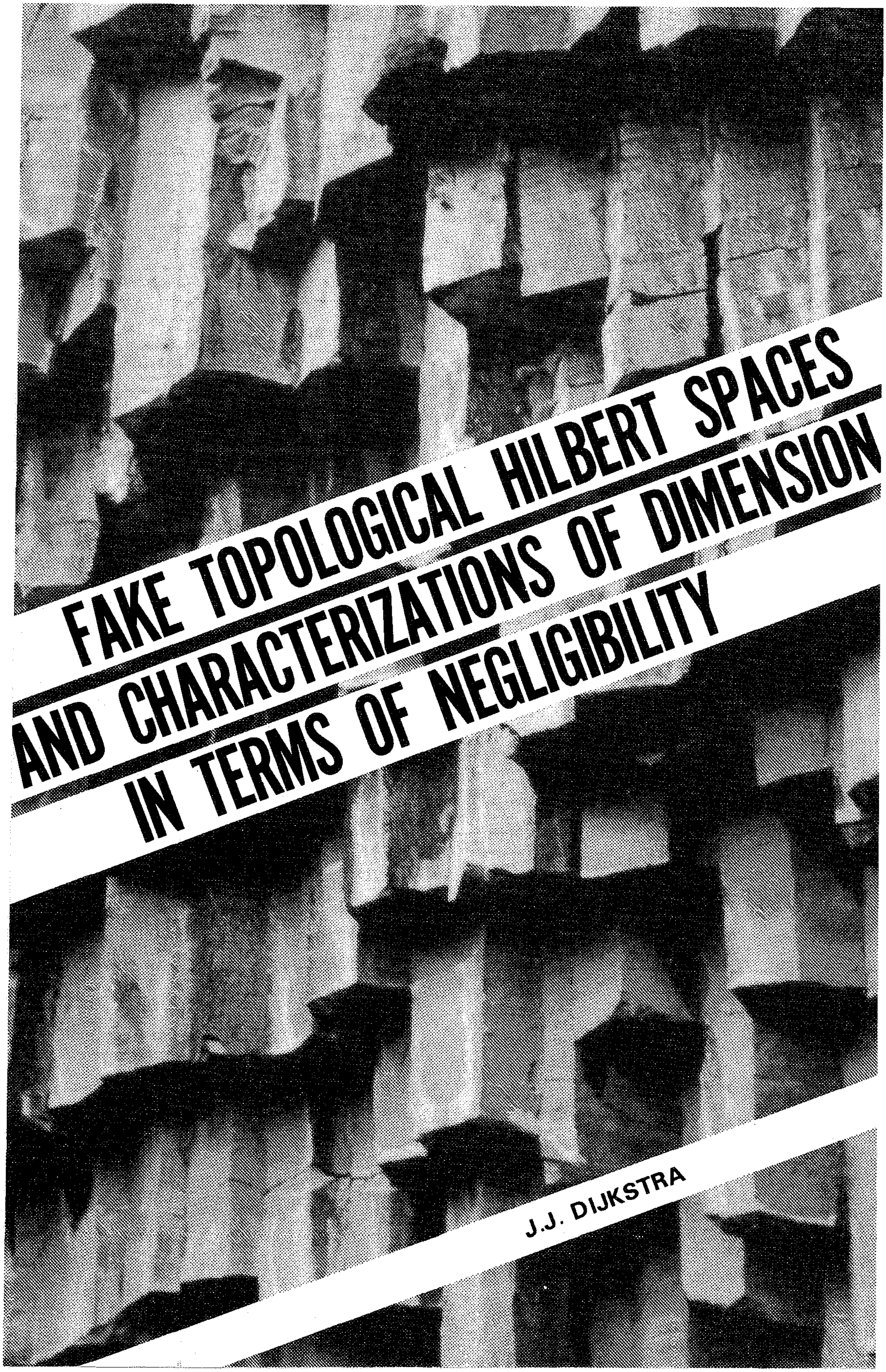

FAKE TOPOLOGICAL HILBERT SPACES AND CHARACTERIZATIONS OF DIMENSION IN TERMS OF NEGLIGIBILITY

J.J. DIJKSTRA

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JAN JAKOBUS DIJKSTRA

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Co-promotor: Dr. J. van Mill

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First, however, she waited for a few minutes to see if she was going to shrink any further: she felt a little nervous about this; "for it might end, you know," said Alice to herself, "in my going out altogether, like a candle. I wonder what I should be like then?"

Lewis Carroll, *Alice's adventures in wonderland.*

Aan de nagedachtenis van mijn vader
Aan mijn moeder

PREFACE

This monograph is an investigation in infinite-dimensional topology. By a fake topological Hilbert space we mean a separable, metrizable space that shares many topological properties with ℓ^2, but yet is not homeomorphic to it. We think of properties like: X is an absolute retract, X is homogeneous, $X \times X$ is homeomorphic to ℓ^2, every compactum in X is a Z-set and X is universal for the class of separable, metrizable spaces. Our aim is to construct a sequence $X_{-1}, X_0, X_1, \ldots$ of fake Hilbert spaces such that an arbitrary σ-compact subspace of X_k has dimension $\leq k$ if and only if it is strongly negligible. In other words X_k has the negligibility properties of ℓ^2 precisely up to dimension k inclusive.

The standard way to obtain spaces with certain negligible subsets is through pseudo-boundaries. We first construct in chapter 2 a k-dimensional pseudo-boundary in $\mathbb{R}^n$. Employing this result we build in chapter 3 a k-dimensional pseudo-boundary in the Hilbert cube for every $k \in \{-1,0,1,\ldots\}$. As basis for our sequence $X_{-1}, X_0, X_1, \ldots$ we use a fake Hilbert space Y, which has been introduced by Anderson, Curtis & Van Mill [ACM]. We show in chapter 4 that Y is homogeneous in a very strong sense and we conclude from this fact that A_k is also a pseudo-boundary in Y. Finally, in chapter 5 the spaces $X_k = Y \setminus A_k$ are analysed.

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CHAPTER 5

FAKE HILBERT SPACES

CHAPTER 1

GENERAL THEORY

1.1 Preliminaries

In this section we introduce basic concepts and we give two simple methods to construct autohomeomorphisms. Our notation is standard, cf. Engelking [E1]. For information concerning infinite-dimensional topology see Bessaga & Pełczyński [BP2] and Chapman [C]. We make the following restriction.

All topological spaces in this treatise are assumed to be separable and metrizable.

We now give a list of definitions and notations. Let X and Y be topological spaces, let $\mathcal{U}$ be a collection of open subsets of X and let d be an admissible metric on X.

(a) $\mathcal{H}(X)$ denotes the group of autohomeomorphisms of X and 1_X or simply 1 is the identity on X.

(b) A continuous mapping is called a *map*.

(c) The symbol $X \approx Y$ means that X and Y are homeomorphic spaces.

(d) If f is a mapping from X into X and A is a subset of X then we say that f is *supported on* A if the restriction $f|X\setminus A$ is equal to $1_{X\setminus A}$.

(e) Mappings $f,g : Y \to X$ are *$\mathcal{U}$-close* if for each $y \in Y$ with $f(y) \neq g(y)$ there exists a $U \in \mathcal{U}$ containing both $f(y)$ and $g(y)$ (note that we did not require $\mathcal{U}$ to cover X). Observe that if $f : X \to X$ is $\mathcal{U}$-close to 1 then f is supported on $\cup\mathcal{U}$.

(f) If f and g are mappings from Y into X then

$$\hat{d}(f,g) = \sup\{d(f(y),g(y)) \mid y \in Y\} \in [0,\infty].$$

(g) $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Q}$ denote the real, natural and rational numbers, respectively.

(h) If C is an n-cell, $n \in \mathbb{N}$, then ∂C denotes the geometric boundary of C. Int C is the set $C \setminus \partial C$.

(i) A *homotopy* is a map $F : Y \times K \to X$, where K is a compact interval in $\mathbb{R}$. Usually, K equals the set $I = [0,1]$ and we define for $t \in K$, $F_t : Y \to X$ by $F_t(y) = F(y,t)$. F is called *limited by* $\mathcal{U}$ if for every $y \in Y$ the path of y, $F(\{y\} \times K)$, is a singleton or is contained in some member of $\mathcal{U}$.

(j) An *isotopy* H of X is a homotopy from $X \times K$ into X such that the function $\hat{H} : X \times K \to X \times K$, defined by $\hat{H}(x,t) = (H(x,t),t)$ is an element of $\mathcal{H}(X \times K)$. For compact X this means that an isotopy H is a homotopy such that each level H_t is in $\mathcal{H}(X)$. Occasionally, we shall also call $\hat{H}$ an isotopy. If $\varepsilon > 0$ then H is an ε-isotopy if the supremum for $x \in X$ of the d-diameter of $H(\{x\} \times K)$ is less than ε.

(k) X is called *homogeneous* if for every pair $x,y \in X$ there is an $f \in \mathcal{H}(X)$ with $f(x) = y$.

We conclude this section with two lemmas which give frequently used methods to construct homeomorphisms.

1.1.1. LEMMA: *If* $H : X \times K \to X$ *is an isotopy of* X *and* α *is a map from* Y *into* K *then the function* f *defined by*

$$f(x,y) = (H(x,\alpha(y)),y) \text{ for } x \in X \text{ and } y \in Y$$

is an element of $\mathcal{H}(X \times Y)$.

PROOF: This is trivial.

1.1.2. LEMMA: *Let* T *be a tree of height* ω, X *a topologically complete space and* $(f_t)_{t\in T}$ *a function from* T *into* $\mathcal{H}(X)$ *such that for every open covering* $\mathcal{U}$ *of* X *and* $t \in T$ *there is an immediate successor* t' *of* t *such that* $f_{t'}$ *and* 1 *are* $\mathcal{U}$*-close. If* d *is an admissible metric on* X *then there is a branch* $t_0,t_1,t_2,\ldots$ *in* T *such that* $(f_{t_i} \circ \ldots \circ f_{t_1} \circ f_{t_0})_{i\in\mathbb{N}}$ *has a uniform* d*-limit that is an element of* $\mathcal{H}(X)$.

Note that for compact X the condition on $(f_t)_{t\in T}$ can be replaced by: for every $\varepsilon > 0$ and $t \in T$ there is an immediate successor t' such that $\hat{d}(f_{t'},1) < \varepsilon$, where d is some fixed metric on X. This lemma is essentially due to Anderson [A2].

PROOF: Let d be an arbitrary admissible complete metric on X. Pick a t_0 in T with rank 0. Assume that a chain $t_0,t_1,\ldots,t_i$ has been chosen. Put $g_i = f_{t_i} \circ \ldots \circ f_{t_1} \circ f_{t_0}$ and define the metric d' on X by:

$$d'(x,y) = d(x,y) + d(g_i^{-1}(x),g_i^{-1}(y)).$$

Let t_{i+1} be an immediate successor of t_i such that $\hat{d}'(f_{t_{i+1}},1) < 2^{-i}$. It is easily verified that the sequence $(g_i)_{i=0}^{\infty}$ constructed in this way has the properties $\hat{d}(g_i,g_{i+1}) < 2^{-i}$ and $\hat{d}(g_i^{-1},g_{i+1}^{-1}) < 2^{-i}$ for $i = 0,1,2,\ldots$.

Since d is complete the uniform limits $g = \lim_{i\to\infty} g_i$ and $h = \lim_{i\to\infty} g_i^{-1}$ exist and are continuous. We have for $x \in X$:

$$d(h \circ g(x),x) = \lim_{i\to\infty} d(h \circ g_i(x),x) = \lim_{i\to\infty} d(h \circ g_i(x),g_i^{-1} \circ g_i(x)) \leq$$

$$\lim_{i\to\infty} \sum_{j=i}^{\infty} 2^{-i} = 0.$$

This means that $h \circ g = 1$. Analogously, one may show that $g \circ h = 1$ and the lemma is proved.

1.2. Negligibility and pseudo-boundaries

We introduce a triple $(X,\mathcal{S},\Gamma)$ that will remain fixed throughout this section. X is a topologically complete space and $(\mathcal{S},\Gamma)$ satisfies the following conditions:

(a) $\mathcal{S}$ is a collection of closed subsets of X,

(b) Γ is a subgroup of $\mathcal{H}(X)$,

(c) $\mathcal{S}$ is *hereditary*, i.e. every closed subset of a member of $\mathcal{S}$ is in $\mathcal{S}$,

(d) $\mathcal{S}$ is invariant under the action of Γ,

(e) There is an admissible metric d on X such that every $f \in \mathcal{H}(X)$ that is the uniform d-limit of a sequence in Γ belongs to Γ.

For convenience we shall call an object that satisfies (a) - (e) a Δ-*pair on* X. Observe that for compact X condition (e) is equivalent to: Γ is closed in the compact-open topology on $\mathcal{H}(X)$. Let $\mathcal{S}_\sigma$ denote the collection of all countable unions of members of $\mathcal{S}$.

1.2.1. DEFINITION: A subset S of X is called *negligible* if $X \approx X \backslash S$. The set S is called *strongly negligible* if for every collection $\mathcal{U}$ of open subsets of X there is a homeomorphism $f : X \to X \backslash (S \cap \cup\mathcal{U})$ that is $\mathcal{U}$-close to to 1.

Obviously, every (relatively) open subset of a strongly negligible set S is negligible; in particular, S itself is negligible. Every negligible subset of X is an F_σ-set. This can be seen as follows. If $X \backslash S \approx X$ then $X \backslash S$ is, like X, topologically complete. This implies that $X \backslash S$ is a G_δ-set in X and hence that S is an F_σ-set ([E1, 4.3.24]). It is also easily verified that a strongly negligible set is always a countable union of nowhere dense sets (indeed, it is a σ-Z-set, see section 3.1). We give more properties of strong negligibility.

1.2.2. PROPOSITION: *Every (relatively) closed subset of a strongly negligible set in* X *is strongly negligible.*

PROOF: Let S be strongly negligible in X and let F be a closed subset of S. There is an open W in X with $S \backslash W = F$. Consider a collection $\mathcal{U}$ of open subsets of X and select an open *star refinement* $\mathcal{V}$ of $\mathcal{U}$, i.e. $\cup\mathcal{V} = \cup\mathcal{U} = O$ and for every $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ such that every $V' \in \mathcal{V}$ that intersects V is contained in U. Since S is strongly negligible there exist homeomorphisms $f : X \to X \backslash (S \cap O)$ and $g : X \to X \backslash (S \cap O \cap W)$ such that f is $\mathcal{V}$-close to 1 and g is $\{V \cap W \mid V \in \mathcal{V}\}$-close to 1. Then $h = g^{-1} \circ f$ is a homeomorphism from X onto $X \backslash (F \cap O)$ which is $\mathcal{U}$-close to 1. This proves that F is strongly negligible in X.

1.2.3. THEOREM: *Strong negligibility is* σ*-additive.*

PROOF: As remarked above every negligible set is an F_σ-set. So proposition 1.2.2 reduces the problem to: if $(S_i)_{i\in\mathbb{N}}$ is a sequence of closed, strongly negligible subsets of X then $S = \bigcup_{i\in\mathbb{N}} S_i$ is strongly negligible.

Let $S_1, S_2, S_3, \ldots$ be all strongly negligible, closed subsets of X and let $\mathcal{U}$ be a collection of open subsets of X. We define $O_1 = \cup\mathcal{U}$ and $O_{i+1} = O_i \setminus S_i$ for $i \in \mathbb{N}$. Select a complete metric d on X and construct a complete metric d_1 on O_1 such that for every $x,y \in O_1$, $d_1(x,y) \geq d(x,y)$ and for some $U \in \mathcal{U}$, $\{z \in O_1 \mid d_1(z,x) < 1\} \subset U$ (see [E1:5.4.H]). Choose for every $i \in \mathbb{N}$ a complete metric d_{i+1} on O_{i+1} such that for $x,y \in O_{i+1}, d_{i+1}(x,y) \geq d_i(x,y)$. We shall construct inductively a sequence $f_1, f_2, f_3, \ldots$ such that for every $i \in \mathbb{N}$, f_i is a homeomorphism from X onto $X\setminus(S_i \cap O_i)$ that is supported on O_i. Since S_1 is strongly negligible there is a homeomorphism $f_1 : X \to X\setminus(S_1 \cap O_1)$ that is supported on O_1 and has the property $\hat{d}_1(f_1|O_1, 1) < \frac{1}{2}$.

Suppose that f_i has been constructed. It follows easily from the induction hypothesis that $g_i = f_i \circ \ldots \circ f_1$ is a homeomorphism from X onto $X\setminus((S_1 \cup \ldots \cup S_i) \cap O_1) = (X\setminus O_1) \cup O_{i+1}$. Define the metric d'_{i+1} on O_{i+1} by

$$d'_{i+1}(x,y) = d_{i+1}(x,y) + d(g_i^{-1}(x), g_i^{-1}(y))$$

and select a homeomorphism $f_{i+1} : X \to X\setminus(S_{i+1} \cap O_{i+1})$ that is supported on O_{i+1} and satisfies

$$\hat{d}'_{i+1}(f_{i+1}|O_{i+1}, 1) < 2^{-i-1}.$$

This completes the induction.

If $S = \bigcup_{i\in\mathbb{N}} S_i$ then $(g_i^{-1}|X\setminus(S \cap O_1))_{i\in\mathbb{N}}$ is a sequence of maps from $X\setminus(S \cap O_1)$ into X that satisfies:

$$\hat{d}(g_1^{-1}|X\setminus(S \cap O_1), 1) < \tfrac{1}{2}$$

and for $i \in \mathbb{N}$,

$$\hat{d}(g_{i+1}^{-1}|X\setminus(S \cap O_1), g_i^{-1}|X\setminus(S \cap O_1)) < 2^{-i-1}.$$

Since d is a complete metric $h = \lim_{i\to\infty} g_i^{-1}|X\setminus(S \cap O_1)$ is a continuous function from $X\setminus(S \cap O_1)$ into X.

Analogously, we can prove that $g = \lim_{i\to\infty} g_i$ is a map from X into X, which is obviously supported on O_1. Let $i \in \mathbb{N}$ and recall that $g_i(X) = (X\setminus O_1) \cup O_{i+1}$. Since $(g_{i+k}|O_{i+1})_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to the complete metric d_{i+1} we have that $g(X) \subset X\setminus O_1 \cup O_{i+1}$. This means that g is a map from X into $X\setminus(S \cap O_1)$. Since both h and g are uniform limits we have that $h \circ g = 1_X$ and $g \circ h = 1_{X\setminus(S \cap O_1)}$ and hence that g is a homeomorphism from X onto $X\setminus(S \cap O_1)$. Obviously, we have that $\hat{d}_1(g|O_1, 1) < 1$ and $g|X\setminus O_1 = 1$, which implies that g and 1 are $\mathcal{U}$-close.

1.2.4. COROLLARY: *A subset of a strongly negligible set* S *in* X *is (strongly) negligible in* X *iff it is an F_σ-set (in* X *or, equivalently, in* S*).*

PROOF: Use proposition 1.2.2, theorem 1.2.3 and the fact that every negligible set is an F_σ-set.

1.2.5. REMARK: One easily verifies that negligibility is neither closed hereditary nor additive (consider for instance the interval I). A more sophisticated counterexample is the space Y which is discussed in chapter 5. This space is universal for the class of separable metric spaces (corollary 5.3.6) and has the property that a compact subspace is negligible iff it has the shape of a finite space (theorems 5.5.4 and 5.5.5).

We now come to the pseudo-boundaries. The first to study this concept were Anderson [A4] and Bessaga & Pełczyński [BP1]. Their notion of a pseudo-boundary was generalized to arbitrary complete metric spaces by Toruńczyk [T1] (these pseudo-boundaries are called skeletoids) and differently by West [W] (called absorbers here). We shall now define these concepts.

1.2.6 DEFINITION: Let $\mathcal{U}$ be a collection of open subsets of a space Z and let $E \subset \mathcal{H}(Z)$. A map h is a $\mathcal{U}$-*push in* E if there is an isotopy $H : Z \times I \to Z$ that is limited by $\mathcal{U}$ and satisfies : $H_0 = 1$, $H_1 = h$ and $H_t \in E$ for every $t \in I$.

1.2.7 DEFINITION: An element A of $\mathcal{S}_\sigma$ is called an $(\mathcal{S},\Gamma)$-*absorber* if for every $S \in \mathcal{S}$ and every collection $\mathcal{U}$ of open subsets of X there is an $h \in \Gamma$ such that h is $\mathcal{U}$-close to 1 while moreover $h(S \cap \cup\mathcal{U}) \subset A$. If we can always choose h in such a way that it is a $\mathcal{U}$-push in Γ then A is an $(\mathcal{S},\Gamma)$-*absorber*$^{\frown}$.

1.2.8 DEFINITION: Let $A_1 \subset A_2 \subset A_3 \subset \ldots$ be a sequence of elements of $\mathcal{S}$. We call $(A_i)_{i \in \mathbb{N}}$ an $(\mathcal{S},\Gamma)$-*skeleton* ($(\mathcal{S},\Gamma)$-*skeleton*$^{\frown}$) if for every open covering $\mathcal{U}$ of X, every $S \in \mathcal{S}$ and every $n \in \mathbb{N}$ there exist an h in $\{\gamma \in \Gamma \mid \gamma|A_n = 1\}$ that is $\mathcal{U}$-close to 1 (a $\mathcal{U}$-push h in $\{\gamma \in \Gamma \mid \gamma|A_n = 1\}$) and an $m \in \mathbb{N}$ such that $h(S) \subset A_m$. The set $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{S}_\sigma$ is called an $(\mathcal{S},\Gamma)$-*skeletoid* ($(\mathcal{S},\Gamma)$-*skeletoid*$^{\frown}$).

Examples of pseudo-boundaries in the Hilbert cube can be found in section 3.1. We now introduce a concept that covers both absorber and skeletoid.

1.2.9 DEFINITION: Let $A_1 \subset A_2 \subset A_3 \subset \ldots$ be a sequence of elements of $\mathcal{S}$. We call $(A_i)_{i \in \mathbb{N}}$ a *strong* $(\mathcal{S},\Gamma)$*-skeleton (strong* $(\mathcal{S},\Gamma)$*-skeleton*$^{\sim}$*)* if for every open covering $\mathcal{U}$ of X, every $S \in \mathcal{S}$, every closed subset F of X with $F \cap S = \emptyset$ and every $n \in \mathbb{N}$ there exist an h in $\{\gamma \in \Gamma | \gamma|A_n \cup F = 1\}$ that is $\mathcal{U}$-close to 1 (a $\mathcal{U}$-push h in $\{\gamma \in \Gamma | \gamma|A_n \cup F = 1\}$) and an $m \in \mathbb{N}$ such that $h(S) \subset A_m$. The set $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{S}_\sigma$ is called a *strong* $(\mathcal{S},\Gamma)$*-skeletoid (strong* $(\mathcal{S},\Gamma)$*-skeletoid*$^{\sim}$*)*.

It is obvious that every strong skeletoid is a skeletoid. With absorbers there is the same connexion.

1.2.10 PROPOSIITON: *Every strong* $(\mathcal{S},\Gamma)$*-skeletoid* $(\sim)$ *is an* $(\mathcal{S},\Gamma)$*-absorber* $(\sim)$.

PROOF: We only prove the proposition for plain strong skeletoids and absorbers; the version with the $\sim$ is completely analogous. Let $(A_i)_{i \in \mathbb{N}}$ be a strong $(\mathcal{S},\Gamma)$-skeleton and put $A = \bigcup_{i \in \mathbb{N}} A_i$. Assume that $\mathcal{U}$ is a collection of open subsets of X and that S is an element of $\mathcal{S}$. Put $O = \bigcup\mathcal{U}$ and select an admissible metric d on O such that $\{U_1(x) | x \in O\}$ refines $\mathcal{U}$, where $U_\varepsilon(x) = \{y \in O | d(y,x) < \varepsilon\}$ for $\varepsilon \geq 0$ and $x \in O$ ([E1 : 5.4.H]). Let $S_0 \subset S_1 \subset S_2 \subset \ldots$ be a sequence of closed subsets of S such that $S_0 = \emptyset$ and $S \cap O = \bigcup_{i=0}^{\infty} S_i$. We shall construct inductively sequences $f_0, f_1, f_2, \ldots$ in Γ and $n_0 < n_1 < n_2 < \ldots$ in $\mathbb{N}$ such that for $i = 1,2,3,\ldots$

$$f_i \circ f_{i-1} \circ \ldots \circ f_0(S_i) \subset A_{n_i}$$

and

$$f_i \text{ is supported on } O \setminus A_{n_{i-1}}.$$

Put $f_0 = 1_X$ and $n_0 = 1$. We shall make sure that every f_i can be chosen arbitrarily close to 1. This implies with lemma 1.1.2 that we may assume that there is an $f \in \mathcal{H}(X)$ which is the uniform d'-limit of $(f_i \circ \ldots \circ f_0)_{i \in \mathbb{N}}$, where d' is a metric on X such that Γ is closed with respect to $\hat{d}'$. So we may assume that $f = \lim_{i \to \infty} f_i \circ \ldots \circ f_0$ is an element of Γ. The other properties that f must satisfy follow easily. We have that f is supported on O and $\hat{d}(f|O,1) \leq \sum_{i=1}^{\infty} \hat{d}(f_i|O,1) < 1$ which means that f and 1 are $\mathcal{U}$-close. Moreover, $f(S \cap O) = \bigcup_{i=1}^{\infty} f(S_i) = \bigcup_{i=1}^{\infty} f_i \circ \ldots \circ f_0(S_i) = \bigcup_{i=1}^{\infty} A_{n_i} \subset A$ and we may conclude that A is an $(\mathcal{S},\Gamma)$-absorber.

It remains to perform the induction. Assume that f_i has been chosen. Let F be a closed neighbourhood of $X \backslash O$ such that $F \cap f_i \circ \ldots \circ f_0(S_{i+1}) = \emptyset$ and in order to show that the f_{i+1} we are about to determine can be chosen arbitrarily close to 1 let $\mathcal{V}$ be an open covering of X that refines $\{\mathrm{Int}_X(F)\} \cup \{U_{2^{-i-2}}(x) \mid x \in O\}$. Since $f_i \circ \ldots \circ f_0\ (S_{i+1})$ is a member of $\mathcal{S}$ there exist an $f \in \Gamma$ and an $n_{i+1} > n_i$ such that $f|F \cup A_{n_i} = 1$, $f_{i+1} \circ f_i \circ \ldots \circ f_0(S_{n+1}) \subset A_{n_{i+1}}$ and f_{i+1} and 1 are $\mathcal{V}$-close. This implies that $\hat{d}(f_{i+1}|O,1) < 2^{-i-1}$ and that f_{i+1} is supported on $O \backslash A_{n_i}$. The proof is completed.

Observe that if $f \in \Gamma$ and A is for instance an $(\mathcal{S},\Gamma)$-absorber then f(A) is also an $(\mathcal{S},\Gamma)$-absorber. Conversely, we have the uniqueness theorem for absorbers:

1.2.11 THEOREM (West [W]): *If* A *and* B *are* $(\mathcal{S},\Gamma)$*-absorbers* $(\tilde{\ })$ *then for every collection* $\mathcal{U}$ *of open subsets of* X *there is an* $f \in \Gamma$ *that is* $\mathcal{U}$*-close to* 1 *(a* $\mathcal{U}$*-push* f *in* Γ*) with* $f(A \cap \bigcup\mathcal{U}) = B \cap \bigcup\mathcal{U}$.

PROOF: Again we only prove the theorem for plain absorbers. Let A and

B be (S,Γ)-absorbers and let $\mathcal{U}$ be a collection of open subsets of X. Put $O = \cup\mathcal{U}$ and write $A = \bigcup_{i\in\mathbb{N}} A_i$ and $B = \bigcup_{i\in\mathbb{N}} B_i$, where $A_1 = B_1 = \emptyset$ and for $i \in \mathbb{N}$, $A_i, B_i \in S$. Select a metric d on O such that the open 1-balls of d form a refinement of $\mathcal{U}$. We construct a sequence $f_1, f_2, f_3, \ldots$ in Γ such that for $i \in \mathbb{N}$:

$$f_i \text{ is supported on } O,$$

$$\hat{d}(f_i|O,1) < 2^{-i},$$

$$f_i \circ g_{i-1}(A_i \cap O) \subset B \cap O,$$

$$B_i \cap O \subset f_i \circ g_{i-1}(A \cap O)$$

and

$$f_i|\bigcup_{j=1}^{i-1} (g_{i-1}(A_j) \cup B_j) = 1,$$

where $g_{i-1} = f_{i-1} \circ \ldots \circ f_1$. We put $f_1 = 1_X$.

