fig. 26b, by simply substituting the representation of src(n,z+1) for src(n,z+1) itself, for each label n of F.
The *frame for* $\varepsilon = e|\varepsilon_F$ *at stage* z+1 is obtained by replacing empty parts by their names in this last representation, or equivalently by substituting $f_n(z+1)$ for each node *n* of F with label n, and deleting the original label
We write $f_F(z+1)$ for the frame for $e|\varepsilon_F$ at stage z+1, and put:
the *restriction for* $\varepsilon = e|\varepsilon_F$ *at stage* z+1 is $(e{:}d_F(z+1),\ f_F(z+1))$.

2.10.6. <u>REMARK</u>. GCC is a subset of GC, the carrier $\varepsilon_{\underline{n}}$ is given by the initial restriction (id,°n). (°n is the frame with a single node, labelled n.) However, there is no extensional distinction between the carriers and the other sequences of GC. We know that for each k, all but finitely many carriers have an initial segment $\overline{\lambda z.0}(k)$. Now let $\varepsilon_{\underline{m}}$ be such a carrier. If we are presented with the sequences $\varepsilon_{\underline{m}} \in$ GC and $s^k|\overline{\varepsilon}_{\underline{m}} \in$ GC (given by the restriction $(s^k,°m)$) there is no way of deciding, looking at their values only, which of the two is the carrier: it may the first one, from which the second one is obtained by deleting the first k zero's (as is actually the case), but it may also be the second one, from which the first one is obtained by prefixing $\overline{\lambda z.0}(k)$.
Thus, the undesired side-effects of our method of guaranteeing that for each n, $\overline{\varepsilon}_{\underline{n}}(z+1)$ is available at stage z+1, are neutralized in GC.

2.11. GCC(C) AND GC(C)

In this section we relativize the notions of GC-carrier and -sequence to special subsets of K.

2.11.1. <u>DEFINITION</u> (of GC-carriers w.r.t. $C \subset K$).
Let C be a subset of K. GCC(C), the universe of GC-carriers w.r.t. C, is defined as GCC, except that if we decide to make a carrier jump at some stage, then our choice of a jumpfunction is restricted to the set C.
Note that GCC itself is GCC(K).

Concepts like the dressing for $\varepsilon_{\underline{n}}$ at stage z, the frame for $\varepsilon_{\underline{n}}$ at stage z and the restriction for $\varepsilon_{\underline{n}}$ at stage z, are defined for $\varepsilon_{\underline{n}} \in GCC(C)$, C arbitrary, exactly as in the special case $\varepsilon_{\underline{n}} \in GCC$.

For any restriction (e,F) we can arrange in GCC, by a proper choice of jumps, jumpfunctions and generated values, the existence of an $\varepsilon_{\underline{k}}$ such that $\varepsilon_{\underline{k}} = e|\varepsilon_F$.
Therefore it makes sense to define GC, the universe of GC-sequences, as the set of sequences of the form $e|\varepsilon_F$ where (e,F) ranges over *all* restrictions.

In GCC(C), the dependencies that can be created between one carrier and a nest of others are limited.
We can achieve that $\varepsilon_{\underline{k}} = e|\varepsilon_{\underline{m}}$ or $\varepsilon_{\underline{k}} = e|(\varepsilon_{\underline{m}},\varepsilon_{\underline{n}})$ for $e \in C$, by making $\varepsilon_{\underline{k}}$ jump at stage 1 with jumpfunction e.
It is also possible to have $\varepsilon_{\underline{k}} = [v] : e : s^x|\varepsilon_{\underline{m}}$, or $\varepsilon_{\underline{k}} = [v] : e : s^x|(\varepsilon_{\underline{m}},\varepsilon_{\underline{n}})$, where $x = lth(v)$, $e \in C$, by making $\varepsilon_{\underline{k}}$ dependent on the empty part $s^x|\varepsilon_{\underline{m}}$ or $s^x|(\varepsilon_{\underline{m}},\varepsilon_{\underline{n}})$ of $\varepsilon_{\underline{m}}$ or $(\varepsilon_{\underline{m}},\varepsilon_{\underline{n}})$ respectively at stage z+1, via the jumpfunction e, after having generated the sequence v for $\varepsilon_{\underline{k}}$.
Combination of these two possiblities can yield the relation

$$\varepsilon_{\underline{k}} = e|([v]:f_1:s^x|\varepsilon_{\underline{m}},[u]:f_2:s^y|\varepsilon_{\underline{n}}) =$$

$$= e: (([v]:f_1:s^x)\wedge([u]:f_2:s^y))|(\varepsilon_{\underline{m}},\varepsilon_{\underline{n}})$$

where e,f_1,f_2 are elements of C.
In general, we can create dependencies

$$\varepsilon_{\underline{k}} = e|\varepsilon_F$$

in GCC(C), where e is constructed from elements $f \in C$ and neighbourhood-functions of the form [v] and s^z, by means of composition and pairing.

2.11.2. DEFINITION (of dependency-closed).
A subset C of K is *dependency-closed* iff
(i) $\forall v([v] \in C)$, whence also $id \in C$,
(ii) $\forall z(s^z \in C)$,
(ii) C is closed under composition :,
(iv) C is closed under pairing $\wedge$.

2.11.3. LEMMA. *If* C *is dependency-closed then:*
(a) *For each* n *and* z, *the dressing for* $\varepsilon_{\underline{n}} \in$ GCC(C) *at stage* z, $d_n(z)$, *belongs to* C.
If $F = f_n(z)$, *the frame for* $\varepsilon_{\underline{n}}$ *at stage* z, *and* x *is the number of values generated through the stages* $z' \leq z$, *for each of the carriers* $\varepsilon_{\underline{k}}$ *that are fresh at stage* z, *then* $\varepsilon_{\underline{n}} = d_n(z) : s^x|\varepsilon_F$, $d_n(z) : s^x \in C$.
(b) *If* $e \in C$, F *an arbitrary frame, then we can arrange for the existence of an* $\varepsilon_{\underline{k}} \in$ GCC(C) *such that* $\varepsilon_{\underline{k}} = e|\varepsilon_F$.

PROOF.
(a) Trivial from the construction of d_n and the definition of dependency-closed. (Note that if C is closed under pairing, then it is also closed under nesting.)
For the equation $\varepsilon_n = d_n z : s^x|\varepsilon_F$ recall that by definition $\varepsilon_{\underline{n}} = d_n z|src(n,z)$. src(n,z) is the nesting of empty parts of carriers. These empty parts can be obtained from the carriers themselves by deleting the values already generated. If the number of these values is x, and $F = f_n(z)$, then $src(n,z) = s^x|\varepsilon_F$.
(b) We give a characteristic example. Let F be the frame of fig. 27a. We shall arrange that

$$\varepsilon_3 = e|\varepsilon_F.$$

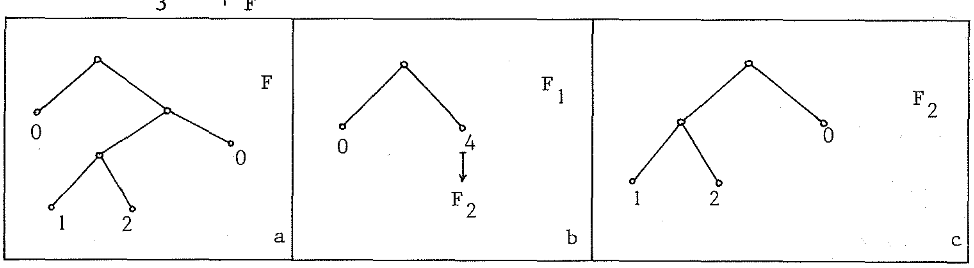


fig. 27

First split F into F_1 and F_2 as in figs. 27b and c, thereby introducing a new label 4.
At stage 1 make $\varepsilon_{\underline{3}}$ jump to $\varepsilon_{\underline{0}},\varepsilon_{\underline{4}}$ with jumpfunction e, i.e.

(1) $\qquad \varepsilon_{\underline{3}} = e|(\varepsilon_{\underline{0}},\varepsilon_{\underline{4}})$.

Choose values for $\varepsilon_{\underline{4}},\varepsilon_{\underline{1}},\varepsilon_{\underline{2}}$ and $\varepsilon_{\underline{0}}$ in such a way that

(2) $\qquad \overline{\varepsilon_{\underline{4}}}(x) = \overline{\varepsilon_{F_2}}(x)$,

where $x = 1 + upb_1$. (I.e. we make the choice of values for $\varepsilon_{\underline{4}}$ dependent on the choices for $\varepsilon_{\underline{1}},\varepsilon_{\underline{2}}$ and $\varepsilon_{\underline{0}}$.)
Now split F_2 into F_3 and F_4 as in figs. 28a and b

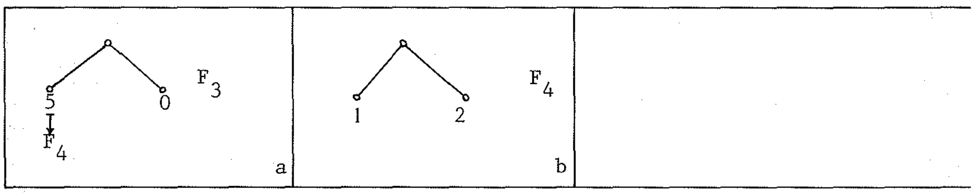


fig. 28

At stage 2 make $\varepsilon_{\underline{4}}$ jump to $(\varepsilon_{\underline{5}},\varepsilon_{\underline{0}})$ with jumpfunction id, i.e. we arrange that

(3) $\qquad \lambda z.\varepsilon_{\underline{4}}(x+z) = j(\lambda z.\varepsilon_{\underline{5}}(x+z),\lambda z.\varepsilon_{\underline{0}}(x+z))$.

Choose values for $\varepsilon_{\underline{0}},\varepsilon_{\underline{1}},\varepsilon_{\underline{2}}$ and $\varepsilon_{\underline{5}}$ in such a way that those for $\varepsilon_{\underline{5}}$ coincide with those for $\varepsilon_{F_4} = j(\varepsilon_{\underline{1}},\varepsilon_{\underline{2}})$, i.e. we arrange that now

(4) $\qquad \overline{\varepsilon_{\underline{4}}}(x+y) = \overline{\varepsilon_{F_2}}(x+y)$,

where $y = 1 + upb_2$.
At stage 3 finally we make $\varepsilon_{\underline{5}}$ dependent on $(\varepsilon_{\underline{1}},\varepsilon_{\underline{2}})$ via id, i.e. we arrange that

(5) $\qquad \lambda z.\varepsilon_{\underline{5}}(x+y+z) = j(\lambda z.\varepsilon_{\underline{1}}(x+y+z),\lambda z.\varepsilon_{\underline{2}}(x+y+z))$.

From (3) and (5) we now read:

(6) $\lambda z.\varepsilon_{\underline{4}}(x+y+z) = \lambda z.\varepsilon_{F_2}(x+y+z).$

From (4) and (6) we find

(7) $\varepsilon_{\underline{4}} = \varepsilon_{F_2}.$

From (1) and (7) we find

(8) $\varepsilon_{\underline{3}} = e|(\varepsilon_{\underline{0}},\varepsilon_{F_2}).$

Obviously $j(\varepsilon_{\underline{0}},\varepsilon_{F_2}) = \varepsilon_F$, i.e. we have the desired relation. □

This lemma justifies the following

2.11.4. DEFINITION (of GC(C), C dependency-closed).
If $C \subset K$ is dependency-closed, then GC(C), the universe of GC sequences w.r.t. C, is defined as

$$GC(C) = \{e|\varepsilon_F : e \in C, F \text{ a frame}\}$$

where ε_F is a nest of GC-carriers w.r.t. C.

2.11.5. REMARKS.
(a) We shall not define GC(C) for arbitrary C.
(b) Since dependency-closed sets contain all mappings [v] and s^z, remark 2.10.6 also holds for GC(C), and GCC(C), C dependency-closed.

2.11.6. LEMMA (closure of GC(C), C dependency-closed, under pairing and $e|.,e \in C$.)
If $\varepsilon,\eta \in GC(C)$, C *dependency-closed, then* $e|\varepsilon \in GC(C)$ *and* $j(\varepsilon,\eta) \in GC(C)$.

PROOF. See 2.10.4, for $e|\varepsilon \in GC(C)$ use that C is closed under composition, for $j(\varepsilon,\eta) \in GC(C)$ use that C is closed under pairing. □

2.12. PROJECTION MODELS FOR GC(C)

In the construction of projection models for GC(C) we shall proceed as follows:

(a) We construct a universe which imitates the behaviour of $\{\lambda z.f_n z : n \in \mathbb{N}\}$, where $f_n z$ is the frame for the carrier $\varepsilon_{\underline{n}} \in GCC(C)$ at stage z.
(b) We define a (class of) universe(s) imitating the behaviour of $\{\lambda z.d_n z : n \in \mathbb{N}\}$, $d_n z$ the dressing for $\varepsilon_{\underline{n}} \in GCC(C)$ at stage z.
(c) From the imitation of dressing sequences under (b), we define the imitation of carriers, using the observation that

$$\varepsilon_{\underline{n}} z = y \leftrightarrow \forall a[(d_n(z+1)|a)(z) = y]$$

cf. 2.9.9 (3) and (4).
(d) From the imitation of carriers we define the imitation of GC(C).

We turn to the projection model construction in chapter 4. First we give the formal theory of frames and nestings in chapter 3.

CHAPTER 3

FRAMES AND NESTINGS

In this chapter we introduce the tools that are needed for the definition of projectionmodels of GC(C)-sequences, and the derivation of their properties. The reader should concentrate on the definitions that are presented, and try to get used to the notation. Once the definitions have been understood, the facts and lemmata will be simple. It suffices to form an impression of their contents. It is not necessary to study them in full detail.

3.1. FRAMES

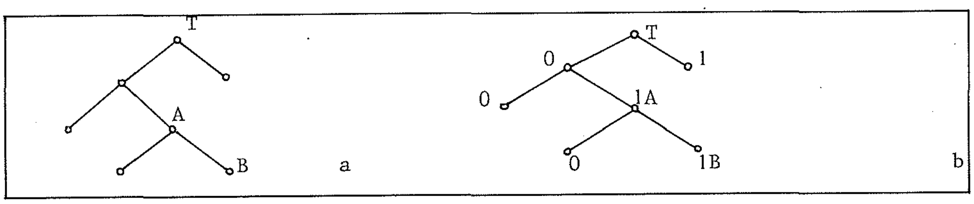


fig. 1

Fig. 1a shows a picture of a finite strictly binary tree. The little circles are the nodes of the tree, the highest node in the picture, marked T, is the top-node. All nodes, except the top-node, immediately descend from (i.e. are connected by a line with) a higher node. A node without descendants is a terminal- or bottom-node (the node marked B in fig. 1). Bottom-nodes will also be called branches; this name is explained by the identification of the node with the path that connects it with the top-node. Each non-terminal node has exactly two immediate descendants (hence *strictly* binary tree).

In fig. 1b all nodes of the tree, except the top-node, are marked by zero or one; zero for left-hand immediate descendants, one for the right-hand immediate descendants. Thus each node is identified by a finite 0-1 sequence: the top-node by < >, and e.g. the nodes marked A and B by <0,1> and <0,1,1> respectively.

We might define a strictly binary tree in the usual manner, i.e. as a finite set S of finite 0-1 sequences, satisfying two closure conditions:

(1) $\exists w(v*w\in S) \to v \in S,$

(2) $v*\langle 0\rangle\in S \leftrightarrow v*\langle 1\rangle \in S.$

However, we shall mainly be interested in the relation 'v is a branch of S', and less in the more general 'v is a node of S'. Therefore it is slightly more economical to define trees as sets of branches, as follows:

3.1.1. DEFINITION (of finite strictly binary tree).

(a) A *finite strictly binary tree* T is a non-empty finite set of finite 0-1 sequences such that

(i) $v\in T \wedge v*w\in T \to w=\langle\ \rangle,$

(ii) $\exists w(v*\langle 0\rangle*w\in T) \leftrightarrow \exists w(v*\langle 1\rangle*w\in T).$

We call the elements of T *branches, terminal-nodes or bottom-nodes.*

(i) states that each branch is maximal w.r.t. $\preccurlyeq$, (ii) corresponds to (2) above: it expresses that T is strictly binary branching. (The tree of fig. 1 e.g. would be formally defined as {<0,0>,<0,1,0>,<0,1,1>,<1>}.)

(b) If T is a finite strictly binary tree, then

$$nT \equiv_{def} \{v : \exists w(v*w\in T)\}.$$

We call the elements of nT the *nodes* of T. If v and w are nodes of T and $v \preccurlyeq w$, then w *descends from, is a descendant of* or *is below* v. If $w = v*\langle x\rangle$ for some $x \in \{0,1\}$, then w is an *immediate descendant* of v.

(c) Equality between finite strictly binary trees is extensional equality between sets, i.e.

$$T = S \equiv_{def} \forall v(v\in T \leftrightarrow v\in S).$$

3.1.2. NOTATION. We use $T,S,T_0,S_0,\ldots$ as variables for strictly binary trees. Script letters b,n with sub- or superscripts are used as syntactic variables for finite 0-1 sequences. b is used especially for branches of trees, *n* for nodes.

3.1.3. FACTS. (a) If T is a finite strictly binary tree, then nT satisfies (1) and (2) above.

(b) The empty sequence is a node of every finite strictly binary tree. We call it the *top-node*.

(c) Branches are nodes, i.e. $T \subset nT$, the only descendant of a branch is the branch itself.

(d) T = S iff nT = nS (the second equality is extensional set equality).

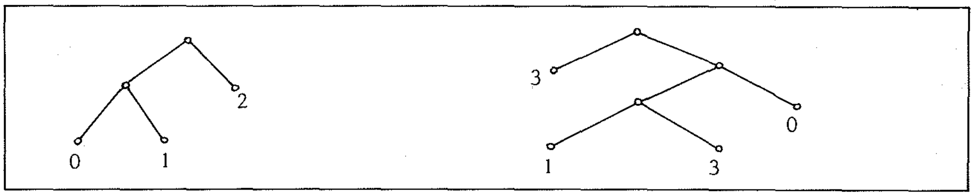


fig. 2

Fig. 2 shows two pictures of frames: finite strictly binary trees with a natural number attached as a label to each of their branches. Formally we put

3.1.4. DEFINITION (of frame).

(a) A *frame* F is a pair $\langle T,\phi\rangle$ consisting of a finite strictly binary tree T, the *tree of* F and a mapping $\phi : T \to \mathbb{N}$, the *labelling of* F.

– $b \in F$, read 'b *is a branch of* F', stands for 'b is a branch of the tree of F'. (If $F \equiv \langle T,\phi\rangle$ then $b\in F \equiv b\in T$.)

– nF, read 'the *nodes of* F', stands for 'the set of nodes of the tree of F' (If $F \equiv \langle T,\phi\rangle$ then $nF \equiv nT$.)

– $\ell_b F$, read 'the *label of* b in F' stands for 'the image of b under the labelling of F'. (If $F \equiv \langle T,\phi\rangle$ then $\ell_b F = \phi b$.)

– ℓF, the set of *labels of* F, is the set $\{n : \exists b \in F(\ell_b F=n)\}$.

(b) Two frames F and G are equal iff their trees and labellings are extensionally equal, i.e.

$$F = G \equiv_{def} \forall bn(b\in F \wedge \ell_b F=n \leftrightarrow b\in G \wedge \ell_b G=n).$$

3.1.5. <u>EXAMPLE</u>. The frames of fig. 2 are formally defined as the pairs $<T,\phi>, <S,\psi>$, where
$T \equiv \{<0,0>,<0,1>,<1>\}$, $\phi(<0,0>) = 0$, $\phi(<0,1>) = 1$, $\phi(<1>) = 2$ and
$S \equiv \{<0>,<1,0,0>,<1,0,1>,<1,1>\}$, $\psi(<0>) = \psi(<1,0,1>) = 3$, $\psi(<1,0,0>) = 1$, $\psi(<1,1>) = 0$.

3.1.6. <u>NOTATION</u>. We use $F,G,H,F_0,G_0,H_0,\ldots$ as variables for frames.

3.1.7. <u>DEFINITION</u>. Let n be a natural number, then $°n$ is the *single-node frame* with label n, i.e. $°n$ satisfies
(i) $b \in (°n) \leftrightarrow b = < >$,
(ii) $\ell_{< >}(°n) = n$.

Note that instead of $°n$ we sometimes write $(°n)$; obviously $\ell(°n) = \{n\}$ and $(°n)=(°m) \leftrightarrow n=m$.
Fig. 3 shows how two frames F and G can be paired into a single frame H, by putting them below a common top-node, F to the left of G. We denote this pairing operation by $\wedge$.

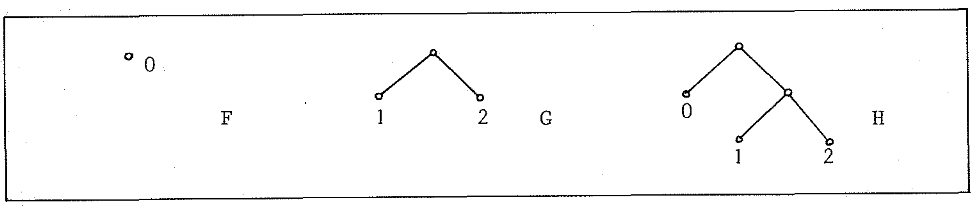


fig. 3

3.1.8. <u>DEFINITION</u> (of $F\wedge G$). Let F and G be frames. $F \wedge G$ is the frame which satisfies:
(i) $b \in F\wedge G \leftrightarrow \exists b_1 \in F(b=<0>*b_1) \vee \exists b_2 \in G(b=<1>*b_2)$,
(ii) $\forall b \in F(\ell_{<0>*b}(F\wedge G) = \ell_b F)$,
(iii) $\forall b \in G(\ell_{<1>*b}(F\wedge G) = \ell_b G)$.

3.1.9. <u>FACTS</u>. $\ell(F\wedge G) = \ell F \cup \ell G$ and $F\wedge G = F'\wedge G' \leftrightarrow (F=F')\wedge(G=G')$.

3.1.10. <u>REMARK</u>. One easily verifies by comparing $F\wedge G$ and $G\wedge F$ (F and G as in fig. 3) that $\wedge$ is not commutative. If one compares $F\wedge(G\wedge F)$ with $(F\wedge G)\wedge F$, it turns out that $\wedge$ is also not associative.

3.1.11. <u>DEFINITION</u> (of ht). Let F be a frame. ht(F), read: the *height* of F, is the length of the longest branch of F, i.e.

$$\mathrm{ht}(F) \equiv_{\mathrm{def}} \max\{\mathrm{lth}(b) : b \in F\}.$$

3.1.12. <u>FACTS</u> (properties of ht).

(a) $\mathrm{ht}(F) = 0$ iff $\exists n(F={}^\circ n)$,

(b) $\mathrm{ht}(F \wedge G) = 1 + \max(\mathrm{ht}(F), \mathrm{ht}(G))$,

(c) $\mathrm{ht}(F)>0 \rightarrow \exists GH(F=G \wedge H)$.

3.1.13. <u>PROPOSITION</u> (induction over frames). *Let* Q *be a property of frames, then*

$$\forall n\, Q({}^\circ n) \wedge \forall FG(Q(F) \wedge Q(G) \rightarrow Q(F \wedge G)) \rightarrow \forall H\, Q(H).$$

<u>PROOF</u>. By induction over $\mathbb{N}$ w.r.t. ht(H). □

3.1.14. <u>DEFINITION</u>. (a) FRAME denotes the set of frames.

(b) A lawlike sequence of frames is a lawlike mapping $f : \mathbb{N} \rightarrow \mathrm{FRAME}$.

3.1.15. <u>NOTATION</u>. We use lower case script letters $f, g, f', g', f_0, g_0, \ldots$ as variables for lawlike sequences of frames.

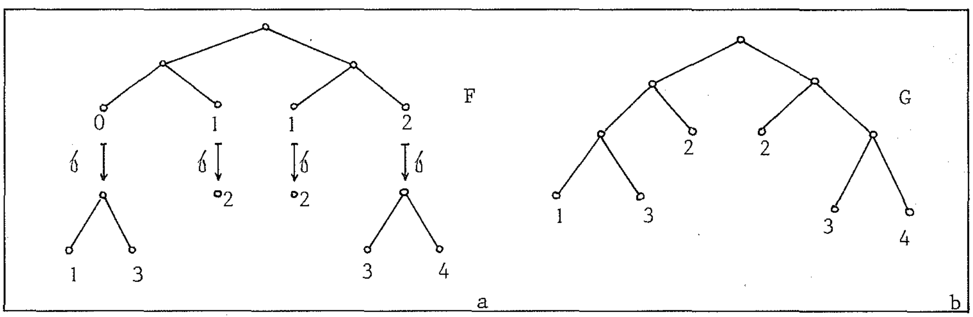


fig. 4

Fig. 4a shows a frame F and $f \upharpoonright \ell F$ for some lawlike sequence f of frames. If we 'replace' each terminal node $b \in F$ by the frame $f(\ell_b F)$ (and delete the original labelling), we obtain a new frame G (see fig. 4b). For the frame G thus constructed from F and f we write $F[f]$.

Note that $({}^\circ n)[f]$ is just fn. Moreover, the replacement of terminal nodes by values of f is distributive over pairing, i.e. $(F \wedge G)[f] = F[f] \wedge G[f]$. This leads us to the following definition by recursion.

3.1.16. DEFINITION (of $F[\mathfrak{f}]$ and $G \geq F$). (a) Let $\mathfrak{f}$ be a lawlike sequence of frames. $F[\mathfrak{f}]$ is the image of F under the mapping from FRAME into FRAME defined by the following recursion equations:

$$(^\circ n)[\mathfrak{f}] = \mathfrak{f}n, \quad (F \wedge G)[\mathfrak{f}] = F[\mathfrak{f}] \wedge G[\mathfrak{f}].$$

If $G = F[\mathfrak{f}]$ we say that $\mathfrak{f}$ *produces* G *from* F.

(b) $$G \geq F \equiv F \leq G \equiv_{def} \exists \mathfrak{f}(G = F[\mathfrak{f}]).$$

If $G \geq F$ then we say that G *can be produced from* F.

3.1.17. FACTS.
(a) $F=G \rightarrow F[\mathfrak{f}]=G[\mathfrak{f}]$.
(b) $\ell(F[\mathfrak{f}]) = \bigcup_{n \in \ell F} \ell(\mathfrak{f}n)$.
(c) $nF \subset n(F[\mathfrak{f}])$, $G \geq F \rightarrow nF \subset nG$, in particular $\forall b \in F(b \in n(F[\mathfrak{f}]))$ and $G \geq F \rightarrow \forall b \in F(b \in nG)$.
(d) $ht(F[\mathfrak{f}]) \geq ht(F)$, $G \geq F \rightarrow ht(G) \geq ht(F)$.

3.1.18. LEMMA (explicit characterization of $F[\mathfrak{f}]$).
Let F *be a frame,* $\mathfrak{f}$ *a lawlike sequence of frames. Then* b *is a branch of* $F[\mathfrak{f}]$ *iff it has the form* $b_1 * b_2$, *where* $b_1 \in F$ *and* $b_2 \in \mathfrak{f}n$, n *the label of* b_1 *in* F. *The label of such a branch* $b = b_1 * b_2$ *in* $F[\mathfrak{f}]$ *is the label of* b_2 *in* $\mathfrak{f}n$.

PROOF. By induction over frames. See also fig. 4. □

3.1.19. LEMMA (properties of $F[\mathfrak{f}]$, $G \geq F$).
(a) $(F[\mathfrak{f}])[g] = F[\lambda n.\mathfrak{f}n[g]]$.
(b) $F[\mathfrak{f}] = F[g] \leftrightarrow \forall n \in \ell F(\mathfrak{f}n = gn)$.
(c) $F[\lambda n.(^\circ n)] = F$.
(d) $F[\mathfrak{f}] = F \leftrightarrow \forall n \in \ell F(\mathfrak{f}n = {}^\circ n)$.
(e) *The* $\geq$*-relation between frames is transitive and reflexive.*

PROOF. For (a), (b) and (c) use induction over frames and 3.1.9: $F \wedge G = F' \wedge G' \leftrightarrow F=F' \wedge G=G'$, $\ell F \subset \ell(F \wedge G)$ and $\ell G \subset \ell(F \wedge G)$, (d) is a corollary of (b) and (c), (e) follows from (a) and (c). □

3.1.20. DEFINITION. $F \approx G \equiv_{def} F \geq G \wedge G \geq F$.
If $F \approx G$ then we call F and G *equivalent*.

E.g. the frames F and G of fig. 5 are equivalent since for f and g satisfying $f0 = {}^\circ 1$, $f1 = {}^\circ 3$ and $g1 = {}^\circ 0$, $g3 = {}^\circ 1$ we have $G = F[f]$ and $F = G[g]$.

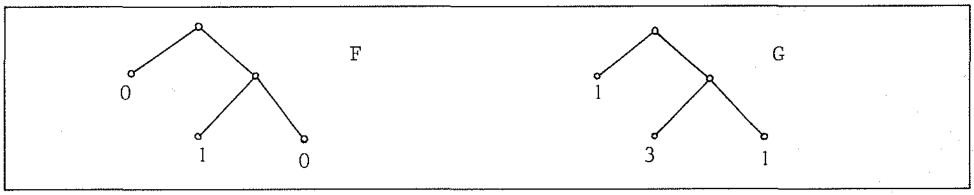


fig. 5

3.1.21. FACTS.

(a) If F and G are both single-node frames, $F = {}^\circ n$ and $G = {}^\circ m$ say, then $F \approx G$ $(F=G[\lambda k.({}^\circ n)],\ G=F[\lambda k.({}^\circ m)])$.

(b) If $F \approx G$ then F and G have the same height, nodes and branches (cf. 3.1.17, (c), (d)). For the relation between their labellings see the next lemma.

3.1.22. LEMMA (alternative characterization of equivalence between frames). *Two frames* F *and* G *are equivalent iff there is a lawlike* $a : \mathbb{N} \to \mathbb{N}$, *which maps* ℓF *one-one onto* ℓG, *such that*

$$G = F[\lambda n.{}^\circ an].$$

PROOF. ($\Leftarrow$) If $G = F[\lambda n.{}^\circ an]$ then $G \geq F$ by definition. If a maps ℓF one-one onto ℓG, then we can find a $b : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \ell F(b(an)=n)$. For this b we have $F = G[\lambda n.{}^\circ bn]$ i.e. $F \geq G$.

($\Rightarrow$) Assume that $F \approx G$, $G = F[f]$, $F = G[g]$.

Then $F = (F[f])[g]$, i.e. $F = F[\lambda n.fn[g]]$, by 3.1.19(a).

Hence $\forall n \in \ell F(fn[g]={}^\circ n)$, by 3.1.19(d).

Hence $\forall n \in \ell F(\mathrm{ht}(fn)=0)$, by 3.1.17(d).

So $\forall n \in \ell F \exists m(fn={}^\circ m)$, and hence $G = F[f] = F[\lambda n.{}^\circ an]$ for some a.

This a maps ℓF onto ℓG by 3.1.17(b), and it is one-one on ℓF, since it satisfies $\forall n \in \ell F(g(an)={}^\circ n)$. □

3.2. NESTINGS

3.2.1. DEFINITION (of pairing w.r.t. $\sim_D$). Let D be a set, $\sim_D$ an equivalence relation on D. A mapping $p:D\times D \to D$ is a *pairing* operation *on* D *w.r.t.* $\sim_D$, iff

$$\forall xyx'y' \in D(p(x,y) = p(x',y') \leftrightarrow x \sim_D x' \wedge y \sim_D y').$$

p is a pairing operation on D iff there is an equivalence relation $\sim_D$ on D such that p is a pairing w.r.t. $\sim_D$.

3.2.2. EXAMPLES.

(a) j is a pairing on $\mathbb{N}$ and N w.r.t. extensional equality.

(b) $\wedge$ is a pairing on K w.r.t. the equivalence $\simeq$, defined by $e \simeq f \equiv \forall a(e|a = f|a)$. (See 1.3.24(e).)

(c) $\wedge$ is a pairing on FRAME w.r.t. extensional equality as defined in 3.1.4(b).

3.2.3. REMARK. The more usual definition of pairing claims the existence of *pairing left-inverses* p_1, p_2, defined on the subset $\{p(x,y) : x \in D, y \in D\}$ of D, satisfying $p_1 p(x,y) = x$ and $p_2 p(x,y) = y$.

In example (a) such pairing left-inverses j_1, j_2 exist. They are in fact *pairing inverses* since $j(j_1 a, j_2 a) = a$ for $a \in \mathbb{N}$ or $a \in N$.

In examples (b) and (c) pairing left inverses can be defined, but their existence is irrelevant for our purposes.

3.2.4. FACT. For each n, the mapping $(\bar{a}n, \bar{b}n) \mapsto \overline{j(a,b)}(n)$, a,b lawlike elements of N, is a pairing on the set of finite sequences with length n, w.r.t. equality; k_1 and k_2 (cf. 1.3.5(d), 1.3.6) are the inverses to this pairing.

Let D be a set with a pairing operation $p : D \times D \to D$. (We shall be interested in the cases $D = \mathbb{N}$, $D = N$ and $D = K$, with $p = j$, $p = j$ and $p = \wedge$ respectively.) Let ϕ be a mapping from $\mathbb{N}$ into D.

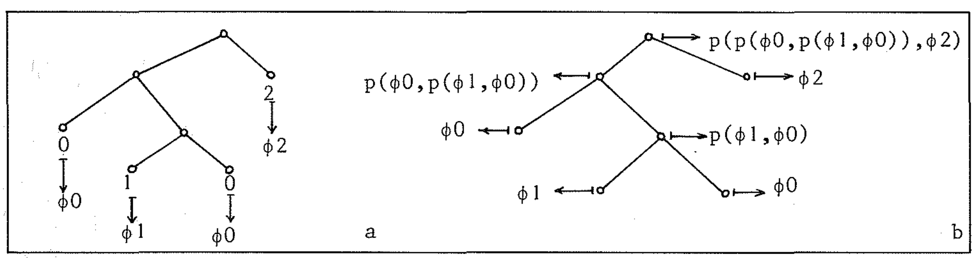


fig. 6

Fig. 6a shows a frame F with $\phi \upharpoonright (\ell F)$. ϕ induces a mapping $b \mapsto \phi(\ell_b F)$ from the terminal nodes of F into D. Fig. 6b shows how this mapping can be

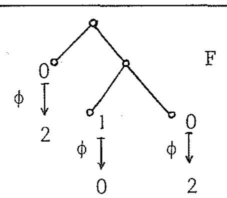
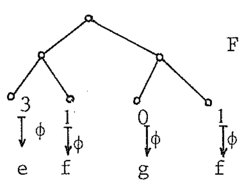
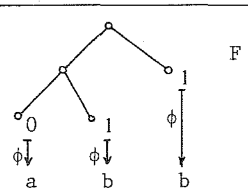
naturally extended to a mapping $\phi' : nF \to D$ by putting:
$\phi' b = \phi(\ell_b F)$ for all branches b of F,
$\phi'(n) = p(\phi'(n*\langle 0\rangle), \phi'(n*\langle 1\rangle))$ for non-terminal nodes n of F,
i.e. the image of a non-terminal node under ϕ' is found by pairing the values assigned to its immediate descendants.
For the image of the top-node under ϕ' we write $\nu_F^{D,p}\phi$, we call it 'the F-nesting of ϕ (w.r.t. p)'. Formally we put:

3.2.5. <u>DEFINITION</u> (of ν_F). Let D be a set with a pairing operation $p : D \times D \to D$, and let ϕ be a mapping from $\mathbb{N}$ into D. By $\nu_F^{D,p}\phi$ we denote the image of F under the mapping from FRAME into D, defined by the recursion equations

$$\nu^{D,p}_{(^\circ n)}\phi = \phi n, \qquad \nu^{D,p}_{F\wedge G}\phi = p(\nu_F^{D,p}\phi,\ \nu_G^{D,p}\phi).$$

If $a \in D$ and $a = \nu_F^{D,p}\phi$, we say that a is the F-*nesting of* ϕ (w.r.t. p).
For $\nu_F^{\mathbb{N},j}\phi$ we write $\nu_F\phi$, for $\nu_F^{N,j}\phi$ we write $\nu_F^1\phi$, and we put $\nu_F^K\phi \equiv \nu_F^{K,\wedge}\phi$.

3.2.6. <u>EXAMPLES</u>.

	$\phi : \mathbb{N}\to\mathbb{N}$ satisfies $\phi 0=2,\ \phi 1=0$	$\nu_F\phi = j(2,j(0,2))$
	$\phi : \mathbb{N}\to K$ satisfies $\phi 0=g,\ \phi 1=f,\ \phi 3=e$	$\nu_F^K\phi = (e\wedge f)\wedge(g\wedge f)$
	$\phi : \mathbb{N}\to N$ satisfies $\phi 0=a,\ \phi 1=b$	$\nu_F^1\phi = j(j(a,b),b)$

3.2.7. REMARKS. (a) Note that the pairing p itself is a special case of F-nesting w.r.t. p: $p(x,y) = \nu_F^{D,p}\phi$, where $F = °0\wedge°1$, and $\phi : \mathbb{N}\to D$ is defined by $\phi n = x$ if $n = 0$ and $\phi n = y$ otherwise.
(b) Let $\mathfrak{f}$ be a lawlike sequence of frames, F a frame. The F-nest of $\mathfrak{f}$ w.r.t. $\wedge$, i.e. $\nu_F^{FRAME,\wedge}\mathfrak{f}$, is exactly the frame produced by $\mathfrak{f}$ from F, i.e. $F[\mathfrak{f}]$. (See def.3.1.16.)

3.2.8. FACTS. (a) Let ϕ map $\mathbb{N}$ into N (i.e. $\phi n\in N$, $\phi n(z)\in\mathbb{N}$). Then $\nu_F^1\phi = \lambda z.\nu_F(\lambda n.\phi n(z))$, since the pairing j on N is defined from the pairing j on $\mathbb{N}$ by $j(\phi,\psi) = \lambda z.j(\phi z,\psi z)$.
(b) If a subset D' of D is closed under the pairing p, then it is closed under F-nesting w.r.t. p.

If $D = \mathbb{N}$ or $D = N$, with the pairing operation j from $D \times D$ *onto* D, and pairing-inverses $j_1,j_2 : D \to D$, we can reverse the construction of nestings as follows.

Let a be an element of D, T a finite strictly binary tree.

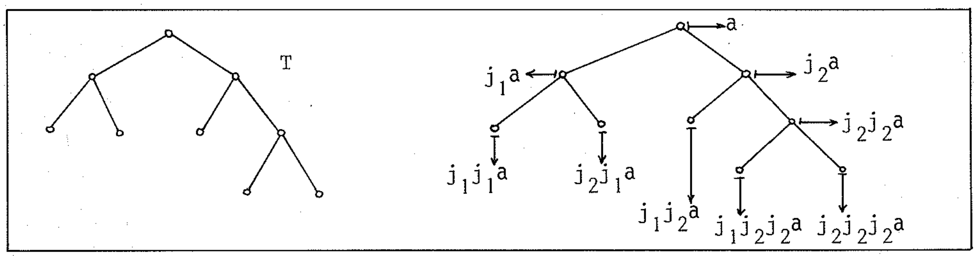


fig. 7

Fig. 7 shows how we can associate with the pair (a,T) a mapping $\phi : nT \to D$, by putting:

$$\phi\langle\ \rangle = a,$$

$$\phi(n*\langle 0\rangle) = j_1(\phi n),\quad \phi(n*\langle 1\rangle) = j_2(\phi n),$$

i.e. ϕ assigns the value a to the top-node of T, to the left-hand immediate descendant of a node n it assigns $j_1(\phi n)$ and to the right-hand immediate descendant of n it assigns $j_2(\phi n)$.

Note that ϕn can be computed independently of the tree T. If $n = \langle x_0,\ldots,x_p\rangle$, $x_i\in\{0,1\}$, for $i = 0,\ldots,p$, then $\phi n = j_{i_p}\cdots j_{i_0}a$, where

$i_q = 1$ iff $x_q = 0$ and $i_q = 2$ iff $x_q = 1$ $(0 \le q \le p)$.

We write $j_n a$ for the value ϕn. The mapping $n \mapsto j_n a$ thus defined on finite 0-1 sequences, can be extended to a mapping $v \mapsto j_v a$ defined on arbitrary finite sequences, by putting $j_v a \equiv j_{\overline{sg}(v)} a$, where $\overline{sg}(<x_0,\dots,x_p>) = <sgx_0,\dots,sgx_p>$. (I.e. for a 0-1 sequence n, $\overline{sg}n = n$.) Formally:

3.2.9. DEFINITION (of $j_v a$, $a \in \mathbb{N}$ or $a \in N$). Let D be $\mathbb{N}$ or N, a an element of D. The mapping $v \mapsto j_v a$ from $\mathbb{N}$ into D is defined by the recursion equations

$$j_{< >} a = a, \quad j_{<x>*v}\, a = \begin{cases} j_v(j_1 a) & \text{if } sg(x) = 0, \\ j_v(j_2 a) & \text{otherwise.} \end{cases}$$

A mapping $a \mapsto j_v a$ from D into D $(v \in \mathbb{N})$, is called a *nesting-inverse*.

3.2.10. REMARK. Since our notation does not distinguish between the indices 1 and 2 and the number terms 1 = S0 and 2 = SS0, we can interpret j_1 and j_2 in two ways: as pairing inverses, where 1 and 2 are indices for the first and the second member of the pair respectively, and as nesting inverses, where 1 and 2 are natural numbers coding finite sequences. We shall assume that 1 codes the sequence <0> and 2 the sequence <1>. Thus we make both readings of j_1, j_2 coincide.

3.2.11. FACTS.

(a) $j_{v*w} a = j_w(j_v a)$,

(b) If $\phi \in N$ then $j_v \phi = \lambda x. j_v(\phi x)$, since the pairing inverses $j_i : N \to N$, $i = 1,2$ are defined by $j_i \phi = \lambda x. j_i(\phi x)$.

3.2.12. DEFINITION (of $k_v : \mathbb{N} \to \mathbb{N}$). $k_v : \mathbb{N} \to \mathbb{N}$ is defined by the equations:

$$k_v< > = < >, \quad k_v(w*\hat{x}) = k_v w * <j_v x>,$$

i.e. $k_v(<x_0,\dots,x_p>) = <j_v x_0,\dots,j_v x_p>$.

3.2.13. FACTS.

(a) $k_{v*w} u = k_w(k_v u)$,

(b)
$$k_{< >}w = w,\ k_{<x>*v}w = \begin{cases} k_v(k_1 w) & \text{if } sg(x) = 0 \\ k_v(k_2 w) & \text{otherwise,} \end{cases}$$

(c) if $\phi \in N$, then $j_v(u*\phi) = k_v u * j_v\phi$,

(d) if $\phi \in N$, then $k_v(\bar{\phi}x) = \overline{j_v\phi}(x)$,

(e) $k_u(v*w) = k_u v * k_u w$.

3.2.14. <u>LEMMA</u>. *Let* F *be a frame. Then*

$$x=y \leftrightarrow \forall b\in F(j_b x = j_b y),$$
$$v=w \leftrightarrow \forall b\in F(k_b v = k_b w),$$
$$\phi=\psi \leftrightarrow \forall b\in F(j_b\phi = j_b\psi),\ \textit{where } \phi,\psi \in N.$$

<u>PROOF</u>. By induction over frames. □

3.2.15. <u>NOTATION</u>. Let $\phi \equiv \phi[n]$ be an element of N for all $n \in \mathbb{N}$. We write $\lambda^1 n.\phi$ for the mapping $x \mapsto \phi[x/n]$ from $\mathbb{N}$ into N.

If $\phi = \phi[n]$ is an element of K for each $n \in \mathbb{N}$, then $\lambda^K n.\phi$ stands for the mapping $x \mapsto \phi[x/n]$ from $\mathbb{N}$ into K.

3.2.16. <u>LEMMA</u> (properties of nestings and nesting-inverses). D *is a set with an equivalence relation* $\sim_D$. $p : D \times D \to D$ *is a pairing w.r.t.* $\sim_D$. *Then*

(a) $\forall\phi\psi\in D^{\mathbb{N}}(\nu_F^{D,p}\phi \sim_D \nu_F^{D,p}\psi \leftrightarrow \forall n\in\ell F(\phi n \sim_D \psi n))$.

(b) $\forall\phi\in N\ \forall b\in F[j_b(\nu_F\phi) = \phi(\ell_b F)]$,
$\forall\psi\in N^{\mathbb{N}}\ \forall b\in F[j_b(\nu_F^1\psi) = \psi(\ell_b F)]$.

(c) $\forall\phi\in K^{\mathbb{N}}\ \forall\psi\in N\ \forall b\in F[j_b(\nu_F^K\phi|\psi) = \phi(\ell_b F)|j_b\psi]$.

(d) *For* $\phi : \mathbb{N} \to D$, $\mathfrak{f}$ *a lawlike sequence of frames,* F *and* G *frames,* $G = F[\mathfrak{f}]$:

$$\nu_G^{D,p}\phi \sim_D \nu_F^{D,p}\psi,$$

where $\psi : \mathbb{N} \to D$ *is defined by* $\psi n = \nu_{\mathfrak{f}n}^{D,p}\phi$.

(e) $\forall\phi\psi\in K^{\mathbb{N}}(\nu_F^K\phi : \nu_F^K\psi \simeq \nu_F^K(\lambda^K n.\phi n : \psi n))$.

(f) $\nu_F^K(\lambda^K n.\mathrm{id}) \simeq \mathrm{id}$, $\nu_F^K(\lambda^K n.s^m) \simeq s^m$.

(g) *For* $\phi : \mathbb{N} \to N$ (*i.e.* $\phi n\in N$, $\overline{\phi n}(m)$ *is the initial segment of the infinite* *sequence* ϕn *with length* m),

$$[\overline{\nu^{1}_{F}\phi(m)}] \simeq \nu^{K}_{F}(\lambda^{K}n.[\overline{\phi n}(m)]).$$

PROOF. All assertions by induction over frames.

(a) and (b) are immediate from the definitions of $\nu^{D,p}_{F}$ and j_{b}. (a) just generalizes the characteristic property of the pairing p, namely $p(x,y) \sim_{D} p(x',y') \leftrightarrow x \sim_{D} x' \wedge y \sim_{D} x'$, (b) formally explains the name nesting-inverse for mappings j_{b}.

(c) is shown in detail below.

(d) states that if G is obtained from F by substituting values of $\mathfrak{f}$ for terminal nodes of F, then the G-nesting of ϕ is obtained by first determining all $\mathfrak{f}n$-nestings of ϕ for values $\mathfrak{f}n$ of $\mathfrak{f}$ and then applying F-nesting.

(e) says that composition of neighbourhood-functions is distributive over nesting, for the proof one uses the corresponding property of : w.r.t. pairing $\wedge$, i.e. $(e\wedge f):(e'\wedge f') \simeq (e:e')\wedge(f:f')$ (cf. 1.3.24.(f)).

(f) says that a nesting of identities is an identity and a nesting of shifts over m is a shift over m. Here use that $id \wedge id \simeq id$, $s^{m} \wedge s^{m} \simeq s^{m}$ (cf. 1.3.24(g)).

(g) is shown in detail below.

The detailed proofs of (c) and (g) can be skipped at first reading.

PROOF of (c):

(i) For $f = {}^{\circ}n$, (c) becomes

(1) $$j_{<\,>}(\nu^{K}_{({}^{\circ}n)}\phi|\psi) = \phi n|\psi.$$

(2) $j_{<\,>}(\nu^{K}_{({}^{\circ}n)}\phi|\psi$ by definition of j_{v} (3.2.9),

(3) $\nu^{K}_{({}^{\circ}n)}\phi = \phi n$ by definition of ν^{K} (3.2.5),

(2) and (3) yield (1).

(ii) For F = G∧H (c) is the conjunction of two statements

(4) $\forall b\in G(j_{<0>*b}(\nu^{K}_{G\wedge H}\phi|\psi) = \phi(\ell_{b}G)|j_{<0>*b}\psi)$ and

(5) $\forall b\in H(j_{<1>*b}(\nu^{K}_{G\wedge H}\phi|\psi) = \phi(\ell_{b}H)|j_{<1>*b}\psi)$.

We show (4).

$j_{<0>*b}(\nu^{K}_{G\wedge H}\phi|\psi) = j_{b}(j_{1}(\nu^{K}_{G\wedge H}\phi|\psi))$ by definition of j_{v} (3.2.9),

$\nu^K_{G\wedge H}\phi|\psi = (\nu^K_G\phi\wedge\nu^K_H\phi)|\psi$ by definition of ν^K (3.2.5),
and $e\wedge f|\psi = j(e|j_1\psi,\ f|j_2\psi)$ by definition of $\wedge$ (1.3.23),
hence $j_{<0>*b}(\nu^K_{G\wedge H}\phi|\psi) = j_b(\nu^K_G\phi|j_1\psi)$.
Moreover: $j_b(\nu^K_G\phi|j_1\psi) = \phi(\ell_b G)|j_b(j_1\psi)$ by induction hypothesis,
and $j_b(j_1\psi) = j_{<0>*b}\psi$ by definition of j_v, which yields (4).

<u>PROOF</u> of (g):

(i) For $F = \circ k$, (g) becomes

(6) $$[\overline{\nu^1_{(\circ k)}\phi(m)}] \simeq \nu^K_{(\circ k)}(\lambda^K n.[\overline{\phi n}(m)]).$$

$\nu^1_{(\circ k)}\phi = \phi k$ by definition of ν^1, hence

(7) $$[\overline{\nu^1_{(\circ k)}\phi(m)}] \simeq [\overline{\phi k}(m)].$$

On the other hand

(8) $$\nu^K_{(\circ k)}(\lambda^K n.[\overline{\phi n}(m)]) = [\overline{\phi k}(m)], \text{ by definition of } \nu^K.$$

(7) and (8) yield (6).

(ii) If $F = G\wedge H$ then

(9) $$[\overline{\nu^1_F\phi}(m)] \simeq [\overline{j(\phi_1,\phi_2)}(m)],$$

with $\phi_1 \equiv \nu^1_G\phi$, $\phi_2 \equiv \nu^1_H\phi$, by definition of ν^1.
On the other hand

(10) $$\nu^K_F(\lambda^K n.[\overline{\phi n}(m)]) = e\wedge f,$$

with $e \equiv \nu^K_G(\lambda^K n.[\overline{\phi n}(m)])$ and $f \equiv \nu^K_H(\lambda^K n.[\overline{\phi n}(m)])$, by definition of ν^K.
By induction hypothesis $e \simeq [\overline{\phi_1}m]$, $f \simeq [\overline{\phi_2}m]$.
$\overline{\phi_i}m = k_i(\overline{j(\phi_1,\phi_2)}(m))$ by 1.3.6, for $i = 1,2$, $[k_1v]\wedge[k_2v]\simeq[v]$ by 1.3.24(g),
hence $e\wedge f \simeq [\overline{j(\phi_1,\phi_2)}(m)]$.
Combining this with (9) and (10) yields the desired result. □

3.2.17. <u>COROLLARIES</u>.

(a) *For* $\psi,\phi : \mathbb{N}\to\mathbb{N}$: $\nu_F\phi = \nu_F\psi \leftrightarrow \forall n\in\ell F(\phi n=\psi n)$,
for $\psi,\phi : \mathbb{N}\to N$: $\nu^1_F\phi = \nu^1_F\psi \leftrightarrow \forall n\in\ell F(\phi n=\psi n)$,
for $\psi,\phi : \mathbb{N}\to K$: $\nu^K_F\phi \simeq \nu^K_F\psi \leftrightarrow \forall n\in\ell F(\phi n\simeq\psi n)$.
[*Special cases of* 3.2.16(a).]

(b) *If* $G = F[\mathfrak{f}]$, *then*

for $\phi : \mathbb{N} \to \mathbb{N}$: $\nu_G\phi = \nu_F(\lambda n.\nu_{\mathfrak{f}n}\phi)$,
for $\phi : \mathbb{N} \to N$: $\nu^1_G\phi = \nu^1_F(\lambda^1 n.\nu^1_{\mathfrak{f}n}\phi)$,
for $\phi : \mathbb{N} \to K$: $\nu^K_G\phi \simeq \nu^K_F(\lambda^K n.\nu^K_{\mathfrak{f}n}\phi)$.

[*Special cases of* 3.2.16(d).]

(c) *For* $\psi : \mathbb{N} \to N$: $\forall b \in F(k_b(\nu^1_F\psi(x)) = \overline{\psi(\ell_b F)}(x))$.

[*By* 3.2.16(b) *and* 3.2.13(d).]

(d) *For* $\phi : \mathbb{N} \to K$: $\forall n \in \ell F(\phi n \simeq id) \leftrightarrow \nu^K_F\phi \simeq id$, *and*
$\forall n \in \ell F(\phi n \simeq s^m) \leftrightarrow \nu^K_F\phi \simeq s^m$.

[*By* 3.2.16(a) *and* (f).]

REMARK. 3.1.19(a) and (b) (properties of $F[\mathfrak{f}]$) are special cases of 3.2.16(d) and (a) respectively, since $F[\mathfrak{f}] = \nu^{FRAME,\wedge}_F \mathfrak{f}$. (See remark 3.2.7(b).)

3.2.18. <u>DEFINITION</u> (of "parallel to"). (a) Let $\phi \in N$, $F \in$ FRAME. ϕ is *parallel to* F, iff there is a $\psi : \mathbb{N} \to N$ such that $\phi = \nu^1_F\psi$, or, equivalently, iff for each pair b, b' of branches of F having the same label in F, $j_b\phi = j_{b'}\phi$. We write $\phi /\!/ F$ for ϕ is parallel to F. In formula:

$$\phi /\!/ F \equiv \forall bb' \in F(\ell_b F = \ell_{b'} F \to j_b\phi = j_{b'}\phi).$$

(b) A finite sequence v is parallel to the frame F iff for all branches b and b' of F with the same label in F, $k_b v = k_{b'} v$. I.e.

$$v /\!/ F \equiv \forall bb' \in F(\ell_b F = \ell_{b'} F \to k_b v = k_{b'} v).$$

(c) An element ϕ of K is C-*parallel to* the frame F, where C is a subset of K, iff there is a $\psi : \mathbb{N} \to C$ such that $\phi \simeq \nu^K_F\psi$. We write $/\!/_C$ for C-parallel to. Formally, we put

$$\phi /\!/_C F \equiv \exists \psi : \mathbb{N} \to C(\phi \simeq \nu^K_F\psi).$$

We denote the negation of parallel to by $\#$.

3.2.19. <u>REMARK</u>. The property of being parallel to F is generally a non-trivial one. E.g. if $a \neq b$, then $j(a,b)$ is not parallel to the frame $°0 \wedge °0$.

On the other hand, all $\phi \in N$ are parallel to $°0 \wedge °1$ (see 3.2.21(e)). A similar observation does not hold for $/\!/_C$, even if we take $C = K$. Consider e.g. the mapping $e \in K$ such that $e|j(a,b) = j(b,a)$. This e is not K-parallel to $°0 \wedge °1$: if $e \simeq f\wedge g$ then $e|j(a,b) = j(f|a,g|b)$, and the assumption that for all a and b $f|a = b$ and $g|b = a$ is obviously contradictory.

3.2.20. LEMMA (properties of // and $//_C$, for consultation when needed).

(a) *For* $\psi : \mathbb{N} \to N$: $\nu^1_F\psi//F$.

(b) $\forall b\in F(j_b\phi=\psi) \to \phi//F$.

(c) $\forall b\in F(k_b v=u) \to v//F$.

(d) $\forall x(\phi//F \leftrightarrow \bar{\phi}x//F \wedge \lambda z.\phi(x+z)//F)$.

(e) $v*w//F \leftrightarrow v//F \wedge w//F$.

(f) $\phi//F\wedge G \to j_1\phi//F \wedge j_2\phi//G$.

(g) $e\wedge f//_C F\wedge G \to e//_C F \wedge f//_C G$.

(h) $\ell F\cap\ell G = \emptyset \to (j_1\phi// F \wedge j_2\phi// G \to \phi// F\wedge G)$.

(i) $\ell F\cap\ell G = \emptyset \to (e//_C F \wedge f//_C G \to e\wedge f//_C F\wedge G)$.

(j) $\phi//G \wedge G \geq F \to \phi//F$.

(k) *If* C *is closed under* $\wedge$ *then* $e//_C G \wedge G \geq F \to e//_C F$.

(l) $F \approx G \to (e//_C F \leftrightarrow e//_C G)$.

(m) *If* C *is closed under* $\wedge$ *then* $e//_C G \to e \in C$.

(n) $\forall e\in C\forall n(e//_C(°n))$.

(o) $id\in C \to id//_C F$.

(p) $s^m\in C \to s^m//_C F$.

(q) *Let* ϕ *be a right-inverse to the labelling of* F*, i.e.* $\forall n\in\ell F(\phi n\in F \wedge \ell_{\phi n}F=n)$, *then* $u//F \to [u] \simeq \nu^K_F(\lambda^K n.[k_{\phi n}u])$.

(r) $e//_C F \wedge \phi//F \to e|\phi//F$.

(s) *If* C *is closed under* : *then* $e//_C F \wedge f//_C F \to e{:}f//_C F$.

PROOF.

(a) by 3.2.16(b).

(b) and (c) by definition of //.

(d) $\phi = \bar{\phi}x*\lambda z.\phi(x+z)$ and $j_b(\bar{\phi}x*\lambda z.\phi(x+z)) = k_b\bar{\phi}x*j_b(\lambda z.\phi(x+z))$ by 3.2.13(c), now apply the definition of //.

(e) by 3.2.13(e).

(f) Assume $\ell_b F = \ell_{b'}F$, $b,b'\in F$, and $\phi//F\wedge G$. Then $\ell_{<0>*b}(F\wedge G) = \ell_{<0>*b'}(F\wedge G)$ by definition of $F\wedge G$, hence $j_{<0>*b}\phi = j_{<0>*b'}\phi$ by definition of //. $j_{<0>*b}\phi = j_b(j_1\phi)$, $j_{<0>*b'}\phi = j_{b'}(j_1\phi)$ by definition of j_v, hence $j_1\phi//F$. By a similar argument we find $j_2\phi//G$.

