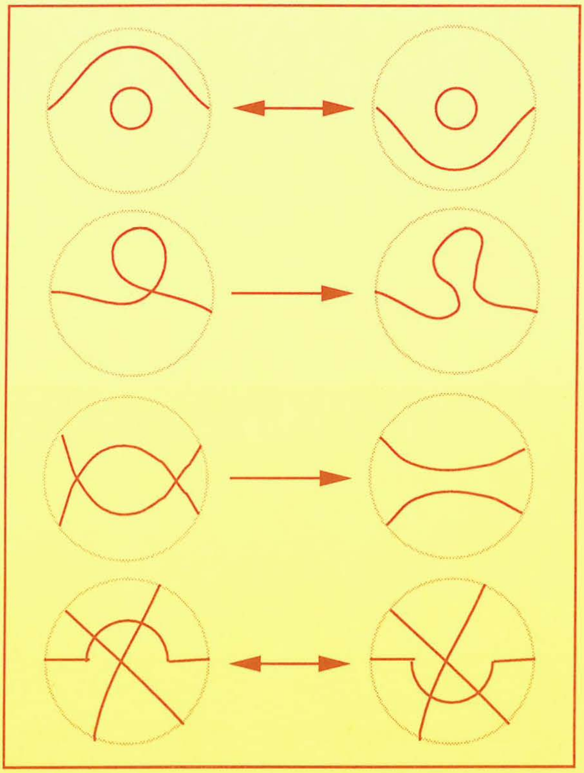
Graphs and curves on surfaces



Maurits de Graaf

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Graphs and curves on surfaces

Chapter 1

Introduction

1.1 Motivation

One of the main reasons to study graphs embedded on surfaces is that they form a natural generalization of planar graphs. Furthermore, their study is interesting because every graph can be embedded on some surface. In this sense graphs on surfaces can give new insights into graphs in general.

The study of graphs on surfaces can also be seen as a means to study surfaces. Every triangulizable surface has natural graphs associated with it, namely its triangulations. Study of these triangulations is one of the methods to investigate the surface. Closed curves on the surface can be regarded as cycles in some triangulation. In this way sometimes problems concerning the behaviour of closed curves on surfaces can be solved by graph theoretical methods.

Finally, it is sometimes possible to transform a difficult problem on a graph embedded on the plane to an easier problem on a graph embedded on a 'more complicated' surface (i.e., a surface of higher genus). The new results on cycles in graphs on surfaces that are presented in this thesis stem from such a transformation. In this thesis we study cycles in graphs embedded on a surface and systems of closed curves on a surface. This first section explains the motivation for our research. It is intended to be informal and illustrative rather than precise. A more formal discussion of the results in this thesis is deferred to the next sections of this chapter.

In many situations one wants to find paths in a graph embedded on the plane in such a way that the paths do not have a vertex or edge in common. For example, suppose that we are designing an electronic circuit as in Figure 1.1. Here the edges of the graph represent channels over which wires can be routed, while the shaded boxes represent modules on which pins with the same number should be interconnected

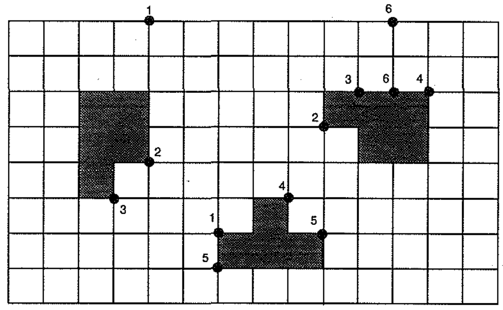


Figure 1.1: Designing a chip.

by wires. None of these connecting wires is allowed to have points in common with another wire, for otherwise a short circuit would be created. So we want to find vertex-disjoint paths connecting the pins in the underlying graph.

For an example of an edge-disjoint paths problem, consider the telecommunication network depicted in Figure 1.2. The vertices of this graph correspond to (telephone)

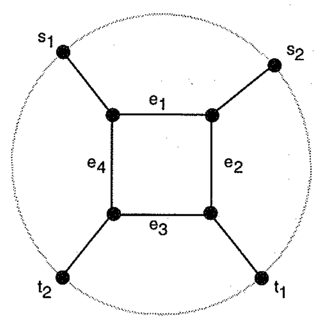


Figure 1.2: A communication network.

exchanges that can handle more than one call at a time. The edges of this graph correspond to connections between the exchanges and can handle at most one call at a time. Suppose that at the same time we want to route a call from s_1 to t_1 and from s_2 to t_2. This corresponds to finding edge-disjoint paths in the graph of Figure 1.2. This is impossible in this case. However, suppose that we want to route many calls, say 100 calls from s_1 to t_1 and from s_2 to t_2, where each connection can handle at most 100 calls simultaneously. Then the problem does have a solution. We can lead half of the calls from s_1 to t_1 over e_1 and e_2, and the other half over e_4 and

e_3. The calls from s_2 to t_2 can be split similarly. This solution can be regarded as a 'fractional' solution to the edge-disjoint path problem.

The previous two problems are so-called 'disjoint paths problems'. Let us explain how to transform a disjoint paths problem to a disjoint cycles problem. Suppose we want to find a path from s to t in a planar graph as in Figure 1.3(a). In this situation

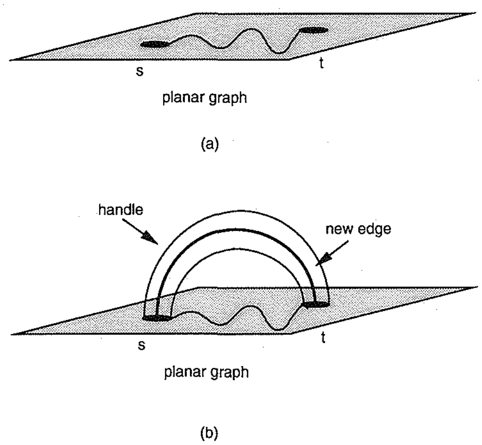


Figure 1.3: Transforming a disjoint path problem to a disjoint cycle problem.

we make a small hole in the plane at s and t and attach a 'handle' between these two holes as in Figure 1.3(b). Moreover, we add a new edge to the graph going over this handle. This process is repeated for all points in the graph that must be connected by disjoint paths. It turns out that in this way the problem of finding disjoint paths in the graph of Figure 1.3(a) can be changed to the problem of finding disjoint cycles going in a specific way over the attached handles in the graph of Figure 1.3(b). This has the advantage that the endpoints of the paths loose their specific role. In a cycle every point behaves equally. Furthermore, we can apply tools from algebraic topology and geometry, where closed curves 'correspond' to distance-preserving mappings on the 'universal covering surface'. A disadvantage of the transformation is that we loose the planarity of the graph. The surface we obtain by adding the handles is more complicated than the surface we started with. It seems, however, that the advantages of the transformation outweigh the disadvantages.

1.2 Terminology and notation

In order to explain the main results of this thesis we introduce some notation. Let S denote a surface. In this thesis a surface is a *triangulizable* (equivalently, metrizable) surface. A *closed curve* on S is a continuous function $C : S^1 \to S$ (where S^1 denotes the unit circle in the complex plane).

A closed curve C is called *orientation-preserving* if passing once through C does not change the meaning of 'left' and 'right'. Otherwise, C is called *orientation-reversing.* A surface S is called *orientable* if every closed curve $C : S^1 \to S$ is orientation-preserving. Otherwise, S is called *nonorientable.*

Two closed curves D and D' are *freely homotopic*, in notation: $D \sim D'$, if there exists a continuous function $\Phi : [0,1] \times S^1 \to S$ such that $\Phi(0,z) = D(z)$ and $\Phi(1,z) = D'(z)$ for all $z \in S^1$. A curve is called *nullhomotopic* if it is freely homotopic to a constant function. For any closed curve C on S, the number of self-intersections (counting multiplicities) of C is denoted by $\mathrm{cr}(C)$. That is,

$$\mathrm{cr}(C) = \frac{1}{2}|\{(w,z) \in S^1 \times S^1 | C(w) = C(z), w \neq z\}|. \tag{1.1}$$

Moreover, the minimal self-crossing number $\mathrm{mincr}(C)$ of C is the minimum number of $\mathrm{cr}(C')$ where C' ranges over all closed curves freely homotopic to C. That is,

$$\mathrm{mincr}(C) = \min\{\mathrm{cr}(C') | C' \sim C\}. \tag{1.2}$$

For any pair of closed curves C, D on S, the number of intersections of C and D (counting multiplicities) is denoted by $\mathrm{cr}(C,D)$. That is,

$$\mathrm{cr}(C,D) = |\{(w,z) \in S^1 \times S^1 | C(w) = D(z)\}|. \tag{1.3}$$

Moreover, $\mathrm{mincr}(C,D)$ denotes the minimum of $\mathrm{cr}(C',D')$ where C' and D' range over all closed curves freely homotopic to C and D, respectively. That is,

$$\mathrm{mincr}(C,D) = \min\{\mathrm{cr}(C',D') | C' \sim C, D' \sim D\}. \tag{1.4}$$

We consider graphs $G = (V,E)$ *embedded* on a surface S. That is, no two edges have a point in common, except possibly for their endpoints. For a graph G embedded on a surface S and a closed curve D on S, we define $\mathrm{cr}(G,D)$ as the number of intersections of G with D.

The *representativity* $\rho(G)$ of a graph G embedded on S is defined as:

$$\rho(G) := \min\{\mathrm{cr}(G,D) | \ D \text{ is non-nullhomotopic}\}. \tag{1.5}$$

So D ranges over all non-nullhomotopic closed curves. It turns out that the representativity of an embedded graph measures how well the graph 'represents' the surface.

A *cycle* in G is a sequence

$$C = (v_0, e_1, v_1, e_2, v_2, \dots, v_{d-1}, e_d, v_d), \tag{1.6}$$

where $v_0, \dots, v_d$ are vertices of G, $v_0 = v_d$ and e_i is an edge connecting v_{i-1} and v_i $(i = 1, \dots, d)$, $(v_1, \dots, v_d, e_1, \dots, e_d$ need not be distinct). With each cycle C in G we can associate in the obvious way a closed curve C on S which we identify with the cycle.

For any cycle C as in (1.6) and any edge e of G let $\mathrm{tr}_C(e)$ denote the number of times C traverses e. So $\mathrm{tr}_C \in \mathbb{R}^E$.

A set of cycles $C_1, \dots, C_k$ in a graph G is called *vertex-disjoint* if $\mathrm{cr}(C_i) = 0$ for $i = 1, \dots, k$ and $\mathrm{cr}(C_i, C_j) = 0$ for $i, j \in \{1, \dots, k\}$ $(i \neq j)$. A set of cycles $C_1, \dots, C_k$ in a graph G is called *edge-disjoint* if $\sum_{i=1}^k \mathrm{tr}_{C_i}(e) \leq 1$ for each $e \in E$.

A graph G is called *Eulerian* if all its vertices have even degree. (We do not assume connectedness of the graph.)

1.3 Edge-disjoint cycles

An important motivation for our research is the problem of finding edge-disjoint cycles that are freely homotopic to given cycles in a graph embedded on a compact surface. More precisely,

> **given:** a graph $G = (V, E)$ embedded on a compact surface S and closed curves $C_1, \dots, C_k$ on S,
> **find:** pairwise edge-disjoint cycles $\widetilde{C}_1, \dots, \widetilde{C}_k$ in G so that $\widetilde{C}_i \sim C_i$ for $i = 1, \dots, k$. (1.7)

For future reference, we call the problem formulated in (1.7) the *edge-disjoint homotopic cycle problem.*

We are particularly interested in necessary and sufficient conditions to guarantee the *existence* of edge-disjoint cycles $\widetilde{C}_1, \dots, \widetilde{C}_k$ as in (1.7). The question of actually *finding* the cycles is not discussed in this thesis.

The corresponding problem for *vertex*-disjoint cycles was solved by A. Schrijver in [28].

A necessary condition for a solution of (1.7) is easily established. If G contains edge-disjoint cycles $\widetilde{C}_1, \dots, \widetilde{C}_k$ freely homotopic to $C_1, \dots, C_k$, then one has for each closed curve D not intersecting vertices of G: $\text{cr}(G, D) \geq \sum_{i=1}^{k} \text{cr}(\widetilde{C}_i, D) \geq \sum_{i=1}^{k} \text{mincr}(C_i, D)$. So a necessary condition for the existence of edge-disjoint cycles freely homotopic to $C_1, \ldots, C_k$ is that for each closed curve D not intersecting vertices of G one has:

$$\text{cr}(G, D) \geq \sum_{i=1}^{k} \text{mincr}(C_i, D). \tag{1.8}$$

Condition (1.8) can be regarded as an analogue of the condition in Menger's theorem [16]. We call condition (1.8) the *cut condition.*

In general, the cut condition is not sufficient for the existence of edge-disjoint cycles homotopic to given cycles. For example, let S be the projective plane and G be the graph depicted in Figure 1.4. Here, the projective plane is represented by a closed unit disk U, where diametrically opposite points on the boundary of U are identified. There are only two free homotopy classes for closed curves: a closed curve is either nullhomotopic or homotopic to a closed curve connecting two diametrically opposite points on the boundary of U.

In Figure 1.4 we have $\text{cr}(G, D) \geq 2$, but G does not contain 2 edge-disjoint non-nullhomotopic closed curves.

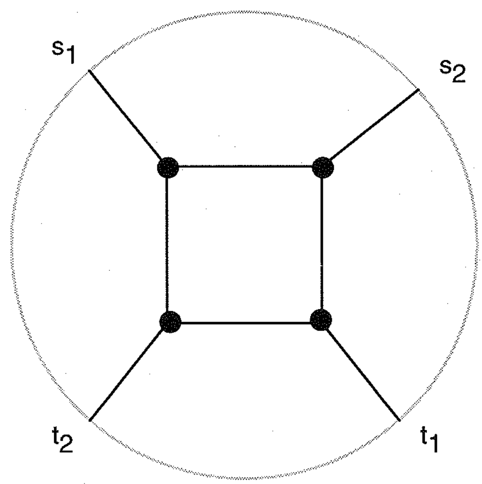


Figure 1.4: The cut condition is not sufficient on the projective plane.

However, S. Lins showed in [15] that the cut condition (1.8) is necessary and sufficient for a solution of the edge-disjoint homotopic cycle problem in case S is the projective plane and the graph G is Eulerian.

Let $G = (V, E)$ be a graph. We call a function $f : E \to \mathbb{R}$ a *circulation* (of value 1) if f is a convex combination of functions tr_C, where C is a cycle in G, and $\mathrm{tr}_C(e)$ denotes the number of times C traverses e. We say that a circulation f is *freely homotopic to* a closed curve C_0, in notation: $f \sim C_0$, if f is a convex combination of functions tr_C, where each C is freely homotopic to C_0.

Note that if f is a circulation freely homotopic to C_0, then for each closed curve D on S one has (denoting by $\mathrm{cr}(e, D)$ the number of times D intersects edge e):

$$\sum_{e \in E} f(e)\mathrm{cr}(e, D) \geq \mathrm{mincr}(C_0, D). \tag{1.9}$$

This follows from the fact that (1.9) holds for $f := \mathrm{tr}_C$ for each C freely homotopic to C_0 (as $\sum_{e\in E} \mathrm{tr}_C(e)\mathrm{cr}(e, D) = \mathrm{cr}(C, D) \geq \mathrm{mincr}(C_0, D)$), and hence also for any convex combination of such functions.

The problem of finding a set of circulations homotopic to given cycles in a graph in such a way that for each edge $e \in E$ the sum of the values of the circulations at e is at most one, can be regarded as a fractional version of the edge-disjoint homotopic cycle problem (1.7). We call problem (1.10) the *homotopic circulation problem.*

given: a graph $G = (V, E)$ embedded on a compact surface S and closed curves $C_1, \ldots, C_k$ on S,

find: circulations $f_1, \ldots, f_k$ in G such that $f_i \sim C_i$ for $i = 1, \ldots, k$ and such that for each edge $e \in E$: $\sum_{i=1}^{k} f_i(e) \leq 1$. (1.10)

If circulations $f_1, \ldots, f_k$ homotopic to $C_1, \ldots, C_k$ exist as in (1.10), then one has for each closed curve D on S not intersecting vertices of G:

$$\mathrm{cr}(G, D) = \sum_{e \in E} \mathrm{cr}(e, D) \geq \sum_{i=1}^{k} \sum_{e \in E} f_i(e)\mathrm{cr}(e, D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D). \tag{1.11}$$

So the cut condition is a necessary condition for a solution of the homotopic circulation problem as well. From the result of Lins [15] it directly follows that for the projective plane the cut condition is sufficient for a solution of the homotopic circulation problem. A. Schrijver showed in [27] that the cut condition is necessary and sufficient for a solution of the homotopic circulation problem in case S is a compact *orientable* surface. In Chapter 3 of this thesis we show that for any compact surface (orientable or non-orientable) the following homotopic circulation theorem holds:

Theorem 1.1 *The cut condition* (1.8) *is necessary and sufficient for a solution of the homotopic circulation problem* (1.10).

The proof method used in proving this is different from the method used in [27] and yields in fact a new and simple proof also for orientable surfaces. Main tool in the proof of Theorem 1.1 is a result about making curve systems minimally crossing by Reidemeister moves. This is explained in the next section.

1.4 Reidemeister moves

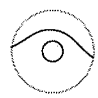
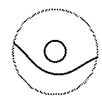
In Chapter 2 we prove a result that in itself is interesting from a topological point of view. In order to formulate it we need a few more definitions.

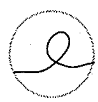
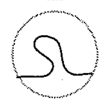
Let $C_1, \ldots, C_k$ be a system of closed curves on S. We call $C_1, \ldots, C_k$ *minimally crossing* if

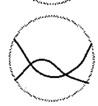
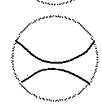
$$\text{(i)} \quad \text{cr}(C_i) = \text{mincr}(C_i) \quad \text{for each } i = 1, \ldots, k \tag{1.12}$$

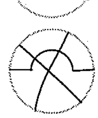
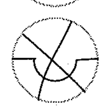
$$\text{(ii)} \quad \text{cr}(C_i, C_j) = \text{mincr}(C_i, C_j) \quad \text{for all } i, j = 1, \ldots, k \text{ with } i \neq j. \tag{1.13}$$

We call $C_1, \ldots, C_k$ a *regular* system of curves if $C_1, \ldots, C_k$ have only a finite number of (self-)intersections, each being a crossing of only two curve parts. That is, no point on S is traversed more than twice by $C_1, \ldots, C_k$ and each point of S traversed twice has a disk-neighborhood on which the curve parts are topologically two crossing straight lines. On such a system of curves, we define the following *Reidemeister moves*:

replacing  by  (*type 0*);

replacing  by  (*type I*);

replacing  by  (*type II*); (1.14)

replacing  by  (*type III*).

These moves were introduced by Reidemeister in the study of knots. In particular he showed that any two knots are 'equivalent' if and only if their diagrams can be moved to one another by a series of moves similar to those in (1.14) (see [13] and [20]).

The pictures in (1.14) represent the intersection of the union of $C_1, \ldots, C_k$ with a closed disk on S. So no other curve parts than the ones shown intersect such a disk.

The main result of Chapter 2 is:

Theorem 1.2 *Any regular system of closed curves on S can be transformed to a minimally crossing system on S by a series of Reidemeister moves.*

It is important to note that the main content of Theorem 1.2 is that we do not need to apply any of the Reidemeister moves in the reverse direction— otherwise the result would follow quite straightforwardly with the techniques of simplicial approximation.

In Chapter 3 we explain how Theorem 1.2 implies Theorem 1.1. This is done by an auxiliary theorem stating that the edges of an Eulerian graph G embedded on a compact surface can be decomposed into cycles $C_1, \ldots, C_t$ in such a way that for each closed curve D on S not intersecting the vertices of G it holds that $\min_{D' \sim D} \mathrm{cr}(G, D') = \sum_{i=1}^{t} \mathrm{mincr}(C_i, D)$.

1.5 Characterizing systems of closed curves on a surface

In Chapter 4 we use the homotopic circulation theorem to derive a result on systems of closed curves on a compact surface.

We call a closed curve C *orientation-primitive* if there do not exist an orientation-preserving closed curve D and an integer $n \geq 2$ such that $C \sim D^n$. For a closed curve D and an integer n, D^n is the closed curve defined by $D^n(z) := D(z^n)$ for $z \in S^1$. So any orientation-reversing closed curve is orientation-primitive.

Two systems of closed curves $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_{k'}$ are called *homotopically equivalent* (in notation: $C_1, \ldots, C_k \cong C'_1, \ldots, C'_{k'}$) if $k = k'$ and there exists a permutation π of $\{1, \ldots, k\}$ such that for each $i = 1, \ldots, k$ one has $C'_{\pi(i)} \sim C_i$ or $C'_{\pi(i)} \sim C_i^{-1}$.

In fact, the main result of Chapter 4 states that the $\mathrm{mincr}(\cdot, \cdot)$-function is 'accurate enough' to distinguish systems of orientation-primitive closed curves that are not homotopically equivalent. More precisely, we show:

Theorem 1.3 *Let $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_{k'}$ be orientation-primitive closed curves on a compact surface S. Then the following are equivalent:*

(i) $C_1, \ldots, C_k \cong C'_1, \ldots, C'_{k'}$

(ii) for each closed curve D on S

$$\sum_{i=1}^{k} \mathrm{mincr}(C_i, D) = \sum_{i=1}^{k'} \mathrm{mincr}(C'_i, D).$$

Theorem 1.3 generalizes a theorem obtained by A. Schrijver in [30] for compact orientable surfaces.

1.6 Poincaré's characterization of homology classes

Theorem 1.3 has a form similar to a theorem of H. Poincaré ([17], [18]). The difference between the two results is that Poincaré's theorem is in terms of *homology* classes instead of homotopy and, moreover, that Poincaré's theorem is straightforwardly reduced to linear algebra, whereas the homotopy theorem seems to require a much more combinatorial analysis.

In this section we discuss Poincaré's theorem.

By $C_1, \ldots, C_k \simeq C'_1, \ldots, C'_{k'}$ we mean that the system $C_1, \ldots, C_k$ is *homologous* to $C'_1, \ldots, C'_{k'}$. Moreover, $\text{pcr}(C, D)$ is the 'Poincaré crossing number' of C and D. Here, the Poincaré crossing number of C and D is the number of times C crosses D 'from left to right' minus the number of times C crosses D 'from right to left'.

In the following we give precise definitions of these notions and reduce Theorem 1.4 to a theorem in linear algebra.

Poincaré's result can be stated as follows:

Theorem 1.4 (Poincaré) *Let $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_{k'}$ be closed curves on a compact orientable surface S. Then the following are equivalent:*

(i) $C_1, \ldots, C_k \simeq C'_1, \ldots, C'_{k'}$.

(ii) for each closed curve D on S

$$\sum_{i=1}^{k} \text{pcr}(C_i, D) = \sum_{i=1}^{k'} \text{pcr}(C'_i, D).$$

The theory of homology was developed by Poincaré in his series of papers on 'Analysis situs'. We restrict ourselves to homology on orientable surfaces here. Poincaré's theorem was formulated in a more general way and is valid for higher dimensional manifolds as well.