Assume that $f_1, \ldots, f_i$ have been selected. Then $g_i(A_{i+1}) = f_i \circ \ldots \circ f_1(A_{i+1})$ is an element of S. It follows from the induction hypothesis that $\bigcup_{j=1}^{i} (g_i(A_j) \cup B_j) \cap O \subset B$. Consequently, there is a $\beta \in \Gamma$ that is supported on $O \backslash \bigcup_{j=1}^{i} (g_i(A_j) \cup B_j)$ and that satisfies $\hat{d}(\beta|O,1) < 2^{-i-2}$ and $\beta(g_i(A_{i+1}) \cap O) \subset B \cap O$. Note that since $\beta \circ g_i \in \Gamma$, $\beta \circ g_i(A)$ is an (S,Γ)-absorber and that $(\bigcup_{j=1}^{i}(g_i(A_j) \cup B_j) \cup \beta \circ g_i(A_{i+1})) \cap O$ is contained in $\beta \circ g_i(A)$. This implies that there is a $\gamma \in \Gamma$ such that γ is supported on $O \backslash (\bigcup_{j=1}^{i} (g_i(A_j) \cup B_j) \cup \beta \circ g_i(A_{i+1}))$, $\hat{d}(\gamma|O,1) < 2^{-i-2}$ and $\gamma(B_{i+1} \cap O) \subset \beta \circ g_i(A) \cap O$. Define $f_{i+1} = \gamma^{-1} \circ \beta$. The map f_{i+1} is obviously supported on O and has the property $\hat{d}(f_{i+1}|O,1) < 2^{-i-1}$. Consider the inclusion

$$f_{i+1} \circ g_i(A_{i+1} \cap O) = \gamma^{-1} \circ \beta(g_i(A_{i+1}) \cap O) =$$

$$= \beta(g_i(A_{i+1}) \cap O) \subset B \cap O$$

and observe that $\gamma(B_i \cap O) \subset \beta \circ g_i(A) \cap O$, whence $B_i \cap O$ is contained in $f_{i+1} \circ g_i(A)$. It is obvious that f_{i+1} restricts to the identity on $\bigcup_{j=1}^{i} (g_i(A_j) \cup B_j)$. This completes the induction.

Observe that every f_i could have been chosen arbitrarily close to 1. Hence, we may assume in view of lemma 1.1.2 that $g = \lim_{i\to\infty} g_i \in \Gamma$. We have that g is supported on O and that

$$\hat{d}(g|O,1) \le \sum_{i=1}^{\infty} \hat{d}(f_i|O,1) < \sum_{i=1}^{\infty} 2^{-i} = 1.$$

This means that g and 1 are $\mathcal{U}$-close. The sets $g(A \cap O)$ and $B \cap O$ coincide because

$$g(A \cap O) = \bigcup_{i\in\mathbb{N}} g(A_i \cap O) = \bigcup_{i\in\mathbb{N}} g_i(A_i \cap O) \subset B \cap O$$

and

$$B \cap O = \bigcup_{i\in\mathbb{N}} B_i \cap O = \bigcup_{i\in\mathbb{N}} g \circ g_i^{-1}(B_i \cap O) \subset g(A \cap O).$$

This proves the theorem.

The same statement could of course have been made about strong skeletoids. For skeletoids a similar result can be obtained (see Bessaga & Pełczyński [BP2 : ch.VI prop.2.2]). We now give the obvious connexion between absorbers and strong negligibility.

1.2.12 THEOREM: *If* A *is an* $(\mathcal{S},\Gamma)$-*absorber and* S *is an element of* $\mathcal{S}_\sigma$ *then* S\A *is strongly negligible in* X\A.

PROOF: Let A be an $(\mathcal{S},\Gamma)$-absorber and let $S \in \mathcal{S}_\sigma$. It is trivial that $A \cup S$ is also an $(\mathcal{S},\Gamma)$-absorber. Let $\mathcal{U}$ be a collection of open subsets of

$X\backslash A$ and construct a collection $\mathcal{U}'$ of open subsets of X such that $\mathcal{U} = \{U\backslash A \mid U \in \mathcal{U}'\}$. Let f be an element of Γ that is $\mathcal{U}'$-close to 1 and that has the property $f(A \cap \cup\mathcal{U}') = (A \cup S) \cap \cup\mathcal{U}'$. Then $f|X\backslash A$ is a homeomorphism from $X\backslash A$ onto $(X\backslash A)\backslash((S\backslash A) \cap \cup\mathcal{U})$ that is $\mathcal{U}$-close to 1.

The next theorem shows that when we omit an absorber the homogeneity properties of the space are preserved.

1.2.13 THEOREM: *Let* A *be an* (S,Γ)*-absorber and let* $\mathcal{U}$ *be a collection of open subsets of* X*. Assume that* f *is an element of* Γ *that is* $\mathcal{U}$*-close to* 1 *and that* F *is a closed subset of* X *with the property that both* F *and* f(F) *are contained in* $X\backslash A$*. Then* $f|F$ *can be extended to an* $h \in \Gamma$ *that is* $\mathcal{U}$*-close to* 1 *and that satisfies* $h|X\backslash A \in \mathcal{H}(X\backslash A)$.

PROOF: Put $O = \cup\mathcal{U}$ and define $\mathcal{V} = \{U \cap f^{-1}(U) \mid U \in \mathcal{U}\}$. Since f and 1 are $\mathcal{U}$-close $\mathcal{V}$ is an open covering of O. Since $f \in \Gamma$, $f^{-1}(A)$ is an (S,Γ)-absorber. Note that F is disjoint from both A and $f^{-1}(A)$. Using theorem 1.2.11 we find a $g \in \Gamma$ that is $\{V\backslash F \mid V \in \mathcal{V}\}$-close to 1, while $g(A \cap O) = f^{-1}(A) \cap O$. Let $h = f \circ g$ and note that $h \in \Gamma$. We have the following situation:

$$h(A) = f \circ g((A \cap O) \cup (A\backslash O)) = f \circ g(A \cap O) \cup f \circ g(A\backslash O)$$

$$= f(f^{-1}(A) \cap O) \cup A\backslash O = (A \cap O) \cup (A\backslash O) = A,$$

$$h|F = f \circ g|F = f|F$$

and

$$h|X\backslash O = 1.$$

If $x \in O$ then there is a $U \in \mathcal{U}$ such that $\{x,g(x)\} \subset U \cap f^{-1}(U)$ and hence

$\{x, f\circ g(x)\} \subset U$. We conclude that h is $\mathcal{U}$-close to 1.

1.2.14 COROLLARY: *If* A *is an* $(\mathcal{S},\Gamma)$*-absorber and* Γ *is such that it makes* X *homogeneous, i.e.* $X = \{\gamma(x) \mid \gamma \in \Gamma\}$ *for any* $x \in X$, *then* $X\backslash A$ *is also homogeneous.*

PROOF: This is trivial.

1.2.15 REMARKS: The concepts we discussed in this section can of course also be defined for non-complete spaces. However, since we then do not have a convergence criterion like lemma 1.1.2 at our disposal this generalization is of limited interest.

The concepts absorber and absorber$^{\sim}$ (or skeletoid and skeletoid$^{\sim}$ etc.) do not coincide. In section 5.3 we discuss a space X_0 with the property that $f,g \in \mathcal{H}(X_0)$ are isotopic iff $f = g$ (remark 5.3.5). This space is, however, homogeneous in a very strong sense (theorem 5.3.3) which implies that every countable, dense subset is a strong $(\mathcal{S}_f,\mathcal{H}(X_0))$-skeletoid, where $\mathcal{S}_f$ is the collection of finite subsets of X_0.

In section 3.1 we give a Δ-pair $(\mathcal{S},\mathcal{H}(Q))$ on the Hilbert cube such that there exists an $(\mathcal{S},\mathcal{H}(Q))$-absorber$^{\sim}$ but no $(\mathcal{S},\mathcal{H}(Q))$-skeletoid.

CHAPTER 2

FINITE DIMENSIONAL SPACES

This chapter is devoted to the construction of k-dimensional skeletoids in I^n and $\mathbb{R}^n$.

2.1 Tame compacta in $\mathbb{R}^n$ and I^n

In their papers [GS1,GS2] Geoghegan & Summerhill have introduced the collection $\mathfrak{M}^n_k$ of "tame" $\leq k$-dimensional compacta in $\mathbb{R}^n$. We shall define this object and discuss its properties and those of the corresponding collection in the n-cube. Let n and k be integers with the properties $n \geq 1$ and $0 \leq k \leq n$. *The numbers* n *and* k *remain fixed throughout this chapter.* We begin with some terminology.

Let X be a subspace of $\mathbb{R}^n$. A *subpolyhedron of* X is a subset of X that is the underlying set of a countable, locally finite simplicial complex in $\mathbb{R}^n$. A subset P of X is called a *tame polyhedron* if there is an $h \in \mathcal{H}(X)$ such that h(P) is a subpolyhedron of X.

2.1.1 DEFINITION: $\mathfrak{M}^n_k$ consists of all compact subsets S of $\mathbb{R}^n$ that satisfy the following property: if P is a subpolyhedron of $\mathbb{R}^n$ with dimension $\leq n - k - 1$ and $\mathcal{U}$ is a collection of open subsets of $\mathbb{R}^n$ that covers $S \cap P$ then there exists a $\mathcal{U}$-push h in $\mathcal{H}(\mathbb{R}^n)$ with $h(S) \cap P = \emptyset$.

$\widetilde{\mathfrak{M}}^n_k$ consists of all compact subsets S of I^n that satisfy the following property: if P is a subpolyhedron of I^n with $\dim(P) \leq n-k-1$ and $\dim(P \cap \partial I^n) < n-k-1$ and $\mathcal{U}$ is a collection of open subsets of I^n that covers $S \cap P$ then there exists a $\mathcal{U}$-push h in $\mathcal{H}(I^n)$ with $h(S) \cap P = \emptyset$.

One sees immediately that $\mathfrak{M}^n_k$ and $\widetilde{\mathfrak{M}}^n_k$ are invariant under PL-homeomorphisms. If P is a $\leq$ k-dimensional subpolyhedron of $\mathbb{R}^n(I^n)$ then by a general position argument we find that $P \in \mathfrak{M}^n_k$ $(P \in \widetilde{\mathfrak{M}}^n_k)$. For information concerning PL-topology see Hudson [H]. The following theorem has been obtained by Geoghegan & Summerhill [GS2].

2.1.2 THEOREM: *$\mathfrak{M}^n_k$ is invariant under the action of $\mathcal{H}(\mathbb{R}^n)$.*

We shall see that an analogous statement can be derived for $\widetilde{\mathfrak{M}}^n_k$.

2.1.3 LEMMA: *If $k \leq n-2$, $x \in \partial I^n$ and $f : \mathbb{R}^{n-1} \to \partial I^n \setminus \{x\}$ is a homeomorphism then for every $S \subset \mathbb{R}^{n-1}$, $S \in \mathfrak{M}^{n-1}_k$ iff $f(S) \in \widetilde{\mathfrak{M}}^n_k$.*

PROOF: Prove the lemma first for a PL-homeomorphism f and use then the invariance of $\mathfrak{M}^{n-1}_k$. The details are left to the reader.

2.1.4 LEMMA: *If S is a subset of Int I^n then it is an element of $\widetilde{\mathfrak{M}}^n_k$ iff it is in $\mathfrak{M}^n_k$.*

PROOF: This is obvious.

2.1.5 LEMMA: *$\mathfrak{M}^n_k$ and $\widetilde{\mathfrak{M}}^n_k$ are hereditary.*

PROOF: We give the proof for $\mathfrak{M}_k^n$. Let S' be a closed subset of an element S of $\mathfrak{M}_k^n$. Assume that P is a subpolyhedron of $\mathbb{R}^n$ with dimension $\leq n-k-1$ and that $\mathcal{U}$ is a collection of open subsets of $\mathbb{R}^n$ that covers $S' \cap P$. Write P as union of two subpolyhedra P_1 and P_2 that satisfy $P_1 \subset \cup\mathcal{U}$ and $P_2 \cap S' = \emptyset$. Let h be a $\{U\backslash P_2 | U \in \mathcal{U}\}$-push in $\mathcal{H}(\mathbb{R}^n)$ with $h(S) \cap P_1 = \emptyset$. We have that $h(S') \cap P = (h(S') \cap P_1) \cup (h(S') \cap P_2) \subset (h(S) \cap P_1) \cup h(S' \cap P_2) = \emptyset$ and hence the lemma is proved.

2.1.6 PROPOSITION: *$\widetilde{\mathfrak{M}}_k^n$ is invariant under the action of $\mathcal{H}(I^n)$.*

PROOF: Let $S \in \widetilde{\mathfrak{M}}_k^n$, $f \in \mathcal{H}(I^n)$, let P be a subpolyhedron of I^n with $\dim(P) \leq n-k-1$ and $\dim(P \cap \partial I^n) \leq n-k-2$ and let $\mathcal{U}$ be an open covering of $P \cap f(S)$ in I^n. We first show that $f(S) \cap \partial I^n \in \widetilde{\mathfrak{M}}_k^n$. If $k = n-1$ then every closed subset of ∂I^n is an element of $\widetilde{\mathfrak{M}}_k^n$. If $k < n-1$ then there is an $x \in \partial I^n \backslash S$. Since $\widetilde{\mathfrak{M}}_k^n$ is invariant under PL-homeomorphisms we may assume that f fixes x. Let $g : \mathbb{R}^{n-1} \to \partial I^n \backslash \{x\}$ be a homeomorphism. Applying lemma 2.1.5, lemma 2.1.3, theorem 2.1.2 and again lemma 2.1.3 we find successively that $S \cap \partial I^n \in \widetilde{\mathfrak{M}}_k^n$, $g^{-1}(S \cap \partial I^n) \in \mathfrak{M}_k^{n-1}$, $g^{-1} \circ f(S \cap \partial I^n) \in \mathfrak{M}_k^{n-1}$ and $f(S \cap \partial I^n) \in \widetilde{\mathfrak{M}}_k^n$.

Let $\mathcal{V}$ be a star refinement of $\mathcal{U}$. There is a $\mathcal{V}$-push h_1 in $\mathcal{H}(I^n)$ with $h_1 \circ f(S \cap \partial I^n) \cap P = \emptyset$. Select an $i \in \mathbb{N}$ such that $h_1 \circ f(S) \cap P \subset O = (1/i, 1-1/i)^n$. Put $C = f^{-1} \circ h_1^{-1}(\mathrm{Cl}_{I^n}(O)) \cap S$ and note that lemma 2.1.5 implies that $C \in \widetilde{\mathfrak{M}}_k^n$. Since $C \subset \mathrm{Int}\, I^n$ we have that $C \in \mathfrak{M}_k^n$, lemma 2.1.4. Since $h_1 \circ f$ can be extended to an element of $\mathcal{H}(\mathbb{R}^n)$ theorem 2.1.2 implies that $h_1 \circ f(C) \in \mathfrak{M}_k^n$. By virtue of lemma 2.1.4 we have that $h_1 \circ f(C) \in \widetilde{\mathfrak{M}}_k^n$. So there is a $\{V \cap O | V \in \mathcal{V}\}$-push h_2 in $\mathcal{H}(I^n)$ such that $h_2 \circ h_1 \circ f(C) \cap P = \emptyset$. This means that $h_2 \circ h_1$ is a $\mathcal{U}$-push in $\mathcal{H}(I^n)$ with $h_2 \circ h_1 \circ f(S) \cap P = \emptyset$.

The following propositions are essentially due to Geoghegan & Summerhill

[GS2]. For the sake of completeness, we have included proofs.

2.1.7 PROPOSITION: *Let* S *be an element of* $\mathfrak{M}_k^n(\tilde{\mathfrak{M}}_k^n)$, *let* $\mathcal{U}$ *be a collection of open subsets of* $\mathbb{R}^n(I^n)$ *and let* $\mathcal{L}$ *be a countable collection of tame polyhedra in* $\mathbb{R}^n(I^n)$ *having dimension* $\leq n-k-1$ *(for* I^n *in addition:* $\dim(\cup\mathcal{L} \cap \partial I^n) \leq n-k-2$). *Then there exists a* $\mathcal{U}$*-push* h *in* $H(\mathbb{R}^n)$ $(H(I^n))$ *such that* $h(X) \cap \cup\mathcal{L} \cap \cup\mathcal{U} = \emptyset$.

PROOF: We prove the proposition for $\mathbb{R}^n$. Put $O = \cup\mathcal{U}$ and write $O \cap \cup\mathcal{L}$ as countable union of tame polyhedra with dimension $\leq n-k-1$: $O \cap \cup\mathcal{L} = \bigcup_{i\in\mathbb{N}} T_i$. Let d be a metric on O such that the 1-balls form a refinement of $\mathcal{U}$. Put $T_0 = \emptyset$. We shall construct inductively a sequence $G^0, G^1, G^2, \ldots$ of isotopies: $\mathbb{R}^n \times I \to \mathbb{R}^n \times I$ such that for $i = 0,1,2,\ldots$

$$G_0^i = 1,$$

$$G_t^i \text{ is supported on } O\setminus\bigcup_{j=1}^{i-1} T_j \text{ for } t \in I,$$

$$\hat{d}(G_t^i|O, 1) < 2^{-i} \text{ for } t \in I$$

and

$$H_1^i(S) \cap T_i = \emptyset,$$

where $H^i = G^i \circ \ldots \circ G^0$. Put $G^0 = 1_{\mathbb{R}^n \times I}$.

If every G^i is chosen close enough to $1_{\mathbb{R}^n \times I}$ then $H = \lim_{i\to\infty} H^i$ is an isotopy of $\mathbb{R}^n$, lemma 1.1.2. It follows easily from the induction hypothesis that H is limited by $\mathcal{U}$ and that $H_1(S) \cap \cup\mathcal{L} \cap O = \emptyset$.

Assume that G^i has been constructed. Let $f \in H(\mathbb{R}^n)$ be such that $f(T_i)$ is a subpolyhedron of $\mathbb{R}^n$. It is a consequence of the induction hypothesis that $f \circ H_1^i(S) \cap f(\bigcup_{j=0}^{i} T_j) = \emptyset$. Since $\mathfrak{M}_k^n$ is invariant we have that

$f \circ H_1^i(S) \in \mathfrak{M}_k^n$. Consequently, there is an isotopy F of $\mathbb{R}^n$ such that $F_0 = 1$, $F_1 \circ f \circ H_1^i(S) \cap f(T_{i+1}) = \emptyset$ and for every $t \in I$, F_t is supported on $f(0 \setminus \bigcup_{j=0}^{i} T_j)$ and $\hat{d}(f^{-1} \circ F_t | f(0), f^{-1} | f(0)) < 2^{-i-1}$. Define the isotopy G^{i+1}: $\mathbb{R}^n \times I \to \mathbb{R}^n \times I$ by $G_t^{i+1} = f^{-1} \circ F_t \circ f$ for $t \in I$. It is clear that G^{i+1} satisfies the induction hypothesis.

2.1.8 PROPOSITION: *If* S *is a compact element of* $\mathfrak{M}_{k\sigma}^n(\tilde{\mathfrak{M}}_{k\sigma}^n)$ *then* S *is an element of* $\mathfrak{M}_k^n(\tilde{\mathfrak{M}}_k^n)$.

PROOF: Consider a compact $S \in \mathfrak{M}_{k\sigma}^n$. Write $S = \bigcup_{i \in \mathbb{N}} S_i$ where each S_i is in $\mathfrak{M}_k^n$ and let P be an $(n-k-1)$-dimensional subpolyhedron of $\mathbb{R}^n$. Let h_1 push S_1 off P. Since $\mathfrak{M}_k^n$ is invariant we have that $h_1(S_2) \in \mathfrak{M}_k^n$. So we can push $h_1(S_2)$ away from P keeping $h_1(S_1)$ fixed. Continue this process. For the epsilonics see the very similar proof of proposition 1.2.10.

Note that lemma 2.1.4, theorem 2.1.2 and proposition 2.1.6 state that $(\mathfrak{M}_k^n, H(\mathbb{R}))$ and $(\tilde{\mathfrak{M}}_k^n, H(I^n))$ are Δ-pairs.

We now introduce a cell structure on I^l for $l \in \mathbb{N}$. If $i \in \{0\} \cup \mathbb{N}$ then $\mathcal{J}_i^l$ is the collection of all cubes in I^l that have the form

$$\prod_{j=1}^{l} [m_j 3^{-i}, (m_j + 1)3^{-i}],$$

where $m_1, m_2, \dots, m_l$ are elements of $\{0, 1, \dots, 3^i - 1\}$. Define furthermore for $i \in \{0\} \cup \mathbb{N}$,

$$K_i = \{\frac{2m+1}{2.3^i} | m \in \{0, 1, \dots, 3^i - 1\}\}$$

and $K = \bigcup_{i=0}^{\infty} K_i$. Note that $K_0 \subset K_1 \subset K_2 \subset \dots$ and that the l-fold product $(K_i)^l$ is the set of centres of members of $\mathcal{J}_i^l$. Let d_l be the maximum metric on $\mathbb{R}^l$ and let $U_\varepsilon^l(\tilde{U}_\varepsilon^l)$ denote the ε-balls in $\mathbb{R}^l(I^l)$ that correspond with d_l.

Let $P_n(\tilde{P}_n)$ be the subgroup of $H(\mathbb{R}^n)(H(I^n))$ that corresponds to permutating the n coordinates. We define the *Menger space* M_k^n by

$$M_k^n = I^n \setminus \cup\{U^n_{\frac{1}{2}3^{-i-1}}(\alpha(\{p\} \times I^{n-k-1}))\,|\,\alpha \in \tilde{P}_n,$$

$$i \in \{0\} \cup \mathbb{N} \text{ and } p \in (K_i)^{k+1}\}.$$

It was proved by Štan'ko [Š] that M_k^n is universal for the k-dimensional compact subsets of $\mathbb{R}^n$. The following fact has been obtained by Geoghegan & Summerhill [GS2]:

2.1.9 PROPOSITION: $M_k^n \in \mathfrak{M}_k^n$.

2.1.10 DEFINITION: If A is a countable dense subset of $\mathbb{R}$ then the *Nöbeling space* N_k^n is the set of all points in $\mathbb{R}^n$ for which at most k coordinates are elements of A. If A is a countable dense subset of (0,1) then $\tilde{N}_k^n(A)$ is the set of all points in I^n for which at most k coordinates are in A. We put $N_k^n = N_k^n(\mathbb{Q})$ and $\tilde{N}_k^n = \tilde{N}_k^n(\mathbb{Q} \cap (0,1))$.

2.1.11 REMARKS: We have the following alternative definitions of N_k^n and $\tilde{N}_k^n$:

$$N_k^n = \mathbb{R}^n \setminus \cup\{\alpha(\{p\} \times \mathbb{R}^{n-k-1})\,|\,\alpha \in P_n \text{ and } p \in \mathbb{Q}^{k+1}\}$$

and

$$\tilde{N}_k^n = I^n \setminus \cup\{\alpha(\{p\} \times I^{n-k-1})\,|\,\alpha \in \tilde{P}_n \text{ and } p \in (\mathbb{Q} \cap (0,1))^{k+1}\}.$$

It is obvious that if A is countable and dense in $\mathbb{R}$ (in (0,1)) then there is an $h \in H(\mathbb{R}^n)$ $(H(I^n))$ such that $h(N_k^n) = N_k^n(A)$ $(h(\tilde{N}_k^n) = \tilde{N}_k^n(A))$. It is known that N_k^n and $\tilde{N}_k^n$ are k-dimensional spaces, see [E2 : 1.5.9].

2.1.12 THEOREM: *If* A *is a countable dense subset of* $\mathbb{R}$ *then*

$$\mathfrak{M}^n_k = \{f(S) \mid f \in H(\mathbb{R}^n) \text{ and } S \text{ compact} \subset N^n_k(A)\}.$$

If A *is a countable dense subset of* (0,1) *then*

$$\widetilde{\mathfrak{M}}^n_k = \{f(S) \mid f \in H(I^n) \text{ and } S \text{ compact} \subset \widetilde{N}^n_k(A)\}.$$

PROOF: In view of 2.1.11 it suffices to prove the theorem for $A = \mathbb{Q}$ respectively $A = \mathbb{Q} \cap (0,1)$. The inclusion $\mathfrak{M}^n_k \subset \{f(S) \mid f \in H(\mathbb{R}^n)$ and S compact $\subset N^n_k\}$ is a consequence of 2.1.7 and 2.1.11. For I^n the same argument applies.

Consider now Bothe's theorem (see Bothe [Be] or [E2 : 1.11.6]) that every compact subset S of N^n_k can be embedded into M^n_k by an $f \in H(\mathbb{R}^n)$. If we combine this result with 2.1.2, 2.1.5 and 2.1.9 we have proved the theorem for $\mathbb{R}^n$.

Let $f \in H(I^n)$ and let S be a compact subset of $\widetilde{N}^n_k$. Define for every $i \in \mathbb{N}$, $S_i = S \cap [2^{-i}, 1-2^{-i}]^n$. If we prove that every element of

$$\{S_i \mid i \in \mathbb{N}\} \cup \{S \cap F \mid F \text{ an } (n-1)\text{-face of } I^n\}$$

is in $\widetilde{\mathfrak{M}}^n_k$ then the propositions 2.1.6 and 2.1.7 imply that $f(S) \in \widetilde{\mathfrak{M}}^n_k$. For every $i \in \mathbb{N}$ we have that $S_i \subset N^n_k$ and hence that $S_i \in \mathfrak{M}^n_k$. This means that $S_i \in \widetilde{\mathfrak{M}}^n_k$. Let F be an $(n-1)$-face of I^n and let $x \in \partial I^n \setminus F$. If $k = n-1$ then every closed subset of ∂I^n is in $\widetilde{\mathfrak{M}}^n_k$ and we are done. If $k < n-1$ select a homeomorphism $h : \partial I^n \setminus \{x\} \to \mathbb{R}^{n-1}$ such that $h(S \cap F) \subset N^n_k(\mathbb{Q} \setminus \{0,1\})$. Then $h(S \cap F) \in \mathfrak{M}^{n-1}_k$ and hence $S \cap F \in \widetilde{\mathfrak{M}}^n_k$. This completes the proof.

2.1.13 COROLLARY: *Every* $S \in \mathfrak{M}^n_k$ $(\widetilde{\mathfrak{M}}^n_k)$ *has dimension* $\leq k$.

PROOF: $\dim (N^n_k) = \dim (\tilde{N}^n_k) = k$, see [E2 : 1.5.9].

2.1.14 COROLLARY: *If* $S \in \mathfrak{M}^n_k$ $(\tilde{\mathfrak{M}}^n_k)$ *and* $S' \in \mathfrak{M}^{n'}_{k'}$ $(\tilde{\mathfrak{M}}^{n'}_{k'})$ *then* $S \times S' \in \mathfrak{M}^{n+n'}_{k+k'}$ $(\tilde{\mathfrak{M}}^{n+n'}_{k+k'})$.

PROOF: There exists an $f \in H(\mathbb{R}^n)$ and an $f' \in H(\mathbb{R}^{n'})$ such that $f(S) \subset N^n_k$ and $f(S') \subset N^{n'}_{k'}$. Consequently, one has that $f \times g(S \times S') \subset N^n_k \times N^{n'}_{k'} \subset N^{n+n'}_{k+k'}$.

2.2 Skeletoids in I^n

In this section we prove that $(\tilde{\mathfrak{M}}^n_k, H(I^n))$-skeletoids exist. Our construction of a skeleton is based on the space M^n_k, which was introduced by Menger [M] and which we modify slightly.

Consider the following collection of $(n-k-1)$-dimensional planes in I^n:

$$L = \{\alpha(\{p\} \times I^{n-k-1}) \mid p \in K^{k+1} \text{ and } \alpha \in \tilde{P}_n\}.$$

Select an enumeration $(L_i)_{i=0}^{\infty}$ of L such that if $L_i = \alpha(\{p\} \times I^{n-k-1})$ then $p \in (K_i)^{k+1}$. Define for $m \in \mathbb{N}$ and $i \in \{0\} \cup \mathbb{N}$ the compact sets

$$F^m_0 = I^m,$$

$$F^m_{i+1} = F^m_i \setminus \tilde{U}^n_{\frac{1}{2}3^{-i-m}}(L_i)$$

and $A_m = \bigcap_{i=0}^{\infty} F^m_i$. It is easily seen that F^m_i can be written as union of members of J^n_{i+m-1}. We obviously have the following situation:

$$F^1_i \subset F^2_i \subset F^3_i \subset \dots$$

and

$$A_1 \subset A_2 \subset A_3 \subset \dots .$$

Note that K is a countable, dense subset of (0,1) and that $\tilde{N}^n_k(K) = I^n \setminus \cup L$. This implies in view of theorem 2.1.12 that every A_i is a member of $\tilde{\mathfrak{M}}^n_k$.

2.2.1 THEOREM: $(A_m)_{m \in \mathbb{N}}$ *is a strong* $(\tilde{\mathfrak{M}}^n_k, H(I^n))$*-skeleton*.

The remaining part of this section is devoted to the proof of this theorem. Before we start with the actual proof we introduce some pushes of $\mathbb{R}^{k+1}$ and I^{k+1}.

Let $\varepsilon \in (0,1/3]$ and define $\varphi_\varepsilon : [0,\infty) \to [1,\infty)$ by

$$\varphi_\varepsilon(r) = \begin{cases} \dfrac{1}{3\varepsilon} & \text{if } 0 \le r \le \varepsilon, \\[2ex] \dfrac{1}{3(1-\varepsilon)}\left(2 + \dfrac{1-3\varepsilon}{r}\right) & \text{if } \varepsilon \le r \le 1, \\[2ex] 1 & \text{if } r \ge 1. \end{cases}$$

Note that that the function $f(r) = r\varphi_\varepsilon(r)$, $r \in [0,\infty)$, is a PL-autohomeomorphism of $[0,\infty)$ with the property $f([0,\varepsilon)) = [0,1/3)$. Using the vector space structure of $\mathbb{R}^{k+1}$ we define for $\varepsilon \in (0,1/3]$ the homeomorphism $\chi_\varepsilon \in H(\mathbb{R}^{k+1})$ by

$$\chi_\varepsilon(x) = \varphi_\varepsilon(d_{k+1}(x,0))x.$$

Note that χ_ε is supported on $U^{k+1}_1(0)$ and satisfies

$$\chi_\varepsilon(U^{k+1}_\varepsilon(0)) = U^{k+1}_{1/3}(0).$$

Section 2.4 is devoted to a proof for the statement:

$$\text{for } x,y \in \mathbb{R}^{k+1},\ d_{k+1}(\chi_\varepsilon(x),\chi_\varepsilon(y)) \geq \frac{2}{3}\, d_{k+1}(x,y).$$

Since $\chi_{1/3} = 1_{\mathbb{R}^{k+1}}$ it is easily seen that for every $\varepsilon \in (0,1/3]$, χ_ε is a $\{U_1^{k+1}(0)\}$-push in $\{\gamma \in H(\mathbb{R}^{k+1}) \mid d_{k+1}(\gamma(x),\gamma(y)) \geq \frac{2}{3}\, d_{k+1}(x,y)$ for $x,y \in \mathbb{R}^{k+1}\}$.

