# A nearly optimal randomized algorithm for explorable heap selection

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#### Abstract

Explorable heap selection is the problem of selecting the nth smallest value in a binary heap. The key values can only be accessed by traversing through the underlying infinite binary tree, and the complexity of the algorithm is measured by the total distance traveled in the tree (each edge has unit cost). This problem was originally proposed as a model to study search strategies for the branch-and-bound algorithm with storage restrictions by Karp, Saks and Widgerson (FOCS '86), who gave deterministic and randomized  $n \cdot \exp(O(\sqrt{\log n}))$  time algorithms using  $O(\log(n)^{2.5})$  and  $O(\sqrt{\log n})$  space respectively. We present a new randomized algorithm with running time  $O(n\log(n)^3)$  using  $O(\log n)$  space, substantially improving the previous best randomized running time at the expense of slightly increased space usage. We also show an  $\Omega(\log(n)n/\log(\log(n)))$  for any algorithm that solves the problem in the same amount of space, indicating that our algorithm is nearly optimal.

# 1 Introduction

Many important problems in theoretical computer science are fundamentally search problems. The objective of these problems is to find a certain solution from the search space. In this paper we analyze a search problem that we call *explorable heap selection*. The problem is related to the famous branch-and-bound algorithm and was originally proposed by Karp, Widgerson and Saks [KSW86] to model node selection for branch-and-bound with low space-complexity. Furthermore, as we will explain later, the problem remains practically relevant to branch-and-bound even in the full space setting.

The explorable heap selection problem<sup>1</sup> is an online graph exploration problem for an agent on a rooted (possibly infinite) binary tree. The nodes of the tree are labeled by distinct real numbers (the key values) that increase along every path starting from the root. The tree can thus be thought of as a min-heap. Starting at the root, the agent's objective is to select the  $n^{\text{th}}$  smallest value in the tree while minimizing the distance traveled, where each edge of the tree has

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<sup>&</sup>lt;sup>1</sup> [KSW86] did not give the problem a name, so we have attempted to give a descriptive one here.

unit travel cost. The key value of a node is only revealed when the agent visits it, and thus the problem has an online nature.

The selection problem for ordinary heaps, which allow for random access (i.e., jumping to arbitrary nodes in the tree for "free"), has also been studied. In this model, it was shown by [Fre93] that selecting the  $n^{\text{th}}$  minimum can be achieved deterministically in O(n) time using O(n) workspace. We note that in both models,  $\Omega(n)$  is a natural lower bound. This is because verifying that a value  $\mathcal{L}$  is the  $n^{\text{th}}$  minimum requires  $\Theta(n)$  time – one must at least inspect the n nodes with value at most  $\mathcal{L}$  – which can be done via straightforward depth-first search.

A simple selection strategy is to use the best-first rule<sup>2</sup>, which repeatedly explores the unexplored node whose parent has the smallest key value. While this rule is optimal in terms of the number of nodes that it explores, namely  $\Theta(n)$ , the distance traveled by the agent can be far from optimal. In the worst-case, an agent using this rule will need to travel a distance of  $\Theta(n^2)$  to find the  $n^{\text{th}}$  smallest value. A simple bad example for this rule is to consider a rooted tree consisting of two paths (which one can extend to a binary tree), where the two paths are consecutively labeled by all positive even and odd integers respectively. Moreover, the space complexity becomes  $\Omega(n)$  in general when using the best-first rule, because essentially all the explored nodes might need to be kept in memory. We note that irrespective of computational considerations on the agent, either in terms of working memory or running time restrictions, minimizing the total travel distance in explorable heap selection remains a challenging online problem.

Improving on the best-first strategy, Karp, Saks and Wigderson [KSW86] gave a randomized algorithm with expected cost  $n \cdot \exp(O(\sqrt{\log(n)}))$  using  $O(\sqrt{\log(n)})$  working space. They also showed how to make the algorithm deterministic using  $O(\log(n)^{2.5})$  space. In this work, our main contribution is an improved randomized algorithm with expected cost  $O(n\log(n)^3)$  using  $O(\log(n))$  space. Given the  $\Omega(n)$  lower bound, our travel cost is optimal up to logarithmic factors. Furthermore we show that any algorithm for explorable heap selection that uses only s units of memory, must take at least  $n \cdot \log_s(n)$  time in expectation. An interesting open problem is the question whether a superlinear lower bound also holds without any restriction on the memory usage.

To clarify the memory model, it is assumed that any key value and  $O(\log n)$  bit integer can be stored using O(1) space. Furthermore, we will assume that key value comparisons and moving across an edge of the tree require O(1) time. Under these assumptions, the running times of the above algorithms are in fact proportional to their travel cost. Throughout the paper, we will thus use travel cost and running time interchangeably.

Motivation The motivation to look at this problem comes from the branch-and-bound algorithm. This is a well-known algorithm that can be used for solving many types of problems. In particular, it is often used to solve integer linear programs (ILP), which are of the form  $\arg\min\{c^{\top}x:x\in\mathbb{Z}^n,Ax\leq b\}$ . In that setting, branch-and-bound works by first solving the linear programming (LP) relaxation, which does not have integrality constraints. The value of the solution to the relaxation forms a lower bound on the objective value of the original problem. Moreover, if this solution only has integral components, it is also optimal for the original problem. Otherwise, the algorithm chooses a component  $x_i$  for which the solution value  $\hat{x}_i$  is not integral. It then creates two new subproblems, by either adding the constraint  $x_i \leq \lfloor \hat{x}_i \rfloor$  or  $x_i \geq \lceil \hat{x}_i \rceil$ . This operation is called branching. The tree of subproblems, in which the children of a problem are created by the branching operation, is called the branch-and-bound tree. Because a subproblem contains more constraints than its parent, its objective value is greater or equal to

<sup>&</sup>lt;sup>2</sup>Frederickson's algorithm [Fre93] is in fact a highly optimized variant of this rule

the one of its parent.

At the core, the algorithm consists of two important components: the branching rule and the node selection rule. The branching rule determines how to split up a problem into subproblems, by choosing a variable to branch on. Substantial research has been done on branching rules, see, e.g., [LS99, AKM05, LZ17, BDSV18].

The node selection rule decides which subproblem to solve next. Not much theoretical research has been done on the choice of the node selection rule. Traditionally, the best-first strategy is thought to be optimal from a theoretical perspective because this rule minimizes the number of nodes that need to be visited. However, a disadvantage of this rule is that searches using it might use space proportional to the number of explored nodes, because all of them need to be kept in memory. In contrast to this, a simple strategy like depth-first search only needs to store the current solution. Unfortunately, performing a depth-first search can lead to an arbitrarily bad running time. This was the original motivation for introducing the explorable heap selection problem [KSW86]. By guessing the number N of branch-and-bound nodes whose LP values are at most that of the optimal IP solution (which can be done via successive doubling), a search strategy for this problem can be directly interpreted as a node selection rule. The algorithm that they introduced can therefore be used to implement branch-and-bound efficiently in only  $O\left(\sqrt{\log(N)}\right)$  space.

Nowadays, computers have a lot of memory available. This usually makes it feasible to store all explored nodes of the branch-and-bound tree in memory. However, many MIP-solvers still make use of a hybrid method that consists of both depth-first and best-first searches. This is not only done because depth-first search uses less memory, but also because it is often faster. Experimental studies have confirmed that the depth-first strategy is in many cases faster than best-first one [CP99]. This seems contradictory, because the running time of best-first search is often thought to be theoretically optimal.