Let $T = (V, E)$ be an oriented triangulation $T = (V, E)$ of a compact orientable surface S. That is, denoting the set of faces by F, to each each edge $e \in E$ and each face $f \in F$ an arbitrary orientation is assigned so that each edge $e \in E$ is adjacent to one face which has an orientation conform to the orientation of e (this face is denoted by $\delta^+(e)$) and to one face with an orientation opposite to the orientation of e (this face is denoted by $\delta^-(e)$). The oriented edge e from v to w is denoted by $e = (v, w)$. For any vertex $v \in V$, we denote by $\delta^+(v)$ the set of edges leaving v. That is, $\delta^+(v) := \{(v, w) \in E | w \in V\}$. Similarly, $\delta^-(v) := \{(w, v) \in E | w \in V\}$.

A *0-chain* is a function from the vertices to the integers. So g is a 0-chain if $g : V \to \mathbb{Z}$. Similarly, a *1-chain* is a function from the edges to the integers and a *2-chain* is a function from the faces to the integers. A 0-, 1-, or 2-chain with all values zero is denoted by 0. (The standard way is to define a 0-, 1-, or 2-chain as a (formal) linear integer combination of the vertices, edges, faces, respectively, see [2] and [7].)

The boundary of a 1-chain $g : E \to \mathbb{Z}$ is the 0-chain $\partial g : V \to \mathbb{Z}$ given by:

$$\partial g(v) := \sum_{e \in \delta^+(v)} g(e) - \sum_{e \in \delta^-(v)} g(e). \tag{1.15}$$

A 1-chain g is called a *1-cycle* if $\partial g = 0$. One can also interpret 1-cycles as kernels of a linear mapping. To that aim we regard a 1-chain $g : E \to \mathbb{Z}$ as a vector in $\mathbb{Z}^E$. Let M denote the $|V| \times |E|$ vertex-edge incidence matrix. That is, $M_{v,e} = 1$ if $e = (w, v)$ for some $w \in V$, $M_{v,e} = -1$ if $e = (v, w)$ for some $w \in V$ and $M_{v,e} = 0$ otherwise. Now $g \in \mathbb{Z}^E$ is a 1-cycle if and only if $Mg = 0$.

The boundary of a 2-chain $h : F \to \mathbb{Z}$ is the 1-chain $\partial h : E \to Z$ given by:

$$\partial h(e) := h(\delta^+(e)) - h(\delta^-(e)). \tag{1.16}$$

A 1-chain $g : E \to \mathbb{Z}$ is called a *boundary-chain* if $g = \partial h$ for some 2-chain $h : F \to \mathbb{Z}$. Boundary-chains can be regarded as images of some linear mapping. Thereto we regard a 2-chain $h : F \to \mathbb{Z}$ as a vector in $\mathbb{Z}^F$. Let N denote the $|E| \times |F|$ edge-face incidence matrix. That is: $N_{e,f} = 1$ if $f = \delta^+(e)$, $N_{e,f} = -1$ if $f = \delta^-(e)$ and $N_{e,f} = 0$ otherwise.

Now $g \in \mathbb{Z}^E$ is a boundary chain if and only if $g = Nh$ for some 2-chain $h \in \mathbb{Z}^F$. As $MN = 0$ it follows that all boundary chains are cycles. From now on the boundary chains are called *boundary cycles*.

Two 1-cycles g and g' are called *homologous*, in notation: $g \simeq g'$, if the 1-cycle $g - g'$ is a boundary cycle.

Note that for any system of closed curves $C_1, \ldots, C_k$, a 1-cycle g can be defined by shifting the curves C_i in such a way that each C_i follows the edges of the triangulation ($i = 1, \ldots, k$). Then $g : E \to \mathbb{Z}$ is defined by $g(e) :=$ the number of times that e is traversed conform its orientation minus the number of times that e is traversed opposite to its orientation. Thus we can speak of homology of curves.

Homology-classes are coarser to homotopy-classes in the sense that any two freely homotopic curves are homologous but the converse is not true in general.

Consider a dual triangulation $T^* = (V^*, E^*)$ of T. That is, each face f of T corresponds to a vertex of T^* and two vertices $f, g \in V^*$ are adjacent if the face corresponding to f and the face corresponding to g are adjacent in T. The edge $e \in E$ incident to both faces $f, g \in F$ and the edge $e^* \in E^*$ incident to the vertices $f, g \in V^*$ are called *associated* edges. The edges of T^* are oriented in such a way that $(f, g) \in E^*$ if $f = \delta^-(e)$ and $g = \delta^+(e)$, where $e \in E$ is the edge incident to both f and g.

Note that $d : E^* \to \mathbb{Z}$ is a 1-cycle in the dual triangulation T^* if and only if $N^T d = 0$, where N is the edge-face incidence-matrix.

Let g, d be cycles in T, T^*, respectively, then the *Poincaré crossing number* of g and d is defined as:

$$\mathrm{pcr}(g, d) = \sum_{e \in E,\ e^* \in E^*} g(e) d(e^*) \tag{1.17}$$

(here e, e^* are associated edges). Note that $\mathrm{pcr}(g, d)$ is invariant under homotopies.

In order to show Poincaré's theorem it is sufficient to show:

$$g \simeq 0 \Leftrightarrow \text{ for each 1-cycle } g \text{ in } T^*\text{: } \mathrm{pcr}(g, d) = 0. \tag{1.18}$$

This is equivalent to the well-known theorem in linear algebra:

$$Nx = g \text{ has a solution} \Leftrightarrow (d^T N = 0 \Rightarrow d^T g = 0 \text{ for all } d \in \mathbb{Z}^E\ ). \tag{1.19}$$

1.7 Kernels

In Chapter 5 we present an application of Theorem 1.3. We show that 'kernels' are 'uniquely determined' (in a sense that will be specified) by a function counting the minimal number of crossings of a graph with homotopy classes of closed curves.

To be precise, let S be a compact surface and let G be a graph embedded on S (without crossing edges). For each closed curve D on S we define

$$\mu_G(D) := \min_{D' \sim D} \mathrm{cr}(G, D'). \tag{1.20}$$

The minimum ranges over all closed curves D' freely homotopic to D, so here the curves D are allowed to traverse vertices of G. In fact, we may assume that the minimum in (1.20) is attained by a closed curve D not traversing the edges of G.

Observe that the function μ_G is invariant under the following operations on G:

$$\begin{array}{rl} \text{(i)} & \text{homotopic shifts of } G \text{ over } S; \\ \text{(ii)} & \text{replacing } G \text{ by a surface dual } G^* \text{ of } G, \\ & \text{(if } G \text{ is cellularly embedded).} \\ \text{(iii)} & \Delta Y\text{-exchanges in } G. \end{array} \tag{1.21}$$

Here we use the following terminology. Graph G' arises by a *homotopic shift* of G over S (or *is homotopic to* G) if there exists a continuous function $\Phi : [0,1] \times G \to S$ so that (i) $\Phi(0,y) = y$ for each $y \in G$; (ii) for each $x \in [0,1]$, $\Phi(x,\cdot)$ is a one-to-one function on G; (iii) $\Phi(1,\cdot)$ maps G onto G'. (We consider G and G' as subspaces of S.)

We say that graph G^* is a (*surface*) *dual* of G if (i) each face of G is an open disk; (ii) each face of G contains exactly one vertex of G^*, and $V(G^*) \cap G = \emptyset$; (iii) each edge of G^* crosses exactly one edge of G, and each edge of G crosses exactly one edge of G^*, while there are no further intersections of G and G^*. (By $V(\cdot)$ and $E(\cdot)$ we mean the vertex set and edge set of $\cdot$.) A surface dual G^* of G can be constructed by putting a vertex in the interior of each face of G and by joining two vertices by an edge if the corresponding faces of G are adjacent. So G has a surface dual if and only if each face of G is an open disk. Moreover, G has only one surface dual up to homotopic shifts.

If v is a vertex of G of degree 3, a ΔY-exchange (at v) replaces v and the three edges incident with v by a triangle connecting the three vertices adjacent to v (thus forming a triangular face). We also call the converse operation (replacing a triangular face by a "star" with three rays) a ΔY-exchange. See Figure 1.5(a) and (b) for an illustration.

Note moreover that if G' is a minor of G then $\mu_{G'} \leq \mu_G$ (i.e., $\mu_{G'}(D) \leq \mu_G(D)$ for each closed curve D). Here a *minor* of G arises by a series of deletions of edges and contractions of non-loop edges. Here a *deletion* of an edge e consists of a removal of e from the graph, a *contraction* of e consists of a deletion of e, followed by an identification of the endpoints of e. If we delete or contract an edge, the graph arising is naturally embedded again on S (unique up to homotopic shifts). Graph G' is called a *proper* minor of G, if G' arises from G by deletion or contraction of at least one edge.

Now we call G a *kernel* on S if $\mu_{G'} \neq \mu_G$ for each proper minor G' of G.

The main result of this chapter is that, if each face of G is an open disk, then kernels are uniquely determined by the function μ_G up to the operations (1.21):

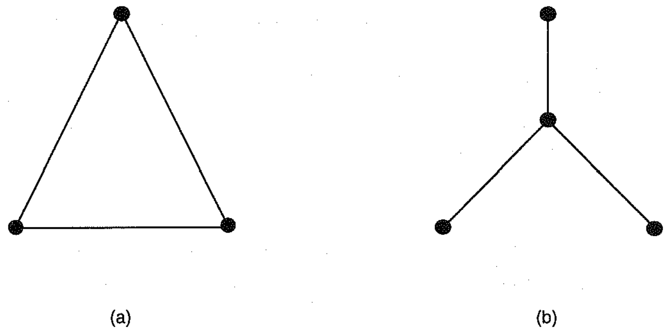


Figure 1.5: ΔY-exchange.

Theorem 1.5 *Let G and G' be kernels on the compact surface S, in such a way that each face of G is an open disk. If $\mu_G = \mu_{G'}$ then G' can be obtained from G by a series of operations* (1.21).

Theorem 1.5 generalizes the main result of [31] to nonorientable surfaces. The only nonorientable surface for which Theorem 1.5 was known to hold is the projective plane (cf. S. Randby [19]).

1.8 Grid minors of graphs on the torus

In the last chapter of this thesis we study graphs embedded on the torus.

The results of the last chapter are connected with the work on graph minors by N. Robertson and P. Seymour. In a series of papers [23], these authors prove the so-called 'Wagners Conjecture' [38]. Their theorem states that in an infinite series of pairwise different graphs, at least one graph is a proper minor of another graph.

In passing, they prove many results and introduce interesting concepts. In [24] they introduce the concept of 'representativity' of a graph, and show the following:

> for each graph H embedded on a compact surface S there exists a number ρ_H so that each graph G embedded on S with $\rho(G) \geq \rho_H$ contains H as a minor. (1.22)

In Chapter 6 of this thesis we determine the minimal value of ρ_H in case S is the torus and H is the toroidal k-grid. The *toroidal k-grid* is the graph-product $C_k \times C_k$ of two circuits with k-vertices. So the toroidal k-grid $C_k \times C_k$ has vertices (i, j) for $0 \leq i, j \leq k-1$, where (i, j) and (i', j') are adjacent if either $i = i'$ and

$j = j' \pm 1 (\text{mod } k)$ or $j = j'$ and $i = i' \pm 1 (\text{mod } k)$. Clearly, each toroidal k-grid can be embedded on the torus. In fact, for $k \geq 4$ there is a unique embedding, up to homeomorphisms (of the torus and of the grid).

With the aid of a result in the geometry of numbers we show that if a graph embedded on the torus has large representativity, it will contain a large toroidal k-grid:

Theorem 1.6 *Any graph G embedded on the torus contains the toroidal $\lfloor \frac{2}{3}\rho(G) \rfloor$-grid as a minor (if $\rho(G) \geq 5$). For each integer $\rho \geq 4$ there exists a graph G embedded on the torus with $\rho(G) = \rho$ and not containing the toroidal $\lfloor \frac{2}{3}\rho \rfloor + 1$-grid as a minor.*

Here $\lfloor r \rfloor$ denotes the largest integer less than or equal to r. This result implies that for the toroidal k-grid $C_k \times C_k$ embedded on the torus, $\rho_H := \lceil \frac{3}{2}k \rceil$ is the smallest integer value one can take for ρ_H in (1.22). (Here $\lceil r \rceil$ denotes the smallest integer greater than or equal to r.)

The problem of finding an embedding of a toroidal k-grid in a graph on the torus relates to the vertex-disjoint cycle problem in the sense that each toroidal k-grid consists of k vertex-disjoint cycles homotopic to a simple closed curve C_1 and k vertex-disjoint cycles homotopic to a simple closed curve C_2 where C_1 and C_2 are not homotopic (cf. [26]).

Chapter 2

Making curve systems minimally crossing by Reidemeister moves

Let $C_1, \ldots, C_k$ be a system of closed curves on a triangulizable surface S with the property that no three curves have a point in common. We show that we can make this system minimally crossing by applying Reidemeister moves in such a way that at each move neither the number of self-crossings of a curve, nor the number of crossings between any two curves increases.

A system $C_1, \ldots, C_k$ is called *minimally crossing* if each curve C_i has a minimal number of self-intersections among all curves C_i' freely homotopic to C_i and if each pair C_i, C_j has a minimal number of intersections among all curve pairs C_i', C_j' freely homotopic to C_i, C_j respectively $(i, j = 1, \ldots, k,\ \ i \neq j)$.

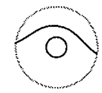
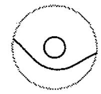
2.1 Introduction and formulation of the theorem

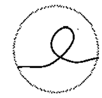
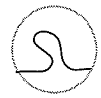
Let $C_1, \ldots, C_k$ be a system of closed curves on a surface S. We call $C_1, \ldots, C_k$ *minimally crossing* if

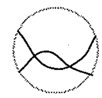
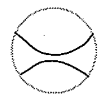
$$\begin{aligned} &\text{(i)}\ \ \mathrm{cr}(C_i) = \mathrm{mincr}(C_i) \text{ for each } i = 1, \ldots, k; \\ &\text{(ii)}\ \ \mathrm{cr}(C_i, C_j) = \mathrm{mincr}(C_i, C_j) \text{ for all } i, j = 1, \ldots, k \text{ with } i \neq j. \end{aligned} \tag{2.1}$$

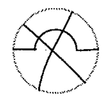
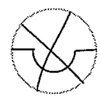
We call $C_1, \ldots, C_k$ a *regular* system of curves if $C_1, \ldots, C_k$ have only a finite number of (self)-intersections, each being a crossing of only two curve parts. That is, no point on S is traversed more than twice by $C_1, \ldots, C_k$ and each point of S traversed twice has a disk-neighborhood on which the curve parts are topologically two crossing straight lines. On such systems of curves we define the following four operations called

Reidemeister moves:

replacing  by  (*type 0*);

replacing  by  (*type I*);

replacing  by  (*type II*); (2.2)

replacing  by  (*type III*).

The pictures here represent the intersection of the union of $C_1, \ldots, C_k$ with an open disk on S. So no other curve parts than the ones shown intersect such a disk. Here and below we take all statements *topologically*. For instance, an open disk is any topological space homeomorphic to an open disk. Pictures are taken up to topological transformations.

The main result of this chapter is:

Theorem 2.1 *Let S be a triangulizable surface. Then any regular system of closed curves on S can be transformed to a minimally crossing system by a series of Reidemeister moves.*

This theorem will be used in Chapter 3 to prove a theorem on decompositions of graphs and a homotopic circulation theorem.

It is important to note that the main content of Theorem 2.1 is that we do not need to apply any of the operations (2.2) in the reverse direction — otherwise the result would follow quite straigthforwardly with the techniques of simplicial approximation.

2.2 Further terminology and notation

Let S be a surface. A *curve* on S is a continuous function $C : I \to S$ where I is a connected subset of S^1. It is *closed* if $I = S^1$ and *simple* if it is one-to-one.

Let C be a curve on a surface S and let $A \subseteq S$. We call L a *chord on A of C* if $L = C|I$ for some connected component I of $C^{-1}[A]$. We call L a *chord on A of $C_1, \ldots, C_k$* if L is a chord on A of one of $C_1, \ldots, C_k$.

We say that the system $C'_1, \ldots, C'_k$ arises from a *homotopic shift* of $C_1, \ldots, C_k$ if there exists a continuous transformation bringing the images of $C_1, \ldots, C_k$ to the images of $C'_1, \ldots, C'_k$ in such a way that during this transformation no new crossings are introduced and no two crossings coincide.

2.3 Reduction to compact surfaces with a finite number of holes

We show that to prove Theorem 2.1 we may restrict ourselves to compact surfaces S with a finite number of holes. (A *hole* arises when we delete a point (equivalenty, a closed disk) from the surface.)

Let S be a surface and let $S' \subseteq S$. For closed curves C and D on S' denote the function mincr by mincr′ if it is with respect to S'. Clearly,

$$\text{mincr}'(C) \geq \text{mincr}(C) \text{ and } \text{mincr}'(C,D) \geq \text{mincr}(C,D). \tag{2.3}$$

Proposition 2.1 *Let S be a triangulizable surface and $C_1, \ldots, C_k$ be a regular system of closed curves on S. Then S contains a compact surface S' with a finite number of holes such that S' contains $C_1, \ldots, C_k$ and such that* $\text{mincr}'(C_i) = \text{mincr}(C_i)$ *and* $\text{mincr}'(C_i, C_j) = \text{mincr}(C_i, C_j)$ *for all i, j.*

Proof. Consider a polygonal decomposition of S such that each vertex has degree 3. For all i, j with $1 \leq i < j \leq k$, let $\Delta_{i,j}$ be the set of all polygons intersected when shifting C_i and C_j to closed curves C'_i and C'_j respectively satisfying $\text{cr}(C'_i, C'_j) = \text{mincr}(C_i, C_j)$. Similarly, for each $i = 1, \ldots, k$ let Δ_i be the set of all polygons intersected when shifting C_i to a closed curve C'_i satisfying $\text{cr}(C'_i) = \text{mincr}(C_i)$. Note that each $\Delta_{i,j}$ and each Δ_i is finite. Let S' be the union of all $\Delta_{i,j}$ and Δ_i. Then S' has the required properties. ∎

Proposition 2.1 shows that in the sequel we may assume:

$$S \text{ is a compact surface with a finite number of holes.} \tag{2.4}$$

2.4 The disk

One important ingredient in our proof is a theorem of Ringel, and an extension of it, on shifting curves in a disk.

Let U be a closed disk. Call a system of curves $C_1, \ldots, C_k$ on U *minimally crossing* if each C_i is simple and C_i and C_j have at most one intersection for all $i \neq j$.

Consider regular minimally crossing systems of curves $C_1, \ldots, C_k$ on U satisfying:

(i) each C_i has two distinct end points on the boundary of U; (2.5)

(ii) if $i \neq j$ the end points of C_i and C_j are distinct.

Ringel [22] (Satz 26) showed:

Theorem 2.2 (Ringel's theorem) *Let U be a closed disk. If $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_k$ are regular minimally crossing systems of curves on U each satisfying* (2.5) *and if C_i and C'_i have the same pair of end points ($i = 1, \ldots, k$), then $C_1, \ldots, C_k$ can be moved to $C'_1, \ldots, C'_k$ by a series of Reidemeister moves of type III and homotopic shifts.*

From this we derive:

Theorem 2.3 *Let U be a closed disk. Then any regular system of curves on U can be transformed to a minimally crossing system by a series of Reidemeister moves.*

Proof. Let $C_1, \ldots, C_k$ be a regular system of curves on U. We apply induction on the number of connected components of $U \setminus \bigcup_{i=1}^{k} C_i$ that do not intersect $\mathrm{bd}(U)$. If this number is 0 the system is minimally crossing. If it is positive, there exists a closed disk $U' \subseteq U$ such that $U \setminus U'$ has exactly one chord C, being a curve connecting two points x_1 and x_2 on $\mathrm{bd}(U')$. Now by induction we know that the chords on U' are minimally crossing. Let D_1 and D_2 be the chords on U' that end at x_1 and x_2 respectively (possibly $D_1 = D_2$). So C, D_1 and D_2 together constitute one of the curves $C_1, \ldots, C_k$, say C_1.

If $D_1 = D_2$ then C_1 is a simple closed curve not intersecting any other curve, so $C_1, \ldots, C_k$ is minimally crossing. So we may assume that $D_1 \neq D_2$.

If D_1 and D_2 cross, we may assume by Ringel's theorem that D_1 and D_2 cross 'close' to C, and moreover, that the area enclosed by the loop in C_1 does not contain any closed curve among $C_2, \ldots, C_k$ (they can be removed with Reidemeister moves of type 0). Hence we can apply a Reidemeister move of type I so as to remove the self-crossing of C_1.

So D_1 and D_2 are disjoint; that is, C_1 is simple. Suppose $C_1, \ldots, C_k$ is not minimally crossing. Then there is a C_i that crosses both D_1 and D_2. Then, by Ringel's theorem applied to U', we may assume that some C_i crosses both D_1 and D_2 in such a way that the crossings are 'close' to C, and moreover, that the area enclosed by C_j and C_1 does not contain any closed curve among $C_2, \ldots, C_k$. Hence we can apply a Reidemeister move of type II so as to remove the two crossings of C_1 and C_j. ∎

2.5 Properties of minimal counterexamples

We now derive some properties of 'minimal counterexamples' to Theorem 2.1. Let S be a triangulizable surface and let $C_1, \ldots, C_k$ be a regular system of closed curves on S. We call $C_1, \ldots, C_k$ a *minimal counterexample* if the following holds:

(i) The system $C_1, \cdots, C_k$ is not minimally crossing; (2.6)

(ii) No series of Reidemeister moves decreases $\mathrm{cr}(C_i)$ for any $i \in \{1, \ldots, k\}$ or $\mathrm{cr}(C_i, C_j)$ for any $i, j \in \{1, \ldots, k\}$ $(i \neq j)$;

(iii) k is minimal (under (i) and (ii)).

It is obvious that any system obtained from a minimal counterexample by applying a series of Reidemeister moves, is a minimal counterexample again. Furthermore, we cannot apply a Reidemeister move of type I or type II to any minimal counterexample.

Proposition 2.2 *Let $C_1, \cdots, C_k$ be a minimal counterexample on S and let A be an open disk on S. Then all chords of $C_1, \ldots, C_k$ on A are simple nonclosed curves and each two chords have at most one intersection.*

Proof. Directly from Theorem 2.3 and (2.6)(ii). ∎

In particular:

Proposition 2.3 *Let $C_1, \ldots, C_k$ be a minimal counterexample on S. Then there is no open disk containing any of the curves C_i for $i = 1, \ldots, k$.*

Proof. Directly from Proposition 2.2. ∎

Next we show:

Proposition 2.4 *Let $C_1, \ldots, C_k$ be a minimal counterexample on S. Then $k \leq 2$ and if $k = 2$ then $\mathrm{cr}(C_i) = \mathrm{mincr}(C_i)$ $(i = 1, 2)$.*

Proof. We first show for any regular system $C_1, \ldots, C_k$ of closed curves on S:

if $C_1, \ldots, C_{k-1}$ can be transformed to closed curves $C'_1, \ldots, C'_{k-1}$ by a series of Reidemeister moves, then $C_1, \ldots, C_k$ can be transformed to $C'_1, \ldots, C'_k$ by a series of Reidemeister moves, for some closed curve C'_k. (2.7)

To see this we may assume that $C'_1, \ldots, C'_{k-1}$ arise from $C_1, \ldots, C_{k-1}$ by one Reidemeister move. We assume this is a Reidemeister move of type III — the other types follow similarly.