Let $m \in \{3,4,5,\dots\}$, $i \in \{0,1,2,\dots\}$, $p \in (K_i)^{k+1}$ and put for every $x \in I^{k+1}$,

$$\psi_{i,p}^m(x) = p + \tfrac{1}{2}3^{-i}\chi_{1/m}(2.3^i(x-p)).$$

It follows that $\psi_{i,p}^m$ is a $\{\tilde{U}^{k+1}_{\frac{1}{2}3^{-i}}(p)\}$-push in

$$E = \{\gamma \in H(I^{k+1}) \mid d_{k+1}(\gamma(x),\gamma(y)) \geq \frac{2}{3}\, d_{k+1}(x,y) \text{ for } x,y \in I^{k+1}\},$$

which satisfies

$$\psi_{i,p}^m(\tilde{U}^{k+1}_{\frac{1}{2}3^{-i}/m}(p)) = \tilde{U}^{k+1}_{\frac{1}{2}3^{-i-1}}(p).$$

PROOF of theorem 2.2.1: Let m be a natural number, ε a positive real number, F a closed subset of I^n and S a member of $\tilde{\mathfrak{M}}_k^n$ that misses F. Since I^n is compact it suffices to consider only one metric: d_n. We have to find a $\{\tilde{U}_\varepsilon^n(x) \mid x \in I^n\}$-push g in $\{\gamma \in H(I^n) \mid \gamma|A_m \cup F = 1\}$ and an $i \in \mathbb{N}$ such that $g(S) \subset A_i$.

Let Γ be the countable subgroup of $H(I^n)$ that is generated by the set

$$\tilde{P}_n \cup \{\psi_{l,p}^r \times 1_{I^{n-k-1}} \mid r \in \{3,4,5,\dots\},\ l \in \{0\} \cup \mathbb{N} \text{ and }$$
$$p \in (K_l)^{k+1}\}.$$

Consider the collection

$$K = \{\gamma(\{p\} \times I^{n-k-1}) \mid p \in K^{k+1} \text{ and } \gamma \in \Gamma\}.$$

Note that L is contained in K. Since K is a countable set of tame polyhedra of dimension $n-k-1$ there exists according to proposition 2.1.7 a $\{U^n_{\varepsilon/2}(x) \mid x \in I^n\}$-push f in $\{\gamma \in H(I^n) \mid \gamma|F \cup A_m = 1\}$ with

$$f(S) \cap \cup K \setminus A_m = \emptyset.$$

Put $S' = f(S)$ and select a $j \in \mathbb{N}$ such that $j > m$, $3^{-j+1} < \varepsilon/2$ and $3^{-j+1} < d_n(S',F)$. Define the compactum

$$C = \cup\{J \in \mathcal{J}^n_j \mid J \cap S' \neq \emptyset\}.$$

Note that C is a neighbourhood of S' that has distance greater than 3^{-j} to F.

We shall construct a 3^{-j+1}-isotopy $H : I^n \times I \to I^n \times I$ that satisfies: $H_0 = 1_{I^n}$, $H_t|F \cup A_m = 1$ for $t \in I$ and $H_1(S') \subset A_{j+1}$. Then the function $H_1 \circ f$ is the push of I^n we need. The isotopy H will be the limit of a sequence $H^0, H^1, H^2, \ldots$ of isotopies of I^n that satisfies for $l = 0,1,2,\ldots$

$$H^l_1(C \setminus \cup K) = C \setminus \cup K$$

and

$$H^l_1(S') \subset F^{j+1}_l.$$

The H^l's are determined inductively with as first step $H^0 = 1_{I^n \times I}$. Moreover, it will be shown that $G^l = H^{l+1} \circ (H^l)^{-1}$ is a 3^{-l-j}-isotopy such that for every $t \in I$, $G^l_t \in E'$, where

$$E' = \{\gamma \in H(I^n) \mid d_n(\gamma(x),\gamma(y)) \geq \tfrac{2}{3}\, d_n(x,y) \text{ for } x,y \in I^n \text{ and}$$
$$\gamma|F \cup A_m = 1\}.$$

Consider now $\lim_{l\to\infty} H^l$. Since G^l is a 3^{-l-j}-isotopy with $G^l_0 = 1$ the sequence $(H^l)_{l=0}^{\infty}$ is uniformly Cauchy. So $H = \lim_{l\to\infty} H^l$ exists and it is a "3^{-j+1}-homotopy" of I^n with $H_0 = 1$. We show that H is an isotopy. Since I^n is compact it suffices to prove that every H_t is onto and one-to-one. Let $t \in I$ and note that H_t is the limit of a sequence of autohomeomorphisms of a compactum and hence it is onto. Let x and y be two arbitrary distinct points in I^n. Select an $l \in \mathbb{N}$ such that $2^l \cdot d_n(x,y) > 1$. Since for every $s \in \{0\} \cup \mathbb{N}$, $G_t^s \in E'$ we have that for $z,z' \in I^n$,

$$d_n(G_t^s(z),\ G_t^s(z')) \geq \frac{2}{3}\, d_n(z,z')$$

and hence that

$$d_n(H_t^l(x),\ H_t^l(y)) \geq (\tfrac{2}{3})^l\, d_n(x,y) > 3^{-l}.$$

Since G^s is a 3^{-s-j}-isotopy with $G_0^s = 1$ it follows that $\hat{d}_{n+1}(H_t \circ (H_t^l)^{-1}, 1) < \frac{3}{2}\, 3^{-j-l}$. Consequently,

$$d_n(H_t(x),\ H_t(y)) > d_n(H_t^l(x),\ H_t^l(y)) - 3.3^{-j-l} > 0$$

and $H_t(x) \neq H_t(y)$. It is obvious that H_t fixes $F \cup A_m$. So we have proved that H is a 3^{-j+1}-isotopy of I^n that satisfies $H_0 = 1$ and $H_t|F \cup A_m = 1$ for $t \in I$. The inclusions $F_0^{j+1} \supset F_1^{j+1} \supset F_2^{j+1} \supset \ldots$ lead, together with $H_1^s(S') \subset F_s^{j+1}$, $s \in \{0\} \cup \mathbb{N}$, to

$$H_1(S') = \lim_{s\to\infty} H_1^s(S') \subset \bigcap_{s=0}^{\infty} F_s^{j+1} = A_{j+1}.$$

Now it remains to perform the construction of the H^l's.

Assume that H^l has been determined. Since $H^l = G^{l-1} \circ \ldots \circ G^0$ we have that H_t^l fixes $F \cup A_m$ for every $t \in I$. Consider the situation:

$$S' \subset C,$$

$$H_1^l(C \backslash \cup K) = C \backslash \cup K$$

and

$$S' \cap \cup K \setminus A_m = \emptyset$$

This implies that

$$H_1^1(S')\setminus A_m \subset C \setminus \cup K$$

and since $L_1 \in L \subset K$ and $L_1 \cap A_m = \emptyset$ we have that $H_1^1(S')$ and L_1 are disjoint. Furthermore, we may derive that

$$H_1^1(S') \subset H_1^1(S'\setminus A_m) \cup H_1^1(S' \cap A_m) \subset C.$$

Since S' is compact there exists an $r \in \{3,4,5,...\}$ such that

$$d_n(H_1^1(S'), L_1) > \frac{1}{2r} 3^{-1-j}.$$

Let L_1 be of the form $\alpha(\{p\} \times I^{n-k-1})$, where $\alpha \in \tilde{P}_n$ and $p \in (K_1)^{k+1}$. Let Ψ be a 3^{-1-j}-isotopy of I^{k+1} such that $\Psi_0 = 1$, $\Psi_1 = \psi^r_{1+j,p}$ and for $t \in I$, Ψ_t is a member of

$$\hat{E} = \{\gamma \in E | \gamma \text{ is supported on } \tilde{U}^{k+1}_{\frac{1}{2}3^{-1-j}}(p)\}.$$

Consider the product $I^{k+1} \times I^{n-k-1} = I^n$ and the projection $\pi : I^{k+1} \times I^{n-k-1} \to I^{n-k-1}$. Let J be the cube in J^{k+1}_{1+j} of which p is the centre. Define $\hat{C} = \pi(J \times I^{n-k-1} \cap \alpha^{-1}(C))$ and $\hat{F} = \pi(J \times I^{n-k-1} \cap \alpha^{-1}(F))$. Since the diameter of J with respect to d_{k+1} is 3^{-1-j} and since $d_n(C,F) > 3^{-j}$ we have that $\hat{C}$ and $\hat{F}$ are disjoint. Let $\beta : I^{n-k-1} \to I$ be a Urysohn function with $\beta(\hat{C}) \subset \{1\}$ and $\beta(\hat{F}) \subset \{0\}$. Define the isotopy $\Theta : I^n \times I \to I^n \times I$ by

$$\Theta_t(x,y) = (\Psi(x,t\beta(y)),y) \text{ for } x \in I^{k+1},\ y \in I^{n-k-1},\ t \in I$$

and put $G^1_t = \alpha \circ \Theta_t \circ \alpha^{-1}$ for $t \in I$. Since $\Psi_t \in \hat{E}$ it follows that G^1 is a 3^{-1-j}-isotopy of I^n such that every level is an element of

$$\{\gamma \in H(I^n) \mid d_n(\gamma(x),\gamma(y)) \geq \frac{2}{3}\, d_n(x,y) \text{ for } x,y \in I^n$$

$$\text{and } \gamma \text{ is supported on } \alpha(\tilde{U}^{k+1}_{\frac{1}{2}3^{-1-j}}(p) \times (I^{n-k-1}\setminus\hat{F}))\}.$$

Since $F \subset I^n\setminus\alpha(J \times (I^{n-k-1}\setminus\hat{F}))$ and since $A_m \subset A_j \subset F^j_{l+1} = F^j_l\setminus\tilde{U}^n_{\frac{1}{2}3^{-1-j}}(L_l)$ this implies that G^l is a 3^{-1-j}-isotopy with each level in E'.

Define now $H^{l+1} = G^l \circ H^l$. We prove that $H^{l+1}_1(C\setminus\cup K) = C\setminus\cup K$ and $H^{l+1}_1(S') \subset F^{j+1}_{l+1}$. Note that for every $t \in I$ and $D \in J^{k+1}_{1+j}$, $\Psi_t(D) = D$. This implies that for each $D \in J^n_{1+j}$, $G^l_1(D) = D$. Both F^{j+1}_l and C can be written as union of members of J^n_{1+j} and hence we have that $G^l_1(F^{j+1}_l) = F^{j+1}_l$ and $G^l_1(C) = C$. Define $g \in H(I^n)$ by

$$g = \alpha \circ (\psi^r_{1+j,p} \times 1_{I^{n-k-1}}) \circ \alpha^{-1}.$$

The function g is a member of Γ and consequently we have that $g(\cup K) = \cup K$. We shall see that $g|C = G^l_1|C$. Let $x \in I^k$ and $y \in I^{n-k-1}$ such that $\alpha(x,y) \in C$. If $x \in J$ then $y \in \hat{C}$ and $\beta(y) = 1$. This implies that $\Theta_1(x,y) = (\psi^r_{1+j,p}(x),y)$ and hence that $G^l_1(\alpha(x,y)) = g(\alpha(x,y))$. If $x \notin J$ then $\Psi_t(x) = x = \psi^r_{1+j,p}(x)$ for every $t \in I$ and consequently $G^l_1(\alpha(x,y)) = \alpha(x,y) = g(\alpha(x,y))$. Now we have that $G^l_1(C\setminus\cup K) = C\setminus\cup K$ and $H^{l+1}_1(C\setminus\cup K) = C\setminus\cup K$. Since $\psi^r_{1+j,p}(\tilde{U}^{k+1}_{\frac{1}{2}3^{-1-j}/r}(p)) = \tilde{U}^{k+1}_{\frac{1}{2}3^{-1-j-1}}(p)$ and $d_n(H^i_1(S'),L_l) \geq \frac{1}{2}3^{-1-j}/r$ we have that $g \circ H^i_1(S')$ and $\tilde{U}^n_{\frac{1}{2}3^{-1-j-1}}(L_l)$ are disjoint. If we combine this with $G^l_1 \circ H^l_1(S') \subset G^l_1(F^{j+1}_l) = F^{j+1}_l$, $g|C = G^l_1|C$, $H^l_1(S') \subset C$ and $F^{j+1}_{l+1} = F^{j+1}_l\setminus\tilde{U}^n_{\frac{1}{2}3^{-1-j-1}}(L_l)$ we find that

$$H^{l+1}_1(S') = G^l_1 \circ H^l_1(S') \subset F^{j+1}_{l+1}.$$

This completes the proof of theorem 2.2.1.

2.3 Skeletoids in $\mathbb{R}^n$

Using the result of the preceeding section 2.2 we construct a k-dimensional skeletoid in $\mathbb{R}^n$. As an application we obtain universal spaces in the class of strongly σ-complete spaces.

2.3.1 THEOREM*): *There exists a strong* $(\mathfrak{M}^n_k, \mathcal{H}(\mathbb{R}^n))$*-skeletoid*$^\sim$.

PROOF: Consider Int $I^n \approx \mathbb{R}^n$ and $\mathcal{S} = \{S \in \tilde{\mathfrak{M}}^n_k \mid S \cap \partial I^n = \emptyset\}$. It is easily seen that it suffices to prove that there is a strong $(\mathcal{S}, \mathcal{H}(\text{Int } I^n))$-skeletoid$^\sim$. Let $(A_i)_{i \in \mathbb{N}}$ be a strong $(\tilde{\mathfrak{M}}^n_k, \mathcal{H}(I^n))$-skeleton$^\sim$, theorem 2.2.1, and define $A_i' = A_i \cap [2^{-i}, 1-2^{-i}]^n$ for $i \in \mathbb{N}$. We show that $(A_i')_{i \in \mathbb{N}}$ is a strong $(\mathcal{S}, \mathcal{H}(\text{Int } I^n))$-skeletoid$^\sim$ ($(\mathcal{S}, \mathcal{H}(\text{Int } I^n))$ is a Δ-pair because $(\mathfrak{M}^n_k, \mathcal{H}(\mathbb{R}^n))$ is a Δ-pair). Let $S \in \mathcal{S}$ and let $\mathcal{U}$ be a collection of open subsets of Int I^n that covers S. If $i \in \mathbb{N}$ then there are a $j \in \mathbb{N}$ and a $\mathcal{U}$-push h in $\{\gamma \in \mathcal{H}(I^n) \mid \gamma | A_i = 1\}$ with $h(S) \subset A_j$. Let $m > j$ such that $2^{-m} < d_n(h(S), \partial I^n)$. Then $h|\text{Int } I^n$ is a $\mathcal{U}$-push in $\{\gamma \in \mathcal{H}(\text{Int } I^n) \mid \gamma | A_i' = 1\}$ with $h(S) \subset A_j'$.

Let B^n_k be a strong $(\mathfrak{M}^n_k, \mathcal{H}(\mathbb{R}^n))$-skeletoid$^\sim$ and put $s^n_k = \mathbb{R}^n \setminus B^n_k$. Note that since B^n_k is σ-compact s^n_k is topologically complete. By the countable sum theorem ([E2 : 3.1.8]) we have that dim $(B^n_k) = k$. Geoghegan & Summerhill [GS2] have shown that there exist $(\mathfrak{M}^n_k, \mathcal{H}(\mathbb{R}^n))$-absorbers. This result follows from theorem 2.3.1. Moreover, theorem 1.2.11 implies that the absorbers constructed in [GS2] are in fact also strong skeletoids.

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*) *This theorem can also be found in Dijkstra* [D1].

2.3.2 PROPOSITION: s_k^n *is homogeneous.*

PROOF: Apply corollary 1.2.14.

Using theorem 1.2.13 we can prove more results in this direction: s_k^n is strongly locally homogeneous and hence countably dense homogeneous (see Anderson, Curtis & van Mill [ACM : sec.5]).

2.3.3 PROPOSITION (Geoghegan & Summerhill [GS2]): $\dim (s_k^n) = n-k-1$ *and every compact subset of* s_k^n *is an element of* $\mathfrak{M}_{n-k-1}^n$.

PROOF: The set $\mathbb{R}^n \setminus N_{n-k-1}^n$ is a countable union of k-dimensional subpolyhedra of $\mathbb{R}^n$ and hence there is an $h \in H(\mathbb{R}^n)$ with $h(B_k^n) = B_k^n \cup (\mathbb{R}^n \setminus N_{n-k-1}^n)$, theorem 1.2.11. Consequently $h(s_k^n) \subset N_{n-k-1}^n$ and hence $\dim (s_k^n) = n-k-1$ ([E2 : 1.5.10]).

Let S be a compact subset of s_k^n. Assume that P is a k-dimensional subpolyhedron of $\mathbb{R}^n$ and that $\mathcal{U}$ is a collection of open subsets of $\mathbb{R}^n$ that covers $S \cap P$. Since $P \in \mathfrak{M}_k^n$ there is a $\mathcal{U}$-push h in $H(\mathbb{R}^n)$ such that $h(B_k^n \cap \cup\mathcal{U}) = (B_k^n \cup P) \cap \cup\mathcal{U}$, theorem 1.2.11. Hence, we have that $h(S) \cap P = \emptyset$.

2.3.4 PROPOSITION (Geoghegan & Summerhill [GS2]): *If* $n \leq 2k+1$ *then every* σ*-compact subset of* s_k^n *is strongly negligible in* s_k^n.

PROOF: According to proposition 2.3.3 every σ-compact subset of s_k^n is an element of $(\mathfrak{M}_{n-k-1}^n)_\sigma \subset \mathfrak{M}_{k\sigma}^n$. Theorem 1.2.12 implies that it is strongly negligible.

2.3.5 DEFINITION: A space is called *strongly* σ*-complete* if it is a countable union of closed, topologically complete subspaces. If $1 \in \{0,1,2,\ldots,\infty\}$ then we define the class

$$V_\sigma^1 = \{X \mid X \text{ is a strongly } \sigma\text{-complete space with dimension } \leq 1\}.$$

A space X is called *universal for* V_σ^1 if

$$V_\sigma^1 = \{Y \mid \text{there is an } F_\sigma\text{-set in X that is homeomorphic to Y}\}.$$

Note that V_σ^∞ is simply the class of all strongly σ-complete spaces. If X is negligible in a complete space then it is an F_σ-set and hence a strongly σ-complete space. We shall see that V_σ^∞ is precisely the class of spaces that can be negligible subsets of a complete space (see theorem 4.5.12).

2.3.6 DEFINITION: A closed subset S of a space X is called *thin* if for every collection U of open subsets of X there is an $f \in H(X)$ that is U-close to 1 and satisfies $h(S \cap \cup U) \cap S = \emptyset$.

Geoghegan & Summerhill [GS2] have shown that every member of $\mathfrak{M}_k^{2k+1}$ is thin in $\mathbb{R}^{2k+1}$. This implies with proposition 2.1.8 that if $S, S' \in \mathfrak{M}_k^{2k+1}$ then there is an $h \in H(\mathbb{R}^n)$, which can be chosen arbitrarily close to 1, with $h(S) \cap S' = \emptyset$. A straightforward application of lemma 1.1.2 gives that if $S, S' \in (\mathfrak{M}_k^{2k+1})_\sigma$ then there is an $h \in H(\mathbb{R}^n)$ such that $h(S) \cap S' = \emptyset$.

2.3.7 THEOREM: *The space* s_k^{2k+1} *is universal for* V_σ^k. *Moreover, an arbitrary space* X *is an element of* V_σ^k *iff it is homeomorphic to a (strongly) negligible set in* s_k^{2k+1}.

PROOF: If X is strongly negligible in s_k^{2k+1} then X is negligible and hence an F_σ-set. Consequently, X is strongly σ-complete.

Let $X \in V_\sigma^k$ and select a compactification C of X with dimension $\leq k$, [E2 : 1.7.2]. There is an embedding f of C in N_k^{2k+1} (see [E2 : 1.11.5]) and hence $f(C) \in \mathfrak{M}_k^{2k+1}$, theorem 2.1.12. Since $B_k^{2k+1} \in \mathfrak{M}_{k\sigma}^{2k+1}$, $f(C)$ can be pushed off B_k^{2k+1}. So we may assume that f embeds C into s_k^{2k+1}. Write $X = \bigcup_{i \in \mathbb{N}} S_i$, where S_i is a closed, topologically complete subset of X. Define for every $i \in \mathbb{N}$, $R_i = f(\mathrm{Cl}_C(S_i)\setminus S_i)$ and furthermore $P = \bigcup_{i \in \mathbb{N}} f(\mathrm{Cl}_C(S_i))$ and $R = \bigcup_{i \in \mathbb{N}} R_i$. For $i \in \mathbb{N}$ we have that R_i is the remainder of a topologically complete space in a compactification and hence a σ-compact space. So R is a σ-compact subset of s_k^{2k+1} and consequently an element of $\mathfrak{M}_{k\sigma}^{2k+1}$. Using theorem 1.2.11 we find an $h \in H(\mathbb{R}^n)$ such that $h(B_k^{2k+1} \cup R) = B_k^{2k+1}$. The σ-compact space $h(P)$ is an element of $\mathfrak{M}_{k\sigma}^{2k+1}$ and hence $h(P)\setminus B_k^{2k+1}$ is strongly negligible in s_k^{2k+1}, theorem 1.2.12. Since S_i is closed in X for every $i \in \mathbb{N}$, we have that

$$h(P)\setminus B_k^{2k+1} = h(P\setminus R) = h \circ f(X).$$

This proves the theorem.

2.3.8 REMARK: The space s_0^1 is homeomorphic to $\mathbb{R}\setminus\mathbb{Q}$. It is easily verified that s_0^1 is nowhere locally compact. The assertion follows then from the Alexandroff & Urysohn [AU] characterization of $\mathbb{R}\setminus\mathbb{Q}$.

2.4 A technical lemma

In this section we consider the functions $\varphi_\varepsilon : [0,\infty) \to [1,\infty)$ and $\chi_\varepsilon \in H(\mathbb{R}^1)$ which are defined by

$$\varphi_\varepsilon(r) = \begin{cases} \frac{1}{3\varepsilon} & \text{if } 0 \le r \le \varepsilon \\ \frac{1}{3(1-\varepsilon)}(2 + \frac{1-3\varepsilon}{r}) & \text{if } \varepsilon \le r \le 1 \\ 1 & \text{if } r \ge 1 \end{cases}$$

and

$$\chi_\varepsilon(x) = \varphi_\varepsilon(\|x\|)X,$$

where $\varepsilon \in (0,1/3]$ and $\|x\| = d_1(x,0) = \max\{|x_i| \,|\, i = 1,2,\ldots,1\}$.

2.4.1 LEMMA: *For every* $x,y \in \mathbb{R}^1$ *we have that*

$$\|\chi_\varepsilon(x) - \chi_\varepsilon(y)\| \ge \frac{2}{3}\|x - y\|.$$

PROOF: We consider four cases.

I. If $\|x\| \le \varepsilon$ or $\|x\| \ge 1$ and $\|y\| \le \varepsilon$ or $\|y\| \ge 1$ then the statement is obvious.

II. Let $\varepsilon \le \|x\|, \|y\| \le 1$. For some $i \le 1$ we have that $\|x - y\| = |x_i - y_i|$. Without loss of generality we may assume that $x_i \ge y_i$ and $x_i \ge 0$. This implies that $\|x\| - \|y\| \le \|x - y\| = x_i - y_i$ and hence we have that $\|x\| - y_i \ge \|x\| - x_i$. Since $\|x\| - x_i \ge 0$, $x_i \ge y_i$ and $x_i \ge 0$ we find that $x_i(\|y\| - y_i) \ge y_i(\|x\| - x_i)$. So we have that

$$\frac{x_i}{\|x\|} \ge \frac{y_i}{\|y\|}.$$

Consider now

$$\chi_\varepsilon(x)_i - \chi_\varepsilon(y)_i = \frac{x_i}{3(1-\varepsilon)}(2 + \frac{1-3\varepsilon}{\|x\|}) - \frac{y_i}{3(1-\varepsilon)}(2 + \frac{1-3\varepsilon}{\|y\|}) =$$

$$(x_i - y_i)\frac{2}{3(1-\varepsilon)} + \frac{1-3\varepsilon}{3(1-\varepsilon)}\left(\frac{x_i}{\|x\|} - \frac{y_i}{\|y\|}\right) \geq \frac{2}{3}\|x-y\|.$$

We may conclude that

$$\|\chi_\varepsilon(x) - \chi_\varepsilon(y)\| \geq |\chi_\varepsilon(x)_i - \chi_\varepsilon(y)_i| \geq \frac{2}{3}\|x-y\|.$$

III. Let $\|y\| \leq \varepsilon$ and $\varepsilon \leq \|x\| \leq 1$. Select an $i \leq 1$ such that $\|x-y\| = |x_i - y_i|$. We may assume that $x_i \geq 0$. We make the following subdivision.

(a) $y_i \geq x_i$. Since φ_ε is a decreasing function we have that $\varphi_\varepsilon(\|y\|) \geq \varphi_\varepsilon(\|x\|)$ and hence that

$$\|\chi_\varepsilon(x) - \chi_\varepsilon(y)\| \geq y_i\,\varphi_\varepsilon(\|y\|) - x_i\,\varphi_\varepsilon(\|x\|) \geq$$

$$y_i - x_i = \|x-y\| \geq \frac{2}{3}\|x-y\|.$$

(b) $x_i \geq y_i$. As above we have that $y_i\|x\| \leq x_i\|y\|$ and consequently,

$$y_i \leq \frac{x_i}{\|x\|}\|y\| \leq \frac{x_i}{\|x\|}\varepsilon.$$

Consider

$$\chi_\varepsilon(x)_i - \chi_\varepsilon(y)_i = \frac{x_i}{3\|x\|} + \frac{2x_i}{3(1-\varepsilon)}\left(\frac{\|x\|-\varepsilon}{\|x\|}\right) - \frac{y_i}{3\varepsilon} =$$

$$\frac{1}{3\varepsilon}\left(\frac{\varepsilon x_i}{\|x\|} - y_i\right) + \frac{2}{3}\,\frac{x_i(\|x\|-\varepsilon)}{(1-\varepsilon)\|x\|} \geq$$

$$\frac{2}{3}\left(\frac{\varepsilon x_i}{\|x\|} - y_i\right) + \frac{2}{3}x_i\left(\frac{\|x\|-\varepsilon}{\|x\|}\right) \geq \frac{2}{3}(x_i - y_i).$$

So the conclusion is that $\|\chi_\varepsilon(x) - \chi_\varepsilon(y)\| \geq \frac{2}{3}\|x-y\|$.

IV. Let $\|y\| \leq$ and $\|x\| \geq 1$ and assume that $\|x-y\| = x_i - y_i$. Again we consider two cases.

(a) $|x_i| \geq 1$. This implies that $x_i \geq 1$. Consider the set $A = \{z \in \mathbb{R}^k \mid z_i = 1\}$. Obviously, there exists an $a \in A$ such that $\|a\| = 1$ and

$d_1(\chi_\varepsilon(y),A) = d_1(\chi_\varepsilon(y),a)$. In view of the results obtained above χ_ε satisfies

$$d_1(\chi_\varepsilon(y),A) = d_1(\chi_\varepsilon(y),a) \geq \frac{2}{3}\, d_1(y,a) \geq \frac{2}{3}\, d_1(y,A).$$

It is easily seen that $d_1(\chi_\varepsilon(y),\chi_\varepsilon(x)) \geq d_1(\chi_\varepsilon(y),A) + d_1(\chi_\varepsilon(x),A)$. This yields:

$$d_1(\chi_\varepsilon(y),\chi_\varepsilon(x)) \geq \frac{2}{3}\, d_1(y,A) + d_1(\chi_\varepsilon(x),A) \geq$$

$$\frac{2}{3}\,(d_1(y,A) + d_1(x,A)) = \frac{2}{3}(1 - y_i + x_i - 1) = \frac{2}{3}\,\|x - y\|.$$

(b) $|x_i| \leq 1$. Define $\tilde{x} \in \mathbb{R}^1$ by

$$\tilde{x}_i = \min\{1, \max\{-1,x_i\}\} \text{ for } 1 \leq i \leq 1.$$

Note that $\|\tilde{x}\| = 1$ and that $\|x-y\| = \|\tilde{x}-y\|$. We have proved that $\|\chi_\varepsilon(\tilde{x}) - \chi_\varepsilon(y)\| \geq \frac{2}{3}\,\|\tilde{x}-y\|$. Using $\chi_\varepsilon(x) = x$ and $\chi_\varepsilon(\tilde{x}) = \tilde{x}$ we find that

$$\|\chi_\varepsilon(x) - \chi_\varepsilon(y)\| \geq \|\chi_\varepsilon(\tilde{x}) - \chi_\varepsilon(y)\| \geq \frac{2}{3}\,\|\tilde{x}-y\| = \frac{2}{3}\,\|x-y\|.$$

Since we have considered all possible choices of x and y this concludes the proof.