(g) Assume $e\wedge f//_C F\wedge G$, i.e. $e\wedge f \simeq \nu^K_{F\wedge G}\phi$ for some $\phi : \mathbb{N} \to C$. $\nu^K_{F\wedge G}\phi = \nu^K_F\phi \wedge \nu^K_G\phi$ by definition of ν^K, and

$e\wedge f \simeq \nu^K_F\phi \wedge \nu^K_G\phi \rightarrow (e\simeq\nu^K_F\phi)\wedge(f\simeq\nu^K_G\phi)$ since $\wedge$ is a pairing w.r.t. $\simeq$ (see 3.2.2(b)). Hence $e//_C F$ and $f//_C G$.

(h) Let $\langle x\rangle * b$, $\langle y\rangle * b'$ be branches of $F\wedge G$ with the same label, assume that $\ell F \cap \ell G = \emptyset$, then either $x = y = 0$ and $b,b' \in F$ or $x = y = 1$ and $b,b' \in G$. In the first case $j_{\langle x\rangle * b}\phi = j_{\langle y\rangle * b'}\phi$ follows from the definition of j_v and the assumption $j_1\phi//F$, in the second case this equality follows from $j_2\phi//G$.

(i) Assume that $e \simeq \nu^K_F\phi_1$ and $f \simeq \nu^K_G\phi_2$, $\phi_1,\phi_2 : \mathbb{N} \rightarrow C$.

$$\text{Define } \psi : \mathbb{N} \rightarrow C \text{ by } \psi n = \begin{cases} \phi_1 n \text{ if } n \in \ell F \\ \\ \phi_2 n \text{ otherwise.} \end{cases}$$

If $\ell F \cap \ell G = \emptyset$ then $\forall n \in \ell F(\psi n = \phi_1 n)$ and $\forall n \in \ell G(\psi n = \phi_2 n)$, whence $e \simeq \nu^K_F\psi$ and $f \simeq \nu^K_G\psi$ by 3.2.17(a). So $e\wedge f \simeq \nu^K_F\psi \wedge \nu^K_G\psi$, $\nu^K_F\psi \wedge \nu^K_G\psi = \nu^K_{F\wedge G}\psi$ by definition of ν^K, and hence $e\wedge f //_C F\wedge G$.

(j) Assume $G = F[\mathfrak{f}]$, $\phi//G$ and let b,b' be branches of F with the same label n. We show that

(1) $$\forall b'' \in \mathfrak{f}n(j_{b''}(j_b\phi) = j_{b''}(j_{b'}\phi)),$$

then $j_b\phi = j_{b'}\phi$ follows by 3.2.14.

To prove (1) we argue as follows:

$j_{b''}(j_b\phi) = j_{b*b''}\phi$, $j_{b''}(j_{b'}\phi) = j_{b'*b''}\phi$ by 3.2.11(a). $b*b''$ and $b'*b''$ are both branches of $G = F[\mathfrak{f}]$, with the same label $\ell_{b''}(\mathfrak{f}n)$, by 3.1.18. Since $\phi//G$ then $j_{b*b''}\phi = j_{b'*b''}\phi$.

(k) Let $G = F[\mathfrak{f}]$, $e \simeq \nu^K_G\phi$ for $\phi : \mathbb{N} \rightarrow C$. Then $e \simeq \nu^K_F(\lambda^K n.\nu^K_{\mathfrak{f}n}\phi)$ by 3.2.17(b). If C is closed under $\wedge$ then $\nu^K_{\mathfrak{f}n}\phi \in C$ by 3.2.8(b), so $e//_C F$.

(l) Let $F \approx G$, then $F = G[\lambda n.(^\circ an)]$ for some a, by 3.1.22. If $e//_C F$ then $e \simeq \nu^K_F\phi$ for some $\phi : \mathbb{N} \rightarrow C$. $\nu^K_F\phi \simeq \nu^K_G(\lambda^K n.\nu^K_{\mathfrak{f}n}\phi)$, where $\mathfrak{f}n = (^\circ an)$, by 3.2.17(b), i.e. $\nu^K_F\phi \simeq \nu^K_G(\lambda^K n.\phi(an))$. $\lambda^K n.\phi(an) : \mathbb{N} \rightarrow C$, so $e//_C G$. The converse implication follows from the symmetry of $\approx$.

(m) by 3.2.8(b).

(n) $e \simeq \nu^K_{(^\circ n)}(\lambda^K m.e)$, if $e \in C$ then $\lambda^K m.e : \mathbb{N} \rightarrow C$.

(o) and (p) by 3.2.16(f).

(q) Assume $u//F$, $\phi : \mathbb{N} \rightarrow \mathbb{N}$ satisfies $\forall n \in \ell F(\phi n \in F \wedge \ell_{\phi n}F = n)$. We show that for all a

$$\forall b \in F(j_b(\nu^K_F(\lambda^K n.[k_{\phi n}u])|a) = j_b([u]|a)),$$

then $[u] \simeq \nu^K_F(\lambda^K n.[k_{\phi n}u])$, i.e. $\forall a([u]|a = \nu^K_F(\lambda^K n.[k_{\phi n}u])|a)$, follows by 3.2.14. $[u]|a = u*a$ by definition of u, $j_b(u*a) = k_b u * j_b a$ by

3.2.13(c). On the other hand $j_b(\nu_F^K\psi|a) = \psi(\ell_b F)|j_b a$ by 3.2.16(c), i.e. for $\psi \equiv \nu_F^K(\lambda^K n.[k_{\phi n}u])$: $j_b(\nu_F^K\psi|a) = [k_{\phi(\ell_b F)}u]|j_b a = k_{\phi(\ell_b F)}u * j_b a$. But $\phi(\ell_b F)$ is a branch of F with label $\ell_b F$, whence, since u//F,
$k_b u = k_{\phi(\ell_b F)}u$.

(r) by 3.2.16(c).

(s) by 3.2.16(e). □

3.2.21. <u>COROLLARIES</u> (for consultation when needed).

(a) *For* $\phi \in N$: $\forall n(\phi//(^\circ n))$. [*By* 3.2.20(b)]

(b) $\forall v \forall n(v//(^\circ n))$. [*By* 3.2.20(c)]

(c) $v//F\wedge G \rightarrow k_1 v//F \wedge k_2 v//G$.
[$v//F\wedge G \rightarrow v*\nu^1_{F\wedge G}(\lambda^1 n.\lambda z.0)//F\wedge G$ *by* 3.2.20(a) *and* (d),
$v*\phi//F\wedge G \rightarrow j_1(v*\phi)//F \wedge j_2(v*\phi)//G$ *by* 3.2.20(f),
$j_1(v*\phi) = k_1 v*j_1\phi$, $j_2(v*\phi) = k_2 v*j_2\phi$ *by* 3.2.13(c) *hence*
$j_i(v*\phi)//H_i \rightarrow k_i v//H_i$ *by* 3.2.20(d), *where* $i = 1,2$, $H_1 = F, H_2 = G$.]

(d) $\ell F \cap \ell G = \emptyset \rightarrow (k_1 v//F \wedge k_2 v // G \rightarrow v//F\wedge G)$.
[*By* 3.2.20 (a), (d) *and* (h), *use a similar argument as for* (c) *above*.]

(e) *If* F *has a* 1-1 *labelling, i.e.* $\forall bb' \in F(\ell_b F = \ell_{b'} F \rightarrow b = b')$, *then* $\forall \phi \in N(\phi//F)$ *and* $\forall v(v//F)$.
[*From corollaries* (a), (b), (d) *and* 3.2.20(h) *by induction over frames*.]

(f) $v//G \wedge G \geq F \rightarrow v//F$.
[*By* 3.2.20(a), (d) *and* (j), *use a similar argument as for corollary* (c).]

(g) $F \approx G \rightarrow (\phi//F \leftrightarrow \phi//G)$. [*By* 3.2.20(j).]

(h) $F \approx G \rightarrow (v//F \leftrightarrow v//G)$. [*By corollary* (f).]

(i) *If* $\forall v([v] \in C)$ *then* $u//F \rightarrow [u]//_C F$. [*By* 3.2.20(q).]

(j) $e//_C F \wedge v//F \rightarrow e\upharpoonright v//F$.
[$e//_C F \wedge v//F \rightarrow e|(v*\nu^1_F(\lambda^1 n.(\lambda z.0)))//F$ *by* 3.2.20 (a), (d) *and* (r),
$e|(v*\phi) \in e\upharpoonright v$ *by definition of* $e\upharpoonright v$, $\psi//F \wedge \psi \in u \rightarrow u//F$ *by* 3.2.20(d).]

(k) $j_1\phi//F \wedge m \notin \ell F \rightarrow \phi//F \wedge (^\circ m)$. [*By* 3.2.20(h) *and corollary* (a).]

(l) $\forall a \exists b((b//(^\circ n) \wedge F) \wedge j_1 b = a)$. [*Take* $b = \nu^1_{(^\circ n)\wedge F}(\lambda^1 m.a)$ *and use* 3.2.20(a) *and* 3.2.16(b).]

(m) $\forall u \exists v((v//(^\circ n) \wedge F) \wedge k_1 v = u)$.
[*Apply corollary* (l) *with* $a = u*\lambda z.0$, *take* $v \equiv \bar{b}(\mathrm{lth}(u))$, *use* 3.2.20(d).]

CHAPTER 4

PROJECTION MODELS FOR GC(C)

4.1. INTRODUCTION

We consider projected universes $\mathcal{U}_\delta^M \equiv \{e|\delta : e \in M\}$, where M is a subset of K. Each $e \in K$ is the neighbourhood-function of a continuous $\Gamma_e : N \to N$. A set $M \equiv \{\Gamma_e : e \in M\}$, $M \subset K$, is (externally) a subset of the Moschovakis model for Baire-space over Baire-space. Validity in $\mathcal{U}_\delta^M$ can be reinterpreted as validity in the submodel M.

We shall not construct a single projected universe imitating GC(C). Instead we define a class $U_\delta(C)$ of universes of the form $\mathcal{U}_\delta^M$, all imitating GC(C), and prove the existence of a $\mathcal{U}_\delta \in U_\delta(C)$ for suitable C.

The lawless sequence δ, the *generator* of the universes $\mathcal{U}_\delta \in U_\delta(C)$, plays the following rôle: the value δx is a numerical code for the choices one makes at stage x+1 in the construction of the universe of GC-carriers. It is convenient to think of δ as a triple of sequences. We put $\alpha \equiv j_1^3\delta$, $\beta \equiv j_2^3\delta$ and $\gamma \equiv j_3^3\delta$. As long as δ does not appear in the same context we can think of α,β and γ as being lawless.

From $\alpha x \equiv j_1^3(\delta x)$, or rather, from αx and $\bar{\alpha}x$, we read whether any carrier jumps at stage x, and if so, which one and where to.

γx codes the preliminary choice of values at stage x, that is to say, the preliminary choice of values for carrier $\underline{n}$ at stage x will be $(\gamma x)_n$. (cf. 2.8.1(a).)

The choice of a jump-function is made (if necessary, i.e. if $\bar{\alpha}(x+1)$ codes the decision to have a jump at stage x) via a lawlike $J : \mathbb{N} \to C$: if there is a jump at stage x, then $J(\beta x)$ is the jump-function.

The imitation of GC(C) in projection models is therefore successful only if there is a J which maps $\mathbb{N}$ *onto* C, at least modulo $\simeq$, i.e. if $\forall e \in C \exists n (Jn \simeq e)$.

4.2. We sketch the construction of $U_\delta(C)$. The detailed explanation of the construction is given in the sections 4.3–4.6 below.

A universe $\mathcal{U}_\delta \in U_\delta(C)$ has the form

$$\mathcal{U}_\delta = \{e|\pi_F\delta : F \in \text{FRAME},\ e \in C\}.$$

For each $F \in$ FRAME, π_F is an element of K, $\pi_F\delta$ abbreviates $\pi_F|\delta$, we put $\pi_n\delta \equiv_{def} \pi_{(\circ n)}\delta$. The universe

$$\{\pi_n\delta : n \in \mathbb{N}\}$$

imitates GCC(C), $\pi_F\delta$ is a nest of carriers, that is to say, $\pi_F\delta$ behaves as ε_F(cf.2.10.1).

Each mapping π_F is related to a sequence $\{d_F v : v \in \mathbb{N}\}$ of elements of K, by

$$\pi_F 0 = 0, \quad \pi_F(\hat{x}*v)=y+1 \leftrightarrow \forall a[(d_F v|a)(x)=y].$$

If $F = (\circ n)$, then $d_F v \equiv d_{(\circ n)}v = d_n v$, where

$$d_n(\bar{\delta}x) \text{ is the dressing for the carrier } \pi_n\delta \text{ at stage } x.$$

The K-element $d_F v$ is the image of the triple $(0,F,v)$ under a mapping $d\colon \mathbb{N} \times \text{FRAME} \times \mathbb{N} \to K$. In general, we write $d_F^w v$ for $d(w,F,v)$, that is to say, $d_F v$ abbreviates $d_F^0 v$.

d belongs to a set DG(J), where J maps $\mathbb{N}$ onto C modulo $\simeq$. If $d \in$ DG(J) we say that d generates a universe of dressing sequences w.r.t. J.

The definition of DG(J) uses the auxiliary mappings jf and gv.
jf (for jump-function) is a mapping from $\mathbb{N}$ into $K^{\mathbb{N}}$:

if $\bar{\alpha}(x+1)$ codes the decision to make carrier n jump at stage x+1, then it jumps with jump-function $jf(\bar{\delta}(x+1))(n) \equiv J(\beta x)$,

if carrier n does not jump at stage x+1 then $jf(\bar{\delta}(x+1))(n) = id$.

gv (for generated values) is a mapping from $\mathbb{N}$ into $K^{\mathbb{N}}$:

$gv(\bar{\delta}(x+1))(n)$ has the form [m], m is the sequence of generated values for carrier n at stage x+1.

d is an element of DG(J) iff it satisfies the following equivalences (some of which are redundant):

$$\begin{cases} d_n 0 \simeq \mathrm{id}, \\ d_n(v*\hat{x}) \simeq d_n v : \nu^K_{\delta_n v}\ jf(v*\hat{x}) : \nu^K_{\delta_n(v*\hat{x})}\ gv(v*\hat{x}), \end{cases}$$

$$\begin{cases} d^v_n 0 \simeq \mathrm{id} \\ d^v_n \hat{x} \simeq \nu^K_{\delta_n v} jf(v*\hat{x}) : \nu^K_{\delta_n(v*\hat{x})} gv(v*\hat{x}) \\ d^v_n(w*\hat{x}) \simeq d^v_n w : d^{v*w}_n \hat{x} \end{cases}$$

$$d^v_F w \simeq \nu^K_{\delta_F v}(\lambda^K n.d^v_n w), \quad \text{if } \mathrm{ht}(F) > 0.$$

In these equivalences, $\delta_n v$ and $\delta_F v$ are frames.

$\delta_F v$ is the image of the pair (F,v) under a mapping from FRAME × $\mathbb{N}$ into FRAME, and $\delta_n v = \delta_{(\circ n)} v$;

$\delta_n(\bar{\delta}x)$ is the frame for the carrier $\pi_n\delta$ at stage x.

The mapping $(F,v) \mapsto \delta_F v$ is defined by the following clauses:

$\delta_n 0 = {}^\circ n$,

$\delta_n(v*\hat{x}) = \delta_n v[\mathrm{jps}(k^3_1(v*\hat{x}))]$,

$\delta_F v = F[\lambda n.\delta_n v]$.

jps (for jumps) is a mapping from $\mathbb{N}$ into $\mathrm{FRAME}^{\mathbb{N}}$:

if $\mathrm{jps}(\bar{\alpha}x)(n) = {}^\circ k$, $k \neq n$, then carrier n jumps to carrier k at stage x,

if $\mathrm{jps}(\bar{\alpha}x)(n) = ({}^\circ k)\wedge({}^\circ m)$, $k \neq n$, $m \neq n$, then carrier n jumps to the carriers k and m at stage x,

if $\mathrm{jps}(\bar{\alpha}x)(n) = {}^\circ n$, then carrier n does not jump at stage x.

Note that $\bar{\alpha}x = k^3_1(\bar{\delta}x)$.

4.3. THE CREATION OF DEPENDENCIES BETWEEN CARRIERS IN PROJECTION MODELS

4.3.1. $\alpha \equiv j^3_1\delta$ governs the creation of dependencies in the GCC-projection models $\{\pi_n\delta : n \in \mathbb{N}\}$. The numerical value αx contains the *suggestion* for a jump at stage x+1. The suggestion is coded as follows:

$\alpha x = \nu_3(0,k,m)$ stands for 'try to make carrier k dependent on carrier m',

$\alpha x = \nu_3(n+1,k,m)$ stands for 'try to make carrier k dependent on the carriers $j_1 m$ and $j_2 m$'.

In other words, each $y \in \mathbb{N}$ can be treated as the code of a suggested jump; $j^3_2 y$ is the name of the carrier which should jump, $j^3_3 y$ contains the name(s) of the carrier(s) it should jump to; if $j^3_1 y = 0$ then a singular jump is suggested: $j^3_2 y$ is to be made dependent on $j^3_3 y$, if $j^3_1 y \neq 0$ then a binary jump is suggested: j_2^3y should be made dependent on $j_1(j_3^3y)$ and $j_2(j_3^3y)$.

We can not always create the dependency that αx suggests, since

(a) it is impossible for a carrier to jump to itself (which might be suggested),

(b) a carrier can only jump to carriers that are still fresh (that is to say, we have to check that the jump which αx suggests, is not in conflict with the dependencies already created, following 'previous suggestions' $\bar{\alpha}x$), and

(c) only fresh carriers can jump.

4.3.2. <u>DEFINITION</u>. $n\epsilon w \equiv \exists i<\mathrm{lth}(w)((w)_i=n)$, $n\notin w \equiv \neg(n\epsilon w)$.

4.3.3. <u>DEFINITION</u>. $A(n,y,w)$ is the formula which expresses:
'y suggests that carrier n should jump. If w is the full list of non-fresh carriers, then we can follow the suggestion, since it is not in conflict with (a), (b) and (c) above'.
Formally:

$$\begin{aligned} A(n,y,w) \equiv_{def}\ & n=j_2^3y \wedge n\notin w \wedge \\ & [(j_1^3y=0 \wedge j_3^3y\neq n \wedge j_3^3y\notin w) \vee \\ & (j_1^3y\neq 0 \wedge \bigwedge_{i=1,2} (j_i(j_3^3y)\neq n \wedge j_i(j_3^3y)\notin w))]. \end{aligned}$$

We use $A(n,y,w)$ to define two mappings: $nf\colon \mathbb{N} \to \mathbb{N}$ and $jps\colon \mathbb{N} \to (\mathrm{FRAME}^{\mathbb{N}})$.

nf stands for 'non-fresh', $nf(\bar{\alpha}x)$ is the full list of names of carriers that have been made dependent on others through the stages $z \le x$.

jps stands for 'jumps', $jps(\bar{\alpha}x)$ is a lawlike sequence of frames.
$jps(\bar{\alpha}x)n = {}^\circ n$ expresses 'carrier n does not jump at stage x',
$jps(\bar{\alpha}x)n = {}^\circ k$, $k \neq n$, expresses 'carrier n jumps to carrier k at stage x',
$jps(\bar{\alpha}x)n = ({}^\circ k)\wedge({}^\circ m)$, $k \neq n$, $m \neq n$, expresses 'carrier n jumps to the carriers k and m at stage x'.

4.3.4. <u>DEFINITION</u> (of nf and jps, see example 4.3.5).

(a) $nf : \mathbb{N} \to \mathbb{N}$ is the mapping which satisfies:

$$nf(0)=\langle\ \rangle,\quad nf(v*\hat{y}) = \begin{cases} nf(v)*\langle j_2^3y\rangle & \text{if } A(j_2^3y,y,nf(v)), \\ nf(v) & \text{otherwise.} \end{cases}$$

(b) jps: $\mathbb{N} \to FRAME^{\mathbb{N}}$ is defined by:

$$jps(0) = \lambda n.(^\circ n),$$

$$jps(v*\hat{y})n = \begin{cases} ^\circ n \text{ if } \neg A(n,y,nf(v)), \\ ^\circ k \text{ if } A(n,y,nf(v)),\ j_1^3 y=0 \text{ and } j_3^3 y=k, \\ (^\circ k)\wedge(^\circ m) \text{ if } A(n,y,nf(v)),\ j_1^3 y\neq 0, \\ \qquad j_1(j_3^3 y)=k \text{ and } j_2(j_3^3 y)=m. \end{cases}$$

4.3.5. EXAMPLE.

x	αx	$jps(\bar{\alpha}(x+1))$	$nf(\bar{\alpha}(x+1))$	comment
0	$\nu_3(0,1,2)$	$n \mapsto \begin{cases} ^\circ 2 \text{ if } n=1 \\ ^\circ n \text{ othw} \end{cases}$	<1>	
1	$\nu_3(1,2,j(2,3))$	$n \mapsto {}^\circ n$	<1>	αx suggests that 2 should jump to 2 and 3, which is impossible.
2	$\nu_3(0,0,0)$	$n \mapsto {}^\circ n$	<1>	αx suggests that 0 should jump to 0. Nothing happens.
3	$\nu_3(1,2,j(3,4))$	$n \mapsto \begin{cases} (^\circ 3)\wedge(^\circ 4) \text{ if } n=2 \\ ^\circ n \text{ othw} \end{cases}$	<1,2>	
4	$\nu_3(0,1,4)$	$n \mapsto {}^\circ n$	<1,2>	αx suggests that 1 should jump to 4, but 1 is non-fresh.
5	$\nu_3(1,3,j(2,5))$	$n \mapsto {}^\circ n$	<1,2>	αx suggests that 3 should jump to 2 and 5, but 2 is non-fresh.

4.3.6. LEMMA (properties of jps and nf).

(a) $jps(v*\hat{y})m \neq {}^\circ m \to m=j_2^3 y \wedge nf(v*\hat{y})=nf(v)*\langle m\rangle$.

(b) $jps(v*\hat{y})m \neq {}^\circ m \to$

$$(jps(v*\hat{y})m={}^\circ j_3^3 y \wedge j_3^3 y\neq m) \vee$$

$$(jps(v*\hat{y})m=({}^\circ j_1(j_3^3 y) \wedge {}^\circ j_2(j_3^3 y)) \wedge \bigwedge_{i=1,2} j_i(j_3^3 y)\neq m).$$

(c) $nf(v*\hat{y})=nf(v)*\langle m\rangle \rightarrow m=j_2^3 y \wedge jps(v*\hat{y})m\neq{}^\circ m$.

(d) $jps(v*\hat{y})m=F \wedge F\neq{}^\circ m \rightarrow \forall k\in\ell F(k\notin nf(v*\hat{y}))$.

PROOF. Trivial by definition. □

4.3.7. COROLLARIES.

(a) $jps(\bar{\alpha}(x+1))m\neq{}^\circ m \rightarrow \forall k(k\neq m \rightarrow jps(\bar{\alpha}(x+1))k={}^\circ k)$.

[*The model has the 'single jump property'* (2.4.4), *by* 4.3.6(a).]

(b) $jps(\bar{\alpha}(x+1))k\neq{}^\circ k \rightarrow \exists mn[m\neq k \wedge n\neq k \wedge (jps(\bar{\alpha}(x+1))k={}^\circ m \vee jps(\bar{\alpha}(x+1))k=({}^\circ m\wedge{}^\circ n))]$

[*The model has 'restriction to binary jumps'* (2.4.4), *by* 4.3.6(b).]

(c) $m\in nf(v) \leftrightarrow \exists u\preccurlyeq v(jps(u)m\neq{}^\circ m)$, *or equivalently*
$m\notin nf(v) \leftrightarrow \forall u\preccurlyeq v(jps(u)m={}^\circ m)$.

[*If* $m\notin nf(\bar{\alpha}(x+1))$ *then carrier* m *is fresh at stage* x+1, *by induction w.r.t.* lth(v) *from* 4.3.6(a) *and* (c).]

(d) $jps(\bar{\alpha}(x+1))k=F \wedge F\neq{}^\circ k \rightarrow \forall m\in\ell F\ \forall y\leq x+1(jps(\bar{\alpha}y)m={}^\circ m)$.

[*If carrier* k *jumps at stage* x+1, *then the carrier(s) it jumps to is (are) fresh at stage* x+1, *by* (c) *above and* 4.3.6(d).]

4.3.8. Fig. 1 shows a possible frame f_0z for the carrier $\varepsilon_{\underline{0}} \in GCC(C)$ at some stage z, and for a number of possible jumps at stage z+1, the resulting frame $f_0(z+1)$ for $\varepsilon_{\underline{0}}$ at stage z+1. (cf. 2.9.7-8.)

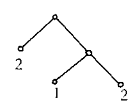
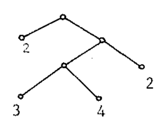
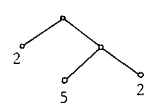
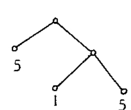
jumps at stage z+1	
	 f_0z
$\varepsilon_{\underline{1}}$ jumps to $\varepsilon_{\underline{3}}$ and $\varepsilon_{\underline{4}}$	 $f_0(z+1)$
$\varepsilon_{\underline{1}}$ jumps to $\varepsilon_{\underline{5}}$	 $f_0(z+1)$
$\varepsilon_{\underline{2}}$ jumps to $\varepsilon_{\underline{5}}$	 $f_0(z+1)$

Fig. 1

The construction of $f_n z$ has been described in 2.9.8. We can rephrase that description, in the terminology of chapter 3, as:
$f_n 0 = {}^\circ n$, and $f_n(z+1)$ is produced from $f_n z$ by a mapping $\mathfrak{f}_{z+1}: \mathbb{N} \to$ FRAME, which satisfies:

$$\mathfrak{f}_{z+1}n = \begin{cases} {}^\circ n \text{ if } \underline{n} \text{ does not jump at stage } z+1, \\ {}^\circ k \text{ if } \underline{n} \text{ jumps to } \underline{k} \text{ at stage } z+1, \\ {}^\circ k \wedge {}^\circ m \text{ if } \underline{n} \text{ jumps to } \underline{k} \text{ and } \underline{m} \text{ at stage } z+1. \end{cases}$$

(For 'produced from F by $\mathfrak{f}$' see 3.1.16.)
In the GCC(C) projection models $\{\pi_n \delta : n \in \mathbb{N}\}$, $jps\bar{\alpha}(z+1)$ plays the rôle of $\mathfrak{f}_{z+1}$.

4.3.9. We introduce a mapping $(n,v) \mapsto \mathfrak{f}_n v$ from $\mathbb{N} \times \mathbb{N}$ into FRAME. $\mathfrak{f}_n v$ is *the frame for* π_n *at* v, $\mathfrak{f}_n(\bar{\delta}x)$ is *the frame for* π_n *at stage* x.

<u>DEFINITION</u>. $\mathfrak{f}_n v$ is the image of a mapping from $\mathbb{N} \times \mathbb{N}$ into FRAME defined by $\mathfrak{f}_n 0 = {}^\circ n$, $\mathfrak{f}_n(v * \hat{x}) = \mathfrak{f}_n v[jps(k_1^3(v * \hat{x}))]$.
(Recall that $\alpha \equiv j_1^3 \delta$, whence $\bar{\alpha}(z+1) = k_1^3(\bar{\delta}(z+1))$.)

4.3.10. <u>LEMMA</u> (properties of $\mathfrak{f}_n v$).
(a) $\forall u \preccurlyeq k_1^3 v(jps(u)n = {}^\circ n) \to \mathfrak{f}_n v = {}^\circ n$.
(*A carrier which has not jumped, is independent of others.*)
(b) $\mathfrak{f}_n v \neq {}^\circ n \to \mathfrak{f}_n(v * \hat{x}) \neq {}^\circ n$.
(*A carrier which depends on others at stage* z, *will not be independent of others at stage* z+1.)
(c) $\forall m \in \mathcal{L}(\mathfrak{f}_n v)(m \notin nf(k_1^3 v))$.
(*The labels of the frame for* π_n *at stage* x, *refer to fresh carriers*.)
(d) $\forall w \exists g \forall n(\mathfrak{f}_n(v * w) = \mathfrak{f}_n v[g])$.
(*With each* y *there is a* g: $\mathbb{N} \to$ FRAME, *which produces the frame for* π_n *at stage* x+y *from the one at stage* x, *for all* n.)

<u>PROOF</u>.
(a) By induction w.r.t. lth(v).
(b) $\mathfrak{f}_n(v * \hat{x}) = \mathfrak{f}_n v[jps(k_1^3(v * \hat{x}))]$ by definition, hence $ht(\mathfrak{f}_n(v * \hat{x})) \geq ht(\mathfrak{f}_n v)$ by 3.1.17(d), so if $ht(\mathfrak{f}_n v) > 0$ then $ht(\mathfrak{f}_n(v * \hat{x})) > 0$ and $\mathfrak{f}_n(v * \hat{x}) \neq {}^\circ n$.
If $ht(\mathfrak{f}_n v) = 0$, i.e. $\mathfrak{f}_n v = {}^\circ m$, $m \neq n$, then $n \in nf(v)$ by (a) and 4.3.7(c). Hence $n \notin \mathcal{L}(jps(k_1^3(v * \hat{x}))m)$, by 4.3.6(d), and hence also $n \notin \mathcal{L}(\mathfrak{f}_n(v * \hat{x}))$, i.e. $\mathfrak{f}_n(v * \hat{x}) \neq {}^\circ n$.
(c) By induction w.r.t. lth(v):

(i) $nf(0) = \langle\ \rangle$, then certainly $\forall k\in\ell(\delta_n 0)(k\notin nf(0))$.

(ii) Assume (induction hypothesis):

(1) $\qquad \forall k\in\ell(\delta_n v)(k\notin nf(k_1^3 v))$.

$m\in\ell(\delta_n(v*\hat{x})) \leftrightarrow \exists k\in\ell(\delta_n v)\ [m\in\ell(jps(k_1^3(v\ \hat{x}))k)]$, by definition of $\delta_n(v*\hat{x})$ and 3.1.17(b). Let $k\in\ell(\delta_n v)$, then by (1) and 4.3.7(c)

(2) $\qquad \forall u\preccurlyeq k_1^3 v(jps(u)k={}^\circ k)$.

Either $jps(k_1^3(v*\hat{x}))k={}^\circ k$, then $k\notin nf(v*\hat{x})$ by (2) and 4.3.7(c), or $jps(k_1^3(v*\hat{x}))k=F$, $F\neq{}^\circ k$, then $\forall m\in\ell F(m\notin nf(v*\hat{x}))$ by 4.3.6(d).

(d) By definition, $\forall n(\delta_n(u*\hat{x})=\delta_n u[g])$, for $g = jps(k_1^3(u*\hat{x}))$.
The desired result now follows from 3.1.19(a) by induction w.r.t. $lth(w)$. □

4.3.11. <u>COROLLARIES</u>.

(a) $n\notin nf(k_1^3 v) \leftrightarrow \delta_n v={}^\circ n$. [$\rightarrow$ *by* 4.3.10(a) *and* 4.3.7(c), $\leftarrow$ *by* 4.3.10(c).]

(b) $\delta_n v={}^\circ n \leftrightarrow \forall u\preccurlyeq k_1^3 v(jps(u)n={}^\circ n)$. [*By* (a) *and* 4.3.7(c).]

(c) $\delta_n(v*\hat{x})={}^\circ n \leftrightarrow \delta_n v={}^\circ n$. [*By* 4.3.10(b).]

(d) $\forall m\in\ell(\delta_n v)(\delta_m v={}^\circ m)$. [*By* 4.3.10(c), 4.3.7(c) *and* 4.3.10(a).]

In 2.10.5 we have defined the frame for the GC-sequence $\varepsilon=e|\varepsilon_F$ at stage z as 'obtained from the initial frame F by substituting $f_n z$ for each label n in F', i.e., in the terminology of chapter 3, as $F[\lambda n.f_n z]$.

4.3.12. <u>DEFINITION</u>. $\delta_F v$ is the image of the pair (F,v) under the mapping from FRAME $\times$ $\mathbb{N}$ $\rightarrow$ FRAME, defined by $\delta_F v = F[\lambda n.\delta_n v]$.
We call $\delta_F v$ *the frame for* π_F *at* v, $\delta_F(\bar{\delta}x)$ is *the frame for* π_F *at stage* x.
Note that $\delta_{({}^\circ n)}v=\delta_n v$, $\delta_{F\wedge G}v=\delta_F v \wedge \delta_G v$ by definition of F[.].

4.3.13. <u>LEMMA</u>. $\delta_F(v*\hat{x}) = \delta_F v[jps(k_1^3(v*\hat{x}))]$.

<u>PROOF</u>. $\delta_F v[jps(k_1^3(v*\hat{x}))] = (F[\lambda n.\delta_n v])[jps(k_1^3(v*\hat{x}))]$ by 4.3.12, $(F[\lambda n.\delta_n v])[jps(k_1^3(v*\hat{x}))] = F[\lambda n.\delta_n v[jps(k_1^3(v*\hat{x}))]]$ by 3.1.19(a), $\lambda n.\delta_n v[jps(k_1^3(v*\hat{x}))] = \lambda n.\delta_n(v*\hat{x})$, by 4.3.9, and finally $F[\lambda n.\delta_n(v*\hat{x})] = \delta_F(v*\hat{x})$, by 4.3.12. □

4.3.14. <u>LEMMA</u> (characteristic properties of $\delta_F v,\delta_n v$).

(a) $\delta_F 0 = F$

(b) $\forall w\exists g\forall F(\delta_F(v*w)=\delta_F v[g])$

(c) $\forall n\in\ell(\delta_F v)(\delta_n v={}^\circ n)$

(d) $\forall v\forall n\exists m>n(\delta_m v={}^\circ m)$.

PROOF.

(a) $\delta_F 0 = F[\lambda n.\delta_n 0]$ by definition, $\delta_n 0 = {}^\circ n$ by definition, and $F[\lambda n.{}^\circ n] = F$ by 3.1.19(c).

(b) $\delta_F(v*\hat{x}) = \delta_F v[g]$ with $g = jps(k^3_1(v*\hat{x}))$ by 4.3.13. Use induction w.r.t. lth(w) and apply 3.1.19(a).

(c) $n\in\ell(\delta_F v) \leftrightarrow \exists k\in\ell F(n\in\ell(\delta_k v))$, by definition of $\delta_F v$ and 3.1.17(b). Now apply 4.3.11(c).

(d) By 4.3.11(a) we find that even $\forall n\notin nf(k^3_1 v)(\delta_n v={}^\circ n)$. □

4.3.15. COROLLARY. $\delta_F(v*w)=\delta_F v[\lambda n.\delta_n(v*w)]$.

PROOF. Let g satisfy $\forall F(\delta_F(v*w)=\delta_F v[g])$ (4.3.14(b)). Then in particular $\delta_m(v*w)=\delta_m v[g]$ for all m. By 4.3.14(c), $\delta_m v={}^\circ m$ for $m\in\ell(\delta_F v)$, whence, for those m, $gm=\delta_m v[g]=\delta_m(v*w)$ (cf. def. $F[g]$, 3.1.16(a)). By 3.1.19(b) it follows that $\delta_F v[g]=\delta_F v[\lambda m.\delta_m(v*w)]$, hence the desired equation. □

4.4. PROJECTED UNIVERSES OF DRESSING SEQUENCES

With each GC-carrier $\varepsilon_{\underline{n}}$ we have associated a sequence $d_n \in K^{\mathbb{N}}$, where $d_n z \equiv$ the dressing for $\varepsilon_{\underline{n}}$ at stage z. d_n will be imitated by a projected sequence $d_n\delta$. Note that $d_n z$ can be determined at stage z, i.e. in the projection model $d_n\delta(z)$ will have the form $d^0_n(\bar{\delta}z)$, where d^0: $(n,v) \mapsto d^0_n v$ is a mapping from $\mathbb{N}\times\mathbb{N}$ into K. With each d^0: $\mathbb{N}\times\mathbb{N} \to K$ we can associate sequences $d_n\delta \equiv \lambda z.d^0_n(\bar{\delta}z)$, but only for special d^0 this will yield faithful imitations of 'the sequence of dressings for $\varepsilon_{\underline{n}}$'.

Our first aim in this section is to define the set $DG^0(J)$ (D for 'dressing', G for 'generate', J a mapping from $\mathbb{N}$ into K; the superscript zero will be explained in 4.4.17). $DG^0(J)$ is to contain exactly those d^0: $\mathbb{N}\times\mathbb{N} \to K$ which yield sequences $\lambda z.d^0_n(\bar{\delta}z)$ imitating 'the sequence of dressings for $\varepsilon_{\underline{n}}$', where $\varepsilon_{\underline{n}} \in GCC(range(J))$ (i.e. jump-functions are $\{Jn : n\in\mathbb{N}\}$).

4.4.1. From 2.9.8 we recall that

$$d_n 0 = id,\ d_n(z+1) = d_n z\colon f_{n,z+1}\colon g_{n,z+1}.$$

Fig. 2 shows an example of the construction of the mappings $f_{n,z+1}$, $g_{n,z+1}$. (See also 2.9.7.)

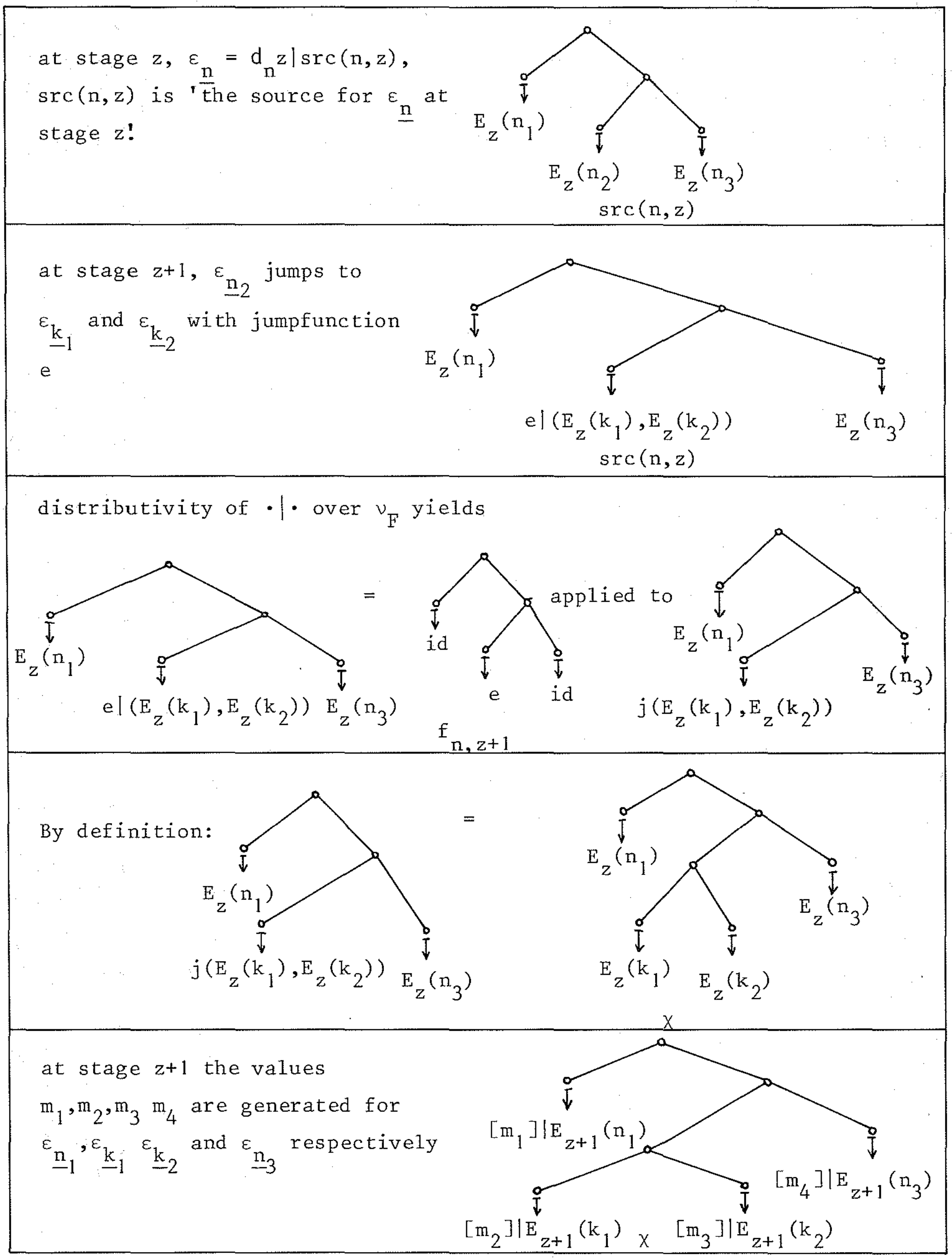


Fig. 2. The construction of $d_n(z+1)$ from $d_n z$ (to be continued.)

By an application of distributivity of $\cdot|\cdot$ over ν_F, χ equals

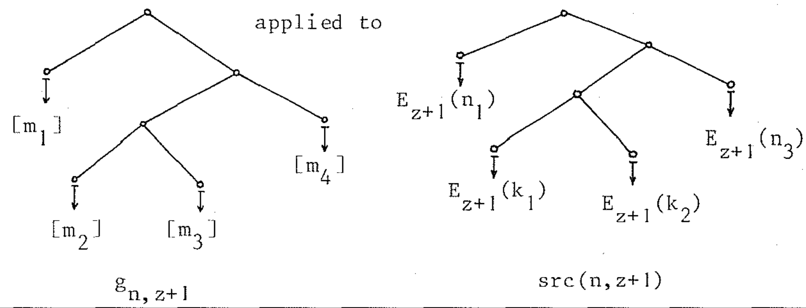


Fig. 2. The construction of $d_n(z+1)$ from $d_n z$.

We can rephrase the definition of $f_{n,z+1}$ and $g_{n,z+1}$, given in 2.9.8, using the terminology of chapter 3, as follows:

$$f_{n,z+1} \equiv \nu^K_{f_n z}\ \phi_{z+1},\ g_{n,z+1} \equiv \nu^K_{f_n(z+1)}\ \psi_{z+1},\ \phi_{z+1},\ \psi_{z+1} \in K^{\mathbb{N}},$$

where

$$\phi_{z+1}m = \begin{cases} e \text{ if } \varepsilon_{\underline{m}} \text{ jumps at stage } z+1 \text{ with jump-function } e \\ \\ id \text{ otherwise,} \end{cases}$$

and

$$\psi_{z+1}m = \begin{cases} [u] \text{ if } \varepsilon_{\underline{m}} \text{ is fresh at stage } z+1, \text{ and } u \text{ is the sequence} \\ \quad \text{of values generated for } \varepsilon_{\underline{m}} \text{ at this stage} \\ \text{arbitrary, if } \varepsilon_{\underline{m}} \text{ is not fresh at stage } z+1. \end{cases}$$

4.4.2. The definition of $DG^0(J)$ will have the form:
$d^0 \in DG^0(J)$ iff d^0 satisfies:

$$d^0_n 0 \simeq id,$$
$$d^0_n(v*\hat{x}) \simeq d^0_n v\colon \nu^K_{\delta_n v} jf(v*\hat{x})\colon \nu^K_{\delta_n(v*\hat{x})} gv(v*\hat{x}),$$

i.e.

$$d^0_n(\bar{\delta}0) \simeq \mathrm{id},$$
$$d^0_n(\bar{\delta}(x+1)) \simeq d^0_n(\bar{\delta}x) : \nu^K_{\mathfrak{f}_n(\bar{\delta}x)} jf(\bar{\delta}(x+1)) : \nu^K_{\mathfrak{f}_n(\bar{\delta}(x+1))} gv(\bar{\delta}(x+1)).$$

Here $\mathfrak{f}_n(\bar{\delta}z)$ is the frame for π_n at stage z as in the previous section, and jf (for jump-function) and gv (for generated values) are mappings from $\mathbb{N}$ into $K^{\mathbb{N}}$ yet to be defined. $jf(\bar{\delta}(x+1))$ is to play the rôle of ϕ_{x+1}, $gv(\bar{\delta}(x+1))$ will play the rôle of ψ_{x+1}.

4.4.3. <u>DEFINITION</u>. jf is the mapping from $\mathbb{N}$ into $K^{\mathbb{N}}$ which satisfies:

$$jf(0) = \lambda^k n.\mathrm{id},$$
$$jf(v*\hat{x})n = \begin{cases} J(j^3_2 x) \text{ if } jps(k^3_1(v*\hat{x}))n \neq {}^\circ n, \\ \\ \mathrm{id} \text{ otherwise,} \end{cases}$$

that is to say: if π_n jumps at stage x+1 then $jf(\bar{\delta}(x+1))n = J(\beta x)$ (recall that $\delta x = \nu_3(\alpha x, \beta x, \gamma x)$, $j^3_2(\delta x) = \beta x$), otherwise $jf(\bar{\delta}(x+1))n = \mathrm{id}$.

It is not so easy to define the mapping $gv : \mathbb{N} \to K^{\mathbb{N}}$ in such a way that $gv(\bar{\delta}(z+1))$ behaves as the ψ_{z+1} which assigns to n the K-element [u], where u is the sequence of generated values for $\varepsilon_{\underline{n}} \in GCC(\mathrm{range}(J))$ at stage z+1 (if $\varepsilon_{\underline{n}}$ is fresh at stage z+1).

From 2.8.1-2 we recall that at each stage, the process of generating values is started by making a preliminary choice of values for all fresh carriers, from which the guiding sequences are constructed.

4.4.4. <u>DEFINITION</u>.

(i) If $\mathfrak{f}_n(\bar{\delta}(x+1)) = {}^\circ n$, i.e. π_n is fresh at stage x+1, then *the preliminary choice of values for* π_n *at stage* x+1 is the finite sequence $(\gamma x)_n$.

(ii) If $\mathfrak{f}_n(v*\hat{x}) = {}^\circ n$ then the *guiding sequence for* π_n *at* $v*\hat{x}$ is

$$gs_n(v*\hat{x}) \equiv (j^3_3 x)_n * \lambda z.0.$$

We call $gs_n(\bar{\delta}(x+1))$ *the guiding sequence for* π_n *at stage* x+1. $gs_n(\bar{\delta}(x+1)) = (\gamma x)_n * \lambda z.0$ (if $\mathfrak{f}_n(\bar{\delta}(x+1)) = {}^\circ n$), since $j^3_3(\delta x) = \gamma x$.

4.4.5. The next step is to determine the upperbound for the relevant values of the guiding sequences.

At stage z we have for each carrier $\varepsilon_{\underline{n}}$ the equation

$$\varepsilon_{\underline{n}} = d_n z | src(n,z)$$

where src is the source for $\varepsilon_{\underline{n}}$ at stage z (cf. 2.9.2-3). src(n,z) is constructed from empty parts of carriers at stage z, in the terminology of chapter 3 we can say:

$$src(n,z) = \nu^1_{f_n z}(\lambda^1 k.E_z(k))$$

(see fig. 3, for $E_z(k)$ see definition 2.9.1).

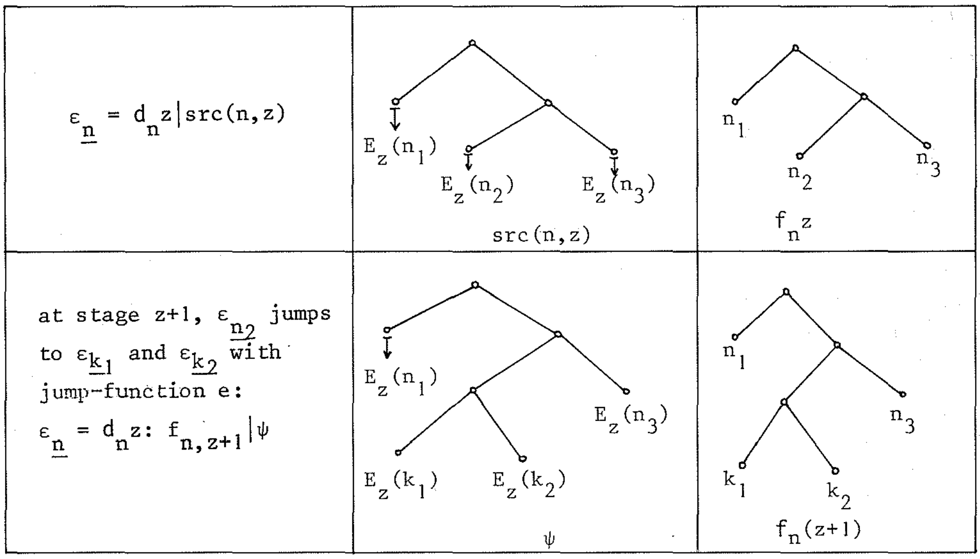


Fig. 3

At stage z+1 we first decide whether there will be a jump and if so, which one and with which jump-function. Then we have, for each carrier n, an equation (cf.2.9.9, see fig.3)

(1) $$\varepsilon_{\underline{n}} = d_n z : f_{n,z+1}|\chi \quad \text{with} \quad \chi \equiv \nu^1_{f_n(z+1)}(\lambda^1 k.E_z(k)).$$

To determine upb_{z+1}, the upperbound for the relevant values of the guiding sequences at stage z+1, we make a list of all the equations (1) for non-fresh carriers n. In these equations we replace empty parts of carriers by guiding sequences, i.e. (1) is replaced by

(2) $\quad \varepsilon_{\underline{n}} = d_n z : f_{n,z+1}|\chi'$

where $j_b\ \chi'$ is the guiding sequence for $\varepsilon_{\underline{k}}$ at stage z+1, if b has label k in $f_n(z+1)$. (See fig.4.)

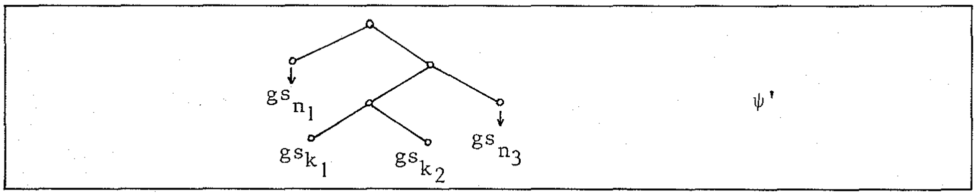


Fig. 4.

From (2) we can determine $\varepsilon_{\underline{n}} z$, the computation of this value requires only an initial segment of χ'. Put

(3) $\quad U_n$ is the minimal k such that $\overline{\chi'}k$ suffices to determine $\varepsilon_{\underline{n}} z$ from (2).

Then

$$upb_{z+1} \equiv \max\{U_n : \text{carrier } \underline{n} \text{ non-fresh at stage } z+1\}.$$

The construction of upb_{z+1} is imitated as follows.

4.4.6. <u>DEFINITION</u> (of guiding sequence for π_n). For each n, gs_n is a mapping from $\mathbb{N}$ into N.

$$gs_n 0 \equiv \lambda z.0,$$

$$gs_n(v*\hat{x}) \equiv \nu^1_{\delta_n(v*\hat{x})}(\lambda^1 k.(j^3_3 x)_k * \lambda z.0).$$

We call $gs_n(v*\hat{x})$ the *guiding sequence for* π_n *at* $v*\hat{x}$, $gs_n(\bar{\delta}(x+1))$ is *the guiding sequence for* π_n *at stage* x+1.

For n satisfying $\delta_n(v*\hat{x}) = {}^\circ n$, this notion has been defined before, in 4.4.4. Note that both definitions coincide. For n satisfying $\delta_n(\bar{\delta}(x+1)) \neq {}^\circ n$ (i.e. π_n is non-fresh at stage x+1), $gs_n(\bar{\delta}(x+1))$ is the sequence χ' of equation (2) above.

4.4.7. DEFINITION (of (d:JF)). Let d be a mapping from $\mathbb{N} \times \mathbb{N}$ into K, d: $(n,v) \mapsto d_n v$. Then

$$(d{:}JF)(n,v*\hat{x}) \equiv d_n v : \nu^K_{\delta_n v} jf(v*\hat{x}),$$

that is to say: if we think of $d_n(\bar{\delta}x)$ as the dressing for carrier n at stage x, then $(d{:}JF)(n,\bar{\delta}(x+1))$ plays the rôle of $d_n x : f_{n,x+1}$ as in equation (2). (For the relation between jf and $f_{n,x+1}$ see 4.4.1-2.)

4.4.8. DEFINITION (of mk(e,x,a)). For $e \in K$, $x \in \mathbb{N}$ and $a \in N$, mk(e,x,a) is the minimal k such that $\bar{a}k$ suffices to determine $e|a(x)$, i.e.

$$mk(e,x,a) \equiv \min_k (e(\langle x\rangle * \bar{a}k) \neq 0).$$

$mk((d{:}JF)(n,\bar{\delta}(z+1)),\ z,\ gs_n(\bar{\delta}(z+1)))$ plays the rôle of U_n in (3).

4.4.9. DEFINITION (of upb). Let d be a mapping from $\mathbb{N} \times \mathbb{N}$ into K.

$$upb(d,v*\hat{x}) \equiv \max\{U_n(v*\hat{x}) \ :\ n \in nf(k^3_1(v*\hat{x}))\},$$

where

$$U_n(v*\hat{x}) \equiv mk((d{:}JF)(n,v*\hat{x}),\ lth(v),\ gs_n(v*\hat{x})).$$

We call $upb(d,v*\hat{x})$ the *upperbound at* $v*\hat{x}$ *w.r.t.* d, $upb(d,\bar{\delta}(x+1))$ is *the upperbound at stage* x+1 *w.r.t.* d.

Once we have upb, the sequence of generated values for the fresh carrier n is easily determined: it is the initial segment with length 1+upb of the guiding sequence for carrier n.

4.4.10. DEFINITION. gv (for generated values) is a mapping which assigns to each pair (d,v), d a mapping from $\mathbb{N} \times \mathbb{N}$ into K, $v \in \mathbb{N}$, an element $gv(d,v) \in K^{\mathbb{N}}$, as follows:

$$gv(d,0) = \lambda^K n.id,$$

$$gv(d,v*\hat{x}) = \lambda^K n.[\overline{(j_3^3 x)_n * \lambda z.0(1+upb(d,v*\hat{x}))}].$$

4.4.11. DEFINITION. $DG^0(J)$ is the set which contains all mappings $d^0: \mathbb{N}\times\mathbb{N} \to K$, with the property that $\lambda z.d^0_n(\bar{\delta}z)$ imitates the behaviour of the sequence of dressings for the carrier $\varepsilon_{\underline{n}}$ in GCC(range(J)), where $J: \mathbb{N} \to K$ and $d^0_n v \equiv d^0(n,v)$.
$d^0 \in DG^0(J)$ iff

$$d^0_n 0 \simeq id,$$

$$d^0_n(v*\hat{x}) \simeq d^0_n v : \nu^K_{\delta_n v} jf(v*\hat{x}) : \nu^K_{\delta_n(v*\hat{x})} gv(d^0,v*\hat{x}).$$

4.4.12. REMARK. Strictly speaking only the $d^0 \in DG^0(J)$ which satisfies the equations

(1) $\quad d^0_n 0 \simeq id$, and

(2) $\quad d^0_n(v*\hat{x}) = d^0_n v : \nu^K_{\delta_n v} jf(v*\hat{x}) : \nu^K_{\delta_n(v*\hat{x})} gv(v*\hat{x})$

imitates the dressing construction as outlined in chapter 2 (2.9.7-8). The other elements of $DG^0(J)$ result so to speak from the choice of a 'non-standard neighbourhood function' for the continuous Γ in the equation

$$\varepsilon_{\underline{n}} = \Gamma(src(n,z)), \quad (cf.2.9.2\text{-}3)$$

for some n and z.

Such a non-standard choice at stage z affects the upb-computation at stage z+1. If d^0 and $d^{0'}$ are elements of $DG^0(J)$ and $d^0_n v \simeq d^{0'}_n v$, but $d^0_n v \neq d^{0'}_n v$, for some n then it is possible that $d^0_n(v*\hat{x})$ is not even equivalent to $d^{0'}_n(v*\hat{x})$.

The existence of a d^0 which satisfies (1) and (2) and hence belongs to $DG^0(J)$ is easily proved by an appeal to the recursion theorem (uniform in J), or by first showing that for each v there is a $d^0(v) \in N$ such that for all $w, w*\hat{x} \leq v$ and for all n $\lambda z.d^0(v)(\nu_3(n,w,z))$ and $\lambda z.d^0(v)(\nu_3(n,w*\hat{x},z))$ belong to K and satisfy the equations (1) and (2) above, then putting these together in a single D by AC-NF, and finally 'diagonalizing' the desired d^0 out of D.

In the appendix we shall show that we can explicitly define an element of $DG^0(J)$, primitive recursive in J. This element however shall not satisfy the equation (2), but only the corresponding equivalence, i.e. it is 'non-standard'. (Note that in the right-hand side of (2) there is an unbounded minimum operator, in the upb construction).

4.4.13. DEFINITION (of UPB). Let $d : \mathbb{N} \times \mathbb{N} \to K$. $\lambda v.UPB(d,v)$ is the mapping from $\mathbb{N}$ into $\mathbb{N}$ which satisfies:

$$UPB(d,0) = 0,$$

$$UPB(d,v*\hat{x}) = UPB(d,v)+(1+upb(d,v*\hat{x})).$$

If no confusion can arise we write gv(v), upb(v) and UPB(v) for gv(d,v), upb(d,v) and UPB(d,v) respectively.

4.4.14. LEMMA. *If a carrier is fresh at stage* z+1, *i.e. if* $\delta_n(\bar{\delta}(x+1)) = {}^\circ n$, *then the dressing* $d_n(\bar{\delta}(x+1))$ *has the form* [w], *where* $lth(w) = UPB(\bar{\delta}(x+1))$. *Formally: if* $d \in DG^0(J)$ *then*

$$\forall v \forall n(\delta_n v = {}^\circ n \to \exists w(d_n^0 v \simeq [w] \wedge lth(w)=UPB(v))).$$

PROOF. By induction w.r.t. lth(v).

(i) For v = < > take w = < >.

(ii) Now let $v = v'*\hat{x}$, assume

(1) $\delta_n v = {}^\circ n$, then

(2) $\delta_n v' = {}^\circ n$ by 4.3.11(c),

whence by induction hypothesis we have a w' such that

(3) $$d_n^0 v' \simeq [w'] \quad \text{and} \quad lth(w') = UPB(v').$$

By definition of $DG^0(J)$,

$$d_n^0 v \simeq d_n^0 v' : \nu^K_{\delta_n v'} jf(v) : \nu^K_{\delta_n v} gv(v), \quad \text{i.e.}$$

(4) $$d_n^0 v \simeq d_n^0 v' : jf(v)n : gv(v)n,$$

by (1), (2) and the definition of ν^K.