Let P, Q, R be the three chords of $C_1, \ldots, C_{k-1}$ on an open disk $A \subset S$ to which the Reidemeister move is applied. By Theorem 2.3 we may assume that the chords of $C_1, \ldots, C_k$ on A are minimally crossing, and by Theorem 2.2 that the triangle enclosed by P, Q and R does not intersect any of the chords. After this we can apply the Reidemeister move to $C_1, \ldots, C_k$, and we obtain (2.7).

It implies:

Let $C_1, \ldots, C_k$ be a minimal counterexample on S. Then for each $r \in \{1, \ldots, k\}$ the system $C_1, \ldots, C_{r-1}, C_{r+1}, \ldots, C_k$ is minimally crossing. (2.8)

For suppose that, say, $C_1, \ldots, C_{k-1}$ is not minimally crossing. By (2.6)(iii) there is a series of Reidemeister moves bringing $C_1, \ldots, C_{k-1}$ to $C'_1, \ldots, C'_{k-1}$ so that for some $i \in \{1, \ldots, k-1\}$, $\mathrm{cr}(C'_i) < \mathrm{cr}(C_i)$ or for some $i, j \in \{1, \ldots, k-1\}$, $\mathrm{cr}(C'_i, C'_j) < \mathrm{cr}(C_i, C_j)$ $(i \neq j)$. By (2.7) there is a curve C'_k and a series of Reidemeister moves bringing $C_1, \ldots, C_{k-1}, C_k$ to $C'_1, \ldots, C'_{k-1}, C'_k$. This contradicts (2.6)(ii).

So we have (2.8), which immediately gives the proposition. ∎

2.6 Sphere, open disk and projective plane

We now have immediately:

Proposition 2.5 *Theorem 2.1 is true in case S is a sphere or an open disk.*

Proof. Directly from Proposition 2.3. ∎

Proposition 2.6 *Theorem 2.1 is true in case S is the projective plane.*

Proof. Let $C_1, \ldots, C_k$ be a minimal counterexample on S. Let D be a simple closed nonnullhomotopic curve on S so that $D, C_1, \ldots, C_k$ is a regular system of curves and so that $\Sigma := \sum_{i=1}^{k} \mathrm{cr}(D, C_i)$ is minimal. Let $A := S \setminus D$. So A is an open disk. We may assume that A is the unit open disk in $\mathbb{C}$ and that S is obtained from the closed unit disk K in $\mathbb{C}$ by identifying opposite points on the boundary of K. By Proposition 2.2 each chord of A is a simple path connecting two points on $\text{bd}(K)$. and each two chords intersect each other at most once. Moreover, by Ringel's theorem and Proposition 2.2 we may assume that all chords are straight line segments with endpoints on $\text{bd}(K)$.

Now if there is a chord l that does *not* connect two opposite points on $\text{bd}(K)$, then there is a straight line segment connecting two opposite points on $\text{bd}(K)$ and not intersecting l. This would give a nonnullhomotopic closed curve on S having fewer intersections with $C_1, \ldots, C_k$ than D — a contradiction.

It follows that each chord corresponds to one nonnullhomotopic closed curve C_i ($i \in \{1, \ldots, k\}$). Hence, by Proposition 2.2, the system $C_1, \ldots, C_k$ is minimally crossing, contradicting (2.6)(i). ∎

2.7 Minimizing the crossing number of permutations

Theorem 2.1 for the special cases of the annulus and the Möbius strip turns out to boil down to statements on permutations. These statements are basic also for our proof for more general surfaces.

Let π be a permutation of $\{1, \ldots, n\}$. A *crossing pair of* π is a pair $\{i, j\}$ with $(i-j)(\pi(i) - \pi(j)) < 0$. The *crossing number* $\text{cr}(\pi)$ of π is the number of crossing pairs of π.

We say π' is a *conjugate* of π if there exists a permutation τ such that $\pi' = \tau^{-1}\pi\tau$.

Let $\text{mincr}(\pi)$ denote the minimum of $\text{cr}(\pi')$ taken over all conjugates π' of π. So $\text{mincr}(\pi)$ only depends on the sizes of the orbits of π.

A *transposition* is any permutation $(k, k+1)$ for some $k \in \{1, \ldots, n-1\}$. Since each permutation σ is a product of transpositions $\tau_1, \ldots, \tau_m$, it is trivial to say that for each permutation π there exist transpositions $\tau_1, \ldots, \tau_m$ such that

$$\text{cr}(\tau_m \cdots \tau_1 \pi \tau_1 \cdots \tau_m) = \text{mincr}(\pi). \tag{2.9}$$

What however can be proved more strongly is:

Theorem 2.4 *For each permutation π of $\{1, \ldots, n\}$ there exist transpositions $\tau_1, \ldots, \tau_m$ such that* (2.9) *holds and such that moreover:*

$$\text{cr}(\tau_{j+1} \cdots \tau_1 \pi \tau_1 \cdots \tau_{j+1}) \leq \text{cr}(\tau_j \cdots \tau_1 \pi \tau_1 \cdots \tau_j) \tag{2.10}$$

for each $j = 1, \ldots, m-1$.

That is, when going step by step to $\text{mincr}(\pi)$ we never have to increase the number of crossings. In Section 2.9 we shall see that a similar statement also holds if we *maximize* the number of crossings.

We first show a lemma.

Lemma 2.1 *Let π be a permutation, let τ be the transposition $(k, k+1)$, and let $\pi' := \tau\pi\tau$. Then:*

$$\text{cr}(\pi') \leq \text{cr}(\pi) \text{ if and only if } \pi' = \pi \text{ or } \pi(k) > \pi(k+1) \text{ or } \pi^{-1}(k) > \pi^{-1}(k+1). \tag{2.11}$$

Proof. To see sufficiency, suppose $\text{cr}(\pi') > \text{cr}(\pi)$. Then clearly $\pi' \neq \pi$. Moreover, by parity, $\text{cr}(\pi') \geq \text{cr}(\pi) + 2$. Hence π' has a crossing pair $\{i, j\} \neq \{k, k+1\}$ such that $\{\tau(i), \tau(j)\}$ is not a crossing pair of π. We may assume that $i < j$, and hence $\tau(i) < \tau(j)$. So $\tau\pi\tau(i) > \tau\pi\tau(j)$ and $\pi\tau(i) < \pi\tau(j)$. Hence $\pi\tau(i) = k$ and $\pi\tau(j) = k+1$. So $\pi^{-1}(k) = \tau(i) < \tau(j) = \pi^{-1}(k+1)$.

One similarly shows that $\pi(k) < \pi(k+1)$ (since $\text{cr}(\pi'^{-1}) = \text{cr}(\pi') > \text{cr}(\pi) = \text{cr}(\pi^{-1})$).

To see necessity, suppose $\pi' \neq \pi$, $\pi(k) < \pi(k+1)$ and $\pi^{-1}(k) < \pi^{-1}(k+1)$. Then for each crossing pair $\{i, j\}$ of π, the pair $\{\tau(i), \tau(j)\}$ is a crossing pair of π'. Indeed, we may assume $i < j$; hence $\pi(i) > \pi(j)$. Since $\pi(k) < \pi(k+1)$ we know $\{i, j\} \neq \{k, k+1\}$. So $\tau(i) < \tau(j)$. If $\{\tau(i), \tau(j)\}$ is not a crossing pair of π', we have $\pi'(\tau(i)) < \pi'(\tau(j))$; that is, $\tau(\pi(i)) < \tau(\pi(j))$. So $\{\pi(i), \pi(j)\} = \{k, k+1\}$, and hence $\pi(i) = k+1$ and $\pi(j) = k$. So $\pi^{-1}(k+1) = i < j = \pi^{-1}(k)$, a contradiction.

Hence $\text{cr}(\pi') \geq \text{cr}(\pi)$. To show strict inequality, we show that $\{k, k+1\}$ is a crossing pair of π'. (Note that it is not a crossing pair of π.)

Suppose $\{k, k+1\}$ is not a crossing pair of π'. So $\pi'(k) < \pi'(k+1)$. That is, $\tau(\pi(k+1)) < \tau(\pi(k))$. As $\pi(k+1) > \pi(k)$, we know $\{\pi(k), \pi(k+1)\} = \{k, k+1\}$. But this would imply that $\pi' = \pi$, contradicting our assumption. ∎

Proof of Theorem 2.4. We put $\pi \to \pi'$ if there exist a transposition τ such that $\pi' = \tau\pi\tau$ and $\text{cr}(\pi) \leq \text{cr}(\pi')$. We put $\pi \preceq \pi'$ if there exist permutations $\pi_0, \ldots, \pi_t$ such that $\pi = \pi_0$ and $\pi' = \pi_t$ and such that $\pi_{i-1} \to \pi_i$ for $i = 1, \ldots, t$ (possibly $t = 0$). So $\preceq$ is reflexive and transitive.

We show that for each permutation π on $\{1, \ldots, n\}$ there exists a permutation $\pi' \preceq \pi$ such that $\pi' = (1, 2, \ldots, j_1)(j_1 + 1, \ldots, j_2) \cdots (j_{s-1} + 1, \ldots, j_s)$ for some $j_1 <$

$j_2 < \cdots < j_s = n$. This proves the theorem, since the number of crossing pairs in π' only depends on the sizes of the orbits.

Consider permutations π' satisfying $\pi' \preceq \pi$ represented as

$$\pi' = (k_1, \ldots, k_{j_1})(k_{j_1} + 1, \ldots, k_{j_2}) \cdots (k_{j_{s-1}} + 1, \ldots, k_{j_s}). \tag{2.12}$$

Choose π' and this representation so that the vector $(k_1, \ldots, k_n)$ is lexicographically minimal. We may assume that $\pi' = \pi$.

We show that $k_j = j$ for $j = 1, \ldots, n$. Suppose this is not the case, and choose r satisfying $k_r \neq r$, with r as small as possible. So $k_j = j$ for all $j < r$ and $k_r > r$.

By the lexicographic minimality of representation (2.12), k_r is not the first of any of the orbits in this representation (otherwise we could choose r as the start of a new orbit). So $\pi^{-1}(k_r) = k_{r-1} = r - 1$.

Define $\pi' := \tau\pi\tau$, where $\tau := (k_r - 1, k_r)$. Then $\pi^{-1}(k_r - 1) \in \{r, \ldots, n\}$, implying $\pi^{-1}(k_r - 1) \geq r > r - 1 = \pi^{-1}(k_r)$. So by Lemma 2.1, $\text{cr}(\pi') \leq \text{cr}(\pi)$. This contradicts the lexicographic minimality of representation (2.12). ∎

Note that from the proof of Theorem 2.4 we immediately obtain:

$$\text{mincr}(\pi) = \text{cr}((1, \ldots, n)) = j_1 + \cdots + j_s - s \tag{2.13}$$

for any permutation π of $\{1, \ldots, n\}$ with orbits of lengths $j_1, \ldots, j_s$. We do not need this explicit expression for $\text{mincr}(\pi)$ in the proof of our main result, but we will come back to (2.13) in Section 2.14.

2.8 The annulus

Theorem 2.4 implies Theorem 2.1 in case S is the annulus. To see this, let $C_1, \ldots, C_k$ be a minimal counterexample on S.

We may assume that S is obtained from the square $K = [0,1] \times (0,1)$ by identifying $(0, x)$ and $(1, x)$ for each $x \in (0,1)$. Let $A_i := i \times (0,1)$, let A denote the curve on S arising after identifying A_0 and A_1, and let $U = (0,1) \times (0,1)$.

We may assume that we have chosen the representation so that $A, C_1, \ldots, C_k$ is regular and so that the number of crossings of A with $C_1, \ldots, C_k$ is as small as possible.

Then each chord of $C_1, \ldots, C_k$ on U connects (when considered in K) A_0 and A_1. (Otherwise we could (with the help of Ringel's theorem) decrease the number of crossings of A with $C_1, \ldots, C_k$.) Let $x_1, \ldots, x_n$ be the crossing points of $C_1, \ldots, C_k$

with A, in order. So there is a permutation π of $\{1,\ldots,n\}$ such that the chord starting at x_i on A_0 ends at $x_{\pi(i)}$ on A_1 $(i=1,\ldots,n)$. Note that $\text{cr}(\pi)$ is equal to the total number of crossings of $C_1,\ldots,C_k$.

Now we have the following:

if τ is a transposition such that $\text{cr}(\tau\pi\tau) \leq \text{cr}(\pi)$, then $C_1,\ldots,C_k$ can be moved to a system of curves with associated permutation $\tau\pi\tau$ by a series of Reidemeister moves. (2.14)

Indeed, let $\tau = (m, m+1)$. By Lemma 2.1, we may assume that $\pi(m) > \pi(m+1)$. Hence the chords starting in x_m and in x_{m+1} cross. Therefore, by Ringel's theorem we can apply Reidemeister moves so that these two chords cross close to A. Then by a topological transformation we can shift the crossing beyond A. This transforms π to $\tau\pi\tau$ showing (2.14).

Now if $k=1$, π has one orbit. Let C_1' be a closed curve on S freely homotopic to C_1 satisfying $\text{cr}(C_1') = \text{mincr}(C_1)$. Then C_1' gives similarly a permutation π'. As C_1' is freely homotopic to C_1, π' is conjugate to π. As $\text{cr}(C_1') < \text{cr}(C_i)$, we know that $\text{cr}(\pi') < \text{cr}(\pi)$.

So by Theorem 2.4 there exist transpositions $\tau_1,\ldots,\tau_m$ such that

$$\text{cr}(\tau_{j+1}\cdots\tau_1\pi\tau_1\cdots\tau_{j+1}) \leq \text{cr}(\tau_j\cdots\tau_1\pi\tau_1\cdots\tau_j)$$

for each $j=1,\ldots,m-1$, with strict inequality for $j=m$. But this would give by (2.14) a series of Reidemeister moves so as to decrease the number of self-crossings of C_1 — contradicting the fact that C_1 is a minimal counterexample.

If $k=2$, then π has two orbits and we consider similarly closed curves C_1', C_2' freely homotopic to C_1, C_2 respectively, satisfying $\text{cr}(C_1', C_2') = \text{mincr}(C_1, C_2)$.

2.9 Maximizing the crossing number of permutations

If we want to apply a similar technique for the Möbius strip, we have to consider *maximizing* the number of crossings of permutations. We define $\text{maxcr}(\pi)$ to be the maximum of $\text{cr}(\pi')$ taken over all permutations π' conjugate to π. Again trivially for any permutation π there exist transpositions $\tau_1,\ldots,\tau_m$ such that

$$\text{cr}(\tau_m\cdots\tau_1\pi\tau_1\cdots\tau_m) = \text{maxcr}(\pi). \tag{2.15}$$

Again this can be sharpened to:

Theorem 2.5 *For each permutation π there exist transpositions $\tau_1, \ldots, \tau_m$ such that (2.15) holds and such that moreover:*

$$\mathrm{cr}(\tau_{j+1} \cdots \tau_1 \pi \tau_1 \cdots \tau_{j+1}) \geq \mathrm{cr}(\tau_j \cdots \tau_1 \pi \tau_1 \cdots \tau_j) \tag{2.16}$$

for each $j = 1, \ldots, m-1$.

Proof for permutations with at most two orbits. We define $\preceq$ as in the proof of Theorem 2.4. Denote the sequence $1, n, 2, n-1, 3, n-2, \ldots$ by $a_1, a_2, a_3, a_4, a_5, \ldots$. So for $1 \leq 2i+1 \leq n$: $a_{2i+1} = i+1$ and and for $1 \leq 2i \leq n$: $a_{2i} = n+1-i$ and $a_n = \lfloor n/2 \rfloor + 1$.

We first show:

Claim 1 *For each permutation π of $\{1, \ldots, n\}$ there exists a permutation π' such that $\pi \preceq \pi'$ and such that π' can be represented as*

$$\pi' = (a_1, \ldots, a_{j_1})(a_{j_1+1}, \ldots, a_{j_2}) \ldots (a_{j_{s-1}} + 1, \ldots, a_{j_s}) \tag{2.17}$$

for some $j_1 < j_2 < \ldots < j_s = n$.

Proof. Consider permutations π' such that $\pi \preceq \pi'$, and represent π' as

$$\pi' = (k_1, \ldots, k_{j_1})(k_{j_1+1}, \ldots, k_{j_2}) \ldots (k_{j_{s-1}} + 1, \ldots, k_{j_s}). \tag{2.18}$$

Choose π' and the representation so that the vector $(k_1, -k_2, k_3, -k_4, \ldots)$ is lexicographically minimal.

We show that $k_j = a_j$ for $j = 1, \ldots, n$. We may assume that $\pi' = \pi$. Let r be such that $k_r \neq a_r$, with r as small as possible. So $k_j = a_j$ for $j = 1, \ldots, r-1$ and $k_r \in \{a_{r+1}, \ldots, a_n\}$. By the lexicographic minimality, r does not belong to $\{j_1+1, j_2+1, \ldots, j_{s-1}+1\}$. That is, $\pi^{-1}(k_r) = k_{r-1} = a_{r-1}$.

First suppose that r is odd, say $r = 2s+1$. By the choice of r we have that $k_r > s+1$, $\{k_r, \ldots, k_n\} = \{s+1, \ldots, n-s\}$ and $\pi^{-1}(k_r - 1) \in \{s+1, \ldots, n-s\}$. Define $\tau := (k_r - 1, k_r)$ and $\pi' := \tau\pi\tau$. Then $\pi'^{-1}(k_r - 1) = \tau\pi^{-1}\tau(k_r - 1) = \tau\pi^{-1}(k_r) = \tau(k_{r-1}) = k_{r-1} \in \{n-s+1, \ldots, n\}$. Moreover, $\pi'^{-1}(k_r) = \tau\pi^{-1}\tau(k_r) = \tau\pi^{-1}(k_r - 1) \in \{s+1, \ldots, n-s\}$. So $\pi'^{-1}(k_r) < \pi'^{-1}(k_r - 1)$, implying by Lemma 2.1 that $\mathrm{cr}(\pi') \geq \mathrm{cr}(\pi)$; so $\pi \preceq \pi'$. This contradicts the lexicographic minimality assumption.

Next suppose that r is even, say $r = 2s$. By the choice of r we have that $k_r \leq n-s$, $\{k_r, \ldots, k_n\} = \{s+1, \ldots, n-s+1\}$ and $\pi^{-1}(k_r + 1) \in \{s+1, \ldots, n-s+1\}$. Define $\tau := (k_r, k_r + 1)$ and $\pi' := \tau\pi\tau$. Then $\pi'^{-1}(k_r + 1) = \tau\pi^{-1}\tau(k_r + 1) = \tau\pi^{-1}(k_r) =$

$\tau(k_{r-1}) = k_{r-1} \in \{1, \ldots, s\}$. Moreover, $\pi'^{-1}(k_r) = \tau\pi^{-1}\tau(k_r) = \tau\pi^{-1}(k_r+1) \in \{s+1, \ldots, n-s+1\}$. So $\pi'^{-1}(k_r) > \pi'^{-1}(k_r+1)$, implying by Lemma 2.1 that $\mathrm{cr}(\pi') \geq \mathrm{cr}(\pi)$; so $\pi \preceq \pi'$. This again contradicts the lexicographic minimality assumption. □

This claim immediately implies the theorem for permutations with one orbit, as there is essentially one maximal element with respect to $\preceq$. It follows that for any permutation π of $\{1, \ldots, n\}$ one has:

$$\mathrm{maxcr}(\pi) = \mathrm{cr}((a_1, \ldots, a_n)) = \frac{1}{2}n(n-1) - \lfloor\frac{n-1}{2}\rfloor. \tag{2.19}$$

Next we consider permutations with two orbits. Let $h + k = n$ and let $\pi_{h,k}$ be the permutation

$$\pi_{h,k} := (a_1, \ldots, a_h)(a_{h+1}, \ldots, a_n). \tag{2.20}$$

Claim 2 *If h is even then* $\mathrm{cr}(\pi_{h,k}) \geq \mathrm{cr}(\pi_{k,h})$.

Proof. Observe that if $i, j \in \{k+1, \ldots, n\}$ and $\{a_i, a_j\}$ is a crossing pair of $\pi_{k,h}$, then $\{a_{i-k}, a_{j-k}\}$ is a crossing pair of $\pi_{h,k}$. Similarly, if $i, j \in \{1, \ldots, k\}$ and $\{a_i, a_j\}$ is a crossing pair of $\pi_{k,h}$, then $\{a_{i+h}, a_{j+h}\}$ is a crossing pair of $\pi_{h,k}$.

Finally, each pair $\{a_i, a_j\}$ with $1 \leq i \leq h < j \leq n$, is a crossing pair of $\pi_{h,k}$. So we obtain the required inequality. □

Claim 2 implies the theorem for permutations with two orbits, of even size each. Indeed, by Claim 1 we have that for each permutation π with two orbits, of even sizes h and k, one has $\pi \preceq \pi_{h,k}$ or $\pi \preceq \pi_{k,h}$. As by Claim 2 one has $\mathrm{cr}(\pi_{h,k}) = \mathrm{cr}(\pi_{k,h})$, both $\pi_{h,k}$ and $\pi_{k,h}$ attain $\mathrm{maxcr}(\pi)$.

Now we consider permutations with at least one odd orbit. Then we have:

Claim 3 *Let h be odd and k be such that k is even or $k \geq h$. Then $\pi_{h,k} \preceq \pi_{k,h}$.*

Proof. It suffices to show that there exists a permutation π such that $\pi_{h,k} \preceq \pi$ and such that the orbit of π containing n has size k.

Indeed, we may assume that $k \geq 2$ (otherwise $k = h = 1$, and the claim is trivial). If the orbit containing n has size k, then also 1 can be brought into this orbit, by applying the techniques used in proving Claim 1. Hence, again by applying the techniques of Claim 1, we know $\pi \preceq \pi_{k,h}$.

Let $u := \lceil n/2 \rceil$. Consider permutations π such that $\pi_{h,k} \preceq \pi$ and such that

$$\pi = (1, k_2, \ldots, k_h)(k_{h+1}, \ldots, k_n) \tag{2.21}$$

where

$$\begin{aligned}&\text{(i) } k_i + k_{i+1} = n+2 \text{ for each even } i < n.\\&\text{(ii) } k_i < k_{i+2} \text{ for each odd } i \leq n-2 \text{ with } i \neq h;\\&\text{(iii) } k_i \leq u \text{ for each odd } i \leq n.\end{aligned} \tag{2.22}$$

Such permutations π exist since (2.20) is of this form. Choose π such that $k_3 + k_5 + \cdots + k_h$ is as large as possible.

Note that the conditions (2.22) imply that $\{k_i | i \text{ odd}, 3 \leq i \leq n\} = \{2, 3, \ldots, u\}$. Then one has:

(2.23) Let $k_j = k_i + 1$ with i, j odd and $3 \leq i \leq h < j \leq n$. Then $i < h$ and $j < n$. Moreover, if $j \leq n-2$, then $k_{j+2} > k_{i+2}$.

Indeed, suppose to the contrary that $i = h$, or $j = n$, or $j \leq n-2$ and $k_{j+2} < k_{i+2}$. Then $\pi(k_i) < \pi(k_j)$. For if $i = h$ then $\pi(k_i) = 1 < \pi(k_j)$. If $i \leq h-2$ and $j = n$ then $k_{i+2} \geq k_i + 1 = k_j \geq k_{h+2}$, and hence $\pi(k_i) = k_{i+1} = n + 2 - k_{i+2} \leq n + 2 - k_{h+2} = k_{h+1} = \pi(k_j)$. If $i \leq h-2$ and $j \leq n-2$ and $k_{j+2} < k_{i+2}$, then $\pi(k_i) = k_{i+1} = n + 2 - k_{i+2} < n + 2 - k_{j+2} = k_{j+1} = \pi(k_j)$. So $\pi(k_i) < \pi(k_j)$.