In part, this contradiction can be explained by the fact that actual IP-solvers often employ complementary techniques and heuristics on top of branch-and-bound, which might benefit from depth-first searches. Additionally, a best-first search can hop between different parts of the tree, while a depth first search subsequently explores nodes that are very close to each other. In the latter case, the LP-solver can start from a very similar state, which is known as warm starting. This is faster for a variety of technical reasons [Ach09]. For example, this can be the case when the LP-solver makes use of the LU-factorization of the optimal basis matrix [MJSS16]. Through the use of dynamic algorithms, computing this can be done faster if a factorization for a similar LP-basis is known [SS93]. Because of its large size, MIP-solvers will often not store the LU-factorization for all nodes in the tree. This makes it beneficial to move between similar nodes in the branch-and-bound tree. Furthermore, moving from one part of the tree to another means that the solver needs to undo and redo many bound changes, which also takes up time. Hence, the amount of distance traveled between nodes in the tree is a relevant metric. The academic solver SCIP also keeps track of this metric.

The explorable heap selection problem captures these benefits of locality by measuring the running time in terms of the amount of travel through the tree. Therefore, we argue that this problem is still relevant for the choice of a node selection rule, even if all nodes can be stored in memory.

**Related work** The explorable heap selection problem was first introduced in [KSW86]. Their result was later applied to prove an upper bound on the parallel running time of branch-and-bound [PPSV15].

When random access to the heap is provided at constant cost, selecting the n'th value in the

heap can be done by a deterministic algorithm in O(n) time by using an additional O(n) memory for auxilliary data structures [Fre93].

The explorable heap selection problem can be thought of as a search game [AG06] and bears some similarity to the cow path problem. In the cow path problem, an agent explores an unweighted unlabeled graph in search of a target node. The location of the target node is unknown, but when the agent visits a node they are told whether or not that node is the target. The performance of an algorithm is judged by the ratio of the number of visited nodes to the distance of the target from the agent's starting point. In both the cow path problem and the explorable heap selection problem, the cost of backtracking and retracing paths is an important consideration. The cow path problem on infinite b-ary trees was studied in [DCD95] under the assumption that when present at a node the agent can obtain an estimate on that node's distance to the target.

Other explorable graph problems exist without a target, where typically the graph itself is unknown at the outset. There is an extensive literature on exploration both in graphs and in the plane. For more information we refer to [Ber98, Tho06] and the references therein.

**Outline** In Section 2 we formally introduce the explorable heap selection problem and any notation we will use. In Section 3 we introduce a new algorithm for solving this problem and provide a running time analysis. In Section 4 we give a lower bound on the complexity of solving explorable heap selection using a limited amount of memory.

# 2 The explorable heap selection problem

We introduce in this section the formal model for the explorable heap selection problem. The input to the algorithm is an infinite binary tree T = (V, E), where each node  $v \in V$  has an associated real value, denoted by  $\operatorname{val}(v) \in \mathbb{R}$ . We assume that all the values are distinct. Moreover, for each node in the tree, the values of its children are larger than its own value. Hence, for every  $v_1, v_2 \in V$  such that  $v_1$  is an ancestor of  $v_2$ , we have that  $\operatorname{val}(v_2) > \operatorname{val}(v_1)$ . The binary tree T is thus a heap.

The algorithmic problem we are interested in is finding the  $n^{\text{th}}$  smallest value in this tree. This may be seen as an online graph exploration problem where an agent can move in the tree and learns the value of a node each time they explore it. At each time step, the agent resides at a vertex  $v \in V$  and may decide to move to either the left child, the right child or the parent of v (if it exists, i.e. if v is not the root of the tree). Each traversal of an edge costs one unit of time, and the complexity of an algorithm for this problem is thus measured by the total traveled distance in the binary tree. The algorithm is also allowed to store values in memory.

We now introduce a few notations used throughout the paper.

- For a node  $v \in V$ , also per abuse of notation written  $v \in T$ , we denote by  $T^{(v)}$  the subtree of T rooted at v.
- For a tree T and a value  $\mathcal{L} \in \mathbb{R}$ , we define the subtree  $T_{\mathcal{L}} := \{v \in T \mid \operatorname{val}(v) \leq \mathcal{L}\}.$
- We denote the  $n^{\text{th}}$  smallest value in T by  $\text{Select}^T(n)$ . This is the quantity that we are interested in finding algorithmically.
- We say that a value  $\mathcal{V} \in \mathbb{R}$  is good for a tree T if  $\mathcal{V} \leq \text{SELECT}^T(n)$  and bad otherwise. Similarly, we call a node  $v \in T$  good if  $\text{val}(v) \leq \text{SELECT}^T(n)$  and bad otherwise.
- We will use [k] to refer to the set  $\{1, \ldots, k\}$ .

• When we write  $\log(n)$ , we assume the base of the logarithm to be 2.

For a given value  $\mathcal{V} \in \mathbb{R}$ , it is easy to check whether it is good in O(n) time: start a depth first search at the root of the tree, turning back each time a value strictly greater than  $\mathcal{V}$  is encountered. In the meantime, count the number of values below  $\mathcal{V}$  found so far and stop the search if more than n values are found. If the number of values below  $\mathcal{V}$  found at the end of the procedure is at most n, then  $\mathcal{V}$  is a good value. This procedure is described in more detail later in the DFS subroutine.

We will often instruct the agent to move to an already discovered good vertex  $v \in V$ . The way this is done algorithmically is by saving val(v) in memory and starting a depth first search at the root, turning back every time a value strictly bigger than val(v) is encountered until finally finding val(v). This takes at most O(n) time, since we assume v to be a good node. If we instruct the agent to go back to the root from a certain vertex  $v \in V$ , this is simply done by traveling back in the tree, choosing to go to the parent of the current node at each step.

In later sections, we will often say that a subroutine takes a subtree  $T^{(v)}$  as input. This implicitly means that we in fact pass it  $\operatorname{val}(v)$  as input, make the agent travel to  $v \in T$  using the previously described procedure, call the subroutine from that position in the tree, and travel back to the original position at the end of the execution. Because the subroutine knows the value  $\operatorname{val}(v)$  of the root of  $T^{(v)}$ , it can ensure it never leaves the subtree  $T^{(v)}$ , thus making it possible to recurse on a subtree as if it were a rooted tree by itself. We write the subtree  $T^{(v)}$  as part of the input for simplicity of presentation.

We will sometimes want to pick a value uniformly at random from a set of values  $\{\mathcal{V}_1, \ldots, \mathcal{V}_k\}$  of unknown size that arrives in a streaming fashion, for instance when we traverse a part of the tree T by doing a depth first search. That is, we see the value  $\mathcal{V}_i$  at the  $i^{\text{th}}$  time step, but do not longer have access to it in memory once we move on to  $\mathcal{V}_{i+1}$ . This can be done by generating random values  $\{X_1, \ldots, X_k\}$  where, at the  $i^{\text{th}}$  time step,  $X_i = \mathcal{V}_i$  with probability 1/i, and  $X_i = X_{i-1}$  otherwise. We define the sample from the stream to be the final random value  $X_k$  and it is easy to check that  $X_k$  is a uniformly distributed sample from  $\{\mathcal{V}_1, \ldots, \mathcal{V}_k\}$ .

# 3 A new algorithm

The authors of [KSW86] presented a deterministic algorithm that solves the explorable heap selection problem in  $n \cdot \exp(O(\sqrt{\log(n)}))$  time and  $O(n\sqrt{\log(n)})$  space. By replacing the binary search that is used in the algorithm by a randomized variant, they are able to decrease the space requirements. This way, they obtain a randomized algorithm with expected running time  $n \cdot \exp(O(\sqrt{\log(n)}))$  and space complexity  $O(\sqrt{\log(n)})$ . Alternatively, the binary search can be implemented using a deterministic routine by [MP80] to achieve the same running time with  $O(\log(n)^{2.5})$  space.

We present a randomized algorithm with a running time  $O(n \log(n)^3)$  and space complexity  $O(\log(n))$ . Unlike the algorithms mentioned before, our algorithm fundamentally relies on randomness to bound its running time. This bound only holds when the algorithm is run on a tree with labels that are fixed before the execution of the algorithm. That is, the tree must be generated by an adversary that is oblivious to the choices made by the algorithm. This is a stronger assumption than is needed for the algorithm that is given in [KSW86], which also works against adaptive adversaries. An adaptive adversary is able to defer the decision of the node label to the time that the node is explored. Note that this distinction does not really matter for the application of the algorithm as a node selection rule in branch-and-bound, since there the node labels are fixed because they are derived from the integer program.