From (1) and 4.3.11(b) we find that $jps(k_1^3 v)n = {}^\circ n$, hence by definition of jf

(5) $\quad jf(v)n = id.$

(6) $\quad gv(v)n = [\overline{(j_3^3 x)_n * \lambda z.0(1+upb(v))}],$

by definition of gv.
From (4), (3), (5) and (6) we find

$$d_n^0 v \simeq [w' * (\overline{(j_3^3 x)_n * \lambda z.0(1+upb(v))})],$$

i.e. $d_n^0 v \simeq [w]$, where $w = w' * (\overline{(j_3^3 x)_n * \lambda z.0(1+upb(v))})$.
So $lth(w) = lth(w') + (1+upb(v))$, while $lth(w') = UPB(v')$ by (3), hence $lth(w) = UPB(v)$ by definition of UPB. □

4.4.15. LEMMA.

$$\nu^K_{\delta_n(v*\hat{x})} gv(v*\hat{x}) \simeq [\overline{gs_n(v*\hat{x})}(1+upb(v*\hat{x}))].$$

PROOF. Put $m \equiv 1+upb(v*\hat{x})$. By definitions 4.4.10 and 4.4.6 of gv and gs:

$$\nu^K_{\delta_n(v*\hat{x})} gv(v*\hat{x}) = \nu^K_{\delta_n(v*\hat{x})}(\lambda^K k.[\overline{(j_3^3 x)_k * \lambda z.0(m)}])$$

and

$$\overline{gs_n(\overline{v*\hat{x}})}(m) = \overline{\nu^1_{\delta_n(v*\hat{x})}(\lambda^1 k.(j_3^3 x)_k * \lambda z.0)}(m).$$

Now apply 3.2.16(g): for $\phi : \mathbb{N} \to N$

$$[\overline{\nu^1_F \phi}(m)] \simeq \nu^K_F(\lambda^K n.[\overline{\phi n}(m)]). \quad □$$

The complex definition of gv was motivated by our wish to achieve the following.

4.4.16. LEMMA. *If* $d^0 \in DG^0(J)$ *then* $d_n^0(\bar{\delta}(z+1))$ *determines a value for* z, *i.e.*

$$\forall n \exists y \forall \phi[(d_n(v*\hat{x}) | \phi)(lth(v)) = y].$$

PROOF. Put $m \equiv 1+\mathrm{upb}(v*\hat{x})$. Let $n\in\mathrm{nf}(k_1^3(v*\hat{x}))$. By the foregoing lemma and the definitions 4.4.7, 4.4.11, we have for $d \in DG^0(J)$

$$d_n^0(v*\hat{x}) \simeq (d^0:JF)(n,v*\hat{x}) : [\overline{gs_n(v*\hat{x})(m)}],$$

i.e. for all ϕ

$$d_n^0(v*\hat{x})|\phi = (d^0:JF)(n,v*\hat{x})|(\overline{gs_n(v*\hat{x})(m)}*\phi).$$

So $(d_n^0(v*\hat{x})|\phi)(\mathrm{lth}(v)) = y$ iff

(1) $$(d^0:JF)(n,v*\hat{x})(\langle\mathrm{lth}(v)\rangle*w) = y+1$$

for some initial segment w of $\overline{gs_n(v*\hat{x})(m)}*\phi$. By definition of upb(=m∸1), there is a y such that (1) holds for $w = \overline{gs_n(v*\hat{x})(m\dot{-}1)}$, i.e. (1) holds for w and y independent of ϕ.

If $n \notin \mathrm{nf}(k_1^3(v*\hat{x}))$ then $d_n^0(v*\hat{x}) \simeq [w]$ for some w with $\mathrm{lth}(w) = \mathrm{UPB}(v*\hat{x})$, by 4.4.14. One easily verifies that $\mathrm{UPB}(v*\hat{x}) > \mathrm{lth}(v)$, i.e. in this case

$$\forall\phi[(d_n^0(v*\hat{x})|\phi)(\mathrm{lth}(v)) = (w)_{\mathrm{lth}(v)}]. \quad \square$$

In the sequel we shall not only be interested in the dressing of a carrier at stage z, but also in the 'difference' between the dressing for carrier n at stage z and the dressing for the same carrier at stage z+z', and in the dressing for a nest of carriers at stage z.

4.4.17. DEFINITION (of DG(J)). Let J be a mapping from $\mathbb{N}$ into K. DG(J) is a set of mappings $d: \mathbb{N} \times \mathrm{FRAME} \times \mathbb{N} \to K$, $d:(v,F,w) \mapsto d_F^v w$.

For $d_{(\circ n)}^v w$ we write $d_n^v w$, for $d_F^0 w$ we write $d_F w$, and we put $d_n w \equiv d_{(\circ n)}^0 w$.

d belongs to DG(J) iff

(a) $\lambda^K n \lambda^K w.d_n w$ belongs to $DG^0(J)$,

(b) $d_n^v w$ is the 'difference' between $d_n v$ and $d_n(v*w)$, and

(c) if ht(F)>0 then $d_F^v w$ is the $\delta_F v$-nesting of $\lambda^K n.d_n^v w$ (i.e. $d_F w$ is the F-nesting of $\lambda^K n.d_n w$, $d_F(\bar{\delta}x)$ behaves as the dressing $d_F x$ for ε_F at stage x, cf.2.10.5).

Formally, $d \in DG(J)$, iff

(a) $\lambda^K n.\lambda^K w.d_n w \in DG^0(J)$, i.e.

(i) $d_n 0 \simeq \mathrm{id}$,

(ii) $d_n(w*\hat{x}) \simeq d_n w : \nu^K_{\delta_n w} jf(w*\hat{x}) : \nu^K_{\delta_n(w*\hat{x})} gv(d^0, w*\hat{x})$,

where $d^0 \equiv \lambda^K n.\lambda^K w.d_n w$;

(b)(i) $d^v_n 0 \simeq \mathrm{id}$,

(ii) $d^v_n \hat{x} \simeq \nu^K_{\delta_n v} jf(v*\hat{x}) : \nu^K_{\delta_n(v*\hat{x})} gv(d^0, v*\hat{x})$,

where $d^0 \equiv \lambda^K n.\lambda^K w.d_n w$, and

(iii) $d^v_n(w*\hat{x}) \simeq d^v_n w : d^{v*w}_n \hat{x}$;

(c) $d^v_F w \simeq \nu^K_{\delta_F v}(\lambda^K n.d^v_n w)$, for frames F with $ht(F)>0$.

If $d \in DG(J)$ then d *generates a universe of dressing sequences w.r.t.* J.

4.4.18. LEMMA. *If* $d \in DG(J)$ *then* $d_F(\bar{\delta}(x+1))$ *determines a value for* x, *i.e. if* $d \in DG(J)$ *then*

(1) $\exists y \forall \phi[(d_F(v*\hat{x})|\phi)(\mathrm{lth}(v))=y]$.

PROOF. In lemma 4.4.16 we have proved this assertion for $F = (°n)$. For F with $ht(F)>0$ we argue as follows:

$d_F(v*\hat{x}) \simeq \nu^K_F(\lambda^K n.d_n(v*\hat{x}))$ by definition 4.4.17(c) and 4.3.14(a) ($\delta_F 0=F$). Hence

$$\forall b \in F(j_b(d_F(v*\hat{x})|\phi) = d_{\ell_b F}(v*\hat{x})|j_b\phi),$$

by 3.2.16(c).

So

$$\forall b \in F \exists z \forall \phi[j_b((d_F(v*\hat{x})|\phi)(\mathrm{lth}(v)))=z],$$

by 4.4.16, which immediately yields (1). □

4.4.19. LEMMA (the extension of a $d^0 \in DG^0(J)$ to a $d \in DG(J)$). *Let* $d^0 \in DG^0(J)$. *Define* $d : \mathbb{N} \times \mathrm{FRAME} \times \mathbb{N} \to K$ *by*:

(1) $d(0,(°n),v) = d^0_n v$,

(2) *if* $\mathrm{lth}(w)>0$: $d(w,(°n),v) = S^{\mathrm{UPB}(w)} : \nu^K_{\delta_n w}(\lambda^K m.d^0_m(w*v))$,

(3) *if* $ht(F)>0$: $d(w,F,v) = \nu^K_{\delta_F w}(\lambda^K n.d^w_n v)$,

where $d^w_n v$ *is* $d(w,(°n),v)$ *as defined in* (2) *and* (1). *Then* $d \in DG(J)$.

PROOF. d fulfills 4.4.17(a) and (c) by (1) and (3).
By (2)

$$d^w_n 0 = s^{UPB(w)} : \nu^K_{\delta_n w}(\lambda^K m. d^0_m w).$$

$$s^{UPB(w)} \simeq \nu^K_{\delta_n w}(\lambda^K m. s^{UPB(w)}) \quad \text{by 3.2.16(f),}$$

hence

(4) $$d^w_n 0 \simeq \nu^K_{\delta_n w}(\lambda^K m. (s^{UPB(w)} : d^0_m w)),$$

by distributivity of: over ν (3.2.16(e)). By lemma 4.3.12(c)
$\forall m \in \ell(\delta_n w)(\delta_n w \neq {}^\circ m)$, hence, by 4.4.14,
$\forall m \in \ell(\delta_n w) \exists u(\text{lth}(u) = UPB(w) \wedge d^0_m w \simeq [u])$, i.e.

(5) $$\forall m \in \ell(\delta_n w)(s^{UPB(w)} : d^0_m w \simeq id).$$

By (4), (5) and 3.2.17(d) we find that d fulfills 4.4.17(b)(i): $d^w_n 0 \simeq id$.
Also by (2):

(6) $$d^w_n(v * \hat{x}) = s^{UPB(w)} : \nu^K_{\delta_n w}(\lambda^K m. d^0_m(w * v * \hat{x})).$$

Since $d^0 \in DG^0(J)$, $d^0_m(w * v * \hat{x})$ is equivalent to

$$d^0_m(w * v) : \nu^K_{\delta_m(w*v)} jf(w * v * \hat{x}) : \nu^K_{\delta_m(w*v*\hat{x})} gv(w * v * \hat{x}).$$

Hence, by distributivity of: over ν (3.2.16(e))

(7) $$\nu^K_{\delta_n w}(\lambda^K m. d^0_m(w * v * \hat{x})) \simeq \nu^K_{\delta_n w}(\lambda^K m. d^0_m(w * v)) : \nu^K_{\delta_n w}\phi : \nu^K_{\delta_n w}\psi,$$

where

(8) $$\phi \equiv \lambda^K m. \nu^K_{\delta_m(w*v)} jf(w * v * \hat{x}),$$

and

(9) $$\psi \equiv \lambda^K m. \nu^K_{\delta_m(w*v*\hat{x})} gv(w * v * \hat{x}).$$

By 4.3.15

$$\delta_n w[\lambda m.\delta_m(w*v)] = \delta_n(w*v),$$

whence

$$\nu^K_{\delta_n(w*v)} jf(w*v*\hat{x}) \simeq \nu^K_{\delta_n w}(\lambda^K m.\nu^K_{\delta_m(w*v)} jf(w*v*\hat{x})) \text{ (3.2.17(b))},$$

i.e.

(10) $$\nu^K_{\delta_n w}\phi \simeq \nu^K_{\delta_n(w*v)} jf(w*v*\hat{x}).$$

Similarly

(11) $$\nu^K_{\delta_n w}\psi \simeq \nu^K_{\delta_n(w*v*\hat{x})} gv(w*v*\hat{x}).$$

By (6), (7), (10) and (11), $d^w_n(v*\hat{x})$ is equivalent to

$$s^{UPB(w)} : \nu^K_{\delta_n w}(\lambda^K m.d^0_m(w*v)) : e,$$

where

$$e \equiv \nu^K_{\delta_n(w*v)} jf(w*v) : \nu^K_{\delta_n(w*v*\hat{x})} gv(w*v*\hat{x}).$$

By (2)

$$s^{UPB(w)} : \nu^K_{\delta_n w}(\lambda^K m.d^0_m(w*v)) = d^w_n v,$$

whence

(12) $$d^w_n(v*\hat{x}) \simeq d^w_n v : \nu^K_{\delta_n(w*v)} jf(w*v*\hat{x}) : \nu^K_{\delta_n(w*v*\hat{x})} gv(w*v*\hat{x}).$$

(12) and 4.4.17(b)(i), which we proved above, yield 4.4.17(b) (ii) and (iii). □

4.4.20. <u>LEMMA</u>. $d : \mathbb{N} \times \text{FRAME} \times \mathbb{N} \to K$ *belongs to* DG(J) *iff for all* F *and* v:

(1) $$d^v_F 0 \simeq \text{id},$$

(2) $$d^v_F(w*\hat{x}) \simeq d^v_F w : \nu^K_{\delta_F(v*w)} jf(v*w*\hat{x}) : \nu^K_{\delta_F(v*w*\hat{x})} gv(d^0, v*w*\hat{x}),$$

where $d^0 : \mathbb{N} \times \mathbb{N} \to K$ *is defined by* $d^0(n,v) = d^0_{(^\circ n)} v$.

PROOF.

($\leftarrow$) If we take $v = 0$, $F = (^\circ n)$ in (1) and (2) we find that $d^0 \in DG^0(J)$.

(b)(i) follows by (1), (b)(ii) by (1) and (2), (b)(iii) by (2) and (b)(ii).

(c) By induction w.r.t. lth(w):

(i) $d^v_F 0 \simeq id$ by (1), $id \simeq \nu^K_{\delta_F v}(\lambda^K n.id)$ by 3.2.16(f), $\lambda^K n.id = \lambda^K n.d^v_n 0$ by (1), hence $d^v_F 0 = \nu^K_{\delta_F v}(\lambda^K n.d^v_n 0)$.

(ii) Assume

(3) $$d^v_F w \simeq \nu^K_{\delta_F v}(\lambda^K n.d^v_n w) \text{ (induction-hypothesis).}$$

$\delta_F(v*w) = \delta_F v[\lambda n.\delta_n(v*w)]$ by 4.3.15, hence

(4) $$\nu^K_{\delta_F(v*w)} jf(v*w*\hat{x}) = \nu^K_{\delta_F v}(\lambda^K n.\nu^K_{\delta_n(v*w)} jf(v*w*\hat{x})) \text{ by 3.2.17(b).}$$

Likewise

(5) $$\nu^K_{\delta_F}(v*w*\hat{x}) gv(d^0, v*w*\hat{x}) = \nu^K_{\delta_F v}(\lambda^K n.\nu^K_{\delta_n(v*w*\hat{x})} gv(d^0, v*w*\hat{x})).$$

Substitute (3), (4) and (5) in (2) and apply distributivity of: over nesting, (3.2.16(e)), this yields

(6) $$d^v_F(w*\hat{x}) \simeq \nu^K_{\delta_n v}(\lambda^K n.d^v_n w : \nu^K_{\delta_n(v*w)} jf(v*w*\hat{x}) : \nu^K_{\delta_n(v*w*\hat{x})} gv(d^0, v*w*\hat{x})),$$

i.e. by (2)

$$d^v_F(w*\hat{x}) \simeq \nu^K_{\delta_n v}(\lambda^K n.d^v_n(w*\hat{x})).$$

($\rightarrow$) If $d \in DG(J)$ then, by 4.4.17(a) and (b)

(7) (1) and (2) hold for $F = (^\circ n)$.

If ht(F)>0, then (1) follows from 4.4.17(b) and (c) by 3.2.16(f). By 4.4.17(c) and (7) we find for F with ht(F)>0:

$$d_F^v(w*\hat{x}) \simeq \nu^K_{\mathfrak{f}_F v}(\lambda^K n.d_n^v w : \nu^K_{\mathfrak{f}_n(v*w)} jf(v*w*\hat{x}) : \nu^K_{\mathfrak{f}_n(v*w*\hat{x})} gv(d^0, v*w*\hat{x}))$$

whence by distributivity of : over ν^K, (4) and (5)

$$d_F^v(w*\hat{x}) \simeq \nu^K_{\mathfrak{f}_F v}(\lambda^K n.d_n^v w) : \nu^K_{\mathfrak{f}_F(v*w)} jf(v*w*\hat{x}) : \nu^K_{\mathfrak{f}_F(v*w*\hat{x})} gv(d^0, v*w*\hat{x})$$

and hence, by 4.4.17(c), (2). □

4.4.21. <u>LEMMA</u>. *If* d ∈ DG(J) *then*

(a) $d_F^v 0 \simeq id$

(b) $d_F^v w \simeq \nu^K_{\mathfrak{f}_F v}(\lambda^K n.d_n^v w)$

(c) $d_F^u(v*w) \simeq d_F^u v : d_F^{u*v} w$

(d) *if* $\forall n(Jn \in C)$, $\forall v([v] \in C)$ *and* C *is closed under* : *and* ∧ *then* $d_n^v w \in C$

(e) $\forall v \exists a \forall n(\mathfrak{f}_n v = {}^\circ n \to d_n v \simeq [an])$.

<u>PROOF</u>. (a) by definition, (d) trivial, (e) by lemma 4.4.14. (b) is a corollary to the proof of 4.4.20: in the proof of 4.4.17(c) from 4.4.20(1) and (2), we do not use the assumption ht(F)>0. For (c) we use the characterization of DG(J) in lemma 4.4.20. We proceed by induction w.r.t. lth(w):

(i) w = 0: $d_F^u(v*0) = d_F^u v \simeq d_F^u v : id$, and $id \simeq d_F^{u*v} 0$ by 4.4.20(1).

(ii) $w = w'*\hat{x}$: by 4.4.20(2)

(1) $$d_F^u(v*w) \simeq d_F^u(v*w') : \nu^K_{\mathfrak{f}_F(u*v*w')} jf(u*v*w) : \nu^K_{\mathfrak{f}_F(u*v*w)} gv(u*v*w).$$

By induction hypothesis

(2) $$d_F^u(v*w') \simeq d_F^u v : d_F^{u*v} w'.$$

By 4.4.20(2)

(3) $$d_F^{u*v} w' : \nu^K_{\mathfrak{f}_F(u*v*w')} jf(u*v*w) : \nu^K_{\mathfrak{f}_F(u*v*w)} gv(u*v*w) \simeq d_F^{u*v} w.$$

If we substitute (2) in (1) and apply (3) we find $d_F^u(v*w) \simeq d_F^u v : d_F^{u*v} w$. □

4.5. PROJECTED UNIVERSES OF NESTS OF GC-CARRIERS

4.5.1. DEFINITION. A mapping $J : \mathbb{N} \to K$ *enumerates the subset* C *of* K *modulo equivalence* (or modulo $\simeq$) iff $e \in C \leftrightarrow \exists n(Jn \simeq e)$.

4.5.2. DEFINITION (of 'to generate nests of GC-carriers' and of $CU_\delta(C)$).

(a) A mapping $\pi : F \to \pi_F$ from FRAME into K *generates nests of* GC-*carriers w.r.t.* $C \subset K$ iff there are a $J : \mathbb{N} \to K$ which enumerates C modulo equivalence and a $d \in DG(J)$ such that, for all F, $\pi_F|\delta$ is the intersection of the ranges of the mappings $d_F(\bar{\delta}x)|\cdot$, more precisely, such that

$$\pi_F(\hat{x}*w) = y+1 \to \forall a[(d_F w|a)(x) = y].$$

(Cf.2.9.9,(3)-(5) and 4.4.18.) We abbreviate $\pi_F|\delta$ to $\pi_F\delta$.

(b) If π generates nests of GC-carriers w.r.t. C, J enumerates C modulo $\simeq$ and $d \in DG(J)$ satisfies

$$\pi_F(\hat{x}*w)=y+1 \to \forall a[(d_F w|a)(x)=y],$$

then $d_F(\bar{\delta}x)$ is the *dressing for* $\pi_F\delta$ *at stage* x, d *generates the dressings for* π. $\mathfrak{f}_F(\bar{\delta}x)$ is the *frame for* $\pi_F\delta$ *at stage* x, and the pair $(d_F(\bar{\delta}x), \mathfrak{f}_F(\bar{\delta}x))$ is the *restriction for* $\pi_F\delta$ *at stage* x.
Instead of dressing, frame and restriction for $\pi_F\delta$, we shall also say dressing, frame and restriction for π_F.

(c) $CU_\delta(C)$ is the set of all universes $\mathcal{U}_\delta$ of the form

$$\mathcal{U}_\delta \equiv \{\pi_F\delta : F \in \text{FRAME}\},$$

where π generates nests of GC-carriers w.r.t. C. An element $\mathcal{U}_\delta \in CU_\delta(C)$ is a *projected universe of nests of GC-carriers w.r.t.* C.

(d) We write π_n for $\pi_{(^\circ n)}$. If $\mathcal{U}_\delta \in CU_\delta(C)$, then the subuniverse

$$\{\pi_n\delta : n \in \mathbb{N}\} \subset \mathcal{U}_\delta$$

is a *projected universe of* GC-*carriers w.r.t.* C. An element $\pi_n\delta \in \mathcal{U}_\delta$ is a *carrier of* $\mathcal{U}_\delta$.

4.5.3. REMARK. The elements $\pi_F\delta$ of a universe $\mathcal{U}_\delta \in CU_\delta(C)$ are to imitate the nests of carriers ε_F (w.r.t.C). This is clear for the carriers $\pi_n\delta$ of

$\mathcal{U}_\delta$. For frames F with ht(F)>0, we have defined

$$\varepsilon_F \equiv \nu_F^1(\lambda^1 n.\varepsilon_{\underline{n}}),$$

(cf. 2.10.1) while here we put

$$\pi_F\delta \equiv \bigcap_x \text{range }(\lambda\phi.d_F(\bar\delta x)|\phi).$$

In lemma 4.5.5 below we shall prove that

$$\pi_F\delta = \nu_F^1(\lambda^1 n.\pi_n\delta).$$

4.5.4. LEMMA. *If* π *generates nests of* GC*-carriers w.r.t.* C *and* d *generates the dressings for* π*, then*

$$\pi_F\delta(x) = y \leftrightarrow \forall a[(d_F(\bar\delta(x+1))|a)(x)=y].$$

PROOF. By lemma 4.4.18

$$\exists z\forall a[(d_F(\bar\delta(x+1))|a)(x)=z],$$

hence it suffices to show that

$$\pi_F\delta(x)=y \wedge \forall a[(d_F(\bar\delta(x+1))|a)(x)=z] \rightarrow y=z.$$

If $\pi_F\delta(x) = y$, then $\pi_F(\hat{x}*\bar\delta(k+1)) = y+1$ for some k, hence

$$\forall a[(d_F(\bar\delta(k+1))|a)(x)=y] \quad \text{(by definition)}.$$

Now assume that we also have

$$\forall a[(d_F(\bar\delta(x+1))|a)(x)=z].$$

If $k \geq x$, then $d_F(\bar\delta(k+1)) \simeq d_F(\bar\delta(x+1)) : e$ for some e. (4.4.21(c)) Hence $d_F(\bar\delta(k+1))|a = d_F(\bar\delta(x+1))|b$ for $b=e|a$, this yields y=z.
If $k < x$ then $d_F(\bar\delta(x+1)) = d_F(\bar\delta(k+1)) : e$ for some e, and then also y=z. □

4.5.5. LEMMA. *If* π *generates nests of* GC-*carriers w.r.t.* C *then*

$$\forall b{\in}F\forall n[\ell_b F{=}n \rightarrow j_b(\pi_F\delta){=}\pi_n\delta],$$

i.e. $\pi_F\delta /\!\!/ F$, $\pi_F\delta = \nu^1_F(\lambda^1 n.\pi_n\delta)$.

PROOF. Let $b \in F$ have the label n, assume that

(1) $\qquad \pi_F\delta(x) = y.$

We show

(2) $\qquad \pi_n\delta(x) = j_b y.$

Let d generate the dressings for π, then (1) is equivalent to

(3) $\qquad \forall a[(d_F(\bar\delta(x+1))|a)(x){=}y],$

by the previous lemma. By lemma 4.4.21(b) and 4.3.14(a)

$$d_F(\bar\delta(x+1)) \simeq \nu^K_F(\lambda^K n.d_n(\bar\delta(x+1))),$$

so

$$j_b(d_F(\bar\delta(x+1))|a) = d_n(\bar\delta(x+1))|j_b a \qquad (3.2.16(c)),$$

whence

$$\forall b[(d_n(\bar\delta(x+1))|b)(x){=}j_b y],$$

by (3), and hence (2) by 4.5.4. □

4.5.6. LEMMA. *Let* $J : \mathbb{N} \rightarrow K$ *enumerate* C *modulo* $\simeq$, *let* d *be an element of* DG(J). *Define* $\pi : F \rightarrow \pi_F$ *from* FRAME *into* K *by*

$$\pi_F 0{=}0, \quad \pi_F(\hat{x}{*}w){=}sg(lth(w)\dot{-}x)\cdot[d_F(\bar{w}(x+1))(\hat{x}{*}w)],$$

where $\bar{w}(x+1) \equiv \overline{w{*}\lambda z.0}(x+1)$. *Then* π *generates nests of* GC-*carriers w.r.t.* C.

PROOF.

(a) ($\pi_F \in K$). Put $e_x \equiv \lambda u.sg(lth(u) \dot{-} x)$, then $e_x \in K$ by 1.3.14 and

$$e_x u \neq 0 \rightarrow \lambda w.\pi_F(\hat{x}*u*w) = \lambda w.[d_F\bar{u}(x+1)(\hat{x}*u*w)] \in K,$$

(since $d_F\bar{u}(x+1) \in K$) hence, by 1.3.13(3) $\forall x(\pi_F(\hat{x}*w) \in K)$, whence, by (K2), $\pi_F \in K$.

(b) ($\pi_F(\hat{x}*w) = y+1 \rightarrow \forall a[(d_F w|a)(x) = y]$). If $\pi_F(\hat{x}*w) = y+1$ then $w = v*\hat{z}*u$, where $lth(v) = x$, and

$$\forall a \in w[(d_F(v*\hat{z})|a)(x) = (d_F(v*\hat{z})|a)(lth(v)) = y].$$

Now apply 4.4.18. □

4.5.7. REMARK. Let π generate nests of GC-carriers w.r.t. C and let d generate dressings for π. From lemma 4.4.21(e) we know that if $\pi_n\delta$ is fresh at stage x, i.e. if $\mathfrak{f}_n(\bar{\delta}x) = {}^\circ n$, then $d_n(\bar{\delta}x) \simeq [a\hat{n}]$ for some $a : \mathbb{N} \rightarrow \mathbb{N}$. That is to say, if $\mathfrak{f}_n(\bar{\delta}x) = {}^\circ n$, then the empty part of $\pi_n\delta$ at stage x, i.e. the part of $\pi_n\delta$ that is not yet available at stage x, is $s^{lth(an)} | \pi_n\delta$.

The source for a carrier ε_m at stage x is represented by substituting the empty part of ε_n at stage x for each occurrence of the label n in the frame for ε_n at stage x (cf.4.4.5). So the source for $\pi_m\delta$ at stage x is

$$\nu^1_{\mathfrak{f}_m(\bar{\delta}x)}(\lambda^1 n.s^{lth(an)} | \pi_n\delta).$$

ε_m is related to its source src(m,x) at stage x via $d_m x$, its dressing at stage x, by the equation $\varepsilon_m = d_m x \mid src(m,x)$. (Cf.4.4.5.)

For $\pi_m\delta$ we can prove the corresponding equation

$$\pi_m\delta = d_m(\bar{\delta}x) \mid (\nu^1_{\mathfrak{f}_m(\bar{\delta}x)}(\lambda^1 n.s^{lth(an)}|\pi_n\delta)).$$

We postpone the proof till chapter 6 (6.3.4(d)).

4.6. PROJECTED UNIVERSES OF GC-SEQUENCES W.R.T. C

4.6.1. DEFINITION. $U_\delta(C)$ is the set of all universes $\mathcal{U}_\delta$ of the form

$$\mathcal{U}_\delta \equiv \{e|\pi_F\delta : e \in C, F \in FRAME\},$$

where π generates nests of carriers w.r.t. C. If C is dependency-closed, then a universe $\mathcal{U}_\delta \in U_\delta(C)$ is a *projected universe of* GC-*sequences w.r.t.* C.

This is completely analogous to the definition of GC(C) from GCC(C).

4.6.2. DEFINITION. Let $\mathcal{U}_\delta \equiv \{e|\pi_F\delta : e \in C, F \in \text{FRAME}\}$ belong to $U_\delta(C)$, and let d generate dressings for π.
(e,F) is the *initial restriction for* $e|\pi_F\delta \in \mathcal{U}_\delta$, e is *the initial dressing for* $e|\pi_F\delta$, F its *initial frame.*
$(e:d_F(\bar{\delta}x), \mathfrak{f}_F(\bar{\delta}x))$ is *the restriction for* $e|\pi_F\delta$ *at stage* x, $e : d_F(\bar{\delta}x)$ is *the dressing for* $e|\pi_F\delta$ *at stage* x, $\mathfrak{f}_F(\bar{\delta}x)$ is *the frame for* $e|\pi_F\delta$ *at stage* x.

4.6.3. LEMMA. *If* $C \subset K$ *is dependency-closed and* $J : \mathbb{N} \to K$ *enumerates* C *modulo* $\simeq$, *then* $U_\delta(C)$ *is not empty: there exists a projected universe of* GC-*sequences w.r.t.* C.

PROOF. It suffices to show that there is a π which generates nests of GC-carriers. By 4.5.6 the problem is reduced to showing that DG(J) contains an element d. This follows from 4.4.19 and the fact that there is a $d^0 \in DG^0(J)$ (4.4.12). □

4.7. At any stage in the construction of the lawless sequence δ, there is only an initial segment of that sequence available to us. If at stage z we have generated the initial segment $\bar{\delta}x$, then we can make no prediction whatsoever about the $\delta(x+y)$ yet to be determined.

Part of the lawless behaviour of δ is reflected in the behaviour of the sequence of restrictions $\lambda x.(d_F(\bar{\delta}x), \mathfrak{f}_F(\bar{\delta}x))$ for $\pi_F\delta$ in a projected universe of nests of GC-carriers, but not all.
E.g. we know that $\mathfrak{f}_F(\bar{\delta}(x+y))$ can be produced from $\mathfrak{f}_F(\bar{\delta}x)$ by a lawlike $g : \mathbb{N} \to \text{FRAME}$, and that

$$d_F(\bar{\delta}(x+y)) \simeq d_F(\bar{\delta}x) : d_F^{\bar{\delta}x}w,\; d_F^{\bar{\delta}x}w \simeq \nu^K_{\mathfrak{f}_F(\bar{\delta}x)}(\lambda^K n.d_n^{\bar{\delta}x}w),$$

where $w = \overline{\lambda z.\delta(x+y)}(z)$. (Cf.4.3.15(b), 4.4.21(b) and (c).) Moreover, we know that $d_F(\bar{\delta}(x+y))$ will determine values for the arguments $0,\dots x+y\dot{-}1$.

The next chapter is devoted to the question of the freedom of continuation for sequences of restrictions $\lambda x.(d_F(\bar{\delta}x), \mathfrak{f}_F(\bar{\delta}x))$.

CHAPTER 5

THE ORDERING OF RESTRICTIONS AND THE OVERTAKE PROPERTY

5.1. THE ORDERING OF RESTRICTIONS

5.1.1. The frame for π_F at $v{*}w(\delta_F(v{*}w))$, can be produced from the frame for π_F at $v(\delta_F v)$, i.e.

(1) $\exists g(\delta_F(v{*}w) = \delta_F v[g])$, or shortly, $\delta_F(v{*}w) \geq \delta_F v$ (4.3.14(b)).

If $d \in DG(J)$, then

(2) $d_F(v{*}w) \simeq d_F v : d^v_F w$ (4.4.21(c)),

and

(3) $d^v_F w \simeq \nu^K_{\delta_F v}(\lambda^K n.d^v_n w)$ (4.4.21(b)).

Moreover, if J enumerates C modulo $\simeq$ and C is dependency-closed, then

(4) $\forall n(d^v_n w \in C)$ (4.4.21(d)).

Hence

(5) $\exists g /\!/_C \, \delta_F v(d_F(v{*}w) \simeq d_F v{:}g)$ (by (2), (3) and (4)).

5.1.2. DEFINITION (of stronger than between restrictions). Let (e,F) and (f,G) be two restrictions. (e,F) is *stronger than* (f,G), or equivalently, (f,G) is *weaker than* (e,F), iff it is consistent with (1) and (5) above that (f,G) is the restriction for a projected nest of carriers $\pi_H\delta$ at stage x, and (e,F) is the restriction for the same sequence at some stage $x' \geq x$.

We denote (e,F) is stronger than (f,G) by $(e,F) \geq (f,G)$ or by

$(f,G) \leq (e,F)$. In formula:

$$(e,F) \geq (f,G) \equiv (f,G) \leq (e,F) \equiv F \geq G \wedge \exists g /\!/_C G(e \simeq f{:}g).$$

5.1.3. <u>REMARK</u>. The terminology and the notation are not quite accurate. Instead of 'stronger than' we should say 'stronger than w.r.t. $C \subset K$', instead of $\geq$ we should use $\geq_C$. Since we shall use $\geq$ only w.r.t. subsets of K denoted by C, this omission will not cause confusion.

5.1.4. <u>FACT</u>. If $d \in DG(J)$, J enumerates C modulo $\simeq$, and C is dependency-closed, then

$$(d_F(v*w), \delta_F(v*w)) \geq (d_F v, \delta_F v) \quad \text{(cf.5.1.1)}.$$

5.1.5. <u>DEFINITION</u> (of equivalence between restrictions). Two restrictions (e,F) and (f,G) are *equivalent*, which we denote by $(e,F) \approx (f,G)$, iff (e,F) is both stronger and weaker than (f,G), i.e.

$$(e,F) \approx (f,G) \equiv (e,F) \geq (f,G) \wedge (e,F) \leq (f,G).$$

5.1.6. <u>LEMMA</u> (properties of $\geq$ and $\approx$).

(a) *If* $id \in C$ *then* $(e \simeq f) \wedge (F \approx G) \rightarrow (e,F) \approx (f,G)$.

(b) *If* C *is closed under* : *and* $\wedge$ *then* $\geq$ *is transitive, i.e.*
$(e,F) \geq (f,G) \wedge (f,G) \geq (g,H) \rightarrow (e,F) \geq (g,H)$.

(c) *If* $\forall v([v] \in C)$ *then* $\forall y /\!/ F\, ((e{:}[y],F) \geq (e,F))$.

(d) $(f,F) \geq (g,G) \rightarrow (e{:}f,F) \geq (e{:}g,G)$.

<u>PROOF</u>.

(a) If $F \approx G$ then $F \geq G$ and $F \leq G$ by definition 3.1.20. If $e \simeq f$ then $e \simeq f{:}id$ and $f \simeq e{:}id$, while if $id \in C$ then $\forall H(id /\!/_C H)$ by 3.2.20(o).

(b) If $F \geq G \geq H$ then $F \geq H$ by 3.1.19(e). Assume $e \simeq f{:}g_1, g_1 /\!/_C G$ and $f \simeq g{:}g_2$, $g_2 /\!/_C H$. Then $g_1 /\!/_C H$, since $G \geq H$ and C is closed under $\wedge$ (3.2.20(k)), and $g_2{:}g_1 /\!/_C H$ (3.2.20(s)), i.e. $e \simeq g{:}(g_2{:}g_1)$, $g_2{:}g_1 /\!/_C H$.

(c) If $\forall v([v] \in C)$ and $y /\!/ F$ then $[y] /\!/_C F$ by 3.2.21(i).

(d) If $f \simeq g{:}g'$, $g' /\!/_C G$, then $e{:}f \simeq (e{:}f){:}g'$, $g' /\!/_C G$. □

5.1.7. <u>COROLLARIES</u>.

(a) *If* id $\in$ C *then* (e,F) $\approx$ (e,F). [By 5.1.6(a).]

(b) *If* C *is closed under pairing and composition, then* $\geq$ *respects* $\approx$, *and* $\approx$ *is transitive.* [*By* 5.1.6(b).]

We shall give more properties of $\geq$ and $\approx$ in chapter 7. Note that the conditions on C in 5.1.6(a)-(c) and 5.1.7 are all fulfilled if C is dependency-closed.

5.2. FREEDOM OF CONTINUATION FOR SEQUENCES OF RESTRICTIONS: THE 'STRONG OVERTAKE PROPERTY'.

5.2.1. First we formulate the (false) principle of 'full freedom of continuation for sequences of restrictions':

Let $C \subset K$ be dependency-closed, let $J: \mathbb{N} \to K$ enumerate C modulo $\simeq$, let $d \in DG(J)$ and let $\mathfrak{f}_F v$ be as defined in 4.3.9, 4.3.12. Then we can find, for each restriction (e,F) stronger than $(d_F(\bar{\delta}x),\mathfrak{f}_F(\bar{\delta}x))$ a lawless sequence $\delta' \in \bar{\delta}x$ and a $y \geq x$ such that $(d_F(\bar{\delta}'y),\mathfrak{f}_F(\bar{\delta}'y)) \approx (e,F)$, i.e. each restriction stronger than the restriction at stage x can be reached at a stage $y \geq x$; in a formula:

$$\forall(e,F)\geq(d_F v,\mathfrak{f}_F v)\exists w((e,F) \approx (d_F(v*w),\mathfrak{f}_F(v*w))).$$

This principle leads to a contradiction. Consider the sequence of restrictions $\{(s^n,{}^\circ m) : n \in \mathbb{N}\}$. By full freedom of continuation for sequences of restrictions, there is a $\phi \in N$ such that

(1) $$\forall n\exists x[(d_m(\bar{\phi}x),\mathfrak{f}_m(\bar{\phi}x)) \approx (s^n,{}^\circ m)].$$

On the other hand, the determination of a value for the argument zero must be guaranteed, i.e.

$$\forall\delta\exists z\exists y\forall a[(d_m(\bar{\delta}z)|a)(0)=y].$$

By the extension principle we find a z such that for the ϕ of (1)

(2) $$\exists y\forall a[(d_m(\bar{\phi}z)|a)(0)=y].$$

By (1) there are $n \in \mathbb{N}$ and $e /\!/_C {}^\circ m$ such that $s^n \simeq d_m(\bar{\phi}z):e$, whence by (2) $\exists y \forall a[(s^n|a)(0) = y]$, which is obviously false.

Note that the contradiction arises from the fact that we have to guarantee the determination of a value for each argument, and not from the method by which this guarantee is provided.

5.2.2. With each $e \in K$ and $n \in \mathbb{N}$ we can find an $f \in K$ such that if w lies in the bar f, i.e. $fw \neq 0$, then $e:[w]$ determines a value for all arguments $m \leq n$, i.e.

$$\forall w[fw \neq 0 \rightarrow \forall m \leq n \exists y \forall a((e:[w]|a)(m)=y)].$$

We might replace the principle of full freedom of continuation for sequences of restrictions by the following:

Let C,J, d and $\delta_F v$ be as above. Then

(1) $$\forall (e,G) \geq (d_F v, \delta_F v) \forall \phi /\!/ G \; \exists xw[(d_F(v*w), \delta_F(v*w)) \approx (e:[\bar{\phi}x],G)],$$

i.e. we can 'overtake' each restriction (e,G) stronger than the restriction $(d_F(\bar{\delta}z), \delta_F(\bar{\delta}z))$ at stage z, and reach a restriction of the form $(e:[\bar{\phi}x],G)$ stronger than (e,G) at some stage $z' \geq z$. The finite sequences u for which $(e:[u],G)$ can be reached form a bar in the set of sequences $\{\phi \in N : \phi /\!/ G\}$.

This principle is valid, as will be shown below. A somewhat weaker formulation is:

Let C,J,d, and $\delta_F v$ be as before. Then

(2) $$\forall (e,G) \geq (d_F v, \delta_F v) \forall \phi /\!/ G \exists xw[(e,G) \leq (d_F(v*w), \delta_F(v*w)) \leq (e:[\bar{\phi}x],G)],$$

which says that we can 'overtake' (e,G) and reach a restriction which lies between (e,G) and $(e:[\bar{\phi}x],G)$. Obviously (1) implies (2), hence this principle is also valid.

5.2.3. If $(e,G) \geq (d_F v, \delta_F v)$ then $G \geq \delta_F v$ and

(3) $$e \simeq d_F v:f, \text{ for some } f /\!/_C \delta_F v.$$

By 4.4.21(c) we have for $d \in DG(J)$

(4) $$d_F(v*w) \simeq d_F v : d_F^v w.$$

So we can replace $(e,G) \le (d_F(v*w), \delta_F(v*w)) \le (e:[\bar{\phi}x],G)$ by

$$(d_F v:f,G) \le (d_F v:d_F^v w, \delta_F(v*w)) \le (d_F v:(f:[\bar{\phi}x]),G),$$

which is equivalent by (3), (4), 5.1.7(a) and (b).

We change 5.2.2(2) into:

Let C,J,d and $\delta_F v$ be as before. Then

(5) $$\forall f //_C\, \delta_F v \forall G \ge \delta_F v \forall \phi // G \exists wx[(f,G) \le (d_F^v w, \delta_F(v*w)) \le (f:[\bar{\phi}x],G)],$$

i.e. instead of $d_F(v*w)$ overtaking e, we now have $d_F^v w$ (the difference between $d_F v$ and $d_F(v*w)$), overtaking the difference between $d_F v$ and e.

(5) implies (2) by the remarks above and 5.1.6(d). (5) is valid, in fact we can prove a stronger form, with $(d_F^v w, \delta_F(v*w)) \approx (f:[\bar{\phi}x],G)$ instead of $(f,G) \le (d_F^v w, \delta_F(v*w)) \le (f:[\bar{\phi}x],G)$.

In the final formulation of the 'overtake property', we replace $\forall\phi//G\exists x A(\bar{\phi}x)$ by the stronger $\exists e \forall u // G\,[eu \ne 0 \to Au]$, i.e.

5.2.4. <u>DEFINITION</u> (of overtake property and strong overtake property). Let d: $\mathbb{N} \times$ FRAME $\times\, \mathbb{N} \to K$, δ: FRAME $\times\, \mathbb{N} \to$ FRAME be two lawlike mappings, put $d_F^v w \equiv d(v,F,w)$, $\delta_F v \equiv \delta(F,v)$.

(a) The pair (d,δ) has the *overtake property* iff

(6) $$\forall f //_C\, \delta_F v \forall G \ge \delta_F v \exists e \forall u // G\,[eu \ne 0 \to \exists w((f,G) \le (d_F^v w, \delta_F(v*w)) \le (f:[u],G))].$$

(b) (d,δ) has the *strong overtake property* iff

(7) $$\forall f //_C\, \delta_F v \forall G \ge \delta_F v \forall g \exists e \forall u // G\,[eu \ne 0 \to$$

$$\exists w(gw \ne 0 \wedge (f,G) \le (d_F^v w, \delta_F(v*w)) \le (f:[u],G))],$$

that is to say, the strong overtake property does not only claim that we can overtake (f,G) by choosing the right w, thereby remaining below a 'bar of restrictions' of the form (f:[u],G), but also that we can choose w in a bar given by g.

5.2.5. LEMMA (the strong overtake property for the projections of chapter 4). *Let* C *be dependency-closed, let* J *enumerate* C *modulo* $\simeq$, *let* d *be an element of* DG(J) *and let* $\mathfrak{f}:(F,v) \mapsto \mathfrak{f}_F v$ *be as defined in* 4.3.9, 4.3.12. *Then* $(d,\mathfrak{f})$ *has the strong overtake property.*

5.3. THE PROOF OF LEMMA 5.2.5

The proof of the validity of the strong overtake property is a long and complicated one. In this section we shall outline the proof, using some examples. We present the details in 5.4. The reader is advised to skip those details at first reading. If one is willing to accept lemma 5.2.5 without proof, one can skip even this section and continue with chapter 6.

5.3.1. Throughout the rest of this chapter
C is a dependency-closed subset of K,
$J: \mathbb{N} \to K$ is lawlike and enumerates C modulo $\simeq$,
d is an element of DG(J), and
for all F and v, $\mathfrak{f}_F v$ is the frame for π_F at v.

5.3.2. We show that for all F, v and g

$$(1)\quad \forall G \geq \mathfrak{f}_F v \forall f /\!/_C\, \mathfrak{f}_F v \forall \phi \in CSL \exists x[\bar{\phi}x /\!/ G \to \exists w(gw \neq 0 \wedge d_F^{\vee} w \simeq f:[\bar{\phi}x] \wedge \mathfrak{f}_F(v*w) \approx G)],$$

where CSL (for 'continuous image of a single lawless sequence') is the set $\{e|\alpha : e \in K,\ \alpha \in LS\}$. In words: $(d_F^{\vee} w, \mathfrak{f}_F(v*w))$ can overtake the restriction (f,G), $f /\!/_C\, \mathfrak{f}_F v$, $G \geq \mathfrak{f}_F v$, and reach a restriction $(f:[\bar{\phi}x],G)$ for any ϕ of the form $e|\alpha$, which has a sufficiently long initial segment $\bar{\phi}x$ parallel to G. In overtaking w reaches the bar g.

The strong overtake property for $(d,\mathfrak{f})$ states that there is a bar given by an $e \in K$, such that $(d_F^{\vee} w, \mathfrak{f}_F(v*w))$ can overtake (f,G), $f /\!/_C\, \mathfrak{f}_F v$, $G \geq \mathfrak{f}_F v$ and reach a restriction which lies between (f,G) and $(f:[u],G)$, for any $u /\!/ G$ in the bar e. Again, in overtaking w reaches the bar g. In formula: for all F, v, and g

$$(2)\quad \forall G \geq \mathfrak{f}_F v \forall f /\!/_C\, \mathfrak{f}_F v \exists e \forall u /\!/ G\,[eu \neq 0 \to \exists w(gw \neq 0 \wedge (f,G) \leq (d_F^{\vee} w, \mathfrak{f}_F(v*w)) \leq (f:[u],G))].$$

LEMMA. (1) *implies* (2).

This is proved by an appeal to the continuity axiom

(3) $\forall\alpha\exists x A(\alpha,x) \to \exists e\forall u[eu\neq 0 \to \forall\alpha\in u A(\alpha, eu\dot{-}1)]$.

The proof is relatively simple. The reader can skip it and continue with 5.3.3.

PROOF. Let $G \geq \delta_F v$ and $f /\!/_C \delta_F v$ be arbitrary and put

$$A(\phi,x) \equiv [\bar{\phi}x /\!/ G \to \exists w(gw\neq 0 \wedge \check{d}_F w \simeq f:[\bar{\phi}x] \wedge \delta_F(v*w)\approx G)].$$

Assume (1), then in particular $\forall\alpha\exists x A(\alpha,x)$ and hence, by (3), there is an e' such that

(4) $\forall u[e'u\neq 0 \to \forall\alpha\in u A(\alpha, e'u \dot{-} 1)]$.

Define e by $eu = e'u\cdot sg(lth(u)\dot{-}e'u)$, then

(5) $eu\neq 0 \to eu=e'u$,

and

(6) $eu\neq 0 \to eu<lth(u)$.

We prove

$$\forall u /\!/ G\, [eu\neq 0 \to \exists w(gw\neq 0 \wedge (f,G)\leq(\check{d}_F w, \delta_F(v*w))\leq(f:[u],G))].$$

Let $u /\!/ G$ be arbitrary and assume that $eu \neq 0$. Then $\forall\alpha\in u A(\alpha, eu\dot{-}1)$ by (4) and (5), i.e.

(7) $\forall\alpha\in u[\bar{\alpha}(eu\dot{-}1) /\!/ G \to \exists w(gw\neq 0 \wedge \check{d}_F w \simeq f:[\bar{\alpha}(eu\dot{-}1)] \wedge \delta_F(v*w)\approx G)]$.

By (6) and the assumptions $eu \neq 0$, $u /\!/ G$ we have $u = u_1 * u_2$, where $lth(u_1) = eu\dot{-}1$, $u_1 /\!/ G$ and $u_2 /\!/ G$. Hence, if $\alpha \in u$ then $\bar{\alpha}(eu\dot{-}1) = u_1$, $u_1 /\!/ G$. I.e. (7) yields a w which satisfies

(8) $gw\neq 0 \wedge \check{d}_F w \simeq f:[u_1] \wedge \delta_F(v*w) \approx G$.

$\check{d}_F w \simeq f:[u_1]$, $u_1 /\!/ G$ and $\delta_F(v*w) \approx G$ imply

$$(f,G) \leq (\check{d}_F w, \delta_F(v*w)) \quad \text{(by 5.1.6(c),(a))};$$

$\check{d}_F w \simeq f:[u_1]$, $f:[u] \simeq f:[u_1]:[u_2]$, $u_2 /\!/ G$ and $\delta_F(v*w) \approx G$ imply

$$(d_F^v w, \delta_F(v*w)) \leq (f:[u],G) \text{ (also by 5.1.6(c),(a))}.$$

So (8) yields $gw \neq 0 \wedge (f,G) \leq (d_F^v w, \delta_F(v*w)) \leq (f:[u],G)$. □

Note that we apply (3) in this proof with a formula A not in the language of LS. A is a formula of LSF, a definitional extension of LS to be discussed in chapter 7, i.e. A can be translated into an LS formula.

5.3.3. We can split 5.3.2(1) into two 'semi-overtake properties' and a 'continuation till bar property': for all F, v and g

(1) $$\forall f /\!/_C \delta_F v \forall \phi \in CSL \exists x[\bar{\phi}x /\!/ \delta_F v \rightarrow \exists w(d_F^v w \simeq f:[\bar{\phi}x] \wedge \delta_F(v*w) \approx \delta_F v)]$$

(i.e. $d_F^v w$ can overtake $f /\!/_C \delta_F v$, while the frame remains equivalent),

(2) $$\forall G \geq \delta_F v \forall \phi \in CSL \exists x[\bar{\phi}x /\!/ G \rightarrow \exists w(d_F^v w \simeq [\bar{\phi}x] \wedge \delta_F(v*w) \approx G)]$$

(i.e. $\delta_F(v*w)$ can overtake $G \geq \delta_F v$, while the dressing follows ϕ),

(3) $$\forall \phi \in CSL \exists x[\bar{\phi}x /\!/ G \rightarrow \exists w(gw \neq 0 \wedge d_F^v w \simeq [\bar{\phi}x] \wedge \delta_F(v*w) = \delta_F v]$$

(i.e. we can leave the frame unchanged and make $d_F^v w$ follow ϕ until w reaches the bar g).

<u>LEMMA</u>. *The universal closures of* (1), (2) *and* (3) *imply* 5.3.2(1).

The proof of this lemma is also simple. It may be skipped. In that case, go on with 5.3.4.

<u>PROOF</u>. Let $G \geq \delta_F v$, $f /\!/_C \delta_F v$ and $\phi \in CSL$ be arbitrary. Apply (1).
Either we find an x_1 such that $\bar{\phi}x_1 \not\!/\!\!/ \delta_F v$, then $\bar{\phi}x_1 \not\!/\!\!/ G$ (3.2.21(f)) and 5.3.2(1) follows trivially,
or we find an x_1 and a w_1 such that

(4) $$d_F^v w_1 \simeq f:[\bar{\phi}x_1] \wedge \delta_F(v*w_1) \approx \delta_F v.$$

Apply (2) with $v*w_1$ for v and $s^{x_1}|\phi$ for ϕ. Since $G \geq \delta_F v$ by assumption and $\delta_F(v*w_1) \approx \delta_F v$ by (4), we have <u>$G \geq \delta_F(v*w_1)$</u> (3.1.19(e)). So either we find an x_2 such that $s^{x_1}|\phi(x_2) \not\!/\!\!/ G$, then $\bar{\phi}(x_1+x_2) \not\!/\!\!/ G$ and 5.3.2(1) follows trivially, or we find an x_2 and a w_2 such that

$$(5) \qquad d_F^{v*w_1}\, w_2 \simeq [\overline{s^{x_1}}|\phi(x_2)] \wedge \delta_F(v*w_1*w_2) \approx G.$$

Combination of (4) and (5) yields (use 4.4.21(c)):

$$(6) \qquad d_F^{v}(w_1*w_2) \simeq f:[\bar{\phi}(x_1+x_2)] \quad \text{and} \quad \delta_F(v*w_1*w_2) \approx G.$$

Finally apply (3) with $v*w_1*w_2$ for v, $\overline{s^{x_1+x_2}}|\phi$ for ϕ and $\lambda w.g(w_1*w_2*w)$ for g. Either we find an x_3 such that $\overline{s^{x_1+x_2}}|\phi(x_3) \not\#\ \delta_F(v*w_1*w_2)$, then $\overline{s^{x_1+x_2}}|\phi(x_3) \not\#\ G$, by (6) and 3.2.21(h), hence $\bar{\phi}(x_1+x_2+x_3) \not\#\ G$ and (1) follows trivially,

or we find x_3 and w_3 such that

$$g(w_1*w_2*w_3) \neq 0,\ d_F^{v*w_1*w_2}\, w_3 \simeq [\overline{s^{x_1+x_2}}|\phi(x_3)]$$

and

$$\delta_F(v*w_1*w_2*w_3) = \delta_F(v*w_1*w_2).$$

Combination of these with (6) yields 5.3.2(1) with $x = x_1+x_2+x_3$ and $w = w_1*w_2*w_3$. □

5.3.4. <u>DEFINITION</u>.

(a) The *jps-part of* y is j_1^3y.

(b) The *jf-part of* y is j_2^3y.

(c) The *gv-part of* y is j_3^3y.

5.3.5. <u>FACTS</u>.

(a) The jps-part of y determines $\delta_F(v*\hat{y})$, that is to say

$$j_1^3y = j_1^3z \rightarrow \delta_F(v*\hat{y}) = \delta_F(v*\hat{z}),$$

since $\delta_F(v*\hat{y}) = \delta_F v[\mathrm{jps}(k_1^3(v*\hat{y}))]$ (4.3.13), and $k_1^3(v*y) = k_1^3 v*\langle j_1^3 y\rangle$ by definition of k_i^p (1.3.5(d)).

(b) If the jps-part of y makes n jump, i.e. if $\mathrm{jps}(k_1^3(v*\hat{y}))n \neq {}^\circ n$, then the jf-part of y determines the jumpfunction, since (cf.4.4.3)

$$jf(v*\hat{y})m = \begin{cases} id \text{ if } jps(k^3_1(v*\hat{y}))m = {}^\circ m, \\ J(j^3_2 y) \text{ otherwise.} \end{cases}$$

(c) The gv-part of y determines the guiding sequences $gs_n(v*\hat{y})$ for n fresh a. $v*\hat{y}$, i.e. $n \notin nf(k^3_1(v*\hat{y}))$; for those n, $gs_n(v*\hat{y}) = (j^3_3 y)_n * \lambda z.0$.

(d) If the jps-part of y is $\nu_3(0,0,0)$ then $jps(k^3_1(v*\hat{y})) = \lambda n.{}^\circ n$ and $\delta_F(v*\hat{y}) = \delta_F v$ (cf. 4.3.4, 4.3.13 and 3.1.19(c)).

(e) Let k,m,n satisfy $k \notin nf(k^3_1 v)$, $m \notin nf(k^3_1 v)$, $n \notin nf(k^3_1 v)$, $k \neq m$ and $k \neq n$.
If the jps-part of y is $\nu_3(0,k,m)$, then $jps(k^3_1(v*\hat{y}))k = {}^\circ m$, and $jps(k^3_1(v*\hat{y}))k' = {}^\circ k'$ for $k' \neq k$.
If the jps-part of y is $\nu_3(1,k,j(m,n))$, then $jps(k^3_1(v*\hat{y}))k = {}^\circ m \wedge {}^\circ n$ and $jps(k^3_1(v*\hat{y}))k' = {}^\circ k'$ for $k' \neq k$.

(f) Let m and n be labels of $\delta_F v$, $m \neq n$. Then $m \notin nf(k^3_1 v)$ and $n \notin nf(k^3_1 v)$ by 4.3.14(c) and 4.3.11(a), hence, if we take $\nu_3(0,n,m)$ for the jps-part of y, then $\delta_F(v*\hat{y})$ is obtained from $\delta_F v$ by erasing all labels n and putting the label m in its place. See fig.1.

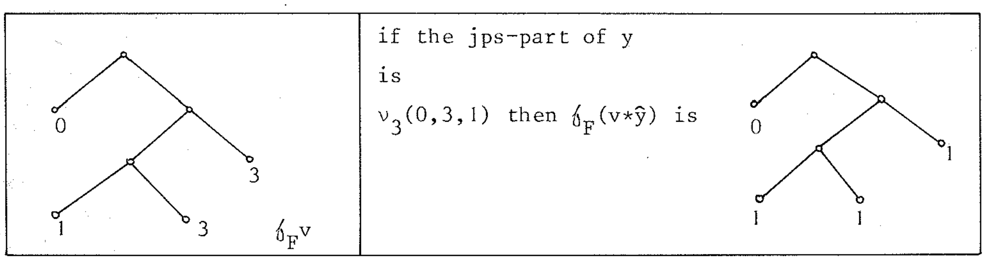


Fig. 1

(g) If m is a label of $\delta_F v$, $k \notin nf(k^3_1 v)$, $k \notin \ell(\delta_F v)$ and the jps-part of y is $\nu_3(0,m,k)$, then $jps(k^3_1(v*\hat{y}))$ has the form $\lambda n.{}^\circ an$, where $am = k$ and $am' = m'$ if $m' \neq m$. Since $k \notin \ell(\delta_F v)$, a is 1-1 on $\ell(\delta_F v)$, hence $\delta_F(v*\hat{y}) \approx \delta_F v$ by 3.1.22.

(h) If the jps-part of y is $\nu_3(0,0,0)$ then $jf(v*\hat{y}) = \lambda^K n.id$, by (d) above and the definition of jf (4.4.3).

(i) If the jps-part of y makes n jump and $e \in C$, then we can choose the jf-part of y in such a way that (an equivalent of) e is generated as the jumpfunction. (By assumption, J enumerates C modulo $\simeq$, cf.5.3.1.).

(j) We can choose a value z for the jf-part of y such that $Jz \simeq$ id. (C is dependency-closed, hence id $\in$ C). In that case $jf(v*\hat{y}) = \lambda^K n.id$, independent of the jps-part of y.

5.3.6. DEFINITION. $JF(F,v*\hat{y}) \equiv \nu^K_{\delta_F v} jf(v*\hat{y})$, $GV(F,v*\hat{y}) \equiv \nu^K_{\delta_F(v*\hat{y})} gv(v*\hat{y})$.

Definitions 5.3.4 and 5.3.6 will not be used outside this chapter.

5.3.7. FACTS.

(a) $d^v_F\hat{y} \simeq JF(F,v*\hat{y}):GV(F,v*\hat{y})$, by 4.4.20.