Now let $\tau := (k_i, k_j)$ and $\pi' := \tau\pi\tau$. As $\pi(k_i) < \pi(k_j)$, Lemma 2.1 gives $\text{cr}(\pi') \geq \text{cr}(\pi)$. So $\pi \preceq \pi'$.

Let $\tau' := (k_{i-1}, k_{j-1})$ and $\pi'' := \tau'\pi'\tau'$. Since $k_{i-1} = k_{j-1} + 1$ and $\pi'(k_{i-1}) = \tau\pi(k_{i-1}) = \tau(k_i) = k_j > k_i = \tau(k_j) = \tau\pi(k_{j-1}) = \pi'(k_{j-1})$, we know (again by Lemma 2.1) that $\text{cr}(\pi'') \geq \text{cr}(\pi')$; so $\pi' \preceq \pi''$. Hence $\pi \preceq \pi''$.

However, in the representation of π'' is obtained from that of π by interchanging k_i and k_j and by interchanging k_{i-1} and k_{j-1}. This contradicts the maximality of $k_3 + k_5 + \cdots + k_h$. Thus we have (2.23).

From this we derive that $k_3 \geq 3$, which finishes the proof, as it implies that $k_{h+2} = 2$ and hence $k_{h+1} = n$.

First we have $k_h = u$. For suppose $k_h < u$. Then there exists an odd $j \in \{h+1, \ldots, n\}$ such that $k_j = k_h + 1$, contradicting (2.23).

Next if k is even, then $k_i = k_{i+2} - 1$ for each odd i in $\{3 \leq i \leq h-2\}$. Otherwise, choose the largest odd i in $\{3, \ldots, h-2\}$ for which $k_{i+2} \geq k_i + 2$. Then there exists an odd $j \in \{h+1, \ldots, n\}$ such that $k_j = k_i + 1$. Then by (2.23), $j \leq n-1$, and hence (as n is odd), $j \leq n-2$. So $k_{j+2} > k_{i+2}$ contradicting the maximality of i.

Hence $k_3 = u - (h-3)/2 \geq 3$ (since $2u = n + 1 = h + k + 1 \geq h + 3$ as $k \geq 2$).

If k is odd, then n is even and $k \geq h$. Then $k_i \geq k_{i+2} - 2$ for each odd i in $\{3 \leq i \leq h-2\}$. Otherwise there exists an odd $j \in \{h+2, \dots, n-3\}$ such that $k_j = k_i + 1$ and $k_{j+2} = k_i + 2$. Then (2.23) implies $k_i + 2 = k_{j+2} > k_{i+2}$, a contradiction.

Therefore, $k_3 \geq u - (h-3) \geq 3$ (since $2u = n = h + k \geq 2h$ as $k \geq h$). □

Now let π be a permutation with two orbits, of size h and k respectively, where h is odd and k is even or $k \geq h$. Then by Claims 1 and 3, $\pi \preceq \pi_{k,h}$. So $\pi_{k,h}$ attains a maximum number of crossings. ∎

In fact, we obtain that $\text{maxcr}(\pi) = \text{cr}(\pi_{h,k})$ for any permutation π with two orbits, of size h and k respectively, where h is odd and k is even or $k \geq h$. So we have:

$$\begin{aligned} \text{maxcr}(\pi) &= n(n-1)/2 - (k-2)/2 - (h-1)/2 \quad &\text{if } k \text{ is even} \\ \text{maxcr}(\pi) &= n(n-1)/2 - (k-1)/2 - (h-1)/2 - h \quad &\text{if } k \text{ is odd.} \end{aligned} \tag{2.24}$$

Remark One can generalize Claim 3 to permutations with more than two orbits as follows.

Let $h_1 + \cdots + h_k = n$ and let $\pi_{h_1,\dots,h_k}$ be the following permutation

$$\pi_{h_1,\dots,h_k} := (a_1, \dots, a_{h_1})(a_{h_1+1}, \dots, a_{h_1+h_2}) \cdots (a_{h_1} + \cdots + a_{h_{k-1}}, \dots, a_n). \tag{2.25}$$

By similar techniques as in the proof of Claim 3 one can show the following statement:

Let π be a permutation of $\{1, \dots, n\}$ with k orbits $h_1, \dots, h_k$. Suppose $h_1, \dots, h_k$ are ordered in such a way that there exists a number l such that $h_1, \dots, h_l$ are even and $h_{l+1}, \dots, h_k$ are odd and, moreover, such that $h_{l+1} \geq \cdots \geq h_k$. Then (2.26)

$$\pi \preceq \pi_{i_1,\dots,i_k}$$

where $(i_1, \dots, i_l)$ is a permutation of $(h_1, \dots, h_l)$ and where $i_j = h_j$ for each $j = l+1, \dots, k$.

2.10 The Mőbius strip

Theorem 2.5 implies Theorem 2.1 in case S is the Mőbius strip in the same way as Theorem 2.4 implies Theorem 2.1 in case S is the annulus as we saw in Section 2.8.

To see this, let $C_1, \dots, C_k$ be a minimal counterexample on S.

We may assume that S is obtained from the square $K = [0,1] \times (0,1)$ by identifying $(0,x)$ and $(1, 1-x)$ for each $x \in (0,1)$. Let $A_i := i \times (0,1)$, let A denote the curve on S arising after identifying A_0 and A_1, and let $U = (0,1) \times (0,1)$.

We may assume that we have chosen the representation so that $A, C_1, \ldots, C_k$ is regular and so that the number of crossings of A with $C_1, \ldots, C_k$ is as small as possible.

Then each chord of $C_1, \ldots, C_k$ on U connects (when considered in K) A_0 and A_1. (Otherwise we could (with the help of Ringel's theorem) decrease the number of crossings of A with $C_1, \ldots, C_k$.) Let $x_1, \ldots, x_n$ be the crossing points of $C_1, \ldots, C_k$ with A, in order. So there is a permutation π of $\{1, \ldots, n\}$ such that the chord starting at x_i on A_0 ends at $x_{\pi(i)}$ on A_1 $(i = 1, \ldots, n)$. Indeed, π has k orbits.

Note that in this case the chord starting at x_i on A_0 and ending at $x_{\pi(i)}$ on A_1 does crosses the chord starting at x_j on A_0 and ending at $x_{\pi(j)}$ on A_1 if and only if $\{i, j\}$ is a *not* a crossing pair for π. So the total number of crossings of $C_1, \ldots, C_k$ equals $n(n-1)/2 - \mathrm{cr}(\pi)$. This is the reason we have to consider maximizing the number of crossing pairs of permutations.

We have the following:

if τ is a transposition such that $\mathrm{cr}(\tau\pi\tau) \geq \mathrm{cr}(\pi)$, then $C_1, \ldots, C_k$ can be brought to a system with associated permutation $\tau\pi\tau$ by Reidemeister moves. (2.27)

Indeed, let $\tau = (m, m+1)$. By Lemma 2.1, we may assume that $\pi(m) < \pi(m+1)$. Hence the chords starting in x_m and in x_{m+1} cross. By Ringel's theorem we can apply Reidemeister moves so that these two chords cross close to A. Then by a topological transformation we can shift the crossing beyond A. This topological transformation transforms π to $\tau\pi\tau$, showing (2.27).

Now if $k = 1$, π has one orbit. Let C_1' be a closed curve on S freely homotopic to C_1 satisfying $\mathrm{cr}(C_1') = \mathrm{mincr}(C_1)$. Then C_1' gives similarly a permutation π'. As C_1' is freely homotopic to C_1, π' is conjugate to π. As $\mathrm{cr}(C_1') < \mathrm{cr}(C_1)$, we know that $\mathrm{cr}(\pi') > \mathrm{cr}(\pi)$.

So by Theorem 2.5 there exist transpositions $\tau_1, \ldots, \tau_m$ such that

$$\mathrm{cr}(\tau_{j+1} \cdots \tau_1 \pi \tau_1 \cdots \tau_{j+1}) \geq \mathrm{cr}(\tau_j \cdots \tau_1 \pi \tau_1 \cdots \tau_j)$$

for each $j = 1, \ldots, m-1$, with strict inequality for $j = m$. This would give by (2.27) a series of Reidemeister moves so as to decrease the number of self-crossings of C_1 — contradicting the fact that C_1 is a minimal counterexample.

If $k = 2$, then π has two orbits and we can consider similarly closed curves C_1', C_2' freely homotopic to C_1, C_2 respectively, satisfying $\mathrm{cr}(C_1', C_2') = \mathrm{mincr}(C_1, C_2)$.

We should also note here that a reverse derivation from Theorem 2.1 for the Möbius strip implies Theorem 2.5 for any permutation.

2.11 Bringing curves close to geodesics

Before considering the torus and the Klein bottle, we show Theorem 2.1 for the so-called *hyperbolic* surfaces. Basic ingredient is the fact there is a series of Reidemeister moves bringing any nonnullhomotopic closed curve arbitrarily close to a geodesic. In order to give a precise formulation and a proof of this statement we need some definitions and basic facts about surfaces and their universal covering surfaces. Most of the definitions and facts can be found in [14], [21], [27], [28], [31] and [36].

Let U be the open unit disk in the complex plane. A set of points of U is called a *hyperbolic line* if it is the intersection of U with a circle or a straight euclidean line in $\mathbb{C}$ crossing the boundary of U orthogonally. The set U together with the set of hyperbolic lines makes the *hyperbolic plane*, denoted by H. Each two distinct points in H are contained in a unique hyperbolic line.

There exists a metric dist on H so that the topology induced by this metric coincides with the usual topology on U and so that the hyperbolic lines in H are the geodesics of this metric. That is, for any three points x, y, z in H one has: x, y, z are, in this order, on a hyperbolic line, if and only if $\text{dist}(x, y) + \text{dist}(y, z) = \text{dist}(x, z)$.

An *isometry* on H is a homeomorphism $\phi : H \to H$ so that $\text{dist}(\phi(x), \phi(y)) = \text{dist}(x, y)$ for all $x, y \in H$. Thus, an isometry maps hyperbolic lines to hyperbolic lines.

Let S be any compact surface with (possibly empty) boundary. If S is not the torus, the annulus, the Klein-bottle, the Möbius strip, the closed disk, the sphere or the projective plane, then S is called *hyperbolic.*

The hyperbolic plane H can be considered as a *universal covering surface* of a hyperbolic surface S. There exists a 'projection' function $\psi : H \to S$ with the following properties:

(i) For each $u \in H$ there is an open disk N containing u so that $\psi|N : H \to S$ is one-to-one. (2.28)

(ii) If $u, u' \in H$ and $\psi(u) = \psi(u')$ then there exists an isometry $\phi : H \to H$ so that $\phi(u) = u'$ and $\psi \circ \phi = \psi$.

(iii) For each closed curve $C : S^1 \to S$ and each $u \in \psi^{-1}[C(1)]$ there exists a unique continuous function $C' : \mathbb{R} \to H$ such that $C'(0) = u$ and $\psi \circ C'(x) = C(e^{2\pi i x})$ for all $x \in \mathbb{R}$. (C' is a *lifting of C to H*.)

A closed curve g on a surface S is called *geodesic* if each lifting of g to H is a hyperbolic line. A basic fact is:

If C_1 and C_2 are geodesic and have a finite number of intersections, then $\mathrm{cr}(C_1, C_2) = \mathrm{mincr}(C_1, C_2)$. (2.29)

Each closed curve on S is freely homotopic to some geodesic curve $g : S^1 \longrightarrow S$. This geodesic curve is unique up to replacing g by $g \circ h$, where $h : S^1 \longrightarrow S^1$ is one-to-one and freely homotopic to the identity function on S^1.

The projection function ψ transmits the distance function dist on H to a distance function dist_S on S given by:

$$\mathrm{dist}_S(x, y) := \min\{\mathrm{dist}(x', y') | x', y' \in H, \psi(x') = x, \psi(y') = y\}. \tag{2.30}$$

for $x, y \in S$. Moreover, we can speak of 'piecewise linear' curves on S, of the length of such curves, and of convex subsets of S (these are the subsets containing with any pair of points x, y also the shortest line segment connecting x and y).

Let K be a convex closed disk on the hyperbolic surface S and let L be a chord of C on K with endpoints on $\mathrm{bd}(K)$. To *straighten L on K* means replacing L by L' so that L' is a (hyperbolic) straight line segment connecting the endpoints of L on $\mathrm{bd}(K)$. (If necessary, L' may be slightly deformed to prevent three chords of C on K from passing through one and the same point).

For each closed curve C on a compact surface S and each convex closed disk K on S not containing C, let C^K be the closed curve obtained from C by straighten each chord of C on K. If $\mathcal{K} = (K_1, \ldots, K_n)$ is a (finite) series of convex closed disks, then $C^{\mathcal{K}} = (\cdots((C^{K_1})^{K_2})^{K_3} \cdots)^{K_n}$.

By Ringel's theorem and Theorem 2.3 we know:

Proposition 2.7 *Let $C_1, \ldots, C_k$ be a regular system of curves on S and let $\mathcal{K} = (K_1, \ldots, K_n)$ be a finite series of convex closed disks on S. Then there is a finite series of Reidemeister moves transforming $C_1, \ldots, C_k$ to $C_1^{\mathcal{K}}, \ldots, C_k^{\mathcal{K}}$.*

The following proposition was proved in [31]. For closed curves $C, D : S^1 \longrightarrow S$ we define $\mathrm{dist}(C, D)$ to be the smallest ε such that for each $z \in S^1$ there exists a $w \in S^1$ satisfying $\mathrm{dist}_S(C(z), D(w)) < \varepsilon$. (Note that this distance function is not symmetric.)

Proposition 2.8 *Let S be a compact hyperbolic surface, let C be a nonnullhomotopic closed curve, and let g be a geodesic closed curve freely homotopic to C. Then for each $\varepsilon > 0$ there exists a finite series $\mathcal{K} = (K_1, \ldots, K_n)$ of convex closed disks such that* $\mathrm{dist}(C^{\mathcal{K}}, g) < \varepsilon$.

A similar result holds for any nullhomotopic closed curve C, which we can bring arbitrarily close to a point on S. Hence:

Theorem 2.6 *Let S be a hyperbolic surface and C be a closed curve on S. If S is nonnullhomotopic, let g be a geodesic freely homotopic to C. Then for each $\varepsilon > 0$ there is a series of Reidemeister moves bringing C to C' so that* $\text{dist}(C', g) < \varepsilon$.

If C is nullhomotopic, then there exists a point p on S such that for each ε there is a series of Reidemeister moves bringing C within distance ε from p.

2.12 Proof of Theorem 2.1 for hyperbolic surfaces

Let $C_1, \ldots, C_k$ be a minimal counterexample. By Proposition 2.4 we know that $k \leq 2$ and that if $k = 2$ then $\text{cr}(C_i) = \text{mincr}(C_i)$ for $i = 1, 2$. Moreover, from Theorem 2.6 and Proposition 2.3 we know that each C_i is nonnullhomotopic.

For $i = 1, \ldots, k$, let g_i be a geodesic freely homotopic to C_i and let G_i be the image of g_i. So G_i is a graph embedded on S. As the g_i are geodesic, if $G_i \neq G_{i'}$, then $G_i \cap G_{i'}$ is finite.

Let G be the graph $G_1 \cup \cdots \cup G_k$. Let V and E denote the vertex set and edge set of G. By introducing some extra vertices of degree 2, we may assume that G does not have loops or multiple edges. Moreover, we may assume that V is also the vertex set of each G_i.

Choose $\varepsilon > 0$ such that the open balls $B(v, \varepsilon)$ for $v \in V$ are pairwise disjoint and such that for each $v \in V$, any edge e of G intersected by $B(v, \varepsilon)$ is incident with v and is intersected by $B(v, \varepsilon)$ only in a neighbourhood on e of v.

Let $H = G \setminus \bigcup_{v \in V} B(v, \varepsilon)$. Then each component of H is contained in a unique edge of G. Choose $\delta > 0$ such that the sets $B(e, \delta) \cap H$ for $e \in E$ are pairwise disjoint.

Note that by Theorem 2.3 the chords of any $B(v, \varepsilon)$ and any $B(e, \delta)$ are minimally crossing. By Theorem 2.6 we may assume that $\text{dist}(C_i, g_i) < \delta$ for each $i = 1, \ldots, k$. We may assume moreover that we have applied Reidemeister moves so as to minimize the number of intersections of the C_i with the boundaries of the $B(v, \varepsilon)$ $(v \in V)$.

For each vertex v and each edge e incident with v let

$$R_{v,e} := \overline{B(v, \varepsilon)} \cap \overline{B(e, \delta)}. \tag{2.31}$$

We call any such set a *rim*. We may assume that $C_1, \ldots, C_k$ do not have any crossing on any rim.

Note that each $B(e, \delta)$ is incident with exactly two rims. Moreover, the rims incident with any $B(v, \varepsilon)$ can be partitioned into 'opposite' pairs. By the minimality condition we know that each C_i, if it enters some $B(v, \varepsilon)$ by crossing a rim then it leaves $B(v, \varepsilon)$ by crossing the opposite rim. Similarly, if it enters $B(e, \delta)$ by crossing a rim, then it leaves $B(e, \delta)$ by crossing the other rim. By Ringel's theorem, we may assume that two chords of $B(v, \varepsilon)$ cross each other only if they connect different pairs of rims.

For $i = 1, \ldots, k$, let J_i be the geodesic curve and t_i be the natural number such that $g_i = J_i^{t_i}$ and such that J_i has only a finite number of self-intersections. For each $v \in V$ and each $i = 1, \ldots, k$, let $d_{v,i}$ be half of the valency of v in G_i.

Suppose first that $k = 2$ and that $G_1 \neq G_2$. Then

$$\mathrm{cr}(C_1, C_2) = \sum_{v \in V} t_1 d_{v,1} t_2 d_{v,2}, \tag{2.32}$$

which is equal to $\mathrm{mincr}(C_1, C_2)$ by (2.29). This contradicts the fact that C_1, C_2 is a minimal counterexample.

So we know that $k = 1$, or $k = 2$ and $G_1 = G_2$. Moreover, if $k = 2$ then without loss of generality $J_1 = J_2$.

Choose an edge e_0 of G, with ends v_1 and v_2 say. By Ringel's theorem, we may assume that $B(e, \delta)$ does not contain any crossing of $C_1, \ldots, C_k$, except if $e = e_0$. (This can be seen as follows. If e and e' are opposite edges of G incident with vertex v of G, then $B(e, \delta) \cup R(v, e) \cup B(v, \varepsilon) \cup R(v, e') \cup B(e', \delta)$ forms an open disk. So we can 'move' crossings from $B(e, \delta)$ to $B(e', \delta)$.)

Let $K_1, \ldots, K_n$ be the chords of $B(e_0, \delta)$ and let $L_1, \ldots, L_n$ be the chords of $S \setminus B(e_0, \delta)$. Let $p_1, \ldots, p_n$ be the crossing points of $C_1, \ldots, C_k$ with rim R_{v_1, e_0}, and let $q_1, \ldots, q_n$ be the crossing points of $C_1, \ldots, C_k$ with rim R_{v_2, e_0}. We number them in such a way that $p_1, \ldots, p_n, q_n, \ldots, q_1$ occur in this order along the boundary of $B(e_0, \delta)$.

We may assume that for each $i = 1, \ldots, n$, K_i and L_i have p_i as end.

First assume that J_1 is orientation-reversing. Let π be the permutation on $\{1, \ldots, n\}$ such that $q_{\pi(i)}$ is the other end point of K_i.

The total number of (self-)crossings of $C_1, \ldots, C_k$ is equal to

$$\frac{1}{2}(t_1 + t_2)^2 \sum_{v \in V} d_{v,1}(d_{v,1} - 1) + \mathrm{cr}(\pi). \tag{2.33}$$

Now if $k = 1$ then π has one orbit. Let C_1' be a closed curve on S freely homotopic to C_1 satisfying $\mathrm{cr}(C_1') = \mathrm{mincr}(C_1)$. By Theorem 3.4 we may assume that $\mathrm{dist}(C_1', g_1) < \delta$. We may assume moreover that we have applied Reidemeister moves so as to minimize the number of intersections of C_1' with the boundaries of the $B(v, \varepsilon)$ $(v \in V)$.

Now C_1' gives similarly a permutation π'. As C_1' is freely homotopic to C_1, π' is conjugate to π. As $\mathrm{cr}(C_1') < \mathrm{cr}(C_i)$, we know that $\mathrm{cr}(\pi') < \mathrm{cr}(\pi)$.

By Theorem 2.4 there exist transpositions $\tau_1, \ldots, \tau_m$ such that

$$\mathrm{cr}(\tau_{j+1} \cdots \tau_1 \pi \tau_1 \cdots \tau_{j+1}) \leq \mathrm{cr}(\tau_j \cdots \tau_1 \pi \tau_1 \cdots \tau_j)$$

for each $j = 1, \ldots, m-1$, with strict inequality for $j = m$. This would give by (2.14) a series of Reidemeister moves so as to decrease the number of self-crossings of C_1 — contradicting the fact that C_1 is a minimal counterexample.

If $k = 2$, then π has two orbits. Then we can consider similarly closed curves C_1', C_2' freely homotopic to C_1, C_2 respectively, satisfying $\mathrm{cr}(C_1', C_2') = \mathrm{mincr}(C_1, C_2)$.

Next, assume that J_1 is orientation-reversing. Let π be the permutation on $\{1, \ldots, n\}$ such that $q_{n+1-\pi(i)}$ is the other end point of K_i.

For this case, the total number of (self-)crossings of $C_1, \ldots, C_k$ is equal to

$$\frac{1}{2}(t_1+t_2)^2 \sum_{v \in V} d_{v,1}(d_{v,1}-1) + \frac{1}{2}(t_1+t_2)(t_1+t_2-1) - \mathrm{cr}(\pi). \tag{2.34}$$

The result follows, similarly to the orientation-preserving case above, from Theorem 2.5.

2.13 The torus and the Klein bottle

We reduce Theorem 2.1 for the torus and the Klein bottle to the case of hyperbolic surfaces. The construction is a little artificial, but we do not know a shorter way of deriving the required result.

Let $C_1, \ldots, C_k$ form a minimal counterexample for the torus or the Klein bottle S. So $k = 1$ or $k = 2$. On S there exist simple closed curves A and B crossing each other exactly once such that the space F left after removing the points in A and B is an open disk. We may assume $A(1) = B(1)$ and that the system $A, B, C_1, \ldots, C_k$ is regular.

The *code* of any closed curve C that forms a regular system with A and B is given as follows. Let a and b be symbols. Each time C crosses A in one direction we put a, and in the other direction we put a^{-1}. (If A is orientation-reversing we fix an orientation of 'left' and 'right' on the open curve $A|(S^1 \setminus \{1\})$.)

Similarly, each time C crosses B in one direction we put b, and in the other direction we put b^{-1}. Putting these symbols in the order given by C, we obtain a word over the alphabet $\{a, a^{-1}, b, b^{-1}\}$. We may assume that if S is the torus then any closed curve with code $aba^{-1}b^{-1}$ is nullhomotopic, and if S is the Klein bottle then any closed curve with code $abab^{-1}$ is nullhomotopic.