**Theorem 1.** There exists a randomized algorithm that solves the explorable heap selection problem, with expected running time  $O(n \log(n)^3)$  and  $O(\log(n))$  space.

As mentioned above, checking whether a value v is good can be done in O(n) time by doing a depth-first search with cutoff value  $\operatorname{val}(v)$  that returns when more than n good nodes are found. For a set of k values, we can determine which of them are good in  $O(\log(k)n)$  time by performing a binary search.

The explorable heap selection problem can be seen as the problem of finding all n good nodes. Both our method and that of [KSW86] function by first identifying a subtree consisting of only good nodes. The children of the leaves of this subtree are called "roots" and the subtree is extended by finding a number of new good nodes under these roots in multiple rounds.

In [KSW86] this is done by running  $O(c^{\sqrt{2\log(n)}})$  different rounds, for some constant c>1. In each round, the algorithm finds  $n/c^{\sqrt{2\log(n)}}$  new good nodes. These nodes are found by recursively exploring each active root and using binary search on the observed values to discover which of these values are good. Which active roots are recursively explored further depends on which values are good. The recursion in the algorithm is at most  $O(\sqrt{\log(n)})$  levels deep, which is where the space complexity bound comes from.

In our algorithm, we take a different approach. We will call our algorithm consecutively with  $n = 1, 2, 4, 8, \dots$  Hence, for a call to the algorithm, we can assume that we have already found at least n/2 good nodes. These nodes form a subtree of the original tree T. In each round, our algorithm chooses a random root under this subtree and finds every good node under it. It does so by doing recursive subcalls to the main algorithm on this root with values  $n = 1, 2, 4, 8, \dots$ As soon as the recursively obtained node is a bad node, the algorithm stops searching the subtree of this root, since it is guaranteed that all the good nodes there have been found. The largest good value that is found can then be used to find additional good nodes under the other roots without recursive calls, through a simple depth-first search. Assuming that the node values were fixed in advance, we expect this largest good value to be greater than half of the other roots' largest good values. Similarly, we expect its smallest bad value to be smaller than half of the other roots' smallest bad values. By this principle, a sizeable fraction of the roots can, in expectation, be ruled out from getting a recursive call. Each round a new random root is selected until all good nodes have been found. This algorithm allows us to effectively perform binary search on the list of roots, ordered by the largest good value contained in each of their subtrees in  $O(\log n)$  rounds, and the same list ordered by the smallest bad values (Lemma 4). Bounding the expected number of good nodes found using recursive subcalls requires a subtle induction on two parameters (Lemma 3): both n and the number of good nodes that have been identified so far.

#### 3.1 The algorithm

We present in this subsection the algorithm for solving the explorable heap selection problem, as well as the subroutines used in it. This algorithm is named SELECT and outputs the  $n^{\text{th}}$  smallest value in the tree T.

A procedure used in Select is the Extend algorithm, which assumes that at least n/2 good nodes have already been found in the tree, and also outputs the  $n^{\text{th}}$  smallest one.

#### Algorithm 1 The SELECT procedure

```
1: Input: n \in \mathbb{N}
 2: Output: Select(n), the n^{\text{th}} smallest value in the heap T.
    procedure Select(n)
          k \leftarrow 1
 4:
          \mathcal{L} \leftarrow \operatorname{val}(v) \ //\ v is the root of the tree T
 5:
          while k < n do
 6:
              if k < n/2 then
 7:
                   k' \leftarrow 2k
 8:
              else
9:
                   k' \leftarrow n
10:
              end if
11:
              \mathcal{L} \leftarrow \text{Extend}(T, k', k, \mathcal{L})
12:
              k \leftarrow k'
13:
14:
          end while
          return \mathcal{L}
15:
16: end procedure
```

#### Algorithm 2 The EXTEND procedure

```
1: Input: T: tree which is to be explored.
                  n \in \mathbb{N}: total number of good values to be found in the tree T, satisfying n \geq 2.
 3:
                  k \in \mathbb{N}: number of good values already found in the tree T, satisfying k \geq n/2.
                  \mathcal{L}_0 \in \mathbb{R}: value satisfying DFS(T, \mathcal{L}_0, n) = k.
 4:
 5: Output: Select<sup>T</sup>(n): the n<sup>th</sup> smallest value in T.
 6: procedure EXTEND(T, n, k, \mathcal{L}_0)
           \mathcal{L} \leftarrow \mathcal{L}_0
 7:
          \mathcal{U} \leftarrow \infty
 8:
           while k < n do
 9:
                r \leftarrow \text{random element from Roots}(T, \mathcal{L}_0, \mathcal{L}, \mathcal{U})
10:
                \mathcal{L}' \leftarrow \max(\mathcal{L}, \operatorname{val}(r))
11:
                k' \leftarrow \text{DFS}(T, \mathcal{L}', n) // counting the number of values \leq \mathcal{L}' in the original tree T
12:
                c \leftarrow \mathrm{DFS}(T^{(r)}, \mathcal{L}', n) // counting the number of values \leq \mathcal{L}' in T^{(r)}
13:
                c' \leftarrow \min(n - k' + c, 2c) // increasing the number of values to be found in T^{(r)}
                while k' < n do // looping until it is certified that SELECT<sup>T</sup>(n) \le \mathcal{L}'
15:
                     \mathcal{L}' \leftarrow \text{Extend}(T^{(r)}, c', c, \mathcal{L}') // computing the (c')^{\text{th}} smallest value in T^{(r)}
16:
                     k' \leftarrow \mathrm{DFS}(T, \mathcal{L}', n)
17:
                     c \leftarrow c'
18:
19:
                     c' \leftarrow \min(n - k' + c, 2c)
                end while
20:
                \tilde{\mathcal{L}}, \tilde{\mathcal{U}} \leftarrow \text{GoodValues}(T, T^{(r)}, \mathcal{L}', n) // finding the good values in T^{(r)}
21:
                \mathcal{L} \leftarrow \max(\mathcal{L}, \tilde{\mathcal{L}})
                \mathcal{U} \leftarrow \min(\mathcal{U}, \mathcal{U})
23:
                k \leftarrow \mathrm{DFS}(T, \mathcal{L}, n) // computing the total number of good values found so far in T
24:
           end while
25:
           return \mathcal{L}
27: end procedure
```

Let us describe a few invariants from the EXTEND procedure.

- $\mathcal{L}$  and  $\mathcal{U}$  are respectively lower and upper bounds on Select<sup>T</sup>(n) during the whole execution of the procedure. More precisely,  $\mathcal{L} \leq \text{Select}^T(n)$  and  $\mathcal{U} > \text{Select}^T(n)$  at any point, and hence  $\mathcal{L}$  is good and  $\mathcal{U}$  is bad. The integer k counts the number of values  $\leq \mathcal{L}$  in the full tree T.
- No root can be randomly selected twice. This is ruled out by the updated values of  $\mathcal{L}$  and  $\mathcal{U}$ , and the proof can be found in Theorem 2.
- After an iteration of the inner while loop,  $\mathcal{L}'$  is set to the  $c^{\text{th}}$  smallest value in  $T^{(r)}$ . The variable c' then corresponds to the next value we would like to find in  $T^{(r)}$  if we were to continue the search. Note that  $c' \leq 2c$ , enforcing that the recursive call to EXTEND satisfies its precondition, and that  $c' \leq n (k' c)$  implies that  $(k' c) + c' \leq n$ , which implies that the recursive subcall will not spend time searching for a value that is known in advance to be bad.
- k' always counts the number of values  $\leq \mathcal{L}'$  in the full tree T. It is important to observe that this is a global parameter, and does not only count values below the current root. Moreover,  $k' \geq n$  implies that we can stop searching below the current root, since it is guaranteed that all good values in  $T^{(r)}$  have been found, i.e.,  $\mathcal{L}'$  is larger than all the good values in  $T^{(r)}$ .