(b) If $jf(v*\hat{y}) \simeq \lambda^K n.id$ then $JF(F,v*\hat{y}) \simeq id$ by 3.2.16(f), and hence $d^v_F\hat{y} \simeq GV(F,v*\hat{y})$ by (a).

(c) If the jps-part of y is $\nu_3(0,0,0)$ then $JF(F,v*\hat{y}) \simeq id$ and $d^v_F\hat{y} \simeq GV(F,v*\hat{y})$ by 5.3.5 (h) and (b) above.

(d) We can choose the jf-part of y in such a way that (independent of the jps-part of y) $JF(F,v*\hat{y}) \simeq id$ and $d^v_F\hat{y} \simeq GV(F,v*\hat{y})$ by 5.3.5(j) and by (b) above.

(e) $JF(F,v*\hat{y})$ is completely determined by the jps- and the jf-part of y, since these two together determine $jf(v*\hat{y})$. The same holds for $(d{:}JF)(n,v*\hat{y})$ as defined in 4.4.7.

5.3.8. LEMMA (freedom of generated values). *Let the* jps- *and the* jf-*part of* y *be given, and let* G *be the frame* $\delta_F(v*\hat{y})$, *as determined by the* jps-*part of* y (*i.e.* $G = \delta_F v[jps(k^3_1(v*\hat{y}))] = \delta_F v[jps(k^3_1 v*<j^3_1 y>)]$). *With any sequence* $\phi \in N$ *we can find*
either an initial segment $\bar{\phi}x$ *which is not parallel to* G,
or an x *and a value for the* gv-*part of* y *such that* $\bar{\phi}x /\!/ G$ *and* $GV(F,v*\hat{y}) \simeq [\bar{\phi}x]$.
In formula

$$\forall y_1 y_2 \forall \phi \exists xy[j^3_1 y = y_1 \wedge j^3_2 y = y_2 \wedge (\bar{\phi}x /\!/ \delta_F(v*\hat{y}) \rightarrow GV(F,v*\hat{y}) \simeq [\bar{\phi}x])].$$

(The formula does not quite match the informal description, but it expresses the same: since $\bar{\phi}x /\!/ \delta_F(v*\hat{y})$ is decidable, $\bar{\phi}x /\!/ \delta_F(v*y) \rightarrow A$ is equivalent to $\bar{\phi}x \not/\!/ \delta_F(v*\hat{y}) \vee (\bar{\phi}x /\!/ \delta_F(v*\hat{y}) \wedge A)$.)

PROOF. See 5.4.1. □

Now we can turn to the proofs of the semi-overtake properties and the continuation to bar property 5.3.3(1)-(3). We consider them in the reverse order.

5.3.9. The continuation to bar property (5.3.3.(3)) states that for all F,v,g and ϕ of the form $e|\alpha$ we can find
either an x such that $\bar{\phi}x \mathrel{\#\!\!\!/} \delta_F v$,
or an x and a w such that $gw \neq 0 \wedge d_F^v w \simeq [\bar{\phi}x] \wedge \delta_F(v*w) = \delta_F v$.

First we show

LEMMA. *For all* F,v *and* ϕ *we can find*
either an x_1 *such that* $\bar{\phi}x_1 \mathrel{\#\!\!\!/} \delta_F v$,
or an x_1 *and a* y *such that* $d_F^v\hat{y} \simeq [\bar{\phi}x_1]$ *and* $\delta_F(v*\hat{y}) = \delta_F v$.
I.e.

$$\forall F v \phi \exists x_1 [\bar{\phi}x_1 /\!/ \delta_F v \rightarrow \exists y (d_F^v\hat{y} \simeq [\bar{\phi}x_1] \wedge \delta_F(v*\hat{y}) = \delta_F v)].$$

(*That is to say: we can take one step towards the bar* g.)

PROOF (can be skipped.)
Choose x_1 and y as follows ((i)-(iii)):
(i) For the jps-part of y take $\nu_3(0,0,0)$, then

(1) $\delta_F(v*\hat{y}) = \delta_F v$ (by 5.3.5(d)) and

(2) $d_F^v y \simeq GV(F, v*\hat{y})$ (by 5.3.7(c)).

(ii) For the jf-part of y take any value you like, the previous choice of the jps-part makes the jf-part irrelevant.
(iii) Now apply lemma 5.3.8:
either we find an x_1 such that $\bar{\phi}x_1 \mathrel{\#\!\!\!/} \delta_F(v*\hat{y})$, then $\bar{\phi}x_1 \mathrel{\#\!\!\!/} \delta_F v$ by (1), which proves the lemma, or we find an x_1 and a value for the gv-part of y such that

(3) $GV(F, v*\hat{y}) \simeq [\bar{\phi}x_1]$,

which, in combination with (1) and (2), also proves the lemma. □

To prove the continuation to bar property itself, one shows that this lemma implies the existence of two mappings $f_1, f_2 \in K$ such that for all ϕ and z

$$\bar{\phi}(\phi_1 z) \;\#\; \delta_F v \;\vee (d_F^v(\overline{\phi_2}(z)) \simeq [\bar{\phi}(\phi_1 z)] \wedge \delta_F(v*\overline{\phi_2}z)=\delta_F v),$$

where $\phi_1 \equiv f_1|\phi$, $\phi_2 \equiv f_2|\phi$. By the extension principle we find a z_0 such that $g(\overline{\phi_2}(z_0)) \neq 0$, the continuation to bar property follows with $\phi_1 z_0$ for x and $\bar{\phi}_2 z_0$ for w. For the details see section 5.4.2.

5.3.10. The semi-overtake property for frames 5.3.3(2) states that for all F, v, ϕ and $G \geq \delta_F v$ we can find
either an x such that $\bar{\phi}x \;\#\; G$,
or an x and a w such that $d_F^v w \simeq [\bar{\phi}x]$ and $\delta_F(v*w) \approx G$.

Recall that $H \approx G$ iff there is an $a: \mathbb{N} \to \mathbb{N}$ such that $G = H[\lambda n.(°an)]$ and $a \upharpoonright \ell H$ is 1-1 (lemma 3.1.22).

First we prove the semi-overtake property for frames under the additional assumption that $G = \delta_F v[\lambda n.(°bn)]$ for some b, i.e.

5.3.11. <u>LEMMA</u>. *Let* F,v *and* ϕ *be arbitrary and assume that for some* $b: \mathbb{N} \to \mathbb{N}$

(1) $$G = \delta_F v[\lambda n.(°bn)].$$

Then
either there is an x *such that* $\bar{\phi}x \;\#\; G$,
or there are x *and* w *such that* $d_F^v w \simeq [\bar{\phi}x]$ and $\delta_F(v*w) \approx G$.

<u>PROOF</u> (in sketch, for details see 5.4.3). Fig.2 shows a possible $\delta_F v$ and two frames G_1, G_2; $G_1 = \delta_F v[\lambda n.(°b_1 n)]$, $G_2 = \delta_F v[\lambda n.(°b_2 n)]$, where $b_1 0 = b_1 2 = 1$ and $b_1 1 = b_1 3 = 0$, while $b_2 0 = 1$ and $b_2 1 = b_2 2 = b_2 3 = 0$.

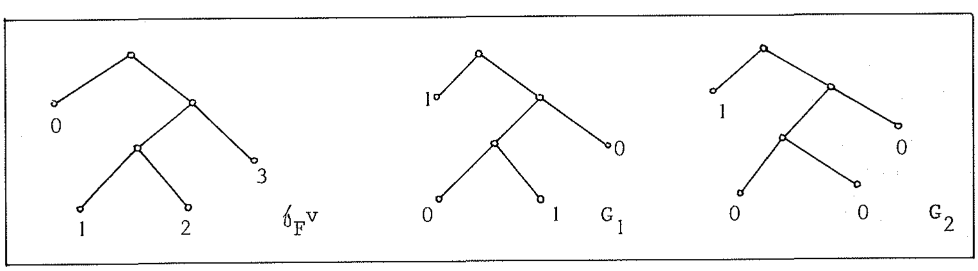


Fig. 2

If the b in assumption (1) is a 1-1 mapping on $\ell(\delta_F v)$, then $\delta_F v \approx G$ by the remark preceding this lemma, hence we can take $x = 0$, $w = 0$.

If b is not 1-1 there is a non-empty set of pairs (n,m), $n \in \ell(\delta_F v)$, $m \in \ell(\delta_F v)$, $n \neq m$, such that $bn = bm$. In the examples we find the set $\{(0,2), (1,3)\}$ for b_1 and $\{(1,2),(2,3),(1,3)\}$ for b_2.
We measure the extent to which b is not 1-1 by counting the members of this set. The formal proof proceeds by induction w.r.t. the resulting number.

In the examples we have $b_1 3 = b_1 1$ and $b_2 3 = b_2 1$. In both cases, y and x_1 can be determined as follows ((i)-(iii)):
(i) For the jps-part of y take $\nu_3(0,3,1)$, then $\delta_F(v*\hat{y})$ is the frame pictured in fig.3 (5.3.5(f)).

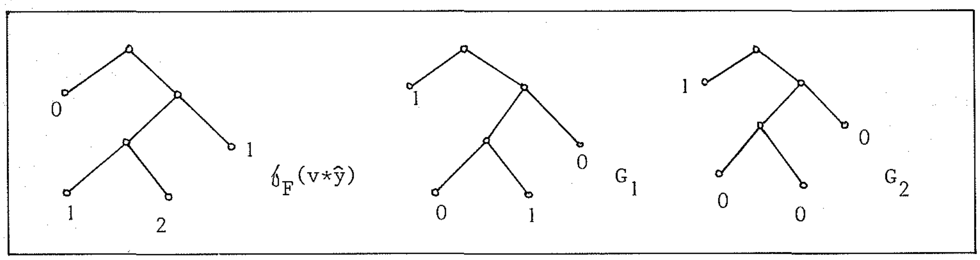


Fig.3

(ii) Choose the jf-part of y in such a way that the jumpfunction id is generated, i.e. such that (5.3.7(d))

(2) $$d_F^v \hat{y} \simeq GV(F, v*\hat{y}).$$

(iii) Apply lemma 5.3.8:

either we find an x_1 such that $\bar{\phi}x \,\#\, \delta_F(v*\hat{y})$, since $G_1 \geq \delta_F(v*\hat{y})$ and $G_2 \geq \delta_F(v*\hat{y})$ (see fig.3) then also $\bar{\phi}x \,\#\, G_1$, $\bar{\phi}x \,\#\, G_2$ and we have the result we want,

or we find an x_1 and a value for the gv-part of y such that

(3) $$GV(F, v*\hat{y}) \simeq [\phi x_1],$$

whence $d_F^v \hat{y} \simeq [\bar{\phi}x_1]$ by (2).

Note that $G_1 = \delta_F(v*\hat{y})[\lambda n.(^\circ b_1'n)]$ and $G_2 = \delta_F(v*\hat{y})[\lambda n.(^\circ b_2'n)]$, where $b_1'0 = b_1'2 = 1$, $b_1'1 = 0$, and $b_2'0 = 1$, $b_2'1 = b_2'2 = 0$; that is to say, for b_1' as well as for b_2' there is only a single pair (n,m) of labels of $\delta_F(v*\hat{y})$ such that $n \neq m$ and $b_i'n = b_i'm$.

If we have found x_1 and y such that $\bar{\phi}x_1 // G_i$, i = 1 or i = 2 respectively, and $d_F^v\hat{y} \simeq [\bar{\phi}x_1]$, then we repeat the construction, with $v*\hat{y}$ for v an $\overline{s^{x_1}|\phi}$ for ϕ, and with the remaining pair (n,m) such that $b_i'n = b_i'm$ instead of (1,3).
Either we find that $s^{x_1}|\phi(x_2) \not\!/\!/ \delta_F(v*<y,y'>)$ for some x_2,
or we find $\delta_F(v*<y,y'>) \approx G_i$ and $d_F^{v*\hat{y}}<y'> \simeq [s^{x_1}|\phi(x_2)]$.
In both cases we obtain the desired result. □

5.3.12. Next we prove a lemma which reduces the semi-overtake property for frames to the property proved in the previous lemma.

<u>LEMMA</u>. *Let* F,v *and* ϕ *be arbitrary and assume that* $G \geq \delta_F v$. *Then we can find either an* x *such that* $\bar{\phi}x \not\!/\!/ G$,
or an x, *a* w *and a* b: $\mathbb{N} \to \mathbb{N}$ *such that* $G = \delta_F(v*w)[\lambda n.(^\circ bn)]$ *and* $d_F^v w \simeq [\bar{\phi}x]$.

<u>PROOF</u> (in sketch, see also 5.4.3). If $G \geq \delta_F v$ then $G = \delta_F v[g]$ for some g: $\mathbb{N} \to$ FRAME. $\delta_F v$, G and g might be e.g. as in fig.4.

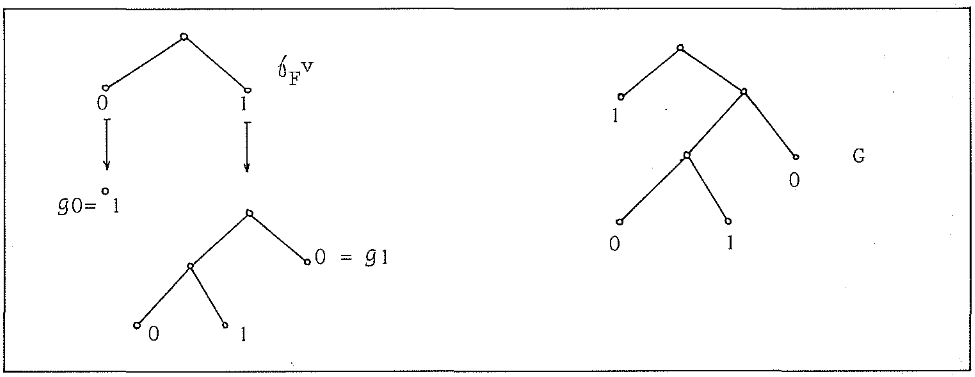


Fig. 4

We measure the extent to which g differs from a mapping of the form $\lambda n.^\circ bn$ by counting for each $m \in \ell(\delta_F v)$ the number of non-empty nodes in gm and adding the results.
The formal proof proceeds by induction w.r.t. to this number.

If it is 0, then, for all $m \in \ell(\delta_F v)$, the only node of gm is the empty one, and g can be replaced by $\lambda n.(°bn)$ for some b.

In the example $g1$ has 4 non-empty nodes, $g0$ has none. Note that a frame which has non-empty nodes is a pair $H_1 \wedge H_2$. In the example $g1 = H_1 \wedge H_2$, with H_1, H_2 as in fig.5(a).

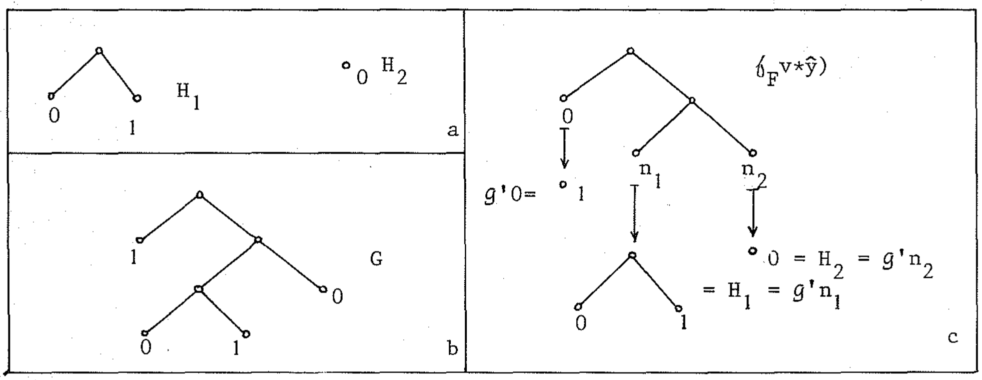


Fig.5

The first step towards constructing x and w such that $G = \delta_F(v*w)[\lambda n.(°bn)]$ for some b and $d_F^v w \simeq [\bar{\phi}x]$ would be to determine y and x_1 as follows ((i)-(ii)):
(i) Choose n_1, n_2 such that $n_1 \neq n_2$, $n_1 \notin nf(k_1^3 v)$, $n_2 \notin nf(k_1^3 v)$, $n_1 \notin \ell(\delta_F v)$, $n_2 \notin \ell(\delta_F v)$, and take $\nu_3(1,1,j(n_1,n_2))$ for the jps-part of y.
Then $jps(k_1^3(v*\hat{y}))1 = °n_1 \wedge °n_2$ by 5.3.5(e) and $\delta_F(v*\hat{y})$ is the frame pictured in fig.5c.
(ii) Choose the jf-part of y, x_1 and the gv-part of y as in the previous lemma, i.e. such that either

(1) $\qquad \bar{\phi}x_1 \,/\!/\, \delta_F(v*\hat{y})$

or

$$d_F^v \hat{y} \simeq [\bar{\phi}x_1].$$

Note that $G = \delta_F(v*\hat{y})[g']$, where $g'0 = °1$, $g'n_1 = H_1$, $g'n_2 = H_2$ (see figs.5b,c), hence if (1) is the case then also $\bar{\phi}x_1 \not\!/\!/ \, G$ and the lemma is proved.

If (2) is the case, then we repeat the construction above with $v*\hat{y}$ for v, $s^{x_1}|\phi$ for ϕ, and g' for g: now we make n_1 jump to k_1 and k_2, $k_1 \neq k_2$, $k_1,k_2 \notin \{0,n_1,n_2\}$.

Note that the distance between g' and a mapping $\lambda n.(°bn)$ is smaller than the one between g and a mapping $\lambda n.(°bn)$: only $g'n_1$ has non-empty nodes, namely two. In our example we need one repetition of the construction given above to reduce the remaining distance to zero; in general, more repetitions will be necessary. □

5.3.13. Now we can prove

<u>LEMMA</u>. *The semi-overtake property for frames holds.*

<u>PROOF</u>. By a simple combination of the foregoing two lemmata (details in 5.4.3). □

5.3.14. The semi-overtake property for dressings (i.e. 5.3.3(1)) states that with all F,v,ϕ and $f/\!/_C\ \delta_F v$ we can find
either an x such that $\bar{\phi}x/\!/\delta_F v$,
or an x and a w such that $d_F^v w \simeq f{:}[\bar{\phi}x]$ and $\delta_F(v*w) \approx \delta_F v$.

We illustrate the proof of this property with a simple example. The formal proof is given in 5.4.4. Let $\delta_F v$ be the frame $(°0)\wedge(°1)$ as in fig.6a.

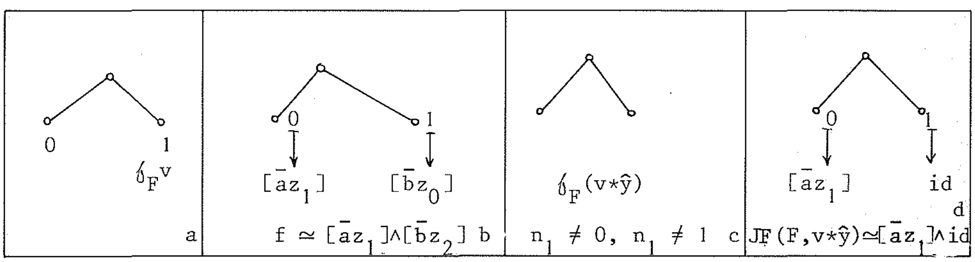


Fig.6.

Since $f/\!/_C\,\delta_F v$ we have a mapping $\psi\colon \mathbb{N} \to C$ such that $f \simeq \nu^K_{\delta_F v}\psi$. For $\delta_F v = (°0)\wedge(°1)$ this yields:

(1) $\qquad f \simeq \psi 0 \wedge \psi 1.$

Now we make an additional assumption, namely

(2) $\forall n \in \mathcal{L}(\delta_F v)\exists u(\psi n \simeq [u])$.

Let e.g. $\psi 0 \simeq [\bar{a}z_1]$ and $\psi 1 \simeq [\bar{b}z_2]$. We find (see fig.6b.):

(3) $f \simeq [\bar{a}z_1]\wedge[\bar{b}z_2]$.

Now determine y and x_1 as follows ((i)-(iii)):
(i) Choose the jps-part of y in such a way that $jps(k_1^3(v*\hat{y}))0 \neq {}^\circ 0$ and

(4) $\delta_F(v*\hat{y}) \approx \delta_F v$ (5.3.5(g)),

then $\delta_F(v*\hat{y})$ has the form $({}^\circ n_1)\wedge({}^\circ 1)$, $n_1 \neq 0$, $n_1 \neq 1$ as in fig.6c.
(ii) Choose the jf-part of y in such a way that $jf(v*\hat{y})0 \simeq [\bar{a}z_1]$ and $jf(v*\hat{y})m \simeq id$ if $m \neq 0$. (Use 5.3.5(i), note that $[\bar{a}z_1] \in C$ since C is dependency-closed.) Then

(5) $JF(F,v*\hat{y}) \equiv \nu^K_{\delta_F v} jf(v*\hat{y}) = jf(v*\hat{y})0\wedge jf(v*\hat{y})1 \simeq [\bar{a}z_1]\wedge id$.

(See figs.6d,7a.)
(iii) Note that $f \simeq [\bar{a}z_1]\wedge[\bar{b}z_2]$ satisfies $f \simeq JF(F,v*\hat{y}):(id\wedge[\bar{b}z_2])$. We incorporate the difference between f and $JF(F,v*\hat{y})$, i.e. $(id\wedge[\bar{b}z_2])$ in the generated values, that is to say: we apply 5.3.8 with $(id\wedge[\bar{b}z_2])|\phi = j(j_1\phi,\bar{b}z_2*j_2\phi)$ for ϕ.
Note that $(id\wedge[\bar{b}z_2])|\phi /\!/ \delta_F(v*\hat{y})$ due to the special structure of $\delta_F(v*\hat{y})$ (by 3.2.21(e), in this respect the example is not quite characteristic). Hence we find an x_1 and a value for the gv-part of y such that

$$GV(F,v*\hat{y}) \simeq \overline{[(id\wedge[\bar{b}z_2])|\phi(x_1)]}$$

or equivalently

(6) $GV(F,v*\hat{y}) \simeq [\overline{j_1\phi(x_1)}]\wedge[\overline{\bar{b}z_2*j_2\phi(x_1)}]$.

(See fig.7b.) Since, by definition,

$$GV(F,v*\hat{y}) = \nu^K_{\delta_F(v*\hat{y})} gv(v*\hat{y}),$$

and

$$\nu^K_{\delta_F(v*\hat{y})} gv(v*\hat{y}) \simeq gv(v*\hat{y})n_1 \wedge g_v(v*\hat{y})1$$

($\delta_F(v*\hat{y}) = (°n_1)\wedge(°1)$), (5) can also be expressed as:

$$gv(v*\hat{y})n_1 \simeq [\overline{j_1\phi}(x_1)], \quad gv(v*\hat{y})1 \simeq [\overline{\bar{b}z_2*j_2\phi(x_1)}].$$

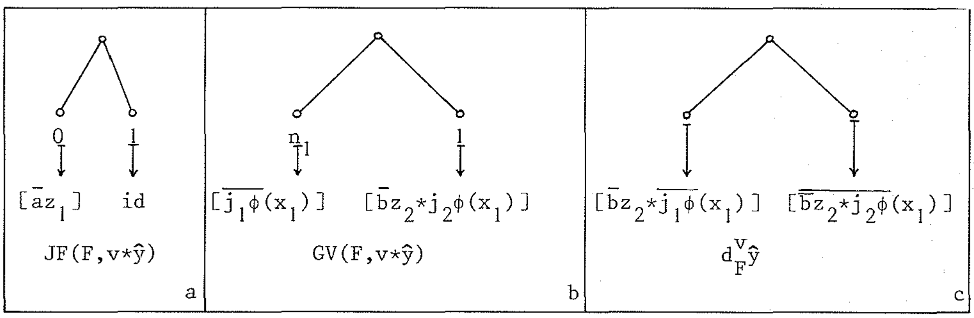


Fig.7

From (5) and (6) we find

(7) $$d^v_F\hat{y} \simeq [\bar{a}z_1*\overline{j_1\phi}(x_1)]\wedge[\overline{\bar{b}z_2*j_2\phi(x_1)}].$$

(Use 5.3.7(a), see fig.7c.)

Note that, since $f \simeq [\bar{a}z_1]\wedge[\bar{b}z_2]$ and $[\bar{\phi}(x_1)] \simeq [\overline{j_1\phi}(x_1)]\wedge[\overline{j_2\phi}(x_2)]$,

$$f:[\bar{\phi}x_1] \simeq [\bar{a}z_1*\overline{j_1\phi}(x_1)]\wedge[\bar{b}z_2*\overline{j_2\phi}(x_1)],$$

whence

(8) $$f:[\bar{\phi}x_1] \simeq d^v_F\hat{y}:(id\wedge u),$$

where u is the finite sequence such that

$$\overline{\bar{b}z_2*j_2\phi}(x_1)*u = \bar{b}z_2*\overline{j_2\phi}(x_1).$$

Recall that our aim is to find a w and an x such that

$$d_F^v w \simeq f:[\bar{\phi}x] \wedge \delta_F(v*w) \approx \delta_F v.$$

($\bar{\phi}x$ is always parallel to $\delta_F v$ due to our special choice of $\delta_F v$.) From (4) and (8) we see that it suffices to construct y' and x_2 such that

(10) $$d_F^{v*\hat{y}}\langle y'\rangle \simeq (id\wedge u):[\overline{s^{x_1}|\phi(x_2)}]$$

and

(11) $$\delta_F(v*\langle y,y'\rangle) \approx \delta_F(v*\hat{y}),$$

for in that case

$$d_F^v\langle y,y'\rangle \simeq d_F^v\hat{y}:d_F^{v*\hat{y}}\langle y'\rangle \simeq f:[\overline{s^{x_1}|\phi(x_2)}] \simeq f:[\bar{\phi}(x_1+x_2)].$$

The construction of y' and x_2 satisfying (10) and (11) is analogous to the construction of y and x_1 above, with the label 1 in the rôle of 0, with u in the rôle of $\bar{a}z_1$ and with $s^{x_1}|\phi$ instead of ϕ.

Now we drop the assumption (2) and consider a more general example, where

(12) $$f \simeq e_1\wedge e_2, \quad e_1,e_2 \in C.$$

We must construct w and x such that

(13) $$d_F^v w \simeq (e_1\wedge e_2):[\bar{\phi}x] \wedge \delta_F(v*w) \approx \delta_F v.$$

It suffices to show that there are w, x and f' such that

(14) $$d_F^v w:f' \simeq (e_1\wedge e_2):[\bar{\phi}x] \text{ and } \delta_F(v*w) \approx \delta_F v,$$

where f' has the form $[u_1]\wedge[u_2]$ since by the argument above we can find w' and x' such that

(15) $$\delta_F(v*w*w') \approx \delta_F(v*w) \text{ and } d_F^{v*w}w' \simeq f':[\overline{s^x|\phi(x')}];$$

combination of (14) and (15) yields

$$\delta_F(v*w*w') \approx \delta_F v$$

and

$$d_F^v(w*w') \simeq d_F^v w : d_F^{v*w} w' \simeq d_F^v w : f'[\overline{s^x|\phi(x')}] \simeq (e_1 \wedge e_2):[\bar{\phi}(x+x')].$$

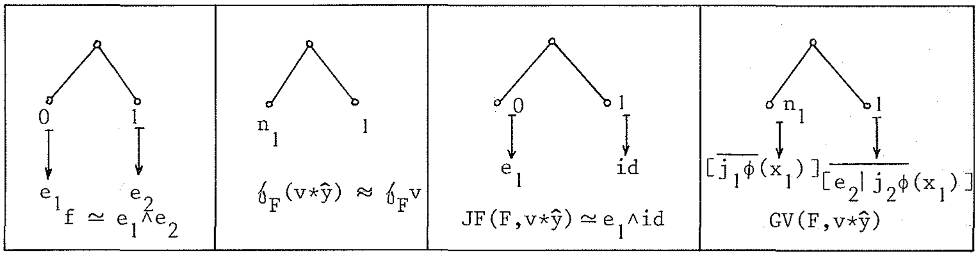


Fig.8

We start our construction of w and x for (14) in the same way as above, i.e. we determine y and x_1 such that ((i)-(iii)):
(i) $jps(k_1^3(v*\hat{y}))0 = {}^\circ n_1$, $n_1 \neq 0$, $n_1 \neq 1$, hence

(16) $$\delta_F(v*\hat{y}) \approx \delta_F v;$$

(ii) $jf(v*\hat{y})0 \simeq e_1$, and $jf(v*\hat{y})m = id$ if $m \neq 0$, hence

(17) $$JF(F,v*\hat{y}) \simeq e_1 \wedge id;$$

(iii) $GV(F,v*\hat{y}) \simeq [\overline{(id \wedge e_2)|\phi}(x_1)]$, where $id \wedge e_2$ is the difference between f and $JF(F,v*\hat{y})$. Then

(18) $$GV(F,v*\hat{y}) \simeq [\overline{j_1\phi}(x_1)] \wedge [\overline{e_2|j_2\phi}(x_1)],$$

where $[\overline{j_1\phi}(x_1)] \simeq gv(v*\hat{y})n_1$, $[\overline{e_2|j_2\phi}(x_1)] \simeq gv(v*\hat{y})1$.

(See fig.8.) Thus we achieve that

(19) $$d_F^v\hat{y} \simeq (e_1:[\overline{j_1\phi}(x_1)]) \wedge [\overline{e_2|j_2\phi}(x_1)]$$

(by (17), (18), 5.3.7(a) and distributivity of: over $\wedge$).

Next we choose y', x_1' and x_2 as follows ((i)-(iv)):

(i) The jps-part of y' is such that $jps(k_1^3(v*<y,y'>))1 = {}^\circ n_2$, $n_2 \neq 1$, $n_2 \neq n_1$, whence

$$(20) \qquad \delta_F(v*<y,y'>) \approx \delta_F(v*\hat{y})$$

(ii) $x_1' \geq x_1$ satisfies $\forall m < x_1(e_2(<m>*\overline{j_2\phi}(x_1')) \neq 0)$, i.e. $\overline{j_2\phi}(x_1')$ suffices to determine $\overline{e_2|j_2\phi}(x_1)$, whence

$$(21) \qquad \forall z \geq x_1'((e_2:[\overline{j_2\phi}(z)])|a \in \overline{e_2|j_2\phi}(x_1)).$$

(iii) The jf-part of y' is such that

$$jf(v*<y,y'>)1 \simeq e_2', \quad \text{and} \quad jf(v*<y,y'>)m \simeq id, \text{ for } m \neq 1,$$

where

$$(22) \qquad e_2' \equiv s^{x_1}:e_2:[\overline{j_2\phi}(x_1')].$$

Then

$$(23) \qquad JF(F,v*<y,y'>) \simeq id\wedge(s^{x_1}:e_2:[\overline{j_2\phi}(x_1')]).$$

Note that $e_2' \in C$ since $e_2 \in C$ and C is dependency-closed. Note also that by (21)

$$(24) \qquad [\overline{e_2|j_2\phi}(x_1)]:e_2' \simeq e_2:[\overline{j_2\phi}(x_1')].$$

So

$$(25) \qquad d_F^V\hat{y}:JF(F,v*<y,y'>) \simeq (e_1:[\overline{j_1\phi}(x_1)])\wedge(e_2:[\overline{j_2\phi}(x_1')]),$$

by (19), (23), distributivity of : over $\wedge$ and (24).

(iv) Finally, the gv-part of y and x_2 are such that

$$GV(F,v*<y,y'>) \simeq [\overline{(s^{x_1}\wedge s^{x_1'})|\phi(x_2)}],$$

i.e.

$$(26)\qquad GV(F,v*\langle y,y'\rangle) \simeq [s^{\overline{x_1}}|\phi(x_2)]\wedge[s^{\overline{x'_1}}|\phi(x_2)].$$

Now $d_F^v\langle y,y'\rangle \simeq d_F^v\hat{y}:d_F^{v*\hat{y}}\langle y'\rangle$, (4.4.21(c)), and

$$d_F^{v*\hat{y}}\langle y'\rangle \simeq JF(F,v*\langle y,y'\rangle):GV(F,v*\langle y,y'\rangle) \quad (5.3.7(a)),$$

hence

$$d_F^v\langle y,y'\rangle \simeq d_F^v\hat{y}:JF(F,v*\langle y,y'\rangle):GV(F,v*\langle y,y'\rangle)$$

whence

$$(27)\qquad d_F^v\langle y,y'\rangle \simeq (e_1:[\overline{j_1\phi}(x_1+x_2)])\wedge(e_2:[\overline{j_2\phi}(x'_1+x_2)]) \quad ((25),(26)).$$

By distributivity of : over $\wedge$

$$(28)\qquad d_F^v\langle y,y'\rangle \simeq (e_1\wedge e_2):([\overline{j_1\phi}(x_1+x_2)]\wedge[\overline{j_2\phi}(x'_1+x_2)]).$$

Now put $u \equiv \langle j_1\phi(x_1+x_2),\ldots, j_1\phi(x'_1+x_2\dot{-}1)\rangle$, (i.e. $u = \langle\ \rangle$ if $x'_1 = x_1$), and $f' \equiv [u]\wedge id$, then

$$(29)\qquad d_F^v\langle y,y'\rangle:f' \simeq (e_1\wedge e_2):[\bar{\phi}(x'_1+x_2)].$$

Moreover

$$\delta_F(v*\langle y,y'\rangle) \approx \delta_F v \quad ((15),(20)),$$

so we have (14) with $\langle y,y'\rangle$ for w and x'_1+x_2 for x. □

5.4. THE PROOF OF THE STRONG OVERTAKE PROPERTY (2)

In this section we give extra details of the proof of lemma 5.2.5, which were left out in the preceding section. This section is to be read only in connection with 5.3. (And at first reading it is to be skipped.)

5.4.1. First we provide a proof for lemma 5.3.8 on the freedom of generated values, which states:

Let the jps- *and the* jf-*part of* y *be fixed, let* G *be* $\mathfrak{f}_F(v*\hat{y})$ *as determined by the* jps-*part of* y (5.3.5(a)), *and let* ϕ *be an element of* N. *Then we can find*
either an x *such that* $\overline{\phi}x \not\# G$,
or an x *and a value for the* gv-*part of* y *such that* $\overline{\phi}x \mathbin{/\!/} G$ *and*
$GV(F,v*\hat{y}) \simeq [\overline{\phi}x]$.

<u>PROOF</u>. By definition 5.3.6, $GV(F,v*\hat{y}) \equiv \nu_G^K gv(v*\hat{y})$.
$gv(v*\hat{y})n = [(j_3^3 y)_n * \lambda z.0(1+upb(v*\hat{y}))]$, by definition 4.4.10.
$(j_3^3 y)_n * \lambda z.0 = gs_n(v*\hat{y})$, for n fresh (i.e. such that $\mathfrak{f}_n(v*\hat{y}) = {}^\circ n$), by definition of gs_n (4.4.4, 4.4.6).
Labels of $G(\equiv \mathfrak{f}_F(v*\hat{y}))$ are fresh (4.3.14(c)), hence for $n \in \mathcal{L}G$
$gv(v*\hat{y})n = [\overline{gs_n(v*\hat{y})}(1+upb(v*\hat{y}))]$, whence

$$GV(F,v*\hat{y}) \simeq \nu_G^K(\lambda^K n.[\overline{gs_n(v*\hat{y})}(1+upb(v*\hat{y}))]).$$

Now put

$$gs_F(v*\hat{y}) \equiv \nu_G^1(\lambda^1 n.gs_n(v*\hat{y})),$$

then

$$GV(F,v*\hat{y}) \simeq [\overline{gs_F(v*\hat{y})}(1+upb(v*\hat{y}))] \quad (3.2.16(g)).$$

Fig. 9 shows an example of G and $gs_F(v*\hat{y})$.

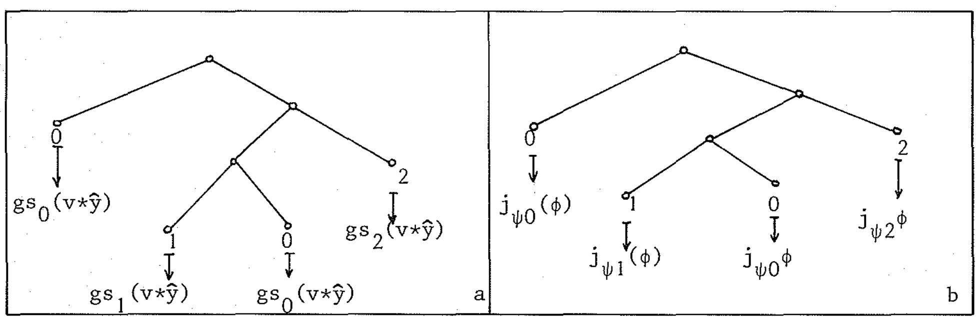


Fig. 9

Let ψ be a mapping from $\mathbb{N}$ into the set of branches of G, which satisfies:

if $n \in \ell G$ then ψn has label n in G.

If ϕ is parallel to G, then $j_b\phi = j_{\psi n}\phi$ for all branches b of G with label n. I.e. in that case, $\phi = \nu^1_G(\lambda^1 n.j_{\psi n}\phi)$ (3.2.14, 3.2.16(b), see fig. 9).

Our problem is to choose the gv-part of y, j^3_3y, in such a way that for all $n \in \ell G$, $gs_n(v*\hat{y})$ coincides with $j_{\psi n}\phi$ over a sufficiently long initial segment.

To make this choice we introduce the pseudo guiding-sequences pgs_n. As an auxiliary we put:

$$\nu^{-1}_G(n,\phi) \equiv \begin{cases} j_{\psi n}\phi \text{ if } n \in \ell G, \\ \lambda z.0 \text{ otherwise.} \end{cases}$$

The pseudo guiding-sequence for n is defined by

$$(1) \qquad pgs_n \equiv \nu^1_{\mathcal{b}_n(v*\hat{y})}(\lambda^1 k.\nu^{-1}_G(k,\phi))$$

i.e. for $n \in \ell G$, $pgs_n = j_{\psi n}\phi$.

Now we perform the upperbound computation (cf.def.4.4.9) with pgs_n instead of gs_n, i.e. we put

$$pupb \equiv \max\{mk((d{:}JF)(n,v*\hat{y}),\ lth(v),\ pgs_n) : n \in nf(k^3_1(v*\hat{y}))\}.$$

Note that pupb can be determined independently of the gv-part of y, $(d{:}JF)(n,v*\hat{y})$ depends on the jps- and jf-part of y only (4.4.7 and 5.3.7(e)). Now take

$$\overline{\lambda n.\overline{\nu^{-1}_G(n,\phi)}(x)(m)}$$

for the gv-part of y, (j^3_3y), where $x \equiv 1+pupb$, and $m \equiv 1 + \max(\ell G)$. Then

$$(j^3_3y)_k = \begin{cases} \overline{\nu^{-1}(k,\phi)}(x) & \text{if } k < m \\ 0 & \text{otherwise.} \end{cases}$$

Since $\nu_G^{-1}(k,\phi) = \lambda z.0$ for $k \geq m$, we find that for all k

$$(j_3^3 y)_k * \lambda z.0 = \overline{\nu_G^{-1}(k,\phi)(x)} * \lambda z.0.$$

Hence

$$\text{(2)} \qquad gs_n(v*\hat{y}) = \nu^1_{\delta_n(v*\hat{y})}(\lambda^1 k.\overline{\nu_G^{-1}(k,\phi)(x)} * \lambda z.0).$$

From (1) and (2) we find (using 3.2.14(b) and 3.2.17(c)) that for all n

$$\overline{gs_n(v*\hat{y})}(x) = \overline{pgs_n}(x).$$

Hence, if we compute upb($v*\hat{y}$), we find

$$upb(v*\hat{y}) = pupb,$$

whence, for all k

$$gv(v*\hat{y})k = [\overline{\nu_G^{-1}(k,\phi)(x)}].$$

For $n \in \ell G$, $\overline{\nu_G^{-1}(n,\phi)(x)} = \overline{j_{\psi n}\phi}(x) = k_{\psi n}(\bar{\phi}x)$. The proof is now completed by observing that hence

$$\bar{\phi}x /\!/ G \rightarrow \nu_G^K gv(v*\hat{y}) \simeq [\bar{\phi}x], \quad \text{by 3.2.20(g)}. \qquad \square$$

5.4.2. Next recall that we have reduced the strong overtake property to two 'semi-overtake' properties and a 'continuation to bar' property (5.3.3). The latter says (cf.5.3.3(3) and 5.3.9):

(A) *Let* F,v,g *and* $\phi \equiv e|\alpha$ *be arbitrary, then*
either there is an x *such that* $\bar{\phi}x \# \delta_F v$,
or there are x *and* w *such that* $gw \neq 0$, $\delta_F(v*w) = \delta_F v$ and $d_F^v w \simeq [\bar{\phi}x]$.

As a first step of the proof we have shown in 5.3.9:

(B) *For all* F,v *and* $\phi \equiv e|\alpha$ *we can find*
either an x *such that* $\bar{\phi}x \# \delta_F v$,
or an x_1 *and a* y *such that* $d_F^v \hat{y} \simeq [\bar{\phi}x_1]$ *and* $\delta_F(v*\hat{y}) = \delta_F v$.

<u>LEMMA</u>. (A) *follows from* (B) *by* AC-NF, $\forall\alpha\exists x$*-continuity and the extension principle* EP.

PROOF. Let F,v,g,e and α be arbitrary and put

$$\phi_{m,\beta} \equiv (s^m{:}e)|\beta \text{ (i.e. } \phi_{m,\beta} = \lambda z.e|\beta(m+z)); \ \phi \equiv \phi_{0,\alpha} = e|\alpha.$$

Let $A(m,w,\beta,x_1,y)$ be the formula

$$\overline{\phi_{m,\beta}}(x_1) /\!/ \delta_F(v*w) \to d_F^{v*w}\hat{y} \simeq [\overline{\phi_{m,\beta}}(x_1)] \wedge \delta_F(v*w*\hat{y}) = \delta_F v.$$

(B) states that

$$\forall mw\beta\exists x_1 y \ A(m,w,\beta,x_1,y),$$

hence

$$\forall mw\exists f_1' f_2' \forall\beta \ A(m,w,\beta,f_1'(\beta),f_2'(\beta)), \quad (\forall\beta\exists z\text{-continuity}),$$

whence

(1) $$\exists f_1 f_2 \forall mw\beta \ A(m,w,\beta,f_1|\beta(m,w),f_2|\beta(m,w)) \text{ (AC-NF)}.$$

(Here $f_i|\beta(m,w)$ abbreviates $f_i|\beta(j(m,w))$, i=1,2.) Let f_1 and f_2 be witnesses to (1). By a simultaneous recursion we define ψ_1 and $\psi_2(\in N)$:

$$\psi_1(0) = f_1|\alpha(0,0), \ \psi_2(0) = f_2|\alpha(0,0);$$
$$\psi_1(n+1) = \psi_1 n + f_1|\alpha(\psi_1 n, \overline{\psi_2}(n+1)),$$
$$\psi_2(n+1) = f_2|\alpha(\psi_1 n, \overline{\psi_2}(n+1)).$$

The reasons for this definition are explained by the following observation

(2) $$\forall n(\bar{\phi}(\psi_1 n) /\!/ \delta_F v \to d_F^v(\overline{\psi_2}(n+1)) \simeq [\bar{\phi}(\psi_1 n)] \wedge \delta_F(v*\overline{\psi_2}(n+1))=\delta_F v),$$

which is proved by induction w.r.t. n.

For n = 0, (2) is simply $A(0,0,\alpha,f_1|\alpha(0,0),f_2|\alpha(0,0))$, which holds by (1).

Now assume (2) to hold for n, and let $\bar{\phi}(\psi_1(n+1))$ be parallel to $\delta_F v$. Then $\bar{\phi}(\psi_1 n) /\!/ \delta_F v$, whence by induction hypothesis

(3) $\qquad \delta_F(v*\overline{\psi_2}(n+1)) = \delta_F v$

and

(4) $\qquad d_F^v(\overline{\psi_2}(n+1)) \simeq [\bar\phi(\psi_1 n)]$.

Moreover, if $\bar\phi(\psi_1(n+1)) \,/\!/\, \delta_F v$, hence

$$\overline{\lambda z.\phi(\psi_1 n+z)}(f_1|\alpha(\psi_1 n,\overline{\psi_2}(n+1))) \,/\!/\, \delta_F v$$

by definition of $\psi_1(n+1)$, whence by (3) and the definitions of $\phi, \phi_{m,w}$:

$$\overline{\phi_{\psi_1 n,\alpha}}(f_1|\alpha(\psi_1 n,\bar\psi_2(n+1))) \,/\!/\, \delta_F(v*\overline{\psi_2}(n+1)).$$

So if we apply (1), with $m = \psi_1 n$, $w = \overline{\psi_2}(n+1)$, and α for β, we find

(5) $\qquad \delta_F(v*\overline{\psi_2}(n+1)*\langle f_2|\alpha(\psi_1 n,\overline{\psi_2}(n+1))\rangle) = \delta_F(v*\overline{\psi_2}(n+1))$

and

(6) $\qquad d_F^{v*\overline{\psi_2}(n+1)}\langle f_2|\alpha(\psi_1 n,\overline{\psi_2}(n+1))\rangle \simeq [\overline{\phi_{\psi_1 n,\alpha}}(f_1|\alpha(\psi_1 n,\overline{\psi_2}(n+1)))]$.

Combining (3) with (5) and (4) with (6) yields (2) for n+1.
All we have to do now in order to prove (A), is to observe that for some n, $g(\overline{\psi_2}(n+1)) \neq 0$ by the extension principle: i.e. (A) holds with $\overline{\psi_2}(n+1)$ for w and $\psi_1 n$ for x. □

Note that we use instances of AC-NF and ∀β∃z-continuity here that are not in the language of LS. They can be translated into that language however, cf. chapter 7.

5.4.3. Of the two semi-overtake properties yet to be proved, the semi-overtake property for frames is the simplest. It states (cf.5.3.3(2) and 5.3.10):
(A) *For all* F,v,ϕ *and* $G \geq \delta_F v$ *there are* x *and* w *such that either* $\bar\phi x \,\#\, G$, *or* $d_F^v w \simeq [\bar\phi x] \wedge \delta_F(v*w) \approx G$.

As shown in 5.3.11-5.3.13, this assertion can be proved in three steps. First one shows (cf. lemma 5.3.11):
(B) *If* $G \geq \delta_F v$ *is replaced by the stronger assumption* $\exists b(G = \delta_F v[\lambda n.(°bn)])$,

then (A) *holds.*

PROOF. (Compare the sketch in 5.3.11.)
For an arbitrary finite set S put

$$eq(b,S) \equiv \{(m,n) : m\in S,\ n\in S,\ m\neq n,\ bn=bm\},$$

and put

$$h_1(b,H) \equiv card(eq(b,\ell H)).$$

Let b be such that $G = \mathfrak{d}_F v[\lambda n.({}^\circ bn)]$, to prove (B) we apply induction w.r.t. $h_1 \equiv h_1(b,\mathfrak{d}_F v)$.

If $h_1 = 0$ then b is 1-1 on $\ell(\mathfrak{d}_F v)$ whence $G \approx \mathfrak{d}_F v$ by 3.1.22, and we can take w = x = 0.

Now let $h_1 = z+1$. Choose a pair $(m,n) \in eq(b,\ell(\mathfrak{d}_F v))$ and determine y and x_1 as follows ((i)-(iii)):

(i) For the jps-part of y take $\nu_3(0,n,m)$, then $\mathfrak{d}_F(v*\hat{y}) = \mathfrak{d}_F v[g]$, where g satisfies: $gk = {}^\circ k$ if $k \neq n$, and $gn = m$ (5.3.5(f)).

(ii) For the jf-part take some value such that

$$d_F^v\hat{y} \simeq GV(F,v*\hat{y}) \quad (5.3.7(d)).$$

(iii) Apply lemma 5.3.8 to find an x_1 and a value for the gv-part of y such that either $\bar{\phi}x_1 \not\parallel \mathfrak{d}_F(v*\hat{y})$ or $\bar{\phi}x_1 \parallel \mathfrak{d}_F(v*\hat{y})$ and $GV(F,v*\hat{y}) \simeq [\bar{\phi}x_1]$, i.e. by (ii)

$$d_F^v\hat{y} \simeq [\bar{\phi}x_1].$$

Once y and x_1 have been determined, we can check whether or not $\bar{\phi}x_1 \parallel G$. If not, then (B) is proved.

Otherwise we note that $G = \mathfrak{d}_F(v*\hat{y})[\lambda n.({}^\circ bn)]$.
(By assumption $G = \mathfrak{d}_F v[\lambda n.({}^\circ bn)]$, by (i) $\mathfrak{d}_F(v*\hat{y}) = \mathfrak{d}_F v[g]$ for a g which maps both n and m on ${}^\circ m$; bn = bm so $(\mathfrak{d}_F v[g])[\lambda n.({}^\circ bn)]$ is simply $\mathfrak{d}_F v[\lambda n.({}^\circ bn)]$.) It follows that $\bar{\phi}x_1 \parallel \mathfrak{d}_F(v*\hat{y})$, whence by (iii) $d_F^v\hat{y} \simeq [\bar{\phi}x_1]$. Moreover, $h_1(b,\ell(\mathfrak{d}_F(v*\hat{y}))) < h_1$, i.e. we can apply induction hypothesis with $v*\hat{y}$ for v and $s^{\bar{\bar{x}}_1}|\phi$ for ϕ, to find either an x_2 such that $\overline{s^{x_1}|\phi}(x_2) \not\parallel G$, which proves (B), or x_2 and w' such that

$$\delta_F(v*\hat{y}*w') = G \quad \text{and} \quad d_F^{v*\hat{y}}w' \simeq [\overline{s^{x_1}|\phi(x_2)}],$$

in which case we also have (B). □

The next step towards proving (A) is to show that (cf.5.3.12)

(C) *For* F,v,ϕ *and* $G \geq \delta_F v$ *we can find* x,w *and* b *such that either* $\bar{\phi}x \not\!/\!/ G$, *or* $\bar{\phi}x /\!/ G$, $d_F^v w \simeq [\bar{\phi}x]$ *and* $G = \delta_F(v*w)[\lambda n.(^\circ bn)]$.

This claim is also proved by induction. If $G \geq \delta_F v$ then $G = \delta_F v[g]$ for some g. We put

$$h_2(g,H) \equiv \sum_{m \in \ell H} ne(gm),$$

where ne(F) is the number of nonempty nodes in F. The induction is w.r.t. $h_2(g,\delta_F v)$. We trust that the reader can find the proof from the sketch given in 5.3.12 and the foregoing proof of (B).

The final step to be taken is to show that (cf.5.3.13)

LEMMA. (B) *and* (C) *imply* (A).

PROOF. Let F,v,ϕ and $G \geq \delta_F v$ be arbitrary. Apply (C) to find x_1,w and b such that either $\bar{\phi}x_1 \not\!/\!/ G$, which yields (A), or $\bar{x}_1 /\!/ G$, $d_F^v w \simeq [\bar{\phi}x_1]$ and $G = \delta_F(v*w)[\lambda n.(^\circ bn)]$.

In the second case apply (B) with $s^{x_1}|\phi$ for ϕ and v*w for v. We find x_2 and w' such that either $\overline{s^{x_1}|\phi} /\!/ G$, which yields (A), or $\overline{s^{x_1}}|\phi(x_2) /\!/ G$, whence (since $\bar{\phi}x_1 /\!/ G$) $\bar{\phi}(x_1+x_2) /\!/ G$, and $d_F^{v*w}w' \simeq [\overline{s^{x_1}|\phi(x_2)}]$. Then by 4.4.21(c)

$$d_F^v(w*w') \simeq [\bar{\phi}x_1]:[\overline{s^{x_1}|\phi(x_2)}] \simeq [\bar{\phi}(x_1+x_2)]$$

and

$$\delta_F(v*w*w') \approx G.$$

I.e. in the second case we also have (A), with x_1+x_2 for x and w*w' for w. □

5.4.4. The most complex part of the verification of the overtake property is the proof of the semi-overtake property for dressings, which states (cf.5.3.3(1) and 5.3.14):

(A) *For all* F,v,ϕ *and* $\delta /\!/_C \delta_F v$ *we can find* w *and* x *such that either* $\bar{\phi}x \,\#\, \delta_F v$, *or* $\delta_F(v*w) \simeq \delta_F v \wedge d_F^v w \simeq f:[\bar{\phi}x]$.

The structure of the proof of this assertion resembles that of the proof of 5.4.3(A): it consists of two auxiliary lemma's, the first one claims that (A) holds under additional assumptions, the second one claims that the general form of (A) can be reduced to the special form of the first lemma. Both lemmata are proved by induction over $\mathbb{N}$. First we show:

(B) *If the assumption* $f /\!/_C \delta_F v$ *is replaced by the stronger* $\exists\psi\in N(f \simeq \nu^K_{\delta_F v}(\lambda^K n.[\psi n]))$, *then* (A) *holds.*

PROOF. We show that

$$(1) \qquad \forall vF\phi\psi[nz(\psi,\delta_F v)=n \rightarrow \exists xw(\bar{\phi}x /\!/ \delta_F v \rightarrow d_F^v w \simeq f:[\bar{\phi}x] \wedge \delta_F(v*w) \approx \delta_F v)],$$

where ϕ,ψ range over N, $f \simeq \nu^K_{\delta_F v}(\lambda^K n.[\psi n])$ and

$$nz(\psi,\delta_F v) \equiv card\{m\in\mathcal{L}(\delta_F v) : \psi m\neq 0\} \quad \text{(nz for non-zero)}.$$

The proof proceeds by induction w.r.t. n.

If n = 0, then $\psi m = 0$ for all $m \in \mathcal{L}(\delta_F v)$, whence $[\psi m] \simeq id$ for all $m \in \mathcal{L}(\delta_F v)$, whence $f \simeq id$, and (1) holds for x = w = 0.

Now assume (1) to hold for n (inductionhypothesis). Let v,F and ϕ be arbitrary, and let ψ be such that $nz(\psi,\delta_F v) = n+1$. Then there is a label, say m, of $\delta_F v$ for which $\psi m \neq 0$. Determine y and x_1 as follows ((i)-(iii)):

(i) For the jps-part of y take $\nu_3(0,m,m')$, $m' \notin \mathcal{L}(\delta_F v)$, then

$$(2) \qquad \delta_F(v*\hat{y}) \approx \delta_F v \quad (5.3.5(g)).$$

(ii) Choose the jf-part of y in such a way that $jf(v*\hat{y})m \simeq [\psi m]$ and $jf(v*\hat{y})k \simeq [0]$ (=id) for $k \neq m$ (cf.5.3.5(i)). Let jf^* be the mapping from $\mathbb{N}$ into $\mathbb{N}$ such that for all k, $jf(v*\hat{y})k \simeq [jf^* k]$. Let JF abbreviate $JF(F,v*\hat{y})$.

(iii) Put $(\psi-jf^*) \equiv \lambda k.(\psi k \dot{-} jf^* k)$, i.e. $(\psi-jf^*)m = 0$, and $(\psi-jf^*)k = \psi k$ if $k \neq m$. Put

$$(f-JF) \equiv \nu^K_{\delta_F v}(\lambda^K k.[(\psi-jf^*)k]).$$

Now apply lemma 5.3.8 with $(f-JF)|\phi$ instead of ϕ. This yields an x_1 and a value for the gv-part of y such that
either $\overline{(f-JF)|\phi}(x_1) \not\!\!/\!\!/ \delta_F(v*\hat{y})$,
or $GV(F,v*\hat{y}) \simeq [\overline{(f-JF)|\phi}(x_1)]$.

We assume that $\bar{\phi}x_1 /\!/ \delta_F v$. Note that for $b \in \delta_F v$,

$$(3) \qquad k_b(\overline{(f-JF)|\phi}(x_1)) = \overline{j_b((f-JF)|\phi)}(x_1) = \overline{[(\psi-jf^*)k]|j_b\phi}(x_1),$$

where $k = \ell_b(\delta_F v)$. Let lk be the length of the finite sequence $(\psi-jf^*)k$. (3) yields

$$(4) \qquad k_b(\overline{(f-JF)|\phi}(x_1)) = ((\psi-jb^*)k)*\overline{j_b\phi}(x_1 \dot{-} lk),$$

whence, by the assumption that $\bar{\phi}x_1 /\!/ \delta_F v$,

$$\overline{(f-JF)|\phi}(x_1) /\!/ \delta_F v,$$

and since $\delta_F v \approx \delta_F(v*\hat{y})$ by (1), $\overline{(f-JF)|\phi}(x_1) /\!/ \delta_F(v*\hat{y})$. Then

$$GV(F,v*\hat{y}) \simeq [\overline{(f-JF)|\phi}(x_1)] \quad \text{(see (iii))},$$

whence

$$d_F^v\hat{y} \simeq JF:GV(F,v*\hat{y}) \simeq JF:[\overline{(f-JF)|\phi(x_1)}].$$

Using (4) one finds that for $b \in \delta_F v$ with label k

$$(5) \qquad j_b(d_F^v\hat{y}|\chi) = jf^*k*((\psi-jf^*)k)*\overline{j_b\phi}(x_1 \dot{-} lk)*j_b\chi.$$

($\chi \in N$ arbitrary). Now define ψ_1 by

$$\psi_1 k = \begin{cases} \langle j_b\phi(x_1 \dot{-} lk),\ldots, j_b\phi(x_1 \dot{-} 1)\rangle, & \text{if } k \in \ell(\delta_F v), \\ 0 & \text{otherwise,} \end{cases}$$

where $b \in \delta_F v$ is such that $\ell_b(\delta_F v) = k$ (which b one chooses is irrelevant, since $\bar{\phi}x_1 /\!/ \delta_F v$). If $k \in \ell(\delta_F v)$ and $lk = 0$, then $\psi_1 k = 0$.

One easily sees that

(6) $\forall k(\psi k=0 \rightarrow \psi_1 k=0)$,

(7) $\psi_1 m = \psi m' = 0$.

(m' replaces m in $\mathscr{b}_F(v*\hat{y})$, see (i), $m' \notin \ell(\mathscr{b}_F v)$). Now put $f' \equiv \nu^K_{\mathscr{b}_F v}(\lambda^K k.[\psi_1 k])$, then

(8) $f' = \nu^K_{\mathscr{b}_F(v*\hat{y})}(\lambda^K k.[\psi_1 k])$ (by (7))

and

(9) $nz(\psi_1, \mathscr{b}_F(v*\hat{y})) = n$ (by (6) and (7)).