By Reidemeister moves we can make $C_1, \ldots, C_k$ so that each C_i has a code that is cyclically reduced. (A word $x_1 \ldots x_t$ over the symbols a, b is *cyclically reduced* if it does not contain any subword aa^{-1}, $a^{-1}a$, bb^{-1}, or $b^{-1}b$, and if it is not the case that $\{x_1, x_t\}$ equals $\{a, a^{-1}\}$ or $\{b, b^{-1}\}$.)

Now first assume that S is the torus. There is the following basic operation:

(2.35) By Reidemeister moves we can modify the system $C_1, \ldots, C_k$ in such a way that in the code of any C_i any occurrence of ab is replaced by ba and any occurrence of $b^{-1}a^{-1}$ is replaced by $a^{-1}b^{-1}$ (also cyclically).

This follows by applying Ringel's theorem to the fundamental region obtained by shifting F slightly.

Similar statements hold for other replacements. Repeating this, we can bring all codes to the form a^pb^q. Indeed, suppose some code contains ab^qa^pb with $p \neq 0, q \neq 0$. If $q > 0$, we can increase p by repeatedly replacing ab by ba and $b^{-1}a^{-1}$ by $a^{-1}b^{-1}$. If $q < 0$ we can increase p by repeatedly replacing ab^{-1} by $b^{-1}a$ and ba^{-1} by $a^{-1}b$. Repeating this, we obtain code a^pb^q.

Now if $k = 1$ then C_1 is not minimally self-crossing. Let C_1' be a minimally self-crossing closed curve freely homotopic to C_1. Again we can apply Reidemeister moves to C_1' so as to make the code of C_1' equal to $a^{p'}b^{q'}$. As C_1 and C_1' are freely homotopic, $p' = p$ and $q' = q$. Hence C_1 and C_1' are freely homotopic also in the space $S' := S \setminus \{A(1)\}$. So C_1 is not minimally crossing in S'. As S' is hyperbolic, we know from the results in Section 2.12 that we can apply Reidemeister moves to C_1 on S' so as to decrease the number of crossings. As this gives Reidemeister moves also on S, this contradicts the fact that C_1 is a minimal counterexample.

A similar argument gives a contradiction if $k = 2$.

If S is the Klein bottle we can apply Reidemeister moves so as to replace in the code of each C_i any occurrence of ab by ba^{-1} and any occurrence of $b^{-1}a^{-1}$ by ab^{-1}. Similarly, we can replace any occurrence of ba by $a^{-1}b$, and $a^{-1}b^{-1}$ by $b^{-1}a$.

The analogue of (2.35) for the Klein bottle gives that we can bring all codes in the form a^pb^q, such that if q is odd then $p \in \{0, 1\}$. Again each free homotopy class of closed curves on the Klein bottle has a unique such code, and the theorem follows like in the torus case.

2.14 Corollaries

We present some corollaries of the previous results on permutations and of Theorem 3.1 and its proof. These corollaries yield more explicit expressions for the minimal crossing number of certain systems of closed curves and will be used later.

Corollary 2.1 *Let C be a closed curve, and let J be the geodesic curve and n be the natural number such that $C \sim J^n$ and such that J has only a finite number of self-intersections. Then*

$$\begin{array}{ll} \text{mincr}(C) = n^2\text{mincr}(J) + n - 1 & \textit{if } J \textit{ is orientation-preserving} \\ \text{mincr}(C) = n^2\text{mincr}(J) + \lfloor\frac{n-1}{2}\rfloor & \textit{if } J \textit{ is orientation-reversing.} \end{array}$$

Proof. Let C be a closed curve such that $\text{cr}(C) = \text{mincr}(C)$. So in particular, no series of Reidemeister moves decreases $\text{cr}(C)$. As in Section 2.12, let g be a geodesic freely homotopic to C and let G be the image of g with vertex set V and edge set E. For each $v \in V$ let d_v denote half of the valency of v in G.

We may assume that Reidemeister moves have been applied to C such that $\text{dist}(C, g) < \delta$ for appropriately chosen $\delta > 0$ and such that the number of intersections of C with the boundaries of $B(v, \epsilon)$ $(v \in V)$ has been minimized. Moreover, we may assume that $B(e, \delta)$ does not contain any crossing of C except if $e = e_0$ for some fixed e_0.

By the fact that $\text{mincr}(J) = \sum_{v \in V} d_v(d_v - 1)$ and (2.13), (2.19) (2.33), (2.34), the assertions follow. ∎

Corollary 2.2 *Let C_1, C_2 be two closed curves, and let J_i be the geodesic curve and t_i be the natural number such that $g_i = J_i^{t_i}$ and such that J_i has only a finite number of self-intersections $(i = 1, 2)$. Suppose moreover that $J_1 = J_2 = J$. Then*

$$\begin{array}{ll} \text{mincr}(C_1, C_2) = 2t_1t_2\text{mincr}(J) + \min\{t_1, t_2\} & C_1 \textit{ and } C_2 \textit{ orientation-reversing} \\ \text{mincr}(C_1, C_2) = 2t_1t_2\text{mincr}(J) & \textit{otherwise.} \end{array}$$

Proof. Similar to the proof of Corollary 2.1. ∎

We call a closed curve C *orientation-primitive* if there is no orientation-preserving closed curve D and integer n such that D^n is freely homotopic to C. So all orientation-reversing closed curves are orientation primitive. For orientation-primitive closed curves we have the following:

Corollary 2.3 *Let C be an orientation-primitive closed curve. Then* $\mathrm{mincr}(C, C) = 2\mathrm{mincr}(C)$ *if C is orientation-preserving and* $\mathrm{mincr}(C, C) = 2\mathrm{mincr}(C) + 1$ *if C is orientation-reversing.*

Proof. Let J be the geodesic with a finite number of selfcrossings such that J^t is freely homotopic to C for some natural number t.

If J is orientation-preserving then $t = 1$ and we obtain by Corollary 2.1 and 2.2 that $\mathrm{mincr}(C) = \mathrm{mincr}(J)$ and $\mathrm{mincr}(C, C) = 2\mathrm{mincr}(J)$.

If J is orientation-reversing then $t = 1$ or $t = 2$ and we obtain that $\mathrm{mincr}(C) = \mathrm{mincr}(J)$ and $\mathrm{mincr}(C, C) = 2\mathrm{mincr}(J) + 1$. This completes the proof. ∎

Corollary 2.4 *Let C be an orientation-reversing closed curve. Then* $\mathrm{mincr}(C, C^2) < 2\mathrm{mincr}(C, C)$.

Proof. Let J be the geodesic with a finite number of selfcrossings such that J^t is freely homotopic to C for some natural number t. Now J is orientation-reversing and t is odd. By Corollary 2.2 it follows that $\mathrm{mincr}(C, C) = 2t^2\mathrm{mincr}(J) + t$ and $\mathrm{mincr}(C, C^2) = 2t^2\mathrm{mincr}(J)$. ∎

The following was proved in [31] for orientable surfaces.

Corollary 2.5 *Let $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_k$ be minimally crossing regular systems of orientation-primitive closed curves on the compact surface S, such that $C_1 \sim C'_1$, $\ldots$, $C_k \sim C'_k$. Then $C_1, \ldots, C_k$ can be moved to $C'_1, \ldots, C'_k$, by Reidemeister moves of type III and homotopic shifts, possibly after permuting indices.*

Proof. As each curve is orientation-primitive, none of the curves $C_1, \ldots, C_k$ is null-homotopic. So any Reidemeister move applied to $C_1, \ldots, C_k$ will be a Reidemeister move of type III.

We assume that S is a hyperbolic surface and adopt the notation of Section 2.12. As in Section 2.12, we bring both sets of curves $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_k$ arbitrarily close to their geodesic representatives. Without loss of generality we may assume that $J_1 = \cdots = J_k = J'_1 \cdots J'_k := J$ (where J'_i corresponds to C'_i). Moreover, we may assume that $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_k$ have the same intersections with $B(v, \epsilon)$ for all $v \in V$.

Choose an edge e_0 of G, with ends v_1 and v_2 say. Again we may assume by Ringel's theorem that if two chords of $C_1, \ldots, C_k$ or of $C'_1, \ldots, C'_k$ connecting the same pair of opposite rims intersect, then they intersect in $B(e_0, \delta)$.

Let $K_1, \ldots, K_n$ be the chords of $C_1, \ldots, C_k$ of $B(e_0, \delta)$ and let $K'_1, \ldots, K'_n$ be the chords of $C'_1, \ldots, C'_k$ on $B(e_0, \delta)$. Let $p_1, \ldots, p_n$ be the crossing points of $C_1, \ldots, C_k$ with rim R_{v_1,e_0}, and let $q_1, \ldots, q_n$ be the crossing points of $C_1, \ldots, C_k$ with rim R_{v_2,e_0}. We number them in such a way that $p_1, \ldots, p_n, q_n, \ldots, q_1$ occur in this order along the boundary of $B(e_0, \delta)$.

We number the chords K_i and K'_i in such a way that K_i, K'_i start in p_i for each $i = 1, \ldots, n$. It is sufficient to show that, after some series of Reidemeister moves, if chord K_i ends in q_j (for some $j \in \{1, \ldots, n\}$), then K'_i ends in q_j as well.

In the case that J is orientation-preserving this follows from the fact that all curves are orientation-primitive which implies that they are all homotopic to J. So $k = n$ and for each $i = 1, \ldots, n$, chord K_i ends in q_i and K'_i ends in q_i.

In the case that J is orientation-reversing we have that each orientation preserving curve C_i is homotopic to J^2. Define π of $C_1, \ldots, C_k$ such that $q_{n+1-\pi(i)}$ is the other endpoint of K_i. The permutation π' of $C'_1, \ldots, C'_k$ is defined similarly. We have to argue that $\pi \preceq \pi'$.

Both permutations have orbits of the same size, where all even-sized orbits have size 2. Say the sizes of the orbits are $h_1, \ldots, h_k$ where the numbers $h_1, \ldots, h_k$ are ordered in such a way that there exists a number l such that $h_1 = \cdots = h_l = 2$ and $h_{l+1} \geq \cdots \geq h_k$ where all numbers $h_{l+1}, \ldots, h_k$ are odd. By (2.25) we have that $\pi \preceq \pi_{h_1,\ldots,h_k}$ and $\pi' \preceq \pi_{h_1,\ldots,h_k}$. From the fact that $\mathrm{cr}(\pi) = \mathrm{maxcr}(\pi) = \mathrm{cr}(\pi_{h_1,\ldots,h_k}) = \mathrm{cr}(\pi')$ we also obtain $\pi_{h_1,\ldots,h_k} \preceq \pi'$. This shows $\pi \preceq \pi'$. The reduction for the torus and the Klein bottle to hyperbolic surfaces is the same as in Section 2.13. ∎

Chapter 3

Decomposition of graphs on surfaces

Let $G = (V, E)$ be an Eulerian graph embedded on a surface S. We show that E can be decomposed into closed curves $C_1, \ldots, C_k$ in G such that $\text{mincr}(G, D) = \sum_{i=1}^{k} \text{mincr}(C_i, D)$ for each closed curve D on S.

Here $\text{mincr}(G, D)$ denotes the minimum number of intersections of G and D' (counting multiplicities), where D' ranges over all closed curves D' freely homotopic to D and not intersecting V. Moreover, $\text{mincr}(C, D)$ denotes the minimum number of intersections of C' and D' (counting multiplicities), where C' and D' range over all closed curves freely homotopic to C and D, respectively. *Decomposing* the edges means that each edge is traversed exactly once by $C_1, \ldots, C_k$.

This result was shown by Lins for the projective plane and by Schrijver for compact orientable surfaces. This chapter presents a shorter proof also for compact orientable surfaces.

We derive the following fractional packing result for closed curves of given homotopies in a graph $G = (V, E)$ on a compact surface S. Let $C_1, \ldots, C_k$ be closed curves on S. Then there exist circulations $f_1, \ldots, f_k \in \mathbb{R}^E$ homotopic to $C_1, \ldots, C_k$ respectively such that $f_1(e) + \cdots + f_k(e) \leq 1$ for each edge e, if and only if $\text{mincr}(G, D) \geq \sum_{i=1}^{k} \text{mincr}(C_i, D)$ for each closed curve D on S.

Here a *circulation homotopic to* a closed curve C_0 is any convex combination of functions $\text{tr}_C \in \mathbb{R}^E$, where C is a closed curve in G freely homotopic to C_0 and where $\text{tr}_C(e)$ is the number of times C traverses e.

3.1 Introduction

Let $G = (V, E)$ be an undirected graph embedded on S. (In this chapter, a graph has a finite number of vertices and edges. We identify G with its embedding on S.) For any closed curve D on S, $\text{cr}(G, D)$ denotes the number of intersections of G and

D (counting multiplicities):

$$\mathrm{cr}(G,D) := |\{z \in S^1 | D(z) \in G\}|. \tag{3.1}$$

Moreover, $\mathrm{mincr}(G,D)$ denotes the minimum of $\mathrm{cr}(G,D')$ where D' ranges over all closed curves freely homotopic to D and not intersecting V:

$$\mathrm{mincr}(G,D) := \min\{\mathrm{cr}(G,D') | D' \sim D, D'(S^1) \cap V = \emptyset\}. \tag{3.2}$$

(It would seem more consistent with the definition of $\mathrm{mincr}(C,D)$ (where C is a closed curve) if we would also allow shifting G so as to obtain $\mathrm{mincr}(G,D)$ by minimizing $\mathrm{cr}(G',D')$, where G' is possibly not one-to-one mapped in S and D' is freely homotopic to D. However, the following theorem implies that this would not change the minimum value.)

The following theorem was proved for the projective plane by Lins [15] and for compact orientable surfaces by Schrijver [27]. (Our present proof is much simpler than that in [27], but uses a lemma on minimizing intersections of closed curves.)

Theorem 3.1 *Let $G = (V,E)$ be an Eulerian graph embedded on a surface S. Then the edges of G can be decomposed into closed curves $C_1, \ldots, C_k$ such that*

$$\mathrm{mincr}(G,D) = \sum_{i=1}^{k} \mathrm{mincr}(C_i, D) \tag{3.3}$$

for each closed curve D on S.

Here a graph is *Eulerian* if each vertex has even degree. (We do not assume connectedness of the graph.) Moreover, *decomposing* the edges into $C_1, \ldots, C_k$ means that for each $s \in G \setminus V$ there is exactly one pair (i,z) such that $C_i(z) = s$; for $s \notin G$ there is no such pair at all, while for $s \in V$ there are $\frac{1}{2}\deg(s)$ such pairs. ($\deg(s)$ denotes the degree of s.)

Note that the inequality $\geq$ in (3.3) trivially holds, for *any* decomposition of the edges into closed curves $C_1, \ldots, C_k$: by definition of $\mathrm{mincr}(G,D)$, there exists a closed curve $D' \sim D$ in $S \setminus V$ such that $\mathrm{mincr}(G,D) = \mathrm{cr}(G,D')$, and hence

$$\mathrm{mincr}(G,D) = \mathrm{cr}(G,D') = \sum_{i=1}^{k} \mathrm{cr}(C_i, D') \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D). \tag{3.4}$$

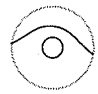
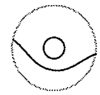
The content of the theorem is that there exists a decomposition attaining equality.

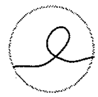
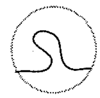
In Section 3.3 we give a proof of Theorem 3.1, and in Sections 3.4 and 3.5 we derive applications, including a 'homotopic circulation theorem'.

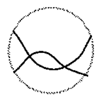
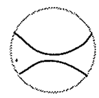
3.2 Making curve-systems minimally crossing by Reidemeister moves

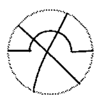
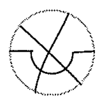
An important tool in our proof is the following result of Chapter 2:

Any regular system of closed curves can be transformed to a minimally crossing system by a series of 'Reidemeister moves': (3.5)

replacing  by  (*type 0*);

replacing  by  (*type I*);

replacing  by  (*type II*); (3.6)

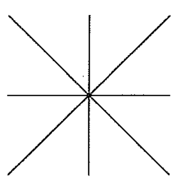
replacing  by  (*type III*).

The pictures in (3.6) represent the intersection of the union of $C_1, \ldots, C_k$ with a closed disk on S. So no other curve parts than the ones shown intersect such a disk.

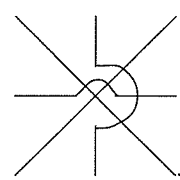
It is important to note that in (3.6) we did not allow to apply the operations in the reverse direction — otherwise the result would follow quite straightforwardly with the techniques of simplicial approximation.

3.3 Proof of the theorem

I. We may assume that each vertex v of G has degree at most 4. If v would have a degree larger than 4, we can replace G in a neighbourhood of v like



by



This modification does not change the value of $\mathrm{mincr}(G, D)$ for any D. Moreover, closed curves decomposing the edges of the modified graph satisfying (3.3), directly yield closed curves decomposing the edges of the original graph satisfying (3.3).

II. For any graph G embedded on S with each vertex having degree 2 or 4, we define the *straight decomposition* of G as the regular system of closed curves $C_1, \ldots, C_k$ such that $G = C_1 \cup \cdots \cup C_k$, where we identify closed curves with their images on S.

So each vertex of G of degree 4 represents a (self-)crossing of closed curves in the straight decomposition $C_1, \ldots, C_k$.

Up to some trivial operations, such a decomposition is unique, and conversely, it uniquely describes G. Moreover, any Reidemeister move applied to $C_1, \ldots, C_k$ carries over a modification of G. So we can speak of Reidemeister moves applied to G.

Note that:

if G' arises from G by one Reidemeister move of type III, then $\operatorname{mincr}(G', D) = \operatorname{mincr}(G, D)$ for each closed curve D. (3.7)

III. We call any graph $G = (V, E)$ that is a counterexample to the theorem with each vertex having degree at most 4 and with a minimal number of faces, a *minimal counterexample.*

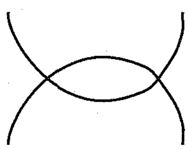
From (3.7) it directly follows that:

if G' arises from a minimal counterexample G by one Reidemeister move of type III, then G' is a minimal counterexample again. (3.8)

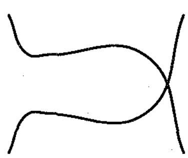
Moreover one has:

if G is a minimal counterexample, then no Reidemeister move of type 0, I or II can be applied to G. (3.9)

Suppose that a Reidemeister move of type II can be applied to G. Then G contains the following subconfiguration:



Replacing this by:



would give a smaller counterexample (since the function $\text{mincr}(G, D)$ does not change by this operation), contradicting the minimality of G.

One similarly sees that no Reidemeister move of type 0 or I can be applied.

IV. We finish the proof by showing that the straight decomposition $C_1, \ldots, C_k$ of any minimal counterexample G satisfies (3.3) — which is a contradiction.

Choose a closed curve D. We may assume that $D, C_1, \ldots, C_k$ form a regular system. By (3.6) we can apply Reidemeister moves so as to obtain a minimally crossing system $D', C'_1, \ldots, C'_k$.

By (3.9) we did not apply Reidemeister moves of type 0, I or II to $C_1, \ldots, C_k$. Hence by (3.7) for the graph G' obtained from the final $C'_1, \ldots, C'_k$ we have

$$\text{mincr}(G', D) = \text{mincr}(G, D).$$

So

$$\text{mincr}(G, D) = \text{mincr}(G', D) \leq \text{cr}(G', D') = \sum_{i=1}^{k} \text{cr}(C'_i, D') \tag{3.10}$$
$$= \sum_{i=1}^{k} \text{mincr}(C'_i, D') = \sum_{i=1}^{k} \text{mincr}(C_i, D).$$

Since the converse inequality holds by (3.4), we have (3.3). ∎

3.4 A corollary on lengths of closed curves

Using surface duality we obtain as in [27] the following. If G is a graph embedded on a surface S and C is a closed curve in G, then $\text{minlength}_G(C)$ denotes the minimum length of any closed curve $C' \sim C$ in G. (The *length* of C' is the number of edges traversed by C', counting multiplicities.)

Corollary 3.1 *Let $G = (V, E)$ be a bipartite graph embedded on a compact surface S and let $C_1, \ldots, C_k$ be closed curves in G. Then there exist closed curves $D_1, \ldots, D_t$ on $S \setminus V$ such that each edge of G is crossed by exactly one D_j and by this D_j only once and such that*

$$\text{minlength}_G(C_i) = \sum_{j=1}^{t} \text{mincr}(C_i, D_j) \tag{3.11}$$

for each $i = 1, \ldots, k$.

Proof. Let

$$d := \max\{\text{minlength}_G(C_i) | i = 1, \ldots, k\}. \tag{3.12}$$

We can extend G to a bipartite graph L embedded on S, so that each face of L is an open disk. By inserting d new vertices on each edge of L not occurring in G, we obtain a bipartite graph H satisfying $\text{minlength}_H(C_i) = \text{minlength}_G(C_i)$ for each $i = 1, \ldots, k$.

Consider a surface dual graph H^* of H. Since H is bipartite, H^* is Eulerian. Hence by Theorem 3.1, the edges of H^* can be decomposed into closed curves $D_1, \ldots, D_t$ such that

$$\text{mincr}(H^*, C) = \sum_{j=1}^{t} \text{mincr}(D_j, C) \tag{3.13}$$

for each closed curve C. Now for each $i = 1, \ldots, k$, $\text{mincr}(H^*, C_i) = \text{minlength}_H(C_i) = \text{minlength}_G(C_i)$, and (3.11) follows. ∎

In [27] an example is given showing that we cannot replace $C_1, \ldots, C_k$ by the set of *all* closed curves occurring in G. However, the proof above also gives that we *can* replace $C_1, \ldots, C_k$ by the set of all closed curves if G is cellularly embedded (i.e., each face is an open disk) — in that case we do not need to extend G to L and H.

3.5 A homotopic circulation theorem

By linear programming duality (Farkas' lemma) we derive from Corollary 3.1 the following 'homotopic circulation theorem' — a fractional packing theorem for cycles of given homotopies in a graph on a compact surface.

Let $G = (V, E)$ be a graph embedded on a compact surface S. For any closed curve C on G and any edge e of G let $\text{tr}_C(e)$ denote the number of times C traverses e. So $\text{tr}_C \in \mathbb{R}^E$.

Call a function $f : E \longrightarrow \mathbb{R}$ a *circulation* (of value 1) if f is a convex combination of functions tr_C. We say that f is *freely homotopic to* a closed curve C_0 if we can take each C freely homotopic to C_0.

Note that if f is a circulation freely homotopic to C_0, then for each closed curve D on S one has (denoting by $\text{cr}(e, D)$ the number of times D intersects edge e):

$$\sum_{e \in E} f(e)\text{cr}(e, D) \geq \text{mincr}(C_0, D). \tag{3.14}$$

This follows from the fact that (3.14) holds for $f := \text{tr}_C$ for each C freely homotopic to C_0 (as $\sum_{e \in E} \text{tr}_C(e)\text{cr}(e, D) = \text{cr}(C, D) \geq \text{mincr}(C_0, D)$), and hence also for any convex combination of such functions.