We now describe the subroutines used in the EXTEND procedure.

The procedure DFS The procedure DFS is a variant of depth first search. The input to the procedure is T, a cutoff value  $\mathcal{L} \in \mathbb{R}$  and an integer  $n \in \mathbb{N}$ . The procedure returns the number of vertices in T whose value is at most  $\mathcal{L}$ .

It achieves that by exploring the tree T in a depth first search manner, starting at the root and turning back as soon as a node  $w \in T$  such that  $\operatorname{val}(w) > \mathcal{L}$  is encountered. Moreover, if the number of nodes whose value is at most  $\mathcal{L}$  exceeds n during the search, the algorithm stops and returns n+1. This subtree visits only nodes in  $T_{\mathcal{L}}$  or its direct descendants and its running time is O(n).

The algorithm output is the following integer.

$$DFS(T, \mathcal{L}, n) := \min \{ |T_{\mathcal{L}}|, n+1 \}.$$

Observe that the DFS procedure allows us to check whether a node  $w \in T$  is a good node, i.e. whether val $(w) \leq \text{SELECT}^T(n)$ . Indeed, w is good if and only if DFS $(T, \text{val}(w), n) \leq n$ .

The procedure Roots The procedure Roots takes as input a tree T as well as an initial fixed lower bound  $\mathcal{L}_0 \in \mathbb{R}$  on the value of  $\mathrm{SELECT}^T(n)$ . We assume that the main algorithm has already found all the nodes  $w \in T$  satisfying  $\mathrm{val}(w) \leq \mathcal{L}_0$ . This means that the remaining values the main algorithm needs to find in T are all lying in the subtrees of the following nodes, that we call the  $\mathcal{L}_0$ -roots of T:

$$R(T, \mathcal{L}_0) := \{ r \in T \setminus T_{\mathcal{L}_0} \mid r \text{ is a child of a node in } T_{\mathcal{L}_0} \}$$

In other words, these are all the vertices in T one level deeper in the tree than  $T_{\mathcal{L}_0}$ , see Figure 1 for an illustration. In addition to that, the procedure takes two other parameters  $\mathcal{L}, \mathcal{U} \in \mathbb{R}$  as input, which correspond to (another) lower and upper bound on the value of Select<sup>T</sup>(n). These

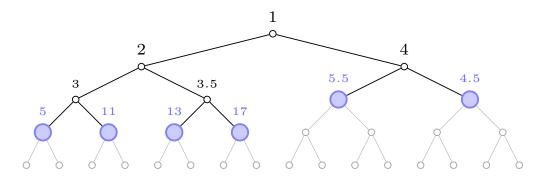


Figure 1: An illustration of  $R(T, \mathcal{L}_0)$  with  $\mathcal{L}_0 = 4$ . The number above each vertex is its value, the blue nodes are  $R(T, \mathcal{L}_0)$ , whereas the subtree above is  $T_{\mathcal{L}_0}$ .

bounds  $\mathcal{L}$  and  $\mathcal{U}$  will be variables being updated during the execution of the main algorithm, where  $\mathcal{L}$  will be increasing and  $\mathcal{U}$  will be decreasing. More precisely,  $\mathcal{L}$  will be the largest value that the main algorithm has certified being at most Select<sup>T</sup>(n), whereas  $\mathcal{U}$  will be the smallest value that the algorithm has certified being at least that. A key observation is that these lower and upper bounds can allow us to remove certain roots in  $R(T, \mathcal{L}_0)$  from consideration, in the sense that all the good values in that root's subtree will be certified to have already been found. The only roots that the main algorithm needs to consider, when  $\mathcal{L}$  and  $\mathcal{U}$  are given, are thus the following.

$$\mathrm{Roots}(T, \mathcal{L}_0, \mathcal{L}, \mathcal{U}) := \left\{ r \in R(T, \mathcal{L}_0) \mid \exists w \in T^{(r)} \text{ with } \mathrm{val}(w) \in (\mathcal{L}, \mathcal{U}) \right\}$$

This subroutine can be implemented as follows. Run a depth first search starting at v. Once a node  $r \in T$  with  $\operatorname{val}(r) > \mathcal{L}_0$  is encountered, the subroutine marks that vertex r as belonging to  $R(T, \mathcal{L}_0)$ . The depth first search continues deeper in the tree until finding a node  $w \in T^{(r)}$  with  $\operatorname{val}(w) > \mathcal{L}$ . At this point, if  $\operatorname{val}(w) < \mathcal{U}$ , then the search directly returns to r without exploring any additional nodes in  $T^{(r)}$  and adds r to the output. If however  $\operatorname{val}(w) \geq \mathcal{U}$ , then the search continues exploring  $T_{\mathcal{L}}^{(r)}$  (and its direct descendants) by trying to find a node w with  $\operatorname{val}(w) \in (\mathcal{L}, \mathcal{U})$ . In case the algorithm explores all of  $T_{\mathcal{L}}^{(r)}$  with its direct descendants, and it turns out that no such node exists (i.e. every direct descendant w of  $T_{\mathcal{L}}^{(r)}$  satisfies  $\operatorname{val}(w) \geq \mathcal{U}$ ), then r is not added to the output.

This procedure takes time  $O(|T_{\mathcal{L}}|)$ , i.e. proportional to the number of nodes in T with value at most  $\mathcal{L}$ . Since the procedure is called only on values  $\mathcal{L}$  which are known to be good, the time is bounded by  $O(|T_{\mathcal{L}}|) = O(n)$ .

The procedure GoodValues The procedure GoodValues takes as input a tree T, a subtree  $T^{(r)}$  for a node  $r \in T$ , a value  $\mathcal{L}' \in \mathbb{R}_{\geq 0}$  and an integer  $n \in \mathbb{N}$ . The procedure then analyzes the set

$$S := \left\{ \operatorname{val}(w) \mid w \in T^{(r)}, \operatorname{val}(w) \le \mathcal{L}' \right\}$$

and outputs both the largest good value and the smallest bad value in that set, that we respectively call  $\mathcal{L}$  and  $\mathcal{U}$ . If no bad values exist in S, the algorithm sets  $\mathcal{U} = \infty$ . Notice that this output determines, for each value in S, whether it is good or not. Indeed, any  $\mathcal{V} \in S$  is good if and only if  $\mathcal{V} \leq \mathcal{L}$ , and is bad if and only if  $\mathcal{V} \geq \mathcal{U}$ .

The implementation is as follows. Start by initializing the variables  $\mathcal{L} = -\infty$  and  $\mathcal{U} = \mathcal{L}'$ . These variables correspond to lower and upper bounds on Select $^T(n)$ . Loop through the values

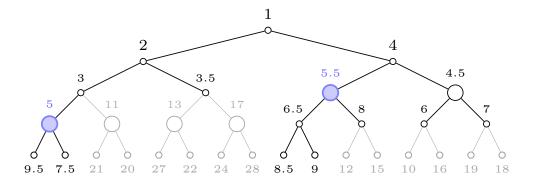


Figure 2: An illustration of the Roots procedure with  $\mathcal{L}_0 = 4$ ,  $\mathcal{L} = 7$  and  $\mathcal{U} = 10$ . Only two active roots remain, and are both colored in blue. The other roots are considered killed since all the good values have been found in their subtrees.