CHAPTER 3

THE HILBERT CUBE

3.1 Introduction

We discuss in this section the connexion between absorbers and skeletoids in the Hilbert cube. Furthermore, we give examples of pseudo-boundaries and related objects.

The Hilbert cube will, except in section 3.2, be represented by

$$Q = \prod_{i \in \mathbb{N}} J_i,$$

where each J_i is the closed interval $J = [-1,1]$. Let π_i be the projection $Q \to J_i$. We use on Q the following convex metric

$$\rho(x,y) = \max_{i \in \mathbb{N}} |x_i - y_i| \frac{1}{2i},$$

where $x = (x_i)_{i \in \mathbb{N}}$ and $y = (y_i)_{i \in \mathbb{N}}$. The open ε-balls ($\varepsilon \geq 0$) in Q with respect to ρ are denoted by U_ε. The symbol ρ is also used for the metric on subproducts of Q: if $P \subset \mathbb{N}$ then for $x,y \in \prod_{i \in P} J_i$, $\rho(x,y) = \max_{i \in P} |x_i - y_i| \frac{1}{2i}$. If A is a subset of $\prod_{i \in P} J_i$ then diam A is the diameter of A with respect to ρ. If $i \in \mathbb{N}$ and $P = \{j \in \mathbb{N} | j \geq i\}$ then we define $Q_i = \prod_{j \in P} J_j$.

Let $J_i^\circ = J^\circ = (-1,1)$ for $i \in \mathbb{N}$ and define *the pseudo-interior* s of Q by $s = \prod_{i \in \mathbb{N}} J_i^\circ$. The space s is homeomorphic to the separable Hilbert space

ℓ^2, Anderson [A1]. Put $0 = (0,0,0,\dots) \in Q$ and $B = Q\setminus s$. The set B is called *the pseudo-boundary* of Q and an element $f \in H(Q)$ is called *boundary preserving* if $f(B) = B$ or, equivalently, $f(s) = s$. We can write B as the union $\cup\{E_i^\theta \mid i \in \mathbb{N} \text{ and } \theta \in \{-1,1\}\}$, where the E_i^θ's are the endfaces of Q:

$$E_i^\theta = \{x \in Q \mid x_i = \theta\}.$$

3.1.1 DEFINITION: A closed subset S of a space X is called a *Z-set in* X if for every open covering $\mathcal{U}$ of X and for every map $f : Q \to X$ there is a map $g : Q \to X\setminus S$ that is $\mathcal{U}$-close to f. A subset A of X is called a *σ-Z-set in* X if it is a countable union of Z-sets. The collections of Z-sets and σ-Z-sets in X are denoted by $Z(X)$ and $Z_\sigma(X)$, respectively.

In complete spaces the following properties are easily proved (see [BP2 : sec.V.2]): $(Z(X), H(X))$ is a Δ-pair, if A is a closed σ-Z-set then A is a Z-set and every Z-set is nowhere dense. It is well known that in Q every Z-set is thin and that every endface and every compactum in s is a Z-set (see [BP2 : sec.V.3]). So B is a σ-Z-set.

Note that since Q is compact, a closed subset S of X is a Z-set iff for every $\varepsilon > 0$ and $f : Q \to X$ there is a map $g : Q \to X\setminus S$ with $\hat{d}(f,g) < \varepsilon$, where d is some fixed metric on X. The following theorem may be derived from Chapman [C : 19.4] and Anderson & Chapman [AC]. We obtain it as a direct consequence of theorem 4.3.6.

3.1.2 THEOREM: *Let* $\mathcal{U}$ *be a collection of open subsets of* Q*, let* A *be a compact space and let* $F : A \times I \to Q$ *be a homotopy that is limited by* $\mathcal{U}$*. If* F_0 *and* F_1 *are embeddings of* A *in* Q *such that their images are Z-sets then there is a* $\mathcal{U}$*-push* h *in* $H(Q)$ *with* $h \circ F_0 = F_1$.

3.1.3 COROLLARY: *If* A *and* A' *are Z-sets in* Q *and* f *is a homeomorphism from* A *onto* A' *with* $\hat{\rho}(f,1) < \varepsilon$ *then there is a* $g \in \mathcal{H}(Q)$ *such that* $g|A = f$ *and* $\hat{\rho}(g,1) < \varepsilon$.

PROOF: Define the straight-line homotopy

$$F(a,t) = (1-t)a + tf(a) \text{ for } a \in A \text{ and } t \in I.$$

Then F is limited by $\mathcal{U} = \{U_{\varepsilon/2}(x) \mid x \in Q\}$. Applying the theorem we find a $\mathcal{U}$-push g in $\mathcal{H}(Q)$ with $g \circ F_0 = F_1$. So $\hat{\rho}(g,1) < \varepsilon$ and $g|A = f$.

Theorem 3.1.2 has the following consequence.

3.1.4 THEOREM: *If* $(\mathcal{S},\mathcal{H}(Q))$ *is a* Δ*-pair such that* $\mathcal{S} \subset \mathcal{Z}(Q)$ *then every* $(\mathcal{S},\mathcal{H}(Q))$*-skeletoid is a strong* $(\mathcal{S},\mathcal{H}(Q))$*-skeletoid*.

PROOF: Let $(A_i)_{i \in \mathbb{N}}$ be an $(\mathcal{S},\mathcal{H}(Q))$-skeleton. Assume that $S \in \mathcal{S}$, $\varepsilon > 0$, $m \in \mathbb{N}$ and that F is a closed set in Q with $\rho(F,S) > \varepsilon$. There are an $n \in \mathbb{N}$ and an $f \in \mathcal{H}(Q)$ such that $\hat{\rho}(f,1) < \varepsilon/2$, $f|A_m = 1$ and $f(S) \subset A_n$. Define the map $F : (S \cup A_m) \times I \to Q \times I$ by

$$F(a,t) = ((1-t)a + tf(a),t).$$

Let π be the projection $Q \times I \to Q$. If $X = (A_n \times I) \cup (S \times \{0,1\})$ then $F|X$ is an embedding. Since $F(X) \subset (A_n \cup A_m \cup S) \times I$, we have that it is a Z-set in $Q \times I$. According to theorem 11.2 in Chapman [C] there exists an embedding $\tilde{F}$ of $(S \cup A_n) \times I$ in $Q \times I$ such that $\tilde{F}|X = F|X$ and $\hat{\rho}(\pi \circ \tilde{F},\pi \circ F) < \varepsilon/2$. Define $G = \pi \circ \tilde{F}$ and note that G is a homotopy from $S \cup A_m$ into Q that is limited by

$$\mathcal{U} = \{U_\varepsilon(x)\backslash(F \cup A_m) \mid x \in Q\}.$$

The functions $G_0 = 1_{S \cup A_m}$ and $G_1 = f|S \cup A_m$ are homeomorphisms from $S \cup A_m$ onto a Z-set in Q. According to theorem 3.1.2 there is a $\mathcal{U}$-push h in $\mathcal{H}(Q)$ with $h(S) = G_1(S) = f(S) \subset A_n$. This proves the theorem.

3.1.5 REMARK: As a corollary to this theorem one has that every $(\mathcal{S},\mathcal{H}(Q))$-skeletoid is an $(\mathcal{S},\mathcal{H}(Q))$-absorber. There are collections $\mathcal{S}$ in Q such that absorbers exist but no skeletoids. Let $\mathcal{S}$ be the collection of all countable Z-sets in Q. It is well known (and easily proved with theorems 3.1.2 and 1.2.11) that every countable dense subset of Q is an $(\mathcal{S},\mathcal{H}(Q))$-absorber. Consider a sequence $A_1 \subset A_2 \subset A_3 \subset \dots$ in $\mathcal{S}$. For every $i \in \mathbb{N}$ there exists a countable ordinal α_i such that the α_i-th derived set $(A_i)^{(\alpha_i)}$ is empty, see Mazurkiewicz & Sierpiński [MS]. If β is a countable ordinal with $\beta > \sup\{\alpha_i | i \in \mathbb{N}\}$ then $[0,\omega^\beta]^{(\beta)} \neq \emptyset$. Hence, the ordered space $[0,\omega^\beta]$, which is of course embeddable as a Z-set in Q, cannot be embedded in any of the A_i's. This means that $(A_i)_{i \in \mathbb{N}}$ is not an $(\mathcal{S},\mathcal{H}(Q))$-skeleton. Note that this idea also works in I^n and $\mathbb{R}^n$.

We shall now discuss some examples of skeletoids in Q. The most important example is B, which is a $(\mathcal{Z}(Q),\mathcal{H}(Q))$-skeletoid (Anderson [A4]). This has the consequence that every σ-compact subset of $s \approx \ell^2$ is strongly negligible. Another example (also due to Anderson) is

$$B_{fd} = \{x \in Q | \text{there is an } i \in \mathbb{N} \text{ such that for every } j > i \; x_j = 0\}.$$

This σ-Z-set is a skeletoid for $\{S \in \mathcal{Z}(Q) | S \text{ is finite dimensional}\}$. Curtis and van Mill [CM] have shown that every dense σ-Z-set in Q that is homeomorphic to the product of $\mathbb{Q}$ and Cantor's discontinuum is a skeletoid for the collection of zero-dimensional Z-sets in Q. We shall construct this skeletoid in the next section. A related concept is that of a boundary set.

3.1.6 DEFINITION: A σ-Z-set A in Q is called a *boundary set* if $Q\backslash A \approx \ell^2$. A σ-Z-set A in Q is called a *deformation boundary set* if there is a homotopy $F : Q \times I \to Q$ with $F_0 = 1$ and $F(Q \times (0,1]) \subset A$.

Curtis [Cs] has shown that every deformation boundary set is a boundary set. Clearly, B and B_{fd} are deformation boundary sets. Van Mill [M1] has obtained a boundary set that contains no arcs. This shows that the concepts boundary set and deformation boundary set do not coincide. Henderson & Walsh [HW] have given an example of a deformation boundary set containing (obviously) arcs but no disks. It was shown by Curtis [Cs] that every boundary set is infinite-dimensional, see also remark 5.4.6.

3.2 k-dimensional skeletoids

Using the main result of section 2.3 we build $(\mathcal{S}_k, \mathcal{H}(Q))$-skeletoids in the Hilbert cube, where

$$\mathcal{S}_k = \{S \mid S \text{ is a Z-set in Q with dimension} \le k\}$$

The number $k \in \{0,1,2,\dots\}$ *remains fixed throughout this section.*

It is convenient to use a different representation for the Hilbert cube here. Let $c\,\mathbb{R}$ be the compactification of $\mathbb{R}$ that is obtained by attaching two endpoints $-\infty$ and ∞. Let d be a convex metric on $c\,\mathbb{R}$ that is bounded by 1. The Hilbert cube Q is represented by

$$\prod_{i \in \mathbb{N}} c\,\mathbb{R}$$

and has metric

$$\rho(x,y) = \max\ \{d(x,y)/i \mid i \in \mathbb{N}\}.$$

Let $\pi_i : Q \to c\ \mathbb{R}$ be the projection on the i-th coordinate.

We construct the skeletoid. Identify for every $n \in \mathbb{N}$, $\mathbb{R}^n$ with $\mathbb{R}^n \times \{(0,0,0,\ldots)\} \subset Q$. This gives us the following situation:

$$\mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \ldots \subset \mathbb{R}^n \subset \ldots \subset Q$$

and in view of corollary 2.1.14:

$$\mathfrak{M}_k^{k+1} \subset \mathfrak{M}_k^{k+2} \subset \mathfrak{M}_k^{k+3} \subset \ldots .$$

Since the elements of $\mathfrak{M}_k^{k+1}$ are compact subsets of the pseudo-interior $s = \prod_{i \in \mathbb{N}} \mathbb{R}$ with dimension $\leq k$, we have that $\mathfrak{M}_k^{k+1} \subset S_k$ for every $1 \in \mathbb{N}$. Let $(C_i^n)_{i \in \mathbb{N}}$ be an $(\mathfrak{M}_k^n, H(\mathbb{R}^n))$-skeleton for $n = 2k+1,\ 2k+2, \ldots,$ theorem 2.3.1. We determine inductively functions $f_1, f_2, f_3, \ldots$ and natural numbers $n_1, n_2, n_3, \ldots$ such that for every $i \in \mathbb{N}$,

$$f_i \in H(\mathbb{R}^{2k+i})$$

and

$$\bigcup_{j=1}^{i} f_j(C_{n_i}^{2k+j}) \subset f_{i+1}(C_{n_{i+1}}^{2k+i+1}),$$

where $n_1 = 1$ and $f_1 = 1_{\mathbb{R}^{2k+1}}$. The construction is straightforward. If $j \leq i$ then $f_j(C_{n_i}^{2k+j})$ is a member of $\mathfrak{M}_k^{2k+j}$, theorem 2.1.2. According to proposition 2.1.8 this implies that $\bigcup_{j=1}^{i} f_j(C_{n_i}^{2k+j}) \in \mathfrak{M}_k^{2k+i+1}$. Since $(C_1^{2k+i+1})_{1 \in \mathbb{N}}$ is an $(\mathfrak{M}_k^{2k+i+1}, H(\mathbb{R}^{2k+i+1}))$-skeleton there exist an $f_{i+1} \in H(\mathbb{R}^{2k+i+1})$ and an $n_{i+1} > n_i$ such that $\bigcup_{j=1}^{i} f_j(C_{n_{i+1}}^{2k+j}) \subset f_{i+1}(C_{n_{i+1}}^{2k+i+1})$. If we define

$$D_i = f_i(C_{n_i}^{2k+i}) \quad \text{for } i \in \mathbb{N},$$

then $D_i \in \mathfrak{M}_k^{2k+i} \subset S_k$ and

$$D_1 \subset D_2 \subset D_3 \subset \dots .$$

In order to prove that $(D_i)_{i=1}^{\infty}$ is a skeleton we need a dimension-theoretic lemma.

3.2.1 DEFINITION: A map f from a metric space (X,δ) into a space Y is called an ε-*mapping* if for every pair $x,y \in X$ with $\delta(x,y) \geq \varepsilon$, $f(x)$ and $f(y)$ are distinct.

3.2.2 LEMMA: *If* X *is a compact metric space with dimension* $\leq k$ *and* L *is a linear* $k+l$-*variety in* $\mathbb{R}^{2k+l+1}$, $l \in \{0\} \cup \mathbb{N}$, *then for every* $\varepsilon > 0$ *the set of* ε-*mappings from* X *into* $\mathbb{R}^{2k+l+1}\setminus L$ *is dense in* $C(X, \mathbb{R}^{2k+l+1})$, *where* $C(X,Y)$ *is the space of continuous functions from* X *into* Y *with the compact-open topology.*

The proof of this lemma is an easy adaptation of [E2 : 1.10.4 and 1.11.3].

3.2.3 THEOREM*): $(D_i)_{i\in\mathbb{N}}$ *is a strong* $(S_k,H(Q))$-*skeleton*.

PROOF: In view of theorem 3.1.4 it suffices to show that $(D_i)_{i\in\mathbb{N}}$ is an $(S_k,H(Q))$-skeleton. Let $\varepsilon > 0$, $m \in \mathbb{N}$ and $S \in S_k$. Since Q is compact we only have to prove that there are a $\gamma \in H(Q)$ and a $j \in \mathbb{N}$ with $\gamma|D_m = 1$, $\gamma(S) \subset D_j$ and $\hat{\rho}(\gamma,1) < \varepsilon$. Corollary 3.1.3 reduces the problem to finding a $j \in \mathbb{N}$ and an embedding f of $S \cup D_m$ in D_j such that $f|D_m = 1$ and $\hat{\rho}(f,1) < \varepsilon$.

Select an $i \in \mathbb{N}$ with $1/i < \varepsilon/2$ and $i > m$. We shall construct a "tame"

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*) *This theorem can also be found in Dijkstra* [D1].

embedding of S in $\mathbb{R}^{2k+i+1}$. Define the function space

$$K = \{\gamma \in C(D_m \cup S, \mathbb{R}^{2k+i+1}) \mid \pi_{2k+i+1} \circ \gamma(S) \subset (-\infty,0]$$

$$\text{and } \gamma|D_m = 1\}.$$

Note that K is a closed subset of the complete metric space $(C(D_m \cup S, \mathbb{R}^{2k+i+1}),\hat{d})$, where $d = d_{2k+i+1}$. Hence, it is a Baire space. Let H be a closed subset of $\mathbb{R}^{2k+i+1}$ and let $\xi > 0$. Define the compactum

$$S_\xi = \{x \in S \mid \rho(x,D_m) \geq \xi\}$$

and the set of functions

$$K(\xi,H) = \{\gamma \in K \mid \gamma|D_m \cup S_\xi \text{ is a } \xi\text{-mapping such that}$$

$$\gamma(S_\xi) \cap H = \emptyset\}.$$

CLAIM: *If* $H = \alpha(\{p\} \times \mathbb{R}^{k+i})$, *where* $\alpha \in P_{2k+i+1}$ *and* $p \in \mathbb{R}^{k+1}$, *then* $K(\xi,H)$ *is open and dense in* K.

PROOF: Showing that $K(\xi,H)$ is open is left as an exercise to the reader. Consider the density. Let $\gamma \in K$ and $\delta > 0$. The set $\gamma(S_\xi)$ is contained in $\mathbb{R}^{2k+1} \times (-\infty,0]$. Select a γ' in $C(S_\xi, \mathbb{R}^{2k+i} \times (-\infty,0))$ with $\hat{d}(\gamma|S_\xi,\gamma') < \delta/2$. Since H is a linear k+i-variety in $\mathbb{R}^{2k+i+1}$ we can find with lemma 3.2.2 a ξ-mapping $\beta \in C(S_\xi, \mathbb{R}^{2k+i} \times (-\infty,0))$ with $\hat{d}(\beta,\gamma') < \delta/2$ and $\beta(S_\xi) \cap H = \emptyset$. Since $D_m \subset \mathbb{R}^{2k+i} \times \{0\}$ the function $\beta' = 1_{D_m} \cup \beta$ is a ξ-mapping from $D_m \cup S_\xi$ into $\mathbb{R}^{2k+i} \times (-\infty,0]$ which satisfies $\hat{d}(\beta',\gamma|D_m \cup S_\xi) < \delta$. If we apply Tietze's theorem coordinate-wise to the function $\beta' - (\gamma|D_m \cup S_\xi)$ we find an extension $\overline{\beta} : D_m \cup S \to \mathbb{R}^{2k+i} \times (-\infty,0]$ with $\hat{d}(\overline{\beta},f) < \delta$. So $\overline{\beta}$ is an element of $K(\xi,H)$ and the claim is proved.

Consider the set $L = \{\alpha(\{p\} \times \mathbb{R}^{k+i}) | \alpha \in P_{2k+i+1}$ and $p \in Q^{k+1}\}$. Select an enumeration $(L_j)_{j \in \mathbb{N}}$ of L such that for each $L \in L$ the set $\{j \in \mathbb{N} | L = L_j\}$ is infinite. Because K is a Baire space the set

$$\mathcal{D} = \bigcap_{j \in \mathbb{N}} K(\tfrac{1}{j}, L_j)$$

is dense in K. It is easily seen that the set $\{\gamma \in K | \hat{\rho}(\gamma,1) < \varepsilon/2\}$ is an open non-empty subset of K. Let h be an element of $\mathcal{D} \cap \{\gamma \in K | \hat{\rho}(\gamma,1) < \varepsilon/2\}$. If x and y are distinct points in $D_m \cup S$ then there is a $j \in \mathbb{N}$ such that $x,y \in D_m \cup S_{1/j}$ and $\rho(x,y) \geq 1/j$. Since $h|D_m \cup S_{1/j}$ is a $1/j$-mapping we may conclude that h is one-to-one and hence an embedding. Note that for every $j \in \mathbb{N}$, $h(S_{1/j}) \cap \cup L = \emptyset$ which means that $h(S_{1/j}) \subset \mathbb{R}^{2k+i+1} \backslash \cup L = N_k^{2k+i+1}$. Theorem 2.1.12 and propositions 2.1.5 and 2.1.8 imply that h(S), which is a compact subset of $D_m \cup \bigcup_{j \in \mathbb{N}} h(S_{1/j})$, is an element of $\mathfrak{M}_k^{2k+i+1}$. Obviously, one has that $\hat{\rho}(h,1) < \varepsilon/2$ and $h|D_m = 1$. The map h is the aforementioned "tame" embedding of S.

Consider now the sequence $(D_j)_{j \in \mathbb{N}}$. The set D_m is contained in $D_{i+1} = f_{i+1}(C_{n_{i+1}}^{2k+i+1})$. Since $(f_{i+1}(C_j^{2k+i+1}))_{j \in \mathbb{N}}$ is an $(\mathfrak{M}_k^{2k+i+1}, H(\mathbb{R}^{2k+i+1}))$-skeleton there exist a $g \in H(\mathbb{R}^{2k+i+1})$ and a $j \in \mathbb{N}$ such that $g|A_m = 1$, $g(h(S)) \subset f_{i+1}(C_j^{2k+i+1})$ and $\hat{\rho}(g,1) < \varepsilon/2$. Let l be such that $n_l > j$ and $l > i+1$. Then $f_{i+1}(C_j^{2k+i+1})$ is a subset of D_{l+1}. The embedding $f = g \circ h$ has the following properties:

$$f|D_m = 1,$$

$$f(S) \subset D_{l+1}$$

and

$$\hat{\rho}(f,1) < \varepsilon.$$

This concludes the proof.

CHAPTER 4

SHRUNKEN ENDFACES

4.1 Preliminaries

The main result of this chapter is a theorem that enables us to manipulate compacta in the Hilbert cube with ambient isotopies without moving certain copies of Q, called "shrunken endfaces". Let us define these objects

Let $\mathcal{R}$ be the set of all sequences $p_1,p_2,p_3,\ldots$ in $(0,1)$ such that $\lim_{i\to\infty} p_i = 1$. We pick a $(p_i)_{i\in\mathbb{N}}$ in $\mathcal{R}$ *that will remain fixed throughout sections* 4.1, 4.2 *and* 4.3. For every $i \in \mathbb{N}$ we define the *shrunken endface in the* i*-coordinate direction* by

$$W_i = \pi_i^{-1}(\{1\}) \cap \bigcap_{j\neq i} \pi_j^{-1}([-p_i,p_i]).$$

Note that W_i is a subset of E_i^1 and hence a Z-set in Q. Observe furthermore that the W_i's are disjoint copies of Q. If $\varepsilon > 0$ then there is an $i \in \mathbb{N}$ such that $1/i < \varepsilon$ and $p_j > 1 - \varepsilon$ for every $j > i$ and hence there exists for every $j > i$ a map $\beta : Q \to W_j$ with $\hat{\rho}(\beta,1) < \varepsilon$. This implies that every union of infinitely many shrunken endfaces, especially $W = \bigcup_{i\in\mathbb{N}} W_i$, is both dense and connected. Moreover, it follows that every compact subset of $Y = Q\backslash W$ is a Z-set in Q. It is easily seen that Γ_W defined by

$$\Gamma_W = \{f \in \mathcal{H}(Q) \mid \text{for every } i \in \mathbb{N},\ f(W_i) = W_i\}$$

is a closed subgroup of the topological group $(\mathcal{H}(Q),\hat{\rho})$.

Anderson, Curtis & van Mill [ACM : sec.4] have shown that Y is homogeneous. We shall prove the following stronger statement*):

Let $\mathcal{U}$ be a collection of open subsets of Q, A a compact space and $F : A \times I \to Q$ a homotopy that is limited by $\mathcal{U}$. If F_0 and F_1 are embeddings of A in Y then there is a $\mathcal{U}$-push h in Γ_W with $h \circ F_0 = F_1$.

The method we use is derived from proofs given in Chapman [C : ch.II] for theorems of this type. Moreover, in lemma 4.2.2 we use an idea of Anderson, Curtis & van Mill [ACM : 4.1].

We conclude this section with some notations. If A is a subset of a space X and $\mathcal{D}$ is a collection of subsets of X then the star of A with respect to $\mathcal{D}$ is defined by

$$St(A,\mathcal{D}) = \cup\{D \in \mathcal{D} | D \cap A \neq \emptyset\}.$$

Furthermore, $St^n(A,\mathcal{D})$, $n = 0,1,2,\dots$, is determined by

$$St^0(A,\mathcal{D}) = A$$

and

$$St^{n+1}(A,\mathcal{D}) = St(St^n(A,\mathcal{D}),\mathcal{D}).$$

4.2 The pseudo-interior

This section is about extending homeomorphisms between compact subsets of s. Consider the factorization $Q = Q_{odd} \times Q_{even}$, where

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*) *This result is taken from Dijkstra* [D2]

$$Q_{odd} = \prod_{i \in \mathbb{N}} J_{2i-1}$$

and

$$Q_{even} = \prod_{i \in \mathbb{N}} J_{2i}.$$

Let $\pi_{odd} : Q \to Q_{odd}$ and $\pi_{even} : Q \to Q_{even}$ be projections and define s_{odd}, s_{even}, 0_{odd} and 0_{even} in the obvious way.

4.2.1 LEMMA: *If* A *is a compact subset of* s *then there is a boundary preserving* $f \in \Gamma_W$ *such that for every* $x,y \in f(A)$ *with* $\pi_{even}(x) = \pi_{even}(y)$ *we have that* $\pi_{odd}(x) = \pi_{odd}(y)$.

PROOF: Let i be odd and m > i even. We may assume that A has the form $\prod_{j \in \mathbb{N}} [-a_j, a_j]$ where $a_j \in (0,1)$. Select a δ such that $a_m < \delta < 1$. Let $\varphi : J_m \times J \to J_m$ be an isotopy of J_m with the following properties:

$\varphi_1 = 1$,

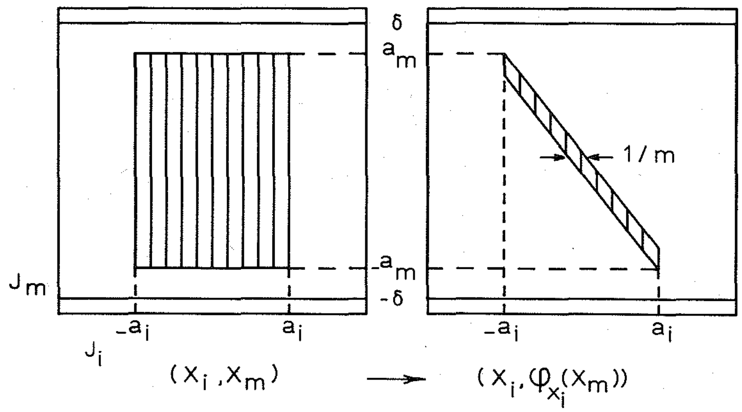
φ_t is supported on $(-\delta,\delta)$ for $t \in J$,

$\varphi_t([-a_m, a_m]) \subset [-a_m, a_m]$ for $t \in J$

and for every $y \in J_m$,

$$\operatorname{diam} \{x \in [-a_i, a_i] \mid \text{there is a } y' \in [-a_m, a_m] \text{ with } \varphi_x(y') = y\} < \frac{1}{m}.$$

See the next page for a picture of φ.



Let k be a natural number such that for every $j > k$, $p_j > \delta$. For $j \in \mathbb{N}$ let $\beta_j : J_j \to I$ be a map that satisfies $\beta_j(1) = 1$ and $\beta_j([-a_j, a_j]) = \{0\}$. Define $\chi_m^i : Q \to Q$ by $\pi_j \circ \chi_m^i = \pi_j$ for $j \neq m$ and

$$\pi_m \circ \chi_m^i(x) = \varphi(x_m, \alpha(x)) \quad \text{for} \quad x \in Q,$$

where

$$\alpha(x) = \min\{1, x_i + 2\max\{\beta_j(x_j) \mid j \in \{1,2,\ldots,k\}\backslash\{m,i\}\}\}.$$

Since α is a continuous function which is independent of x_m we have with lemma 1.1.1 that χ_m^i is a homeomorphism. Since diam $J_m = 1/m$ it is obvious that $\hat{\rho}(\chi_m^i, 1) \leq 1/m$. Furthermore, we have that $\chi_m^i(A) \subset A$ and for every endface E_n^θ, $\chi_m^i(E_n^\theta) = E_n^\theta$. We verify that $\chi_m^i \in \Gamma_W$.

(a) If $x \in W_i$ then $x_i = 1$ and hence $\alpha(x) = 1$. This implies that $\chi_m^i(x) = x$.

(b) If $x \in W_m$ then $x_m = 1$. Since $\varphi_t(1) = 1$ for every $t \in J$ this yields that $\chi_m^i(x) = x$.

(c) Let $x \in W_j$ with $j \leq k$ and $j \neq i,m$. In this case $x_j = 1$, whence $\alpha(x) = 1$ and $\chi_m^i(x) = x$.

(d) Assume that $j > k$ and $j \neq m$. This means that $p_j > \delta$. Since φ_t is supported on $(-\delta,\delta)$ we have that $\varphi_t([-p_j,p_j]) = [-p_j,p_j]$ and hence $\chi_m^i(W_j) = W_j$.