Corollary 3.2 (homotopic circulation theorem) *Let $G = (V, E)$ be an undirected graph embedded on a compact surface S and let $C_1, \ldots, C_k$ be closed curves on S. Then there exist circulations $f_1, \ldots, f_k$ such that f_i is freely homotopic to C_i ($i = 1, \ldots, k$) and such that $\sum_{i=1}^k f_i(e) \leq 1$ for each edge e, if and only if for each closed curve D on $S \setminus V$ one has*

$$\mathrm{cr}(G, D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D). \tag{3.15}$$

Proof. *Necessity.* Suppose there exist circulations $f_1, \ldots, f_k$ as required. Let D be a closed curve on $S \setminus V$. Then by (3.14):

$$\begin{aligned}\mathrm{cr}(G, D) &= \sum_{e \in E} \mathrm{cr}(e, D) \geq \sum_{e \in E} \mathrm{cr}(e, D) \sum_{i=1}^{k} f_i(e) = \sum_{i=1}^{k} \sum_{e \in E} f_i(e)\mathrm{cr}(e, D) \\ &\geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D).\end{aligned} \tag{3.16}$$

Sufficiency. Suppose (3.15) is satisfied for each closed curve D on $S \setminus V$. Let $I := \{1, \ldots, k\}$, and let K be the convex cone in $\mathbb{R}^I \times \mathbb{R}^E$ generated by the vectors

$$\begin{array}{ll}(\varepsilon_i; \mathrm{tr}_C) & (i \in I; C \text{ closed curve in } G \text{ with } C \sim C_i); \\ (0_I; \varepsilon_e) & (e \in E).\end{array} \tag{3.17}$$

Here ε_i denotes the ith unit basis vector in $\mathbb{R}^I$ and ε_e denotes the eth unit basis vector in $\mathbb{R}^E$. Moreover, 0_I denotes the all-zero vector in $\mathbb{R}^I$.

Although generally there are infinitely many vectors (3.17), K is finitely generated. This can be seen by observing that we can restrict the vectors $(\varepsilon_i; \mathrm{tr}_C)$ in the first line of (3.17) to those that are minimal with respect to the usual partial order $\leq$ on $\mathbb{Z}_+^I \times \mathbb{Z}_+^E$ (with $(x, y) \leq (x', y') \Leftrightarrow x_i \leq x'_i$ for all $i \in I$ and $y_e \leq y'_e$ for all $e \in E$). They form an 'antichain' in $\mathbb{Z}_+^I \times \mathbb{Z}_+^E$ (i.e., a set of pairwise incomparable vectors), and since each antichain in $\mathbb{Z}_+^I \times \mathbb{Z}_+^E$ is finite, K is finitely generated.

We must show that the vector $(1_I; 1_E)$ belongs to K. Here 1_I and 1_E denote the all-one vectors in $\mathbb{R}^I$ and $\mathbb{R}^E$, respectively. By Farkas' lemma, it suffices to show that each vector $(d; l) \in \mathbb{Q}^I \times \mathbb{Q}^E$ having nonnegative inner product with each of the vectors (3.17), also has nonnegative inner product with $(1_I; 1_E)$. Thus let $(d; l) \in \mathbb{Q}^I \times \mathbb{Q}^E$ have nonnegative inner product with each vector among (3.17). This is equivalent to:

$$\begin{array}{lll}\text{(i)} & d_i + \sum_{e \in E} l(e)\mathrm{tr}_C(e) \geq 0 & (i \in I; C \text{ curve in } G \text{ with } C \sim C_i); \\ \text{(ii)} & l(e) \geq 0 & (e \in E).\end{array} \tag{3.18}$$

Suppose now that $(d;l)^T(1_I;1_E) < 0$. By increasing l slightly, we may assume that $l(e) > 0$ for each $e \in E$. Next, by blowing up $(d;l)$ we may assume that each entry in $(d;l)$ is an even integer.

Let G' be the graph arising from G by replacing each edge e of G by a path of length $l(e)$. That is, we insert $l(e) - 1$ new vertices on e. Then by (3.18)(i),

$$-d_i \leq \text{minlength}_{G'}(C_i) \tag{3.19}$$

for each $i \in I$. Since G' is bipartite, by Corollary 3.1 there exist closed curves $D_1, \ldots, D_t$ not intersecting any vertex of G' such that each edge of G' is intersected by exactly one D_j and only once by that D_j and such that

$$\text{minlength}_{G'}(C_i) = \sum_{j=1}^{t} \text{mincr}(C_i, D_j) \tag{3.20}$$

for each $i \in I$. So

$$l(e) = \sum_{j=1}^{t} \text{cr}(e, D_j) \tag{3.21}$$

for each edge e of G. Hence (3.15), (3.19) and (3.20) give

$$\begin{aligned}\sum_{e\in E} l(e) &= \sum_{j=1}^{t}\sum_{e\in E} \text{cr}(e, D_j) = \sum_{j=1}^{t} \text{cr}(G, D_j) \geq \sum_{j=1}^{t}\sum_{i=1}^{k} \text{mincr}(C_i, D_j) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{t} \text{mincr}(C_i, D_j) = \sum_{i=1}^{k} \text{minlength}_{G'}(C_i) \geq -\sum_{i=1}^{k} d_i.\end{aligned} \tag{3.22}$$

So $(d;l)^T(1_I;1_E) \geq 0$. ∎

So if (3.15) is satisfied for each closed curve D on $S \backslash V$, then there exist circulations $f_1, \ldots, f_k$ homotopic to $C_1, \ldots, C_k$. That is, for each $i \in \{1, \ldots, k\}$ there exist rational numbers $x_{i,C}$, $0 \leq x_{i,C} \leq 1$ such that $f_i = \sum_{C \sim C_i} x_{i,C}\text{tr}_C$ and $\sum_{C \sim C_i} x_{i,C} = 1$.

In [27] it is shown that generally we cannot take the $x_{i,C}$ 0,1-valued, even not if the following 'parity conditions' hold for each curve D on $S \setminus V$:

$$\sum_{i=1}^{k} \text{cr}(C_i, D) + \text{cr}(G, D) \text{ is even.} \tag{3.23}$$

Lins [15] showed that the $x_{i,C}$ can be taken 0,1-valued if S is the projective plane and the graph G is Eulerian.

A. Frank and A. Schrijver showed [6] that if S is the torus and $G, C_1, \ldots, C_k$ satisfies the parity conditions (3.23) then condition (3.15) is necessary and sufficient to take the $x_{i,C}$ 0,1-valued.

Here we present an example on a nonorientable surface satisfying condition (3.15) and condition (3.23), but not allowing a 0,1-solution.

Let G be the graph embedded on surface S, where S is a sphere with 4 cross-caps adjoined and let $C_1 = C_2$ be the dashed closed curve indicated in Figure 3.1. Then both (3.15) and (3.23) are satisfied, and indeed there exists circulations as in Corollary 3.2. However, no edge-disjoint cycles homotopic to C_1, C_2 exist in G.

We do not know if for graphs embedded on the Klein bottle the parity condition (3.23) together with (3.15) are sufficient for the existence of edge-disjoint cycles homotopic to given cycles.

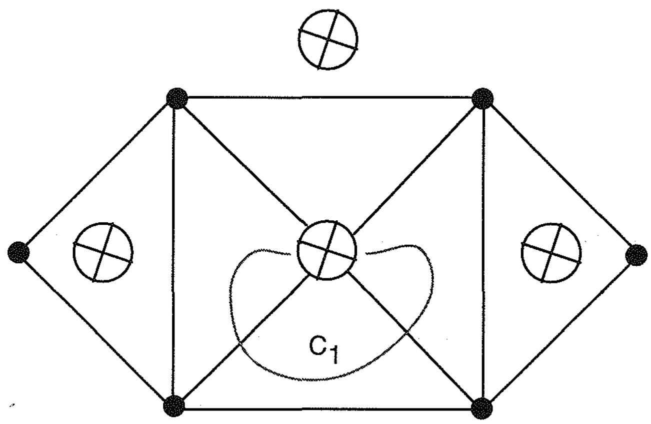


Figure 3.1: No 0,1-solution

3.6 Circulations homotopic to one given curve

For any graph G embedded on a compact surface S and any closed curve C_0 on S we define the set $P_G(C_0)$ as:

$$P_G(C_0) := \{x \geq 0 | \sum_{e \in E} x_e \text{cr}(e, D) \geq \text{mincr}(C_0, D) \text{ for all } D \text{ on } S \backslash V\}. \qquad (3.24)$$

The homotopic circulation theorem implies that $P_G(C_0) = Q_G(C_0)$ where $Q_G(C_0)$ is defined as:

$$Q_G(C_0) := \{y \geq 0 | \exists x \in \text{conv.hull}\{\text{tr}_C,\ C \sim C_0\} \text{ such that } y \geq x\}, \tag{3.25}$$

where C ranges over all closed curves in G freely homotopic to C_0.

From (3.25) it follows that $P_G(C_0)$ is a polyhedron, i.e. $P_G(C_0) = \{x | Ax \leq b\}$ for some matrix A and vector b. We say that the system $Ax \leq b$ *determines* $P_G(C_0)$.

In fact, we have that $P_G(C_0)$ is a *blocking polyhedron*, that is $P_G(C_0) \subset \mathbb{R}^n_+$ and $x \in P_G(C_0)$ and $y \geq x$ implies $y \in P_G(C_0)$. This implies that there is a system of inequalities $Ax \geq 1$ determining $P_G(C_0)$, where A is a nonnegative $m \times n$-matrix and 1 denotes the all-one vector in $\mathbb{R}^m$.

Similarly, we say that $P_G(C_0)$ is *determined* by $\mathcal{D}$ if $\mathcal{D}$ is a collection of closed curves on $S \backslash V$ such that

$$P_G(C_0) = \{x \geq 0 | \sum_{e \in E} x_e \text{cr}(e, D) \geq \text{mincr}(C_0, D) \text{ for all } D \text{ in } \mathcal{D}\}. \tag{3.26}$$

From Farkas' lemma and Caratheodory's theorem (cf. [32] Chapter 7), we derive the following:

Proposition 3.1 *Let G be a graph embedded on a compact surface S and let C_0 be a closed curve on S. Then there exists a finite collection of closed curves determining $P_G(C_0)$.*

Proof. It follows from the homotopic circulation theorem that $P_G(C_0)$ is a blocking polyhedron, hence there is a finite set $Ax \geq 1$ of inequalities determining $P_G(C_0)$, where A is a nonnegative $m \times n$-matrix. Denote the row-vecors of A by a_i, $i = 1, \ldots, m$.

We assert that each vector $(a_i\ 1) \in \mathbb{R}^{E+1}$ $(i = 1, \ldots, m)$ is an element of the convex cone K. Here K denotes the cone generated by the vectors

$$\{(\text{cr}_D\ \text{mincr}(C_0, D)) | D \subset S \backslash V\}$$

($\text{cr}_D \in \mathbb{R}^E$ denotes the vector with entries $\text{cr}(e, D)$ for each $e \in E$).

For otherwise, by Farkas' lemma and the fact that K is closed there exists a vector $(x\ y) \in \mathbb{R}^{E+1}$ such that $\sum_{e \in E} x_e \text{cr}(e, D) \geq -y \cdot \text{mincr}(C_0, D)$ and $a_i^T x < -y$. It follows that $x_e \geq 0$ $(e \in E)$ (for any edge e, let D be a nullhomotopic closed curve crossing e twice). Hence $a_i^T x \geq 0$ so $y < 0$. We obtain that $-\frac{1}{y} x \in P_G(C_0)$ contradicting the fact that $Ax \geq 1$ determines $P_G(C_0)$.

It follows by Caratheodory's theorem that for each $i \in \{1, \ldots, m\}$ and each vector $(a_i\ 1)$ there exist curves $D_1, \ldots, D_t$ with $t \leq E$ such that $(a_i\ 1)$ is an element of the cone generated by the vectors $\{(\mathrm{cr}_{D_i}\ \mathrm{mincr}(C_0, D_i))|\ i = 1, \ldots, t\}$.

Hence there exists a collection of at most mE closed curves determining $P_G(C_0)$.

∎

It would be interesting to find a more concrete description of collections of curves determining $P_G(C_0)$.

Assume G is embedded so that each face of G is an open disk. Then each closed curve D on $S \backslash V$ corresponds to a cycle in the dual graph G^*. Moreover, $\mathrm{cr}(e, D) = \mathrm{tr}_D(e^*)$, where e^* is the edge in G^* associated with e. As before, we call a cycle D in graph G^* *minimal* if there is no cycle $D' \sim D$ with $\mathrm{tr}_{D'}(e) \leq \mathrm{tr}_D(e)$ with strict inequality for at least one edge $e \in G^*$.

We say that C_0 satisfies the *triangle inequality* if for any two curves D_1,D_2 on S one has:

$$\mathrm{mincr}(C_0, D_1 \cdot D_2) \leq \mathrm{mincr}(C_0, D_1) + \mathrm{mincr}(C_0, D_2). \tag{3.27}$$

On the torus and on the projective plane, every curve C satisfies the triangle inequality (cf. [6]).

Proposition 3.2 *Let G be a graph embedded on S such that each face is an open disk, let C_0 be a curve on S satisfying the triangle inequality. Moreover, let $\mathcal{D}$ denote the set of minimal cycles in G^* traversing each edge at most once. Then $\mathcal{D}$ determines $P_G(C_0)$.*

Proof. We show that for any cycle D in G^* traversing some edges of G^* more than once, the inequality

$$\sum_{e \in E} x_e \mathrm{tr}_D(e) \geq \mathrm{mincr}(C_0, D) \tag{3.28}$$

is implied by a positive linear combination of inequalities obtained by $D \in \mathcal{D}$. So, by contradiction, let D be a minimal cycle in G^* such that this is not the case and such that tr_D is minimal among all such cycles, i.e. for any other such cycle D' we have $\mathrm{tr}_D \leq \mathrm{tr}_{D'}$ and $\mathrm{tr}_{D'} \neq \mathrm{tr}_D$.

Without loss of generality, e_1 is traversed at least twice by D.

Write:

$$D = (e_1, \ldots, e_r, e_1, \ldots, e_s, e_1) \tag{3.29}$$

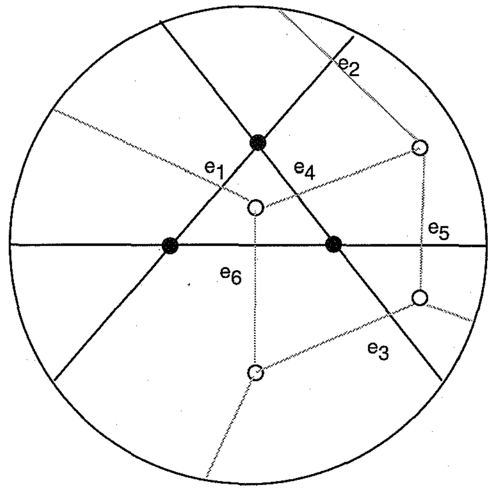


Figure 3.2: Example on the projective plane

Define $D' := (e_1, \dots, e_r, e_1)$ and $D'' := (e_1, \dots, e_s, e_1)$. Then $D \sim D' \cdot D''$ and $\mathrm{tr}_{D'}(e) \leq \mathrm{tr}_D(e)$, $\mathrm{tr}_{D''}(e) \leq \mathrm{tr}_D(e)$ with strict inequality for e_1.

By minimality of tr_D both inequalities are implied by a positive linear combination of inequalities obtained by $D \in \mathcal{D}$. Hence the have the inequality

$$\sum_{e \in E} x_e(\mathrm{cr}(e, D') + \mathrm{cr}(e, D'')) \geq \mathrm{mincr}(C_0, D') + \mathrm{mincr}(C_0, D''). \tag{3.30}$$

As C_0 satisfies the triangle inequality, we have that inequality (3.28) is implied by inequalities obtained by $D \in \mathcal{D}$, contradicting the assumption. ∎

Example

Let S be the projective plane and let G be the graph depicted in Figure 3.2. Moreover let C_0 denote any non-nullhomotopic curve. The set $Q_G(C_0)$ as in (3.25) is the set of

$y \geq 0$ such that there is an x in

$$\text{conv.hull}\{\begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\\0\\0\\1\end{pmatrix}\}$$

such that $y \geq x$.

Indeed $Q_G(C_0)$ is also described by the inequalities: $x_i \geq 0$ $(i = 1, \ldots, 6)$ and

$$\begin{aligned}x_1 + x_4 + x_5 &\geq 0\\ x_1 + x_3 + x_6 &\geq 0\\ x_2 + x_3 + x_5 &\geq 0\\ x_2 + x_4 + x_6 &\geq 0.\end{aligned}$$

∎

In polyhedral combinatorics a rational system of linear inequalities $x \geq 0, Ax \geq b$, with integral A, is called *totally dual integral*, if the maximum in the linear programming-duality equation

$$\min\{cx|x \geq 0; Ax \geq b\} = \max\{yb|y \geq 0; yA \leq c\} \tag{3.31}$$

has an integral optimum solution y for each integral vector c with finite maximum.

So the system defined by inequalities

$$\begin{aligned}&x \geq 0\\ &\textstyle\sum_{e\in E} x_e \mathrm{tr}_D(e) \geq \mathrm{mincr}(C_0, D)\ D \in \mathcal{D},\end{aligned} \tag{3.32}$$

is totally dual integral if the maximum

$$\max\{\sum_{D\in\mathcal{D}} y_D \mathrm{mincr}(C_0, D) \mid y \geq 0 \text{ and } c_e \geq \sum_{D\in\mathcal{D}} y_D \mathrm{tr}_D(e)\ (e \in E)\ \} \tag{3.33}$$

has an integral optimum solution y for each integral vector $c \geq 0$ with finite maximum. Here $\mathcal{D}$ is a finite collection of curves as in Proposition 3.1. A system is called totally dual $\frac{1}{k}$-integral if there is an optimum solution y so that ky is integral for each integral vector $c \geq 0$ with finite maximum.

Note that for $c_e = 1$ for all $e \in E$, an integral optimum solution y corresponds to edge-disjoint cycles $D_1, \ldots, D_s$ in G^* maximizing $\sum_{D \in \mathcal{D}} \text{mincr}(C_0, D)$.

In general, the system (3.32) will not be totally dual integral as is shown by the example in Figure 3.2. Here we have:

$$\max\{\sum_{i=1}^{4} y_i \mid y \geq 0 \text{ and } 1 \geq \sum_{i=1}^{4} y_i + y_j \text{ for all } i < j,\ i,j = 1, \ldots, 6\ \} \tag{3.34}$$

where $y_i := y_{D_i}$ and $D_1 = (e_1, e_4, e_5)$, $D_2 = (e_1, e_3, e_6)$, $D_3 = (e_2, e_4, e_6)$ and $D_4 = (e_2, e_3, e_5)$.

Clearly, the maximum in (3.34) is at most 2, and can be attained by taking $y_i = \frac{1}{2}$ $(i = 1, \ldots, 6)$. However, no integral solution attains this maximum. This corresponds to the fact that in the dual graph G^* there exist no two edge-disjoint cycles that are not nullhomotopic.

Lins' theorem [15] implies that:

Proposition 3.3 *Let G be a graph embedded on the projective plane so that each face of G is an open disk. Let C_0 be non-nullhomotopic cycle in G. Then the system*

$$x \geq 0$$
$$\sum_{e \in E} x_e \text{cr}(e, D) \geq \text{mincr}(C_0, D) \quad D \in \mathcal{D},$$

where $\mathcal{D}$ denotes the set of minimal non-nullhomotopic cycles in G^*, is totally dual $\frac{1}{2}$-integral.

Chapter 4

Homotopy and crossings of systems of curves on a surface

Let $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_k$ be closed curves on a compact surface S. We characterize (in terms of counting crossings) when there exists a permutation π of $\{1, \ldots, k\}$ such that $C'_{\pi(i)}$ is freely homotopic to C_i or C_i^{-1} for each $i = 1, \ldots, k$.

4.1 Introduction

We call a closed curve C *orientation-primitive* if there do not exist an orientation-preserving curve D and an integer $n \geq 2$ so that $C \sim D^n$. So each orientation-reversing curve is orientation-primitive. For a closed curve D and an integer n, D^n is the closed curve defined by $D^n(z) := D(z^n)$ for $z \in S^1$.

Two systems of closed curves $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_{k'}$ are called *homotopically equivalent* (in notation: $C_1, \ldots, C_k \cong C'_1, \ldots, C'_{k'}$) if $k = k'$ and there exists a permutation π of $\{1, \ldots, k\}$ such that for each $i = 1, \ldots, k$ one has $C'_{\pi(i)} \sim C_i$ or $C'_{\pi(i)} \sim C_i^{-1}$.

The main result of this chapter is:

Theorem 4.1 *Let $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_{k'}$ be orientation-primitive closed curves on a compact surface S. Then the following are equivalent:*

$$(i)\ C_1, \ldots, C_k \cong C'_1, \ldots, C'_{k'} \tag{4.1}$$

(ii) for each closed curve D on S

$$\sum_{i=1}^{k} \mathrm{mincr}(C_i, D) = \sum_{i=1}^{k'} \mathrm{mincr}(C'_i, D). \tag{4.2}$$

Theorem 4.1 generalizes a theorem of A. Schrijver [30] for compact orientable surfaces.

4.2 Cycles in graphs

In the sequel $G = (V, E)$ denotes a graph without loops or parallel edges embedded on S where degree$(v) \in \{2, 4\}$ for each vertex $v \in V$. The set of vertices of degree 4 in G is denoted by W.

We recall that a *cycle* in G is a sequence

$$C = (v_0, e_1, v_1, e_2, v_2, \ldots, v_{d-1}, e_d, v_d), \tag{4.3}$$

where $v_0, \ldots, v_d$ are vertices of G, $v_0 = v_d$, and e_i is an edge connecting v_{i-1} and v_i ($i = 1, \ldots, d$), ($v_1, \ldots, v_d, e_1, \ldots, e_d$ need not be distinct). With each cycle in G we can associate in the obvious way a closed curve on S— unique up to homotopy. For any cycle (4.3) and any edge e of G,

$$\mathrm{tr}_C(e) := \text{ number of } i \in \{1, \ldots, d\} \text{ with } e_i = e. \tag{4.4}$$

We call a cycle (4.3) *nonreturning* if $e_i \neq e_{i-1}$ for $i = 1, \ldots, d$, and $e_1 \neq e_d$.

For each vertex v of degree 4, we fix a cyclic ordering of e_1, e_2, e_3, e_4 of the edges incident with v.

We call edges e_1 and e_3 *opposite in v*, and similarly we call e_2 and e_4 *opposite in v*. If C is a cycle such that e_i and e_{i+1} are opposite for all $i \in \{1, \ldots, d\}$ (taking indices modulo d), then we say that C *follows the straight decomposition of G*. If, additionally, each edge $e \in E$ is traversed at most once by C then we say that C *belongs to the straight decomposition of G*.

For any cycle C in G, any vertex of degree 4 in G and any $i, j \in \{1, 2, 3, 4\}$, let

$$\alpha^v_{ij}(C) := \text{number of times } C \text{ traverses } v \text{ by going from } e_i \text{ to } e_j \text{ or from } e_j \text{ to } e_i. \tag{4.5}$$

Proposition 4.1 *For any nonreturning cycle C in G,*

$$\mathrm{mincr}(C) \leq \sum_{v\in W} [\alpha^v_{13}(C)\alpha^v_{24}(C) + \frac{1}{4} \sum_{1\leq g<h\leq 4} \sum_{\substack{1\leq k<l\leq 4 \\ |\{g,h\}\cap\{k,l\}|=1}} \alpha^v_{gh}(C)\alpha^v_{kl}(C)]. \tag{4.6}$$

Proof. Similar to the proof of Proposition 4.2. ∎

Proposition 4.2 *For any pair of nonreturning cycles C, D in G with $C \neq D$,*

$$\begin{aligned}\mathrm{mincr}(C,D) \leq \sum_{v\in W} [&\alpha^v_{13}(C)\alpha^v_{24}(D) + \alpha^v_{13}(D)\alpha^v_{24}(C) \\ &+\frac{1}{2} \sum_{1\leq g<h\leq 4} \sum_{\substack{1\leq k<l\leq 4 \\ |\{g,h\}\cap\{k,l\}|=1}} \alpha^v_{gh}(C)\alpha^v_{kl}(D)].\end{aligned} \tag{4.7}$$

Proof. This proposition was proved in [30]. For completeness we include this proof here.