in

$$S' := \left\{ \operatorname{val}(w) \mid w \in T^{(r)}, \ \mathcal{L} < \operatorname{val}(w) < \mathcal{U} \right\}$$

using a depth first search starting at r and sample one value  $\mathcal{V}$  uniformly randomly from that set. Check whether  $\mathcal{V}$  is a good value by calling DFS $(T, \mathcal{V}, n)$ . If it is good, update  $\mathcal{L} = \mathcal{V}$ . If it is bad, update  $\mathcal{U} = \mathcal{V}$ . Continue this procedure until S' is empty, i.e. |S'| = 0. If, at the end of the procedure,  $\mathcal{L} = \mathcal{L}' = \mathcal{U}$ , then set  $\mathcal{U} = \infty$ . The output is thus:

GOODVALUES
$$(T, T^{(r)}, \mathcal{L}', n) := \{\mathcal{L}, \mathcal{U}\}$$

where

$$\begin{split} \mathcal{L} &:= \max \left\{ \mathcal{V} \in S \mid \mathcal{V} \leq \text{Select}^T(n) \right\}, \\ \mathcal{U} &:= \min \left\{ \mathcal{V} \in S \mid \mathcal{V} > \text{Select}^T(n) \right\}. \end{split}$$

Sampling a value from S' and checking whether the sampled value is good takes O(n) time (under the assumption that |S'| = O(n), which will always be the case when this procedure is called in the main algorithm, since  $|\{v \in T \mid \text{val}(v) \leq \mathcal{L}'\}| \leq 2n$ ). Moreover, in expectation, the number of updates before the set S' is empty is  $O(\log(n))$ , leading to a total running time of  $O(n \log(n))$ .

### 3.2 Proof of correctness

**Theorem 2.** At the end of the execution of Algorithm 1,  $\mathcal{L}$  is set to the  $n^{th}$  smallest value in T. Moreover, the algorithm is guaranteed to terminate.

Proof. We show  $\mathcal{L} = \text{Select}^T(n)$  holds at the end of Algorithm 2, i.e. the EXTEND procedure. Correctness of Algorithm 1, i.e. the Select procedure, then clearly follows from that. First, notice that  $\mathcal{L}$  is always set to the first output of the procedure GoodValues, which is always the value of a good node in T, implying

$$\mathcal{L} < \text{Select}^T(n)$$

at any point during the execution of the algorithm. Since the outer while loop ends when at least n good nodes in T have value at most  $\mathcal{L}$ , we get

$$\mathcal{L} \geq \text{SELECT}^T(n),$$

which implies that when the algorithm terminates it does so with the correct value.

It remains to prove that the algorithm terminates. We observe that every recursive call  $\mathcal{L}' \leftarrow \text{Extend}(T^{(r)}, c', c, \mathcal{L}')$  strictly increases the value of  $\mathcal{L}'$ , ensuring that at least one extra value in T is under the increased value. This implies that k' strictly increases every iteration of the inner while loop, thus ensuring that it terminates.

Next, we prove that the outer while loop terminates by considering the number of roots  $s := |\text{Roots}(T, \mathcal{L}_0, \mathcal{L}, \mathcal{U})|$ . At the start of the algorithm, we know that s = O(n) is finite because  $\mathcal{L}_0$  is a good value. When s = 1, the outer loop terminates since  $k \geq k' \geq n$ . By definition of Roots $(T, \mathcal{L}_0, \mathcal{L}, \mathcal{U})$ , we see that once a root r is selected in one iteration of the outer while loop, it can never again be a member of Roots $(T, \mathcal{L}_0, \mathcal{L}, \mathcal{U})$  in later iterations. Hence s strictly decreases every iteration, proving that the outer while loop takes at most O(n) iterations.

Last, we observe that k > c', which implies c' < n, ensuring that the depth of recursion is finite.

### 3.3 Running time analysis

The main challenge in analyzing the running time of the algorithm is dealing with the cost of the recursive subcalls. For this we rely on two important ideas.

Firstly, remember that n is the index of the node value that we want to find, while k is the index of the node value that is passed to the procedure. In particular this means that the procedure needs to find only n - k new good nodes. Because of this, our runtime bound for the recursive subcalls that are performed does not just depend on n, but also on n - k.

We will show that the amount of travel done in the non-recursive part of a call to the algorithm with parameters n and k is bounded by  $O(n \log(n)^2)$ . We will charge this travel to the parent call that makes these recursive calls. Hence, a parent call that does z recursive calls with parameters  $(n_1, k_1), \ldots, (n_z, k_z)$  will be charged a cost of  $\sum_{i=1}^{z} n_i \log(n_i)^2$ . In our analysis we will show that this sum can be upper bounded by  $(n-k)\log(n)^2$ . So, for every recursive call with parameters n and k, a cost of at most  $(n-k)\log(n)^2$  is incurred by the caller.

So, we only have to bound the sum over  $(n-k)\log(n)^2$  for all calls with parameters n and k that are done. We do this by first considering a single algorithm call with parameters n and k that makes z recursive subcalls with parameters  $(n_1, k_1), \ldots, (n_z, k_z)$ . For such a subcall we would like to bound the sum  $\sum_{i=1}^{z} (n_i - k_i) \log(n_i)^2$  by  $(n-k) \log(n)^2$ . However this bound does not hold deterministically. Instead we show that this bound does hold in expectation.

Now we know that every layer of recursion incurs an expected cost of at most  $(n-k)\log(n)^2$ . Because the parameter n will decrease by at least a constant factor in each layer of recursion, there can be at most  $O(\log(n))$  layers. The upper bound of  $O(n\log(n)^3)$  on the expected running time of our algorithm now follows.

Let us now prove these claims. First we will show that the expectation of  $\sum_{i=1}^{z} (n_i - k_i)$  is bounded.

**Lemma 3.** Let z be the number of recursive calls that are done in the main loop of EXTEND(T,  $n^*$ ,  $k^*$ ,  $\mathcal{L}$ ) with parameter  $k \geq 1$ . For  $i \in [z]$ , let  $n_i$  and  $k_i$  be the values of n and k that are given as parameters to the ith such subcall. Then:

$$\mathbb{E}\left[\sum_{i=1}^{z} n_i - k_i\right] \le n^* - k^*.$$

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*Proof.* Assume we have m roots, whose order is fixed. For  $i \in [z]$ , let  $r_i \in [m]$  be such that the ith recursive subcall is done on the root with index  $r_i$ . For  $t \in [m]$ , let  $s_t = \sum_{i=1}^z \mathbf{1}_{r_i=t} (n_i - k_i)$ . From the algorithm we see that when  $r_i = t$ , all successive recursive calls will also be on root t, until all good nodes under this root have been found. The updated values of  $\mathcal L$  and  $\mathcal U$  ensure this root is never selected again after this, hence all iterations i with  $r_i = t$  are consecutive. Now let  $a_t, b_t$  be variables that respectively denote the first and last indices i with  $r_i = t$ . When there is no iteration i with  $r_i = t$ , then  $a_t = b_t = \infty$ .

For two calls i and i + 1 with  $r_i = t = r_{i+1}$ , observe that after call i already  $n_i$  good nodes under root t have been found. Note that on line 16, c' corresponds to  $n_i$  and c corresponds to  $k_i$ , hence  $k_{i+1} = n_i$ . Therefore, the definition of  $s_t$  is a telescoping series and can be rewritten as  $s_t = n_{b_t} - k_{a_t}$ , when we define  $k_{\infty} = n_{\infty} = 0$ .

Let  $p = n^* - k^*$  and let  $W = \{w_1, \dots, w_p\}$  denote the p smallest values under T that are larger than  $\mathcal{L}_0$ , in increasing order. Now each of these values in W will be part of a subtree generated by one of the roots. For the  $j \in [p]$ , let  $d_j \in [m]$  be such that value  $w_j$  is part of the subtree of root  $d_j$ . Let  $S_t = \{j \in [p] : d_j = t\}$ .