So χ_m^i is a member of Γ_W. Consider now a point z in A. Then all $\beta_j(z_j)$'s vanish and hence $\alpha(z) = z_i$ and $\pi_m \circ \chi(z) = \varphi(z_m,z_i)$. We have for every $y \in J_m$ that

$$\operatorname{diam}\{z_i \mid z \in A \text{ with } \pi_m \circ \chi_m^i(z) = y\} < \frac{1}{m}.$$

Now, let ξ be a function from $\mathbb{N}$ onto $\{2j-1 \mid j \in \mathbb{N}\}$ such that every fibre is infinite. Select with lemma 1.1.2 a strictly increasing sequence of even numbers $(m(j))_{j\in\mathbb{N}}$ such that $m(j) > \xi(j)$ and

$$f = \lim_{i\to\infty} \chi_{m(j)}^{\xi(j)} \circ \ldots \circ \chi_{m(1)}^{\xi(1)} \in H(Q).$$

It is obvious that $f(A) \subset A$, f is boundary preserving and that $f \in \Gamma_W$. Observe that $\pi_{odd} \circ f = \pi_{odd} \circ \chi_{m(j)}^{\xi(j)} = \pi_{odd}$ for every $j \in \mathbb{N}$. Let i be an odd number, $\varepsilon > 0$ and $x,y \in f(A)$ with $\pi_{even}(x) = \pi_{even}(y)$. Select a $j \in \mathbb{N}$ such that $\xi(j) = i$ and $1/j < \varepsilon$. We have the following estimate for $\rho(x_i,y_i)$:

$$\rho(x_i,y_i) \leq \operatorname{diam}\{z_i \mid z \in f(A) \text{ with } \pi_{even}(z) = \pi_{even}(x)\} \leq$$

$$\operatorname{diam}\{z_i \mid z \in \chi_{m(j-1)}^{\xi(j-1)} \circ \ldots \circ \chi_{m(1)}^{\xi(1)}(A) \text{ with } \pi_{m(j)} \circ \chi_{m(j)}^{\xi(j)}(z) =$$

$$= x_{m(j)}\} \leq \operatorname{diam}\{z_i \mid z \in A \text{ with } \pi_{m(j)} \circ \chi_{m(j)}^{\xi(j)}(z) = x_{m(j)}\} <$$

$$\frac{1}{m(j)} \leq \frac{1}{j} < \varepsilon.$$

Consequently, $\rho(x_i,y_i) = 0$ and the lemma is proved.

4.2.2 LEMMA: *If* A *is a compact subset of* s *such that for every* $x,y \in A$, $\pi_{\mathrm{even}}(x) = \pi_{\mathrm{even}}(y)$ *implies that* $\pi_{\mathrm{odd}}(x) = \pi_{\mathrm{odd}}(y)$ *then there is a boundary preserving* $h \in \Gamma_W$ *with* $\pi_{\mathrm{even}} \circ h = \pi_{\mathrm{even}}$ *and* $\pi_{\mathrm{odd}} \circ h(A) \subset \{0_{\mathrm{odd}}\}$.

PROOF: Let A be such a set. Select for every $i \in \mathbb{N}$ an $a_i \in (0,1)$ with $\pi_i(A) \subset (-a_i,a_i)$. Construct a continuous mapping $H^i : J_i \times (-a_i,a_i) \to J_i$ that satisfies for $t \in (-a_i,a_i)$: $H_0^i = 1$, $H_t^i(t) = 0$ and H_t^i is an element of $\mathcal{H}(J_i)$ that is supported on $(-a_i,a_i)$. Let $\beta_i : J_i \to I$ be a map with $\beta_i(1) = 0$ and $\beta_i([-a_i,a_i]) = \{1\}$. Select an arbitrary j in $\mathbb{N}$ and consider $\overline{A} = \pi_{\mathrm{even}}(A) \subset Q_{\mathrm{even}}$. We have that if $x,y \in A$ and $\pi_{\mathrm{even}}(x) = \pi_{\mathrm{even}}(y)$ then $x_{2j-1} = y_{2j-1}$. Since $\pi_{\mathrm{even}}|A : A \to \overline{A}$ is a quotient map this implies that there exists a continuous $g_j : \overline{A} \to (-a_{2j-1},a_{2j-1})$ such that $g_j \circ \pi_{\mathrm{even}}|A = \pi_{2j-1}|A$. Let $\tilde{g}_j : Q_{\mathrm{even}} \to (-a_{2j-1},a_{2j-1})$ be a continuous extension of g_j. Select a $\tilde{j} \in \mathbb{N}$ such that for every $k > \tilde{j}$, $a_{2\tilde{j}-1} < p_k$ and define $\alpha_j : Q \to (-a_{2j-1},a_{2j-1})$ by

$$\alpha_j(x) = \tilde{g}_j \circ \pi_{\mathrm{even}}(x) \cdot \prod_{\substack{k=1 \\ k \neq 2j-1}}^{\tilde{j}} \beta(x_k).$$

Let $h_j : Q \to Q$ be determined by $\pi_k \circ h_j = \pi_k$ if $k \neq 2j-1$ and

$$\pi_{2j-1} \circ h_j(x) = H^{2j-1}(x_{2j-1},\alpha_j(x)).$$

Since α_j is independent of x_{2j-1} we have that $h_j \in \mathcal{H}(Q)$. That h_j is an element of Γ_W follows from:

(a) If $x \in W_{2j-1}$ then $x_{2j-1} = 1$ and $H^{2j-1}(x_{2j-1},\alpha_j(x)) = 1$. This yields that $h_j(x) = x$.

(b) If $k \le \tilde{j}$ and $k \neq 2j-1$ then for $x \in W_k$, $\beta_k(x_k) = 0$. Consequently, we have that $\alpha_j(x) = 0$ and $h_j(x) = x$.

(c) Let $k > \tilde{j}$ and $k \neq 2j-1$. In this case $[-a_{2j-1}, a_{2j-1}] \subset [-p_k, p_k]$.

Since H_t^{2j-1} is supported on $(-a_{2j-1}, a_{2j-1})$ we have that $h_j(W_k) = W_k$.

It is clear that $\pi_{even} \circ h_j = \pi_{even}$ and that for every E_n^θ, $h_j(E_n^\theta) = E_n^\theta$.

Define $h = \lim_{j\to\infty} h_j \circ \ldots \circ h_1$. Obviously, h is a boundary preserving map onto Q with $\pi_{even} \circ h = \pi_{even}$. We show that h is one-to-one and hence a homeomorphism. Let x and y be distinct points in Q. If $\pi_{even}(x) \neq \pi_{even}(y)$ then also $h(x) \neq h(y)$. Assume therefore that $\pi_{even}(x) = \pi_{even}(y)$. Let $i = 2j-1$ be a coordinate with $x_i \neq y_i$ and define $x' = h_{j-1} \circ \ldots \circ h_1(x)$ and $y' = h_{j-1} \circ \ldots \circ h_1(x)$. If $\alpha_j(x') = \alpha_j(y')$ then

$$\pi_i \circ h(x) = H^i(x_i', \alpha_j(x')) = H^i(x_i, \alpha_j(x')) \neq$$

$$H^i(y_i, \alpha_j(y')) = \pi_i \circ h(y).$$

and therefore $h(x) \neq h(y)$. If, however, $\alpha_j(x') \neq \alpha_j(y')$ then in view of $\tilde{g}_j \circ \pi_{even}(x') = \tilde{g}_j \circ \pi_{even}(y')$ there is a $k \leq \tilde{j}$ with $\beta_k(x_k') \neq \beta_k(y_k')$. Consequently, $x_k' \neq y_k'$ and $\{x_k', y_k'\}$ is not contained in $[-a_k, a_k]$. We can have the following situations:

(i) $\pi_k \circ h(x) = x_k'$ and $\pi_k \circ h(y) = y_k'$ or

(ii) For some $t, r \in (-a_k, a_k)$, $\pi_k \circ h(x) = H_t^k(x_k')$ and $\pi_k \circ h(x) = H_r^k(y_k')$.

Since H_r^k and H_t^k are supported on $(-a_k, a_k)$ we may conclude in both cases that $\pi_k \circ h(x) \neq \pi_k \circ h(y)$. So $h \in \mathcal{H}(Q)$ and since h is the limit of a sequence in the closed group Γ_W we have that $h \in \Gamma_W$.

Let $x \in A$ and $i = 2j-1$. If $x' = h_{j-1} \circ \ldots \circ h_1(x)$ then $\pi_i \circ h(x) = \pi_i \circ h_j(x')$. Since $\pi_{even}(x) = \pi_{even}(x')$ and $x_i = x_i'$ we have that

$$\tilde{g}_j \circ \pi_{even}(x') = \tilde{g}_j \circ \pi_{even}(x) = g_j \circ \pi_{even}(x) = x_i = x_i'.$$

For every $k \in \mathbb{N}$, x_k is an element of $(-a_k, a_k)$ and since H_t^k is supported on

$(-a_k,a_k)$ this implies that $x' \in \prod_{k\in\mathbb{N}} (-a_k,a_k)$. Consequently, $\alpha_j(x') = x_i'$ and $\pi_i \circ h(x) = H^i(x_i',\alpha_j(x')) = 0$. So $\pi_{odd} \circ h(A) \subset \{0_{odd}\}$ and the lemma is proved.

We are now ready to prove that homeomorphisms between compacta in s can be extended.

4.2.3 LEMMA: *If* A *and* A' *are compact subsets of* s *and* h *is a homeomorphism from* A *onto* A' *then there is a boundary preserving* f *in* Γ_W *with* $f|A = h$.

PROOF: Lemma 4.2.1 and 4.2.2 reduce the problem to the statement: if A and A' are compacta in respectively s_{even} and s_{odd} and h is a homeomorphism from A onto A' then there is an $f \in \Gamma_W$ such that $f(B) = B$ and for every $a \in A$, $f(a,0_{odd}) = (0_{even},h(a))$. Define the compact subset C of s by

$$C = \{(a,h(a))|a \in A\} = \{(h^{-1}(b),b)|b \in A'\}.$$

We can apply lemma 4.2.2 to C: there is a $\gamma_1 \in \Gamma_W$ with $\gamma_1(B) = B$, $\pi_{even} \circ \gamma_1 = \pi_{even}$ and $\pi_{odd} \circ \gamma(C) \subset \{0_{odd}\}$. Analogously, there is a $\gamma_2 \in \Gamma_W$ with $\gamma_2(B) = B$, $\pi_{odd} \circ \gamma_2 = \pi_{odd}$ and $\pi_{even} \circ \gamma_2(C) \subset \{0_{even}\}$. Then $\gamma_2 \circ \gamma_1^{-1} \in \Gamma_W$ has the properties $\gamma_2 \circ \gamma_1^{-1}(B) = B$ and for every $a \in A$,

$$\gamma_2 \circ \gamma_1^{-1}(a,0_{odd}) = \gamma_2(a,h(a)) = (0_{even},h(a)).$$

Before we prove an estimated version of this lemma we give two technical lemmas.

4.2.4 LEMMA: *Let $\mathcal{U}$ be a collection of open subsets of* s *and let* A *be a compact space. If* $f : A \to s$ *is a map and* A_0 *is a closed subset of* A *such that* $f|A_0$ *is an embedding and* $f(A\backslash A_0) \subset \cup\mathcal{U}$, *then there is an embedding* g *of* A *into* s *that is* $\mathcal{U}$*-close to* f *and coincides with* f *on* A_0.

REMARK: This lemma is essentially Chapman [C : 8.1]. We have included a more elementary proof.

PROOF: Let $(F_i)_{i\in\mathbb{N}}$ and $(G_i)_{i\in\mathbb{N}}$ be sequences of compact subsets of $A\backslash A_0$ with the properties

$$F_i \cap G_i = \emptyset \text{ for every } i \in \mathbb{N},$$

$$\bigcup_{i\in\mathbb{N}} F_i = A\backslash A_0$$

and for all distinct x and y in $A\backslash A_0$ there is an $i \in \mathbb{N}$ such that $x \in F_i$ and $y \in G_i$. Select for every $i \in \mathbb{N}$ a closed neighbourhood V_i of A_0 with $V_i \cap (F_i \cup G_i) = \emptyset$. Note that $f(A\backslash V_i)$ has compact closure in $\cup\mathcal{U}$. This enables us to select a strictly increasing sequence $(m_i)_{i\in\mathbb{N}}$ of natural numbers with the property that for every $x \in f(A\backslash V_i)$ there is a $U \in \mathcal{U}$ such that $U_{2/m_i}(x) \subset U$. Observing that $\pi_{m_i} \circ f(V_i)$ is a compact subset of $\overset{\circ}{J}_{m_i} = (-1,1)$ select with Tietze's extension theorem for every $i \in \mathbb{N}$ a continuous $g_i : A \to \overset{\circ}{J}_{m_i}$ with the properties:

$$g_i|V_i = \pi_{m_i} \circ f|V_i$$

and

$$g_i(V_i) \cap (g_i(F_i) \cup g_i(G_i)) = g_i(F_i) \cap g_i(G_i) = \emptyset.$$

Define the map $g : A \to s$ by $\pi_{m_i} \circ g = g_i$ for $i \in \mathbb{N}$ and $\pi_i \circ g = \pi_i \circ f$ for $i \in \mathbb{N}\backslash\{m_j | j \in \mathbb{N}\}$. Obviously, we have that $g|A_0 = f|A_0$. The properties of $(F_i)_{i\in\mathbb{N}}$ and $(G_i)_{i\in\mathbb{N}}$ imply that g is one-to-one and hence an embedding. Let $x \in A$ and assume that m_i is the first coordinate with $\pi_{m_i} \circ f(x) \neq \pi_{m_i} \circ g(x)$. Then $x \notin V_i$ and since diam $\prod_{j=m_i}^{\infty} \overset{\circ}{J_j} = 1/m_i$, we have that $\rho(f(x),g(x)) < 2/m_i$. Consequently, there is a $U \in \mathcal{U}$ with $\{f(x),g(x)\} \subset U_{2/m_i}(f(x)) \subset U$. This means that f and g are $\mathcal{U}$-close.

The following lemma is folklore.

4.2.5 LEMMA: *Let* (X,d) *be a metric space and* $\mathcal{U}$ *a collection of open subsets of* X. *Then there is a map* $\varepsilon : X \to I$ *such that* $\varepsilon^{-1}((0,1]) = \cup\mathcal{U}$ *and for every* $x \in X$, $\{y \in X \mid d(y,x) < \varepsilon(x)\}$ *is contained in some member of* $\mathcal{U}$.

PROOF: We may assume without loss of generality that $\mathcal{U}$ is locally finite and that d is bounded by 1. Define for every $U \in \mathcal{U}$ the map $f_U : X \to I$ by

$$f_U(x) = d(x,X\setminus U).$$

Since $\mathcal{U}$ is locally finite the function $\varepsilon : X \to I$ defined by

$$\varepsilon(x) = \max\{f_U(x) \mid U \in \mathcal{U}\}$$

is continuous. It is obvious that ε meets the requirements.

We now come to the estimated extension theorem for s.

4.2.6 THEOREM: *Let* $\mathcal{U}$ *be a collection of open subsets of* Q, A *a compact space and* $F : A \times I \to s$ *a homotopy that is limited by* $\mathcal{U}$. *If* F_0 *and* F_1 *are embeddings then there is a* $\mathcal{U}$*-push* h *in* $\{\gamma \in \Gamma_W \mid \gamma(s) = s\}$ *with* $h \circ F_0 = F_1$.

PROOF: We first introduce a notation. If $\alpha : X \to I$ is continuous then the *variable product of* X *and* I is the space

$$X \times_\alpha I = \{(x,t) | x \in X \text{ and } t \in [0,\alpha(x)]\} \subset X \times I.$$

Let A_0 be the closed subset of A that is determined by $A_0 \times I = F^{-1}(Q \setminus \cup \mathcal{U})$. We have that $F_t|A_0 = F_0|A_0$ for $t \in I$ and that $\mathcal{U}$ covers $F((A\setminus A_0) \times I)$. Select an open covering $\mathcal{V}$ of $F((A\setminus A_0) \times I)$ in Q such that for every $a \in A\setminus A_0$, $St^4(F(\{a\} \times I),\mathcal{V})$ is contained in some element of $\mathcal{U}$. We may assume that every member of $\mathcal{V}$ has a non-empty intersection with $F((A\setminus A_0) \times I)$.

CLAIM 1: *There exists an isotopy* $G : Q \times I \to Q$ *that is limited by* $\mathcal{V}$ *and has the properties*: $G_t \in \Gamma_W$ *and* $G_t(s) = s$ *for* $t \in I$, $G_0 = 1$ *and* $G_1 \circ F_1(A\setminus A_0) \cap F_0(A) = \emptyset$.

A proof of this assertion can be found below. Since $\cup\mathcal{V} \subset \cup\mathcal{U}$ we have that $G_t|F_0(A_0) = 1$ for each $t \in I$. We may assume that A is a subset of the pseudo-interior of Q_2. Let η be an element of $(0,1)$ with $\eta < \min_{i \in \mathbb{N}} p_i$ and define $\alpha : Q_2 \to I$ by $\alpha(x) = \rho(x,A_0)\cdot\eta/2$. Let $\tilde{F} : A \times_\alpha I \to s$ be given by

$$\tilde{F}(a,t) = G_{t/\alpha(a)} \circ F_{t/\alpha(a)}(a) \text{ if } a \in A\setminus A_0$$

and

$$\tilde{F}(a,0) = F_0(a) \text{ if } a \in A_0.$$

It is easily verified that $\tilde{F}$ is a continuous mapping that satisfies $\tilde{F}(\{a\} \times [0,\alpha(a)]) \subset St(F(\{a\} \times I),\mathcal{V})$ for every $a \in A\setminus A_0$. Define the compact subset X of $A \times_\alpha I$ by

$$X = \{(a,t) \in A \times_\alpha I | t = 0 \text{ or } t = \alpha(a)\}.$$

Since $\tilde{F}_0 = F_0$, $F(a,\alpha(a)) = G_1 \circ F_1(a)$ for $a \in A$ and $G_1 \circ F_1(A\setminus A_0) \cap F_0(A) = \emptyset$ we have that $\tilde{F}|X$ is an embedding. According to lemma 4.2.4 there is an embedding P of $A \times_\alpha I$ in s such that $\tilde{F}$ and P are $\mathcal{V}$-close and $\tilde{F}|X = P|X$. Note that we have for every $a \in A\setminus A_0$:

$$P(\{a\} \times [0,\alpha(a)]) \subset St(\tilde{F}(\{a\} \times [0,\alpha(a)]),\mathcal{V}) \subset St^2(F(\{a\} \times I),\mathcal{V}).$$

CLAIM 2: *There exists an isotopy* $H : Q \times I \to Q$ *that is limited by* $\mathcal{W} = \{St(P(\{a\} \times [0,\alpha(a)]),\mathcal{V}) | a \in A\setminus A_0\}$ *and that satisfies moreover* $H_t \in \Gamma_W$ *and* $H_t(s) = s$ *for* $t \in I$, $H_0 = 1$ *and* $H_1 \circ F_0 = G_1 \circ F_1$.

Define the isotopy $\tilde{H} : Q \times I \to Q$ by

$$\tilde{H}_t = (G_t)^{-1} \circ H_t \text{ for } t \in I.$$

One readily sees that $\tilde{H}_0 = 1$, $\tilde{H}_1 \circ F_1 = F_0$ and for $t \in I$, $\tilde{H}_t \in \Gamma_W$ and $\tilde{H}_t(s) = s$. We shall see that $\tilde{H}$ is limited by $\{St^4(F(\{a\} \times I),\mathcal{V}) | a \in A\setminus A_0\}$ and hence by $\mathcal{U}$. Let $x \in Q$ and assume firstly that $H(\{x\} \times I) = \{x\}$. Pick an arbitrary $t \in I$ and let y be such that $G_t(y) = x$. If $x \in \cup\mathcal{V}$ then there is a $V \in \mathcal{V}$ with $\{G_0(y),G_t(y)\} = \{y,x\} \subset V$. Consequently, $H(\{x\} \times I)$ is contained in $St(\{x\},\mathcal{V})$ and since every element of $\mathcal{V}$ intersects $F((A\setminus A_0) \times I)$, $\tilde{H}(\{x\} \times I) \subset St^2(F(\{a\} \times I),\mathcal{V})$ for some $a \in A\setminus A_0$. If $x \notin \cup\mathcal{V}$ then $G(\{x\} \times I) = \{x\}$ and hence $\tilde{H}(\{x\} \times I) = \{x\}$.

Consider now the second case that $H(\{x\} \times I)$ is contained in $St(P(\{a\} \times [0,\alpha(a)]),\mathcal{V})$ for some $a \in A\setminus A_0$. If $t \in I$ then we have as above that there is a $V \in \mathcal{V}$ such that $\{\tilde{H}_t(x),H_t(x)\} \subset V$. This means that $\tilde{H}(\{x\} \times I)$ is contained in $St^2(P(\{a\} \times [0,\alpha(a)]),\mathcal{V})$ and hence that

$$\tilde{H}(\{x\} \times I) \subset St^4(F(\{a\} \times I),\mathcal{V}).$$

So we may conclude that $\tilde{H}_1$ is the $\mathcal{U}$-push we need. It remains to prove the claims.

PROOF of claim 1: According to 4.2.1 and 4.2.2 there is a boundary preserving χ in Γ_W such that $\pi_1 \circ \chi \circ F(A \times I) \subset \{0\}$. Let $\hat{A}_{(0)}$ be the projection of $\chi \circ F_1(A_{(0)})$ on Q_2 and select a θ in $(0, \min_{i \in \mathbb{N}} p_i)$. According to lemma 4.2.5 there is a map $\varepsilon : Q_2 \to [0,\theta]$ such that $\varepsilon(\hat{A}\backslash\hat{A}_0) \subset (0,\theta]$ and for every $x \in Q_2$, $U_{\varepsilon(x)}(0,x)$ is contained in some element of $\chi(\mathcal{V})$. Let $\varphi : J_1 \times [0,\theta] \to J_1$ be an isotopy of J_1 such that $\varphi_0 = 1$, $\varphi_t(0) = \frac{1}{2}t$ and φ_t is supported on $(-t,t)$ for $t \in [0,\theta]$. Define the isotopy $G : Q \times I \to Q$ by

$$G_t(x,y) = (\varphi_{t\varepsilon(y)}(x),y) \text{ for } x \in I,\ y \in Q_2 \text{ and } t \in I.$$

The maps G_t are obviously boundary preserving and since $\theta < \min_{i \in \mathbb{N}} p_i$ they are elements of Γ_W. It is easily seen that G is limited by $\chi(\mathcal{V})$ and that $G_1(\{0\} \times (\hat{A}\backslash\hat{A}_0))$ misses $\{0\} \times Q_2$. This means that $\chi^{-1} \circ G_t \circ \chi$ is the isotopy we need.

PROOF of claim 2: Note that since A is a subset of the pseudo-interior of Q_2 the variable product $A \times_\alpha I$ is contained in s (write $Q = Q_2 \times J_1$). So P is a homeomorphism between two compact subset of s. According to lemma 4.2.3 there is a boundary preserving $h \in \Gamma_W$ such that for each $(a,t) \in A \times_\alpha I$ we have that $h(a,t) = P(a,t)$. Consider the following open covering of $(A\backslash A_0) \times_\alpha I$ in Q:

$$\begin{aligned}\mathcal{W}' = \{U_\varepsilon(\{a\} \times [0,\alpha(a)]) \mid\ & a \in A\backslash A_0,\ \varepsilon > 0 \text{ and} \\ & U_\varepsilon(\{a\} \times [0,\alpha(a)]) \subset h^{-1}(W) \text{ for some } W \in \mathcal{W}\}.\end{aligned}$$

By virtue of lemma 4.2.5 there is a map $\delta : Q_2 \to [0,\eta/2]$ such that $\delta(A\backslash A_0) \subset (0,\eta/2]$ and for every $x \in Q_2$, $U_{\delta(x)}(x,\alpha(x))$ is contained in some element of $\mathcal{W}'$. Define the open set $O = \{x \in Q_2 | \delta(x) > 0\}$ and construct with Tietze's theorem a continuous $\beta : Q_2\backslash A_0 \to [0,\eta\backslash 2]$ that extends $\alpha|A\backslash A_0$ and satisfies $\beta(x) = 0$ for $x \notin O$ and $\beta(x) \leq \alpha(x)$ for $x \in Q_2\backslash A_0$. Since $\alpha(a) = 0$ for $a \in A_0$ the function $\overline{\beta} : Q_2 \to [0,\eta/2]$ that is defined by $\overline{\beta}(x) = \beta(x)$ if $x \notin A_0$ and $\overline{\beta}(x) = 0$ if $x \in A_0$, is continuous.

Let C be the space $([0,\eta/2] \times (0,\eta/2]) \cup \{(0,0)\} \subset I^2$ and construct a continuous function $\psi : J_1 \times C \to J_1$ with the properties

$$\psi_{t,r} \in \mathcal{H}(J_1),$$

$$\psi_{t,0} = 1,$$

$$\psi_{t,r} \text{ is supported on } (-t,r+t)$$

and

$$\psi_{t,r}(0) = r,$$

where we used the notation $\psi_{t,r}(x) = \psi(x,t,r)$ for $x \in J_1$ and $(t,r) \in C$. Just as if ψ were an isotopy we can construct an isotopy $H : Q \times I \to Q$ by $\pi_i \circ H_t = \pi_i$ if $i > 1$ and

$$\pi_1 \circ H_t(y,x) = \psi(x,\delta(y),t\overline{\beta}(y)) \text{ for } x \in J_1 \text{ and } y \in Q_2.$$

The following properties of H are easily verified:

$$H_0 = 1,$$

$$H_t \in \{\gamma \in \Gamma_W | \gamma(s) = s\} \text{ for } t \in I$$

and

$$H_1(a,0) = (a,\alpha(a)) \text{ for } a \in A.$$

We prove that H is limited by $h^{-1}(\mathcal{W})$. Let $(y,x) \in Q_2 \times J_1$ and select an

$\varepsilon > 0$ and an $a \in A\backslash A_0$ such that

$$U_{\delta(y)}(y,\alpha(y)) \subset U_\varepsilon(\{a\} \times [0,\alpha(a)]) \in \mathcal{W}'.$$

Then $\delta(y) \le \varepsilon$ and hence $\{y\} \times (-\delta(y),\alpha(y) + \delta(y))$ is contained in $U_\varepsilon(\{a\} \times [0,\alpha(a)])$ which is in turn a subset of an element $h^{-1}(W)$ of $h^{-1}(\mathcal{W})$. Recall that $\psi_{\delta(y),t\bar{\beta}(y)}$ is supported on $(-\delta(y), t\bar{\beta}(y) + \delta(y))$ and hence on $(-\delta(y),\alpha(y) + \delta(y))$. This implies that $H(\{(y,x)\} \times I) = \{(y,x)\}$ or that $H(\{y,x\} \times I) \subset \{y\} \times (-\delta(y),\alpha(y) + \delta(y))$. So we have shown that H is limited by $h^{-1}(\mathcal{W})$.

Let us now introduce the isotopy

$$H'_t = h \circ H_t \circ h^{-1} \text{ for } t \in I.$$

Obviously, we have that $H'_0 = 1$, $H'_t \in \{\gamma \in \Gamma_W | \gamma(s) = s\}$ for $t \in I$ and that H' is limited by $\mathcal{W}$. H'_1 is a $\mathcal{W}$-push in Γ_W with the property that for every $a \in A$:

$$H'_1 \circ F_0(a) = h \circ H_1 \circ h^{-1} \circ P(a,0) = h \circ H_1(a,0) =$$

$$h(a,\alpha(a)) = P(a,\alpha(a)) = \tilde{F}(a,\alpha(a)) = G_1 \circ F_1(a).$$

This proves claim 2.

4.2.7 COROLLARY: *Let* A *and* A' *be compact subsets of* s. *If* $h : A \to A'$ *is a homeomorphism with* $\hat{\rho}(h,1) < \varepsilon$ *then there is an* $\tilde{h} \in \Gamma_W$ *with* $\hat{\rho}(\tilde{h},1) < \varepsilon$, $\tilde{h}|A = h$ *and* $\tilde{h}(s) = s$.

PROOF: Define the map $F : A \times I \to s$ by $F(a,t) = (1-t)a + th(a)$. The straight-line homotopy F is limited by $\mathcal{U} = \{U_{\varepsilon/2}(x) | x \in Q\}$. Apply theorem 4.2.6 to F. The $\mathcal{U}$-push $\tilde{h}$ we get has the properties $\tilde{h} \in \Gamma_W$, $\tilde{h}|A = h$, $\hat{\rho}(\tilde{h},1) < \varepsilon$ and $\tilde{h}(s) = s$.

4.3 The estimated extension theorem

In this section we reduce our problems to compacta in s so that theorem 4.2.6 can be applied. We prove that any compact set that is disjoint from W can be homeomorphed into s. We conclude the section with an observation that shows that Y is not quite as homogeneous as ℓ^2.

4.3.1 LEMMA: *Let* A *be a compact subset of an endface* E_n^θ *such that* $A \cap W = \emptyset$. *Then there are for each* $\varepsilon > 0$ *an* $h \in \Gamma_W$ *and an* $m > n$ *such that* $h(A) \cap \cup\{E_i^\mu \mid i < m$ *and* $\mu \in \{-1,1\}\} = \emptyset$, $h(A) \subset E_m^{-1}$ *and* $\hat{\rho}(h,1) < \varepsilon$.