Moreover

$$j_b(f'|\chi) = \langle j_b\phi(x \dot{-} 1k), \ldots, j_b\phi(x \dot{-} 1)\rangle * j_b\chi,$$

whence

$$j_b(d^v_F\hat{y}:f'|\chi) = \psi k * \overline{j_b\phi}(x_1) * j_b\chi$$

for any $b \in \mathscr{b}_F v$ with label k (by (5)), i.e.

(10) $d^v_F\hat{y}:f' \simeq f:[\bar{\phi}x_1]$.

By induction hypothesis, applied with $v*\hat{y}$ for v, $s^{x_1}|\phi$ for ϕ and ψ_1, f' for ψ and f respectively (cf. (8),(9)), we find x_2 and w' such that either $\overline{s^{x_1}|\phi}(x_2) \# \mathscr{b}_F(v*\hat{y})$, which proves (B), or $d^{v*\hat{y}}_F w \simeq f':[\overline{s^{x_1}|\phi}(x_2)]$ and $\mathscr{b}_F(v*\hat{y}*w') \approx \mathscr{b}_F(v*\hat{y})$. In that case (B) follows from (2) and (10) with x_1+x_2 for x and $\hat{y}*w'$ for w. □

Finally we prove

(C) *For arbitrary* F,v,ϕ *and* $\psi\colon \mathbb{N} \rightarrow C$, *we can find* x,w *and* $\psi' \in N$ *such that either* $\bar{\phi}x_1 \# \mathscr{b}_F v$,
or $\mathscr{b}_F(v*w) \approx \mathscr{b}_F v$ *and* $d^v_F w:f' \simeq f:[\bar{\phi}x]$,
where $f' \simeq \nu^K_{\mathscr{b}_F(v*w)}(\lambda^K n.[\psi' n])$ *and* $f \simeq \nu^K_{\mathscr{b}_F v}\psi$.

(We leave it to the reader to verify that (B) and (C) imply (A)).

PROOF. The proof is very similar to the proof of (B) above. We show that

$$(1) \qquad \forall v F\phi \forall \psi \in C^{\mathbb{N}} [\mathrm{nix}(\psi, \delta_F v) \leq n \rightarrow$$
$$\exists \psi' \in N \exists xw(\bar{\phi}x /\!/ \delta_F v \rightarrow d_F^v w : f' \simeq f : [\bar{\phi}x] \wedge \delta_F(v*w) \approx \delta_F v)],$$

where $f \simeq \nu^K_{\delta_F v}\psi$ and $f' \simeq \nu^K_{\delta_F(v*w)}(\lambda^K n.[\psi' n])$, and where $\mathrm{nix}(\psi, \delta_F v) \leq n$ is the formula which expresses that we have a subset of $\ell(\delta_F v)$ with $\mathrm{card}(\ell(\delta_F v)) \dot{-} n$ labels to which ψ assigns a value of the form [u]. ([u]|• prefixes the finite sequence u to elements of N, nix is a contraction of non-prefix.)

The proof proceeds by induction w.r.t. n.

If n = 0 then ψk has the form [u] for all $k \in \ell(\delta_F v)$, i.e. there is a ψ' such that $\forall k \in \ell(\delta_F v)(\psi k \simeq [\psi' k])$, and (1) follows trivially with x = w = 0.

Now assume (1) to hold for n. Let v, F and ϕ be arbitrary, and assume $\mathrm{nix}(\psi, \delta_F v) \leq n+1$. Let m be a label of $\delta_F v$ outside the given set of labels k for which ψk has the form [u].

Determine x_1 and y as follows ((i)-(iii)):

(i) For the jps-part of y take $\nu_3(0,m,m')$, $m' \notin \ell(\delta_F v)$, then

$$(2) \qquad \delta_F(v*\hat{y}) \approx \delta_F v.$$

(ii) Choose the jf-part of y in such a way that jf(m) = ψm, jf(k) = id for k ≠ m, where jf ≡ jf(v*ŷ). JF will abbreviate JF(F,v*ŷ).

(iii) Let (ψ-jf) be the mapping from $\mathbb{N}$ into C such that (ψ-jf)m = id and (ψ-jf)k = ψk for k ≠ m. Put (f-JF) ≡ $\nu^K_{\delta_F v}(\psi\text{-jf})$.

Apply lemma 5.3.8 with (f-JF)|ϕ for ϕ. We find x_1 and a value for the gv-part of y such that either $(f\text{-}JF)|\phi \,\#\, \delta_F(v*\hat{y})$,

or $GV(F,v*\hat{y}) \simeq [\overline{(f\text{-}JF)|\phi}(x_1)]$.

By 1.3.12 there is an $x_1' \geq x_1$ such that $\overline{(f\text{-}JF)|\phi}(x_1)$ is an initial segment of $(f\text{-}JF)\restriction\bar{\phi}(x_1')$.

We assume that $\bar{\phi}(x_1') /\!/ \delta_F v$.

Since $(f\text{-}JF) /\!/_C \delta_F v$ by definition, then also $(f\text{-}JF)\restriction\bar{\phi}(x_1') /\!/ \delta_F v$ (3.2.21(j)), whence $\overline{(f\text{-}JF)|\phi}(x_1) /\!/ \delta_F v$ (3.2.20(e)). Then $\overline{(f\text{-}JF)|\phi}(x_1) /\!/ \delta_F(v*\hat{y})$, by (2), so $GV(F,v*\hat{y}) \simeq [\overline{(f\text{-}JF)|\phi}(x_1)]$ (see (iii) above), i.e.

$$d_F^v\hat{y} \simeq JF:[\overline{(f-JF)}\,|\,\phi(x_1)].$$

We put

$$f_1 \equiv s^{x_1}:(f-JF):[\bar{\phi}(x_1')].$$

$f_1|\cdot$ is the composition of three mappings, $[\bar{\phi}(x_1')]|\cdot$, $(f-JF)|\cdot$ and $s^{x_1}|\cdot$. For all χ, $[\bar{\phi}x_1']|\chi = \bar{\phi}x_1'*\chi$; $(f-JF)|\cdot$ maps all sequences with initial segment $\bar{\phi}x_1'$ onto sequences with initial segment $(f-JF)\restriction\bar{\phi}x_1'$; $s^{x_1}|\cdot$ deletes the first x_1 values of all sequences, for sequences with initial segment $(f-JF)\restriction\bar{\phi}x'$ these first x_1 values are $\overline{(f-JF)}\,|\,\phi(x_1)$. That is to say $[\overline{(f-JF)}\,|\,\phi(x_1)]:f_1 \simeq (f-JF):[\bar{\phi}x_1']$, and

(3) $$d_F^v\hat{y}:f_1 \simeq JF:(f-JF):[\bar{\phi}x_1'] \simeq f:[\bar{\phi}x_1'].$$

(The equivalence $JF:(f-JF) \simeq f$ is easily verified.)

Define $\psi_1 : \mathbb{N} \to C$ by

$$\psi_1 k = \begin{cases} s^{x_1}:\psi k:[k_{\chi k}(\bar{\phi}x_1')] \text{ if } k \neq m', \\ s^{x_1}:[k_{\chi m'}(\bar{\phi}x_1')] \text{ for } k = m', \end{cases}$$

where χ is a labelling-inverse for $\mathcal{f}_F(v*\hat{y})$, such that for all $k \in \mathcal{L}(\mathcal{f}_F(v*\hat{y}))$ χk is a branch of $\mathcal{f}_F(v*\hat{y})$ with label k. One may verify that

$$f_1 \simeq \nu^K_{\mathcal{f}_F(v*\hat{y})}\psi_1,$$

$$\forall k\in\mathcal{L}(\mathcal{f}_F(v*\hat{y}))(\exists u(\psi k \simeq [u]) \to \exists v(\psi_1 k \simeq [v])),$$

and

$$\exists u(\psi_1 m' \simeq [u]).$$

So $nix(\psi_1, \mathcal{f}_F(v*\hat{y})) \leq n$, and we can apply induction hypothesis with $v*\hat{y}$ for v, $s^{x_1'}|\phi$ for ϕ and ψ_1, f_1 for ψ and f respectively, to find x_2, w' and $\psi' \in N$ such that either $s^{x_1}|\phi(x_2) \# \mathcal{f}_F(v*\hat{y})$, which proves (C) or

(4) $\delta_F(v*\hat{y}*w') \approx \delta_F(v*\hat{y})$ and $d_F^{v*\hat{y}}w':f' \simeq f_1:[\overline{s^{x_1'}|\phi(x_2)}]$,

where $f' \simeq \nu^K_{\delta_F(v*\hat{y}*w)}(\lambda^K n.[\psi' n])$.
In that case (C) follows immediately by (2) and (3), with $\hat{y}*w'$ for w, $x_1'+x_2$ for x; in particular we have

$$d_F^v w:f' \simeq d_F^v\hat{y}:d_F^{v*\hat{y}}w':f' \quad \text{(by 4.4.21(c))},$$

and

$$d_F^v\hat{y}:d_F^{v*\hat{y}}w:f' \simeq d_F^v\hat{y}:f_1:[\overline{s^{x_1'}|\phi(x_2)}] \quad \text{(by (4))},$$

whence

$$d_F^v\hat{y}:f_1:[\overline{s^{x_1'}|\phi(x_2)}] \simeq f:[\bar{\phi}(x_1'+x_2)] \quad \text{(by (3))}. \quad \Box$$

CHAPTER 6

THE CONCEPT OF A DOMAIN

In the next chapters we intend to show that for a special class of dependency-closed subsets of K, each projected universe $\mathcal{U}_\delta$ of GC-sequences w.r.t. C is a model for $\underset{\sim}{CS}$(C) (cf.1.3.29).

In the definition of U_δ(C), the set of projected universes of GC-sequences w.r.t. C, we have used natural numbers to 'code' all kinds of information: αx codes the jumps at stage x+1, βx codes the choice of a jump-function and γx codes the preliminary choice of values at stage x+1.

The coding which we use is fairly arbitrary. E.g. $\alpha x = \nu_3(0,k,m)$ expresses 'if possible, make k jump to m', and $\alpha x = \nu_3(z+1,k,j(n,m))$ expresses 'if possible, make k jump to n and m'. For the same purpose we could also have used $\alpha x = j(2k,m)$ and $\alpha x = j(2k+1,j(n,m))$ respectively.

Moreover, the concept of GC-sequence that is imitated in universes $\mathcal{U}_\delta \in U_\delta(C)$ has some special features like the single jump property, the restriction to binary jumps and the guarantee that at stage x for all carriers the initial segment with length x is available.

It would be most satisfactory if we could show that the validity of $\underset{\sim}{CS}$(C) in universes $\mathcal{U}_\delta \in U_\delta(C)$ is independent of our choice of coding and of the special features of our concept of GC-sequence.

To achieve this we introduce for each $C \subset K$ a class $D_\delta(C)$ of *domains* w.r.t. C. The definition of $D_\delta(C)$ is coding-independent, and does not require any of the special features of the universes $\mathcal{U}_\delta \in U_\delta(C)$. For dependency-closed C, $U_\delta(C) \subset D_\delta(C)$. For suitable dependency-closed C, all $\mathcal{D}_\delta \in D_\delta(C)$ are models of $\underset{\sim}{CS}$(C).

6.1. THE DEFINITION OF DOMAIN

A domain w.r.t. C (to be defined formally in 6.1.1 below) is a universe of the form

$$\mathcal{D}_\delta \equiv \{e|\pi_F\delta : e \in C,\ F \in \mathrm{FRAME}\},$$

where $\pi_F\delta \equiv \pi_F|\delta$, $\pi_F \in K$ the image of $F \in$ FRAME under the mapping π, i.e. a domain w.r.t. C has the same structure as a universe $\mathcal{U}_\delta \in \mathrm{U}_\delta(C)$.

With each domain there are two lawlike mappings d and $\mathfrak{f}$ from $\mathbb{N}\times\mathrm{FRAME}\times\mathbb{N}$ into K and from FRAME $\times\ \mathbb{N}$ into FRAME respectively. We put $d_F^v w \equiv d(v,F,w)$, $d_F w \equiv d(0,F,w)$, $d_n^v w \equiv d(v,(^\circ n),w)$, $\mathfrak{f}_F w \equiv \mathfrak{f}(F,w)$, $\mathfrak{f}_n w \equiv \mathfrak{f}((^\circ n),w)$ and $\pi_n \equiv \pi_{(^\circ n)}$, and we call $d_F(\bar{\delta}x)$ the dressing for $\pi_F\delta$ at stage x', $\mathfrak{f}_F(\bar{\delta}x)$ the 'frame for $\pi_F\delta$ at stage x', just as for universes $\mathcal{U}_\delta \in \mathrm{U}_\delta(C)$.

The mappings π,d and $\mathfrak{f}$ associated with a domain, satisfy the following 'axioms':

(a) For the relation between π and d:

(1) $$\pi_F(\hat{x}*w)=y+1 \rightarrow \forall a[(d_F w|a)(x)=y]$$

which expresses that $\pi_F\delta$ is the intersection of the ranges of the mappings $\lambda\phi.d_F(\bar{\delta}x)|\phi$.

(b) For d:

(2) $$d_F^v\langle\ \rangle \simeq \mathrm{id},$$

(3) $$d_F^v w \in C$$

and

(4) $$d_F^u(v*w) \simeq d_F^u v : d_F^{u*v} w.$$

The last axiom expresses that $d_F^{u*v}w$ is the difference between $d_F^u(v*w)$ and $d_F^u v$, in particular $d_F^v w$ is the difference between $d_F v$ and $d_F(v*w)$. Note that $d_F\bar{\delta}x$ is (modulo equivalence) completely determined by the values $d_F^{\bar{\delta}y}\langle\delta y\rangle$, $y < x$.

(c) For the relation between d and $\mathfrak{f}$((5),(6)):

(5) $$d_F^v w \simeq \nu^K_{\mathfrak{f}_F v}(\lambda^K n.d_n^v w).$$

From this axiom and (1) one finds that there is a relation between $\pi_F\delta$ and the sequences $\pi_n\delta$, $n \in \mathcal{L}(\mathfrak{f}_F\bar{\delta}x)$. From (5) and (4) it follows that $d_F(\bar{\delta}x)$ is completely determined by the values $\{d_n^{\bar{\delta}y}\langle\delta y\rangle : n \in \ell(\mathfrak{d}_F\bar{\delta}y)\}$ for $y < x$. The axioms do not specify any further properties of $d_n^{\bar{\delta}y}\langle\delta y\rangle$ for $n \in \ell(\mathfrak{d}_F(\bar{\delta}y))$, in particular it is not required that $d_n^{\bar{\delta}y}\langle\delta y\rangle$ is built from jf and gv or similar mappings.

(6) $$\mathfrak{d}_n v = {}^\circ n \rightarrow \exists u(d_n v \simeq [u]),$$

which expresses that if $\pi_n\delta$ is independent of others at stage x, i.e. $\mathfrak{d}_n(\bar{\delta}x) = {}^\circ n$, then there is only an initial segment of $\pi_n\delta$ available at stage x.

(d) For $\mathfrak{d}$((7)-(10)):

(7) $$\mathfrak{d}_F\langle\ \rangle = F,$$

(8) $$\forall w \exists g \forall F(\mathfrak{d}_F(v * w) = \mathfrak{d}_F v[g]),$$

which expresses that $\mathfrak{d}_F\bar{\delta}(x+y)$ is produced from $\mathfrak{d}_F(\bar{\delta}x)$ by the same g for all F. This axiom is equivalent to

$$\forall x \exists g \forall F(\mathfrak{d}_F(v * \hat{x}) = \mathfrak{d}_F v[g]).$$

The axioms do not require that g has the properties of jps like the restriction to binary jumps and the single jump property.

(9) $$\forall n \in \ell(\mathfrak{d}_F v)(\mathfrak{d}_n v = {}^\circ n),$$

i.e. if n occurs as a label in $\mathfrak{d}_F v$ then it is itself independent of others.

(10) $$\forall m \exists n > m(\mathfrak{d}_n v = {}^\circ n),$$

i.e. there are infinitely many n which are independent of others

(e) For d and $\mathfrak{d}$ finally:

(11) the strong overtake property (5.2.4)

which expresses the freedom of continuation of sequences of restrictions $\lambda z.(d_F(\bar{\delta}z), \mathfrak{d}_F(\bar{\delta}z))$.

All these axioms hold for projected universes of GC-sequences w.r.t. C, see definition 4.5.2 and the lemmata 4.3.14, 4.4.21 and 5.2.5.

6.1.1. DEFINITION (of domain).

Let π and d be mappings from FRAME into K and from $\mathbb{N}\times$FRAME $\times\ \mathbb{N}$ into K respectively. Let π_F be the image of F under π, and $d_F^v w$ the image of (v,F,w) under d. Let $\mathfrak{f}$ be a mapping from FRAME $\times\ \mathbb{N}$ into FRAME, and let $\mathfrak{f}_F v$ be the image of (F,v) under $\mathfrak{f}$.

Put $\pi_n \equiv \pi_{(\circ n)}$, $d_n^v w \equiv d_{(\circ n)}^v w$, $\mathfrak{f}_n v \equiv \mathfrak{f}_{(\circ n)} v$, $d_F v \equiv d_F^0 v$ and $\pi_F\delta \equiv \pi_F|\delta$.

π,d and $\mathfrak{f}$ *define a domain w.r.t.* C iff the following hold:

(D1) $\forall v(\pi_F(\hat{x}*v)=y+1 \rightarrow \forall a[(d_F v|a)(x)=y])$,

(D2a) $\mathfrak{f}_F 0 = F$,

(D2b) $\forall vw\exists g\forall F(\mathfrak{f}_F(v*w)=\mathfrak{f}_F v[g])$,

(D3a) $d_F 0 \simeq id$,

(D3b) $d_F^u(v*w) \simeq d_F^u v : d_F^{u*v} w$,

(D3c) $d_F^v w \simeq \nu^K_{\mathfrak{f}_F v}(\lambda^K n.d_n^v w)$,

(D3d) $d_F^v w \in C$,

(D4) $\forall n\in\ell(\mathfrak{f}_F v)(\mathfrak{f}_n v=\circ n)$,

(D5) $\forall n\exists m>n(\mathfrak{f}_m v=\circ m)$,

(D6) $\exists a\forall n(\mathfrak{f}_n v=\circ n \rightarrow d_n v \simeq [an])$,

(D7) the strong overtake property for d and $\mathfrak{f}$, i.e.

$$\forall f//_C\mathfrak{f}_F v\forall G\geq\mathfrak{f}_F v\forall g\exists e\forall u//G\,[eu\neq 0 \rightarrow$$

$$\exists w(gw\neq 0 \wedge (f,G)\leq(d_F^v w,\mathfrak{f}_F(v*w))\leq(f:[u],G))].$$

We call (D1)-(D7) the *domain axioms*.

A universe $\mathcal{D}_\delta$ projected from the single lawless sequence δ is a *domain w.r.t.* C iff there are π,d and $\mathfrak{f}$ which define a domain w.r.t. C such that $\mathcal{D}_\delta = \{e|\pi_F\delta:e \in C,\ F \in \text{FRAME}\}$.

The sequences $\pi_n\delta$ are *the carriers of the domain* $\mathcal{D}_\delta$, $d_F(\bar{\delta}x)$ is the dressing, $\mathfrak{f}_F(\bar{\delta}x)$ is the frame and $(d_F(\bar{\delta}x),\mathfrak{f}_F(\bar{\delta}x))$ is the restriction for $\pi_F\delta$ at stage x. $D_\delta(C)$ is the set of all domains w.r.t. C

6.2. THEOREM (models are domains).

If $\mathcal{U}_\delta \in U_\delta(C)$, C *dependency-closed, then* $\mathcal{U}_\delta \in D_\delta(C)$, *i.e. if* π *generates* *nests of* GC-*carriers and* d *generates the dressings for* π *(cf. definition* 4.5.2) *and* $\mathfrak{f}$ *is the mapping which assigns to* (F,v) *the frame for* π_F *at* v *(cf. definition* 4.3.12) *then* π,d *and* $\mathfrak{f}$ *define a domain.*

<u>PROOF</u>. Immediate from definition 4.5.2 and the lemmata 4.3.14, 4.4.21 and 5.2.5. □

6.3. PROPERTIES OF DOMAINS

6.3.1. <u>LEMMA</u>. *Let* π,d *and* $\mathfrak{f}$ *define a domain, then* $\mathfrak{f}$ *satisfies:*

(a) $\mathfrak{f}_F v = F[\lambda n.\mathfrak{f}_n v]$,

(b) $\mathfrak{f}_{F\wedge G} v = \mathfrak{f}_F v \wedge \mathfrak{f}_G v$,

(c) $\mathfrak{f}_F v = \mathfrak{f}_G v \rightarrow \mathfrak{f}_F(v*w) = \mathfrak{f}_G(v*w)$,

(d) $\mathfrak{f}_{\mathfrak{f}_F v}(v*w) = \mathfrak{f}_F(v*w)$.

<u>PROOF</u>.

(a) Let g satisfy

(1) $\forall F(\mathfrak{f}_F v = \mathfrak{f}_F < >[g])$,

such a g exists by (D2b). By (D2a), (1) yields

(2) $\forall F(\mathfrak{f}_F v = F[g])$,

whence in particular, for all n, $\mathfrak{f}_n v = ({}^\circ n)[g] = gn$ (by definition of $F[\mathfrak{f}]$, 3.1.16), i.e. $g = \lambda n.\mathfrak{f}_n v$. Hence (2) becomes

(3) $\forall F(\mathfrak{f}_F v = F[\lambda n.\mathfrak{f}_n v])$.

(Compare the proof of corollary 4.3.15.)

(b) $\mathfrak{f}_{F\wedge G} v = (F\wedge G)[\lambda n.\mathfrak{f}_n v] = F[\lambda n.\mathfrak{f}_n v] \wedge G[\lambda n.\mathfrak{f}_n v] = \mathfrak{f}_F v \wedge \mathfrak{f}_G v$, the first and last equality by (a), the second one by definition of $F[\mathfrak{f}]$, 3.1.16.

(c) Assume $\mathfrak{f}_F v = \mathfrak{f}_G v$, let g satisfy $\forall H(\mathfrak{f}_H(v*w) = \mathfrak{f}_H v[g])$. Then $\mathfrak{f}_F(v*w) = \mathfrak{f}_F v[g] = \mathfrak{f}_G v[g] = \mathfrak{f}_G(v*w)$.

(d) In view of (c) it suffices to show that $\mathfrak{f}_{\mathfrak{f}_F v} v = \mathfrak{f}_F v$. We find $\mathfrak{f}_{\mathfrak{f}_F v} v = \mathfrak{f}_F v[\lambda n.\mathfrak{f}_n v]$ by (a), and $\forall m \in \mathcal{L}(\mathfrak{f}_F v)(\mathfrak{f}_m v = {}^\circ m)$ by (D4), hence $\mathfrak{f}_F v[\lambda n.\mathfrak{f}_n v] = \mathfrak{f}_F v$ by 3.1.19(d). □

6.3.2. LEMMA. *Let* π,d *and* $\mathfrak{f}$ *define a domain, then* d *satisfies:*

(a) $d^v_{F\wedge G}w \simeq d^v_F w \wedge d^v_F w$,

(b) $d^v_{\mathfrak{f}_F v} w \simeq d^v_F w$,

(c) $d^v_F w //_C \mathfrak{f}_F v$, $d_F w //_C F$,

(d) $\forall a(d^v_F w | a\in u) \to u // \mathfrak{f}_F v$,

(e) $\forall a(d_F v | a(x)=y) \to \forall a(d_F(v*w) | a(x)=y)$.

PROOF.

(a) The following equivalences hold by (D3c), 6.3.1(b), the definition of ν^K_F (3.2.5) and (D3c) respectively:

$$d^v_{F\wedge G}w \simeq \nu^K_{\mathfrak{f}_{F\wedge G}v}(\lambda^K n.d^v_n w) \simeq \nu^K_{\mathfrak{f}_F v\wedge \mathfrak{f}_G v}(\lambda^K n.d^v_n w) \simeq$$

$$\nu^K_{\mathfrak{f}_F v}(\lambda^K n.d^v_n w)\wedge\nu^K_{\mathfrak{f}_G v}(\lambda^K n.d^v_n w) \simeq d^v_F w \wedge d^v_G w.$$

(b) Like (a) above, now using (D3c) and 6.3.1(d).

(c) The first assertion is immediate from (D3c) and (D3d), the second assertion follows from the first one by (D2a).

(d) Assume $\forall a(d^v_F w|a\in u)$, then in particular $\forall a//\mathfrak{f}_F v(d^v_F w|a\in u)$. By (c) and 3.2.20(r) we find $\forall a//\mathfrak{f}_F v(d^v_F w|a//\mathfrak{f}_F v$, hence $u//\mathfrak{f}_F v$, by 3.2.20(d).

(e) Immediate from (D3b). □

6.3.3. COROLLARY. *If* π,d *and* $\mathfrak{f}$ *define a domain, then*
$\forall e\forall w[(e:d_F(v*w),\ \mathfrak{f}_F(v*w))\geq(e:d_F v, \mathfrak{f}_F v)]$.

PROOF. By (D3b), $e:d_F(v*w) \simeq e:d_F v:d^v_F w$, by 6.3.2(c) $d^v_F w//_C \mathfrak{f}_F v$, and $\mathfrak{f}_F(v*w) = \mathfrak{f}_F v[g]$ for some g by (D2b). □

6.3.4. LEMMA. *Let* π,d *and* $\mathfrak{f}$ *define a domain. Then*

(a) $\pi_F\delta(x)=y \leftrightarrow \exists v(\delta\in v \wedge \forall a(d_F v|a(x)=y))$,

(b) $\forall n\forall b(\ell_b F=n \to j_b(\pi_F\delta)=\pi_n\delta)$, *and hence* $\pi_F\delta//F$,

(c) $\pi_F\delta\in u \leftrightarrow \exists v(\delta\in v \wedge \forall a(d_F v|a\in u))$,

(d) $\exists g\forall\delta\in v(\pi_F\delta=d_F v|(g|\delta) \wedge g|\delta//\mathfrak{f}_F v)$.

(d) states that $\pi_F\delta \in \text{range}(\lambda\phi.d_F(\bar{\delta}x)|\phi)$, and that the sequence ψ such that $\pi_F\delta = d_F(\bar{\delta}x)|\psi$ has the form $g|\delta$ and is parallel to $\mathfrak{f}_F(\bar{\delta}x)$. Inspection of the proof will show that

$$\psi = \nu^1_{\not{\delta}_F(\bar{\delta}x)}(\lambda^1 n.s^{bn}|\pi_n\delta),$$

where bn is the length of the initial segment of $\pi_n\delta$ that is available to us at stage x (for fresh n). I.e. in projection models ψ is the source for $\pi_F\delta$ at stage x, and this result is the one that was announced in 4.5.7.

PROOF.

(a) The implication from left to right follows trivially from (D1). From right to left: let v be an initial segment of δ which satisfies

(1) $\forall a(d_F v|a(x)=y)$,

let w be an initial segment of δ such that

(2) $\pi_F(\hat{x}*w) \neq 0$.

Since v and w are initial segments of the same sequence δ, we have $v = w*u$ or $w = v*u'$. In both cases we find $\pi_F(\hat{x}*w) = y+1$:
if $v = w*u$, then (2) implies (by (D1)) $\forall a(d_F w|a(x)=\pi_F(\hat{x}*w)\dot{-}1)$, hence $\forall a(d_F w|(d^v_F u|a)(x)=\pi(\hat{x}*w)\dot{-}1)$, hence (by (D3b)) $\forall a(d_F v|a(x)=\pi(\hat{x}*w)\dot{-}1)$, hence, by (1), $\pi(\hat{x}*w) = y+1$; if on the other hand $w = v*u'$, then by (1) and (D3b) $\forall a(d_F w|a(x)=y)$, while by (2) and (D1) $\forall a(d_F w|a(x)=\pi(\hat{x}*w)\dot{-}1)$, hence $\pi(\hat{x}*w) = y+1$.

(b) The second assertion is an immediate corollary of the first one. To prove the first assertion let n be a label of F, and $b \in F$ a branch such that $\ell_b F = n$. Let x and y be such that $\pi_F\delta(x) = y$, we show that $\pi_n\delta(x) = j_b y$.
By (a) above, there is an initial segment v of δ such that $\forall a(d_F v|a(x)=y)$. Hence $\forall a(j_b(d_F v|a)(x)=j_b y)$. By (D3c), (D2a) $d_F v \simeq \nu^K_F(\lambda^K n.d_n v)$, hence (by 3.2.16(c)), $j_b(d_F v|a) = d_n v|j_b a$, and we find $\forall a(d_n v|j_b a(x)=j_b y)$, i.e. $\forall b(d_n v|b(x)=j_b y)$, whence by (a): $\pi_n\delta(x) = j_b y$. (Cf.4.5.5.)

(c) This is an easy corollary of (a) and (D3b).

(d) By (D6) and (D4) there is an a such that $\forall n\in\ell(\not{\delta}_F v)(d_n v\simeq[an])$. Put $b \equiv \lambda n.\ \mathrm{lth}(an)$, $e \equiv \nu^K_{\not{\delta}_F v}(\lambda^K n.s^{bn})$, $f \equiv \pi_{\not{\delta}_F v}$ and $g \equiv e{:}f$.
Then g satisfies (i) $\forall\delta(g|\delta /\!/ \not{\delta}_F v)$, (ii) $\forall\delta\in v(d_F v|(g|\delta)=\pi_F\delta)$.

(i) By (b), $f|\delta(=\pi_{\not{\delta}_F v}\delta)$ is parallel to $\not{\delta}_F v$, $e /\!/_K F$ by definition, hence $e{:}f|\delta /\!/ F$ by 3.2.20(r).

(ii) Let δ have initial segment v, and assume $(d_F v{:}g|\delta)(x) = y$, i.e. there is a u such that

(1) $\quad d_F v(\hat{x}*u) = y+1$

and

(2) $\quad g|\delta \in u.$

We show that there is a w such that $\delta \in v*w$ and

(3) $\quad \forall a(d_F(v*w)\,|\,a(x)=y),$

whence (by (a)) $\forall\delta'\in v*w(\pi_F\delta'(x)=y)$, hence $\pi_F\delta(x) = y$.
From (2) and the definition of g ($g \equiv e{:}f$), we find a u' such that

(4) $\quad \forall c\in u'(e\,|\,c\in u)$

and

(5) $\quad f|\delta \in u'.$

$f|\delta \equiv \pi_{\delta_F v}\delta$, so by (5) and (c) we find a w such that $\delta \in v*w$ and

(6) $\quad \forall a(d_{\delta_F v}(v*w)\,|\,a\in u').$

Hence by (4)

(7) $\quad \forall a(e{:}d_{\delta_F v}(v*w)\,|\,a\in u).$

By (D3b)

$$e{:}d_{\delta_F v}(v*w) \simeq e{:}d_{\delta_F v}v{:}d^{\vee}_{\delta_F v}w.$$

Now

(8) $\quad e{:}d_{\delta_F v}v \simeq \nu^K_{\delta_F v}(\lambda^K n.s^{bn}){:}\nu^K_{\delta_F v}(\lambda^K n.d_n v) \simeq \nu^K_{\delta_F v}(\lambda^K n.s^{bn}{:}d_n v),$

the first equivalence by definition of e, 6.3.2(b) and (D3c), the second one by 3.2.16(e). By definition of b, $s^{bn}{:}d_n v \simeq id$ for all $n \in \ell(\mathfrak{f}_F v)$, hence (by (8) and 3.2.19(d)) $e{:}d_{\mathfrak{f}_F v} v \simeq id$, whence

$$e{:}d_{\mathfrak{f}_F v}(v*w) \simeq id{:}d^v_{\mathfrak{f}_F v} w \simeq d^v_F w$$

the second equality by 6.3.2(b)). So (7) yields

(9) $\quad \forall a(d^v_F w \mid a \in u)$.

Then

(10) $\quad \forall a(d_F v{:}d^v_F w \mid a(x)=y)$

follows by (1), hence (3) holds by (D3b). □

6.3.5. <u>LEMMA</u>. *Let* π,d *and* $\mathfrak{f}$ *define a domain. Then*

(a) $(f,G) \geq (e{:}d_F v, \mathfrak{f}_F v) \rightarrow$

$$\exists e_2 \forall u /\!/ G\, [e_2 u \neq 0 \rightarrow \exists w((f,G) \leq (e{:}d_F(v*w), \mathfrak{f}_F(v*w)) \leq (f{:}[u],G))],$$

(b) *if* id ∈ C *then*

$\forall e_1 \exists e_2 \forall u /\!/ \mathfrak{f}_F v\, [e_2 u \neq 0 \rightarrow$

$$\exists w(e_1 w \neq 0 \wedge (e{:}d_F(v*w), \mathfrak{f}_F(v*w)) \leq (e{:}d_F v{:}[u], \mathfrak{f}_F v))],$$

(c) *if* [u] ∈ C *for all* u, *then* $\forall a /\!/ \mathfrak{f}_F v \forall x \exists \delta \in v(\overline{\pi_F \delta(x)} = \overline{d_F v \mid a(x)})]$,

(d) $\forall v \exists f \forall w [f \restriction w /\!/ \mathfrak{f}_F v \ \wedge \ \forall x(f(\hat{x}*w) \neq 0 \rightarrow \forall a(d^v_F w \mid a(x) = f(\hat{x}*w) \dot{-} 1))]$,

(e) *if* $s^n \in$ C *for all* n *and* C *is closed under composition, then*

$\forall e_2 \exists e_1 \forall w[e_1 w \neq 0 \rightarrow$

$$\exists u /\!/ \mathfrak{f}_F v(e_2 u \neq 0 \wedge (e{:}d_F v{:}[u], \mathfrak{f}_F v) \leq (e{:}d_F(v*w), \mathfrak{f}_F(v*w)))],$$

(f) *if* $s^n \in$ C *for all* n *and* C *is closed under composition and pairing, then*

$\forall g \in C \forall H \forall F \forall u \exists f \in C \exists G$

$$[((e{:}d_F u) \wedge (f{:}d_G u),\ \mathfrak{f}_F u \wedge \mathfrak{f}_G u) \geq ((e{:}d_F u) \wedge g,\ \mathfrak{f}_F u \wedge H)].$$

Note that the conditions on C in (b), (c), (e) and (f) are automatically fulfilled if C is dependency-closed.

(a), (b), (c) and (e) are corollaries of the strong overtake property (D7).

(a) says that with any restriction (f,G) stronger than $(e{:}d_Fv,\mathfrak{f}_Fv)$ we can find a bar e_2 such that with all u parallel to G in this bar there is a w such that $(e{:}d_F(v{*}w)\mathfrak{f}_F(v{*}w))$ overtakes (f,G), but remains weaker than $(f{:}[u],G)$.

(b) says that with each bar e_1 we can find a bar e_2 such that if $u/\!/\mathfrak{f}_Fv$ lies in the bar e_2 then there is a w in the bar e_1 such that the restriction $(e{:}d_F(v{*}w),\mathfrak{f}_F(v{*}w))$, which is stronger than $(e{:}d_Fv,\mathfrak{f}_Fv)$, is still weaker than $(e{:}d_Fv{:}[u],\mathfrak{f}_Fv)$.

(c) says that we can choose $\delta \in v$ such that the initial segment $\overline{\pi_F\delta}(x)$ of $\pi_F\delta$ equals $\overline{d_Fv|a}(x)$, for any $a/\!/\mathfrak{f}_Fv$. (Recall that by 6.3.4(d) for all $\delta \in v$, $\pi_F\delta = d_Fv|\psi$, for some $\psi/\!/\mathfrak{f}_Fv$.)

(e) says that with any bar e_2 there is a bar e_1 such that if w lies in the second bar then $(e{:}d_F(v{*}w),\mathfrak{f}_F(v{*}w))$ is stronger than $(e{:}d_Fv{:}[u],\mathfrak{f}_Fv)$ for some $u/\!/\mathfrak{f}_Fv$ in the first bar.

(d) says that there is a mapping f such that for all ϕ $f|\phi$ is the intersection of the ranges of $\lambda^1\psi.\ d_F^v(\bar{\phi}x)|\psi$. (E.g. for $v = 0$, we can take $f = \pi_F$ by (D1).) f satisfies $\forall\phi(f|\phi/\!/\mathfrak{f}_Fv)$.

(f) finally says something about the existence of restrictions of the form $(f{:}d_Gu,\mathfrak{f}_Gu)$.

Let $(e{:}d_F(\bar{\delta}x),\mathfrak{f}_F(\bar{\delta}x))$ be the restriction for $e|\pi_F\delta$ at stage x and let (g,H) be an arbitrary restriction, $g \in C$.

Note that $((e{:}d_F(\bar{\delta}x))\wedge(f{:}d_G(\bar{\delta}x)),\ \mathfrak{f}_F(\bar{\delta}x)\wedge\mathfrak{f}_G(\bar{\delta}x))$ is equivalent to $((e\wedge f){:}d_{F\wedge G}(\bar{\delta}x),\ \mathfrak{f}_{F\wedge G}(\bar{\delta}x))$, by distributivity of: over $\wedge$, 6.3.2(a) and 6.3.1(b). The second restriction is the restriction for $e\wedge f|\pi_{F\wedge G}\delta(=j(e|\pi_F\delta,f|\pi_G\delta))$ at stage x.

The claim is that we can choose f and G in such a way that this restriction is stronger than $((e{:}d_F(\bar{\delta}x))\wedge g,\ \mathfrak{f}_F(\bar{\delta}x)\wedge H)$.

PROOF (of 6.3.5).

(a) Assume $(f,G) \geq (e{:}d_Fv,\mathfrak{f}_Fv)$, then we can find an f' and a g such that

(1) $$G = \mathfrak{f}_Fv[g],$$

(2) $$f \simeq e{:}d_Fv{:}f',\quad f'/\!/_C\,\mathfrak{f}_Fv,$$

i.e. we have $(f',G) \geq (id,\mathfrak{f}_Fv)$.

By (D7) we find an e_2 such that if $u/\!/\mathfrak{f}_Fv$ and $e_2u \neq 0$ then for some w $(f',G) \leq (d_F^vw,\mathfrak{f}_F(v{*}w)) \leq (f'{:}[u]G)$.

But then

$$(e:d_Fv:f',G) \le (e:d_Fv:d_F^{v}w,\delta_F(v*w)) \le (e:d_Fv:f':[u],G),$$

by 5.1.6(d), and hence by (2) and (D3b)

$$(f,G) \le (e:d_F(v*w),\delta_F(v*w)) \le (f:[u],G).$$

(b) If id $\in$ C, then $(id,\delta_Fv) \ge (id,\delta_Fv)$. Hence, by (D7) we find

$$\forall e_1 \exists e_2 \forall u /\!/ \delta_F v\,[e_2u\neq 0 \rightarrow \exists w(e_1w\neq 0 \wedge (d_F^{v}w,\delta_F(v*w))\le([u],\delta_Fv))].$$

But if $(d_F^{v}w,\delta_F(v*w)) \le ([u],\delta_Fv)$, then also (by 5.1.6(d))
$(e:d_Fv:d_F^{v}w,\ \delta_F(v*w)) \le (e:d_Fv:[u],\delta_Fv)$, i.e. (by (D3b))
$(e:d_F(v*w),\ \delta_F(v*w)) \le (e:d_Fv:[u],\delta_Fv)$.

(c) Let $a /\!/ \delta_F v$ and x be arbitrary. Let y satisfy

(3) $\quad d_Fv\restriction(\bar{a}y) \not\succ \overline{d_Fv|a}(x)$ (1.3.12).

Since [v] $\in$ C for all v,

$$(d_Fv:[\bar{a}y],\delta_Fv) \ge (d_Fv,\delta_Fv) \quad (5.1.6(c)),$$

hence, by (a) above, there is a w such that

$$(d_F(v*w),\delta_F(v*w)) \ge (d_Fv:[\bar{a}y],\delta_Fv),$$

whence $d_F(v*w) \simeq d_Fv:[\bar{a}y]:f$ for some f. But then we find that for all b

(4) $\quad d_F(v*w)|b = d_Fv:[\bar{a}y]|(f|b) \in d_Fv\restriction(\bar{a}y)$ (1.3.11),

hence ((3), (4)):

$$\forall b(d_F(v*w)|b \in \overline{d_Fv|a}(x)).$$

By 6.3.4(a) this yields

$$\forall\delta'\in v*w(\overline{\pi_F\delta'\in d_F v|a(x)}),$$

whence by LS1 (which implies $\exists\delta(\delta\in v*w)$)

$$\exists\delta\in v(\overline{\pi_F\delta}(x) = \overline{d_F v|a}(x)).$$

(d) Let v be arbitrary. It suffices to show that there exists an $f \in K$ such that

$$\forall xw[f(\hat{x}*w)\neq 0 \rightarrow \forall a(d_F^v w|a(x)=f(\hat{x}*w)\dot{-}1)],$$

for such an f will automatically satisfy $\forall w\forall a(d_F^v w|a\in f\restriction w)$, whence, by 6.3.2(d), $f\restriction w//\delta_F v$.

Let a be such that $\forall n\in\mathcal{L}(\delta_F v)(d_n v\simeq[an])$ (cf.(D6) and (D4)), put $b \equiv \lambda n.\mathrm{lth}(an)$, $e \equiv \nu^K_{\delta_F v}(\lambda^K n.s^{bn})$, $g \equiv \pi_{\delta_F v}$. (See the proof of 6.3.4(d).)
Define f by $f0 = 0$, $f(\hat{x}*w) = (e:g)(\hat{x}*v*w) = e(\hat{x}*g\restriction(v*w))$. Obviously, f is an element of K. Now assume $f(\hat{x}*w) = y+1$, i.e. $e(\hat{x}*g\restriction(v*w)) = y+1$. By (D1) we have:

$$\forall a(d_{\delta_F v}(v*w)|a \in g\restriction(v*w)) \quad (g \equiv \pi_{\delta_F v}!)$$

hence

$$\forall a(e(\hat{x}*d_{\delta_F v}(v*w)|a)=y), \text{ i.e. } \forall a(e:d_{\delta_F v}(v*w)|a(x)=y).$$

As in the proof of 6.3.4(d) we have $e:d_{\delta_F v}(v*w) \simeq d_F^v w$, so we find $\forall a(d_F^v w|a(x)=y)$, where $y = f(\hat{x}*w)\dot{-}1$, as desired.

(e) Let e_2 be arbitrary. Put $e_1 \equiv e_2;f \equiv \lambda w.e_2(f\restriction w)$, f as in (d). Assume that $e_1 w \neq 0$, i.e. $e_2(f\restriction w) \neq 0$, we must find a $u//\delta_F v$ such that $e_2 u \neq 0$ and $(e:d_F v:[u],\delta_F v) \leq (e:d_F(v*w),\delta_F(v*w))$.

We take $u = f\restriction w$, then $u//\delta_F v$ (by (d)) and $e_2 u \neq 0$. In order to prove that

$$(e:d_F v:[f\restriction w],\delta_F v) \leq (e:d_F(v*w),\delta_F(v*w)),$$

it suffices to show that there are g and f' such that

(5) $\qquad \mathfrak{f}_F(v*w) = \mathfrak{f}_F v[g]$,

(6) $\qquad [f{\restriction}w]:f' \simeq d^v_F w,\ f' /\!/_C \mathfrak{f}_F v$ (use (D3b)).

(5) follows immediately from (D2b).

For (6), take $f' \simeq s^n : d^v_F w$, where $n \equiv \mathrm{lth}(f{\restriction}w)$. We find that by (d)

$$\forall a(d^v_F w|a = (f{\restriction}w)*\lambda z.d^v_F w|a(n+z))$$

hence $[f{\restriction}w]:f' \simeq d^v_F w$.

Moreover $s^n \simeq \nu^K_{\mathfrak{f}_F v}(\lambda^K m.s^n)$ by 3.2.16(f), $d^v_F w \simeq \nu^K_{\mathfrak{f}_F v}(\lambda^K m.d^v_m w)$ by (D3c), hence $f' \simeq \nu^K_{\mathfrak{f}_F v}(\lambda^K m.s^n:d^v_m w)$ by 3.2.16(e).

$s^n \in C$ by assumption, $\forall m(d^v_m w \in C)$ by (D3d), hence $\forall m(s^n:d^v_m w \in C)$ (by assumption C is closed under composition) and this yields $f' /\!/_C \mathfrak{f}_F v$.

(f). Let $\mathrm{g} \in C$, H, F and u be arbitrary. We first construct a g and a G such that $\mathfrak{f}_F u \wedge \mathfrak{f}_G u = (\mathfrak{f}_F u \wedge H)[g]$.

Let m be a label of $\mathfrak{f}_F u$. Let g satisfy

$$g n = \begin{cases} {}^\circ m \text{ if } n \notin \ell(\mathfrak{f}_F u) \\ {}^\circ n \text{ otherwise.} \end{cases}$$

Put $G \equiv H[g]$.

By definition of G and g, $k \in \ell(G) \to k \in \ell(\mathfrak{f}_F u)$, hence (by (D4)) $k \in \ell(G) \to \mathfrak{f}_k u = {}^\circ k$ and hence $\mathfrak{f}_G u = G[\lambda k.\mathfrak{f}_k u] = G$ (the first equality by 6.3.1(a)). By definition of g we have $k \in \ell(\mathfrak{f}_F u) \to g k = {}^\circ k$, hence $\mathfrak{f}_F u[g] = \mathfrak{f}_F u$. So

(7) $\qquad \mathfrak{f}_F u \wedge \mathfrak{f}_G u = \mathfrak{f}_F u[g] \wedge G = \mathfrak{f}_F u[g] \wedge H[g] = (\mathfrak{f}_F u \wedge H)[g]$.

(The first equality is immediate from the foregoing, the second from the definition of G, the last one holds by definition of $F[\mathfrak{f}]$, (3.1.16).)

Next we construct an $f \in C$ such that $f:d_G u \simeq \mathrm{g}$.

Let a be such that $\forall n \in \ell(\mathfrak{f}_F u)(d_n u \simeq [an])$, this a exists by (D4) and (D6). Put $b \equiv \lambda n.\mathrm{lth}(an)$, $f' \equiv \nu^K_G(\lambda^K n.s^{bn})$ and $f \equiv \mathrm{g}:f'$.

By assumption, C contains all functions s^n and is closed under pairing, hence $f' \in C$. By assumption $\mathrm{g} \in C$, and C is closed under composition, hence $f \in C$. Moreover

$$f:d_G u \simeq g:\nu_G^K(\lambda^K n.s^{bn}):\nu_G^K(\lambda^K n.d_n u),$$

by definition of f, f', (D3c), and the fact that $\delta_G u = G$. By 3.2.16(e)

$$\nu_G^K(\lambda^K n.s^{bn}):\nu_G^K(\lambda^K n.d_n u) \simeq \nu_G^K(\lambda^K n.s^{bn}:d_n u).$$

All labels of G are labels of $\delta_F u$, hence by definition of b $s^{bn}:d_n u \simeq id$ for all $n \in \mathcal{L}(G)$, whence $\nu_G^K(\lambda^K n.s^{bn}:d_n u) \simeq id$ and

(8) $\qquad f:d_G u \simeq g.$

From (7) and (8) we find

$$((e:d_F u)\wedge(f:d_G u),\ \delta_F u\wedge\delta_G u) \geq ((e:d_F u)\wedge g,\ \delta_F u\wedge H). \quad \square$$

CHAPTER 7

FORMAL SYSTEMS; SUMMARY OF TECHNICAL RESULTS

7.1. OUTLINE

This chapter consists of two parts, 7.2 and 7.3.

In 7.2 we shall show that the results we have obtained so far can be formally expressed and derived in $\underset{\sim}{IDB}_1$ and $\underset{\sim}{LS}$. More precisely: we shall introduce definitional extensions $\underset{\sim}{IDBF}^*$ and $\underset{\sim}{LSF}^*$ of $\underset{\sim}{IDB}_1$ and $\underset{\sim}{LS}$ respectively (F for frame) in which the foregoing can be formalized. The fact that these extensions are definitional means that we can translate our results into $\underset{\sim}{IDB}_1$ and $\underset{\sim}{LS}$.

In 7.3 we have listed the lemmata and facts of the previous chapters to which we shall refer in the sequel, supplemented with some properties of the $\geq$-relation between restrictions which have not been proved before.

This chapter does not claim to contribute to a better understanding of projected universes of GC-sequences and of domains and their properties.

The reader is advised to glance through 7.2 and to skip 7.3 altogether (it is to be used merely as a source of reference) except maybe subsection 7.3.7 which contains the new results on the $\geq$-relation.

7.2. FORMAL SYSTEMS

The system $\underset{\sim}{IDBF}$ (7.2.1-7.2.7).

7.2.1. $\underset{\sim}{IDBF}$ is a definitional extension of $\underset{\sim}{IDB}_0$ (i.e. without K-variables, cf.1.3.8, [KT70] section 3.1) in which the theory of frames and nestings of chapter 3 can be formalized.

(a) Symbols of the language of $\underset{\sim}{IDBF}$ are those of $\underset{\sim}{IDB}_0$ and in addition:

(i) two countable sets of variables, for frames ($F,G,H,F_0,G_0,H_0,\ldots$) and for lawlike sequences of frames ($\mathfrak{f},g,\mathfrak{f}_0,g_0,\ldots$) respectively;

(ii) the constants nodes, ℓ, $\circ$, $\wedge$, prod (for the definition of frames

$F[\mathfrak{f}])$, Ψ (to be explained below), Π_F (for the definition of frames by recursion), ν, λ^F, $=_F$ and branch-of.

(b) Besides number- and function terms (Tm and F-Tm), IDBF has frame-terms (Fr-Tm, meta-variables F, G etc.) and frame-function-terms (Frf-Tm, meta-variables F^1, G^1 etc.). The term-formation rules are those of IDB_0 and

(i) if $F \in$ Fr-Tm, $t \in$ Tm, then nodes(F), $\ell_t F$, $\Psi(F)$ and $\nu_F \phi$ are number terms;

(ii) frame-variables are frame-terms; if $F, G \in$ Fr-Tm, $t \in$ Tm, $F^1 \in$ Frf-Tm then $F \wedge G$, prod(F, F^1) (or shortly $F[F^1]$), and $\Pi_F(F, F^1, t)$ are frame-terms;

(iii) frame-function-variables are element of Frf-Tm; the constant $\circ$ belongs to Frf-Tm; if $F \in$ Fr-Tm, n a number variable then $\lambda^F n.F \in$ Frf-Tm. (We shall omit the superscript F below.)

(c) Prime-formulae and formulae are defined as usual, with two additional prime-formula clauses: if $t \in$ Tm, $F, G \in$ Fr-Tm then branch-of(t, F) (or shortly $t \in F$) and $F =_F G$ are prime-formulae. (We shall omit the subscript F below.)

(d) The axioms of IDBF are those of IDB_0 (schemata extended to the new language), AC-NF (also in the language of IDBF) and

(i) the defining axioms for the constants branch-of, nodes, ℓ, $=_F$, $\circ$, $\wedge$, prod and ν as given in chapter 3;

(ii) the defining axioms for Π_F (which allow a special kind of definition of frames by recursion): $\Pi_F(F, \mathfrak{f}, 0) = F$, $\Pi_F(F, \mathfrak{f}, n+1) = (\Pi_F(F, \mathfrak{f}, n))[\lambda m.\mathfrak{f}j(n,m)]$;

(iii) the axioms for Ψ:

$$\Psi(F) = \Psi(G) \leftrightarrow F = G, \quad (\Psi \text{ is a 1-1 extensional mapping})$$

$$\exists F(n=\Psi F) \wedge \neg\exists F(n=\Psi F) \quad (\text{range } (\Psi) \text{ is decidable});$$

(iv) the λ^F-conversion rule;

(v) the choice-axiom (AC-NFrf)

$$\forall n \exists \mathfrak{f}\, A(n, \mathfrak{f}) \rightarrow \exists g \forall n\, A(n, \lambda m.gj(n,m)).$$

7.2.2. FACTS.

(a) The properties of $\circ n$, $F \wedge G$, $F[\mathfrak{f}]$, $\nu_F \phi$ which we derived in chapter 3 are provable in IDBF.

(b) If v and w are two finite sequences of equal length, v is without repetitions and the relation $\exists n < lth(v)(b=(v)_n)$ between b and v satisfies the axioms of the relation branch-of, then there is an F such that $b \in F$ iff $\exists n<lth(v)(b=(v)_n)$ and $\ell_b F = m$ for $b \in F$ iff $\exists n<lth(v)(b=(v)_n \wedge m=(w)_n)$. This is provable in IDBF by induction w.r.t. the length of v.

(c) The properties of ht given in 3.1.12 are derivable in IDBF. To prove $ht(F)>0 \rightarrow \exists GH(F=G\wedge H)$ we need fact (b).

(d) The principle of induction over frames is provable in IDBF by a reduction to ordinary induction over $\mathbb{N}$ via ht as indicated in 3.1.13.

(e) All properties of frames expressible in IDBF are extensional, i.e. IDBF $\vdash F=G \rightarrow (A(F) \leftrightarrow A(G))$.

7.2.3. It is easy to see that IDBF is indeed a definitional extension of IDB_0. One can define in IDB_0 a subset FRAMECODE of $\mathbb{N}$ with a primitive recursive characteristic function, which may serve as the range of the frame-variables and frame-terms. Frame-function-variables and -terms can then be interpreted as lawlike mappings from $\mathbb{N}$ into $\mathbb{N}$ whose range is a subset of FRAMECODE, constants like $\smile, \ell, \wedge$ etc. are interpreted by (suitably chosen) mappings from $\mathbb{N}$ into $\mathbb{N}$, and definition of frames by recursion reduces to ordinary definition by recursion. The constant Ψ can be interpreted by the identity mapping.

7.2.4. The addition of the constant Π_F to IDBF and its defining axioms are completely ad hoc: they make it possible to construct terms jps[v,n], δ[n,v] and δ[F,v] which satisfy the defining equations for jps(v)(n), $\delta_n v$ and $\delta_F v$ of chapter 4. (Of course nf(v) is definable already in IDB_0.)

7.2.5. Via the constant Ψ we can reinterpret mappings from $\mathbb{N}$ into $\mathbb{N}$ as mappings from FRAME into $\mathbb{N}$. With $a:\mathbb{N} \rightarrow \mathbb{N}$ we associate ϕ: FRAME $\rightarrow \mathbb{N}$ where ϕ is defined by $\phi(F) = a(\Psi F)$. That is to say, in IDBF we can quantify indirectly over lawlike mapping from FRAME $\rightarrow \mathbb{N}$, and if we combine the use of Ψ with pairing also over $\mathbb{N}^{(\mathrm{FRAME}\times\mathbb{N})}$, $\mathbb{N}^{\mathrm{FRAME}\times\mathrm{FRAME}}$ etc.

7.2.6. Pairing (as we have seen before) makes it possible to reinterpret a lawlike $b:\mathbb{N} \rightarrow \mathbb{N}$ as a lawlike $\phi:\mathbb{N} \rightarrow N$. With b we associate the mapping $\phi: n \mapsto (b)_n$. Hence we can discuss (and quantify over) lawlike mappings from $\mathbb{N}$ into the lawlike part of N in IDBF. In particular we can put for $\phi \in$ F-Tm, $F \in$ Fr-Tm: $\lambda^1 n.\phi[n] \equiv_{def} \lambda z.\phi[j_1 z](j_2 z)$ (cf.3.2.15), then $\lambda^1 n.(\phi)_n = \phi$, and $\nu^1_F\phi \equiv \nu^1_F(\lambda^1 n.(\phi)_n) \equiv_{def} \lambda z.\nu_F(\lambda n.(\phi)_n(z))$ (cf.3.2.8(a)).

Using Ψ as in 7.2.5 we can also talk about lawlike mappings from FRAME into the lawlike part of N inside IDBF.

Just as we use $b \in N$ to 'code' mappings $\phi: \mathbb{N} \to N$ we can use $\mathfrak{f} \in \mathrm{FRAME}^{\mathbb{N}}$ to 'code' mappings $\phi: \mathbb{N} \to \mathrm{FRAME}^{\mathbb{N}}$. I.e. in IDBF we can indirectly discuss and quantify over lawlike mappings from $\mathbb{N}$ (or FRAME or $\mathbb{N} \times \mathrm{FRAME}$ etc.) into $\mathrm{FRAME}^{\mathbb{N}}$.

7.2.7. FACT. The nesting-and //- properties not involving ν^k or $//_C$ can be expressed and proved in IDBF.

The system IDBF_1 (7.2.8-7.2.11).

7.2.8. IDBF_1 is obtained from IDBF by adding K-variables and constants for elements of K and operations on K to the language, and specifying term-formation rules for a set of K-terms. (I.e. the relation between IDBF and IDBF_1 is like the relation between IDB and IDB_1.) The full description of IDBF_1 is in 7.2.9 below.

Note that we can associate with each $e \in K$ a mapping $\phi: \mathbb{N} \to K$, putting $\phi(n) = \lambda v.e(\langle n\rangle * v)$. In IDBF_1 we can quantify indirectly over $K^{\mathbb{N}}$, and, if we use Ψ as in 7.2.5-6, also over K^{FRAME}, $K^{\mathbb{N} \times \mathrm{FRAME}}$ etc.

IDBF_1 has constants λ^K and ν^K, and the rules for term-formation specify that if $F \in$ Fr-Tm, $\phi \in$ K-Tm, n is a numerical variable then $\lambda^K n.\phi$ and $\nu^K_F\phi$ are K-terms (see 7.2.9).

$\lambda^K n.\phi[n]$ is the element of K which represents the mapping $n \mapsto \phi[n] \in K^{\mathbb{N}}$, i.e. $\lambda^K n.\phi[n]$ is defined by the axioms $\lambda^K n.\phi[n](0) = 0$, $\lambda^K n.\phi[n](\hat{x} * v) = \phi[x](v)$. It follows that $e \simeq \lambda^K n.(\lambda v.e(\langle n\rangle * v))$.

$\nu^K_F e$ is the F-nesting of the mapping $n \mapsto \lambda v.e(\langle n\rangle * v) \in K^{\mathbb{N}}$ represented by e, i.e. as axioms we have

$$\nu^K_{\circ_n} e = \lambda v.e(\langle n\rangle * v),\quad \nu^K_{F \wedge G} e = \nu^K_F e \wedge \nu^K_G e.$$

7.2.9. The complete definition of IDBF_1 is as follows:

(a) The language of IDBF_1 consists of the language of IDBF plus

(i) a set of K-variables e, f, g, e_0, f_0, g_0 etc.