We will now show that for each root  $r_t$ , we have:

$$\mathbb{E}[s_t] \le |S_t|.$$

This will imply that  $\mathbb{E}\left[\sum_{i=1}^{z} n_i - k_i\right] = \sum_{t=1}^{m} \mathbb{E}[s_t] \leq \sum_{t=1}^{m} |S_t| = n^* - k^*$ . First, let us consider a root t with  $t \neq d_p$ . On line 10 each iteration a random root is chosen. Because in every iteration root  $d_p$  will be among the active roots, the probability that this root is chosen before root t, is at least a half. In that case, after the iteration of root  $d_n$ ,  $\mathcal{L}$  will be set to  $w_p$ . Then DFS $(T, \mathcal{L}, n)$  will return n, and the algorithm will terminate. Because no subcalls will be done on root t, in this case  $s_i = 0$ .

If the algorithm does do subcalls i with  $r_i = t$ , then consider iteration  $b_t$ , the last iteration i that has  $r_i = t$ . Before this iteration, already  $k_{b_t}$  good nodes under the root have been found by the algorithm. It can be seen in the algorithm on lines 14 and 19 that  $n_{b_t} \leq 2k_{b_t}$ . Hence  $s_t = n_{b_t} - k_{a_t} \le n_{b_t} \le 2k_{b_t} \le 2|S_t|$ . We therefore have:

$$\mathbb{E}[s_t] \le \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 2|S_t| = |S_t|.$$

Now consider the root  $d_p$ . If  $S_{d_p} = [p]$ , then  $s_p = n_{b_{d_p}} - k_{a_{d_p}} \le n^* - k^* = |S_{d_p}|$ , because  $n_{b_{d_p}} \leq n^*$  and  $k_{a_{d_p}} \geq k^*$ .

If  $S_{d_p} \subsetneq [p]$ , then there exists a j with  $d_j \neq d_p$ . Thus, we can define  $j^* = \max\{j \in [p] : d_j \neq j \in [p]\}$  $d_p$ . With probability a half, root  $d_{j^*}$  is considered before root  $d_p$ . If this happens,  $\mathcal{L}$  will be equal to  $w_{j^*}$  when root  $d_p$  is selected by the algorithm. In particular, this means that  $k_{a_{d_p}}$  will be equal to  $j^*$ . Recall the stated invariant that  $c' \leq n^* - k^* = p$ , and hence  $n_{b_{d_p}} = c' \leq p$ . Now we can see that  $s_{d_p} = n_{b_{d_p}} - k_{a_{d_p}} \le p - j^*$ . If root  $d_p$  is chosen before root  $d_{j^*}$ , then consider the last recursive call  $b_{d_p}$  to EXTEND that

we do on root  $d_p$ . Define  $A = [k' - k^*] \cap S_{d_p}$ , i.e. the set of all good values under root  $d_p$  that have been found so far. We distinguish two cases.

If  $k' - k^* \ge j^*$ , i.e., when all good values under  $d_{j^*}$  have been found, then by the definition of  $j^*$  we have  $[p] \setminus [k'-k^*] \subseteq [p] \setminus [j^*] \subseteq S_{d_p}$ . Because A and  $[p] \setminus [k'-k^*]$  are disjoint, we have  $|A| + (n^* - k') = |A| + |[p] \setminus [k'-k^*]| \le |S_{d_p}|$ . Hence:

$$c' \le n^* - k' + c = n^* - k' + |A| \le |S_{d_n}|.$$

Therefore,  $s_{d_p} \leq n_{b_{d_p}} = c' \leq |S_{d_p}|$ .

If  $k' - k^* < j^*$  at the time of subcall  $b_{d_n}$ , then the last good value under  $d_{j^*}$  has yet to be found, implying that  $A \subseteq [j^*]$ . From the definition of  $j^*$  we get  $[p] \setminus [j^*] \subseteq S_{d_p}$ . Hence,  $|A| \le |S_{d_p}| - |[p] \setminus [j^*]| = |S_{d_p}| - (p - j^*)$ . Thus  $c' \le 2c = 2|A| \le 2(|S_{d_p}| - (p - j^*))$ . So, in this case we have  $s_{d_p} \le n_{b_{d_p}} = c' \le 2(|S_{d_p}| - (p - j^*))$ . Collecting the three cases above, we find that

$$\mathbb{E}[s_{d_p}] \le \frac{1}{2} \cdot (p - j^*) + \frac{1}{2} \cdot \max\left(|S_{d_p}|, 2(|S_{d_p}| - (p - j^*))\right)$$

$$\le \max\left(\frac{1}{2}|S_{d_p}| + \frac{1}{2}(p - j^*), |S_{d_p}| - \frac{1}{2}(p - j^*)\right).$$

Lastly, by definition of  $j^*$  we have  $[p] \setminus [j^*] \subseteq S_{d_p}$ , from which it follows that  $p - j^* \leq |S_{d_p}|$ . We finish the proof by observing that this implies

$$\max\left(\frac{1}{2}|S_{d_p}| + \frac{1}{2}(p-j^*), |S_{d_p}| - \frac{1}{2}(p-j^*)\right) \le |S_{d_p}|,$$

which finishes the proof.

Now we will bound the expected number of iterations of the outermost while-loop.

**Lemma 4.** The expected number of times that the outermost while-loop (at line 9) is executed by the procedure EXTEND is at most  $O(\log(n))$ .

*Proof.* Let  $r_1, \ldots, r_m$  denote the roots returned by ROOTS $(T, \mathcal{L}_0, \mathcal{L}_0, \infty)$ . For  $j \in [m]$ , let  $\ell_j$  and  $u_i$  respectively denote the largest good value and the smallest non-good value under root  $r_i$ . Let  $A_{\ell}(\mathcal{L}) := \{r_j : \ell_j > \mathcal{L}\}$  and  $A_{u}(\mathcal{U}) := \{r_j : u_j < \mathcal{U}\}$ . Observe that Roots $(T, \mathcal{L}_0, \mathcal{L}, \mathcal{U}) = \mathcal{L}_0$  $A_{\ell}(\mathcal{L}) \cup A_{u}(\mathcal{U})$  for any  $\mathcal{L} \leq \mathcal{U}$ .

Let  $\mathcal{L}_i$  and  $\mathcal{U}_i$  denote the values of  $\mathcal{L}$  and  $\mathcal{U}$  at the start of the *i*th iteration. After an iteration i in which root  $r_i$  was selected, the algorithm updates  $\mathcal{L}$  and  $\mathcal{U}$  such that  $\mathcal{L}_{i+1} = \max(\mathcal{L}, \ell_i)$  and  $\mathcal{U}_{i+1} = \min(\mathcal{U}, u_i)$ . Observe that  $\mathcal{L}_i$  is nondecreasing and that  $\mathcal{U}_i$  is nonincreasing.

We will now show that if a root from  $A_{\ell}(\mathcal{L}_i)$  is selected in iteration i, then the expected size of  $A_{\ell}(\mathcal{L}_{i+1})$  is at most half that of  $A_{\ell}(\mathcal{L}_i)$ . This will imply that in expectation only  $\log(n)$ iterations are needed to make  $|A_{\ell}(\mathcal{L})| = 1$ .

Let  $\mathcal{F}_i$  be the filtration containing all information up till iteration i. Let  $X_i$  denote the value of  $|A_{\ell}(\mathcal{L}_i)|$ . Let  $s_k$  be the subsequence consisting of iteration indices i in which a root from  $A_{\ell}(\mathcal{L}_i)$ 

is selected. Because roots are selected uniformly at random, we have  $\mathbb{E}[X_{s_{k+1}} \mid \mathcal{F}_{s_k}] \leq \frac{1}{2}X_{s_k}$ . Let  $Y_i = \max(\log(X_i), 0)$ . Note that when  $Y_{s_k} \geq 2$ , we have  $\mathbb{E}[Y_{s_{k+1}} \mid \mathcal{F}_{s_k}] = \mathbb{E}[\log(X_{s_{k+1}}) \mid \mathcal{F}_{s_k}] \leq \log(\mathbb{E}[X_{s_{k+1}} \mid \mathcal{F}_{s_k}]) \leq Y_{s_k} - 1$ . Let  $\tau$  be the smallest k such that  $Y_{s_k} = 0$ . Note that  $\tau$  is the number of iterations i in which a root from  $A_{\ell}(\mathcal{L}_i)$  is selected, and hence  $\tau \leq n$ . The sequence  $(Y_{s_k}+k)_{k=1,\ldots,\tau}$  is therefore a supermartingale and  $\tau$  is a stopping time. By the martingale stopping theorem [MU05, Theorem 12.2], we have  $\mathbb{E}[\tau] = \mathbb{E}[Y_{s_{\tau}} + \tau] \leq \mathbb{E}[Y_{s_{1}} + 1] = \log(m) + 1$ .