We can represent C and D as

$$C = (v_0, f_1, v_1, f_2, v_2, \ldots, v_{s-1}, f_s, v_s), \tag{4.8}$$

and

$$D = (w_0, g_1, w_1, g_2, w_2, \ldots, w_{t-1}, g_t, w_t), \tag{4.9}$$

where $v_0, \ldots, v_s$ and $w_0, \ldots, w_t$ are vertices of G with $v_s = v_0$ and $w_t = w_0$, f_i is an edge of G connecting v_{i-1} and v_i ($i = 1, \ldots, s$), and g_i is an edge of G connecting w_{i-1} and w_i ($i = 1, \ldots, t$), so that $f_i \neq f_{i-1}$ for $i = 1, \ldots, s$ and $g_i \neq g_{i-1}$ for $i = 1, \ldots, t$ (taking indices of v and f mod s, and indices of w and g mod t).

Let λ be the number of pairs $(i, j) \in \{1, \ldots, s\} \times \{1, \ldots, t\}$ such that

(4.10) $v_i = w_j \in W$, f_i and f_{i+1} are opposite in v_i, and g_j and g_{j+1} are opposite in w_j, while $\{f_i, f_{i+1}\} \neq \{g_j, g_{j+1}\}$.

Let μ be the number of pairs $(i, j) \in \{1, \ldots, s\} \times \{1, \ldots, t\}$ such that

$$v_i = w_j \in W,\ f_{i+1} = g_{j+1},\ \text{and}\ f_i \neq g_j. \tag{4.11}$$

So μ is also equal to the number of pairs $(i,j) \in \{1,\ldots,s\} \times \{1,\ldots,t\}$ such that

$$v_i = w_j \in W,\ f_i = g_j,\ \text{and}\ f_{i+1} \neq g_{j+1}. \tag{4.12}$$

Similarly, let ν be the number of pairs $(i,j) \in \{1,\ldots,s\} \times \{1,\ldots,t\}$ such that

$$v_i = w_j \in W,\ f_{i+1} = g_j,\ \text{and}\ f_i \neq g_{j+1}. \tag{4.13}$$

Again, ν is also equal to the number of pairs $(i,j) \in \{1,\ldots,s\} \times \{1,\ldots,t\}$ such that

$$v_i = w_j \in W,\ f_i = g_{j+1},\ \text{and}\ f_{i+1} \neq g_j. \tag{4.14}$$

Note that the right-hand side of (4.7) is equal to $\lambda + \mu + \nu$. Moreover, note that $C \neq D$ implies that $\lambda + \mu + \nu = 0$ only if $\mathrm{cr}(C, D) = 0$.

To see that $\mathrm{mincr}(C, D) \leq \lambda + \mu + \nu$, note that μ is equal to the number of pairs $(i,j) \in \{1,\ldots,s\} \times \{1,\ldots,t\}$ such that there exists a number $b \geq 1$ with

$$\begin{array}{llll} f_i \neq g_j, & v_i = w_j, & f_{i+1} = g_{j+1}, & v_{i+1} = w_{j+1}, \\ f_{i+2} = g_{j+2}, & \ldots, & v_{i+b} = w_{j+b}, & f_{i+b+1} \neq g_{j+b+1}, \end{array} \tag{4.15}$$

Which corresponds to pictures of the type as in Figure 4.1.

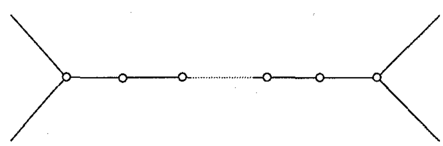


Figure 4.1: Crossing of C and D.

Similarly, ν is equal to the number of pairs $(i,j) \in \{1,\ldots,s\} \times \{1,\ldots,t\}$ such that there exists a number $b \geq 1$ with

$$\begin{array}{llll} f_i \neq g_{j+1}, & v_i = w_j, & f_{i+1} = g_j, & v_{i+1} = w_{j-1}, \\ f_{i+2} = g_{j-1}, & \ldots, & v_{i+b} = w_{j-b}, & f_{i+b+1} \neq g_{j-b}, \end{array} \tag{4.16}$$

Again, this corresponds to pictures of the type as in Figure 4.1.

Since each of the intersections of the type of Figure 4.1. can be replaced by parts that have one crossing of none at all, we obtain $\mathrm{mincr}(C, D) \leq \lambda + \mu + \nu$. ∎

If $C_1, \ldots, C_s$ are edge-disjoint cycles in a graph G with vertices of degree at most 4 and W denotes the set of vertices of degree 4, then clearly $|W| \geq \sum_{i=1}^{s} \text{mincr}(C_i) + \sum_{i<j} \text{mincr}(C_i, C_j)$. The next proposition gives a lower bound for $|W|$ in case the cycles $C_1, \ldots, C_s$ are 'fractionally' edge-disjoint as in (4.17).

Proposition 4.3 *Let $C_1, \ldots, C_s$ be a set of nonreturning cycles in G and let $\lambda_1, \ldots, \lambda_s > 0$ be so that,*

$$\sum_{j=1}^{s} \lambda_j \text{tr}_{C_j}(e) \leq 1 \quad (e \in E). \tag{4.17}$$

Then

$$\sum_{i=1}^{s} \lambda_i^2 \text{mincr}(C_i) + \sum_{\substack{i,j=1 \\ i<j}}^{s} \lambda_i \lambda_j \text{mincr}(C_i, C_j) \leq |W|. \tag{4.18}$$

Equality in (4.18) *implies that each cycle C_i ($i = 1, \ldots, s$) follows the straight decomposition and that* (4.17) *holds with equality.*

Proof. By Proposition 4.1 and Proposition 4.2 we obtain:

$$\sum_{i=1}^{s} 2\lambda_i^2 \text{mincr}(C_i) + \sum_{\substack{i,j=1 \\ i\neq j}}^{s} \lambda_i \lambda_j \text{mincr}(C_i, C_j) \leq$$

$$\sum_{v\in W} \sum_{i,j=1}^{s} \lambda_i \lambda_j [\alpha_{13}^v(C_i)\alpha_{24}^v(C_j) + \alpha_{13}^v(C_j)\alpha_{24}^v(C_i)$$

$$+\frac{1}{2} \sum_{g<h} \sum_{\substack{k<l \\ |\{g,h\}\cap\{k,l\}|=1}} \alpha_{gh}^v(C_i)\alpha_{kl}^v(C_j)]. \tag{4.19}$$

For any vertex $v \in W$ and $g, h \in \{1, 2, 3, 4\}$, define

$$\alpha_{gh}^v := \sum_{j=1}^{s} \lambda_j \alpha_{gh}^v(C_j). \tag{4.20}$$

The sum in the righthandside of (4.19) is equal to

$$\sum_{v\in W} [2\alpha_{13}^v \alpha_{24}^v + \frac{1}{2} \sum_{g<h} \sum_{\substack{k<l \\ |\{g,h\}\cap\{k,l\}|=1}} \alpha_{gh}^v \alpha_{kl}^v].$$

So it is sufficient to show that for any fixed vertex $v \in W$ with edges e_1, e_2, e_3, e_4 incident with v:

$$2\alpha_{13}\alpha_{24} + \frac{1}{2} \sum_{g<h} \sum_{\substack{k<l \\ |\{g,h\}\cap\{k,l\}|=1}} \alpha_{gh}\alpha_{kl} \leq 2, \tag{4.21}$$

where $\alpha_{ij} := \alpha^v_{ij}$. Define for $g \in \{1,2,3,4\}$:

$$\delta_g := \sum_{i=1}^{s} \lambda_i \mathrm{tr}_{C_i}(e_g).$$

So by (4.17) $\delta_g \leq 1$ for each $g \in \{1,2,3,4\}$. Moreover:

$$\begin{aligned} \delta_1 = \alpha_{12} + \alpha_{13} + \alpha_{14}, \quad \delta_2 = \alpha_{12} + \alpha_{23} + \alpha_{24}, \\ \delta_3 = \alpha_{13} + \alpha_{23} + \alpha_{34}, \quad \delta_4 = \alpha_{14} + \alpha_{24} + \alpha_{34}. \end{aligned} \tag{4.22}$$

After adding $2\alpha_{12}\alpha_{34} + 2\alpha_{14}\alpha_{23}$ to the sum in (4.21) and subtituting (4.22) we obtain that the sum in equation (4.21) is not larger than:

$$\frac{1}{2} \sum_{\substack{|\{g,h\}\cap\{k,l\}|=0 \\ g<h,k<l}} \alpha_{gh}(\delta_k + \delta_l) \leq \sum_{g<h} \alpha_{gh} = \frac{1}{2}(\delta_1 + \delta_2 + \delta_3 + \delta_4) \leq 2. \tag{4.23}$$

This proves inequality (4.21). In order to have equality in (4.21) we should have

$$\alpha_{12}\alpha_{34} = 0, \ \alpha_{14}\alpha_{23} = 0, \ \delta_1 = \delta_2 = \delta_3 = \delta_4 = 1. \tag{4.24}$$

Now (4.22) and (4.24) imply

$$\begin{aligned} \alpha_{12} &= \frac{1}{2}(\delta_1 + \delta_2 - \delta_3 - \delta_4) + \alpha_{34} = \alpha_{34}, \\ \alpha_{14} &= \frac{1}{2}(\delta_1 + \delta_4 - \delta_2 - \delta_3) + \alpha_{23} = \alpha_{23}. \end{aligned}$$

Hence $\alpha_{12} = \alpha_{34} = \alpha_{14} = \alpha_{23} = 0$ and $\alpha_{13} = \alpha_{24} = 1$. So for any $v \in W$, inequality (4.21) holds with equality only if $\alpha^v_{13} = \alpha^v_{24} = 1$ and $\alpha^v_{12} = \alpha^v_{14} = \alpha^v_{23} = \alpha^v_{34} = 0$. This shows the proposition. ∎

4.3 Cycles on surfaces

We define for a closed curve C on S, the function $\mathrm{odd}(C) := 1$ if C is orientation-reversing. Otherwise, $\mathrm{odd}(C) := 0$.

In Chapter 2, Corollary 2.3, we showed:

Proposition 4.4 *Let C be an orientation-primitive closed curve on S. Then*

$$\mathrm{mincr}(C, C) = 2\mathrm{mincr}(C) + \mathrm{odd}(C). \tag{4.25}$$

In Chapter 2, Corollary 2.4 we showed:

Proposition 4.5 *Let C be an orientation-reversing closed curve on S. Then*

$$\mathrm{mincr}(C, C^2) < 2\mathrm{mincr}(C, C). \tag{4.26}$$

Here we show:

Proposition 4.6 *Let C, D be closed curves on S. Then*

$$\mathrm{mincr}(C, D^2) \leq 2\mathrm{mincr}(C, D). \tag{4.27}$$

Proof. Let C, D be closed curves with the property that $\mathrm{cr}(C, D) = \mathrm{mincr}(C, D)$. Now $\mathrm{mincr}(C, D^2) \leq \mathrm{cr}(C, D^2) = 2\mathrm{cr}(C, D) = 2\mathrm{mincr}(C, D)$. ∎

4.4 Proof of Theorem 4.1

The implication (4.1) ⇒ (4.2) as stated in Theorem 4.1 is trivial as $\mathrm{mincr}(C^{-1}, D) = \mathrm{mincr}(C, D)$ for any pair of closed curves C, D on S. We show (4.2) ⇒ (4.1).

Suppose by contradiction that $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_{k'}$ are two sets of curves satisfying (4.2) but not (4.1) such that $k + k'$ is minimal. This implies that:

$$\text{there is no pair } i, j \text{ with } i \in \{1, \ldots, k\},\ j \in \{1, \ldots, k'\} \text{ such that } C_i \sim C'_j \text{ or } C_i^{-1} \sim C'_j. \tag{4.28}$$

By symmetry we may assume that

$$\sum_{i=1}^{k'} \mathrm{mincr}(C'_i) + \sum_{\substack{i,j=1 \\ i<j}}^{k'} \mathrm{mincr}(C'_i, C'_j) \leq \sum_{i=1}^{k} \mathrm{mincr}(C_i) + \sum_{\substack{i,j=1 \\ i<j}}^{k} \mathrm{mincr}(C_i, C_j). \tag{4.29}$$

It is a basic fact (cf. [1], [21] and [36]), that there exist $\widetilde{C}_1 \sim C'_1, \ldots, \widetilde{C}_{k'} \sim C'_{k'}$ such that

$$\mathrm{cr}(\widetilde{C}_i, \widetilde{C}_j) = \mathrm{mincr}(C'_i, C'_j) \text{ for } i, j = 1, \ldots, k' \text{ and } i \neq j \tag{4.30}$$

$$\mathrm{cr}(\widetilde{C}_i) = \mathrm{mincr}(C'_i) \text{ for } i = 1, \ldots, k'. \tag{4.31}$$

The result being invariant under homotopies, we may assume that $\widetilde{C}_i = C'_i$ for $i = 1, \ldots, k'$, that $C'_i \neq C'_j$ if $i \neq j$, and that each point of S is traversed at most twice by the C'_i (so no two crossings of the C'_i coincide).

Let $G = (V, E)$ be the graph made up by the curves C_i'. So G is a graph embedded on S. Each point of S traversed twice by the C_i' is a vertex of degree 4 of G. Moreover, we take as vertices some of the points of S traversed exactly once by the C_i', in such a way that G will be a graph without loops or parallel edges. So each vertex of G has degree 2 or 4 and $C_1', \ldots, C_{k'}'$ is the straight decomposition of G. Let W denote the set of vertices of degree 4. We obtain:

$$|W| = \sum_{i=1}^{k'} \text{mincr}(C_i') + \sum_{\substack{i,j=1\\ i<j}}^{k'} \text{mincr}(C_i', C_j'). \tag{4.32}$$

By (4.2) for each closed curve $D : S^1 \to S \backslash V$,

$$\text{cr}(G, D) = \sum_{i=1}^{k'} \text{cr}(C_i', D) \geq \sum_{i=1}^{k'} \text{mincr}(C_i', D) = \sum_{i=1}^{k} \text{mincr}(C_i, D), \tag{4.33}$$

where $\text{cr}(G, D) := |\{z \in S^1 | D(z) \in G\}|$. Hence, by the 'homotopic circulation theorem' in Chapter 3, there exist cycles $D_1, \ldots, D_s$, with rationals $\lambda_1, \ldots, \lambda_s > 0$ and a partition $S_1, \ldots, S_k$ of $\{1, \ldots, s\}$ such that

$$D_j \sim C_i \quad (i = 1, \ldots, k; j \in S_i), \tag{4.34}$$

$$\sum_{j \in S_i} \lambda_j = 1 \quad (i = 1, \ldots, k), \tag{4.35}$$

$$\sum_{j=1}^{s} \lambda_j \text{tr}_{D_j}(e) \leq 1 \quad (e \in E). \tag{4.36}$$

Clearly, we may assume the D_j to be nonreturning. We obtain using (4.34),(4.35),(4.36), Proposition 4.3 and Proposition 4.4:

$$2\sum_{i=1}^{k} \text{mincr}(C_i) + \sum_{\substack{i,j=1\\ i\neq j}}^{k} \text{mincr}(C_i, C_j) = \sum_{i,j=1}^{k} \text{mincr}(C_i, C_j) - \sum_{i=1}^{k} \text{odd}(C_i) =$$

$$\sum_{g,h=1}^{s} \lambda_g \lambda_h \text{mincr}(D_g, D_h) - \sum_{i=1}^{k} \text{odd}(C_i) =$$

$$\sum_{\substack{g,h=1\\ h\neq g}}^{s} \lambda_g \lambda_h \text{mincr}(D_g, D_h) + \sum_{g=1}^{s} \lambda_g^2 \text{mincr}(D_g, D_g) - \sum_{i=1}^{k} \text{odd}(C_i) = \tag{4.37}$$

$$\sum_{\substack{g,h=1\\h\neq g}}^{s} \lambda_g\lambda_h \text{mincr}(D_g, D_h) + 2\sum_{g=1}^{s} \lambda_g^2 \text{mincr}(D_g) + \sum_{g=1}^{s} \lambda_g^2 \text{odd}(D_g) - \sum_{i=1}^{k} \text{odd}(C_i) \leq$$

$$2|W| + \sum_{i=1}^{k} \text{odd}(C_i)(-1 + \sum_{g\in S_i} \lambda_g^2) \leq 2|W|.$$

(The first inequality follows from Proposition 4.3.)

By our assumption (4.29) and by (4.32), we should have equality throughout in (4.37). By Proposition 4.3, each curve D_j ($j = 1, \ldots, s$) follows the straight decomposition and there is equality in (4.17). So $s = k'$ and there exists a permutation π of $\{1, \ldots, s\}$ together with positive integers $n_1, \ldots, n_{k'}$ so that

$$C_j'^{n_j} \sim D_{\pi(j)} \text{ or } C_j'^{n_j} \sim D_{\pi(j)}^{-1} \quad (j = 1, \ldots, k'). \tag{4.38}$$

Without loss of generality, we may assume that $\pi(j) = j$. Now $D_j \sim C_i$ for all $j \in S_i$ and hence, by (4.28), $n_j \geq 2$. As C_i is orientation-primitive, it follows that C_j' is orientation-reversing for $j = 1, \ldots, k'$.

Suppose that C_i is an orientation-reversing curve for some $i \in \{1, \ldots, k\}$. It follows from

$$\sum_{i=1}^{k} \text{odd}(C_i)(-1 + \sum_{g\in S_i} \lambda_g^2) = 0 \tag{4.39}$$

that $|S_i| = 1$, say $S_i = \{j_i\}$. We now obtain $\lambda_{j_i} = 1$ and $D_{j_i} \sim C_i$ or $D_{j_i} \sim C_i^{-1}$ contradicting (4.28). Hence C_i is orientation-preserving for $i = 1, \ldots, k$.

So $n = 2$ in (4.38) and, by (4.35) and equality in (4.17), we have $|S_i| = 2$ and $\lambda_j = \frac{1}{2}$ ($j = 1, \ldots, s$). Hence $s = 2k$ and after renumbering we may assume that $C_i \sim C_i'^2$ or $C_i^{-1} \sim C_i'^2$ and $C_i' \sim C_{i+k}'$ for $i = 1, \ldots, k$.

We obtain using Proposition 4.4 and Proposition 4.5:

$$\sum_{i=1}^{k} \text{mincr}(C_i, C_1') = \text{mincr}(C_1, C_1') + \sum_{i=2}^{k} \text{mincr}(C_i, C_1') < \tag{4.40}$$

$$2\text{mincr}(C_1', C_1') + \sum_{i=2}^{k} \text{mincr}(C_i, C_i') \leq \tag{4.41}$$

$$2\sum_{i=1}^{k} \text{mincr}(C_i', C_1') = \sum_{i=1}^{2k} \text{mincr}(C_i', C_1'). \tag{4.42}$$

This contradicts (4.2). ∎

Chapter 5

Uniqueness of kernels on compact surfaces

Let S be a compact surface. For any graph G embedded on S and any closed curve D on S we define $\mu_G(D)$ as the minimum number of intersections of G and D', where D' ranges over all closed curves freely homotopic to D. We call G a *kernel* if $\mu_{G'} \neq \mu_G$ for each proper minor G' of G. We prove that if G and G' are kernels with $\mu_G = \mu_{G'}$ (in such a way that each face of G is an open disk), then G' can be obtained from G by a series of the following operations: (i) homotopic shifts over S; (ii) taking the surface dual graph; (iii) ΔY-exchange (i.e. replacing a vertex v of degree 3 by a triangle connecting the three vertices adjacent to v, or conversely). This was shown for orientable surfaces by A. Schrijver.

5.1 Introduction

Let S be a compact surface and let G be a graph embedded on S (without crossing edges). For each closed curve D on S we define

$$\mu_G(D) := \min_{D' \sim D} \operatorname{cr}(G, D'). \tag{5.1}$$

Here $\operatorname{cr}(G, D')$ denotes the number of times D' intersects G (i.e., $\operatorname{cr}(G, D') = |\{z \in S^1 \mid D'(z) \in G\}|$). The minimum ranges over all closed curves D' freely homotopic to D.

Observe that the function μ_G is invariant under the following operations on G:

$$\begin{array}{rl} \text{(i)} & \text{homotopic shifts of } G \text{ over } S; \\ \text{(ii)} & \text{replacing } G \text{ by a surface dual } G^* \text{ of } G; \\ & \text{(if } G \text{ is cellularly embedded).} \\ \text{(iii)} & \Delta Y\text{-exchanges in } G. \end{array} \tag{5.2}$$

Here we use the following terminology. Graph G' arises by a *homotopic shift* of G over S (or *is homotopic to* G) if there exists a continuous function $\Phi : [0,1] \times G \to S$

so that (i) $\Phi(0,y) = y$ for each $y \in G$; (ii) for each $x \in [0,1]$, $\Phi(x,\cdot)$ is a one-to-one function on G; (iii) $\Phi(1,\cdot)$ maps G onto G'. (We consider G and G' as subspaces of S.)

We say that graph G^* is a (*surface*) *dual* of G if (i) each face of G is an open disk; (ii) each face of G contains exactly one vertex of G^*, and $V(G^*) \cap G = \emptyset$; (iii) each edge of G^* crosses exactly one edge of G, and each edge of G crosses exactly one edge of G^*, while there are no further intersections of G and G^*. (By $V(\cdot)$ and $E(\cdot)$ we mean the vertex set and edge set of $\cdot$.) So G has a surface dual if and only if each face of G is an open disk. Moreover, G has only one surface dual up to homotopic shifts.

If v is a vertex of G of degree 3, a ΔY-exchange (at v) replaces v and the three edges incident with v by a triangle connecting the three vertices adjacent to v (thus forming a triangular face). We also call the converse operation (replacing a triangular face by a "star" with three rays) a ΔY-exchange.

Note moreover that if G' is a minor of G then $\mu_{G'} \leq \mu_G$ (i.e., $\mu_{G'}(D) \leq \mu_G(D)$ for each closed curve D). Here a *minor* of G arises by a series of deletions of edges and contractions of non-loop edges. If we contract an edge, the graph arising is naturally embedded again on S (unique up to homotopic shifts).

Now we call G a *kernel* on S if $\mu_{G'} \neq \mu_G$ for each proper minor G' of G. *Proper* means that we delete or contract at least one edge of G.

The main result of this chapter is that, if each face of G is an open disk, then kernels are uniquely determined by the function μ_G up to the operations (5.2):

Theorem 5.1 *Let G and G' be kernels on the compact surface S, in such a way that each face of G is an open disk. If $\mu_G = \mu_{G'}$ then G' can be obtained from G by a series of operations* (5.2).

Theorem 5.1 generalizes the result of A. Schrijver [31] to nonorientable surfaces. The only nonorientable surface for which Theorem 5.1 was known to hold is the projective plane (cf. S. Randby [19]).