(ii) constants app_0, app_1 (for neighbourhood-function-application $\cdot(\cdot)$ and $\cdot|\cdot)$, λ' (for K-abstraction), λ^K (for the formation of $K^{\mathbb{N}}$-elements), shift, prix, nestinv, dpl and nest (to form neighbourhood-functions in K for the shift- ($a \mapsto \lambda z.a(n+z)$), prefix- ($a \mapsto v*a$), nesting-inverse- ($a \mapsto \lambda z.j_b a$), duplicate- ($a \mapsto j(a,a)$) and F-nest-mapping ($a \mapsto \nu^1_F(\lambda^1 n.a)$) respectively), and constants for operations on K namely ;,:,∧,×,ν^K (nesting).

(b) The term formation rules for $\underset{\sim}{IDBF}_1$ are those of $\underset{\sim}{IDBF}$ plus

(i) the formation rules for the set of K-terms (K-Tm, ϕ and ψ are used as meta-variables for this set), namely:
K-variables are in K-Tm; the constant dpl is in K-Tm; if $t \in$ Tm, $F \in$ Fr-Tm, n and m are distinct numerical variables and ϕ and ψ are K-terms then $\lambda'n.Sm$, $\lambda'v.\phi(t*v)$, $\lambda'v.h(\phi,v)$ (h as defined in 1.3.19), $\lambda^K n.\phi$, shift(t) (shortly s^t), prix(t) (shortly[t]), nestinv(t) (shortly j_t), nest(F) (written as $nest_F$), $\phi;\psi$, $\phi:\psi$, $\phi\wedge\psi$, $\phi\times\psi$ and finally $\nu^K_F\phi$ are elements of K-Tm;

(ii) the following new formation rules for Tm and F-Tm: if $t \in$ Tm, $\psi_1,\ldots,\psi_p \in$ F-Tm and $\phi \in$ K-Tm then $\phi t \in$ Tm, $app_0(\phi,\psi_1,\ldots,\psi_p) \in$ Tm and $app_1(\phi,\psi_1,\ldots,\psi_p) \in$ F-Tm. For $app_0(\phi,\psi_1,\ldots,\psi_p)$ we write $\phi(\psi_1,\ldots,\psi_p)$ $app_1(\phi,\psi_1,\ldots,\psi_p)$ is abbreviated to $\phi|(\psi_1,\ldots,\psi_p)$.

(c) Formulae and prime-formulae are constructed as in $\underset{\sim}{IDBF}$.

(d) The axioms of $\underset{\sim}{IDBF}_1$ are those of $\underset{\sim}{IDBF}$ (schemata extended to the new language) and

(i) the defining equivalences for app_0 and app_1 (1.3.10):
$e(a_1,\ldots,a_p)=y \leftrightarrow \exists v(\nu_p(a_1,\ldots,a_p)\epsilon v \wedge ev=y+1)$
$e|(a_1,\ldots,a_p)(x)=y \leftrightarrow \exists v(\nu_p(a_1,\ldots,a_p)\epsilon v \wedge e(\hat{x}*v)=y+1)$;

(ii) the λ'-conversion-rule: $\lambda'n.t[n](x) = t[x]$, the λ^K-conversion rules (see 7.2.8);

(iii) the defining equations for the remaining constants (for dpl and nest these are given in 9.2.1, for s^n, [v],j_b the precise choice is irrelevant (cf.1.3.16), for ;,:,∧,× the definitions are given in 1.3.17,18, 21 and 23, for ν^K finally the defining axioms are specified in 7.2.8 above);

(iv) the axiom expressing that K-variables range over K, i.e.
$\forall a\forall e(\forall z(az=ez) \leftrightarrow K(a))$.

7.2.10 <u>REMARKS</u>.

(a) We shall omit the superscript 'in λ' below, i.e. we do not make the syntactic distinction between e.g. the K-element $\lambda'v.e(\langle n\rangle*v)$ and the mapping $\lambda v.e(\langle n\rangle*v) \in N$ as $\underset{\sim}{IDBF}_1$ does.

(b) So for we have not used the mappings dpl and $nest_F$. They will play a rôle only in chapter 9.

(c) Until now $j_v a$ was used to abbreviate $\lambda x.j_v(ax)$. From now on we put $j_v a \equiv j_v|a$ (which is extensionally equal to $\lambda x.j_v(ax)$), i.e. we treat

j_v in $j_v a$ as a neighbourhood-function.

(d) Our choice of K-Tm is a matter of convenience. It makes it possible to express the properties which we are interested in, without much circumscription, in the language of $\underset{\sim}{IDBF}_1$.

7.2.11. FACT. The systems $\underset{\sim}{IDBF}_1$ and $\underset{\sim}{IDBF}$ are equivalent: there is a translation from $\underset{\sim}{IDBF}_1$ into $\underset{\sim}{IDBF}$ which preserves derivability and which maps each sentence A of $\underset{\sim}{IDBF}_1$ to a sentence A' of $\underset{\sim}{IDBF}$ which is equivalent (in $\underset{\sim}{IDBF}_1$), moreover the range of the K-variables and all constants of $\underset{\sim}{IDBF}_1$ are definable in $\underset{\sim}{IDBF}$.

PROOF. The only problem is to eliminate the constant ν^K. The axioms of $\underset{\sim}{IDBF}_1$ define this constant by recursion over frames, but such a definition is not generally possible in $\underset{\sim}{IDBF}$. Combining the ν^K-axioms with the axioms for $\wedge$ (definition 1.3.23) we find that

$$\nu^K_F e(0) = \begin{cases} e(\langle n\rangle) \text{ if } F = {}^\circ n, \\ \\ 0 \quad \text{otherwise;} \end{cases}$$

$$\nu^K_F e(\hat{x}*v) = \phi(F,e,\hat{x}*v)\cdot(1+\nu_{F'}(\lambda w.\psi(w,F,e,\hat{x}*v))).$$

Here $\phi(F,e,\hat{x}*v) \equiv sg(\Pi_{b\in F} e(\langle \ell_b F\rangle*\hat{x}*k_b v))$,
$\psi(w,F,e,\hat{x}*v) \equiv e(\langle \ell_w F\rangle*\hat{x}*k_w v)\dot{-}1$, and F' is the frame with the same branches as F, but satisfying $\forall b\in F(\ell_b F'=b)$ (each branch is labelled with itself). There is a term $t_1[e,F,0]$ of $\underset{\sim}{IDBF}_1$ which satisfies the equation for $\nu^K_F e(0)$, there is a term $s[e,F,\hat{x}*v]$ which satisfies the equation for $\phi(F,e,\hat{x}*v)$ (use nodes(F) to construct a term card(F) and an enumeration of the branches of F, then $\Pi_{b\in F}$ can be defined by an ordinary primitive recursion), there is a term $s'[w,e,F,\hat{x}*v]$ which satisfies the equation for $\psi(w,F,e,\hat{x}*v)$, it remains to show that there is a frame-term $F(F)$ such that $\forall b\in F(\ell_b(F(F))=b)$.

This term is constructed as follows.

(a) Using nodes(F) construct a mapping χ such that $\chi n = 0$ if $n \notin$ nodes(F) or $n*\langle 0\rangle \notin$ nodes(F) or $n*\langle 1\rangle \notin$ nodes(F) and which gives the value 1 otherwise.

(b) Put $g \equiv \lambda n.({}^\circ(n*\langle 0\rangle)\wedge{}^\circ(n*\langle 1\rangle))$, $\mathfrak{f} \equiv \lambda m.g(j_2 m)$ (i.e. $\mathfrak{f}(k,n) = gn$), and put $\mathfrak{f}' \equiv \lambda n.\Pi_F({}^\circ n,\mathfrak{f},\chi n)$; then $\mathfrak{f}n = {}^\circ n$ if n is not in n(F) or n is a terminal

node of F, and $\mathfrak{f}'n = {}^\circ(n*\langle 0\rangle) \wedge {}^\circ(n*\langle 1\rangle)$ if n is a non-terminal node of F.
(c) Put $\mathcal{F}(F) \equiv \Pi_F({}^\circ 0, \lambda m.\mathfrak{f}'(j_2 m), ht(F))$.
The proof of the correctness of this definition is given by introducing $\mathcal{F}(F,k)$, defined as $\mathcal{F}(F)$ but with k instead of ht(F), and then showing that $b \in \mathcal{F}(F,k)$ iff $lth(b) \leq k$ and $b \in nodes(F)$, while $\ell_b(\mathcal{F}(F,k)) = b$. This is done by induction w.r.t. k using the explicit characterization of $F[\mathfrak{f}]$ (3.1.18). □

For the formulation and proofs of the $/\!/_C$-properties and the properties of $\geq$ between restrictions and for the treatment of models and domains, we enrich $IDBF_1$ to the system $IDBF^*$.

7.2.12. $IDBF^*$ is $IDBF_1$ with two additional constants in its language, C and J, a term-formation rule $J \in$ K-Tm and a new type of prime-formulae: if $\phi \in$ F-Tm, then $C(\phi)$ is a prime-formula. Axioms to be added are:
C is a subset of K: $C(a) \to \exists e(a=e)$,
C is closed under $\simeq$ (cf.1.3.26): $C(\lambda z.ez) \wedge e \simeq f \to C(\lambda z.fz)$.

For $C(\lambda z.ez)$ we shall simply write $e \in C$. J will be used only as representative of the mapping $n \mapsto \lambda v.J(\langle n\rangle * v) \in K^{\mathbb{N}}$. Therefore Jn will mean $\lambda v.J(\langle n\rangle * v)$.

All properties of $\nu_F, \nu_F^1, \nu_F^K, /\!/, /\!/_C$ and $\geq$ (between restrictions) that have been stated so far can be formulated and proved in $IDBF^*$.

7.2.13. Models and domains in $IDBF^*$.

There is a frame-term jps[v,x] of IDBF such that for all v the defining equations for jps(v) (4.3.4) are provable in IDBF for $\lambda x.jps[v,x]$.

Using jps[v,x] we can express by a formula $GFS(\mathfrak{f})$ (GFS for 'generates frame-sequences') that the mappings $(n,v) \mapsto \mathfrak{f}(\Psi({}^\circ n),v)$ and $(F,v) \to \mathfrak{f}(\Psi(F),v)$ satisfy the defining equations for $\mathfrak{f}_n v, \mathfrak{f}_F v$ respectively (4.3.9,4.3.12).

In fact there is a frame-term $\mathcal{F}(n,v)$ of IDBF which satisfies the equations for $\mathfrak{f}_n v$ (4.3.9), hence for the mapping $\mathfrak{f}$ such that $\mathfrak{f}(\Psi(F),v) = F[\lambda n.\mathcal{F}(n,v)]$ we can prove $GFS(\mathfrak{f})$ in IDBF.

The properties of jps and $\mathfrak{f}_F v$ that are derived in 4.3 are provable in IDBF for the corresponding term jps[v,x] and the mappings $\mathfrak{f}$ satisfying $GFS(\mathfrak{f})$.

In the language of $IDBF^*$ there are formulae JPF(e), $UP(a,e,f,\mathfrak{f})$ and GEV(g,a) which express the following: JPF(e): the mapping $\lambda w.e(\langle v,n\rangle * w)$

behaves as jf(v)n (4.4.3) (in JPF the constant J occurs),
UP(a,e,f,$\mathfrak{f}$): a(v) behaves as upb(d,v) (4.4.9) if λw.e(<v,n>*w) is used as jf(v)(n) and $\mathfrak{f}$j(ΨF,v) as $\mathfrak{f}_F v$ while the rôle of d: $\mathbb{N} \times \mathbb{N} \to K$ is played by f, i.e. $d_n v \in K$ is λw.f(<v,n>*w),
GEV(g,a): λw.g(<v,n>*w) behaves as gv(d,v)n (4.4.10) if a(v) is used as upb(d,v).

In IDBF* one can prove $\exists$e JPF(e), $\forall$ef$\mathfrak{f}\exists$a UP(a,e,f,$\mathfrak{f}$) and $\forall$a$\exists$g GEV(g,a). For JPF one easily defines an F-Tm ϕ such that IDBF* $\vdash$ $\exists$e(e$\simeq\phi$ $\wedge$ JPF(e)).

We can use JPF, UP and GEV to construct formulae DG^0(f,$\mathfrak{f}$) and DG(g,$\mathfrak{f}$) which express that the mappings d^0: $\mathbb{N} \times \mathbb{N} \to K$ represented by f(i.e. λw.f(<n,v>*w) $\equiv d_n v$) and d: $\mathbb{N} \times$ FRAME $\times\ \mathbb{N} \to K$ represented by g (i.e. λu.g(<v,Ψ(F),w>*u) $\equiv d_F^v w$) belong to DG^0(J) and DG(J) respectively (the formulae DG^0 and DG contain the constant J in JPF), if $\mathfrak{f}$ plays the rôle of frame-sequence-generator. (cf.4.4.11,4.4.17.)

In IDBF* we can prove $\exists$f DG^0(f,$\mathfrak{f}$) (4.4.12). In the appendix we show that there is an F-Tm ϕ[n,v] of IDBF* such that $\exists$f($\forall$nvw(ϕ[n,v](w) = = f(<n,v>*w))$\wedge DG^0$(f,$\mathfrak{f}$)). Once we have an f such that DG^0(f,$\mathfrak{f}$) we can construct a g such that DG(g,$\mathfrak{f}$) (4.4.19). All properties of $d^0 \in DG^0$(J) and d $\in$ DG(J) mentioned in chapter 4 can be derived (assuming GFS($\mathfrak{f}$), DG^0(f,$\mathfrak{f}$), DG(g,$\mathfrak{f}$)) for the mappings (n,v) $\mapsto$ λw.f(<n,v>*w) and (v,F,w) $\mapsto$ $\mapsto$ λu.g(<v,Ψ(F),w>*u) respectively in IDBF*.

There also is a formula GNGC(e,g,$\mathfrak{f}$) which expresses that π: F $\mapsto \pi_F \in K$ defined by $\pi_F \equiv \lambda$w.e(<ΨF>*w) generates nests of GC-carriers, that d: $\mathbb{N} \times$ FRAME $\times\ \mathbb{N} \to K$ defined by $d_F^v w \equiv$ d(v,F,w) $\equiv$ λu.g(<v,Ψ(F),w>*u) generates the dressings for π, and that $\mathfrak{f}$j(Ψ(F),v) is the frame for π_F at v (4.5.2). GNGC(e,g,$\mathfrak{f}$) has 'J enumerates C modulo $\simeq$' as a sub-formula. The existence of g,e, and $\mathfrak{f}$ such that GNGC(e,g,$\mathfrak{f}$) is provable in IDBF* from the assumption 'J enumerates C modulo $\simeq$'. (It suffices to construct π_F from d as in 4.5.6.)

We shall continue to use π,d,$\mathfrak{f}$ and expressions like π_F, $d_F^v w$, $\mathfrak{f}_F v$ etc. as in chapter 4 but now as abbreviations for K-terms in IDBF*. E.g. for GNGC(e,g,$\mathfrak{f}$) we write GNGC(π,d,$\mathfrak{f}$).

From GNGC(π,d,$\mathfrak{f}$) we can derive the properties of π mentioned in 4.5.

Obviously there is an IDBF* sentence dclosed(C) which expresses that C is dependency-closed. For dclosed(C) $\wedge$ GNGC(π,d,$\mathfrak{f}$) we write model(π,d,$\mathfrak{f}$)

In 5.2.4 we have given the formula which expresses that the pair (d,$\mathfrak{f}$) has the strong overtake property. The proof of model(π,d,$\mathfrak{f}$) $\to$ strong overtake(d,$\mathfrak{f}$) (5.2.5) as given in 5.3, 5.4 can be formalized in LSF* (to be discussed below) hence this implication is provable in IDBF* (via the elimination theorem).

Finally we can express in IDBF* that π,d and $\mathfrak{f}$ satisfy the domain-axioms (6.1.2). The formula which does so is denoted by domain(π,d,$\mathfrak{f}$). The properties of domains derived in 6.3 can be formally proved in IDBF*. The same holds for the theorem that models are domains (6.2):

IDBF* $\vdash$ model(π,d,$\mathfrak{f}$) $\to$ domain(π,d,$\mathfrak{f}$).

We conclude section 7.2 with the introduction of LSF* (7.2.14-15).

7.2.14. Over IDBF* we define a formal system for the theory of lawless sequences LSF* as follows (cf. the description of LS in [T77]).

(a) To the language of IDBF* we add variables for lawless sequences α,α_0,α_1 etc.

(b) We introduce a set L-Tm of lawless sequence terms, which contains only the lawless variables.

(c) We leave the definitions of Tm, F-Tm, Fr-Tm, FrF-Tm and K-Tm unchanged, so these contain only terms with lawlike parameters, and add a set Tm* of terms which may contain lawless variables. Tm* contains the same expressions and is closed under the same term-formation rules as Tm (with one exception, see below), and satisfies in addition:

if $\alpha,\alpha_1,\ldots,\alpha_p \in$ L-Tm, $t \in$ Tm* then $\alpha t \in$ Tm*, $e(\alpha_1,\ldots,\alpha_p) \in$ Tm* and $e|(\alpha_1,\ldots,\alpha_p)(t) \in$ Tm*.

(d) The formation rule for recursion terms in Tm* is slightly changed w.r.t. the corresponding rule for Tm (the exception mentioned above) as follows:

if $t_1,t_2,t_3 \in$ Tm* and x is a numerical variable, then

$\Pi(t_1,(\lambda x.t_2),t_3) \in$ Tm*.

Thus we introduce expressions for natural numbers defined by recursion w.r.t. a lawless parameter (like e.g. $\bar{\alpha}x$), without having function-terms for constructs of lawless sequences.

(e) Prime formulae and formulae are defined as usual.

(f) Axioms for the theory are:

(i) The axioms of IDBF* (schemata in the new language, but with the stipulation that instances of AC-NF cannot contain a lawless parameter, and terms now ranging over Tm* instead of Tm.

(ii) The defining axioms for $e(\alpha_1,\ldots,\alpha_p)$, $e|(\alpha_1,\ldots,\alpha_p)(x)$, similar to those for the lawlike case.

(iii) Axioms for the new recursion terms (obvious).

(iv) The usual LS-axioms, in the new language.

7.2.15. REMARKS.

(a) Elements of K-Tm and F-Tm in LSF^* cannot contain lawless variables, so $[\alpha x]$, $s^{\alpha x}$ are not in K-Tm. Such K-functions can be discussed only indirectly in the language of LSF^*. Moreover, in the prime formulae $K(\phi)$ and $C(\phi)$, ϕ is an element of F-Tm, hence these formulae are lawlike.

(b) LSF^* does not contain expressions for constructs of lawless sequences like $e|\alpha$, but it does contain expressions for the values of such constructs. Still we use expressions of the form $e|\alpha$, $e(f|\alpha)$frequently below. For the formalization of our arguments this is harmless, eventually we are interested only in the values of such sequences.

(c) Note that we can formally define what we mean by the substitution of an expression $e|\beta$ for α (and of $e|(f|\beta)$ for α, etc.) in a term $t[\alpha]$. Some examples:

$\alpha x[(e|\beta)/\alpha] \equiv e|\beta(x)$,

$e|\alpha(x)[(f|\beta)/\alpha] \equiv e{:}f|\beta(x)$,

$e(\alpha)[(f|\beta)/\alpha] \equiv e;f(\beta)$,

$e(\alpha_1,\alpha_2)[(f|\beta)/\alpha_1] \equiv e;(f\wedge id)(\beta,\alpha_2)$,

$e|(\alpha_1,\alpha_2)(x)[(f|\beta)/\alpha_2] \equiv e{:}(id\wedge f)|(\alpha_1,\beta)$ etc.

(d) All theorems in the sequel can be formalized in the monadic part of LSF^* (domains and models are projected from a single lawless sequence δ).

7.2.16. LEMMA.

(a) *The following continuity schemata are derivable in* LSF^*:

$\forall\alpha\exists F\ A(\alpha,F) \rightarrow \exists e\forall v(ev\neq 0 \rightarrow \exists F\forall\alpha\in v\ A(\alpha,F))$,

$\forall\alpha\exists \mathfrak{f}\ A(\alpha,\mathfrak{f}) \rightarrow \exists e\forall v(ev\neq 0 \rightarrow \exists \mathfrak{f}\forall\alpha\in v\ A(\alpha,\mathfrak{f}))$.

(b) *The elimination theorem for* LS *relative to* IDB_1 *can be extended to* LSF^* *relative to* $IDBF^*$.

PROOF.

(a) $\forall\alpha\exists F\ A(\alpha,F)$ is equivalent to $\forall\alpha\exists n\exists F(\Psi(F)=n \wedge A(\alpha,F))$. By the ordinary $\forall\alpha\exists n$-continuity axiom we find that for some e if $ev \neq 0$ then $\exists n\forall\alpha\in v\exists F(\Psi(F)=n \wedge A(\alpha,F))$. But $\psi(F) = n$ uniquely determines F, so we can interchange $\forall\alpha\in v$ and $\exists F$. $\forall\alpha\exists\mathfrak{f}\ A(\alpha,\mathfrak{f})$ is treated similarly.

(b) By a straightforward adaption of the original proof of the elimination theorem. Note that the new classes of terms Fr-Tm and Frf-Tm will pose no problem because their elements are lawlike. □

7.3. SUMMARY OF LEMMATA

In this section we have put together the technical results of the previous chapters which remain important in the sequel, supplemented with some properties of the $\geq$-relation between restrictions which have not yet been discussed but will be used later on.

<u>K-functions and related topics</u> (7.3.1-7.3.5).

7.3.1. With each $e \in K$ and finite sequence w we have associated (in 1.3.11) a finite sequence $e{\restriction}w$ such that

$\mathrm{lth}(e{\restriction}w) = \min_{k<\mathrm{lth}(w)}(e(\langle k\rangle * w)=0)$ $[=\mathrm{lth}(w)$ if $\forall k<\mathrm{lth}(w)(e(\langle k\rangle * w)\neq 0)]$,

$\forall x<\mathrm{lth}(e{\restriction}w)((e{\restriction}w)_x = e(\hat{x}*w)\dot{-}1)$.

Properties of $e{\restriction}w$ are (1.3.12):

(a) $\forall x\exists y\geq x(\overline{e|a}(x) \preccurlyeq e{\restriction}(\bar{a}y))$,

(b) $\forall y\exists x \leq y(e{\restriction}(\bar{a}y) = \overline{e|a}(x))$.

Remember that $e;f \equiv_{\mathrm{def}} \lambda w.e(f{\restriction}w)$ (1.3.17).

7.3.2. $\wedge$ is a pairing operation on K w.r.t. $\simeq$, which satisfies

(a)	$j_1(e\wedge f|a) = e|j_1a, \quad j_2(e\wedge f|a) = f|j_2a$	(1.3.23)
(b)	$(e\wedge f):(e'\wedge f') \simeq (e:e')\wedge(f:f')$	(1.3.24(f))
(c)	$e\wedge e' \simeq f\wedge f'$ iff $e \simeq f \wedge e' \simeq f'$	(1.3.24(e)).

7.3.3. $[v]$ denotes the neighbourhoodfunction such that $[v]|a = v*a$, s^n is an element of K satisfying $s^n|a = \lambda z.a(n+z)$ (1.3.16).

$[\ ]$ satisfies:

(a)	$[k_1z]\wedge[k_2z] \simeq [z]$	(1.3.24(g))
(b)	$\forall a(e|a\in v) \rightarrow e \simeq [v]:s^{\mathrm{lth}(v)}:e$	(1.3.24(c))

Note that as a corollary of (b) and 7.3.1(b) we have

(c) $f:[v] \simeq [f{\restriction}v]:s^m:f:[v]$, where $m = \mathrm{lth}(f{\restriction}v)$. (1.3.24(d))

7.3.4. The mapping $e\times f$ (composition of the bars e and f) is defined as $\lambda u.sg(eu).f(\langle h(e,u)\dot{-}1\rangle * h_c(e,u))$ (1.3.21), whence $e\times f(u)\neq 0 \rightarrow eu\neq 0$ and even $e\times f(u)=m+1 \rightarrow \exists vw(u=v*w \wedge ev\neq 0 \wedge f(\langle v\rangle * w)=m+1)$.

7.3.5. As an important property of K-nestings we recall

$\forall b\in F(j_b(\nu_F^K\phi|a) = \phi(\ell_b F)|j_b a)$ (3.2.16(c)).

7.3.6. The relations $//$ and $//_C$ (3.2.18-21).

A sequence $\phi \in N$ is parallel to the frame F, notation $\phi//F$, iff $\forall bb'\in F(\ell_b F=\ell_{b'} F \to j_b\phi=j_{b'}\phi)$.

Likewise, if v is a finite sequence, then
$v//F \equiv_{def} \forall bb'\in F(\ell_b F=\ell_{b'} F \to k_b v=k_{b'} v)$.

An element e of K is C-parallel to F ($e//_C F$) iff there is a ϕ: $\mathbb{N}\to C$ (represented by f $\in$ K through $\phi n = \lambda v.f(\langle n\rangle * v)$) such that $e \simeq \nu^K_F\phi$.

Properties of $//$ and $//_C$

(a)	If F has a 1-1 labelling, then $\forall a(a//F)$ and $\forall v(v//F)$	(3.2.21(e))
(b)	$\forall a(a//(^\circ n))$, $\forall v(v//(^\circ n))$, $\forall e\in C(e//_C(^\circ n))$	(3.2.20(n),21(a),(b))
(c)	$a//F\wedge G \to j_1 a//F \wedge j_2 a//G$, $v//F\wedge G \to k_1 v//F \wedge k_2 v//G$	(3.2.20(f),21(c))
(d)	$a//F \wedge m\notin\ell F \to \forall b(j(a,b)//F\wedge(^\circ m))$	(3.2.21(k))
(e)	$\forall v\exists v'v''(k_1 v'=v \wedge k_2 v'=v'' \wedge v'//(^\circ 0)\wedge H)$	(3.2.21(m))
(f)	$a//F \leftrightarrow \forall x(\bar{a}x//F \wedge \lambda z.a(x+z)//F)$	(3.2.20(d))
(g)	$a//G \wedge G\geq F \to a//F$,	(3.2.20(j))
	$v//G \wedge G\geq F \to v//F$	(3.2.21(f))
	if C is closed under pairing then $e//_C G \wedge G \geq F \to e//_C F$	(3.2.20(k))
(h)	$F\approx G \to \forall a(a//G \leftrightarrow a//F)$	(3.2.21(g))
(j)	$e//_C F \wedge a//F \to e\vert a//F$, $e//_C F \wedge v//F \to e\upharpoonright v//F$	(3.2.20(r),21(j))
(k)	if C is closed under pairing then $e//_C F \to e \in C$	(3.2.20(m)).

7.3.7. The $\geq$-relation between restrictions (5.1.2-7).

$$(f,G) \geq (e,F) \equiv_{def} \exists g//_C F(f\simeq e:g) \wedge G\geq F,$$

$$(f,G) \approx (e,F) \equiv_{def} (f,G)\geq(e,F) \wedge (e,F)\geq(f,G).$$

Properties of $\geq$

(a) If id $\in$ C then $e\simeq f \wedge F\approx G \to (e,F)\approx(f,G)$ whence in particular
$(e,F) \approx (e,F)$, $e\simeq f \to (e,F)\approx(f,F)$, $F \approx G \to (e,F) \approx (e,G)$.
(5.1.6(a),7(a))

(b) If C is closed under pairing and composition, then the $\geq$-relation is transitive. (5.1.6(b))

(c) If $[v] \in C$ for all v then $y//F \to (e:[y],F)\geq(e,F)$ (5.1.6(c))

(d) (i) $\forall e\in C\forall F[(e,F)\geq(id,{}^\circ n)]$,
(ii) $\forall e\in C[(f:e,{}^\circ n)\geq(f,{}^\circ n)]$.

(e) If $s^n \in C$, then $(f:s^n,F) \geq (f,F)$.

(f) $(f,G)\geq(e,F) \wedge n\notin \ell F \rightarrow \forall g\in C\forall H[(f\wedge g,G\wedge H) \geq (e\wedge id,F\wedge(°n))]$,
if $id \in C$ and $n \notin \ell G$, then $\forall g'\in C\forall H[(f\wedge(g:g'),G\wedge H) \geq (f\wedge g,G\wedge(°n))]$.

(g) $(f,G)\geq(e,F) \rightarrow \forall e'F'\exists f'G'[(f\wedge f',G\wedge G') \geq (e\wedge e',F\wedge F')]$, where if C is closed under pairing and composition then $e'\in C \rightarrow f'\in C$.

(h) If C is closed under composition, $\forall n(s^n\in C)$ and $\forall v([v]\in C)$, and if $G \geq F$ then $g/\!/_C F \wedge y/\!/F \rightarrow (e:g:[y],G)\geq(e:[g\restriction y],F)$.
If C is also closed under pairing, then we may replace the premiss $g/\!/_C F \wedge y/\!/F$ of the implication by $g/\!/_C G \wedge y/\!/G$
(or $g/\!/_C F \wedge y/\!/G$, $g/\!/_C G \wedge y/\!/F$), by 7.3.6(g).

Note that the conditions on C occurring in (a), (b), (c), (e), (f), (g) and (h) are fulfilled if C is dependency-closed.

<u>PROOF</u> (of (d)-(h)).

(d) and (e) are trivial, observe that $F = (°n)[\lambda m.F]$, $e\in C \rightarrow e/\!/_C(°n)$ (cf.7.3.6(b)) and $s^n\in C \rightarrow s^n/\!/_C F$ by 3.2.20(p).

(f)(i) if $n \notin \ell F$ and $G = F[\mathfrak{f}]$, then $G\wedge H = (F\wedge(°n))[g]$, where $gm = \mathfrak{f}m$ if $n \neq m$ and $gn = H$. If $f \simeq e:\nu^K_F\phi$, $\phi:\mathbb{N}\rightarrow C$, and $g \in C$, then $f\wedge g \simeq (e\wedge id):\nu^K_{F\wedge(°n)}\psi$, where $\psi m = \phi m$ if $m \neq n$, and $\psi n = g$.

(f)(ii) $G\wedge H \geq G\wedge(°n)$ by the same argument as above;
$f\wedge(g:g') \simeq (f\wedge g):\nu^K_{G\wedge(°n)}\phi$, where $\phi m = id$ if $m \neq n$, and $\phi n = g$.

(g) If $G = F[\mathfrak{f}]$ then $G\wedge F'[\mathfrak{f}] = (F\wedge F')[\mathfrak{f}]$, so take $G' = F'[\mathfrak{f}]$. If $f \simeq e:\nu^K_F\phi$, $\phi: \mathbb{N} \rightarrow C$, then $f\wedge(e':\nu^K_{F'}\phi) \simeq (e\wedge e'):\nu^K_{F\wedge F'}\phi$, so take $f' \equiv e':\nu^K_{F'}\phi$. If C is closed under pairing, then $\nu^K_{F'}\phi \in C$, if C is closed under composition and $e' \in C$, then $f' \equiv e':\nu^K_{F'}\phi \in C$.

(h) Note that $g:[y] \simeq [g\restriction y]:s^m:g:[y]$, where $m = lth(g\restriction y)$ (7.3.3(c)). If $s^m \in C$ then $s^m/\!/_C F$, (3.2.20(p)), if $\forall v([v]\in C)$ and $y/\!/F$ then $[y]/\!/_C F$ (3.2.21(i)). $g/\!/_C F$ by assumption, so if C is closed under composition then $s^m:g:[y]/\!/_C F$ (3.2.20(s)). □

7.3.8. Finally we recall a number of the domain properties of section 6.3: let π,d and $\mathfrak{f}$ define a domain then

(a) $\mathfrak{f}_{F\wedge G}v = \mathfrak{f}_F v\wedge\mathfrak{f}_G v$ (6.3.1(b))

(b) $d^v_{F\wedge G}w \simeq d^v_F w\wedge d^v_G w$ (6.3.2(a))

(c) $\forall w((e:d_F(v*w),\mathfrak{f}_F(v*w)) \geq (e:d_F v,\mathfrak{f}_F v))$ (6.3.3)

(d) $\forall n\forall b(\ell_b F=n \rightarrow j_b(\pi_F\delta)=\pi_n\delta)$ (6.3.4(b))

(e) $\exists g\forall\delta\in v(\pi_F\delta=d_F v|(g|\delta) \wedge g|\delta/\!/\mathfrak{f}_F v)$ (6.3.4(d))

(f) $(f,G) \geq (e{:}d_F v, \vartheta_F v) \rightarrow$
$$\exists e_2 \forall u /\!/ G[e_2 u \neq 0 \rightarrow \exists w((f,G) \leq (e{:}d_F(v*w), \vartheta_F(v*w)) \leq (f{:}[u],G))]$$
(6.3.5(a))

(g) If id $\in$ C then
$$\forall e_1 \exists e_2 \forall u /\!/ \vartheta_F v[e_2 u \neq 0 \rightarrow \exists w(e_1 w \neq 0 \wedge (e{:}d_F(v*w), \vartheta_F(v*w)) \leq (e{:}d_F v{:}[u], \vartheta_F v))]$$
(6.3.5(b))

(h) If $\forall u([u] \in C)$, then $\forall a /\!/ \vartheta_F v \forall x \exists \delta \in v(\overline{\pi_F \delta}\,(x) = \overline{d_F v \restriction a}(x))$ (6.3.5(c))

(j) If $s^n \in$ C for all n and C is closed under composition, then
$$\forall e_2 \exists e_1 \forall w[e_1 w \neq 0 \rightarrow$$
$$\exists u /\!/ \vartheta_F v(e_2 u \neq 0 \wedge (e{:}d_F v{:}[u], \vartheta_F v) \leq (e{:}d_F(v*w), \vartheta_F(v*w)))]$$
(6.3.5(e))

(k) If $\forall n(s^n \in C)$ and C is closed under composition and pairing, then
$$\forall g \in C \forall H F u \exists f \in C \exists G[((e{:}d_F u) \wedge (f{:}d_G u),\ \vartheta_F u \wedge \vartheta_G u) \geq ((e{:}d_F u) \wedge g,\ \vartheta_F u \wedge H)]$$
(6.3.5(f)).

Note that the conditions on C in (g)-(k) are fulfilled if C is dependency-closed.

CHAPTER 8

THE ELIMINATION THEOREM FOR DOMAINS

8.1. OUTLINE

In this section we shall take the first step towards proving that suitable domains are models for the system **CS**(C), by deriving an elimination theorem for domains.

First we introduce the language $\mathcal{L}_\varepsilon$ (in which **CS**(C) is formulated). $\mathcal{L}_\varepsilon$ is the same as the language of $\mathbf{LSF}^*$, except that it has choice variables $\varepsilon,\eta,\varepsilon_0,\eta_0$ etc., instead of the lawless variables α,β etc.

With each formula $A(\varepsilon_1,\ldots,\varepsilon_p)$ of $\mathcal{L}_\varepsilon$ we associate a formula $A^\delta(e_1|\pi_{F_1},\ldots,e_p|\pi_{F_p})$ of $\mathbf{LSF}^*$, which expresses that A holds if we let its choice quantifiers range over the domain $\mathcal{D}_\delta = \{e|\pi_F\delta : e \in C,\ F \in \text{FRAME}\}$ and interpret the choice parameters ε_i in A by $e_i|\pi_{F_i}\delta \in \mathcal{D}_\delta$ $(i = 1,\ldots,p)$.

Next we expand $\mathcal{L}_\varepsilon$ to a language $\mathcal{L}^*_\varepsilon$ by adding a clause to the formula-definition, saying that if A is a formula then so is $\forall\varepsilon\epsilon(\phi,\mathcal{F})A$, where $\phi \in$ K-Tm, $\mathcal{F} \in$ Fr-Tm (i.e. $(\phi,\mathcal{F})$ denotes a restriction).

Then we define an elimination translation which maps formulae of $\mathcal{L}^*_\varepsilon$ onto formulae of $\mathbf{IDBF}^*$. For this translation τ we derive two lemmata, stating properties that are essential for all its further uses.

The proof of the elimination theorem concludes this chapter.

8.2. THE LANGUAGES $\mathcal{L}_\varepsilon$ AND $\mathcal{L}^*_\varepsilon$, THE SYSTEM **CS**(C)

8.2.1. <u>DEFINITION</u> (of $\mathcal{L}_\varepsilon,\mathcal{L}^*_\varepsilon$).

(a) $\mathcal{L}_\varepsilon$ is the language of $\mathbf{LSF}^*$ with choice variables $\varepsilon,\eta,\zeta,\varepsilon_0,\eta_0,\zeta_0$ etc. instead of the lawless variables α,β etc.

(b) $\mathcal{L}^*_\varepsilon$ is the language obtained from $\mathcal{L}_\varepsilon$ by adding the clause: 'if $\phi \in$ K-Tm, $\mathcal{F} \in$ Fr-Tm and A is a formula, then $\forall\varepsilon\epsilon(\phi,\mathcal{F})A$ is a formula' to the clauses defining the set of formulae (see 8.2.5).

In L_ε we formulate the axioms of CS(C) (cf.1.3.29).

8.2.2. DEFINITION. CS(C) is the system with the following axioms and axiom schemata:

CS(C)1 $\forall\varepsilon\eta\forall e\in C\exists\zeta(\zeta=e|(\varepsilon,\eta))$,

CS(C)2 $A(\varepsilon) \to \exists e\in C(\exists\eta(\varepsilon=e|\eta) \wedge \forall\zeta\ A(e|\zeta))$,

CS(C)3 $\forall\varepsilon\exists a\ A(\varepsilon,a) \to \exists e\forall u(eu\neq 0 \to \exists a\forall\varepsilon\ A([u]|\varepsilon,a))$,

CS(C)4 $\forall\varepsilon\exists\eta\ B(\varepsilon,\eta) \to \forall\varepsilon\exists f\in C\ A(\varepsilon,f|\varepsilon)$,

where A and B are formulae of L_ε containing no choice parameters besides those shown in notation, and a is a meta-variable for 'any lawlike variable of L_ε'.

From now on we shall frequently use the meta-variable a for the same purpose as in definition 8.2.2, namely to abbreviate 'any lawlike variable of L_ε (L_ε^*).'

8.2.3. DEFINITION. A is a *closed formula* of L_ε (L_ε^*), if it contains no *choice* parameters.

Convention

If we denote a formula of L_ε, L_ε^* by $A(\varepsilon_1,\ldots,\varepsilon_p)$, we mean that it contains no choice parameters besides $\varepsilon_1,\ldots,\varepsilon_p$.

8.2.4. DEFINITION. Let π,d and δ define a domain, put $\mathcal{D}_\delta \equiv \{e|\pi_F\delta: e \in C, F \in \text{FRAME}\}$. With each formula A of L_ε we associate a formula A^δ in the language of LSF*, which expresses that $\mathcal{D}_\delta$ fulfills A, as follows: A^δ is obtained from A by replacing, for each $i \in \mathbb{N}$, all occurrences of the i-th choice variable u_i in A by $v_{i+j}|\pi_{w_{i+k}}\delta$ and all quantifiers $\forall u_i$, $\exists u_i$ by $\forall v_{i+j}\forall w_{i+k}$, $\exists v_{i+j}\exists w_{i+k}$ respectively, where v_{i+j} is the i+j-th K-variable, w_{i+k} is the i+k-th frame-variable, j is 1 plus the maximum of the indices of the K-variables occurring in A and k is 1 plus the maximum of the indices of the frame-variables occurring in A.

For $(A(\varepsilon_1,\ldots,\varepsilon_p))^\delta$ we write $A^\delta(e_1|\pi_{F_1},\ldots,e_p|\pi_{F_p})$, to indicate that $e_i|\pi_{F_i}\delta$ replaces ε_i $(i = 1,\ldots,p)$.

NOTE: when we replace a choice variable ε by a term $e|\pi_F\delta$, we follow the conventions of 7.2.15(c).

8.2.5. We introduce the language L_ε^* for purely formal reasons: it is easier to describe a translation which eliminates choice quantifiers from L_ε^* than to describe such a translation directly for L_ε. (It is an elimination translation for L_ε which interests us.)

Yet, it would be convenient if we could assign some meaning to the restricted quantifiers $\forall\varepsilon\epsilon(e,F)$. To do so we consider another expansion L^0_ε, of L_ε, obtained by adding the clause

"if $\phi \in$ K-Tm, $F \in$ Fr-Tm and ε is a choice-variable then $\varepsilon \in (\phi,F)$ is a prime formula"

to the formula definition.

The δ translation of definition 8.2.4 above, which gives us the interpretation of a formula A of L_ε in the domain $\mathcal{D}_\delta$, can be extended to L^0_ε by requiring that subformulae $\varepsilon \in (\phi,F)$ of a formula A are replaced by $\exists x[(e{:}d_F(\bar{\delta}x),\not{\delta}_F(\bar{\delta}x))\geq(\phi,F)]$, where $e|\pi_F\delta$ replaces ε everywhere else.

That is to say, $\varepsilon \in (\phi,F)$ is interpreted as: 'there is an x such that the restriction for ε at stage x is stronger than (ϕ,F)'. We abbreviate this to: 'ε meets the restriction (ϕ,F)' (where ε ranges over the sequences $e|\pi_F\delta$ in the domain $\mathcal{D}_\delta$).

L^*_ε can be defined as a sublanguage of L^0_ε; we can put

$$\forall\varepsilon\epsilon(\phi,F)\ A \equiv_{def} \forall\varepsilon(\varepsilon\epsilon(\phi,F) \to A).$$

Thus $\forall\varepsilon\epsilon(\phi,F)$ A says: all sequences ε which meet the restriction (ϕ,F) satisfy A.

8.3. THE ELIMINATION TRANSLATION

8.3.1. The translation τ to be defined in this section maps closed formulae of L^*_ε onto formulae of $L(\text{IDBF}^*)$, i.e. it eliminates choice quantifiers. The idea behind the translation is (in complete analogy with the elimination translations for LS and CS) to replace quantifiers $\exists\varepsilon$ not in the scope of a universal choice quantifier by $\exists e\epsilon C\exists F\forall\varepsilon\epsilon(e,F)$, to contract pairs of universal choice quantifiers into a single one, and to push universal choice quantifiers not in the scope of other universal choice quantifiers inwards over the other logical signs $\wedge,\vee,\to,\forall a$ $\exists a$ and $\exists\varepsilon$, until we are left with a formula which contains only universal choice quantifiers in front of prime formulae, which are then replaced by lawlike quantifiers.

As will become clear on inspection of the definition of the elimination mapping τ (8.3.3-7), the translated sentence τA is equivalent to A if we assume the following principles (in the language L^0_ε):

(a) $\forall\varepsilon\epsilon(e,F)\forall\eta\epsilon(f,G)\exists\zeta(\zeta\epsilon(e\wedge f,F\wedge G) \wedge j_1\zeta=\varepsilon \wedge j_2\zeta=\eta)$,

(b) $\forall\zeta\in(e\wedge f,F\wedge G)\exists\varepsilon\exists\eta(\varepsilon\in(e,F)\wedge\eta\in(f,G)\wedge j_1\zeta=\varepsilon\wedge j_2\zeta=\eta)$,

(c) $\forall e\in C\forall F\exists\varepsilon(\varepsilon\in(e,F))$,

(d) $\varepsilon\in(e,F)\wedge\varepsilon\in(f,G)\rightarrow(e,F)\geq(f,G)\vee(f,G)\geq(e,F)$,

(e) $A\varepsilon\rightarrow\exists e\in C\exists F(\varepsilon\in(e,F)\wedge\forall\eta\in(e,F)\ A\eta)$,

(f) $\forall\varepsilon\in(e,F)\exists a\ A(\varepsilon,a)\leftrightarrow\exists e\forall u/\!/F\,[eu\neq 0\rightarrow\exists a\forall\varepsilon\in(e:[u],F)\ A(\varepsilon,a)]$,

(g) $\forall\varepsilon\in(e,F)\exists\eta\ B(\varepsilon,\eta)\leftrightarrow\exists e\forall u/\!/F\,[eu\neq 0\rightarrow\exists f\in C\exists G\forall\zeta\in((e:[u])\wedge f,F\wedge G)\ B(j_1\zeta,j_2\zeta)]$,

(h) $\forall\varepsilon\in(e,F)(t[\varepsilon]=s[\varepsilon])\leftrightarrow\forall a/\!/F\ (t[e|a]=s[e|a])$.

We shall prove the elimination theorem without relying on (a)-(h). However, these principles may help to explain the successful use of the elimination translation: in content they are close to the CS-axioms, in form they resemble the axioms for lawless sequences (in particular (e), (f) and (g)).

8.3.2. The translation τ below is obtained by reworking a notion of forcing introduced by Dragalin in [Dr74]. In fact, in [Dr74] a whole range of notions of forcing is introduced, generalizing both the elimination translations for LS and for CS. It is proved that one of these notions provides a model for the CS-axioms (our theorem 9.2.10) but without using the key-lemma 9.2.9 which is essential for our proof.

Dragalin seems to claim that his forcing is 'essentially' Beth-forcing. From our point of view the reduction to Beth-forcing is far from trivial, this reduction is proved in the elimination-theorem 8.4.2 below. Though Dragalin's forcing is obviously inspired by Troelstra's description of GC-sequences, it does not provide a notion of sequence which fulfills the CS-axioms.

Before we define the actual elimination translation, we introduce an auxiliary mapping $\mapsto$. In 8.3.3 and 8.3.4 ϕ and ψ range over K-Tm, F and G range over Fr-Tm.

8.3.3. <u>DEFINITION.</u> $\mapsto$ is a partial mapping from the set of closed formulae of L^*_ε into itself. A closed formula Φ is in the domain of $\mapsto$ iff $\Phi\equiv\forall\varepsilon\in(\phi,F)\ A\varepsilon$, $\Phi\equiv\forall\varepsilon\ A\varepsilon$ or, $\Phi\equiv\exists\varepsilon\ A\varepsilon$ for some formula A of L^*_ε. The image of Φ under $\mapsto$ is constructed as follows:

(i)	$\forall\varepsilon\in(\phi,F)(t[\varepsilon]=s[\varepsilon])$	$\mapsto$	$\forall a/\!/F(t[\phi\|a]=s[\phi\|a])$,
(ia)	$\forall\varepsilon\in(\phi,F)\ K\psi$	$\mapsto$	$K\psi$,
(ib)	$\forall\varepsilon\in(\phi,F)\ C\psi$	$\mapsto$	$C\psi$,

(ii)	$\forall\varepsilon\epsilon(\phi,F)(A\wedge B)$	$\mapsto$	$\forall\varepsilon\epsilon(\phi,F)\ A \wedge \forall\varepsilon\epsilon(\phi,F)\ B$,
(iii)	$\forall\varepsilon\epsilon(\phi,F)(A\vee B)$	$\mapsto$	$\exists e\forall y /\!/ F[ey\neq 0 \rightarrow \forall\varepsilon\epsilon(\phi{:}[y],F)\ A \vee \forall\varepsilon\epsilon(\phi{:}[y],F)\ B]$,
(iv)	$\forall\varepsilon\epsilon(\phi,F)(A\rightarrow B)$	$\mapsto$	$\forall(f,G)\geq(\phi,F)[\forall\varepsilon\epsilon(f,G)\ A \rightarrow \forall\varepsilon\epsilon(f,G)B]$,
(v)	$\forall\varepsilon\epsilon(\phi,F)\forall aA$	$\mapsto$	$\forall a\forall\varepsilon\epsilon(\phi,F)\ A$,
(v)C1	$\forall\varepsilon\epsilon(\phi,F)\forall\eta\ A(\varepsilon,\eta)$	$\mapsto$	$\forall e\epsilon C\forall G\forall\zeta\epsilon(\phi\wedge e,F\wedge G)\ A(j_1\zeta,j_2\zeta)$,
(v)C2	$\forall\varepsilon\epsilon(\phi,F)\forall\eta\epsilon(\psi,G)\ A(\varepsilon,\eta)$	$\mapsto$	$\forall\zeta\epsilon(\phi\wedge\psi,F\wedge G)\ A(j_1\zeta,j_2\zeta)$,
(vi)	$\forall\varepsilon\epsilon(\phi,F)\exists a\ A$	$\mapsto$	$\exists e\forall y /\!/ F[ey\neq 0 \rightarrow \exists a\forall\varepsilon\epsilon(\phi{:}[y],F)\ A]$,
(vi)C	$\forall\varepsilon\epsilon(\phi,F)\exists\eta\ A(\varepsilon,\eta)$	$\mapsto$	$\exists e\forall y /\!/ F[ey\neq 0 \rightarrow \exists f\epsilon C\exists G\forall\zeta\epsilon((\phi{:}[y])\wedge f,\ F\wedge G)\ A(j_1\zeta,j_2\zeta)]$,
(vii)	$\forall\varepsilon\ A\varepsilon$	$\mapsto$	$\forall e\epsilon C\forall F\forall\varepsilon\epsilon(e,F)\ A\varepsilon$,
(viii)	$\exists\varepsilon\ A\varepsilon$	$\mapsto$	$\exists e\epsilon C\exists F\forall\varepsilon\epsilon(e,F)\ A\varepsilon$.

8.3.4. <u>REMARKS</u>.

(a) The choice-quantifier in $\forall\varepsilon\epsilon(\phi,F)\ K\psi, \forall\varepsilon\epsilon(\phi,F)\ C\psi$ is void, since ψ in this context must be lawlike. The mapping $\mapsto$ deletes such quantifiers (see (ia), (ib) above). In proofs by induction w.r.t. the logical complexity of formulae, involving $\mapsto$, we shall omit these (trivial) cases.

(b) Note that $\mapsto$ treats disjunction as if it were defined as follows: $A \vee B \equiv \exists x[(x=0 \rightarrow A) \wedge (x\neq 0 \rightarrow B)]$. This means that we can omit the disjunction-case in inductive proofs too.

8.3.5. <u>DEFINITION</u>. Let Φ be a closed formula of L_ε^*. Let A be a subformula of Φ in the domain of $\mapsto$, let B be such that $A \mapsto B$. Φ' *is obtained from* Φ *by an application of* $\mapsto$, if Φ' is the result of a replacement of an occurrence of A in Φ, not in the scope of a choice-quantifier, by an occurrence of B.

8.3.6. <u>FACTS</u>.

(a) If Φ is closed and Φ' is obtained from Φ by an application of $\mapsto$, then Φ' is closed.

(b) $c(\Phi) \equiv$ the number of logical operations (connectives and quantifiers) occurring in Φ in the scope of a choice quantifier + the number of restricted choice quantifiers in Φ + twice the number of unrestricted choice quantifiers in Φ.

We find that

(i) if Φ' is obtained from Φ by an application of $\mapsto$, then $c(\Phi') < c(\Phi)$,

(ii) if Φ is closed, $c(\Phi) > 0$, then there is a Φ' that can be obtained from Φ by an application of $\mapsto$, and

(iii) if Φ is closed, $c(\Phi) = 0$, then Φ is lawlike.

(c) Let Φ', Φ'' be distinct formulae, obtained from Φ by an application of $\mapsto$, Φ' resulting from a replacement of an occurrence of A, Φ'' from a replacement of an occurrence of B. Then these occurrences of A and B must be disjoint, hence there is a formula Φ''' which can be obtained from Φ' as well as from Φ'' by an application of $\mapsto$.

(d) From (a)-(c) we can conclude that with each closed formula Φ of L_ε^* there is a unique formula Ψ such that

(i) Ψ is lawlike, and

(ii) there is a finite sequence $\Phi \equiv \Phi_0, \ldots, \Phi_p \equiv \Psi$ of closed formulae of L_ε^* such that for all $i < p$, Φ_{i+1} is obtained from Φ_i by an application of $\mapsto$.

8.3.7. <u>DEFINITION</u> (of $\ulcorner\Phi\urcorner$ and τ). Let Φ be a closed formula of L_ε^*, then $\ulcorner\Phi\urcorner$ is the unique lawlike formula Ψ which satisfies 8.3.6(d)(ii).
τ is the translation which carries Φ into $\ulcorner\Phi\urcorner$.
Since τ eliminates choice variables from closed formulae of L_ε^* we call it an elimination translation for L_ε^*.

8.3.8. <u>FACTS</u>.

(a) $\ulcorner A \overset{\vee}{\underset{\wedge}{\rightarrow}} B\urcorner \equiv \ulcorner A\urcorner \overset{\vee}{\underset{\wedge}{\rightarrow}} \ulcorner B\urcorner$, A and B closed.

(b) $\ulcorner Q\mathrm{a}\ A\urcorner \equiv Q\mathrm{a}\ulcorner A\urcorner$, A closed, $Q \equiv \exists$ or $Q \equiv \forall$, a a lawlike variable of any sort.

(c) $\ulcorner\forall\varepsilon\ A\urcorner \equiv \forall e\epsilon C \forall F \ulcorner\forall\varepsilon\epsilon(e,F)\ A\urcorner$, see 8.3.3 (the definition of $\mapsto$).

(d) $\ulcorner\exists\varepsilon\ A\urcorner \equiv \exists e\epsilon C \exists F \ulcorner\forall\varepsilon\epsilon(e,F)\ A\urcorner$, see 8.3.3.

(e) If $\Phi \equiv \forall\varepsilon\epsilon(\phi,F)\ A$, then the structure of $\ulcorner\Phi\urcorner$ depends on the main logical sign of Φ, see 8.3.3.

The next two lemmata, 8.3.9 and 8.3.11, state important properties of τ. The reader is advised to skip their proofs at first reading.

8.3.9. <u>LEMMA</u> (monotonicity of τ).
Let $A\varepsilon$ *be a formula of* L_ε *with at most one choice parameter*: ε. *Let* (e,F) *and* (f,G) *be restrictions. Assume*

(a) C is *dependency-closed*,

(b) $(f,G) \geq (e,F)$,

(c) $\ulcorner\forall\varepsilon\epsilon(e,F)\ A\varepsilon\urcorner$.

Then

(d) $\ulcorner\forall\varepsilon\epsilon(f,G)\ A\varepsilon\urcorner$

is derivable; the derivation can be formalized in IDBF*, *i.e.*

$$\text{IDBF}^* \vdash \text{dclosed}(C) \rightarrow$$
$$\forall(e,F)[\ulcorner\forall\varepsilon\epsilon(e,F)\ A\varepsilon\urcorner \rightarrow \forall(f,G)\geq(e,F)\ulcorner\forall\varepsilon\epsilon(f,G)\ A\varepsilon\urcorner].$$

PROOF. We proceed by induction w.r.t. the logical complexity of A. The proof is subdivided into cases, one for each possible main logical sign in A. The numbering of these cases corresponds to that of 8.3.3. By assumption (a) we can apply all $\geq$-properties (7.3.7.).

case (i) $A\varepsilon \equiv t[\varepsilon]=s[\varepsilon]$.
Assumption (c) becomes in this case

(1) $\forall a/\!/F(t[e|a] = s[e|a])$.

To derive (d), i.e. in this case

(2) $\forall b/\!/G(t[f|b] = s[f|b])$,

it suffices to show that for each $b/\!/G$ there is an $a/\!/F$ such that $f|b = e|a$. Let $b/\!/G$ be arbitrary. By assumption (b) there is an $\mathfrak{f}$ such that $G = F[\mathfrak{f}]$, so (by $/\!/$-property 7.3.6(g)) $b/\!/F$.
Also by assumption (b) there is an e' such that $f \simeq e{:}e'$ and $e'/\!/_C F$.
Put $a \equiv e'|b$. Then $a/\!/F$ by 7.3.6(j) and $f|b = e{:}e'|b = e|a$.

case (ii) $A\varepsilon \equiv B\varepsilon \wedge C\varepsilon$,
trivial by induction-hypothesis.

case (iii) $A\varepsilon \equiv B\varepsilon \vee C\varepsilon$,
can be treated as $A\varepsilon \equiv \exists x\ D(\varepsilon,x)$, see 8.3.4(b).

case (iv) $A\varepsilon \equiv B\varepsilon \rightarrow C\varepsilon$.
By assumption (a), C is dependency-closed, hence the relation $\geq$ between restrictions is transitive ($\geq$-property 7.3.7(b)). (d) immediately follows from (c) by this transitivity.

case (v) $A\varepsilon \equiv \forall a\ B(\varepsilon,a)$,
trivial by induction-hypothesis.

case (v)C $A\varepsilon \equiv \forall\eta\ B(\varepsilon,\eta)$.
In this case assumption (c) reads $\forall g\epsilon C\forall H\ulcorner\forall\zeta\epsilon(e\wedge g,F\wedge H)\ B(j_1\zeta,j_2\zeta)\urcorner$.

id $\in$ C by assumption (a), hence this specializes to

(3) $\quad \forall n \notin \ell F \ulcorner \forall \zeta \in (e \wedge id, F \wedge (°n))\ B(j_1\zeta, j_2\zeta) \urcorner$.

By assumption (b) and $\geq$-property 7.3.7(f)

$$n \notin \ell F \rightarrow \forall g \in C \forall H[(f \wedge g, G \wedge H) \geq (e \wedge id, F \wedge (°n))],$$

whence (3) yields by induction-hypothesis, $\forall g \in C \forall H \ulcorner \forall \zeta \in (f \wedge g, G \wedge H)\ B(j_1\zeta, j_2\zeta) \urcorner$, i.e. $\ulcorner \forall \varepsilon \in (f,G) A\varepsilon \urcorner$.

case (vi) $\quad A\varepsilon \equiv \exists a\ B(\varepsilon, a)$.
By assumption (c) we have an $e_1 \in K$ such that

(4) $\quad \forall y /\!/ F[e_1 y \neq 0 \rightarrow \exists a \ulcorner \forall \varepsilon \in (e:[y], F)\ B(\varepsilon, a) \urcorner]$.

To derive (d) we must construct an $e_2 \in K$ such that

(5) $\quad \forall y /\!/ G[e_2 y \neq 0 \rightarrow \exists a \ulcorner \forall \varepsilon \in (f:[y], G)\ B(\varepsilon, a) \urcorner]$.

By assumption (b), $(f,G) \geq (e,F)$, there is a g such that $g /\!/_C F$ and $f \simeq e{:}g$. Put $e_2 \equiv e_1;g$. (7.3.1.) To show that e_2 fulfills (5), let $y /\!/ G$ satisfy $e_2 y \neq 0$, i.e. $e_1(g \restriction y) \neq 0$. By (4) we find an a such that

(6) $\quad \ulcorner \forall \varepsilon \in (e:[g \restriction y], F)\ B(\varepsilon, a) \urcorner$.