Now we have shown that in expectation at most  $\log(m) + 1$  iterations i are needed in which roots from  $A_{\ell}(\mathcal{L}_i)$  are considered. The same argument can be repeated for  $A_u(\mathcal{U})$ . Since in every iteration a root from  $A_{\ell}(\mathcal{L})$  or  $A_{u}(\mathcal{U})$  is selected, the expected total number of iterations is at most  $2\log(m) + 2$ . This directly implies the lemma as  $m \leq |T_{\mathcal{L}}| + 1 \leq n + 1$ .

Finally we are able to prove the running time bound.

**Lemma 5.** Let R(T,n,k) be denote the running time of a call EXTEND $(T, n, k, \mathcal{L}_0)$ . Then there exists C > 0 such that

$$\mathbb{E}[R(T, n, k)] \le 5C(n - k)\log(n)^3 + Cn\log(n)^2.$$

*Proof.* We will prove this with induction on  $r := \lceil \log(n) \rceil$ . For r = 1, we have  $n \leq 2$ . In this case R is constant, proving our induction base.

Now consider a call EXTEND $(T, n, k, \mathcal{L}_0)$  and assume the induction claim is true when  $\lceil \log(n) \rceil \le r - 1$ . Let p be the number of iterations of the outer-most while-loop that are executed.

We will first consider the running time induced by the base call itself, excluding any recursive subcalls. Note that all of this running time is incurred by the calls to the procedures DFS, ROOTS and GOODVALUES, plus the cost of moving to the corresponding node before each of these calls. In the base call, the procedure will only move between nodes that are among the ones with the n smallest values, or the nodes directly below them. For this reason, we can upper bound the cost of each movement action by a running time of O(n).

- On line 12, 13, 24 each call DFS incurs a running time of at most O(n). Each of these lines will be executed p times, incurring a total running time of O(pn).
- On line 17 each call DFS $(T, \mathcal{L}', n)$  incurs a running time of at most O(n). The code will be executed  $O(p\log(n))$  times since c' doubles after every iteration of the inner loop and never grows larger than n, thus incurring a total running time of  $O(pn \log(n))$ .
- The call to GOODVALUES on line 21 takes  $O(n \log(n))$  time. The line is executed at most p times, so the total running time incurred is  $O(pn \log(n))$ .

Adding up all the running times listed before, we see that the total running time incurred by the non-recursive part is  $O(pn\log(n))$ . So, we can choose C such that this running time is bounded by  $C \cdot pn \log(n)$ . By Lemma 4,  $\mathbb{E}[p] \leq \log(n)$ . Hence the expected running time of the non-recursive part is bounded by  $Cn \log(n)^2$ .

Now we move on to the recursive part of the algorithm. All calls to EXTEND $(T, n, k, \mathcal{L}_0)$ with k=0 will have n=1, so each of these calls takes only O(1) time. Hence we can safely ignore these calls.

Let z be the number of of recursive calls to EXTEND $(T, n, k, \mathcal{L}_0)$  that are done from the base call with  $k \geq 1$ . Let  $T_i$ ,  $k_i$ ,  $n_i$  for  $i \in [z]$  be the arguments of these function calls. Note that  $n_i \geq 2$  for all i. By the induction hypothesis the expectation of the recursive part of the running time is:

$$\mathbb{E}\left[\sum_{i=1}^{z} R(T_i, n_i, k_i)\right] \leq \mathbb{E}\left[\sum_{i=1}^{z} 5C \log(n_i)(n_i - k_i) \log(n_i)^2 + Cn_i \log(n_i)^2\right]$$

$$\leq 5C(\log(n) - 1) \mathbb{E}\left[\sum_{i=1}^{r} n_i - k_i\right] \log(n)^2 + C \log(n)^2 \sum_{i=1}^{r} n_i$$

$$\leq 5C(\log(n) - 1)(n - k) \log(n)^2 + 5C \log(n)^2 (n - k)$$

$$\leq 5C(n - k) \log(n)^3.$$

Here we used Lemma 3 as well as the fact that  $\sum_{i=1}^{r} n_i \leq 4(n-k)$ . To see why this latter inequality is true, consider an arbitrary root q that has s good values under it. Now  $\sum_{i=1}^{z} \mathbf{1}_{T_i = T(q)} n_i \leq \sum_{i=2}^{\lceil \log(s+1) \rceil} 2^i \leq 2^{\lceil \log(s+1) \rceil + 1} \leq 4s$ . In total there are n-k good values under the roots, and hence  $\sum_{i=1}^{z} n_i \leq 4(n-k)$ .

Adding the expected running time of the non-recursive part, we see that  $\mathbb{E}[R(T, n, k)] \leq 1$ 

 $5Cn\log(n)^3 + Cn\log(n)^2$ . This proves the induction step.

### 4 Lower bound

No lower bound is known for the running time of the selection problem on explorable heaps. However, we will show that any algorithm with space complexity at most s, has a running time of at least  $\Omega(n \log_s(n))$ . Somewhat surprisingly, the tree that is used for the lower bound construction is very simple: a root with two trails of length O(n) attached to it.

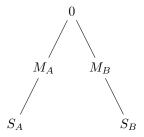
We will make use of a variant of the communication complexity model. In this model there are two agents A and B, that both have access to their own sets of values in  $S_A$  and  $S_B$  respectively. These sets are the input. We have  $|S_A| = n+1$  and  $|S_B| = n$ . Assume that all values  $S_A$  and  $S_B$  are different. Now consider the problem where player A wants to compute the median of  $S_A \cup S_B$ .

Because the players only have access to their own values, they need to communicate. For this purpose they use a protocol, that can consist of multiple rounds. In every odd round, player A can do computations and send units of information to player B. In every even round, player B does computations and sends information to player A. We assume that sending one value from  $S_A$  or  $S_B$  takes up one *unit of information*. Furthermore, we assume that, except for comparisons, no operations can be performed on the values. For example, the algorithm cannot do addition or multiplication on the values.

We will now reduce the median computation problem to the explorable heap selection problem.

**Lemma 6.** If there is a algorithm for that solves SELECT(3n) in f(n)n time and g space, then there is a protocol for median computation that uses f(n)/2 rounds in each of which at most g units of information are sent.

Proof. Consider arbitrary sets  $S_A$  and  $S_B$  with  $|S_A| = n+1$  and  $|S_B| = n$  and  $S_A \cap S_B = \emptyset$ . Without loss of generality we can assume x>0 for all  $x\in S_A\cup S_B$ . Let  $M_A$  and  $M_B$  be two sets with  $|M_A|=|M_B|=n$  and y< x for all  $y\in M_A\cup M_B$  and  $x\in S_A\cup S_B$ . Write  $S_A=\{a_1,\ldots,a_{n+1}\}$ . Now consider a subtree for which the root node has value  $a_1$ . Let every node that has value  $a_i$  have a child with value  $a_{i+1}$  and another child with some value that is larger than any value in  $S_A\cup S_B\cup M_A\cup M_B$ . We will call this a trail of  $S_A$ .