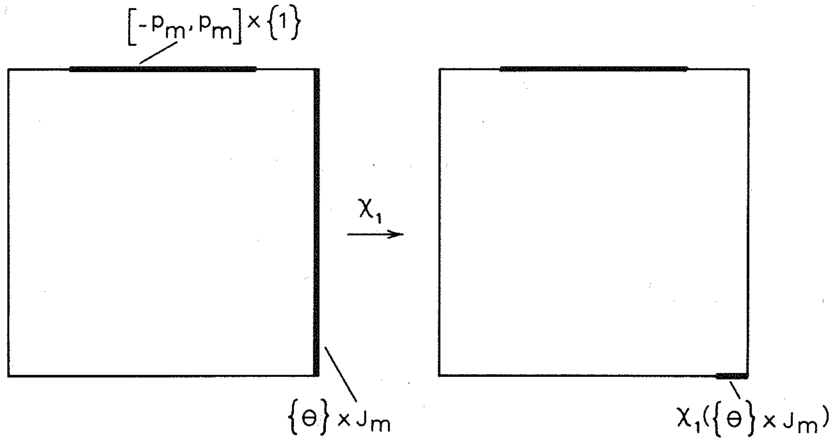
PROOF: Let $\varepsilon > 0$ and select an $m > n$ with $1/m < \rho(A,W_n)$ and $1/m < \varepsilon/2$. We first push A into E_m^{-1} and then away from the endfaces in the lower coordinate directions. Noting that diam $(J_m) = 1/m$ it is geometrically obvious that there exists an $\varepsilon/2$-isotopy $\chi : \partial(J_n \times J_m) \times I \to \partial(J_n \times J_m)$ such that $\chi_0 = 1$,

$$\chi_t|([-p_m,p_m] \times \{1\}) \cup (\{-\theta\} \times J_m) = 1 \text{ for } t \in I$$

and

$$\chi_1(\{\theta\} \times J_m) \subset J_n \times \{-1\}.$$

See the facing page for a picture of χ_1.



Noting that $J_n \times J_m$ is a subset of the linear space $\mathbb{R}^2$ define the $\varepsilon/2$-isotopy $\hat{\chi}$ of $J_n \times J_m$ by $\hat{\chi}_t(0) = 0$ and

$$\hat{\chi}_t(x) = \|x\| \, \chi_t(x/\|x\|) \text{ if } x \neq 0 \text{ and } t \in I.$$

Observe that $\hat{\chi}_t$ is *norm preserving*, i.e. $\|\hat{\chi}(x)\| = \|x\|$ for every x. Define $h \in H(Q)$ by $\pi_i \circ h = \pi_i$ for $i \neq m,n$ and

$$\pi_i \circ h(x) = \pi_i \circ \hat{\chi}_{\alpha(x)}(x_n, x_m) \text{ for } i = m,n,$$

where

$$\alpha(x) = \min\{1, m \,.\, \max(\{-\theta\} \cup \{\rho(x_j, [-p_n, p_n]) \mid$$
$$j \in \{1,\ldots,m-1\}\backslash\{n\}\})\}.$$

It is obvious that $\hat{\rho}(h,1) < \varepsilon/2$. The function h is a member of Γ_W because:

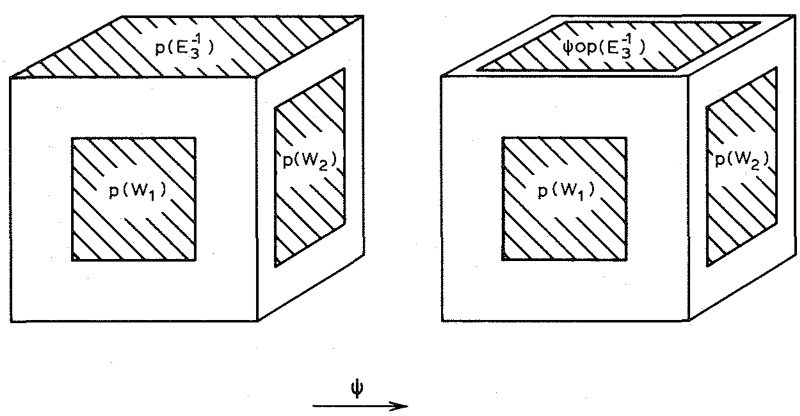
(a) Let $x \in W_n$. If $\theta = -1$ then $x_n = -\theta$ and $\hat{\chi}_t(x_n, x_m) = (x_n, x_m)$ for every $t \in I$. This means that $h(x) = x$. Let now $\theta = 1$. For every $i \neq n$ we have that $x_i \in [-p_n, p_n]$ and hence $\alpha(x) = 0$. So again $h(x) = x$.

(b) If $x \in W_m$ then $(x_n, x_m) \in [-p_m, p_m] \times \{1\}$. Since this set is fixed by χ_t and $\hat{\chi}_t$ we have that $h(x) = x$.

(c) Let $i \neq m,n$. Since $\hat{\chi}_t$ is norm preserving we have that $\chi_t([-p_i, p_i]^2) = [-p_i, p_i]^2$ and hence that $h(W_i) = W_i$.

If $x \in A$ and $\theta = -1$ then $\alpha(x) = 1$ which yields that $h(x) \in E_m^{-1}$. If $\theta = 1$ then $\rho(x, W_n) > 1/m$ implies that there is a $j < m$ such that $j \neq n$ and $\rho(x_j, [-p_n, p_n]) > 1/m$. Consequently, $\alpha(x) = 1$ and $h(x) \in E_m^{-1}$. The conclusion is that $h(A) \subset E_m^{-1}$.

Consider now $B_m = \prod_{j=1}^{m} J_j$ and the projection $p : Q \to B_m$. There is a homeomorphism ψ of ∂B_m such that $\hat{\rho}(\psi, 1) < \varepsilon/2$, $\psi(p(E_m^{-1})) \subset (\prod_{j=1}^{m-1} J_j^\circ) \times \{-1\}$ and for every $j \leq m$, $\psi | p(W_j) = 1$ (the picture gives the situation for $m = 3$).



Let $\hat{\psi} \in H(B_m)$ be given by $\hat{\psi}(0) = 0$ and $\hat{\psi}(x) = \|x\| \, \psi(x/\|x\|)$ for $x \neq 0$. Define $g \in H(B_m)$ by $g(x,y) = (\hat{\psi}(x),y)$ for $x \in B_m$ and $y \in Q_m$. We show that $g \in \Gamma_W$. If $j \leq m$ then $\hat{\psi}|p(W_j) = \psi(p(W_i)) = 1$ and hence $g|W_j = 1$. If $j > m$ then, since $\hat{\psi}$ is norm preserving, we have that $\hat{\psi}([-p_j,p_j]^m) = [-p_j,p_j]^m$ and $g(W_j) = W_j$. If x is an element of E_m^{-1} then $\pi_i \circ \hat{\psi} \circ p(x) \in \overset{\circ}{J_i}$ for $i < m$. This means that $g(E_m^{-1})$ and $\cup\{E_i^{\mu} | i < m \text{ and } \mu \in \{-1,1\}\}$ are disjoint. Also we have that $g(E_m^{-1}) \subset E_m^{-1}$ and $\hat{\rho}(g,1) < \varepsilon/2$. It is now obvious that $g \circ f$ is the homeomorphism we need.

4.3.2 LEMMA: *If* A *is a compact subset of* $E_n^{\theta}\backslash W$ *then there is for every* $\varepsilon > 0$ *an* $f \in \Gamma_W$ *with* $\hat{\rho}(f,1) < \varepsilon$ *and* $f(A) \subset s$.

PROOF: Using the convergence criterion 1.1.2 we can find sequences $(f_i)_{i \in \mathbb{N}}$ in Γ_W and $m_1 < m_2 < m_3 < \ldots$ in $\mathbb{N}$ such that $f = \lim_{i\to\infty} f_i \circ \ldots \circ f_1 \in \Gamma_W$ and $f_i \circ \ldots \circ f_1(A) \cap \cup\{E_j^{\theta} | j < m_i$ and $\theta \in \{-1,1\}\} = \emptyset$. If we take care that for every $i \in \mathbb{N}$,

$$\sum_{j=i+1}^{\infty} \hat{\rho}(f_j,1) < \rho(f_i \circ \ldots \circ f_1(A), \cup\{E_j^{\theta} | j < m_i \text{ and } \theta \in \{-1,1\}\})$$

then $f(A) \subset s$.

4.3.3 LEMMA: *If* A *is a compact subset of* Y *then for every* E_n^{θ} *and* $\varepsilon > 0$ *there is an* $f \in \Gamma_W$ *with* $\hat{\rho}(f,1) < \varepsilon$ *and* $f(A) \cap E_n^{\theta} = \emptyset$.

PROOF: Let A be a compactum in Y, let $\varepsilon > 0$ and put $\delta = \min\{\frac{1}{2}\rho(A,W_n),\varepsilon\}$. Define the compact set $\hat{A} = \{x \in E_n^{\theta} | \rho(x,A) \leq \delta\}$. According to lemma 4.3.2 there is a $\chi \in \Gamma_W$ with $\hat{\rho}(\chi,1) < \delta/4$ and $\chi(\hat{A}) \subset s$. If m is a natural number such that $1 - p_m < \delta/4$ and $1/m < \delta/4$ then there is a map $h : Q \to W_m$ with $\hat{\rho}(h,1) < \delta/4$. Note that $h \circ \chi(\hat{A}) \cap A = \emptyset$ and construct a continuous $g : Q \to s$ such that

$$\hat{\rho}(g,1) < \min\{\delta/4, \rho(h \circ \chi(\hat{A}), A)\}.$$

Since $g \circ h \circ \chi(\hat{A}) \subset s$ and $g \circ h \circ \chi(\hat{A}) \cap A = \emptyset$ there exists by virtue of lemma 4.2.4 an embedding β of $\chi(\hat{A})$ in s that satisfies

$$\hat{\rho}(g \circ h|\chi(\hat{A}), \beta) < \min\{\delta/4, \rho(g \circ h \circ \chi(\hat{A}), A)\}.$$

We now have the following situation: $\hat{\rho}(\beta,1) < 3\delta/4$, β is a homeomorphism between compact subsets of s and $\beta \circ \chi(A) \cap A = \emptyset$. In view of corollary 4.2.7 there is an extension $\overline{\beta} \in \Gamma_W$ of β with $\hat{\rho}(\overline{\beta},1) < 3\delta/4$. Consider $f = (\overline{\beta} \circ \chi)^{-1} \in \Gamma_W$. We have that $\hat{\rho}(f,1) < \varepsilon$ and $f(A) \cap \hat{A} = \emptyset$. If $x \in f(A)$ then $\rho(x,A) < \delta$ and $x \notin \hat{A}$. This implies that $x \notin E_n^\theta$ and the conclusion is that $f(A) \cap E_n^\theta = \emptyset$.

4.3.4 LEMMA: *If* A *is a compactum in* Y *then for every* $\varepsilon > 0$ *there exists an* $f \in \Gamma_W$ *such that* $\hat{\rho}(f,1) < \varepsilon$ *and* $f(A) \subset s$.

PROOF: This is a straightforward application of the convergence criterion, see lemma 4.3.2.

Before we prove the main result a technical lemma.

4.3.5 LEMMA: *Let* $\mathcal{U}$ *be a collection of open subsets of* Q *and let* A *be a compact space. If* f *is a continuous function from* A *into* Q *and* A_0 *is a closed subset of* A *such that* $f(A \setminus A_0) \subset \cup\mathcal{U}$ *and* $f(A_0) \subset s$, *then there is a continuous* $g : A \to s$ *that is* $\mathcal{U}$*-close to* f *and that coincides with* f *on* A_0.

PROOF: Select for every $i \in \mathbb{N}$ a compact neighbourhood V_i of A_0 with

$\pi_i \circ f(V_i) \subset \overset{\circ}{J}_i$. Let $(\varepsilon_i)_{i \in \mathbb{N}}$ be a decreasing sequence of numbers from $(0,\frac{1}{2})$ such that for every $x \in f(A \backslash V_i)$ there is a $U \in \mathcal{U}$ with $U_{\varepsilon_i}(x) \subset U$. Select for every $i \in \mathbb{N}$ a continuous $g_i : A \to \overset{\circ}{J}_i$ such that $\hat{\rho}(g_i, \pi_i \circ f) < \varepsilon_i$ and $g_i | V_i = \pi_i \circ f | V_i$. Let $g : A \to s$ be defined by $\pi_i \circ g = g_i$ for $i \in \mathbb{N}$. Assume that x is an element of A with $f(x) \neq g(x)$. If i is the first coordinate with $\pi_i \circ f(x) \neq g_i(x)$ then $x \notin V_i$ and there is a $U \in \mathcal{U}$ such that $U_{\varepsilon_i}(f(x)) \subset U$. Since $\rho(f(x),g(x)) \leq \sup \{\rho(\pi_j \circ f(x), g_j(x)) | j \geq i\} < \varepsilon_i$ we have that both $f(x)$ and $g(x)$ are in U. This shows that f and g are $\mathcal{U}$-close and since it is obvious that $g|A_0 = f|A_0$, the proof is completed.

4.3.6 THEOREM: *Let* $\mathcal{U}$ *be a collection of open subsets of* Q, A *a compact space and* $F : A \times I \to Q$ *a homotopy that is limited by* $\mathcal{U}$. *If* F_0 *and* F_1 *are embeddings of* A *in* Y *then there is a* $\mathcal{U}$*-push* h *in* Γ_W *with* $h \circ F_0 = F_1$.

PROOF: Let A_0 be the closed subset of A that is determined by $A_0 \times I = F^{-1}(Q \backslash \cup \mathcal{U})$. Since $F_0(A) \cup F_1(A)$ is a compact subset of Y there exists by virtue of lemma 4.3.4 an $f \in \Gamma_W$ with $f(F_0(A) \cup F_1(A)) \subset s$. Let $\tilde{F}$ be the homotopy $f \circ F$. Select an open covering $\mathcal{V}$ of $F((A \backslash A_0) \times I)$ in Q such that for every $a \in A \backslash A_0$, $\mathrm{St}(\tilde{F}(\{a\} \times I), \mathcal{V})$ is contained in some element of $f(\mathcal{U})$. Note that $\tilde{F}_0(A) \cup \tilde{F}_1(A) = \tilde{F}_0(A) \cup \tilde{F}_1(A) \cup \tilde{F}(A_0 \times I)$. According to lemma 4.3.5 there is a homotopy $G : A \times I \to s$ that is $\mathcal{V}$-close to $\tilde{F}$ and that coincides with $\tilde{F}$ on $(A \times \{0,1\}) \cup (A_0 \times I)$. Since G is also limited by $f(\mathcal{U})$ we find with theorem 4.2.6 an $f(\mathcal{U})$-push g in Γ_W such that $g \circ G_0 = G_1$. Then $h = f^{-1} \circ g \circ f$ is a $\mathcal{U}$-push in Γ_W with $h \circ F_0 = F_1$.

4.3.7 COROLLARY: *If* h *is a homeomorphism between compacta in* Y *with* $\hat{\rho}(h,1) < \varepsilon$ *then it has an extension* $\hat{h} \in \Gamma_W$ *such that* $\hat{\rho}(\hat{h},1) < \varepsilon$.

PROOF: See corollary 4.2.7.

The next corollary has already been introduced as theorem 3.1.2. It is essentially due to Anderson & Chapman [AC].

4.3.8 COROLLARY: *Let* $\mathcal{U}$ *be a collection of open subsets of* Q, A *a compact space and* $F : A \times I \to Q$ *a homotopy that is limited by* $\mathcal{U}$. *If both* F_0 *and* F_1 *are embeddings such that their image is a Z-set then there exists a* $\mathcal{U}$*-push* h *in* $\mathcal{H}(Q)$ *with* $h \circ F_0 = F_1$.

PROOF: According to Chapman [C : 10.2] there is an $f \in \mathcal{H}(Q)$ with $f(F_0(A) \cup F_1(A)) \subset s \subset Y$. Apply theorem 4.3.6 to the homotopy $f \circ F$.

As is well known theorem 4.3.6 holds also for $\ell^2 \approx s$ (cf. theorem 4.2.6). In ℓ^2 we can also extend homeomorphisms between non-compact Z-sets, Anderson [A2]. This is not the case for Y. To show this we need the following lemma that we took from Anderson, Curtis & van Mill [ACM : 3.6].

4.3.9 LEMMA: *Let* B_1 *and* B_2 *be* σ*-Z-sets in* Q *and let* $f : Q\backslash B_1 \to Q\backslash B_2$ *be a homeomorphism. Then there exist a compact space* M *and monotone maps* $\gamma_1, \gamma_2 : M \to Q$ *such that* $\gamma_1^{-1}(B_1) = \gamma_2^{-1}(B_2)$ *and* $f \circ \gamma_1 | \gamma_1^{-1}(Q\backslash B_1) = \gamma_2 | \gamma_2^{-1}(Q\backslash B_2)$.

Recall that a map h is *monotone* if it is onto, closed and has the property that every fibre is connected or, equivalently, the pre-image under h of every connected set is connected.

PROOF: Let M be the closure of the graph of f in $Q \times Q$ and take for γ_1 an γ_2 the restrictions to M of the projections $Q \times Q \to Q$. By symmetry, it suffices to prove that γ_1 is monotone. Since M is compact and $Q\backslash B_1$ is dense in Q, γ_1 is closed and onto. Let $x \in Q$ and consider the ε-ball $U_\varepsilon(x)$. Since every path in $U_\varepsilon(x)$ connecting two points of $U_\varepsilon(x)\backslash B_1$, can be pushed off the σ-Z-set B_1 we have that $U_\varepsilon(x)\backslash B_1$ is connected. So

$$C = \{Cl_M\{(a,h(a))|a \in U_\varepsilon(x)\backslash B_1\}|\varepsilon > 0\}$$

is a collection of continua that is linearly ordered by $\subset$. Since $\gamma_1^{-1}(\{x\})$ equals $\cap C$ it is also a continuum. The other properties of γ_1 and γ_2 are obvious.

Now let L_1 and L_2 be two copies of $(0,1)$ that are embedded in Y as Z-sets such that $L_1 \cup W_1 \cup W_2$ and $L_2 \cup W_1$ are continua. So L_1 and L_2 are paths going from W_1 to W_2 and from W_1 to W_1, respectively.

4.3.10 PROPOSITION: *There is no* $h \in H(Y)$ *that throws* L_1 *onto* L_2.

PROOF: Assume that $h \in H(Y)$ has the property that $h(L_1) = L_2$. There are a compact space M and monotone maps $\gamma_1, \gamma_2 : M \to Q$ such that $\gamma_1^{-1}(W) = \gamma_2^{-1}(W)$ and $h \circ \gamma_1|\gamma_1^{-1}(Y) = \gamma_2|\gamma_2^{-1}(Y)$. Since $W_1 \cup W_2 \cup L_1$ is a continuum and γ_1 is monotone we have that $\gamma_1^{-1}(W_1 \cup W_2 \cup L_1)$ and hence $\gamma_2(\gamma_1^{-1}(W_1 \cup W_2 \cup L_1))$ is a continuum. Note that $\gamma_2(\gamma_1^{-1}(W_1 \cup W_2 \cup L_1))$ is covered by the disjoint collection $\{L_2 \cup W_1\} \cup \{W_i|i \geq 2\}$. Applying the Sierpiński theorem, see section 5.2, we find that $\gamma_2(\gamma_1^{-1}(W_1 \cup W_2 \cup L_1))$ is contained in $L_2 \cup W_1$. Since $\gamma_1^{-1}(W) = \gamma_2^{-1}(W)$ this means that $\gamma_1^{-1}(W_1 \cup W_2) \subset \gamma_2^{-1}(W_1)$. If we apply the same argument to the continuum $\gamma_1(\gamma_2^{-1}(W_1))$ we find that $\gamma_1(\gamma_2^{-1}(W_1)) = W_1 \cup W_2$ which is obviously false.

4.4 Shifting shrunken endfaces

In this section we prove that whatever choice we make for $p \in \mathcal{R}$, the space Y is topologically always the same. Furthermore, it is shown that subsets of Y that are homeomorphic to Q are negligible. In order to prove the first assertion we need a notation that distinguishes between representations of Y.

4.4.1 NOTATION: If $r \in (0,1)$ and $i \in \mathbb{N}$ then we define the shrunken endface $W_i(r)$ by

$$W_i(r) = \pi_i^{-1}(\{i\}) \cap \bigcap_{j \neq i} \pi_j^{-1}([-r,r]).$$

If $p = (p_i)_{i \in \mathbb{N}} \in \mathcal{R}$ then $W(p) = \bigcup_{i \in \mathbb{N}} W_i(p_i)$; $\Gamma_{W(p)}$ and Y_p are defined in the obvious way. The set $\mathcal{R}^{\uparrow}$ is given by

$$\mathcal{R}^{\uparrow} = \{p \in \mathcal{R} \mid p_1 < p_2 < p_3 < \ldots\}.$$

4.4.2 LEMMA: *If* $p \in \mathcal{R}$ *then there is a* $q \in \mathcal{R}^{\uparrow}$ *and an* $f \in \mathcal{H}(Q)$ *such that* $f(Y_p) = Y_q$.

PROOF: Let $p \in \mathcal{R}$. We show that there are a $q \in \mathcal{R}$ and an $f \in \mathcal{H}(Q)$ such that for $i \neq j$, $q_i \neq q_j$ and $f(Y_p) = Y_q$. If we have established this then the lemma follows by simply applying a permutation of coordinates.

We construct inductively a sequence $f_1, f_2, f_3, \ldots$ in $\mathcal{H}(Q)$ and a sequence $q_1, q_2, q_3, \ldots$ in $(0,1)$ such that for every $i \in \mathbb{N}$:

$$q_i \notin \{q_1, \ldots, q_{i-1}\},$$

$$p_i \leq q_i,$$

$$f_i(W_j(q_j)) = W_j(q_j) \quad \text{for } j < i,$$

$$f_i(W_i(p_i)) = W_i(q_i)$$

and

$$f_i(W_j(p_j)) = W_j(p_j) \quad \text{for } j > i.$$

In order to obtain that $f = \lim_{i\to\infty} f_i \circ \ldots \circ f_1 \in \mathcal{H}(Q)$ we make sure that every f_i can be chosen arbitrarily close to 1. It is obvious that f and $q = (q_i)_{i\in\mathbb{N}}$ meet the requirements.

Put $f_1 = 1$ and $q_1 = p_1$. Suppose that h_i and q_i have been selected. Let $\varepsilon > 0$ be such that $(p_{i+1}, p_{i+1} + \varepsilon) \cap \{q_1,\ldots,q_i\} = \emptyset$ and $p_{i+1} + \varepsilon < 1$. Pick an element q_{i+1} of $(p_{i+1}, p_{i+1} + \varepsilon)$ and define $r \in \mathcal{R}$ by $r_j = q_j$ for $j \leq i$ and $r_j = p_j$ for $j > i$. Let $\chi \in \mathcal{H}(Q)$ be defined by $\chi(x) = (x_1,\ldots,x_i, -x_{i+1}, x_{i+2}, x_{i+3},\ldots)$. Note that $\chi(W_{i+1}(p_{i+1}))$ and $\chi(W_{i+1}(q_{i+1}))$ are subsets of Y_r and that there exists a homeomorphism $g : \chi(W_{i+1}(p_{i+1})) \to \chi(W_{i+1}(q_{i+1}))$ with $\hat{\rho}(g,1) < q_{i+1} - p_{i+1}$. In view of corollary 4.3.7 there is an extension $\overline{g} \in \Gamma_{W(r)}$ of g such that $\hat{\rho}(\overline{g},1) < q_{i+1} - p_{i+1}$. Then $f_{i+1} = \chi \circ \overline{g} \circ \chi$ has the following properties:

$$f_{i+1}(W_j(q_j)) = W_j(q_j) \quad \text{for } j \leq i,$$

$$f_{i+1}(W_{i+1}(p_{i+1})) = W_{i+1}(q_{i+1}),$$

$$f_{i+1}(W_j(p_j)) = W_j(p_j) \quad \text{for } j > i+1$$

and

$$\hat{\rho}(f_{i+1},1) < q_{i+1} - p_{i+1}.$$

This completes the induction.

4.4.3 THEOREM: *If* $p,q \in \mathcal{R}$ *then there is an* $f \in \mathcal{H}(Q)$ *such that* $f(Y_p) = Y_q$.

PROOF: In view of lemma 4.4.2 it suffices to prove the theorem for $p,q \in \mathcal{R}^{\uparrow}$. Let β be an element of $\mathcal{H}(J)$ such that for every $i \in \mathbb{N}$, $\beta(p_i) = q_i$ and $\beta(-p_i) = -q_i$. If $f = \prod_{i \in \mathbb{N}} \beta \in \mathcal{H}(Q)$ then $f(Y_p) = Y_q$.

4.4.4 LEMMA*): *If* $p \in \mathcal{R}^{\uparrow}$ *then there is an* $f \in \mathcal{H}(Q)$ *such that for every* $i \in \mathbb{N}$, $f(W_i(p_i)) = W_{i+1}(p_{i+1})$.

PROOF: Let $p \in \mathcal{R}^{\uparrow}$ and construct for every $i \in \mathbb{N}$ a norm preserving $\beta_i \in \mathcal{H}(J \times J)$ such that

$$\beta_i(\{1\} \times [-p_{2i-1},p_{2i-1}]) = \{1\} \times [-p_{2i-1},p_{2i-1}]$$

and

$$\beta_i([p_{2i},p_{2i}] \times \{1\}) = \{-1\} \times [-p_{2i},p_{2i}].$$

If we define $\chi \in \mathcal{H}(Q)$ by

$$\chi(x) = (\beta_1(x_1,x_2),\ \beta_2(x_3,x_4),\ \beta_3(x_5,x_6),\ \ldots)$$

then we have for every $i \in \mathbb{N}$, $\chi(W_{2i-1}(p_{2i-1})) = W_{2i-1}(p_{2i-1})$ and

$$\chi(W_{2i}(p_{2i})) = \pi_{2i-1}^{-1}(\{-1\}) \cap \bigcap_{j \neq 2i-1} \pi_j^{-1}([-p_{2i},p_{2i}]).$$

Let γ be the homeomorphism of Q that interchanges adjacent odd and even

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*) *This lemma is due to R.D. Anderson (unpublished).*

coordinates:

$$\gamma(x) = (x_2, x_1, x_4, x_3, x_6, x_5, \ldots).$$

Define $\varphi \in H(Q)$ by

$$\varphi(x) = (x_1,\ \beta_1^{-1}(x_2,x_3),\ \beta_2^{-1}(x_4,x_5),\ \beta_3^{-1}(x_6,x_7),\ \ldots)$$

Observe that for every $i \in \mathbb{N}$ we have that $\varphi(W_{2i}(p_{2i-1})) = W_{2i}(p_{2i-1})$ and

$$\varphi(\pi_{2i}^{-1}(\{-1\}) \cap \bigcap_{j \neq 2i} \pi_j^{-1}([-p_{2i}, p_{2i}])) = W_{2i+1}(p_{2i}).$$

Since $(p_i)_{i \in \mathbb{N}}$ is strictly increasing there is an $\alpha \in H(J)$ such that for every $i \in \mathbb{N}$, $\alpha(p_i) = p_{i+1}$ and $\alpha(-p_i) = -p_{i+1}$. If we put $\psi = \prod_{i \in \mathbb{N}} \alpha$ then it is easily verified that $f = \psi \circ \varphi \circ \gamma \circ \chi$ has the property:

$$f(W_i(p_i)) = W_{i+1}(p_{i+1}) \quad \text{for every } i \in \mathbb{N}.$$

4.4.5 THEOREM: *Any subset of* Y *that is homeomorphic to* Q *is negligible.*

PROOF: Let Y be represented by Y_p, where $p \in R^{\uparrow}$, and let $f \in H(Q)$ be a "shift" on the shrunken endfaces: $f(W_i) = W_{i+1}$ for $i \in \mathbb{N}$. Then $f^{-1}(W_1)$ is a negligible subset of Y and in view of the homeomorphism extension theorem 4.3.7 this implies that every copy of Q is negligible in Y.

CHAPTER 5

FAKE HILBERT SPACES

5.1 Introduction

The study of "fake Hilbert spaces" has been inspired by Toruńczyk's characterization of ℓ^2. Before we state it some definitions.

5.1.1 DEFINITION: A space X is called an *absolute retract* (*AR*) if for every space Z, every map into X that is defined on a closed subset of Z can be extended over Z. A space X is called an *absolute neighbourhood retract* (*ANR*) if for every space Z and every map f from a closed subset Z_0 of Z into X there is a neighbourhood of Z_0 in Z over which f can be extended. For information concerning A(N)R's see Borsuk [B1].

5.1.2 DEFINITION: A collection $\mathcal{D}$ of subsets of a space X is *discrete* if each point of X has a neighbourhood intersecting at most one member of $\mathcal{D}$. A space X is said to have the *strong discrete approximation property* (*SDAP*) if for every admissible metric d on X, every $\varepsilon > 0$ and every map f from the countable free union of Hilbert cubes $\bigoplus_{i \in \mathbb{N}} Q_i$ into X there is a map $g\colon \bigoplus_{i \in \mathbb{N}} Q_i \to X$ such that $\hat{d}(f,g) < \varepsilon$ and $\{g(Q_i) \mid i \in \mathbb{N}\}$ is discrete.

5.1.3 THEOREM (Toruńczyk [T2]): *A topologically complete AR is homeomorphic to ℓ^2 iff it has the SDAP.*

This extremely useful characterization has now become the standard method for recognizing topological Hilbert spaces. In Anderson, Curtis & van Mill [ACM] it was shown that the SDAP cannot be relaxed by considering only one metric on the space. Specifically, they constructed a topologically complete AR space X with the following properties:

(1) There is an admissible metric d on X such that for every $\varepsilon > 0$ and continuous $f : \bigoplus_{i \in \mathbb{N}} Q_i \to X$ there is a map $g : \bigoplus_{i \in \mathbb{N}} Q_i \to X$ that satisfies $\hat{d}(g,f) < \varepsilon$ while $\{g(Q_i) \mid i \in \mathbb{N}\}$ is discrete (this is called the *weak discrete approximation property, WDAP*).