By $\geq$-property 7.3.7(h) $(f:[y],G) \geq (e:[g \restriction y],F)$, so (6) yields, by induction-hypothesis $\ulcorner \forall \varepsilon \in (f:[y], G)\ B(\varepsilon, a) \urcorner$.

case (vi)C $\forall\varepsilon \equiv \exists \eta\ B(\varepsilon, \eta)$.
By assumption (c) we have an $e_1 \in K$ such that

(7) $\quad \forall y /\!/ F[e_1 y \neq 0 \rightarrow \exists g \in C \exists H \ulcorner \forall \zeta \in ((e:[y]) \wedge g,\ F \wedge H)\ B(j_1\zeta, j_2\zeta) \urcorner]$.

To derive (d), an $e_2 \in K$ must be constructed which satisfies

(8) $\quad \forall y /\!/ G[e_2 y \neq 0 \rightarrow \exists g' \in C \exists H' \ulcorner \forall \zeta \in ((f:[y]) \wedge g',\ G \wedge H')\ B(j_1\zeta, j_2\zeta) \urcorner]$.

By assumption (b), $(f,G) \geq (e,F)$, there is an e' such that

$e' /\!/_C F$ and $f \simeq e{:}e'$.

Put $e_2 \equiv e_1;e'$, then e_2 fulfills (8):

Let $y /\!/ G$ be such that $e_2 y \neq 0$, i.e. $e_1(e' \upharpoonright y) \neq 0$. By (7) we find a $g \in C$ and an H such that

(9) $$\ulcorner \forall \zeta \in ((e{:}[e' \upharpoonright y]) \wedge g,\ F \wedge H)\ B(j_1\zeta, j_2\zeta) \urcorner.$$

By $\geq$-property 7.3.7(h) $(f{:}[y],G) \geq (e{:}[e' \upharpoonright y],F)$.

By $\geq$-property 7.3.7(g) we find $g' \in C$ and H' such that $((f{:}[y]) \wedge g',\ G \wedge H') \geq ((e{:}[e' \upharpoonright y]) \wedge g,\ F \wedge H)$, so (g) yields by induction-hypothesis

$$\ulcorner \forall \zeta \in ((f{:}[y]) \wedge g',\ G \wedge H')\ B(j_1\zeta, j_2\zeta) \urcorner. \quad \Box$$

8.3.10 COROLLARIES. *Let* $A\varepsilon$ *be a formula with at most one choice parameter:* ε, *let* $B(\varepsilon,\eta)$ *have no choice parameters besides* ε *and* η. *Then, if* C *is dependency-closed:*

(a) $\forall n(\ulcorner \forall \varepsilon\ A\varepsilon \urcorner \leftrightarrow \ulcorner \forall \varepsilon \in (\mathrm{id}, {}^\circ n)\ A\varepsilon \urcorner)$,
[From left to right by definition, from right to left by monotonicity and $\geq$-property 7.3.7(d).]

(b) $\forall n \notin \ell F(\ulcorner \forall \varepsilon \in (e,F) \forall \eta\ B(\varepsilon,\eta) \urcorner \leftrightarrow \ulcorner \forall \zeta \in (e \wedge \mathrm{id}, F \wedge ({}^\circ n))\ B(j_1\zeta, j_2\zeta) \urcorner)$,
[From left to right by definition, from right to left by monotonicity and $\geq$-property 7.3.7(f).]

(c) $e \simeq f \rightarrow (\ulcorner \forall \varepsilon \in (e,F)\ A\varepsilon \urcorner \leftrightarrow \ulcorner \forall \varepsilon \in (f,F)\ A\varepsilon \urcorner)$,
[By monotonicity and $\geq$-property 7.3.7(a).]

(d) $\ulcorner \forall \varepsilon (A\varepsilon \rightarrow B\varepsilon) \urcorner \leftrightarrow \forall e \in C \forall F(\ulcorner \forall \varepsilon \in (e,F)\ A\varepsilon \urcorner \rightarrow \ulcorner \forall \varepsilon \in (e,F)\ B\varepsilon \urcorner)$.
[By (a) and $\geq$-property 7.3.7(d).]

8.3.11. LEMMA (bar-property of τ).

Let $A\varepsilon$ *be a formula of* L^*_ε *with at most one choice parameter:*ε. *Let* f *be an element of* K, (e,F) *a restriction. Assume*

(a) C *is dependency-closed, and*

(b) $\forall y /\!/ F[fy \neq 0 \rightarrow \ulcorner \forall \varepsilon \in (e{:}[y],F)\ A\varepsilon \urcorner]$.

Then

(c) $\ulcorner \forall \varepsilon \in (e,F)\ A\varepsilon \urcorner$

is derivable, the derivation can be formalized in IDBF*, *i.e.*

$$\underset{\sim}{\text{IDBF}}^{*} \vdash \text{dclosed}(C) \rightarrow$$

$$\forall(e,F)\forall f(\forall y/\!/F[fy\neq 0 \rightarrow \ulcorner\forall\varepsilon\epsilon(e{:}[y],F)\ A\varepsilon\urcorner] \rightarrow \ulcorner\forall\varepsilon\epsilon(e,F)\ A\varepsilon\urcorner).$$

PROOF. By induction w.r.t. the logical complexity of $A\varepsilon$, cf. the proof of the monotonicity of τ. Because we assume C to be dependency-closed, monotonocity of τ can be applied, as well as the $\geq$-properties.

case (i) $A\varepsilon \equiv t[\varepsilon] = s[\varepsilon]$.
By assumption (b), f satisfies

$$(1) \qquad \forall y/\!/F[fy\neq 0 \rightarrow \forall a/\!/F(t[e{:}[y]|a] = s[e{:}[y]|a])].$$

This yields

$$(2) \qquad \forall b/\!/F\forall x[f(\bar{b}x)\neq 0 \rightarrow t[e{:}[\bar{b}x]|(\lambda z.b(x+z))] = s[e{:}[\bar{b}x]|(\lambda z.b(x+z))]].$$

Since $f \in K$, we have $\forall b\exists x(f(\bar{b}x)\neq 0)$, by definition $[\bar{b}x]|(\lambda z.b(x+z)) = b$ for all b and x, so (2) yields $\forall b/\!/F(t[e|b] = s[e|b])$, i.e. $\ulcorner\forall\varepsilon\epsilon(e,F)\ A\varepsilon\urcorner$.

case (ii) $A\varepsilon \equiv B\varepsilon\wedge C\varepsilon$,
trivial by induction-hypothesis.

case (iii) $A\varepsilon \equiv B\varepsilon\vee C\varepsilon$,
can be treated as $A\varepsilon \equiv \exists x\ D(\varepsilon,x)$, cf. remark 8.3.4(b).

case (iv) $A\varepsilon \equiv B\varepsilon \rightarrow C\varepsilon$.
By assumption (b), f satisfies:

$$(3) \qquad \forall y/\!/F[fy\neq 0 \rightarrow \forall(g,H)\geq(e{:}[y],F)[\ulcorner\forall\varepsilon\epsilon(g,H)\ B\varepsilon\urcorner \rightarrow \ulcorner\forall\varepsilon\epsilon(g,H)\ C\varepsilon\urcorner]].$$

We want to derive $\ulcorner\forall\varepsilon(e,F)\ A\varepsilon\urcorner$, i.e.

$$\forall(e',F')\geq(e,F)[\ulcorner\forall\varepsilon\epsilon(e',F')\ B\varepsilon\urcorner \rightarrow \ulcorner\forall\varepsilon\epsilon(e',F')\ C\varepsilon\urcorner].$$

To this end, let $(e',F') \geq (e,F)$ be arbitrary, let $g' \in K$ satisfy $g'/\!/_C F$ and $e' \simeq e{:}g'$ and assume

$$(4) \qquad \ulcorner\forall\varepsilon\epsilon(e',F')\ B\varepsilon\urcorner.$$

Put $f' \equiv f;g'$, let y be parallel to F'. Then (4) yields, by monotonicity and $\geq$-property 7.3.7(c)

(5) $\ulcorner \forall \varepsilon \in (e':[y],F')\ B\varepsilon \urcorner$,

while by $\geq$-property 7.3.7(h)

(6) $(e':[y],F') \geq (e:[g' \restriction y],F)$.

Now assume $f'y \neq 0$, i.e. $f(g' \restriction y) \neq 0$.
$y /\!/ F'$, $F' \geq F$ hence $y /\!/ F$ by $/\!/$-property 7.3.6(g); $g' /\!/_C F$, so $g' \restriction y /\!/ F$ by $/\!/$-property 7.3.6(j). Hence, (by (3), (5), (6))

$$f'y \neq 0 \rightarrow \ulcorner \forall \varepsilon \in (e':[y],F')\ C\varepsilon \urcorner.$$

By induction-hypothesis we conclude $\ulcorner \forall \varepsilon \in (e',F')\ C\varepsilon \urcorner$.

case (v) $A\varepsilon \equiv \forall a\ B(\varepsilon,a)$,
trivial by induction-hypothesis.

case (v)C $A\varepsilon \equiv \forall \eta\ B(\varepsilon,\eta)$.
By assumption (b), f satisfies

(7) $\forall y /\!/ F[fy \neq 0 \rightarrow \forall g \in C \forall H \ulcorner \forall \zeta \in ((e:[y]) \wedge g,\ F \wedge H)\ B(j_1\zeta, j_2\zeta) \urcorner]$.

Let $g' \in C$ and H' be arbitrary. We want to derive

(8) $\ulcorner \forall \zeta \in (e \wedge g', F \wedge H')\ B(j_1\zeta, j_2\zeta) \urcorner$

Put $f' \equiv \lambda z.f(k_1 z)$, one easily sees that $f' \in K$. Let $z /\!/ F \wedge H'$, then $k_1 z /\!/ F$ (7.3.6(c)); suppose $f'z \neq 0$, i.e. $f(k_1 z) \neq 0$. Then (7) yields

$$\forall g \in C \forall H \ulcorner \forall \zeta \in ((e:[k_1 z]) \wedge g,\ F \wedge H)\ B(j_1\zeta, j_2\zeta) \urcorner,$$

which specializes to

(9) $\forall n \notin \ell F \ulcorner \forall \zeta \in ((e:[k_1 z]) \wedge g',\ F \wedge (\circ n))\ B(j_1\zeta, j_2\zeta) \urcorner$.

By assumption (a) C is dependency-closed, whence $[k_2 z] \in C$. By 7.3.7(f) we find for $n \notin \ell F$:

(10) $((e:[k_1z])\wedge(g':[k_2z]),\ F\wedge H') \geq ((e:[k_1z])\wedge g',\ F\wedge(°n))$.

By 7.3.2(b), 7.3.3(a)

(11) $(e:[k_1z])\wedge(g':[k_2z]) \simeq (e\wedge g'):[z]$.

If we combine (9), (10), (11) with the monotonicity of τ and the corollary 8.3.10(c), we find $\ulcorner\forall\zeta\epsilon((e\wedge g'):[z],\ F\wedge H')\ B(j_1\zeta,j_2\zeta)\urcorner$. By induction-hypothesis, (8) follows.

case (vi) $A\varepsilon \equiv \exists a\ B(\varepsilon,a)$.
By assumption (b) f satisfies

$$\forall y//F[fy\neq 0 \rightarrow \exists e_1\forall z//F[e_1z\neq 0 \rightarrow \exists a\ulcorner\forall\varepsilon\epsilon(e:[y]:[z],F)\ B(\varepsilon,a)\urcorner]].$$

Hence, by AC-NF, there is an $e' \in K$ such that

$$\forall y//F[fy\neq 0 \rightarrow \forall z//F[e'(\langle y\rangle * z)\neq 0 \rightarrow \exists a\ulcorner\forall\varepsilon\epsilon(e:[y]:[z],F)\ B(\varepsilon,a)\urcorner]].$$

We must derive $\ulcorner\forall\varepsilon\epsilon(e,F)\ A\varepsilon\urcorner$, i.e. we have to construct an $e_2 \in K$ such that

(12) $\forall w//F[e_2w\neq 0 \rightarrow \exists a\ulcorner\forall\varepsilon\epsilon(e:[w],F)\ B(\varepsilon,a)\urcorner]$.

Take $e_2 \equiv f\times e'$, i.e. if $e_2w \neq 0$ then there are u and v such that $w = u*v$, $fu \neq 0$ and $e'(\langle u\rangle * v) \neq 0$ (7.3.4). Then e_2 clearly satisfies (12).

case (vi)C $A\varepsilon \equiv \exists\eta\ B(\varepsilon,\eta)$,
can be treated exactly like case (vi). □

For the proof of the elimination theorem, we need the following three propositions.

8.3.12. PROPOSITION. *With each equation* t=s *of* $\underset{\sim}{LSF}^*$, *there is a formula* $(t=s)^*$ *of* $\underset{\sim}{LSF}^*$, *provably equivalent to* t=s, *but which contains only prime-formulae of the form* t'=s', *where* s' *is lawlike and* t' *is either lawlike or of the form* $\alpha t''$, t" *lawlike*.

8.3.13. PROPOSITION. *If* t=s *is an equation of* L^*_ε *in a single choice parameter* ε, *and* $(t=s)^*$ *is its translation as in* 8.3.12, *then*

$$\underset{\sim}{IDBF}^* \vdash \ulcorner\forall\varepsilon\epsilon(e,F)(t=s)\urcorner \leftrightarrow \ulcorner\forall\varepsilon\epsilon(e,F)(t=s)^{*}\urcorner.$$

8.3.14. PROPOSITION (extensionality of $\underset{\sim}{LSF}^*$). *If* $A(\alpha_1,\ldots,\alpha_p)$ *is a formula of* $\underset{\sim}{LSF}^*$*, which may contain more choice parameters besides* $\alpha_1,\ldots,\alpha_p$*, then*

$$\underset{\sim}{LSF}^* \vdash \Lambda^p_{i=1}[\forall x(e_i|\beta_i(x)=f_i|\gamma_i(x))] \rightarrow$$
$$(A(e_1|\beta_1,\ldots,e_p|\beta_p) \leftrightarrow A(f_1|\gamma_1,\ldots,f_p|\gamma_p)),$$

where $e_i|\beta_i$, $f_i|\gamma_i$ *are substituted for* α_i $(i = 1,\ldots,p)$ *following the conventions of* 7.2.15(c).

8.3.14 is proved by formula-induction (straightforward). To give an idea of the translation $(\ )^*$ in 8.3.12 we state some clauses:
if s is not lawlike then $(t=s)^* \equiv \exists x((t=x)^* \wedge (s=x)^*)$,
if s is lawlike then e.g.:

$$(e|\alpha(t)=s)^* \equiv \exists yv((t=y)^* \wedge \forall n<lth(v)(\alpha n=(v)_n) \wedge e(\hat{y}*v)=s+1),$$
$$(\Pi(t_1,\lambda z.t_2,t_3)=s)^* \equiv \exists y_1y_2v((t_1=y_1)^* \wedge (t_3=y_2)^* \wedge (v)_0=y_1 \wedge$$
$$(v)_{y_2}=s \wedge \forall n<y_2(t_2[j((v)_n,n)/z]=(v)_{n+1})^*).$$

The completion of the definition of $(\ )^*$ is simple. 8.3.12 is easily proved. For the proof of 8.3.13 finally, one needs the observation that with each term $t[a]$ of $\underset{\sim}{IDBF}^*$ there is an element $e_t \in K$ such that for all a $t[a] = e_t(a)$. For terms of $\underset{\sim}{IDB}_1$ this fact is proved in [KT70]. We leave it to the reader to verify that this result also holds for $\underset{\sim}{IDBF}^*$.

8.4. THE ELIMINATION THEOREM

The hard work for the proof of the elimination theorem is done in the following lemma. The elimination theorem itself is then easily proved in 8.4.2.

8.4.1. LEMMA. *Let* $A\varepsilon$ *be a formula of* L_ε *with at most one choice parameter:* ε. *Assume*
(a) C *is dependency-closed, and*
(b) π, d *and* δ *define a domain.*
Then we can derive
(c) $\forall\delta\epsilon v\ A^\delta(e|\pi_F) \leftrightarrow \ulcorner\forall\varepsilon\epsilon(e:d_Fv,\delta_Fv)\ A\varepsilon\urcorner$.
This derivation can be formalized in $\underset{\sim}{LSF}^*$, *i.e.*

$$\underset{\sim}{\text{LSF}}^{*} \vdash \text{dclosed}(C) \wedge \text{domain}(\pi,d,\ell) \rightarrow$$
$$[\forall\delta\epsilon v\ A^{\delta}(e|\pi_F) \leftrightarrow \ulcorner\forall\varepsilon\epsilon(e{:}d_F v,\ell_F v)\ A\varepsilon\urcorner].$$

<u>PROOF</u>. The proof of (c) from (a) and (b) proceeds by induction w.r.t. the logical complexity of A. Like the proofs of 8.3.9 and 8.3.11 it is subdivided into cases. Each nontrivial case consists of two parts, part (→) for the implication from left to right, part (←) for the converse implication. By assumption (a), we can use the monotonicity and the bar-property for τ, and all the $\geq$- and domain-properties (7.3.7 and 7.3.8).

case (i) $A\varepsilon \equiv t'[\varepsilon]=s'[\varepsilon]$.

By propositions 8.3.12 and 8.3.13 we may restrict our attention to formulae of the form $A\varepsilon \equiv \varepsilon t = s$, t and s lawlike terms.

(→) We assume $\forall\delta\epsilon v\ A^{\delta}(e|\pi_F)$, i.e.

(1) $$\forall\delta\epsilon v\forall z(e(\langle t\rangle * \overline{\pi_F|\delta}(z)) \neq 0 \rightarrow e(\langle t\rangle * \overline{\pi_F|\delta}(z)) = s+1).$$

Let $a /\!/ \ell_F v$ be arbitrary, and let z be such that $e(\langle t\rangle * \overline{d_F v|a}(z)) \neq 0$. By domain-property 7.3.8(h) there is a $\delta \epsilon v$ such that $\overline{\pi_F|\delta}(z) = \overline{d_F v|a}(z)$, hence (by (1)) $e(\langle t\rangle * \overline{d_F v|a}(z)) = s+1$.

(←) For the converse implication we assume $\ulcorner\forall\varepsilon\epsilon(e{:}d_F v,\ell_F v)\ A\varepsilon\urcorner$, i.e.

(2) $$\forall a /\!/ \ell_F v\forall z(e(\langle t\rangle * \overline{d_F v|a}(z)) \neq 0 \rightarrow e(\langle t\rangle * \overline{d_F v|a}(z)) = s+1).$$

In order to derive (1), let $\delta \epsilon v$ and z satisfy

(3) $$e(\langle t\rangle * \overline{\pi_F\delta}(z)) \neq 0.$$

By domain-property 7.3.8(e) we find a $g \epsilon K$ such that

(4) $$\pi_F\delta = d_F v|(g|\delta)$$

and

(5) $$g|\delta /\!/ \ell_F v.$$

By 7.3.1(a) and (4) there is a y such that $\overline{\pi_F\delta}(z) \preccurlyeq d_F v\upharpoonright(g|\delta(y))$, so by (3)

$$e(\langle t\rangle * \overline{\pi_F\delta(z)}) = e(\langle t\rangle * d_F v \restriction \overline{g|\delta(y)}).$$

From this equation and (5) we find an $a /\!/ \mathfrak{d}_F v$ such that

(6) $$e(\langle t\rangle * \overline{\pi_F\delta(z)}) = e(\langle t\rangle * d_F v \restriction \bar{a}y).$$

By 7.3.1(b) there is an x such that $d_F v \restriction \bar{a}y = \overline{d_F v|a}(x)$, whence by (6) and (3) $e(\langle t\rangle * \overline{\pi_F\delta}(z)) = e(\langle t\rangle * \overline{d_F v|a}(x)) \neq 0$. Now apply (2), this yields $e(\langle t\rangle * \overline{\pi_F\delta}(z)) = s+1$.

case (ii) $A\varepsilon \equiv B\varepsilon\ C\varepsilon$,
trivial by induction-hypothesis.

case (iii) $A\varepsilon \equiv B\varepsilon \vee C\varepsilon$,
can be treated as $A\varepsilon \equiv \exists x\, D(\varepsilon,x)$.

case (iv) $A\varepsilon \equiv B\varepsilon \to C\varepsilon$.
($\to$) We assume $\forall\delta\in v\ A^{\delta}(e|\pi_F)$, or equivalently

(7) $$\forall w(\forall\delta\in v*w\ B^{\delta}(e|\pi_F) \to \forall\delta\in v*w\ C^{\delta}(e|\pi_F)).$$

We want to derive $\ulcorner\forall\varepsilon\in(e{:}d_F v, \mathfrak{d}_F v)A\varepsilon\urcorner$, i.e.

(8) $$\forall(f,G) \geq (e{:}d_F v, \mathfrak{d}_F v)[\ulcorner\forall\varepsilon\in(f,G)B\varepsilon\urcorner \to \ulcorner\forall\varepsilon\in(f,G)C\varepsilon\urcorner].$$

Let (f,G) be stronger than $(e{:}d_F v, \mathfrak{d}_F v)$, and assume $\ulcorner\forall\varepsilon\in(f,G)\ B\varepsilon\urcorner$.
Then by monotonicity

$$\forall w[(e{:}d_F(v*w), \mathfrak{d}_F(v*w)) \geq (f,G) \to \ulcorner\forall\varepsilon\in(e{:}d_F(v*w), \mathfrak{d}_F(v*w))\ B\varepsilon\urcorner].$$

By induction-hypothesis (applied to $B\varepsilon$), assumption (7), and induction-hypothesis, now applied to $C\varepsilon$, this yields

$$\forall w[(e{:}d_F(v*w), \mathfrak{d}_F(v*w)) \geq (f,G) \to \ulcorner\forall\varepsilon\in(e{:}d_F(v*w), \mathfrak{d}_F(v*w))\ C\varepsilon\urcorner],$$

whence by monotonicity

(9) $$\forall uw[(f,G) \leq (e{:}d_F(v*w), \mathfrak{d}_F(v*w)) \leq (f{:}[u],G) \to \ulcorner\forall\varepsilon\in(f{:}[u],G)\ C\varepsilon\urcorner].$$

By domain property 7.3.8(f), there is an $e_1 \in K$ such that $\forall u /\!/ G[e_1 u \neq 0 \rightarrow \exists w((f,G) \leq (e:d_F(v*w), \delta_F(v*w)) \leq (f:[u],G))]$. For this e_1 we find (by (9)) $\forall u /\!/ G[e_1 u \neq 0 \rightarrow \ulcorner \forall \varepsilon \in (f:[u],G)\ C\varepsilon \urcorner]$. But then $\ulcorner \forall \varepsilon \in (f,G)\ C\varepsilon \urcorner$ follows immediately by the bar-property of τ.

($\leftarrow$) The derivation of (7) from (8) is trivial, since by domain property 7.3.8(c) $\forall w[(e:d_F(v*w), \delta_F(v*w)) \geq (e:d_F v, \delta_F v)]$.

case (v) $A\varepsilon \equiv \forall a\ B(\varepsilon,a)$,
trivial by induction-hypothesis.

case (v)C $A\varepsilon \equiv \forall \eta\ B(\varepsilon,\eta)$.

($\rightarrow$) We assume $\forall \delta \in v\ A^{\delta}(e|\pi_F)$, i.e.

$$(10) \qquad \forall \delta \in v \forall f \in C \forall G\ B^{\delta}(e|\pi_F, f|\pi_G).$$

We must derive $\forall f \in C \forall G \ulcorner \forall \zeta \in ((e:d_F v)\wedge f, \delta_F v \wedge G)\ B(j_1\zeta, j_2\zeta) \urcorner$ or equivalently (by monotonicity, corollary 8.3.10(b))

$$(11) \qquad \exists n \notin \ell(\delta_F v) \ulcorner \forall \zeta \in ((e:d_F v)\wedge \mathrm{id}, \delta_F v \wedge (^\circ n))\ B(j_1\zeta, j_2\zeta) \urcorner.$$

By definition of domain ((D6)), there are infinitely many m and u such that

$$(12) \qquad \delta_{(^\circ m)} v = \delta_m v = {}^\circ m \quad \text{and} \quad d_{(^\circ m)} v = d_m v = [u],$$

so in particular there are $n \notin \ell(\delta_F v)$ and u which satisfy (12). Let z be lth(u); since C is dependency-closed (assumption (a)) $s^z \in C$, so (10) specializes to

$$(13) \qquad \forall \delta \in v\ B^{\delta}(e|\pi_F, s^z|\pi_{(^\circ n)}).$$

Put $\psi \equiv_{def} (e \wedge s^z)|\pi_{F\wedge(^\circ n)}$, then $j_1\psi = e|\pi_F$, $j_2\psi = s^z|\pi_{(^\circ n)}$, so (13) yields, by extensionality (8.3.14) $\forall \delta \in v\ B^{\delta}(j_1\psi, j_2\psi)$, which, by induction-hypothesis, is equivalent to

$$(14) \qquad \ulcorner \forall \zeta \in ((e\wedge s^z):d_{F\wedge(^\circ n)} v,\ \delta_{F\wedge(^\circ n)} v)\ B(j_1\zeta, j_2\zeta) \urcorner.$$

By 7.3.8(b) $d_{F\wedge(^\circ n)} v \simeq d_F v \wedge d_{(^\circ n)} v$, hence, by choice of n,
$d_{F\wedge(^\circ n)} v \simeq d_F v \wedge [u]$.
By 7.3.2(b) $(e\wedge s^z):(d_F v \wedge [u]) \simeq (e:d_F v)\wedge(s^z:[u])$, hence, by choice of z,

$(e\wedge s^z):(d_F v\wedge[u]) \simeq (e:d_F v)\wedge id$.

By 7.3.8(a) $\delta_{F\wedge(\circ n)}v = \delta_F v\wedge\delta_{(\circ n)}v$, hence, by choice of n, $\delta_{F\wedge(\circ n)}v = \delta_F v\wedge(\circ n)$. So $((e\wedge s^z):d_{F\wedge(\circ n)}v,\ \delta_{F\wedge(\circ n)}v) \approx ((e:d_F v)\wedge id,\ \delta_F v\wedge(\circ n))$, whence (14) yields (11) by monotonicity of τ.

($\leftarrow$) Now we assume $\ulcorner\forall\varepsilon\epsilon(e:d_F v,\delta_F v)A\varepsilon\urcorner$, i.e.

$$(15) \qquad \forall g\epsilon C\forall H\ulcorner\forall\zeta\epsilon((e:d_F v)\wedge g,\ \delta_F v\wedge H)\ B(j_1\zeta,j_2\zeta)\urcorner.$$

By domain axiom (D3d), $\forall G(d_G v\epsilon C)$; since C is dependency-closed then also $\forall f\epsilon C\forall G(f:d_G v\epsilon C)$, so (15) specializes to

$$(16) \qquad \forall f\epsilon C\forall G\ulcorner\forall\zeta\epsilon((e:d_F v)\wedge(f:d_G v),\ \delta_F v\wedge\delta_G v)\ B(j_1\zeta,j_2\zeta)\urcorner.$$

By an argument similar to the one we used to show that (13) implies (11), but now applied in the reverse direction, (10) is derived from (16).

case (vi) $A\varepsilon \equiv \exists a\ B(\varepsilon,a)$.

($\rightarrow$) We assume $\forall\delta\epsilon v\ A^\delta(e|\pi_F)$, i.e. we have an $e_1 \in K$ such that

$$(17) \qquad \forall w[e_1 w\neq 0 \rightarrow \exists a\forall\delta\epsilon v{*}w\ B^\delta(e|\pi_F,a)],$$

or equivalently (by induction-hypothesis), such that

$$(18) \qquad \forall w[e_1 w\neq 0 \rightarrow \exists a\ulcorner\forall\varepsilon\epsilon(e:d_F(v{*}w),\delta_F(v{*}w))\ B(\varepsilon,a)\urcorner].$$

We must derive $\ulcorner\forall\varepsilon\epsilon(e:d_F v,\delta_F v)\ A\varepsilon\urcorner$, so we must find an $e_2 \in K$ such that

$$(19) \qquad \forall u/\!/\delta_F v[e_2 u\neq 0 \rightarrow \exists a\ulcorner\forall\varepsilon\epsilon(e:d_F v:[u],\delta_F v)B(\varepsilon,a)\urcorner].$$

By domain property 7.3.8(g) there is an e_2 such that

$$\forall u/\!/\delta_F v[e_2 u\neq 0 \rightarrow \exists w[e_1 w\neq 0\ \wedge\ (e:d_F(v{*}w),\delta_F(v{*}w))\leq(e:d_F v:[u],\delta_F v)]].$$

By (18) and monotonicity of τ, this e_2 will fulfill (19).

($\leftarrow$) Now we assume to have an e_2 which fulfills (19), we must find an e_1 which fulfills (17).

By domain property 7.3.8(j), we have an e_1 such that

(20) $e_1w\neq 0 \rightarrow \exists u /\!/ \delta_F v(e_2u\neq 0 \wedge (e:d_F(v*w),\delta_F(v*w))\geq(e:d_Fv:[u],\delta_Fv))$.

This e_1 satisfies (17), for let $e_1w \neq 0$, then by (20) we have a $u/\!/\delta_Fv$ such that

(21) $(e:d_Fv:[u],\delta_Fv) \leq (e:d_F(v*w),\delta_F(v*w))$

and $e_2u \neq 0$, whence by (19) there is an a such that

(22) $\ulcorner\forall\varepsilon\in(e:d_Fv:[u],\delta_Fv)\ B(\varepsilon,a)\urcorner$.

By monotonicity, (21) and (22) yield $\ulcorner\forall\varepsilon\in(e:d_F(v*w),\delta_F(v*w))\ B(\varepsilon,a)\urcorner$, whence by induction-hypothesis $\forall\delta\in v*w\ B^{\delta}(e|\pi_F,a)$.

case (vi)C $A\varepsilon \equiv \exists\eta\ B(\varepsilon,\eta)$.

($\rightarrow$) We assume $\forall\delta\in v\ A^{\delta}(e|\pi_F)$, i.e. we have an $e_1 \in K$ such that

(23) $\forall w[e_1w\neq 0 \rightarrow \exists f\in C\exists G\forall\delta\in v*w\ B^{\delta}(e|\pi_F,f|\pi_G)]$.

We must find an $e_2 \in K$ such that

(24) $\forall u/\!/\delta_Fv[e_2u\neq 0\rightarrow \exists g\in C\exists H\ulcorner\forall\zeta\in((e:d_Fv:[u])\wedge g,\ \delta_Fv\wedge H)\ B(j_1\zeta,j_2\zeta)\urcorner]$.

Take e_2 such that it satisfies (domain property 7.3.8(g))

(25) $\forall u/\!/\delta_Fv[e_2u\neq 0\rightarrow \exists w(e_1w\neq 0 \wedge (e:d_F(v*w),\delta_F(v*w))\leq(e:d_Fv:[u],\delta_Fv))]$.

e_2 fulfills (24). Let $u/\!/\delta_Fv$ be such that $e_2u \neq 0$. By (25) we find a w such that

(26) $(e:d_F(v*w),\delta_F(v*w))\leq(e:d_Fv:[u],\delta_Fv)$

and $e_1w \neq 0$, whence by (23) we have $f \in C$ and G such that $\forall\delta\in v*w\ B^{\delta}(e|\pi_F,f|\pi_G)$, and hence, by induction-hypothesis, extensionality and monotonicity of τ:

(27) $\ulcorner\forall\zeta\in((e:d_F(v*w))\wedge(f:d_G(v*w)),\ \delta_F(v*w)\wedge\delta_G(v*w))\ B(j_1\zeta,j_2\zeta)\urcorner$.

From (26) and $\geq$-property 7.3.7(g), we find a g and an H such that

$$(28) \qquad ((e:d_F v:[u])\wedge g,\ \delta_F v\wedge H) \geq$$
$$((e:d_F(v*w))\wedge(f:d_G(v*w)),\ \delta_F(v*w)\wedge\delta_G(v*w)).$$

$f:d_G(v*w) \in C$ (because $f \in C$, $d_G(v*w) \in C$ (domain axiom (D3d)) and C is dependency-closed), hence ($\geq$-property 7.3.7(g)) $g \in C$. By monotonicity of τ we conclude from (27) and (28) $\ulcorner\forall\zeta\in((e:d_F v:[u])\wedge g,\ \delta_F v\wedge H)\ B(j_1\zeta,j_2\zeta)\urcorner$.

($\rightarrow$) Now we assume to have an e_2 which satisfies (24). Let e_1 satisfy (domain property 7.3.8(j)):

$$(29) \qquad e_1 w\neq 0 \rightarrow \exists u/\!/\delta_F v(e_2 u\neq 0 \wedge (e:d_F(v*w),\delta_F(v*w))\geq(e:d_F v:[u],\delta_F v)).$$

Then e_1 satisfies (23). Let $e_1 w \neq 0$, then by (29) we have a $u/\!/\delta_F v$, such that

$$(30) \qquad (e:d_F(v*w),\delta_F(v*w))\geq(e:d_F v:[u],\delta_F v)$$

and $e_2 u \neq 0$, whence by (24) we have $g \in C$ and H such that

$$(31) \qquad \ulcorner\forall\zeta\in((e:d_F:v:[u])\wedge g,\ \delta_F v\wedge H)\ B(j_1\zeta,j_2\zeta)\urcorner.$$

From (30) and $\geq$-property 7.3.7(g), we find an $f' \in C$ (since $g \in C$) and a G' such that

$$(32) \qquad ((e:d_F(v*w))\wedge f',\ \delta_F(v*w)\wedge G') \geq ((e:d_F v:[u])\wedge g,\ \delta_F v\wedge H).$$

By domain property 7.3.8(k) we can find an $f \in C$ and a G such that

$$(33) \qquad ((e:d_F(v*w))\wedge(f:d_G(v*w)),\ \delta_F(v*w)\wedge\delta_G(v*w)) \geq$$
$$((e:d_F(v*w))\wedge f',\ \delta_F(v*w)\wedge G').$$

From (33), (32), transitivity of $\geq$ (7.3.7(b)), (31) and monotonicity of τ we find $\ulcorner\forall\zeta\in((e:d_F(v*w))\wedge(f:d_G(v*w)),\ \delta_F(v*w)\wedge\delta_G(v*w))\ B(j_1\zeta,j_2\zeta)\urcorner$ whence by monotonicity of τ, induction-hypothesis and extensionality $\forall\delta\in v*w\ B^\delta(e|\pi_F,f|\pi_G)$. □

Note that as a corollary to this lemma and the monotonicity of τ we have the following 'permutability property': if $A\varepsilon$ is a formula of L_ε with at most one parameter ε, then

$$(e{:}d_F v, \mathfrak{f}_F v) \approx (f{:}d_G w, \mathfrak{f}_G w) \rightarrow (\forall\delta\in v\ A^\delta(e|\pi_F) \leftrightarrow \forall\delta\in w\ A^\delta(f|\pi_G))$$

and

$$(e{:}d_F v, \mathfrak{f}_F v) \leq (f{:}d_G w, \mathfrak{f}_G w) \rightarrow (\forall\delta\in v\ A^\delta(e|\pi_F) \rightarrow \forall\delta\in w\ A^\delta(f|\pi_G)).$$

8.4.2. <u>THEOREM</u> (the elimination theorem for domains).
Let Φ *be a closed formula of* L_ε. *Assume*
(a) C *is dependency-closed, and*
(b) π, d *and* $\mathfrak{f}$ *define a domain.*
Then
(c) $\Phi^\delta \leftrightarrow \tau\Phi$.
This is provable in $\underset{\sim}{\text{LSF}}^*$ *i.e.*

$$\underset{\sim}{\text{LSF}}^* \vdash \text{dclosed}(C) \wedge \text{domain}(\pi,d,\mathfrak{f}) \rightarrow (\Phi^\delta \leftrightarrow \tau\Phi).$$

<u>PROOF</u>. The proof proceeds by induction w.r.t. the logical complexity of A. Most cases are trivial: closed prime formulae are lawlike, hence for those $\Phi^\delta \equiv \Phi \equiv \tau\Phi$; if the main logical sign in Φ is $\wedge,\vee,\rightarrow$, or a lawlike quantifier, then we can simply apply induction-hypothesis. The interesting cases are $\Phi \equiv \forall\varepsilon\ A\varepsilon$, $\Phi \equiv \exists\varepsilon\ A\varepsilon$.

(i) $\Phi \equiv \forall\varepsilon\ A\varepsilon$.

Assume Φ^δ, i.e. $\forall e\in C \forall F\ A^\delta(e|\pi_F)$. Then, by open data, there is a v such that

$$(1) \qquad \forall\delta\in v \forall e\in C \forall F\ A^\delta(e|\pi_F).$$

Let n be such that $\mathfrak{f}_n v = {}^\circ n$ (exists by (D5)) and let u satisfy $d_n v \simeq [u]$ (this u exists by (D6)). Since C is dependency-closed, $s^m \in C$, where $m = \text{lth}(u)$. Hence (1) specializes to

$$\forall\delta\in v\ A^\delta(s^m|\pi_n).$$

By lemma 8.4.1 this is equivalent to $\ulcorner\forall\varepsilon\in(s^m{:}d_n v, \delta_n v)\ A\varepsilon\urcorner$, but by choice of m and n, $s^m{:}d_n v \simeq id$, $\delta_n v = {}^\circ n$, hence $\ulcorner\forall\varepsilon\in(id,{}^\circ n)\ A\varepsilon\urcorner$, which is equivalent to $\tau\Phi$ by 8.3.10(a).

For the converse implication we assume $\tau\Phi$, i.e. $\forall e\in C\forall F\ulcorner\forall\varepsilon\in(e,F)\ A\varepsilon\urcorner$. By the preceding lemma, (D2a) and (D3a), this is equivalent to $\forall\delta\forall e\in C\forall F\ A^\delta(e|\pi_F)$, whence in particular Φ^δ.

(ii) $\Phi \equiv \exists\varepsilon\ A\varepsilon$.

For the implication from left to right we assume Φ^δ, i.e. we have an $e \in C$ and an F such that $A^\delta(e|\pi_F)$, whence by open data for some v $\forall\delta\in v\ A^\delta(e|\pi_F)$. By lemma 8.4.1 this is equivalent to $\ulcorner\forall\varepsilon\in(e{:}d_F v, \delta_F v)\ A\varepsilon\urcorner$, hence (since $e \in C$, $d_F v \in C$ (by (D3d)) and C is closed under composition) $\exists f\in C\exists G\ulcorner\forall\varepsilon\in(f,G)\ A\varepsilon\urcorner$, i.e. $\tau\Phi$.

For the converse implication, we assume to have an $f \in C$ and a G such that $\ulcorner\forall\varepsilon\in(f,G)\ A\varepsilon\urcorner$. By the preceding lemma, (D2a) and (D3a) this yields $\forall\delta\ A^\delta(f|\pi_G)$, whence in particular Φ^δ. □

CHAPTER 9

THE MAIN THEOREM AND ITS COROLLARIES

9.1. OUTLINE

In this chapter we prove the main theorem, which states that for suitable dependency-closed $C \subset K$, $\underset{\sim}{IDBF}^* \vdash \ulcorner\Psi\urcorner$ for all axioms and instances of axiom-schemata Ψ of $\underset{\sim}{CS}(C)$. Combined with the elimination theorem for domains this yields that each domain w.r.t. a suitable C is a model of $\underset{\sim}{CS}(C)$, from which we derive (by theorem 6.2) that each projection model for GC-sequences w.r.t. a suitable C is a model of $\underset{\sim}{CS}(C)$.

It is not so that each domain w.r.t. a dependency-closed C is a model for $\underset{\sim}{CS}(C)$. E.g. the set C defined by $e \in C$ iff $e = \nu_F^K\phi$ for some frame F and mapping ϕ with the property that for all n, ϕn has the form $[u]:s^m$, is dependency-closed. (To prove this use the fact that $\nu_F^K\phi \simeq \nu_{F'}^K\phi'$ for some F',ϕ' where F' has a 1-1 labelling (a corollary to 3.2.17(b)), 3.2.16(e), (f) and 3.2.20(g).) This set, which is in fact the smallest dependency-closed subset of K, does not contain (equivalents of) the pairing inverse j_1. In a domain w.r.t. this C the formula

$$\exists e \in C \exists \varepsilon(\varepsilon = j_1(e|\pi_n\delta))$$

does not hold. (ε ranges over the sequences $f|\pi_F\delta$ ($f \in C$) in the domain.) But the formula $\exists\varepsilon(\varepsilon = j_1\eta)$ does hold in the domain e.g. for $\eta = \pi_{\circ_n \wedge \circ_m}\delta$. That is to say, in this domain analytic data is not fulfilled.

The set C defined by: $e \in C$ iff either there is an $f \in K$ such that $\forall a(j_1(e|a) = f|j_1a)$ or there is an $f \in K$ such that $\forall a(j_1(e|a) = f|j_2a)$, is also dependency-closed. It is richer than the previous one since it contains j_1 and j_2. A domain w.r.t. this C does not fulfill $\underset{\sim}{CS}(C)4$: it satisfies $\forall\varepsilon\exists\eta(\eta = j(\varepsilon,\varepsilon))$, but there is no $e \in C$ such that $e|\pi_n\delta = j(\pi_n\delta,\pi_n\delta)$.

It turns out that domains w.r.t. a $C \subset K$ which is dependency-closed and contains j_1, j_2 and a neighbourhood-function for the mapping $a \mapsto j(a,a)$ are $\underset{\sim}{CS}(C)$-models. We shall call such a C 'CS-closed' (definition 9.2.3).

The first step towards the main theorem (for CS-closed C) is the introduction of subsets C[F] of C for each frame F. e is an element of C[F] iff $\forall a(e|a//F)$ and $\exists f \in C \forall a//F(e|(f|a)=a)$ (cf.9.2.5). We derive some properties of the sets C[F], which are used to prove the key lemma for the main theorem, stating that for CS-closed C

$$\forall f \in C[F](\ulcorner \forall \varepsilon \in (e,F)\ A\varepsilon \urcorner \leftrightarrow \ulcorner \forall \varepsilon\ A(e:f|\varepsilon)\urcorner).$$

The main theorem follows simply from the key lemma.

In the final section of this chapter we show that each subset of K which can be enumerated modulo $\simeq$ is contained in a CS-closed $C \subset K$ which can be enumerated modulo $\simeq$. That is to say: with each $J: \mathbb{N} \to K$ there are $C \subset K$ and a $\underset{\sim}{CS}(C)$-model $\mathcal{U}_\delta \in U_\delta(C)$ which satisfies the closure axiom $\forall \varepsilon \eta \forall e \in \text{range}(J)\ \exists \zeta(\zeta = e|(\varepsilon,\eta))$.

9.2. THE VALIDITY OF $\underset{\sim}{CS}(C)$ UNDER τ

9.2.1. <u>DEFINITION</u> (of dpl and nest_F, cf.7.2.9, 7.2.10(b)).

(a) dpl (for duplicate) is the element of K which satisfies

$$\text{dpl}(0) = 0,\ \text{dpl}(\hat{x}*u) = \text{sg}(\text{lth}(u)\dot{-}x)\cdot(1+j((u)_x,(u)_x)).$$

(b) nest_F is the element of K which satisfies

$$\text{nest}_F 0 = 0,\ \text{nest}_F(\hat{x}*u) = \text{sg}(\text{lth}(u)\dot{-}x)\cdot(1+\nu_F(\lambda n.(u)_x)).$$

9.2.2. <u>FACTS</u>.

(a) For all a and x, $\text{dpl}(\hat{x}*\bar{a}(x+1)) = j(ax,ax)+1$. Hence $\forall a(\text{dpl}|a=j(a,a))$, or equivalently $\forall a(j_1(\text{dpl}|a)=j_2(\text{dpl}|a)=a)$.

(b) For all a and x, $\text{nest}_F(\hat{x}*\bar{a}(x+1)) = 1+\nu_F(\lambda n.ax)$. Hence $\forall a(\text{nest}_F|a = \nu^1_F(\lambda^1 n.a))$, or equivalently $\forall a \forall b \in F(j_b(\text{nest}_F|a)=a)$, i.e. $\text{nest}_F|\cdot$ maps a onto an F-nest of copies of a.

(c) One easily verifies that a sequence b is parallel to $F[\lambda z.0]$ (the frame obtained from F by substituting 0 for all its labels) iff $\exists c \forall b \in F(j_b b=c)$. From (b) it follows that $\forall a(\text{nest}_F|a//F[\lambda z.0])$; since

$F[\lambda z.0] \geq F$ then also $\forall a(\text{nest}_F|a//F)$ (7.3.6(g)). In fact: if F and G have the same branches, then $\text{nest}_F|a//G$.

(d) With the help of (a) and (b) one easily verifies that $\forall n(\text{nest}_{(\circ n)} \simeq \text{id})$ and $\forall F \forall G(\text{nest}_{F\wedge G} \simeq (\text{nest}_F \wedge \text{nest}_G):\text{dpl})$.

9.2.3. DEFINITION (of CS-closed).
We call a subset C of K CS-*closed* iff
(a) C is dependency-closed,
(b) $\text{dpl} \in C$, and
(c) $j_1 \in C$ and $j_2 \in C$.

9.2.4. FACTS. (a) By 9.2.2(d) a CS-closed $C \subset K$ contains nest_F for all F (proof by induction over frames).
(b) By induction w.r.t. lth(v) one proves that a CS-closed $C \subset K$ contains all functions j_v.

9.2.5. DEFINITION (of C[F]). Let C be a subset of K, let F be a frame. C[F] is the subset of K defined by

$$e \in C[F] \quad \text{iff} \quad e \in C,\ \forall a(e|a//F) \quad \text{and} \quad \exists f \in C \forall a//F(e:f|a=a),$$

i.e. an $e \in C$ belongs to C[F] iff the functional $\lambda\phi.e|\phi$
(a) maps N onto the set of sequences parallel to F, and
(b) has a continuous right-inverse on this set, with a neighbourhood-function $f \in C$.

9.2.6. LEMMA (properties of C[F]).

(a) $F \approx G \rightarrow C[F] = C[G]$.

(b) *Let* F *be a frame with a* 1-1 *labelling, i.e.* $b \neq b'$ *implies* $\ell_b F \neq \ell_{b'} F$ *for all* $b,b' \in F$. *In that case,* $\text{id} \in C$ *implies* $\text{id} \in C[F]$. *In particular, if* $\text{id} \in C$ *then* $\text{id} \in C[^\circ 0]$ *and* $\text{id} \in C[^\circ 0 \wedge {}^\circ 1]$.

(c) *If* C *is* CS-*closed and* F *is a frame in which all branches have the same label, then* $\text{nest}_F \in C[F]$.

(d) *If* C *is* CS-*closed and* $\ell F \subset \{0,1\}$ *then* $\nu^K_F(\lambda^K n.j_{\langle n\rangle}):\text{nest}_F \in C[F]$.

(e) *Let* C *be* CS-*closed, let* F *and* G *be frames and assume that* $e \in C[F]$. *Then there are* H, f *and* g *such that*
(i) $(e \wedge g):f \in C[F \wedge G]$,
(ii) $f \in C[^\circ 0 \wedge H]$ *and*
(iii) $g \in C$.

(f) $f \in C[F] \to \forall v(f{\restriction}v/\!/F)$.

PROOF.

(a) follows immediately from 7.3.6(h): $F \approx G \to \forall a(a/\!/F \leftrightarrow a/\!/G)$.

(b) follows immediately from 7.3.6(a): if F has a 1-1 labelling then $\forall a(a/\!/F)$, and the fact that id is its own inverse.

(c) if C is CS-closed then $nest_F \in C$ by 9.2.4(a); $\forall a(nest_F|a/\!/F)$ by 9.2.2(c); if $\ell F = \{m\}$ then $\forall a/\!/F(nest_F : j_b|a=a)$ for any branch b of F (as is easily verified) and if C is CS-closed then $j_b \in C$ by 9.2.4(b).

(d) Put $e \equiv \nu_F^K(\lambda^K n.j_{<n>}):nest_F$. If C is CS-closed then $nest_F \in C$, $\forall n(j_{<n>} \in C)$ and C is closed under pairing and composition, hence $e \in C$. $\forall a(nest_F|a/\!/F)$ by 9.2.2(c), $\nu_F^K(\lambda^K n.j_{<n>})/\!/_C F$ by definition, hence $\forall a(e|a = \nu_F^K(\lambda^K n.j_{<n>})|(nest_F|a)/\!/F)$ by 7.3.6(j).
To construct the right inverse to e, let $b: \mathbb{N} \to F$ be a labelling inverse, i.e. $\forall n \in \ell F(\ell_{bn}F=n)$. Put $f \equiv (j_{b0} \wedge j_{b1}):dpl$. Then $f \in C$ since j_{b0}, j_{b1} and $dpl \in C$, and C is closed under composition and pairing. Moreover, if $a/\!/F$ then $e:f|a = a$, because $j_b a = j_b(e:f|a)$ for arbitrary $b \in F$:
Let $m \in \{0,1\}$ be the label of b, then
$j_b(e:f|a) = j_b(e|(f|a)) = j_{<m>}(j_b(nest_F|(f|a)))$ by 7.3.5;
$j_{<m>}(j_b(nest_F|(f|a))) = j_{<m>}(f|a)$ by 9.2.2(b);
$j_{<m>}(f|a) = j_{<m>}((j_{b0} \wedge j_{b1})|(dpl|a)) = j_{bm}(j_{<m>}(dpl|a))$, by definition of $\wedge$ (recall that $m \in \{0,1\}$ i.e. $j_{<m>} = j_1$ or $j_{<m>} = j_2$);
$j_{bm}(j_{<m>}(dpl|a)) = j_{bm}a$ by 9.2.2(a); and finally
$j_{bm}a = j_b a$ since $a/\!/F$ and $m = \ell_b F = \ell_{bm}F$.

(e) Define a by

$$an = \begin{cases} 0 \text{ if } n \in \ell F, \\ 1 \text{ otherwise.} \end{cases}$$

Put $H \equiv G[\lambda n.{}^\circ an]$, $H' \equiv {}^\circ 0 \wedge H$, $f \equiv \nu_{H'}^K(\lambda^K n.j_{<n>}):nest_{H'}$.
Let b_1, b_2 be labelling inverses for F and G respectively, i.e. $\forall n \in \ell F(\ell_{b_1 n}F=n)$ and $\forall m \in \ell G(\ell_{b_2 m}G=m)$.
Define $\phi: \mathbb{N} \to C$ by

$$\phi n = \begin{cases} j_{b_1 n}:e \quad \text{if } n \in \ell F \text{ (i.e. } an = 0), \\ j_{b_2 n} \quad \text{otherwise.} \end{cases}$$

Put $g \equiv \nu^K_G\phi$.

(iii) $g \in C$ since C is closed under pairing.

(ii) $f \in C[°0\wedge H]$ by (d) above (obviously $\ell(°0\wedge H) \subset \{0,1\}$).

(i) $(e\wedge g):f \in C[F\wedge G]$ is shown as follows.

Firstly $(e\wedge g):f \in C$, since e,f and g belong to C and C is closed under pairing and composition.

Secondly $\forall c((e\wedge g):f|c /\!/ F\wedge G)$. To prove this let b,b' be branches of $F\wedge G$ with the same label, m say.

Case 1. $b = \langle 0\rangle * b_1$, $b' = \langle 0\rangle * b_2$, $b_1,b_2 \in F$. Then
$j_b((e\wedge g):f|c) = j_{b_1}(e|j_1(f|c))$ by definition of j_b and 7.3.2(a);
$j_{b'}((e\wedge g):f|c) = j_{b_2}(e|j_1(f|c))$ analogously.
$e|j_1(f|c)/\!/F$ since $e \in C[F]$, and hence $j_{b_1}(e|j_1(f|c)) = j_{b_2}(e|j_1(f|c))$.

Case 2. $b = \langle 0\rangle * b_1$, $b' = \langle 1\rangle * b_2$, $b_1 \in F$, $b_2 \in G$. Then $m \in \ell F$, hence $am = 0$.
$j_b((e\wedge g):f|c) = j_{b_1}(e|j_1(f|c))$ as in case 1, but now
$j_{b'}((e\wedge g):f|c) = j_{b_2}(g|j_2(f|c))$.
$j_{b_2}(g|j_2(f|c)) = (j_{b_1m}:e)|j_{b_2}j_2(f|c)$ by 7.3.5, the definition of g and the definition of ϕ.
$\langle 0\rangle$ and $b' = \langle 1\rangle * b_2$ are both branches of $°0\wedge H$. Obviously $\ell_{\langle 0\rangle}(°0\wedge H) = 0$, but also $\ell_{b'}(°0\wedge H) = 0$ since $\ell_{b'}(°0\wedge H) = \ell_{b_2}H = \ell_{b_2}(G[\lambda n.°an]) = a(\ell_{b_2}G) = am = 0$.
Since $f \in C[°0\wedge H]$ (by (ii)), $j_1(f|c) = j_{b'}(f|c) = j_{b_2}j_2(f|c)$. I.e. we find that
$j_b((e\wedge g):f|c) = j_{b_1}(e|c')$ and $j_{b'}((e\wedge g):f|c) = j_{bm}(e|c')$ for $c' = j_1(f|c)$.
By the same argument as in the last step of case 1 we have
$j_{b_1}(e|c') = j_{bm}(e|c')$.

Case 3. $b = \langle 1\rangle * b_1$, $b' = \langle 1\rangle * b_2$, $b_1,b_2 \in G$.
If $m \in \ell F$ i.e. if there is a $b_3 \in F$ such that $\ell_{b_3}F = m$, then we can apply the argument of case 2 twice: to the pairs b, $\langle 0\rangle * b_3$ and b', $\langle 0\rangle * b_3$.
Assume $m \notin \ell F$, $am = 1$.
$j_b((e\wedge g):f|c) = j_{b_1}(g|j_2(f|c))$, $j_{b'}((e\wedge g):f|c) = j_{b_2}(g|j_2(f|c))$. By 7.3.5, the definition of g and the definition of ϕ
$j_{b_1}(g|j_2(f|c)) = j_{b_2m}(j_{b_1}j_2(f|c))$, $j_{b_2}(g|j_2(f|c)) = j_{b_2m}(j_{b_2}j_2(f|c))$.
b and b' are branches of $°0\wedge H$ with the same label 1, $f|c/\!/°0\wedge H$ by (ii), hence
$j_{b_1}j_2(f|c) = j_b(f|c) = j_{b'}(f|c) = j_{b_2}j_2(f|c)$.

Finally we must show that $(e\wedge g):f$ has a right-inverse in C. One may verify the following claims:

if $b \in F \wedge G$, $b = \langle 0\rangle * b_1$, $b_1 \in F$ then $j_b((e\wedge g):f|c) = j_{b_1}(e|j_1c)$,
if $b \in F\wedge G$, $b = \langle 1\rangle * b_2$, $b_2 \in G$ then

$$j_b((e\wedge g):f|c) = \begin{cases} j_{b_3}(e|j_1c) & \text{if } \ell_{b_2}G \in \ell F\text{, where } b_3 \in F \text{ has label } \ell_{b_2}G, \\ j_{b_2}(j_2c) & \text{otherwise.} \end{cases}$$

With these observations one easily proves that the desired right-inverse is $e^{-1}\wedge id$, i.e. $\forall c /\!/ F\wedge G((e\wedge g):f:(e^{-1}\wedge id)|c=c)$, where e^{-1} is such that $\forall c/\!/F(e:e^{-1}|c=c)$.

(f) follows immediately from the fact that for $f \in C[F]$ we have $f|(v*\lambda z.0)/\!/F$, while by 7.3.1(b), $f\restriction v = \overline{f\restriction(v*\lambda z.0)}(x)$ for some x, whence $f\restriction v/\!/F$ by 7.3.6(f). □

9.2.7. <u>COROLLARY</u>. *If* C *is* CS-*closed then* $\forall F \exists e \in C(e \in C[F])$.
[By induction over frames from 9.2.6(b) and (e).]

To prove the key lemma 9.2.9 we need one more fact, namely

9.2.8. <u>PROPOSITION</u> (extensionality of τ). *Let* $A(\varepsilon_1,\ldots,\varepsilon_p)$ *be a formula of* L_ε, *with no other choice parameters than* $\varepsilon_1,\ldots,\varepsilon_p$. *Then*

$$IDBF^* \vdash \bigwedge_{i=1}^{p} (f_i \simeq g_i) \to (\ulcorner \forall \zeta \in (e,F)\ A(f_1|\zeta,\ldots,f_p|\zeta)\urcorner \leftrightarrow \ulcorner \forall \zeta \in (e,F)\ A(g_1|\zeta,\ldots,g_p|\zeta)\urcorner),$$

where $f_i|\zeta$, $g_i|\zeta$ *are substituted for* ε_i, $i = 1,\ldots,p$ *according to the conventions of* 7.2.15(c).

<u>PROOF</u>. Is left to the reader. □

9.2.9. <u>LEMMA</u>. *Let* C *be a* CS-*closed subset of* K, *and let* $A\varepsilon$ *be a formula of* L_ε *with at most one choice parameter:* ε. *If* F *is a frame and* f *is an element of* C[F] *then*

$$\ulcorner \forall \varepsilon \in (e,F)\ A\varepsilon \urcorner \leftrightarrow \ulcorner \forall \varepsilon\ A(e:f|\varepsilon)\urcorner.$$

This is provable in $\underset{\sim}{IDBF}^*$ *i.e.*

$$\underline{\text{IDBF}}^* \vdash \text{CSclosed}(C) \rightarrow$$
$$\forall F \forall f \in C[F](\ulcorner \forall \varepsilon \in (e,F)\ A\varepsilon \urcorner \leftrightarrow \ulcorner \forall \varepsilon\ A(e{:}f|\varepsilon)\urcorner).$$

PROOF. By induction w.r.t. the logical complexity of $A\varepsilon$. The proof is subdivided into cases, most of the non-trivial cases consist of a part ($\rightarrow$) for the implication from left to right and a part ($\leftarrow$) for the converse implication. The numbering of the cases corresponds to the numbering of definition 8.3.3. In each case we assume $f \in C[F]$. Since CS-closed implies dependency-closed, we can use all $\geq$-and $/\!/$- properties, as well as monotonicity and the bar-property of τ. Throughout the proof, 'extensionality' refers to proposition 9.2.8.

case (i) $A\varepsilon \equiv t[\varepsilon] = s[\varepsilon]$.
Then

$$\ulcorner \forall \varepsilon \in (e,F)\ A\varepsilon \urcorner \leftrightarrow \forall b /\!/ F(t[e|b] = s[e|b]) \leftrightarrow \forall a(t[e{:}f|a]=s[e{:}f|a]) \leftrightarrow \ulcorner \forall \varepsilon\ A(e{:}f|\varepsilon)\urcorner,$$

the first equivalence holds by definition of τ (8.3.3-7), the second one by definition of $C[F]$, the last one follows from the observations that $\ulcorner \forall \varepsilon\ B\varepsilon \urcorner \leftrightarrow \ulcorner \forall \varepsilon \in (id,{}^\circ n)\ B\varepsilon \urcorner$ (8.3.10(a)) and that $\forall a(id|a=a/\!/({}^\circ n))$ (7.3.6(b)).

case (ii) $A\varepsilon \equiv B\varepsilon \wedge C\varepsilon$,
trivial by induction-hypothesis.

case (iii) $A\varepsilon \equiv B\varepsilon \vee C\varepsilon$,
can be treated as $A\varepsilon \equiv \exists x\ D(\varepsilon,x)$.

case (iv) $A\varepsilon \equiv B\varepsilon \rightarrow C\varepsilon$.
($\rightarrow$) First we assume $\ulcorner \forall \varepsilon \in (e,F)\ A\varepsilon \urcorner$, i.e.