Now we will construct a labeled tree in the following way: create a tree with a root node of value 0. Attach a trail of  $M_A$  as the left child of this root and a trail of  $M_B$  as the right child. Attach a trail of  $S_A$  as a child of the largest node in  $M_A$  and do the same for a trail of  $M_B$  under the largest node of  $S_B$ . The resulting tree will now looks as shown in the above picture.

Observe that the 3nth smallest value in this tree is the median of  $S_A \cup S_B$ . Now we can view the selection algorithm as an algorithm for median computation if we consider moving between  $S_A$  and  $S_B$  in the tree as sending the g units of information that are in memory to the other player. Because moving between the two sets takes at least 2n steps, the number of rounds of

rounds in the corresponding communication protocol is at most  $\frac{f(n)n}{2n} = f(n)/2$ , proving the statement.

**Lemma 7.** Let  $S \subseteq [n]$  be a randomly distributed subset of [n] with sizes  $|S| \le k \le n$ . Then for  $\ell \le \frac{n}{8k}$  there exists an interval  $I = \{i, \ldots, i + \ell\} \subseteq [n]$  such that:  $\Pr[S \cap I \ne \emptyset] \le \frac{1}{4}$ .

*Proof.* Let  $\mathcal{I}_{\ell}$  be the set of length- $\ell$  intervals in [n]. We have  $|\mathcal{I}_{\ell}| = n - \ell + 1$ . Observe that any value in [n] is contained in at most  $\ell$  elements of  $\mathcal{I}_{\ell}$ . Hence, for any set S of size at most k, there are at most  $k \cdot \ell$  elements of  $\mathcal{I}_{\ell}$  that contain any of the elements of S. That is:  $|\{I \in \mathcal{I}_{\ell} : I \cap S \neq \emptyset\}| \leq k \cdot \ell$ . This implies that for a randomly distributed set  $S \subseteq [n]$  we also have:

$$\sum_{I \in \mathcal{I}_{\ell}} \Pr_{S}[I \cap S \neq \emptyset] = \mathbb{E}_{S}[|\{I \in \mathcal{I}_{\ell} : I \cap S \neq \emptyset\}|] \leq k \cdot \ell.$$

So there must be an  $I \in \mathcal{I}_{\ell}$  with:

$$\Pr_{S}[I \cap S \neq \emptyset] \leq \frac{k \cdot \ell}{|\mathcal{I}_{\ell}|} \leq \frac{k \cdot \ell}{n - \ell + 1} \leq \frac{k \cdot \frac{n}{8k}}{\frac{1}{2}n} = \frac{1}{4}.$$

**Theorem 8.** Any randomized protocol for median computation that sends at most g units of info per round, takes at least  $\Omega(\log_{g+1}(n))$  rounds in expectation.

*Proof.* We will instead prove the following result for a symmetric version of median computation, because this makes the proof a bit easier. In this setting, we have  $|S_A| = |S_B| = n$  and the objective is to find both the *n*th and the (n+1)th smallest element of  $S_A \cup S_B$ . We will call the set consisting of these two values the 2-median of  $S_A \cup S_B$  and we will denote it by 2median $(S_A \cup S_B)$ . Because this problem can be trivially solved by appending two rounds to any median-computation algorithm, proving a lower bound for this case is sufficient.

Let g' = g + 1. We can assume that  $g \ge 1$ , and hence  $g' \ge 2$ . We will prove with induction on n that the expected number of rounds to compute the median is at least  $\frac{1}{10} \log_{g'}(n) - 9$ . For  $n < 2^8(g')^2$ , this is trivial. Now let  $n \ge 2^8(g')^2$ . Assume that the claim is true for values strictly smaller than n. We will now prove the claim for n.

Consider an arbitrary randomized algorithm. Let  $V_i$  be the set of indices in  $S_{p_i}$  of the values that are emitted during the round i. Observe that the distribution of the set  $V_1$  does not depend on the input, because player A only has access to his own set of n values that he can compare to each other. Order the values in  $S_A$  by their values as  $x_1, \ldots, x_n$ . Order the values of  $S_B$  in decreasing order as  $y_1, \ldots, y_n$ .

Let  $\ell = \lfloor \frac{n}{8g} \rfloor$ . From Lemma 7 it follows that there exists an interval  $I = \{a, \ldots, a+\ell-1\} \subseteq [n]$  such that  $\Pr[V_1 \cap I \neq \emptyset] \leq \frac{1}{4}$ . Now let  $L = \{1, \ldots, a-1\}$  and  $U = \{a+\ell, \ldots, n\}$ . Observe that  $\{L, I, U\}$  forms a partition of [n]. Now order the values in the sets such that we have  $y_u < x_l < y_i < x_u < y_l$  for all  $l \in L, u \in U, i \in I$ . Note that this fixes the ordinal index of any element in  $S_A \cup S_B$ , except for the elements  $x_i$  and  $y_i$  for  $i \in I$ .

Condition on the event that  $I \cap V_1 = \emptyset$ . Observe that in this case, the results of all comparisons that player 2 can do in the second round have been fixed. Hence,  $V_2$  will be a random subset of [n], whose distribution will not depend on the order of the values  $x_a, \ldots, x_{a+\ell+1}$  with respect to  $y_1, \ldots, y_n$ .

Let  $\ell' = \lfloor \frac{\ell}{8g} \rfloor$ . From Lemma 7 there exist an interval  $I' = \{a', \ldots, a' + \ell' - 1\} \subseteq I$  such that  $\Pr[I' \cap V_2 \neq \emptyset \mid I \cap V_1 = \emptyset] \leq \frac{1}{4}$ . Define  $L' = \{a, \ldots, a' - 1\}$  and  $U' = \{a' + \ell', \ldots, a + \ell - 1\}$ .

Observe that  $\{L', I', U'\}$  forms a partition of I. We now order the values in the sets such that we have  $y_u < x_l < y_i < x_u < y_l$  for all  $l \in L', u \in U', i \in I'$ . Note that we have now fixed the ordinal index of any element in  $S_A \cup S_B$ , except for the elements  $x_i$  and  $y_i$  for  $i \in I'$ .

Because  $I' \subseteq I$ , we have  $\Pr[I' \cap (V_1 \cup V_2) \neq \emptyset] \leq \Pr[I \cap V_1 \neq \emptyset] + \Pr[I' \cap V_2 \neq \emptyset \mid S \cap V_1 = \emptyset] \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ . Now define  $S'_A = \{x_i : i \in I'\}$  and  $S'_B = \{y_i : i \in I'\}$ . Observe that  $2 \operatorname{median}(S_A \cup S_B) = 2 \operatorname{median}(S'_A \cup S'_B)$ . So the algorithm can now be seen as an algorithm to compute the 2-median of  $S'_A \cup S'_B$ . However with some probability  $\phi := \Pr[I' \cap (V_1 \cup V_2) = \emptyset] \geq \frac{1}{2}$ , no information about  $S'_A$  and  $S'_B$  is transmitted in the first two rounds. So, we can consider the algorithm that leaves out these two first rounds whenever this happens. If R' is the number of rounds that this algorithm takes, then  $\mathbb{E}[R'] = \phi \mathbb{E}[R-2] + (1-\phi) \mathbb{E}[R] = \mathbb{E}[R] - 2\phi \leq \mathbb{E}[R] - 1$ .

By our induction hypothesis it follows that R' will satisfy  $\mathbb{E}[R'] \geq \frac{1}{10} \log_{g'}(|S'_B|) - 9 = \frac{1}{8} \log_{g'}(\ell') - 9 = \frac{1}{10} (\log_{g'}(n) - 2\log_{g'}(8g) - 2) - 9 \geq \frac{1}{10} \log_{g'}(n) - 10$ . So  $\mathbb{E}[R] \geq \mathbb{E}[R'] + 1 \geq \frac{1}{10} \log_{g'}(n) - 9$ .

By Lemma 7, this directly implies:

**Theorem 9.** The time complexity for SELECT(n) with at most g units of storage is  $\Omega(\log_{q+1}(n))$ .

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