(2) Every compact subset of X is a Z-set.

(3) X embeds as a linearly convex subset of ℓ^2.

(4) $X \times X \approx \ell^2$.

(5) X is homogeneous.

(6) Every countable subset of X is strongly negligible.

(7) No Cantor set is negligible in X.

Since in ℓ^2 every σ-compact set is strongly negligible, Anderson [A3], property (7) shows that $X \not\approx \ell^2$. The space X is a "fake topological Hilbert space" since it has many of the familiar topological properties of ℓ^2 but yet is not homeomorphic to it. As an "application" we get that the properties (1) through (6) do not characterize ℓ^2. It is useful to push this point further. Every "fake topological Hilbert space" blocks a possible generalization of Toruńczyk's theorem.

The aim of this chapter is to construct spaces that "approximate" ℓ^2 closer than the space above. We are interested in dimension theory and negligibility properties and we shall obtain a characterization of dimension in terms of negligibility.

Consider the space Y defined in section 4.1. Recall that we proved in section 3.2 that there is for every $k \in \{0,1,2,\dots\}$ a strong $(\mathcal{S}_k, \mathcal{H}(Q))$-skeletoid A_k in Q, where $\mathcal{S}_k$ is the collection of Z-sets in Q with dimension $\leq k$. For convenience, we put $A_{-1} = \emptyset$ and $\mathcal{S}_{-1} = \{\emptyset\}$. The skeletoids A_k were constructed in the pseudo-interior s of Q which is a subset of Y (indeed, we may always assume this, because every σ-Z-set can be pushed into s). *Let* $k \in \{-1,0,1,\dots\}$ *and* A_k *be fixed in the remaining part of this chapter.* The space X_k is defined as

$$X_k = Y \setminus A_k.$$

We shall prove that X_k is a topologically complete AR, which is *not* homeomorphic to ℓ^2 but which has the following properties*):

(1) X_k has the WDAP.

(2) Every compact subset of X_k is a Z-set.

(3) X_k embeds as linearly convex subset of ℓ^2.

(4) $X_k \times X_k \approx \ell^2$.

(5) Let $\mathcal{U}$ be a collection of open subsets in X_k, A a compact space and $F : A \times I \to X_k$ a homotopy that is limited by $\mathcal{U}$. If F_0 and F_1 are embeddings then there is an $h \in \mathcal{H}(X_k)$ that is $\mathcal{U}$-close to 1 and has the property $h \circ F_0 = F_1$. Since X_k is an AR this implies that X_k is homogeneous.

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*) *This result was established in Dijkstra & van Mill* [DM].

(6) If $A \subset X_k$ is σ-compact, then A is strongly negligible iff $\dim(A) \le k$ (in particular, $X_k \not\approx X_{k'}$, if $k \ne k'$).

(7) If $A \subset X_k$ is a compactum of fundamental dimension at most k, then A is negligible (in particular, if $C \subset X_k$ is an n-cell, then C is negligible and C is strongly negligible iff $n \le k$).

5.2 A generalization of the Sierpiński theorem

The aim of this section is to prove a generalization of Sierpiński's theorem that no continuum (i.e. a compact connected space) can be partitioned into countably many pairwise disjoint non-empty closed subsets, see Sierpiński [S] or [E1 : p.440]. This generalization plays a key role in deciding whether a subset of X_k is strongly negligible. Since we feel that the result is of independent interest we have put it in a separate section.

As usual, S^n denotes the n-sphere, $n \in \{0,1,2,\ldots\}$.

5.2.1 THEOREM: *Let* n *be a nonnegative integer and* X *a compact space. If* $\{F_i \mid i \in \mathbb{N}\}$ *is a closed covering of* X *such that for each pair of distinct natural numbers* i *and* j, $\dim(F_i \cap F_j) < n$ *then every map* $f : F_1 \to S^n$ *can be extended over* X.

The theorem is also valid outside the class of metric spaces, see Dijkstra [D3]. The reader is encouraged to verify that Sierpiński's theorem follows easily if one substitutes $n = 0$.

PROOF: We shall work with the following induction hypothesis for $n = 0,1,2,\ldots$

Let X be a compact space and M an AR. If $\{F_i | i \in \mathbb{N}\}$ is a closed covering of X such that for every i and j with $i \neq j$, $\dim(F_i \cap F_j) < n$ then every map $f : F_1 \to S^n \times M$ is extendable over X.

Consider the case $n = 0$, where we have that S^n is the discrete doubleton $\{-1,1\}$ and $\{F_i | i \in \mathbb{N}\}$ is a pairwise disjoint collection. Assume that the closed set $A = f^{-1}(\{-1\} \times M) \subset F_1$ is non-empty. Let $\tilde{X}$ be the space we obtain from X by identifying A to a single point a and let $q : X \to \tilde{X}$ be the decomposition map. If C is the component of a in $\tilde{X}$ then it is a continuum with the following pairwise disjoint, closed covering:

$$\{\{a\},\hat{A} \cap C\} \cup \{F_i \cap C | i \geq 2\},$$

where $\hat{A} = f^{-1}(\{1\} \times M)$. According to Sierpiński we have that $C = \{a\}$. Since $\tilde{X}$ is a compact Hausdorff space there is a clopen neighbourhood O of a in $\tilde{X}$ that misses $\hat{A}$. Because M is an AR we can find maps $g_1 : q^{-1}(O) \to \{-1\} \times M$ and $g_2 : q^{-1}(\tilde{X}\backslash O) \to \{1\} \times M$ such that $g_1|A = f|A$ and $g_2|\hat{A} = f|\hat{A}$. Then $g_1 \cup g_2$ is the required extension of f.

Assume now that the induction hypothesis holds for n. Let $\{F_i | i \in \mathbb{N}\}$ be a closed covering of X such that for $i \neq j$, $\dim(F_i \cap F_j) \leq n$ and let $f : X \to S^{n+1} \times M$ be continuous. According to the countable sum theorem (see [E2 : 3.1.8]) the set $R = \cup\{F_i \cap F_j | i,j \in \mathbb{N} \text{ with } i \neq j\}$ has dimension $\leq n$. Select two distinct points x_1 and x_2 in S^{n+1} and note that $S^{n+1}\backslash\{x_1,x_2\}$ is homeomorphic to $S^n \times \mathbb{R}$. Using the separation theorem (see [E2 : 4.1.13]) we find a closed covering $\{H_1,H_2\}$ of X such that for $j \in \{1,2\}$, $H_j \cap f^{-1}(\{x_j\} \times M) = \emptyset$ and

$$\dim(H_1 \cap H_2 \cap R) < n.$$

Consider the compact space $X' = H_1 \cap H_2$ and its closed covering $\{F_i \cap X' | i \in \mathbb{N}\}$. Obviously, we have for $i \neq j$ that

$\dim(F_i \cap F_j \cap X') \le \dim(R \cap X') < n$. Observe that $f|F_1 \cap X'$ is a continuous mapping into $(S^{n+1}\setminus\{x_1,x_2\}) \times M$, which space is homeomorphic to $S^n \times \mathbb{R} \times M$. Since $\mathbb{R} \times M$ is, as product of AR's, itself an AR we may apply the induction hypothesis to find a continuous $g : X' \to (S^{n+1}\setminus\{x_1,x_2\}) \times M$ with $g|F_1 \cap X' = f|F_1 \cap X'$. Observing that $S^{n+1}\setminus\{x_j\}$ is homeomorphic to $\mathbb{R}^{n+1}$ select for $j \in \{1,2\}$ a continuous extension $h_i : H_j \to (S^{n+1}\setminus\{x_j\}) \times M$ of $(f|F_1 \cap H_j) \cup g$. Then $h = h_1 \cup h_2$ is a map from X into $S^{n+1} \times M$ which extends f and the theorem is proved.

5.3 Some topological properties of X_k

In this section we give a number of properties that X_k shares with ℓ^2; we show that X_k is a "fake Hilbert space".

5.3.1 THEOREM:

(1) X_k *is topologically complete.*

(2) X_k *embeds as a linearly convex set in* ℓ^2 *and hence it is an AR.*

(3) X_k *has the WDAP.*

(4) *Every compact subset of* X_k *is a Z-set.*

(5) $X_k \times X_k \approx \ell_2$.

PROOF: It is proved in Anderson, Curtis & van Mill [ACM : sec.3] that if A is a σ-Z-set in Q such that for every $\varepsilon > 0$ there is a map $\beta : Q \to A$ with $\hat{\rho}(\beta,1) < \varepsilon$ then $Q\setminus A$ satisfies (1) through (5).

We now turn to the homogeneity properties of X_k. Put

$$\mathcal{S}_{kW} = \{S \subset Y | S \text{ is compact and } \dim(S) \leq k\}.$$

Since every compact subset of Y is a Z-set in Q it follows that

$$\mathcal{S}_{kW} = \{S \in \mathcal{S}_k | S \cap W = \emptyset\}.$$

We have the following proposition:

5.3.2 PROPOSITION: A_k *is a strong* $(\mathcal{S}_{kW},\Gamma_W)$*-skeletoid*$^\frown$ *in* Q *and a strong* $(\mathcal{S}_{kW},\mathcal{H}(Y))$*-skeletoid*$^\frown$ *in* Y.

PROOF: Since $A_k \cap W = \emptyset$, A_k is a member of $(\mathcal{S}_{kW})_\sigma$. Let S be in $\mathcal{S}_{kW}$ and assume that $\mathcal{U}$ is a collection of open subsets of Q that covers S. Put $O = \cup\mathcal{U}$ and select a closed neighbourhood F of $Q\backslash O$ that misses S. Let $(A_k^i)_{i\in\mathbb{N}}$ be the skeleton that corresponds with A_k and let $n \in \mathbb{N}$. There are an $m \in \mathbb{N}$ and an isotopy H of Q such that H is limited by $\{\mathrm{Int}_Q(F)\} \cup \mathcal{U}$ $H_0 = 1$, $H_1(S) \subset A_m$ and $H_t|F \cup A_n = 1$ for every $t \in I$. So $H|S \times I$ is a homotopy that is limited by $\{U\backslash A_n | U \in \mathcal{U}\}$ and with the property that $H_0|S$ and $H_1|S$ are embeddings of S into Y. According to theorem 4.3.6 there is a $\{U\backslash A_n|U \in \mathcal{U}\}$-push h in Γ_W with $h(S) \subset A_m$. This proves that A_k is a strong $(\mathcal{S}_{kW},\Gamma_W)$-skeletoid$^\frown$. Since $h|Y$ is a $\{U \cap Y|U \in \mathcal{U}\}$-push in $\{\gamma \in \mathcal{H}(Y)|\gamma|A_n = 1\}$ we have also proved that A_k is a strong $(\mathcal{S}_{kW},\mathcal{H}(Y))$-skeletoid$^\frown$.

5.3.3 THEOREM: *Let* $\mathcal{U}$ *be a collection of open subsets in* Q, A *a compact space and* $F : A \times I \to Q$ *a homotopy that is limited by* $\mathcal{U}$. *If* F_0 *and* F_1 *are embeddings of* A *in* X_k *then there is an* $h \in \Gamma_W$ *that is* $\mathcal{U}$*-close to* 1 *and that has the properties* $h \circ F_0 = F_1$ *and* $h|X_k \in \mathcal{H}(X_k)$.

PROOF: According to theorem 4.3.6 there is an $f \in \Gamma_W$ that is $\mathcal{U}$-close to 1 and satisfies $f \circ F_0 = F_1$. Using theorem 1.2.13 we find an $h \in \Gamma_W$ that extends $f|F_0(A)$ and has the properties that it is $\mathcal{U}$-close to 1 and $h(A_k) = A_k$.

5.3.4 COROLLARY: *Let* $\mathcal{U}$ *be a collection of open subsets of* X_k, A *a compact space and* $F : A \times I \to X_k$ *a homotopy that is limited by* $\mathcal{U}$. *If* F_1 *and* F_0 *are embeddings then there is an* $h \in \mathcal{H}(X_k)$ *that is* $\mathcal{U}$*-close to* 1 *and has the property* $h \circ F_0 = F_1$.

PROOF: This is trivial.

5.3.5 REMARK: In view of theorem 4.3.6 it is natural to ask whether the homeomorphism of corollary 5.3.4 can be chosen in such a way that it is isotopic to the identity of X_k. This is not the case for $k = 0$. We believe that for $k > 0$ the spaces X_k also behave "badly" in this respect, but we have no proof of this assertion.

Consider an isotopy $H : X_0 \times I \to X_0 \times I$ such that $H_0 = 1$. We shall show that $H_1 = 1$ for every $t \in I$. Pick an arbitrary point x in A_0 and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X_0 that converges to x in Q. There is a copy L of $[0,1)$ in X_0 such that $\{x_n | n \in \mathbb{N}\} \subset L$ and $L \cup \{x\} \approx I$ (use the fact that every Z-set in Q is thin). If we put $D = H(L \times I)$ then D is a closed subset of $X_0 \times I$ that is homeomorphic to $[0,1) \times I$. Let $K = \mathrm{Cl}_{Q \times I}(D) \backslash D$ and let $\tilde{K}$ be the projection of K into the first factor of the product $Q \times I$. Then K and $\tilde{K}$ are continua which are contained in $(W \cup A_0) \times I$ and $W \cup A_0$, respectively. Since $A_0 \cup W$ can be written as a disjoint union of compacta and since $x \in \tilde{K} \cap A_0$, Sierpiński's theorem gives that $\tilde{K} \subset A_0$. Now A_0 is totally disconnected and hence $\tilde{K} = \{x\}$. This implies that $\lim_{i \to \infty} H_t(x_i) = x$

for every $t \in I$ and hence H_t can be extended over Y with the identity on A_0. Since A_0 is dense in Y we have that $H_t = 1$ for every $t \in I$.

So we may conclude that if f and g are isotopic members of $\mathcal{H}(X_0)$ then $f = g$ (cf. remark 1.2.15).

5.3.6 COROLLARY: *Let* A *be compact and* $f : A \to X_k$ *continuous. If* A' *is a closed subset of* A *such that* $f|A'$ *is an embedding and if* $\mathcal{U}$ *is an open covering of* X_k*, then there is an embedding* g *of* A *in* X_k *such that* g *and* f *are* $\mathcal{U}$*-close and* $g|A' = f|A'$.

PROOF: It is no problem to find a subset R of X_k that is homeomorphic to s; put for instance $R = \{-1\} \times \prod_{i=2}^{\infty} (-1,1)$. Let C be a subset of R that is homeomorphic to $f(A)$. Both embeddings of $f(A)$ in X_k are of course homotopic in Q and hence there is an $h \in \mathcal{H}(X_k)$ such that $h \circ f(A) \subset R$. Since $R \approx s$, there is according to lemma 4.2.4 an embedding g of A in R such that g and $h \circ f$ are $h(\mathcal{U})$-close and $g|A' = h \circ f|A'$. If $\tilde{g} = h^{-1} \circ g$ then $\tilde{g}$ and f are $\mathcal{U}$-close and $\tilde{g}|A' = f|A'$.

5.4 Negligibility and dimension

In this section we shall prove the connexions that exist between (strong) negligibility in X_k and dimension.

5.4.1 THEOREM: *Every* σ*-compact subset of* X_k *with dimension at most* k *is strongly negligible.*

PROOF: As observed in the preceeding section, A_k is a strong

$(S_{kW}, H(Y))$-skeletoid$^{\sim}$. Now apply proposition 1.2.10 and theorem 1.2.12.

We identify S^{n-1} and the boundary ∂I^n for every natural number n. Let X be a space. A map $f : X \to I^n$ is called *essential* if $f|f^{-1}(S^{n-1})$ cannot be extended to a map $g : X \to S^{n-1}$.

5.4.2 LEMMA: *Let* n *be a natural number with* $n > k$. *If* A *is a compact subset of* X_k *and* $f : A \to I^n$ *is essential then* $f^{-1}(\text{Int } I^n)$ *is not negligible in* X_k.

PROOF: Let $R = f^{-1}(S^{n-1})$ and $O = A\backslash R$. In view of corollary 5.3.6 we may assume that $A \times I$ is a subset of X_k such that $A \times \{0\}$ coincides with A. Suppose that O is a negligible subset of X_k. This implies that $Z = (A \times I)\backslash O$ can be embedded as a closed subset in X_k. Assume that Z is reembedded as a closed subset in X_k and let $\overline{Z}$ be the closure of Z in Q. Put $Z^* = \overline{Z}\backslash Z$ and note that the local compactness of $A \times (0,1]$ implies that $Z^* \cup R$ is compact. Also, Z^* is a closed subset of $Q\backslash X_k = A_k \cup W$. Since $Z^* \cap A_k$ is σ-compact and at most (n-1)-dimensional, we can find a sequence $(F_i)_{i\in\mathbb{N}}$ of compact subsets of $Z^* \cap A_k$ such that $Z^* \cap A_k = \bigcup_{i\in\mathbb{N}} F_i$ and $F_i \cap F_j$ is at most (n-2)-dimensional for all distinct $i,j \in \mathbb{N}$. In addition, observe that $Z^* \cap W$ is a countable disjoint union of compacta and that $W \cap A_k = \emptyset$. Theorem 5.2.1 implies that the map $g = f|R$ can be extended to a map $\overline{g} : (Z^* \cup R) \to S^{n-1}$. Since S^{n-1} is an ANR there is an open U containing $Z^* \cup (R \times I)$ such that the map h, defined by

$$h(x) = \overline{g}(x) \quad \text{if} \quad x \in Z^* \cup R$$

and

$$h(x,t) = f(x) \quad \text{if} \quad (x,t) \in R \times I,$$

can be extended to a continuous $\overline{h} : U \to S^{n-1}$. Since $(A \times (0,1])\setminus U$ is compact there is an $\varepsilon \in (0,1]$ such that $A \times \{\varepsilon\} \subset U$. Define the function $\eta : A \to S^{n-1}$ by $\eta(a) = \overline{h}(a,\varepsilon)$, $a \in A$. Then $\eta|R = f|R$ and $\eta(A) \subset S^{n-1}$, which means that f is not essential.

5.4.3 COROLLARY: *If* $n \in \mathbb{N}$ *and* $n > k$ *then there exist copies of* $\mathbb{R}^n$ *in* X_k *that are not negligible.*

PROOF: I^n is embedded in X_k, corollary 5.3.6, and 1_{I^n} is essential.

5.4.4 COROLLARY: X_k *is not homeomorphic to* ℓ^2.

PROOF: As remarked in section 3.1, every σ-compact subset of ℓ^2 is strongly negligible.

5.4.5 COROLLARY: X_k *does not admit the structure of a topological group.*

PROOF: ℓ^2 is the only infinite dimensional topological group that is a complete AR (Dobrowolski & Toruńczyk [DT]).

5.4.6 REMARK: With the method of lemma 5.4.2 and corollaries we can prove that if C is a compact space containing ℓ^2 and $C\setminus\ell^2 = \bigcup_{i \in \mathbb{N}} F_i$, where the F_i's are compacta, then there is for every $n \in \mathbb{N}$ an infinite set $\{i_m | m \in \mathbb{N}\}$ of natural numbers greater than n such that for every $m \in \mathbb{N}$, $\dim (F_{i_m} \cap F_{i_{m+1}}) \geq n$.

We sketch a proof. Define the following equivalence relation on $N = \{i \in \mathbb{N} | i > n\}$: $m \sim 1$ if there is a sequence $m = i_1, i_2, \ldots, i_j = 1$ in N with dim $(F_{i_r}, F_{i_{r+1}}) \geq n$ for $r = 1,2,\ldots, j-1$. If there is an infinite equivalence class we are done. If every class is finite we define new compacta $G_{[i]} = \cup\{F_j \mid j \sim i\}$, where $[i]$ is the class of $i \in \mathbb{N}$. Note that if $[i] \neq [j]$ then dim $(G_{[i]} \cap G_{[j]}) < n$. Let U be an open, non-empty subset of ℓ^2 which closure in C misses $\bigcup_{i=1}^{n} F_i$. If $Z = I^{n+2}\backslash(\text{Int } I^{n+1}) \times \{0\}$ then we can embed Z as a closed subset in ℓ^2 such that $Z \subset U$. The proof of lemma 5.4.2 shows that we cannot do this in $C\backslash(\bigcup_{i\in N} G_{[i]} \cup \bigcup_{i=1}^{n} F_i) = C\backslash\bigcup_{i\in \mathbb{N}} F_i$.

We now come to the announced characterizations of dimension in terms of negligibility.

5.4.7 THEOREM: *Let* $k \neq -1$. *For every* σ*-compact space* A, *the following statements are equivalent:*

(1) dim (A) $\leq$ k.

(2) *There is an embedding* f *of* A *in* X_k *such that for every open* O *in* A, f(O) *is negligible in* X_k.

(3) *Every embedding* f *of* A *in* X_k *has the property that for every open* O *in* A, f(O) *is negligible in* X_k.

PROOF: (1) $\rightarrow$ (3). If dim (A) $\leq$ k then by theorem 5.4.1 f(A) is strongly negligible. Consequently, every relatively open subset of f(A) is negligible.

(3) $\rightarrow$ (2). By corollary 5.3.6, X_k is universal.

(2) $\rightarrow$ (1). Assume that A satisfies (2) for some embedding f. Write A as a countable union of compacta $F_1, F_2, F_3, \ldots$. We show that F_i also satisfies (2). Let $i \in \mathbb{N}$ and let O be a relatively open subset of F_i. Choose an open $\tilde{O}$ in A with $\tilde{O} \cap F_i = O$. Since A satisfies (2) there exist two homeomorphisms $\alpha : X_k \to X_k\backslash f(\tilde{O})$ and $\beta : X_k \to X_k\backslash f(\tilde{O}\backslash F_i)$. In view of the homeomorphism extension theorem 5.3.4 there is a $\gamma \in H(X_k)$ with

$\gamma \circ f|F_i = \beta^{-1} \circ f|F_i$. Then $\gamma^{-1} \circ \beta^{-1} \circ \alpha$ is a homeomorphism from X_k onto

$$\gamma^{-1} \circ \beta^{-1} \circ \alpha(X_k) = \gamma^{-1} \circ \beta^{-1}(X_k \backslash f(\tilde{O})) = \gamma^{-1}(X_k \backslash \beta^{-1} \circ f(F_i \cap \tilde{O}))$$

$$= X_k \backslash f(F_i \cap \tilde{O}) = X_k \backslash f(O),$$

which proves the claim that F_i satisfies (2). Since F_i is compact lemma 5.4.2 implies that no map from F_i into I^{k+1} is essential. This means that $\dim(F_i) \le k$, see [E2 : 1.9.A]. According to the countable sum theorem, see [E2 : 3.1.8], we have that $\dim(A) \le k$.

5.4.8 REMARK: As for the case $k = -1$, we shall show in the next section that a space A satisfies (2) or (3) iff it is finite.

5.4.9 LEMMA: *If* A *is a nonempty, compact subset of* $Y = X_{-1}$ *and if* $f : Y \to Y\backslash A$ *is a homeomorphism then* $\{x \in Y | f(x) = x\}$ *is a* Z*-set in* Y.

PROOF: According to lemma 4.3.9 there exist a compact space M and monotone maps g,h from M onto Q with $g^{-1}(Y) = h^{-1}(Y\backslash A)$ and $f \circ g|g^{-1}(Y) = h|g^{-1}(Y)$. Consider a shrunken endface W_i. Since h is monotone we have that $g(h^{-1}(W_i))$ is a continuum in W. By Sierpiński's theorem there is an $\alpha(i) \in \mathbb{N}$ with $g(h^{-1}(W_i)) \subset W_{\alpha(i)}$. Analogously we can show that $h(g^{-1}(W_{\alpha(i)})) \subset W_i$. So for every $i \in \mathbb{N}$, $h^{-1}(W_i) = g^{-1}(W_{\alpha(i)})$ and hence α is one-to-one. Since $g(h^{-1}(A))$ is a non-empty subspace of W, $\alpha(\mathbb{N}) \neq \mathbb{N}$. Put $Z = \{x \in Y | f(x) = x\}$. Let γ be a map from Q into Y and let $\varepsilon > 0$. Since $\alpha : \mathbb{N} \to \mathbb{N}$ is one-to-one but not onto there exist an $i \in \mathbb{N}$ and a map $\beta : Q \to W_{\alpha(i)}$ such that $\hat{\rho}(\beta,1) < \varepsilon/2$ and $i \neq \alpha(i)$. Put $\delta = \frac{1}{2}\rho(W_i, W_{\alpha(i)})$. Since $g^{-1}(W_{\alpha(i)}) = h^{-1}(W_i)$, the set $O = U_\delta(W_{\alpha(i)}) \backslash g(h^{-1}(Q \backslash U_\delta(W_i)))$ is a neighbourhood of $W_{\alpha(i)}$. Since $f \circ g|g^{-1}(Y) = h|g^{-1}(Y)$ the sets Z and O are disjoint. Let δ' be an element of $(0,\varepsilon/2)$ such that $U_{\delta'}(W_{\alpha(i)}) \subset O$ and construct a map $\eta : Q \to s$ with

$\hat{\rho}(\eta,1) < \delta'$. Then the map $\gamma' = \eta \circ \beta \circ \gamma$ has the properties:
$\hat{\rho}(\gamma',\gamma) \leq \hat{\rho}(\eta,1) + \hat{\rho}(\beta,1) < \varepsilon$ and

$$\gamma'(Q) \subset \eta(W_{\alpha(i)}) \subset O \cap s \subset Y \backslash Z.$$

This proves that Z is a Z-set in Y.

5.4.10 THEOREM: *Let A be a σ-compact space. The following statements are equivalent:*

(1) $\dim(A) \leq k$.

(2) *There is an embedding f of A in X_k such that $f(A)$ is strongly negligible in X_k.*

(3) *Every subset of X_k that is homeomorphic to A is strongly negligible.*

PROOF: (1) $\rightarrow$ (3). Apply theorem 5.4.1.

(3) $\rightarrow$ (2). This is trivial.

(2) $\rightarrow$ (1). Note that every relatively open subset of a strongly negligible set is negligible. If $k \neq -1$, apply theorem 5.4.7. Let A satisfy (2) for $k = -1$. If A is non-empty then there is an $a \in A$ such that $\{f(a)\}$ is strongly negligible in X_{-1}, proposition 1.2.2. This means that for every neighbourhood U of $f(a)$ there is a homeomorphism $g : X_{-1} \rightarrow X_{-1} \backslash \{f(a)\}$ that is supported on U. Since a Z-set is always nowhere dense this contradicts lemma 5.4.9. So we may conclude that $A = \emptyset$ and $\dim(A) = -1$. Note that we did not use the σ-compactness of A here: the empty set is the only strongly negligible subset of X_{-1}.

We conclude this section with discussing a generalization of σ-compactness, strongly σ-complete spaces (cf. section 2.3). Note that every negligible subset of a complete space is strongly σ-complete. So strongly σ-complete spaces are the most general type of spaces for which it makes sense to consider negligibility in X_k.

5.4.11 PROPOSITION: *Every strongly σ-complete space with dimension $\leq k$ has a strongly negligible embedding in X_k.*

PROOF: Let S be a space with dimension $\leq k$ and let $(S_i)_{i \in \mathbb{N}}$ be a sequence of closed, topologically complete subsets of S with $S = \bigcup_{i \in \mathbb{N}} S_i$. Select a $\leq k$-dimensional compactification C of S (see [E2 : 1.7.2]) and assume that C is embedded in X_k. Define for $i \in \mathbb{N}$, $R_i = \mathrm{Cl}_C(S_i)\backslash S_i$ and $P = \bigcup_{j \in \mathbb{N}} \mathrm{Cl}_C(S_j)$, $R = \bigcup_{j \in \mathbb{N}} R_j$. Since S_i is closed in S we have that $R_i = \mathrm{Cl}_C(S_i)\backslash S$ and hence $S = P\backslash R$. The set R_i is the remainder of a topologically complete space in a compact space and hence a σ-compact space. So also R is a σ-compact space with dimension $\leq k$. Consequently, $R \cup A_k$ is an $(S_{kW}, \mathcal{H}(Y))$-absorber in Y. According to the uniqueness theorem 1.2.11 there is an $f \in \mathcal{H}(Y)$ with $f(R \cup A_k) = A_k$. This means that $f(S) = f(P)\backslash A_k \subset X_k$. The space f(P) is an element of $(S_{kW})_\sigma$ and hence theorem 1.2.12 implies that f(S) is a strongly negligible subset of X_k.

We do not know whether the converse of this proposition holds. Note that every non-σ-compact space has a nonnegligible embedding in X_k (embed a compactification of the space in X_k and observe that it is not an F_σ-set). If we apply the argument of proposition 5.4.11 to the pseudo-boundary B in Q (see also theorem 2.3.7) we find that ℓ^2 is universal for V_σ^∞.

5.4.12 THEOREM: *Let X be a space. The following statements are equivalent:*

(1) X *is strongly σ-complete.*

(2) X *is homeomorphic to a (strongly) negligible subset of ℓ^2.*

(3) X *is homeomorphic to an F_σ-set in ℓ^2.*

5.5. Negligibility and shape

In this section we shall discuss a connexion between negligibility of compacta in X_k and fundamental dimension. We begin by giving the definition of shape in the sense of Borsuk [B2].