(1) $\forall (e',F') \geq (e,F)(\ulcorner \forall \varepsilon \in (e',F')\ B\varepsilon \urcorner \rightarrow \ulcorner \forall \varepsilon \in (e',F')\ C\varepsilon \urcorner).$

We must show that (cf.8.3.10(d))

(2) $\forall g \in C \forall H(\ulcorner \forall \varepsilon \in (g,H)\ B(e{:}f|\varepsilon)\urcorner \rightarrow \ulcorner \forall \varepsilon \in (g,H)\ C(e{:}f|\varepsilon)\urcorner).$

Let $g \in C$ and H be arbitrary and assume

(3) $\ulcorner \forall \varepsilon \in (g,H)\ B(e{:}f|\varepsilon)\urcorner.$

Let f' be an element of C H , then by induction-hypothesis, (3) is equivalent to

(4) $\ulcorner\forall\varepsilon\ B(e{:}f{:}g{:}f'|\varepsilon)\urcorner$.

Let a be a labelling-inverse for F, i.e. $\forall n\in\ell F(\ell_{an}F=n)$. (a assigns to each label of F a branch of F which has this label.) Put $f'' \equiv \nu^K_F(\lambda^K n.j_{an}{:}f{:}g{:}f')$. Then $f''{:}nest_F \simeq f{:}g{:}f'$, which is seen as follows: let b be an arbitrary branch of F, let n be $\ell_b F$, then
$j_b(f''{:}nest_F|b) = j_{an}{:}f{:}g{:}f'|j_b(nest_F|b)$ by 7.3.5;
$j_{an}{:}f{:}g{:}f'|j_b(nest_F|b) = j_{an}|(f|(g{:}f'|b))$ by 9.2.2(b); and
$j_{an}|(f|(g{:}f'|b)) = j_b(f|(g{:}f'|b))$ since $\forall c(f|c/\!/F)$ and $\ell_{an}F = \ell_b F = n$.
Hence (4) is, by extensionality, equivalent to

(5) $\ulcorner\forall\varepsilon\ B(e{:}f''{:}nest_F|\varepsilon)\urcorner$.

Put $F[0] \equiv F[\lambda z.({}^\circ 0)]$, then $nest_F = nest_{F[0]} \in C[F[0]]$ (9.2.2(c)) so (5) is, by induction-hypothesis, equivalent to

(6) $\ulcorner\forall\varepsilon\in(e{:}f'',F[0])\ B\varepsilon\urcorner$.

Obviously $F[0] \geq F$, moreover $f'' = \nu^K_F(\lambda^K n.j_{an}{:}f{:}g{:}f')/\!/_C F$ (since j_{an},f,g and f' are elements of C and C is closed under composition), hence $(e{:}f'',F[0]) \geq (e,F)$ and we can apply (1) to (6) yielding

(7) $\ulcorner\forall\varepsilon\in(e{:}f'',F[0])\ C\varepsilon\urcorner$.

But by the same argument which showed the equivalence between (3) and (6) above, (7) is equivalent to

(8) $\ulcorner\forall\varepsilon\in(g,H)\ C(e{:}f|\varepsilon)\urcorner$.

(←) To prove the converse implication, assume (2), let $(e',F') \geq (e,F)$ be arbitrary and suppose that

(9) $\ulcorner\forall\varepsilon\in(e',F')\ B\varepsilon\urcorner$.

Let f' be an element of C[F'], then (9) is equivalent to

(10) $\ulcorner \forall \varepsilon\ B(e':f'|\varepsilon) \urcorner$

by induction-hypothesis. Since $(e',F') \geq (e,F)$ we have that (i) $F' \geq F$ and for some g (ii) $e' \simeq e:g$, where (iii) $g/\!/_C F$. Moreover, the f of (2) is an element of C[F], whence for some $f^{-1} \in C$ (iv) $\forall a/\!/F(f:f^{-1}|a=a)$ (cf. definition of C[F], 9.2.5).
It follows that $e:f:f^{-1}:g:f' \simeq e:g:f' \simeq e':f'$, in fact we even have $f:f^{-1}:g:f' \simeq g:f'$. (This is seen as follows: let a be arbitrary, then $f'|a/\!/F'$ (since $f' \in C[F']$), hence $f'|a/\!/F$ (by (i) and 7.3.6(g)), hence $g:f'|a = g|(f'|a)/\!/F$ (by (iii) and 7.3.6(j)) whence $f:f^{-1}:g:f'|a = g:f'|a$ (by (iv)).) So by extensionality, (10) is equivalent to

$$\ulcorner \forall \varepsilon\ B(e:f:f^{-1}:g:f'|\varepsilon) \urcorner,$$

which (by induction-hypothesis) is equivalent to

(11) $\ulcorner \forall \varepsilon \in (f^{-1}:g,F')\ B(e:f|\varepsilon) \urcorner$.

$g/\!/_C F$ by (iii), C is closed under pairing, hence $g \in C$. $f^{-1} \in C$ by definition of C[F], C is closed under composition, hence $f^{-1}:g \in C$. So we can apply (2) to (11) yielding

$$\ulcorner \forall \varepsilon \in (f^{-1}:g,F')\ C(e:f|\varepsilon) \urcorner.$$

But this is equivalent to $\ulcorner \forall \varepsilon \in (e',F')\ C\varepsilon \urcorner$: simply replace B by C in the equivalence (9) $\leftrightarrow$ (11).

case (v) $A\varepsilon \equiv \forall a\ B(\varepsilon,a)$,
trivial by induction-hypothesis.

case (v)C $A\varepsilon \equiv \forall \eta\ B(\varepsilon,\eta)$.
Let m be a natural number, $m \notin \ell F$, then $\ulcorner \forall \varepsilon \in (e,F) \forall \eta\ B(\varepsilon,\eta) \urcorner$ is equivalent to

(12) $\ulcorner \forall \zeta \in (e \wedge id, F \wedge {}^\circ m)\ B(j_1\zeta, j_2\zeta) \urcorner$

by 8.3.10(b). If $f \in C[F]$ then $f \wedge id \in C[F \wedge {}^\circ m]$, for
(i) $f \in C$, $id \in C$, C is closed under pairing, hence $f \wedge id \in C$;
(ii) $\forall a(f \wedge id|a = j(f|j_1 a, j_2 a)/\!/F \wedge {}^\circ m)$ since $f|j_1 a/\!/F$ (cf.7.3.6(d));

(iii) let $f^{-1} \in C$ be such that $\forall a//F(f:f^{-1}|a=a)$, then $f^{-1}\wedge id \in C$ (cf.(i)) and $(f\wedge id):(f^{-1}\wedge id)|a = (f:f^{-1})\wedge id|a = j(f:f^{-1}|j_1a, j_2a)$; if $a//F\wedge{}^\circ m$ then $j_1a//F$ (7.3.6(c)). So $f:f^{-1}|j_1a = j_1a$, whence $\forall a//F\wedge{}^\circ m((f\wedge id):(f^{-1}\wedge id)|a=a)$, i.e. $f^{-1}\wedge id$ is a right-inverse to $f\wedge id$.

So (12) is (by induction-hypothesis) equivalent to

$$\ulcorner\forall\zeta\ B(j_1((e\wedge id):(f\wedge id)|\zeta), j_2((e\wedge id):(f\wedge id)|\zeta))\urcorner,$$

which (by extensionality) is equivalent to

$$\ulcorner\forall\zeta\ B(e:f|j_1\zeta, j_2\zeta)\urcorner \tag{13}$$

$id \in C[{}^\circ 0\wedge{}^\circ 1]$ by 9.2.6(b), so (13) is equivalent to

$$\ulcorner\forall\zeta\in(id, {}^\circ 0\wedge{}^\circ 1)\ B(e:f|j_1\zeta, j_2\zeta)\urcorner$$

by extensionality and induction-hypothesis. The desired $\ulcorner\forall\varepsilon\forall\eta\ B(e:f|\varepsilon,\eta)\urcorner$ follows by 8.3.10(a) and (b).

case (vi) $A\varepsilon \equiv \exists a\ B(\varepsilon, a)$.

(→) First we assume $\ulcorner\forall\varepsilon\in(e,F)\ A\varepsilon\urcorner$, i.e. we have an e_1 such that

$$\forall u//F[e_1u\neq 0 \rightarrow \exists a\ulcorner\forall\varepsilon\in(e:[u],F)\ B(\varepsilon,a)\urcorner]. \tag{14}$$

Since for all n and e $(e:s^n,F) \geq (e,F)$, (7.3.7(e)), (14) yields (by monotonicity)

$$\forall u//F[e_1u\neq 0 \rightarrow \exists a\forall n\ulcorner\forall\varepsilon\in(e:[u]:s^n,F)\ B(\varepsilon,a)\urcorner],$$

whence by induction-hypothesis and 8.3.10(a)

$$\forall u//F[e_1u\neq 0 \rightarrow \exists a\forall n\ulcorner\forall\varepsilon\in(id,{}^\circ 0)\ B(e:[u]:s^n:f|\varepsilon,a)\urcorner]. \tag{15}$$

Since $([v],{}^\circ 0) \geq (id,{}^\circ 0)$ for all v (by 7.3.7(c) and 7.3.6(b)), (15) yields (by monotonicity)

$$\forall u//F[e_1u\neq 0 \rightarrow \exists a\forall n\forall w\ulcorner\forall\varepsilon\in([w],{}^\circ 0)\ B(e:[u]:s^n:f|\varepsilon,a)\urcorner].$$

$id \in C[0]$, (9.2.6(b)), hence, by induction-hypothesis and extensionality

(16) $\forall u /\!/ F[e_1 u \neq 0 \to \exists a \forall n \forall w \ulcorner \forall \varepsilon\ B(e:[u]:s^n:f:[w] | \varepsilon, a) \urcorner]$.

Now put $e_2 \equiv e_1;f$, let v satisfy $e_2 v \neq 0$, i.e. $e_1(f \upharpoonright v) \neq 0$. $f \upharpoonright v /\!/ F$ by 9.2.6(f), so

$$\exists a \forall n \forall w \ulcorner \forall \varepsilon\ B(e:[f \upharpoonright v]:s^n:f:[w] | \varepsilon, a) \urcorner$$

follows from (16), whence in particular

(17) $\exists a \ulcorner \forall \varepsilon\ B(e:[f \upharpoonright v]:s^m:f:[v] | \varepsilon, a) \urcorner$,

where $m = \text{lth}(f \upharpoonright v)$. $[f \upharpoonright v]:s^m:f:[v] \simeq f:[v]$ by 7.3.3(c), hence (17) is equivalent to

$$\exists a \ulcorner \forall \varepsilon\ B(e:f:[v] | \varepsilon, a) \urcorner$$

by extensionality, which in turn is equivalent to

$$\exists a \ulcorner \forall \varepsilon \in ([v], {}^\circ 0)\ B(e:f | \varepsilon, a) \urcorner$$

by induction-hypothesis and 9.2.6(b): $id \in C[0]$. Thus we have shown that

(18) $\forall v[e_2 v \neq 0 \to \exists a \ulcorner \forall \varepsilon \in ([v], {}^\circ 0)\ B(e:f | \varepsilon, a) \urcorner]$,

i.e. we have $\ulcorner \forall \varepsilon \in (id, {}^\circ 0) \exists a\ B(e:f | \varepsilon, a) \urcorner$ or equivalently, by 8.3.10(a) $\ulcorner \forall \varepsilon \exists a\ B(e:f | \varepsilon, a) \urcorner$.

(←) For the converse implication assume e_2 to satisfy (18). Let $f^{-1} \in C$ be such that $\forall a /\!/ F(f:f^{-1} | a = a)$. $f \in C$, $s^n \in C$, $[w] \in C$ and C is closed under composition, hence $\forall n \forall w(s^n:f^{-1}:[w]:f \in C)$, so $([v]:s^n:f^{-1}:[w]:f, {}^\circ 0) \geq ([v], {}^\circ 0)$ for all n and w, by 7.3.7(d). By monotonicity, (18) yields

$$\forall v[e_2 v \neq 0 \to \exists a \forall n \forall w \ulcorner \forall \varepsilon \in ([v]:s^n:f^{-1}:[w]:f, {}^\circ 0)\ B(e:f | \varepsilon, a) \urcorner],$$

which (by induction-hypothesis and 9.2.6(b) ($id \in C[0]$)) is equivalent to

(19) $\forall v[e_2 v \neq 0 \to \exists a \forall n \forall w \ulcorner \forall \varepsilon\ B(e:f:[v]:s^n:f^{-1}:[w]:f | \varepsilon, a) \urcorner]$.

Now put $e_1 \equiv e_2:f^{-1}$, let $u//F$ be arbitrary and assume that $e_1u \equiv e_2(f^{-1}\restriction u) \neq 0$. By (19) we find an a such that

$$\forall n\forall w \ulcorner\forall\varepsilon\ B(e:f:[f^{-1}\restriction u]:s^n:f^{-1}:[w]:f|\varepsilon,a)\urcorner$$

whence in particular

$$(20) \qquad \ulcorner\forall\varepsilon\ B(e:f:[f^{-1}\restriction u]:s^m:f^{-1}:[u]:f|\varepsilon,a)\urcorner,$$

where $m = lth(f^{-1}\restriction u)$. But then $[f^{-1}\restriction u]:s^m:f^{-1}:[u] \simeq f^{-1}:[u]$ (7.3.3(c)), so (20) is equivalent to

$$(21) \qquad \ulcorner\forall\varepsilon\ B(e:f:f^{-1}:[u]:f|\varepsilon,a)\urcorner$$

by extensionality. Since $f \in C[F]$ whence $\forall a(f|a//F)$, $u//F$ whence $\forall b//F([u]|b = u*b//F)$ (by 7.3.6(f)), and $\forall c//F(f:f^{-1}|c=c)$, we have $f:f^{-1}:[u]:f \simeq [u]:f$. Hence (by extensionality) (21) is equivalent to $\ulcorner\forall\varepsilon\ B(e:[u]:f|\varepsilon,a)\urcorner$, which is equivalent to $\ulcorner\forall\varepsilon\epsilon(e:[u],F)\ B(\varepsilon,a)\urcorner$ by induction-hypothesis. Thus we have shown that

$$\forall u//F[e_1u\neq 0 \rightarrow \exists a\ulcorner\forall\varepsilon\epsilon(e:[u],F)\ B(\varepsilon,a)\urcorner],$$

i.e. we have $\ulcorner\forall\varepsilon\epsilon(e,F)\ A\varepsilon\urcorner$.

case (vi)C $A\varepsilon \equiv \exists\eta\ B(\varepsilon,\eta)$.

($\rightarrow$) We assume $\ulcorner\forall\varepsilon\epsilon(e,F)\ A\varepsilon\urcorner$, i.e. we have an $e_1 \in K$ such that

$$\forall u//F[e_1u\neq 0 \rightarrow \exists g\epsilon C\exists G\ulcorner\forall\zeta\epsilon((e:[u])\wedge g,\ F\wedge G)\ B(j_1\zeta,j_2\zeta)\urcorner].$$

As in case (vi)($\rightarrow$) above we find (by monotonicity)

$$(22) \qquad \forall u//F[e_1u\neq 0 \rightarrow \exists g\epsilon C\exists G\forall n\ulcorner\forall\zeta\epsilon((e:[u]:s^n)\wedge(g:s^n),\ F\wedge G)\ B(j_1\zeta,j_2\zeta)\urcorner].$$

Now put $e_2 \equiv e_1;f$ ($f \in C[F]$), let v be such that $e_2v \equiv e_1(f\restriction v) \neq 0$. Since $f\restriction v//F$, (22) yields us a $g \in C$ and a G such that

$$(23) \qquad \forall n\ulcorner\forall\zeta\epsilon((e:[f\restriction v]:s^n)\wedge(g:s^n),\ F\wedge G)\ B(j_1\zeta,j_2\zeta)\urcorner.$$

Let f', f", g' and H satisfy (i) $f'\simeq(f\wedge g'):f"$, (ii) $f'\in C[F\wedge G]$, (iii) $f"\in C[°0\wedge H]$ and (iv) $g'\in C$; such f', f", g' and H exist by 9.2.6(e). Then (23) is equivalent to

(24) $$\forall n\ulcorner\forall\zeta\ B(e:[f\restriction v]:s^n:f|j_1(f"|\zeta),\ g:s^n:g'|j_2(f"|\zeta))\urcorner$$

by induction-hypothesis, (ii), (i) and extensionality; (24) in turn is (by (iii) and induction-hypothesis) equivalent to

(25) $$\forall n\ulcorner\forall\zeta\in(id,°0\wedge H)\ B(e:[f\restriction v]:s^n:f|j_1\zeta,\ g:s^n:g'|j_2\zeta)\urcorner.$$

Let v', v" be such that $v'/\!/ °0\wedge H$, $k_1v'=v$, $k_2v'=v"$ (7.3.6(e)), then (25) yields (by monotonicity):

$$\forall n\ulcorner\forall\zeta\in([v'],°0\wedge H)\ B(e:[f\restriction v]:s^n:f|j_1\zeta,\ g:s^n:g'|j_2\zeta)\urcorner.$$

$[v']\simeq[k_1v']\wedge[k_2v']$ (7.3.3(a)), so it follows by induction-hypothesis and extensionality that

(26) $$\forall n\ulcorner\forall\zeta\ B(e:[f\restriction v]:s^n:f:[v]|j_1(f"|\zeta),\ g:s^n:g':[v"]|j_2(f"|\zeta))\urcorner.$$

If $n=\text{lth}(f\restriction v)$ then $[f\restriction v]:s^n:f:[v]\simeq f:[v]$ (7.3.3(c)), hence we have, as a special case of (26) (by extensionality):

$$\ulcorner\forall\zeta\ B(e:f:[v]|j_1(f"|\zeta),\ g:s^n:g':[v"]|j_2(f"|\zeta))\urcorner$$

where $n=\text{lth}(f\restriction v)$. By induction-hypothesis, this is equivalent to

$$\ulcorner\forall\zeta\in([v]\wedge(g:s^n:g':[v"]),\ °0\wedge H)\ B(e:f|j_1\zeta,j_2\zeta)\urcorner.$$

Thus we have shown that

$$\forall v[e_1v\neq 0\rightarrow\exists g"\in C\exists H\ulcorner\forall\zeta\in([v]\wedge g",°0\wedge H)\ B(e:f|j_1\zeta,j_2\zeta)\urcorner]$$

(note that $g:s^n:g':[v"]\in C$ since g,s^n,g' and $[v"]$ are elements of C and C is closed under composition), i.e. we have $\ulcorner\forall\varepsilon\in(id,°0)\exists\eta\ B(e:f|\varepsilon,\eta)\urcorner$.

($\leftarrow$) Conversely, assume $\ulcorner\forall\varepsilon\ A(e:f|\varepsilon)\urcorner$, i.e.

$$\forall g\in C\forall H\exists e_2\forall u/\!/H[e_2u\neq 0 \to \exists g'\in C\exists G\ulcorner\forall\zeta\in((g:[u]\wedge g'),\ H\wedge G)\ B(e:f|j_1\zeta,j_2\zeta)\urcorner].$$

Take for $g \in C$ the mapping f^{-1} such that $\forall a/\!/F(f:f^{-1}|a=a)$, take F for H, then we find an e_2 such that

$$\forall u/\!/F[e_2u\neq 0 \to \exists g\in C\exists G\ulcorner\forall\zeta\in((f^{-1}:[u])\wedge g,\ F\wedge G)\ B(e:f|j_1\zeta,j_2\zeta)\urcorner].$$

Let $u/\!/F$ be such that $e_2u \neq 0$, then we have a $g \in C$ and a G such that

$$\ulcorner\forall\zeta\in((f^{-1}:[u])\wedge g,\ F\wedge G)\ B(e:f|j_1\zeta,j_2\zeta)\urcorner.$$

Let $f' \in C[F\wedge G]$ (f' exists by 9.2.7); apply induction-hypothesis and extensionality, this yields

$$(27)\qquad \ulcorner\forall\zeta\ B(e:f:f^{-1}:[u]|j_1(f'|\zeta),\ g|j_2(f'|\zeta))\urcorner.$$

$f' \in C[F\wedge G]$, hence $f'|a/\!/F\wedge G$ for all a, i.e. $\forall a(j_1(f'|a)/\!/F)$, (7.3.6(c)), since $[u]/\!/F$ then also $[u]|j_1(f'|a)/\!/F$ for all a. Hence $\forall a(f:f^{-1}:[u]|j_1(f'|a) = [u]|j_1(f'|a))$, so (27) is equivalent to

$$(28)\qquad \ulcorner\forall\zeta\ B(e:[u]|j_1(f'|\zeta),\ g|j_2(f'|\zeta))\urcorner$$

by extensionality. But (28) yields $\ulcorner\forall\zeta\in((e:[u])\wedge g,\ F\wedge G)\ B(j_1\zeta,j_2\zeta)\urcorner$ by induction-hypothesis. I.e. we have shown that

$$\forall u/\!/F[e_2u\neq 0 \to \exists g\in C\exists G\ulcorner\forall\zeta\in((e:[u])\wedge g,\ F\wedge G)\ B(j_1\zeta,j_2\zeta)\urcorner]$$

so we have $\ulcorner\forall\varepsilon\in(e,F)\exists\eta\ B(\varepsilon,\eta)\urcorner$. □

9.2.9 is the key-lemma for the derivation of the main theorem:

9.2.10. <u>THEOREM</u>. *If* C *is* CS-*closed, then* $\underset{\sim}{\mathrm{CS}}$(C) *is valid under* τ, *i.e. from the assumption* CSclosed(C) *we can prove in* $\underset{\sim}{\mathrm{IDBF}}^*$

(a) $\ulcorner\underset{\sim}{\mathrm{CS}}(C)1\urcorner$, i.e. $\forall e\in C\ulcorner\forall\varepsilon\eta\exists\zeta(\zeta=e|(\varepsilon,\eta))\urcorner$,

(b) $\ulcorner\underset{\sim}{\mathrm{CS}}(C)2\urcorner$, i.e. $\ulcorner\forall\varepsilon(A\varepsilon \to \exists e\in C(\exists\eta(\varepsilon=e|\eta)\ \wedge\ \forall\zeta\ A(e|\zeta)))\urcorner$,

(c) $\ulcorner\underset{\sim}{\mathrm{CS}}(C)3\urcorner$, i.e. $\ulcorner\forall\varepsilon\exists a A(\varepsilon,a)\urcorner \to \exists e\forall u[eu\neq 0 \to \exists a\ulcorner\forall\varepsilon\ A([u]|\varepsilon,a)\urcorner]$,

(d) $\ulcorner\underset{\sim}{\mathrm{CS}}(C)4\urcorner$, i.e. $\ulcorner\forall\varepsilon\exists\eta\ B(\varepsilon,\eta)\urcorner \to \ulcorner\forall\varepsilon\exists e\in C\ B(\varepsilon,e|\varepsilon)\urcorner$,

for all formulae A *and* B *of* L_ε *which contain no choice parameters besides* ε *and* ε,η *respectively.*

<u>PROOF</u>.

(a) By 8.3.10(a) and (b), 1.3.24(g) ($id \wedge id \simeq id$) and 8.3.10(c) we have $\ulcorner CS(C)1 \urcorner \leftrightarrow \forall e \in C \ulcorner \forall \zeta' \in (id, {}^\circ 0 \wedge {}^\circ 1)\ A(e,\zeta')\urcorner$, where $A(e,\zeta') \equiv \exists \zeta(\zeta = e|(j_1\zeta', j_2\zeta'))$. By definition of τ and 7.3.6(a) (which implies $\forall u(u /\!/ {}^\circ 0 \wedge {}^\circ 1))$ we have

$\ulcorner \forall \zeta' \in (id, {}^\circ 0 \wedge {}^\circ 1) A(e,\zeta')\urcorner \leftrightarrow \exists e_1 \forall u[e_1 u \neq 0 \to \exists f \in C \exists G \ulcorner B(u,f,G,e)\urcorner]$, where $B(u,f,G,e) \leftrightarrow \forall \zeta \in ([u] \wedge f, ({}^\circ 0 \wedge {}^\circ 1) \wedge G)(j_2\zeta = e|(j_1 j_1 \zeta, j_2 j_1 \zeta))$ (by 8.3.10(c) and the definition of τ). To prove $\ulcorner \underset{\approx}{CS}(C)1\urcorner$ it suffices to show that $\forall e \in C \exists e_1 \forall u[e_1 u \neq 0 \to \exists f \in C \exists G \ulcorner B(u,f,G,e)\urcorner]$. We shall show that in fact $\forall e \in C \forall u \exists f \in C \exists G \ulcorner B(u,f,G,e)\urcorner$: let $e \in C$ and u be arbitrary, put $f \equiv e:[u]$ ($f \in C$) and $G \equiv {}^\circ 0 \wedge {}^\circ 1$.

Then $\ulcorner B(u,f,G,e)\urcorner$ is equivalent to

$\ulcorner \forall \zeta \in ([u] \wedge (e:[u]), G \wedge G)\ (j_2\zeta = e|(j_1 j_1 \zeta, j_2 j_1 \zeta))\urcorner$ which is (by definition of τ and 7.3.2(a): $j_1(e \wedge f|a) = e|j_1 a, j_2(e \wedge f|a) = f|j_2 a$) equivalent to $\forall a /\!/ G \wedge G(e:[u]|j_2 a = e|([u]|j_1 a))$. This is obviously true, since $a /\!/ G \wedge G$ implies $a /\!/ {}^\circ 0 \wedge {}^\circ 0$ (by 7.3.6(g)) and $a /\!/ {}^\circ 0 \wedge {}^\circ 0$ iff $j_1 a = j_2 a$ by definition of $/\!/$.

(b) By 8.3.10(d), $\ulcorner \underset{\approx}{CS}(C)2\urcorner$ is equivalent to $\forall f \in C \forall F(\ulcorner \forall \varepsilon \in (f,F)\ A\varepsilon\urcorner \to \ulcorner \forall \varepsilon \in (f,F)\ B\varepsilon\urcorner)$ where $B\varepsilon \equiv \exists e \in C\ D(\varepsilon,e)$, and $D(\varepsilon,e) \equiv \exists \eta(\varepsilon = e|\eta) \wedge \forall \zeta\ A(e|\zeta)$.

Let $f \in C$ and F be arbitrary and assume $\ulcorner \forall \varepsilon \in (f,F)\ A\varepsilon\urcorner$. We have to show that $\ulcorner \forall \varepsilon \in (f,F)\ B\varepsilon\urcorner$ follows, i.e. (by definition of τ) we must find an e_1 such that

$\forall u /\!/ F[e_1 u \neq 0 \to \exists e \in C \ulcorner \forall \varepsilon \in (f:[u],F)\ D(\varepsilon,e)\urcorner]$.

We take $e_1 \equiv \lambda z.S0$, i.e. now we have to find for each $u /\!/ F$ an $e \in C$ such that $\ulcorner \forall \varepsilon \in (f:[u],F)\ D(\varepsilon,e)\urcorner$. For e we take $e \equiv f:[u]:f'$, where f' is an (arbitrarily chosen) element of $C[F]$.

By definition of τ, $\ulcorner \forall \varepsilon \in (f:[u],F)\ D(\varepsilon,e)\urcorner$ is the conjunction of $\ulcorner \forall \varepsilon \in (f:[u],F) \exists \eta(\varepsilon = e|\eta)\urcorner$ and $\ulcorner \forall \varepsilon \in (f:[u],F) \forall \zeta\ A(e|\zeta)\urcorner$ (where ε does not occur in A). If we apply the key-lemma 9.2.9 to the first conjunct we find that it is equivalent to $\ulcorner \forall \varepsilon \exists \eta(e|\varepsilon = e|\eta)\urcorner$ which is easily seen to be true.

Also by 9.2.9 the second conjunct is equivalent to $\ulcorner \forall \varepsilon \forall \zeta\ A(e|\zeta)\urcorner$.

$\ulcorner \forall \varepsilon \forall \zeta A(e|\zeta)\urcorner \leftrightarrow \ulcorner \forall \zeta \in (id, {}^\circ 0 \wedge {}^\circ 1) A(e|j_2\zeta)\urcorner$ by 8.3.10(a), (b), (c),

$\ulcorner \forall \zeta \in (id, {}^\circ 0 \wedge {}^\circ 1)\ A(e|j_2\zeta)\urcorner \leftrightarrow \ulcorner \forall \zeta\ A(e|j_2\zeta)\urcorner$ by 9.2.9 and 9.2.6(b): $id \in C[{}^\circ 0 \wedge {}^\circ 1]$.

$\ulcorner \forall \zeta\ A(e|j_2\zeta)\urcorner$ follows immediately from the assumption $\ulcorner \forall \varepsilon \in (f,F)\ A\varepsilon\urcorner$:

$\ulcorner \forall \varepsilon \in (e,F)\ A\varepsilon\urcorner \to \ulcorner \forall \varepsilon \in (f:[u],F)\ A\varepsilon\urcorner$ by monotonicity of τ and 7.3.7(c) ($[u] /\!/ F$);

$\ulcorner\forall\varepsilon\in(f{:}[u],F)\ A\varepsilon\urcorner \to \ulcorner\forall\varepsilon\ A(e|\varepsilon)\urcorner$ by 9.2.9 ($e = f{:}[u]{:}f'$, $f' \in C[F]$);
$\ulcorner\forall\varepsilon\ A(e|\varepsilon)\urcorner \to \ulcorner\forall\varepsilon\in(j_2,{}^\circ 0)\ A(e|\varepsilon)\urcorner$ by definition of τ ($j_2 \in C$);
$\ulcorner\forall\varepsilon\in(j_2,{}^\circ 0)\ A(e|\varepsilon)\urcorner \to \ulcorner\forall\varepsilon\ A(e|j_2\varepsilon)\urcorner$ by 9.2.9 and 9.2.6(b): $id \in C[{}^\circ 0]$.

(c) Assume $\ulcorner\forall\varepsilon\exists a\ A(\varepsilon,a)\urcorner$ then in particular $\ulcorner\forall\varepsilon\in(id,{}^\circ 0)\exists a\ A(\varepsilon,a)\urcorner$ whence, by definition of τ and 7.3.6(b) ($\forall u(u /\!/ {}^\circ 0)$).
$\exists e\forall u[eu{\neq}0 \to \exists a\ulcorner\forall\varepsilon\in([u],{}^\circ 0)\ A(\varepsilon,a)\urcorner]$. By 9.2.9 and 9.2.6(b) ($id \in C[{}^\circ 0]$) $\ulcorner\forall\varepsilon([u],{}^\circ 0)\ A(\varepsilon,a)\urcorner$ is equivalent to $\ulcorner\forall\varepsilon\ A([u]|\varepsilon,a)\urcorner$.

(d) Assume $\ulcorner\forall\varepsilon\exists\eta\ B(\varepsilon,\eta)\urcorner$, then in particular $\ulcorner\forall\varepsilon\in(id,{}^\circ 0)\exists\eta\ B(\varepsilon,\eta)\urcorner$, i.e. we have an e_1 such that
$\forall u[e_1u{\neq}0 \to \exists f\in C\exists F\ulcorner\forall\zeta\in([u]\wedge f,{}^\circ 0\wedge F)\ B(j_1\zeta,j_2\zeta)\urcorner]$ by definition of τ and 7.3.6(b): $\forall u(u/\!/{}^\circ 0)$.
We must derive $\ulcorner\forall\varepsilon\exists e\in C\ B(\varepsilon,e|\varepsilon)\urcorner$ or equivalently (by 8.3.10(a)) $\ulcorner\forall\varepsilon\in(id,{}^\circ 0)\exists e\in C\ B(\varepsilon,e|\varepsilon)\urcorner$, i.e. (by definition of τ and 7.3.6(b)) we must find an e_1 such that $\forall u[e_1u{\neq}0 \to \exists e\in C\ulcorner\forall\varepsilon\in([u],{}^\circ 0)\ B(\varepsilon,e|\varepsilon)\urcorner]$.
For e_1 we take the one we have by assumption. Let u be arbitrary, $e_1u \neq 0$, then we have an $f \in C$ and an F such that $\ulcorner\forall\zeta\in([u]\wedge f,{}^\circ 0\wedge F)\ B(j_1\zeta,j_2\zeta)\urcorner$. By monotonicity of τ then also $\ulcorner\forall\zeta\in([u]\wedge f,G)\ B(j_1\zeta,j_2\zeta)\urcorner$ where $G = ({}^\circ 0\wedge F)[\lambda z.{}^\circ 0]$. By 9.2.6(c) $nest_G \in C[G]$, i.e. we find that $\ulcorner\forall\zeta\ B([u]|j_1(nest_G|\zeta),\ f|j_2(nest_G|\zeta))\urcorner$ by extensionality and 9.2.9. By 9.2.2(d), $nest_G \simeq (nest_{{}^\circ 0} \wedge nest_{F'})$: dpl, where $F' = F[\lambda z.{}^\circ 0]$. Hence $\forall a(j_1(nest_G|a) = nest_{{}^\circ 0}|j_1(dpl|a) = nest_{{}^\circ 0}|a{=}a)$ (9.2.2(a),(b)) and $\forall a(j_2(nest_G|a) = nest_{F'}|j_2(dpl|a) = nest_{F'}|a)$ (9.2.2(a)). I.e. by extensionality we obtain $\ulcorner\forall\zeta\ B([u]|\zeta,f{:}nest_{F'}|\zeta)\urcorner$.
Our aim is to find an $e \in C$ such that $\ulcorner\forall\varepsilon\in([u],{}^\circ 0)\ B(\varepsilon,e|\varepsilon)\urcorner$. We take $e \equiv f{:}nest_{F'}{:}s^n$, where $n = lth(u)$, then $e{:}[u] \simeq f{:}nest_{F'}$, hence the foregoing yields (by extensionality) $\ulcorner\forall\zeta\ B([u]|\zeta,e|([u]|\zeta))\urcorner$, from which the desired result follows by one more application of 9.2.9 (again using $id \in C[{}^\circ 0]$). □

9.3. CONCLUSIONS

Combining the results of the previous chapters with theorem 9.2.10 we obtain the following theorems.

9.3.1. THEOREM. *If $\mathcal{U}_\delta$ is a domain w.r.t. a* CS-*closed* $C \subset K$, *then $\mathcal{U}_\delta$ is a model for* CS(C). *This can be shown formally in* LSF^*, *i.e.*

$$LSF^* \vdash \text{CSclosed}(C) \wedge \text{domain}\ (\pi,d,\delta) \to \Phi^\delta$$

for each axiom and instance of an axiom schema Φ *of* CS(C).

PROOF. Immediately from the main theorem 9.2.10 and the elimination theorem 8.4.2. Observe that CSclosed(C) → dclosed(C) by definition. □

9.3.2. THEOREM. *If* U_δ *is a projected universe of* GC-*sequences w.r.t. a* CSlcosed C ⊂ K (*which means in particular that* J *enumerates* C *modulo* ≃) *then* U_δ *is a model for* CS(C). *This can be proved in* LSF*, *i.e.*

$$LSF^* \vdash CSclosed(C) \wedge model(\pi,d,\mathfrak{f}) \rightarrow \Phi^\delta$$

for each axiom and instance of an axiom schema Φ *of* CS(C).

PROOF. Combine theorem 6.2 (models are domains) with the previous theorem. Note that 6.2 can be formalized in IDBF* (cf.7.2.13). (Note also that dclosed(C) and 'J enumerates C modulo ≃' are subsentences of model $(\pi,d,\mathfrak{f})$ (cf.7.2.13).) □

9.3.3. THEOREM. *With each mapping* I:ℕ→ K *there exists a universe* U_δ *of projections of lawless sequences which satisfies* $\varepsilon\in U_\delta \rightarrow \forall n(In|\varepsilon\in U_\delta)$ *and which is a model for* CS(C).

PROOF. It suffices to show that with each mapping I:ℕ → K we can find a J:ℕ → K such that

(a) range(I) ⊂ range(J),

(b) C ≡ {e ∈ k: ∃n(Jn ≃ e)} is CS-closed,

for then the desired result follows immediately from 9.3.2 above and the observation that there exist π,d and $\mathfrak{f}$ which generate a projected universe of nests of GC-carriers and the corresponding dressings and frames respectively, whatever J is (cf.7.2.13). (Note that J enumerates C modulo ≃ by definition of C.)

To make J fulfill (a) and (b) we must ensure that:

(i) $\forall n\exists m(In \simeq Jm)$,

(ii) $\forall v\exists n(Jn \simeq [v])$,

(iii) $\forall n\exists m(Jm \simeq s^n)$,

(iv) $\exists m_0 m_1 m_2(Jm_0 \simeq j_{<0>} \wedge Jm_1 \simeq j_{<1>} \wedge Jm_2 \simeq dpl)$,

(v) $\forall k\forall m\exists n(Jn \simeq Jk{:}Jm)$,

(vi) $\forall k\forall m\exists n(Jn \simeq Jk\wedge Jm)$.

This is achieved if we construct J such that J(j(0,n)) = In, J(j(1,v)) = [v],

$J(j(2,n)) = s^n$, $J(j(3,0)) = j_{<0>}$, $J(j(3,1)) = j_{<1>}$, $J(j(3,2)) = dp1$, $J(j(3,n+3)) = id$, $J(j(n+4,2m)) \simeq Jn{:}Jm$, and $J(j(n+4,2m+1)) \simeq Jn \wedge Jm$.
In IDB we can construct an F-Tm ϕ such that $K(\phi)$, $\phi 0 = 0$ and $\lambda v.\phi(\langle n\rangle * v)$ behaves as desired for Jn, relative to any ψ such that $\forall n\ K(\lambda v.\psi(\langle n\rangle * v))$, i.e. such that $n \mapsto \lambda v.\psi(\langle n\rangle * v)$ can play the rôle of I. □

APPENDIX

In 4.4.11 we introduced the set $DG^0(J)$ of mappings d: $\mathbb{N} \times \mathbb{N} \to K$ satisfying

(1) $d_n 0 \simeq id$

(2) $d_n(v*\hat{x}) \simeq d_n v\colon JF(n,v*\hat{x}):GV(n,v*\hat{x})$

where $JF(n,v*\hat{x}) \equiv \nu^K_{\delta_n v}\ jf(v*\hat{x})$ and $GV(n,v*\hat{x}) \equiv \nu^K_{\delta_n(v*\hat{x})}\ gv(v*\hat{x})$.
In this appendix we shall show that $DG^0(J)$ has elements which are primitive recursive in J.

Since each element $e \in K$ is a mapping from $\mathbb{N}$ to $\mathbb{N}$, a mapping d: $\mathbb{N} \times \mathbb{N} \to K$ can be viewed as a mapping d: $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. To construct the desired d, we use an auxiliary mapping D, which assigns to each $k \in \mathbb{N}$ a finite sequence Dk with length k. The finite sequence D(k+1) is to contain the 'initial segment of d', i.e.
$D(k+1) = \langle d_{j^3_1 0} j^3_2 0(j^3_3 0),\ldots,d_{j^3_1 k} j^3_2 k(j^3_3 k)\rangle$. That is to say, once D has been defined we shall put

(3) $d_n v \equiv \lambda u.(D(au+1))_{au}$,

where $au \equiv \nu_3(n,v,u)$.

D is defined by an ordinary recursion, its definition has the form

$$D0 = 0, \quad D(k+1) = Dk*\langle\phi(Dk,k)\rangle.$$

$\phi(Dk,k)$ will be the value of $d_n v(z)$, where $\nu_3(n,v,z) = k$. We define $\phi(Dk,k)$ as follows ($k = \nu_3(n,v,z)$).

(a) If $v = 0$ then we put $\phi(Dk,k) = id(z)$. Thus we achieve that for all n, $d_n 0$ will be equal to id eventually.

(b) If $z = 0$ then also $\phi(Dk,k) = 0$. It follows that $d_n v(0) = 0$ for all n and v, this is consistent with (a) above and with equivalence (2), if we write $f[n,v*\hat{x}]$ for the right-handside of (2) then $f[n,v*\hat{x}](0) = 0$.

(c) If both z and v are unequal zero, say $z = \hat{y}*u$, $v = w*\hat{x}$, then we proceed as follows: we put

$$f[n,w*\hat{x}] \equiv d_n w\colon JF(n,w*\hat{x}):GV(n,w*\hat{x})$$

(i.e. the right-hand side of (2) with w for v) and we try to establish the value $f[n,w*\hat{x}]$ $(\hat{y}*u)$, using only information that is to be found in Dk. If we succeed we put $\phi(Dk,k) = f[n,w*\hat{x}](\hat{y}*u)$, otherwise $\phi(Dk,k) = 0$.

In order to find $f[n,w*\hat{x}](\hat{y}*u)$ we must first try to compute $upb(d,w*\hat{x})$. $upb(d,w*\hat{x})$ is defined as

$$upb(d,w*\hat{x}) \equiv \max\{U_m(w*\hat{x}):m \in nf(k_1^3(w*\hat{x}))\}$$

where

$$U_m(w*\hat{x}) \equiv mk((d:JF)(m,w*\hat{x}),lth(w),gs_m(w*\hat{x}))$$

(see 4.4.9). $mk((d:JF)(m,w*\hat{x}),lth(w),gs_m(w*\hat{x}))$ is the smallest z such that $((d:JF)(m,w*\hat{x}))(<lth(w)>*\overline{gs_m(w*\hat{x})}(z)) \neq 0$ (see 4.4.8) and this inequality is equivalent to

$$(4) \qquad d_m w(<lth(w)>*JF_m \restriction \overline{gs_m}(z)) \neq 0,$$

where $JF_m \equiv JF(m,w*\hat{x})$ and $gs_m \equiv gs_m(w*\hat{x})$ (see 4.4.7 and the definition of: in 1.3.18). In computing $upb(d,w*\hat{x})$ from the information on d contained in Dk, we shall first make lists $\{w_m : m \in nf(k_1^3(v*\hat{x}))\}$ satisfying

$$z \in w_m \quad \text{iff} \quad \nu_3(m,w,<lth(w)>*JF_m \restriction \overline{gs_m}(z)) < k,$$

i.e. if $z \in w_m$ then we can use Dk to check whether or not (4) holds. If there is an m such that (4) does not hold for any of the $z \in w_m$, then Dk gives us too little information to determine $upb(d,w*\hat{x})$ and we shall put $\phi(Dk,k) = 0$. Otherwise we compute $upb(d,w*\hat{x})$. (We tacilty assume here that the lists w_m are initial segments of $\mathbb{N}$. This will be the case if $z < z*\hat{n}$ for all z and n, and if ν_3 is monotone in all its arguments. We can do without such assumptions, the construction of $\phi(Dk,k)$ will remain essentially the same, but we shall have to proceed with more care.)

Once we have succeeded in finding $upb(d,w*\hat{x})$ from Dk we can easily determine the K-function $GV(n,w*\hat{x})$. By definition of:, $f[n,w*\hat{x}](\hat{y}*u)$ (the value that we want to assign to $\phi(Dk,k)$) is equal to

(5) $d_n w(\hat{y}*(JF_n:GV_n)\upharpoonright u)$,

where $JF_n \equiv JF(n,w*\hat{x})$, $GV_n \equiv GV(n,w*\hat{x})$. In order to compute (5) from Dk we make a list $v_0,v_1,\ldots,v_i$ (possibly empty) of initial segments of $(JF_n:GV_n)\upharpoonright u$, in which v_j occurs iff $\nu_3(n,w,\hat{y}*v_j) < k$. If for some v_j in the list $(Dk)_{\nu_3(n,w,\hat{y}*v_j)} = m+1$ then (5) will yield m+1 and we put $\phi(Dk,k) = m+1$, otherwise $\phi(Dk,k) = 0$.

We have to check the following facts for the mappings $d_n v$ defined by $d_n v \equiv \lambda u.(D(1+\nu_3(n,v,u)))_{\nu_3(n,v,u)}$:

(i) $d_n v \in K$,

(ii) $d_n 0 \simeq id$,

(iii) $d_n(v*\hat{x}) \simeq d_n v:JF(n,v*\hat{x}):GV(n,v*\hat{x})$.

(ii) is trivial, by (a) above we have $d_n 0 = id$, whence also $d_n 0 \in K$.

(i) is proved by induction w.r.t. lth(v), in this proof we shall establish (iii). The basis-step of the proof of (i) (v = 0) is in the proof of (ii). For the induction step we show that

(6) $d_n(v*\hat{x})(\hat{y}*u) = sg(e(\hat{y}*u))\cdot(d_n v:(JF_n:GV_n))(\hat{y}*u)$

for some $e \in K$. Since $d_n(v*\hat{x})(0) = 0$ (by (b)) this proves that $d_n(v*\hat{x}) \in K$, at the same time it shows (iii).

The left-hand side of (6) is $\phi(Dk,k)$ for $k = \nu_3(n,v*\hat{x},\hat{y}*u)$. From (c) above it follows that we must choose e such that $e(\hat{y}*u) \neq 0$ iff Dk contains sufficient information to determine a value for $(d_n v:(GS_n:JF_n))(\hat{y}*u)$. The existence of such an e follows from the induction-hypothesis: $\forall m(d_m v \in K)$.

First one proves that there is an $e_1 \in K$ such that $e_1(\hat{y}*u) \neq 0$ iff

$$\forall m \in nf(k_1^3(v*\hat{x}))\exists z[d_m v(<lth(v)>*JF_m\upharpoonright\overline{gs_m}(z)) \neq 0 \wedge$$

$$\nu_3(m,v,<lth(v)>*JF_m\upharpoonright\overline{gs_m}(z)) < k],$$

then one shows that there is an e_2 in K such that $e_2(\hat{y}*u) \neq 0$ iff

$$\exists w \preceq (JF_n:GV_n)\upharpoonright u(d_n v(\hat{y}*w) \neq 0 \wedge \nu_3(n,v,\hat{y}*w) < k).$$

Then e can be defined by $e(\hat{y}*u) = e_1(\hat{y}*u)\cdot e_2(\hat{y}*u)$.

e_1 is found as a product of mappings $e_{1,m}$, where $e_{1,m}(\hat{y}*u) \neq 0$ iff

$$\exists z[d_m v(\langle 1th(v)\rangle * JF_m \restriction \overline{gs_m}(z)) \neq 0 \wedge \nu_3(m,v,\langle 1th(v)\rangle * JF_m \restriction \overline{gs_m}(z)) < k];$$

since $d_m v \in K$ there is a shortest w of the form $\langle 1th(v)\rangle * JF_m \restriction \overline{gs_m}(z)$ such that $d_m v(w) \neq 0$, and we can put $e_{1,m} \equiv \lambda u.sg(\nu_3(n,v*\hat{x},u) \dot{-} \nu_3(m,v,w))$. Since $\nu_3(m,v,w)$ is a constant, there is a k such that for all u' with $1th(u') > k$ $e_{1,m}(u'*u) = 1$, together with the monotonicity of ν_3 in its third argument this yields $e_{1,m} \in K$ (see 1.3.13,14).

e_2 is the product of $e_{2,1}$ and $e_{2,2}$, where $e_{2,1}(\hat{y}*u) = h(\lambda z.e(\hat{y}*z), JF_n:GV_n \restriction u)$ and $e_{2,2}(\hat{y}*u) = sg(\nu_3(n,v*\hat{x},\hat{y}*u) \dot{-} \nu_3(n,v,\hat{y}*(e_{2,1}(\hat{y}*u) \dot{-} 1)))$. $e_{2,1}(\hat{y}*u) \neq 0$ means that there is an initial segment w of $JF_n:GV_n \restriction u$ such that $d_n v(\hat{y}*w) \neq 0$, if $e_{2,1}(\hat{y}*u) \neq 0$ then $e_{2,2}(\hat{y}*u) \neq 0$ means that the shortest $w \preccurlyeq JF_n:GV_n \restriction u$ such that $d_n v(\hat{y}*w) \neq 0$ satisfies $\nu_3(n,v,\hat{y}*w) < k$, i.e. $d_n v(\hat{y}*w)$ can be found in Dk. We leave it to the reader to verify that $e_{2,1} \in K$ and $e_{2,2} \in K$.

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INDEX

AXIOMS AND SCHEMATA

AC-NF	axiom of choice from numbers to (lawlike) functions $\forall x \exists a\, A(x,a) \rightarrow \exists b \forall x\, A(x,(b)_x)$.	1.1, 1.3.27, 1.3.28.
AC-NFrf	axiom of choice from numbers to lawlike sequences of frames: $\forall x \exists \mathfrak{f}\, A(x,\mathfrak{f}) \rightarrow \exists g \forall x\, A(x,(g)_x)$.	7.2.1.
CSi, i = 1,...,4	CS-axioms	1.1.
CS(C)i, i = 1,...,4	CS(C)-axioms	1.1, 1.3.29.
ECT_0	extended Church's thesis	1.1.
EP	extension principle: $\forall e \in K_{LS} \forall \phi \in N \exists x (e(\bar{\phi}x) \neq 0)$.	1.1, 1.3.28.
LSi, i = 1,...,4	LS-axioms	1.1, 1.3.28.
QF-AC	quantifier-free axiom of choice: for A quantifier-free $\forall x \exists y\, A(x,y) \rightarrow \exists a \forall x\, A(x,ax)$.	1.3.5.

FORMAL LANGUAGES

$L(X)$	X any formal system; the language of X, *see* formal systems	
L_ε	the language of CS(C)	8.2.1.
$L^0_\varepsilon, L^*_\varepsilon$	extensions of L_ε	8.2.1, 8.2.5.

FORMAL SYSTEMS

CS	The Kreisel-Troelstra system for the foundation of intuitionistic analysis	1.1.
CS(C)	Relativized CS	1.1, 1.3.29, 8.2.2.
EL	Elementary analysis	1.3.5.
IDB_0	EL + inductively defined set K of neighbourhood functions	1.3.8.
IDB	IDB_0 + the axiom of choice from numbers to functions	
IDB_1	IDB with K-terms	1.3.27.
IDBF	IDB + theory of frames	7.2.1-7.2.7.
$IDBF_1$	IDBF with K-terms	7.2.8-7.2.11.
$IDBF^*$	$IDBF_1$ with additional constants C and J	7.2.12.
LS	The theory of lawless sequences	1.1, 1.3.28.
LS^K	The theory of lawless sequences of K-functions	1.1.
LSF^*	The theory of lawless sequences with lawlike part $IDBF^*$.	7.2.14

SETS, UNIVERSES AND CLASSES

$CU_\delta(C)$	The class of projected universes of nests of GC-carriers w.r.t. C	4.5.2.
C[F]	The subset of $C \subset K$ which contains exactly those e such that $\{e\|\phi:\phi\in N\} = \{\psi\in N:\psi//F\}$.	9.2.5.
$DG^0(J)$	The set of mappings $d^0:\mathbb{N}\times\mathbb{N}\to K$ which generate dressings for carriers	4.4.2, 4.4.10
DG(J)	The set of mappings $d:\mathbb{N}\times FRAME\times\mathbb{N}\to K$ which generate universes of dressing sequences	4.2, 4.4.17
$D_\delta(C)$	The class of domains w.r.t. C	6.1.1.
FRAME	The set of frames	3.1.14.
GC	The universe of GC-sequences	2.2, 2.10.2.
GCC	The universe of GC-carriers	2.2-2.8.
GC(C)	The universe of GC-sequences w.r.t. C	2.11.4.
GCC(C)	The universe of GC-carriers w.r.t. C	2.11.1.
K	The inductively defined set of neighbourhood functions	1.1, 1.3.1, 1.3.7-1.3.27.
K_{LS}	The set of neighbourhood functions for continuous mappings with domain LS	1.1, 1.3.28
LS	The universe of lawless sequences	1.1, 1.3.1.
$\mathbb{N}$	The natural numbers	
N	Intuitionistic Baire-space	1.1, 1.3.1.
PLS	The universe of proto-lawless sequences	2.10.3.
$\mathcal{U}^M_\delta$	The projected universe $\{e\|\delta:e\in M\}$	4.1.
$U_\delta(C)$	The class of projected universes of GC-sequences w.r.t. C	4.1, 4.2, 4.6.

SYMBOLS, TERMS, RELATIONS AND SPECIAL FORMULAE

SAMENVATTING

Dit proefschrift behandelt de volgende drie nauw samenhangende vragen uit het onderzoek naar deelverzamelingen van de intuitionistische Baire-ruimte:

(a) Geef een nauwkeurige beschrijving van het door TROELSTRA geïntroduceerde informele begrip GC-rij.

(b) Construeer een verzameling continue afbeeldingen van de Baire-ruimte naar zichzelf, zodanig dat de beelden van een vaste wetteloze rij onder de operaties uit deze verzameling zich als GC-rijen gedragen. D.w.z. construeer een projectiemodel voor de GC-rijen.)

(c) Bewijs dat het onder (b) geconstrueerde universum een model is voor het axiomasysteem $\underset{\sim}{CS}$ (uit KREISEL TROELSTRA 1970).

In hoofdstuk 1 wordt de achtergrond van deze vragen uiteengezet. Bovendien bevat dit hoofdstuk een opsomming van (merendeels uit de literatuur bekende) definities, feiten en lemma's die voor het vervolg van belang zijn.
In hoofdstuk 2 wordt vraag (a) beantwoord. In aansluiting daarop wordt een relativering van het begrip GC-rij geïntroduceerd, de GC(C)-rij, waar C een verzameling continue afbeelding van de Baire-ruimte naar zichzelf is.
Hoofdstuk 3 bevat de technische hulpmiddelen die nodig zijn voor het beantwoorden van vraag (b).
In hoofdstuk 4 laten we zien hoe voor een aantal soorten GC(C)-rijen een projectiemodel kan worden geconstrueerd. Deze constructie werkt alleen in gevallen waar de verzameling C aftelbaar is. Het antwoord op vraag (b) dat hier gegeven wordt is derhalve onvolledig, voor de GC-rijen zelf vinden we geen model. (Overigens valt te verwachten dat een kleine aanpassing van de constructie, onder de aanname van de zogeheten uitgebreide these van Church, wel een model voor het gedrag van de GC-rijen zal geven.)
In hoofdstuk 5 wordt een lemma bewezen dat van wezenlijk belang is voor de beantwoording van vraag (c), in hoofdstuk 6 wordt de in hoofdstuk 4 geïntroduceerde klasse projectiemodellen gegeneraliseerd tot de klasse van domeinen.
Hoofdstuk 7 geeft een samenvatting van de tot dan toe gevonden resultaten (met name die, die in het vervolg nog een rol spelen). Bovendien worden in dit hoofdstuk de formele systemen beschreven waarbinnen deze resultaten kunnen worden afgeleid.
In hoofdstuk 8 behandelen we een eliminatie vertaling die geïntroduceerd door DRAGALIN, en we bewijzen dat een zin waar is in een domein dan en

slechts dan als hij waar is onder deze eliminatievertaling.
In hoofdstuk 9 tenslotte wordt bewezen dat alle CS(C)-axioma's (CS(C) is een gerelativeerde variant van CS) waar zijn onder de eliminatievertaling uit hoofdstuk 8. Daaruit volgt dat alle domeinen modellen zijn van de CS(C)-axioma's en daaruit volgt weer dat de projectiemodellen van GC(C)-rijen uit hoofdstuk 4 modellen zijn van de CS(C) axioma's. Daarmee is ook vraag (c) beantwoord.

STELLINGEN

bij het proefschrift *Projections of Lawless Sequences*
van G.F. van der Hoeven.

I. Het is mogelijk een beperkte versie van het begrip GC-rij tot het begrip wetteloze rij te reduceren. In de projectiemodellen die voor deze reductie gebruikt worden, gelden varianten van de $\underset{\sim}{CS}$-axioma's "analytic data", "$\forall\varepsilon\exists n$-continuiteit", "$\forall\varepsilon\exists\eta$-continuiteit". Bovendien zijn deze modellen gesloten onder een aftelbare verzameling continue operaties.

Dit proefschrift.

II. De rechtvaardiging van de continuiteitsaxioma's voor de theorie der wetteloze rijen zoals die te vinden is in TROELSTRA (1977) gaat voorbij aan de vraag of een formule waarin geen vrije keuzevariabelen voorkomen altijd wetmatig is. Een eenvoudig voorbeeld uit de theorie der protowetteloze rijen laat zien dat dit niet altijd het geval hoeft te zijn, met name niet als in de formule existentiële kwantificatie over een niet dicht liggend deeluniversum van de Baire-ruimte voorkomt.

TROELSTRA, A.S.(1977) Choice Sequences, Clarendon Press, Oxford.
(1981) Analysing choice sequences, Report 81-05, Dept. of Math. Univ. A'dam.

III. Het projectiemodel corresponderend met KROL's model voor zwakke continuiteit in parameters, is het universum CL van de vorm
$\{e|(\alpha_1,\ldots,\alpha_p) : e \in K, \#(\alpha_1,\ldots,\alpha_p)\}$.

KROL', M.D.(1978) Distinct variants of Kripke's scheme in intuitionistic analysis, Soviet Mathematics 19 I, 474-477.

IV. De parallel die bestaat tussen geldigheid in Beth modellen en geldigheid in een wetteloze parameter, bestaat ook tussen geldigheid in topologische modellen over [0,1] en geldigheid in een parameter lopend over de vrije reële getallen in [0,1].

van der HOEVEN, G.F.(1981) To appear in the Proceedings of the Brouwer Centenary Conference.

X. In het artikel "Analysis without actual infinity" geeft J. MYCIELSKI te kennen dat naar zijn mening intuitionistische logica onhandig is, en de Platonistische filosofie van de wiskunde kinderachtig. Hij wekt de indruk te geloven dat door het toekennen van deze twee adjectieven het intuitionisme en het Platonisme voldoende gediskwalificeerd zijn. Een nadere toelichting op dit oordeel was echter op zijn plaats geweest.

MYCIELSKI, J., Analysis without actual infinity, The Journal of Symbolic Logic 46 (1981) 625-633.