Let A and A' be compacta in Q. A *shape map* $\mathfrak{f}$ from A to A' is a sequence $f_n : Q \to Q$, $n \in \mathbb{N}$, of maps with the following property: for every neighbourhood V of A' there are a neighbourhood U of A and a natural number n such that for every $m > n$, $f_m|U$ and $f_{m+1}|U$ are homotopic in V, i.e. there is a map $F : U \times I \to V$ with $f_m|U = F_0$ and $f_{m+1}|U = F_1$. We write $\mathfrak{f} = (f_n,A,A')$. If $\mathfrak{f} = (f_n,A,A')$ and $\mathfrak{g} = (g_n,A,A')$ are two shape maps from A to A' we say that $\mathfrak{f}$ and $\mathfrak{g}$ are *homotopic* if there are for every neighbourhood V of A' an $n \in \mathbb{N}$ and a neighbourhood U of A such that $f_m|U$ and $g_m|U$ are homotopic in V for $m > n$.

The *identity* shape map is $\mathfrak{1}_A = (1_Q,A,A)$. If $\mathfrak{f} = (f_n,A,A')$ and $\mathfrak{g} = (g_n,A',A'')$ are shape maps then their composition is the shape map $\mathfrak{g} \circ \mathfrak{f} = (g_n \circ f_n,A,A'')$. We say that A and A' *have the same shape*, notation $Sh(A) = Sh(A')$, if there exist a shape map $\mathfrak{f}$ from A to A' and a shape map $\mathfrak{g}$ from A' to A such that $\mathfrak{g} \circ \mathfrak{f}$ and $\mathfrak{f} \circ \mathfrak{g}$ are homotopic to $\mathfrak{1}_A$ and $\mathfrak{1}_{A'}$, respectively. One may show that this notion is independent of the given embeddings of A and A' in Q.

We now state the complement theorem that is due to Chapman [C : sec.25].

5.5.1 THEOREM: *If* A *and* A' *are* Z*-sets in* Q *then* $Sh(A) = Sh(A')$ *iff* $Q\backslash A \approx Q\backslash A'$.

5.5.2 COROLLARY: *If* A *is a non-empty* Z*-set in* Q *then* A *has trivial shape (i.e. the shape of a singleton) iff* $Q/A \approx Q$, *where* Q/A *is the space we obtain by identifying* A *to a point.*

PROOF: If $Q/A \approx Q$ then $Q\backslash A \approx Q\backslash\{p\}$ for some $p \in Q$ and hence A and $\{p\}$ have the same shape.

If A has trivial shape then for every $p \in Q$, $Q\backslash A \approx Q\backslash\{p\}$. Observe that Q/A and Q are one-point compactifications of $Q\backslash A$ and $Q\backslash\{p\}$. Since one-point compactifications are unique this implies that $Q/A \approx Q$.

We have for X_k the following analogue of Chapman's theorem.

5.5.3 LEMMA: *If* A *and* A' *are compacta in* X_k *with the same shape then there is a homeomorphism* $h : Q\backslash A \to Q\backslash A'$ *with* $h(A_k) = A_k$ *and* $h(W_i) = W_i$ *for every* $i \in \mathbb{N}$.

PROOF: The method is based on Chapman's proof for theorem 5.5.1. Let $\mathcal{f} = (f_n, A, A')$ and $g = (g_n, A', A)$ be shape maps such that $\mathcal{f} \circ g$ and $g \circ \mathcal{f}$ are homotopic to $1_{A'}$ and 1_A, respectively. Since $W \cup A_k$ is a σ-Z-set we may assume that for every $n \in \mathbb{N}$ both $f_n(Q)$ and $g_n(Q)$ are contained in X_k. It is left as an exercise to the reader to verify this. We shall construct inductively a sequence $\chi_1, \chi_2, \chi_3, \ldots$ in $\{\gamma \in \Gamma_W | \gamma(X_k) = X_k\}$ and a sequence $O_1 \supset O_2 \supset O_3 \supset \ldots$ of open neighbourhoods of A in Q such that for every $i \in \mathbb{N}$, $\chi_i(O_i)$ contains A' and there exist an $n \in \mathbb{N}$ and an open neighbourhood V of A' in Q with the property that $V \subset \chi_i(O_i)$ and 1_V is in $\chi_i(O_i)$ homotopic to $\chi_i \circ g_m|V$ for every $m \geq n$. The basis step of the induction is $\chi_1 = 1$ and $O_1 = Q$.

Assume that χ_i and O_i have been constructed and that they satisfy the induction hypothesis. Since $\mathcal{f}$ is a shape map and since $g \circ \mathcal{f}$ and 1_A are homotopic there exist an $m > n$ and an open neighbourhood P of A in Q such that $P \subset O_i$, $g_m \circ f_m|P$ and 1_P are homotopic in O_i and $f_m|P$, $f_{m+1}|P$, $f_{m+2}|P$, ... are all homotopic in $V' = V \cap U_{2/(i+1)}(A')$. Since $f_m(A) \subset V' \cap X_k$ there is in view of corollary 5.3.6 an embedding α of A in $V' \cap X_k$ that is in V' homotopic to $f_m|A$. We have that the following maps are homotopic to each other in $\chi_i(O_i)$:

$$\alpha,\ f_m|A,\ \chi_i \circ g_m \circ f_m|A \text{ and } \chi_i|A.$$

Using theorem 5.3.3 we find a $\beta \in \{\gamma \in \Gamma_W|\gamma(X_k) = X_k\}$ that is supported on $\chi_i(O_i)$ and satisfies $\alpha = \beta \circ \chi_i|A$. So $\beta \circ \chi_i|A$ and $f_m|A$ are homotopic in V'. Since V' is, as open subset of Q, an ANR there is an open neighbourhood O_{i+1} of A in Q such that $\beta \circ \chi_i|O_{i+1}$ and $f_m|O_{i+1}$ are homotopic in V'. We may assume in addition that $O_{i+1} \subset U_{2/(i+1)}(A) \cap P$. Note that O_{i+1} and $\beta \circ \chi_i(O_{i+1})$ are contained in O_i and V', respectively.

Since g is a shape map and since $\mathcal{f} \circ g$ is homotopic to $1_{A'}$, there is an open P' in Q and an $m' > m$ such that $A' \subset P' \subset V'$, $f_{m'} \circ g_{m'}|P'$ and $1_{P'}$ are homotopic in V' and $g_{m'}|P'$, $g_{m'+1}|P'$, $g_{m'+2}|P'$, ... are all homotopic to each other in O_{i+1}. Since $\beta \circ \chi_i \circ g_{m'}(P') \subset \beta \circ \chi_i(O_{i+1}) \cap X_k$ there is in view of corollary 5.3.6 an embedding α' of A' in X_k that is in $\beta \circ \chi_i(O_{i+1})$ homotopic to $\beta \circ \chi_i \circ g_{m'}|A'$. It is easily verified that

$$\alpha',\ \beta \circ \chi_i \circ g_{m'}|A',\ f_m \circ g_{m'}|A',\ f_{m'} \circ g_{m'}|A' \text{ and } 1_{A'}$$

are homotopic in V'. Using theorem 5.3.3. we find a $\beta' \in \{\gamma \in \Gamma_W|\gamma(X_k) = X_k\}$ that is supported on V' and satisfies $\beta' \circ \alpha' = 1_{A'}$. Put $h_{i+1} = \beta' \circ \beta$ and

$\chi_{i+1} = h_{i+1} \circ \chi_i$. Since $\alpha'(A') \subset \beta \circ \chi_i(O_{i+1})$ we have that

$$A' = \beta' \circ \alpha'(A') \subset \chi_{i+1}(O_{i+1}).$$

One readily sees that $\chi_{i+1} \circ g_{m'}|A'$ is in $\chi_{i+1}(O_{i+1})$ homotopic to $\beta' \circ \alpha' = 1_{A'}$. Since $\chi_{i+1}(O_{i+1})$ is an ANR there is an open set $\tilde{V}$ such that $A' \subset \tilde{V} \subset P'$ and $\chi_{i+1} \circ g_{m'}|\tilde{V}$ and $1_{\tilde{V}}$ are homotopic in $\chi_{i+1}(O_{i+1})$. If $j \geq m'$ then $g_{m'}|P'$ and $g_j|P'$ are homotopic in O_{i+1} and hence $\chi_{i+1} \circ g_j|\tilde{V}$ is in $\chi_{i+1}(O_{i+1})$ homotopic to $1_{\tilde{V}}$. This completes the induction.

Note that every h_{i+1} is supported on $\chi_i(O_i)$ and is a member of $\{\gamma \in \Gamma_W | \gamma(A_k) = A_k\}$. Observe furthermore that for $i \in \mathbb{N}$, $O_i \subset U_{2/i}(A)$ and $\chi_i(O_i) \subset U_{2/i}(A')$. If $x \in Q\backslash A$ and i is such that $2/i < \rho(x,A)$ then $Q\backslash O_i$ is a neighbourhood of x such that $\chi_i(Q\backslash O_i) \subset Q\backslash A'$ and for every $j > i$ $\chi_j|Q\backslash O_i = \chi_i|Q\backslash O_i$. Consequently, if we define for $x \in Q\backslash A$, $h(x) = \lim_{i\to\infty} \chi_i(x)$ then h is a local homeomorphism from $Q\backslash A$ into $Q\backslash A'$. Since $O_i \subset U_{2/i}(A)$ and $\chi_i(O_i) \subset U_{2/i}(A')$ for $i \in \mathbb{N}$, h is one-to-one and onto and hence a homeomorphism. Since for every $x \in Q\backslash A$ there is an $i \in \mathbb{N}$ such that $h(x) = \chi_i(x)$ we have that $h(A_k) = A_k$ and $h(W_j) = W_j$ for $j \in \mathbb{N}$. This completes the proof.

It is natural to ask whether strong negligibility in theorem 5.4.10 can be replaced by negligibility. The following theorem shows that that is not the case. If X is compact then the *fundamental dimension* Fd(X) of X is defined by

$$\mathrm{Fd}(X) = \min\{n | \text{there is a compact } Z \text{ with } \mathrm{Sh}(Z) = \mathrm{Sh}(X) \text{ and } \dim(Z) = n\}.$$

5.5.4 THEOREM: *If* S *is a compactum in* X_k *with* $\mathrm{Fd}(S) \leq k$ *then* S *is negligible. If* S *is a compactum in* Y *with the shape of a finite space then*

S *is negligible.*

PROOF: If $Fd(S) \leq k$ we can choose by corollary 5.3.6 a compact $S' \subset X_k$ such that $Sh(S) = Sh(S')$ and $\dim (S') \leq k$. By lemma 5.5.3 and theorem 5.4.1 we have that $X_k \backslash S \approx X_k \backslash S' \approx X_k$.

According to theorem 4.4.5 every copy of Q is negligible in Y. Since Q has trivial shape lemma 5.5.3 implies that every singleton is negligible in Y. Consequently, every finite subset of Y is negligible. Applying once more lemma 5.5.3 we find that every space with the shape of a finite set is negligible.

So every cube is negligible in any X_k. We can prove a partial converse of theorem 5.5.4.

5.5.5 THEOREM: *If* S *is a negligible compactum in* X_0 *then* $Fd(S) \leq 0$. *If* S *is a negligible compactum in* Y *then* S *has the shape of a finite space.*

PROOF: Let k be either -1 or 0 and assume that S is a negligible compactum in X_k. Let h be a homeomorphism from $X_k \backslash S$ onto X_k. According to lemma 4.3.9 there exist a compact space M and monotone maps γ_1 and γ_2 from M onto Q with $\gamma_1^{-1}(X_k \backslash S) = \gamma_2^{-1}(X_k)$ and $h \circ \gamma_1 | \gamma_1^{-1}(X_k \backslash S) = \gamma_2 | \gamma_2^{-1}(X_k)$. Let $\mathcal{C}$ be the collection of components of S and define

$$\mathcal{P} = \{W_i | i \in \mathbb{N}\} \cup \{\{a\} | a \in A_k\}.$$

Let $C \in \mathcal{C}$ and consider the non-empty continuum $\alpha(C) = \gamma_2(\gamma_1^{-1}(C))$, which is a subset of $A_k \cup W$. Since A_k is a σ-compactum with dimension ≤ 0 Sierpiński's theorem implies that there is a $P \in \mathcal{P}$ with $\alpha(C) \subset P$. Analogously we can prove that the continuum $\gamma_1(\gamma_2^{-1}(P))$ is contained in S and hence in C. So α is a function from $\mathcal{C}$ into $\mathcal{P}$ such that for every $C \in \mathcal{C}$, $\gamma_1^{-1}(C) = \gamma_2^{-1}(\alpha(C))$.

Consider the compact set $\tilde{S} = \gamma_2(\gamma_1^{-1}(S))$, which is equal to $\cup\{\alpha(C) | C \in \mathcal{C}\} \subset A_k \cup W$. Observe that $\gamma_1^{-1}(S) = \gamma_2^{-1}(\tilde{S})$. Since any union of infititely many shrunken endfaces is dense in Q, $\tilde{S}$ can intersect only finitely many W_i's. Let $i_1,\ldots,i_l$ be such that $\tilde{S} \cap W = \bigcup_{j=1}^{l} W_{i_j}$. Define the quotient space $\tilde{Q}$ of Q by identifying every W_{i_j} to a point a_j and let p be the natural map from Q onto $\tilde{Q}$. We show that S and $p(\tilde{S})$ have the same shape (cf. Chapman [C : 25.1] and Kozlowski [K]).

It is easily verified that if Z is a Z-set in Q then p(Z) is a Z-set in $\tilde{Q}$. According to corollary 5.5.2 $\tilde{Q}$ is homeomorphic to Q. Note that $S \cup A_k \cup W$ and $p(A_k \cup W)$ are σ-Z-sets in Q and $\tilde{Q}$, respectively. Consequently there exist homotopies $F : Q \times I \to Q$ and $G : \tilde{Q} \times I \to \tilde{Q}$ such that $F_0 = 1$, $G_0 = 1$, $F(Q \times (0,1]) \subset Q\backslash(S \cup A_k \cup W)$ and $G(\tilde{Q} \times (0,1]) \subset \tilde{Q}\backslash p(A_k \cup W)$. Observe that $p|Y : Y \to Y \subset \tilde{Q}$ is a homeomorphism and define for $n \in \mathbb{N}$, $f_n = p \circ h \circ F_{1/n}$ and $g_n = h^{-1} \circ p^{-1} \circ G_{1/n}$. We shall prove that $\mathfrak{f} = (f_n,S,p(\tilde{S}))$ and $\mathfrak{g} = (g_n,p(\tilde{S}),S)$ are shape maps such that $\mathfrak{f} \circ \mathfrak{g}$ and $\mathfrak{g} \circ \mathfrak{f}$ are homotopic to $1_{p(\tilde{S})}$ and 1_S, respectively.

Let V be an open neighbourhood of S in Q. Since $\gamma_1^{-1}(S) = \gamma_2^{-1}(\tilde{S}) = \gamma_2^{-1}(p^{-1}(p(S)))$ we have that $C = p \circ \gamma_2(\gamma_1^{-1}(Q\backslash V))$ is a compact set that is disjoint from $p(\tilde{S})$. Then there is a neighbourhood U of $p(\tilde{S})$ in $\tilde{Q}$ and an $n \in \mathbb{N}$ such that $G(U \times [0,\frac{1}{n}]) \cap C = \emptyset$. Since $p \circ f(V \cap X_k\backslash S) = X_k\backslash C$ and $G(\tilde{Q} \times (0,1]) \subset X_k$ we see that $g_n|U$, $g_{n+1}|U$, $g_{n+2}|U$, ... are homotopic in V. So $\mathfrak{g}$ is a shape map. The proof that $\mathfrak{f}$ is a shape map is analogous.

To see that $\mathfrak{g} \circ \mathfrak{f}$ is homotopic to 1_S choose an open neighbourhood U of S in Q. Select a neighbourhood V of $p(\tilde{S})$ in $\tilde{Q}$ and an $n_1 \in \mathbb{N}$ such that

$$h^{-1} \circ p^{-1} \circ G((V \cap X_k) \times [0,\frac{1}{n_1}]) \subset U$$

and select subsequently a neighbourhood W of S in Q and an $n_2 > n_1$ with for every $m > n_2, f_m(W) \subset V$ and

$$F(W \times [0,\frac{1}{n_2}]) \subset U.$$

If $m > n_2$ then $g_m \circ f_m|W = h^{-1} \circ p^{-1} \circ G_{1/m} \circ f_m|W$ and $h^{-1} \circ p^{-1} \circ f_m|W$ are homotopic in U. Furthermore, we have that $h^{-1} \circ p^{-1} \circ f_m|W = F_{1/m}|W$ and 1_W are homotopic in U. So we may conclude that $\mathcal{g} \circ \mathcal{f}$ is homotopic to $\mathit{1}_S$. The proof for $\mathcal{f} \circ \mathcal{g}$ is similar.

So we have shown that $Sh(S) = Sh(p(\tilde{S}))$. Consider first the case $k = -1$. Then $A_k = \emptyset$ and $p(\tilde{S}) = \{a_1,\ldots,a_l\}$. If $k = 0$ then A_k is a zero-dimensional σ-compactum. Here the countable sum theorem implies that $\dim (p(A_k) \cup \{a_1,\ldots,a_l\}) = 0$. Consequently, $\dim (p(\tilde{S})) \leq 0$ and the theorem is proved.

We believe that the converse of theorem 5.5.4 is also true for $k > 0$ but we have no proof of this.

5.5.6 CONJECTURE: *Let* $k \geq 0$ *and let* $S \subset X_k$ *be compact. Then* S *is negligible iff* $Fd(S) \leq k$.

According to theorem 5.4.10 a σ-compact subset of X_k is strongly negligible iff its dimension is at most k. So strong negligibility depends only on topological properties of the space itself and not on the way that it is embedded in X_k. This is not surprising for compact spaces since they have essentially only one embedding in X_k, cf. corollary 5.3.4. For non-compact spaces, however, there are many non-equivalent embeddings. Negligibility of a σ-compact space in X_k is dependent on the way the space is embedded. Let $k \geq 0$. By corollary 5.4.3 there are copies of $\mathbb{R}^{k+1}$ in X_k that are not negligible. According to theorem 5.5.4 every subset of X_k that is homeomorphic to I^{k+1} is negligible. Also the boundary of I^{k+1} is negligible because it is k-dimensional. This implies that it is possible to embed $I^{k+1}\backslash\partial I^{k+1} \approx \mathbb{R}^{k+1}$ in X_k in such a way that it is negligible.

It remains to prove remark 5.4.8.

5.5.7 PROPOSITION: *An arbitrary subspace* S *of* Y *is finite iff every relatively open subset of* S *is negligible in* Y.

PROOF: One direction of the equivalence follows from theorem 5.5.4.

Consider now a subspace S of Y such that every open subset of S is negligible. Precisely as in theorem 5.4.7 we can prove that every compact subset C of S is negligible in Y and has dimension ≤ 0. This implies in view of theorem 5.5.5 that C has the shape of a finite set. So C has finitely many components which are singletons because $\dim(C) \leq 0$. We have shown that every compact subset of S is finite and hence S is a countable, discrete space. If S is finite we are done.

We shall see that S cannot be infinite (cf. Anderson, Curtis & van Mill [ACM : 6.2]). Let $f : Y\backslash S \to Y$ be a homeomorphism. According to lemma 4.3.9 there exist a compact M and monotone maps γ_1 and γ_2 from M onto Q such that $\gamma^{-1}(Y\backslash S) = \gamma_2(Y)$. We construct in the usual way a one-to-one function $\alpha : S \to \mathbb{N}$ such that for every $a \in S$, $\gamma_1^{-1}(\{a\}) = \gamma_2^{-1}(W_{\alpha(a)})$. Note that $D = \cup\{W_{\alpha(a)} | a \in S\}$ is connected if S is infinite. Consequently, $S = \gamma_1(\gamma_2^{-1}(D))$ is connected which is obviously false.

BIBLIOGRAPHY

[AU] ALEXANDROFF, P. and P. URYSOHN, *Uber null-dimensionale Punktmengen*, Math. Ann. 98 (1928) 89-106.

[A1] ANDERSON, R.D., *Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. 72 (1966) 515-519.

[A2] ANDERSON, R.D., *On topological infinite deficiency*, Mich. Math. J. 14 (1967) 365-383.

[A3] ANDERSON, R.D., *Strongly negligible sets in Fréchet manifolds*, Bull. Amer. Math. Soc. 75 (1969) 64-67.

[A4] ANDERSON, R.D., *On sigma-compact subsets of infinite-dimensional spaces*, unpublished manuscript.

[AC] ANDERSON, R.D. and T.A. CHAPMAN, *Extending homeomorphisms to Hilbert cube manifolds*, Pacific J. Math. 38 (1971) 281-293.

[ACM] ANDERSON, R.D., D.W. CURTIS and J. van MILL, *A fake topological Hilbert space*, Trans. Amer. Math. Soc. 272 (1982) 311-321.

[BP1] BESSAGA, C. and A. PEŁCYŃSKI, *The estimated extension theorem, homogeneous collections and their application to the topological classification of linear metric spaces and convex sets*, Fund. Math. 69 (1970) 153-190.

[BP2] BESSAGA, C. and A. PEŁCYŃSKI, *Selected topics in infinite-dimensional topology*, PWN, Warsaw, 1975.

[B1] BORSUK, K., *Theory of retracts*, PWN, Warsaw, 1967.

[B2] BORSUK, K., *Theory of shape*, PWN, Warsaw, 1975.

[Be] BOTHE, H.G., *Eine Einbettung* m*-dimensionaler Mengen in einen* (m + 1)*-dimensionalen absoluten Retrakt*, Fund. Math. 52 (1963) 209-224.

[C] CHAPMAN, T.A., *Lectures on Hilbert cube manifolds*, CMBS Regional Conf. Series in Math. no.28, Amer. Math. Soc., Providence, R.I., 1976.

[CM] CURTIS, D.W. and J. van MILL, *Zero-dimensional countable dense unions of Z-sets in the Hilbert cube*, to appear in Fund. Math.

[Cs] CURTIS, D.W., *Boundary sets in the Hilbert cube*, to appear.

[D1] DIJKSTRA, J.J., k-*dimensional skeletoids in* $\mathbb{R}^n$ *and the Hilbert cube*, to appear in Topology Appl.

[D2] DIJKSTRA, J.J., *Shrunken endfaces and isotopies of the Hilbert cube*, to appear.

[D3] DIJKSTRA, J.J., *A generalization of the Sierpiński theorem*, to appear in Proc. Amer. Math. Soc.

[DM] DIJKSTRA, J.J. and J. van MILL, *Fake topological Hilbert spaces and characterizations of dimension in terms of negligibility*, to appear.

[DT] DOBROWOLSKI, T. and H. TORUŃCZYK, *Separable complete ANR's admitting a group structure are Hilbert manifolds*, Topology Appl. 12 (1981) 229-235.

[E1] ENGELKING, R., *General topology*, PWN, Warsaw, 1977.

[E2] ENGELKING, R., *Dimension theory*, PWN, Warsaw, 1978.

[GS1] GEOGHEGAN, R. and R.R. SUMMERHILL, *Concerning the shapes of finite-dimensional compacta*, Trans. Amer. Math. Soc. 179 (1973) 281-292.

[GS2] GEOGHEGAN, R. and R.R. SUMMERHILL, *Pseudo-boundaries and pseudo-interiors in euclidean spaces and topological manifolds*, Trans. Amer. Math. Soc. 194 (1974) 141-165.

[HW] HENDERSON, J.P. and J.J. WALSH, *Examples of cell-like decompositions of the infinite dimensional manifolds* σ *and* Σ, to appear.

[H] HUDSON, J.F.P., *Piecewise linear topology*, University of Chicago lecture notes, Benjamin, New York, 1969.

[K] KOZLOWSKI, G., *Images of ANR's*, unpublished manuscript.

[MS] MAZURKIEWICZ, S. and W. SIERPIŃSKI, *Contributions á la topologie des ensembles dénombrables*, Fund. Math. 1 (1920) 17-27.

[M] MENGER, K., *Allgemeine Räume und Cartesische Räume, zweite Mitteilung: Über umfassendste* n*-dimensionale Mengen*, Proc. Kon. Ned. Akad. Wetensch. 29 (1926) 1125-1128.

[Ml] MILL, J. van, *A boundary set for the Hilbert cube containing no arcs*, to appear in Fund. Math.

[S] SIERPIŃSKI, W., *Un théorème sur les continus*, Tôhoku Math. J. 13 (1918) 300-303.

[Š] ŠTAN'KO, M.A., *Solution of Menger's problem in the class of compacta*, Soviet Math. Dokl. 12 (1971) 1846-1849; from: Dokl. Akad. Nauk SSSR 201 (1971) 1299-1302.

[T1] TORUŃCZYK, H., *Skeletonized sets in complete metric spaces and homeomorphisms of the Hilbert cube*, Bull. Acad. Pol. Sci. Sér. Math. 18 (1970) 119-126.

[T2] TORUŃCZYK, H., *Characterizing Hilbert space topology*, Fund. Math. 111 (1981) 247-262.

[W] WEST, J.E., *The ambient homeomorphy of an incomplete subspace of infinite-dimensional Hilbert spaces*, Pacific J. Math. 34 (1970) 257-267.

LIST OF SYMBOLS

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SAMENVATTING

Het hoofdresultaat van dit proefschrift is de constructie van een rij separabele, metriseerbare ruimten $X_{-1}, X_0, X_1, \ldots$ met onder andere de volgende eigenschappen:

(1) X_k is een absoluut retract.

(2) X_k is homogeen.

(3) $X_k \times X_k$ is homeomorf met de hilbertruimte ℓ^2.

(4) Elk compactum in X_k is een Z-verzameling.

(5) X_k is universeel element van de klasse van separabele, metriseerbare ruimten.

(6) Een willekeurige σ-compacte deelruimte van X_k heeft dimensie $\leq k$ dan en slechts dan als zij sterk verwaarloosbaar is.

Deze ruimten worden topologische schijnhilbertruimten genoemd aangezien (1) - (5) bekende topologische eigenschappen zijn van ℓ^2 terwijl uit (6) blijkt dat zij niet homeomorf zijn met ℓ^2. Wij bereiken dit resultaat via de constructie van k-dimensionale pseudoranden in $\mathbb{R}^n$ (hoofdstuk 2) en de hilbertkubus (hoofdstuk 3). Als basis voor onze rij wordt een schijnhilbertruimte gebruikt die geïntroduceerd is door Anderson, Curtis & Van Mill [ACM]. De homogeniteit van deze ruimte wordt in hoofdstuk 4 onderzocht.

STELLINGEN

I De procedure voor het berekenen van de relativistische transportcoëfficiënten van een ijl gas die voorgesteld wordt in [2:sec.5, ex.1], en die op natuurlijke wijze voortkomt uit een wiskundige analyse van de gelineariseerde transportvergelijking, verdient de voorkeur boven de gebruikelijke methode (zie [7]).

II Elke Lebesgue meetbare relativistische sominvariant is bijna overal gelijk aan een functie van de vorm $\alpha + \beta_\mu p^\mu$ ([3]).

III Het bewijs van Grad [6] van de oplosbaarheid van de gelineariseerde Boltzmannvergelijking vertoont een leemte. Het non-relativistische analogon van stelling II brengt hier uitkomst.

IV De gelineariseerde transportvergelijking voor een neutrinogas is oplosbaar en de transportcoëfficiënten kunnen met behulp van een polynomiale benadering van de oplossing bepaald worden ([2]).

V De overdekkingsdimensie van het kwadraat van de rechte van Sorgenfrey is oneindig ([4]).

VI Metriseerbaarheid van reëelcompacte ruimten is geen eerste orde begrip in de ring van continue functies.

VII Er bestaat een compacte, metriseerbare ruimte met inductieve dimensie $\omega + 1$ die geen essentiële afbeelding toelaat naar Hendersons [8] $(\omega + 1)$-dimensionale absolute retract $J^{\omega+1}$ ([1]).

VIII De stelling van Sierpiński [9] laat de volgende generalisatie toe. Zij n een niet-negatief geheel getal en zij X een compacte Hausdorffruimte. Indien $\{F_i \mid i \in \mathbb{N}\}$ een gesloten overdekking is van X zodanig dat voor elk paar verschillende natuurlijke getallen i en j, $\dim(F_i \cap F_j) < n$, dan is elke continue afbeelding van F_1 naar de n-sfeer S^n uit te breiden over geheel X ([5]).

REFERENTIES

[1] P. BORST en J.J. DIJKSTRA, *Essential mappings and transfinite dimension*, verschijnt in Fund.Math.

[2] J.J. DIJKSTRA en W.A. van LEEUWEN, *Mathematical aspects of relativistic kinetic theory*, Physica 90A (1978) 450-486.

[3] J.J. DIJKSTRA, *Mathematical aspects of relativistic kinetic theory II. The summational invariants*, Proc.Kon.Ned.Akad.Wetensch.ser.B 81 (1978) 265-289.

[4] J.J. DIJKSTRA, *A space with maximal discrepancy between its Katětov and covering dimension*, Topology Appl. 12 (1981) 45-48.

[5] J.J. DIJKSTRA, *A generalization of the Sierpiński theorem*, verschijnt in Proc.Amer.Math.Soc.

[6] H.GRAD, *Asymptotic theory of the Boltzmann equation II*, in: Third International Rarefied Gas Symposium vol. 1, ed. J.A. Laurmann (Academic Press, New York, 1963) 26-59.

[7] S.R. de GROOT, W.A. van LEEUWEN en P.H. MELTZER, *Transport coefficients of a neutrino gas*, Il Nuovo Cimento 25A (1975) 229-251.

[8] D.W. HENDERSON, *A lower bound for transfinite dimension*, Fund.Math. 63 (1968) 167-173.

[9] W. SIERPIŃSKI, *Un théorème sur les continus*, Tôhoku Math.J. 13 (1918) 300-303.