5.2 Tight Graphs

We next formulate an analogous result for so-called tight graphs. This result actually implies Theorem 5.1. Tight graphs were introduced in [27].

Let H be a graph embedded on the compact surface S. For each closed curve D

on S we define

$$\mu'_H(D) := \min_{\substack{D' \sim D \\ D' \subset S \setminus V(H)}} \operatorname{cr}(H, D'). \tag{5.3}$$

Here the minimum ranges over all closed curves D' freely homotopic to D so that D' does not intersect the vertex set $V(H)$ of H.

Let H be 4-regular. The function μ'_H is clearly invariant under the following operations on H:

$$\begin{array}{ll} \text{(i)} & \text{homotopic shifts of } H \text{ over } S, \\ \text{(ii)} & \Delta\nabla\text{-exchanges in } H. \end{array} \tag{5.4}$$

A $\Delta\nabla$-exchange replaces a triangular face, adjacent to r, s, t, u, v, w as in Figure 5.1(a) by a configuration as in Figure 5.1(b).

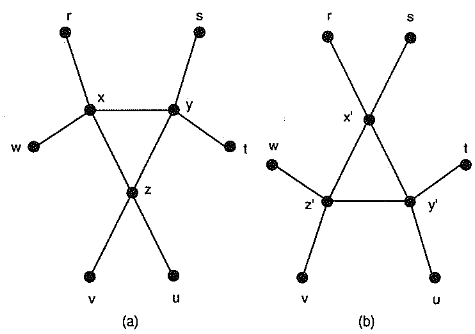


Figure 5.1: $\Delta\nabla$-exchange.

Moreover, we define an *opening* (at v) as replacing a neighborhood of vertex v of H of degree 4 in Figure 5.2(a) by the configuration as in Figure 5.2(b). (So at one vertex v of degree 4 there are two possible openings.) If this operation creates a loop without a vertex, we add a new vertex on the loop.

We call a graph H' an *opening* of H if H' arises from H by a series of openings. Note that if H' is an opening of H, then $\mu'_{H'} \leq \mu'_H$. We call a 4-regular graph H *tight* (on S) if $\mu'_{H'} \neq \mu'_H$ for each proper opening H' of H. (*Proper* means that we open H at least once.)

Note that in [27] tight graphs were defined for all eulerian graphs, but in this chapter we restrict ourselves to tight 4-regular graphs.

The following proposition states that tightness is invariant under $\Delta\nabla$-exchanges.

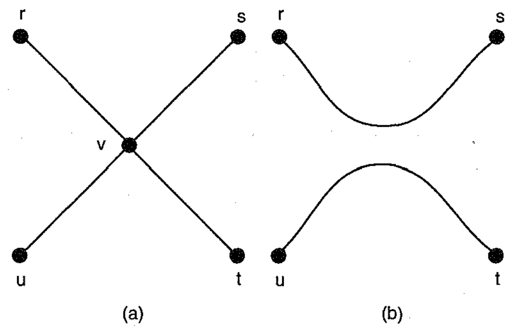
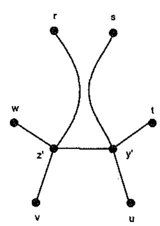


Figure 5.2: Opening at vertex v.

Proposition 5.1 *Let G be a tight graph and let G' be a graph arising from G by one $\Delta\nabla$-exchange. Then G' is tight.*

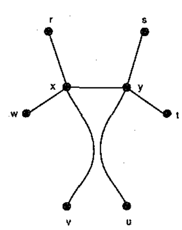
Proof. Consider the effect of $\Delta\nabla$-exchange to the graph G, replacing the configuration of of Figure 5.1(a) by the configuration of Figure 5.1(b).

Consider an opening at one of the three new vertices, say without loss of generality at x'. There are two possibilities. Suppose first that we open x' as in:



(5.5)

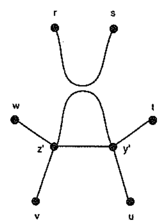
and that $\mu'_{G'}$ would be equal to μ'_G. However, as the original graph G is tight, opening z as in:



(5.6)

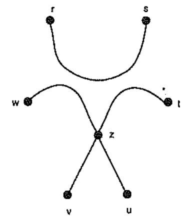
does change μ'_G. As replacing (5.6) by (5.5) clearly does not change the function μ'_G, we have a contradiction.

Suppose second that we open x' as in:



(5.7)

and that it would not change the function μ'_G. As the original graph G is tight, opening x and y as in:



(5.8)

does change the function μ'_G. Again, as replacing (5.8) by (5.7) does not change the function μ'_G we have a contradiction. ∎

The following theorem, to be proved in the last section of this chapter, says that, if S is a compact surface, then 4-regular tight graphs are uniquely determined by the function μ'_H, up to the operations (5.4):

Theorem 5.2 *Let H and H' be tight 4-regular graphs on the compact surface S. If $\mu'_H = \mu'_{H'}$ then H' can be obtained from H by a series of operations* (5.4).

5.3 Reduction of Theorem 5.1 to Theorem 5.2

We show a relation between kernels and tight graphs, which allows us to reduce Theorem 5.2 to Theorem 5.1. This reduction was shown first in [31] (Proposition 4). For reasons of completeness we repeat the arguments here.

The reduction is based on constructing the *medial graph* $H(G)$ of G, introduced by Steinitz [34], who called it the ω-process, and in reverse form by Tait [37].

For any graph G embedded on a surface S, $H(G)$ is constructed as follows. Choose an arbitrary point $w(e)$ 'in the middle of' e, for each edge e of G. These points form the vertex set of $H(G)$. For each vertex v of G, there will be edges of $H(G)$ forming a circuit connecting the points $w(e)$ on edges e incident with v. That is, we consider a neighbourhood N (homeomorphic to an open disk) of v. If $e_1, \ldots, e_k$

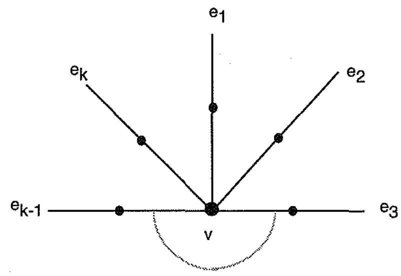


Figure 5.3: Neighbourhood of a vertex v.

denote the edges incident with v in cyclic order, $H(G)$ has edges connecting the pairs $\{w(e_1), w(e_2)\}, \{w(e_2), w(e_3)\}, \ldots, \{w(e_{k-1}, w(e_k)\}, \{w(e_k), w(e_1)\}$, drawn in N as in Figure 5.4. We do this for every vertex $v \in V(G)$ and adopt the convention that if

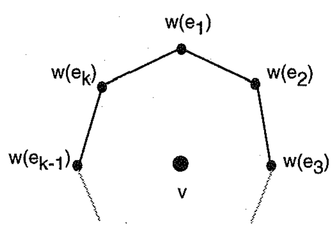


Figure 5.4: Construction of the medial graph.

v is an isolated vertex, then $H(G)$ has a loop around v without a vertex. This gives the 4-regular graph $H(G)$ (possibly with some loops without vertices). Note that $\mu'_{H(G)} = 2\mu_G$.

In fact, $H(G)$ determines G up to homotopy and duality:

Proposition 5.2 *Let G and G' be graphs embedded on the compact surface S so that each face of G is an open disk. Then $H(G)$ and $H(G')$ are homotopic, if and only if G' is homotopic to G or to its dual G^*.*

Proof. This follows directly from the fact that G can be reconstructed from $H(G)$, up to homotopy and duality. ∎

Moreover, we have:

Proposition 5.3 *Let G be a graph embedded on the compact surface S so that each face of G is an open disk. Then G is a kernel, if and only if $H(G)$ is tight.*

Proof. One easily checks that for any edge e, if G' arises from G by deleting e or contracting e (if e is not a loop), then $H(G')$ arises from $H(G)$ by an opening at vertex $w(e)$ and conversely, if H' arises from $H(G)$ by opening at a vertex $w(e)$, then either $H' = H(G')$ where G' arises from G' by deleting e or contracting e (if e is not a loop), or e is a loop which has no points in common with H'.

So if G' is a proper minor of G then $H(G')$ is (homotopic to) a proper opening of $H(G)$. This implies that if $H(G)$ is tight, then for each proper minor G' of G,

$$\mu_{G'} = \frac{1}{2}\mu'_{H(G')} \neq \frac{1}{2}\mu'_{H(G)} = \mu_G. \tag{5.9}$$

So G is a kernel.

Conversely, if G is a kernel, then $H(G)$ is tight. For suppose to the contrary that we can open $H(G)$ at a vertex $w(e)$, say, obtaining a graph H' with $\mu'_{H'} = \mu'_{H(G)}$. In this case either $H' = H(G')$ where G' arises from G by a deletion or a contraction of a non-loop edge e in G and we obtain:

$$\mu_{G'} = \frac{1}{2}\mu'_{H(G')} = \frac{1}{2}\mu_{H'} = \frac{1}{2}\mu'_{H(G)} = \mu_G, \tag{5.10}$$

contradicting the fact that G is a kernel; or e is a loop which has no points in common with H'. Let D be the closed curve following loop e. Since G is a kernel, D is not nullhomotopic (otherwise we could delete e from G without modifying μ_G). Now $\mathrm{cr}(H', D) = 0$. Hence $\mu_G(D) = \frac{1}{2}\mu'_{H(G)}(D) = \frac{1}{2}\mu'_{H'}(D) = 0$, contradicting the fact that each face of G is an open disk. ∎

We cannot delete the condition in Proposition 5.3 that each face of G is an open disk, as on the torus S, the graph G consisting of one vertex with one non-nullhomotopic loop attached, is a kernel, but $H(G)$ is not tight ($H(G)$ consists of one vertex with two non-nullhomotopic loops (of the same homotopy-type) attached).

Finally we have:

Proposition 5.4 *Let G and G' be graphs embedded on the compact surface S. If $H(G')$ arises from $H(G)$ by one $\Delta\nabla$-exchange, then G' arises from G by one ΔY-exchange, up to homotopy and duality.*

Proof. This follows from Proposition 5.2 and by considering Figures 5.5 (where the uninterrupted lines are edges of $H(G)$ or $H(G')$, and the dashed lines are edges of G or G'). ∎

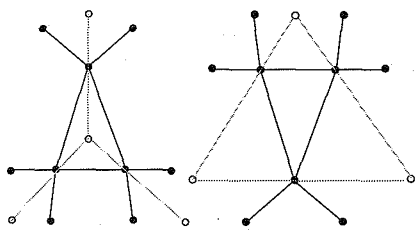


Figure 5.5: A ΔY-exchange and a $\Delta\nabla$-exchange.

Proposition 5.3 and 5.4 directly yield:

Proposition 5.5 *Theorem 5.2 implies Theorem 5.1.*

Proof. If G and G' are kernels on the compact surface S, so that each face of G and of G' is an open disk, then by Proposition 5.3, $H(G)$ and $H(G')$ are tight graphs. If $\mu_G = \mu_{G'}$ then $\mu'_{H(G)} = 2\mu_G = 2\mu_{G'} = \mu'_{H(G)}$. So by Theorem 5.2, $H(G)$ and $H(G')$ arise from each other by homotopic shifts and $\Delta\nabla$-exchanges. So by Proposition 5.4, G and G' arise from each other by homotopic shifts, duality and ΔY-exchanges. ∎

5.4 Minimally crossing collections of curves

In Corollary 2.3 we showed:

Proposition 5.6 *Let C be an orientation-primitive closed curve on S. Then*

$$\mathrm{mincr}(C, C) = 2\,\mathrm{mincr}(C) + \mathrm{odd}(C). \tag{5.11}$$

If D_1, D_2 are closed curves with $D_1(1) = D_2(1)$, then $D_1 \cdot D_2$ is the closed curve given by:

$$(D_1 \cdot D_2)(z) := \begin{cases} D_1(z^2) & \text{if} \quad \mathrm{Im}(z) \geq 0 \\ D_2(z^2) & \text{if} \quad \mathrm{Im}(z) < 0. \end{cases} \tag{5.12}$$

We call $D_1(1) = D_2(1)$ the *concatenation point* of D_1 and D_2.

Proposition 4 of [27] immediately implies the following split-off phenomenon:

Proposition 5.7 *Let C and D be orientation-preserving closed curves on a compact surface S and let $n \in \mathbb{N}$ so that $C \sim D^n$. Then there exists an orientation-preserving homeomorphism $\phi : S^1 \to S^1$ so that the closed curve $C \circ \phi$ is equal to $\widetilde{D} \cdot E$, where $\widetilde{D} \sim D$ and $E \sim D^{n-1}$.*

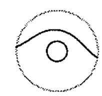
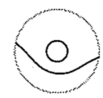
This proposition has the following consequence (see Proposition 5 in [27]).

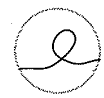
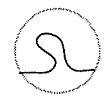
Proposition 5.8 *If C and D are orientation-preserving closed curves on the compact surface S and $n \in \mathbb{N}$ then*

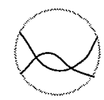
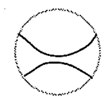
$$\mathrm{mincr}(C, D^n) = n\,\mathrm{mincr}(C, D). \tag{5.13}$$

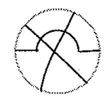
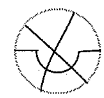
We call a system $C_1, \ldots, C_k$ of closed curves *minimally crossing* if if $\mathrm{cr}(C_i) = \mathrm{mincr}(C_i)$ and $\mathrm{cr}(C_i, C_j) = \mathrm{mincr}(C_i, C_j)$ for all i, j with $i \neq j$.

We call $C_1, \ldots, C_k$ a *regular* system of closed curves if $C_1, \ldots, C_k$ have only a finite number of (self-)intersections, each being a crossing of only two curve parts. That is, no point on S is traversed more than twice by $C_1, \ldots, C_k$ and each point of S traversed twice has a disk-neighborhood on which the curve parts are topologically two crossing straight lines. On such a system of curves we define four operations called *Reidemeister moves*, depicted below in (5.14).

replacing  by  (*type 0*);

replacing  by  (*type I*);

replacing  by  (*type II*); (5.14)

replacing  by  (*type III*).

The pictures in (5.14) represent the intersection of the union of $C_1, \ldots, C_k$ with a closed disk on S. So no other curve parts than the ones shown intersect such a disk.

In Chapter 2 we showed:

Proposition 5.9 *Any regular system of closed curves on S can be transformed to a minimally crossing system on S by a series of Reidemeister moves.*

and (Corollary 2.5):

Proposition 5.10 *Let $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_k$ be minimally crossing regular systems of orientation-primitive closed curves on the compact surface S, such that $C_1 \sim C'_1, \ldots, C_k \sim C'_k$. Then $C_1, \ldots, C_k$ can be moved to $C'_1, \ldots, C'_k$, by Reidemeister moves of type III and homotopic shifts, possibly after permuting indices.*

Let H be a 4-regular graph on a compact surface S. The *straight decomposition* of H is the decomposition of the edges of H into closed curves $C_1, \ldots, C_k$ in such a way that each edge is traversed exactly once by these curves, and that in each vertex w of H, if e_1, e_2, e_3, e_4 are the edges incident with w in cyclic order, then e_1, w, e_3 are traversed consecutively (in one way or the other), and similarly, e_2, w, e_4 are traversed consecutively (in one way or the other).

The straight decomposition is unique up to the choice of the beginning vertex of the curves, up to reversing the curves, and up to permuting the indices of $C_1, \ldots, C_k$.

Note that a $\Delta\nabla$-exchange on the graph corresponds to a Reidemeister move of type III on the corresponding curve parts of the straight decomposition.

As is shown in Chapter 3 it is not difficult to derive from Proposition 5.9:

Proposition 5.11 *Let H be a 4-regular graph on a compact surface S, with straight decomposition $C_1, \ldots, C_k$ so that the collection $C_1, \ldots, C_k$ is a minimally crossing collection of regular closed curves. Then for each closed curve D on S,*

$$\mu'_H(D) = \sum_{i=1}^{k} \text{mincr}(C_i, D). \tag{5.15}$$

Cycles in a graph G correspond in an obvious way to closed curves on S. For a cycle C in G, the number of times that C traverses edge $e \in E(G)$ is denoted by $\text{tr}_C(e)$.

5.5 Characterization of tight graphs

The following proposition has been proved in Chapter 4.

Proposition 5.12 *Let G be a 4-regular graph embedded on a compact surface S and let $C_1, \ldots, C_s$ and $\lambda_1, \ldots, \lambda_s > 0$ be a set of cycles in G and rationals so that*

$$\sum_{j=1}^{s} \lambda_j \text{tr}_{C_j}(e) \leq 1 \quad (e \in E). \tag{5.16}$$

Then

$$\sum_{i=1}^{s} \lambda_i^2 \text{mincr}(C_i) + \sum_{\substack{i,j=1 \\ i<j}}^{s} \lambda_i \lambda_j \text{mincr}(C_i, C_j) \leq |V(G)|. \tag{5.17}$$

We characterize tight graphs as follows:

Theorem 5.3 *Let $G = (V, E)$ be a 4-regular graph embedded on the compact surface S. Then G is tight if and only if the straight decomposition of G is a minimally crossing collection of orientation-primitive regular closed curves on S.*

Proof. ($\Rightarrow$) Let G be a tight graph with straight decomposition $C_1, \ldots, C_k$. First we show:

$$\textit{the system } C_1, \ldots, C_k \textit{ is minimally crossing.} \tag{5.18}$$

For suppose not. Then, by Proposition 5.9, there is a series of Reidemeister moves bringing the system $C_1, \ldots, C_k$ to $C'_1, \ldots, C'_k$ so that $\text{cr}(C'_i, C'_j) < \text{cr}(C_i, C_j)$ or $\text{cr}(C'_i) <$

$\mathrm{cr}(C_i)$ for some $i, j \in \{1, \ldots, k\}$ $(i \neq j)$ or $i \in \{1, \ldots, k\}$. This means that after some series of Reidemeister moves of type 0 and type III on $C_1, \ldots, C_k$ we obtain by Proposition 5.1 a tight graph G' where a Reidemeister move of type I or of type II can be applied on the straight decomposition of G'. This is in contradiction with the tightness of G'. In both cases the graph can be opened at a vertex without changing $\mu'_{G'}$.

Next we show:

$$\textit{each curve } C_i \ (i = 1, \ldots, k) \textit{ is orientation-primitive.} \tag{5.19}$$

To see (5.19), suppose one of the curves — C_1, say — is not orientation-primitive. So $C_1 \sim E^n$ for some orientation-preserving curve E and an integer $n \geq 2$. By Proposition 5.7 there is an orientation-preserving homeomorphism $\phi : S^1 \to S^1$ so that $C_1 \circ \phi = \widetilde{E} \cdot F$ with $\widetilde{E} \sim E$ and $F \sim E^{n-1}$. We may assume that ϕ is the identity. As C_1 traverses each edge $e \in E(G)$ at most once, the concatenation point $w := \widetilde{E}(1) = F(1)$ is a vertex of degree 4 of G. One of the two possible openings at w splits C_1 into the two curves $\widetilde{E}$ and F. Let G' denote the graph resulting from this opening. Clearly $\mu'_{G'} \leq \mu'_G$. By Proposition 5.11 we have for any closed curve D:

$$\mu'_G(D) = \sum_{i=1}^{k} \mathrm{mincr}(C_i, D). \tag{5.20}$$

Furthermore we have by Proposition 5.8:

$$\mathrm{mincr}(\tilde{E}, D) + \mathrm{mincr}(F, D) = \mathrm{mincr}(C_1, D). \tag{5.21}$$

We obtain:

$$\mu'_G(D) = \sum_{i=1}^{k} \mathrm{mincr}(C_i, D) = \mathrm{mincr}(\tilde{E}, D) + \mathrm{mincr}(F, D) + \sum_{i=2}^{k} \mathrm{mincr}(C_i, D) \leq \mu'_{G'}(D). \tag{5.22}$$

So $\mu'_G = \mu'_{G'}$, contradicting the tightness of G.

($\Leftarrow$) Let G be a 4-regular graph such that the straight decomposition $C_1, \ldots, C_k$ of G is a minimally crossing collection of orientation-primitive closed curves. So

$$|V(G)| = \sum_{i=1}^{k} \mathrm{mincr}(C_i) + \sum_{\substack{i,j=1 \\ i<j}}^{k} \mathrm{mincr}(C_i, C_j). \tag{5.23}$$

Suppose by contradiction that G is not tight and let G' arise from G by opening G at some vertex $w \in V(G)$ so that $\mu'_{G'} = \mu'_G$. Then:

$$|V(G')| = |V(G)| - 1. \tag{5.24}$$

Since for each closed curve D on S we have:

$$\text{mincr}(G', D) = \text{mincr}(G, D) \geq \sum_{i=1}^{k} \text{mincr}(C_i, D), \tag{5.25}$$

by the 'homotopic circulation theorem' in [8] (Chapter 3), there exist cycles $D_1, \ldots, D_s$ in G' together with rationals $\lambda_1, \ldots, \lambda_s > 0$ and a partition $S_1, \ldots, S_k$ of $\{1, \ldots, s\}$ such that

$$D_j \sim C_i \ \ (i = 1, \ldots, k; j \in S_i), \tag{5.26}$$

$$\sum_{j \in S_i} \lambda_j = 1 \ \ (i = 1, \ldots, k), \tag{5.27}$$

$$\sum_{j=1}^{s} \lambda_j \text{tr}_{D_j}(e) \leq 1 \ \ (e \in E). \tag{5.28}$$

By Proposition 5.6 and Proposition 5.12 we obtain:

$$2\sum_{i=1}^{k} \text{mincr}(C_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \text{mincr}(C_i, C_j) = \sum_{i,j=1}^{k} \text{mincr}(C_i, C_j) - \sum_{i=1}^{k} \text{odd}(C_i) =$$

$$\sum_{g,h=1}^{s} \lambda_g \lambda_h \text{mincr}(D_g, D_h) - \sum_{i=1}^{k} \text{odd}(C_i) =$$

$$\sum_{g=1}^{s} \lambda_g^2 \text{mincr}(D_g, D_g) + \sum_{\substack{g,h=1 \\ g \neq h}}^{s} \lambda_g \lambda_h \text{mincr}(D_g, D_h) - \sum_{i=1}^{k} \text{odd}(C_i) = \tag{5.29}$$

$$2\sum_{g=1}^{s} \lambda_g^2 \text{mincr}(D_g) + \sum_{\substack{g,h=1 \\ g \neq h}}^{s} \lambda_g \lambda_h \text{mincr}(D_g, D_h) + \sum_{g=1}^{s} \lambda_g^2 \text{odd}(D_g) - \sum_{i=1}^{k} \text{odd}(C_i) \leq$$

$$2|V(G')| + \sum_{i=1}^{k} \text{odd}(C_i)(-1 + \sum_{g \in S_i} \lambda_g^2) \leq 2|V(G)| - 2.$$

This contradicts (5.23). ∎

5.6 Proof of the Theorem

By Proposition 5.5 it suffices to prove Theorem 5.2. Let H and H' be tight 4-regular graphs with $\mu'_H = \mu'_{H'}$. By Theorem 5.3 the straight decompositions of H and H', say $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_{k'}$ respectively, are minimally crossing collections of orientation-primitive closed curves. By Proposition 5.11 and by Theorem 4.1 we have that $k = k'$ and that there exists a permutation π of $\{1, \ldots, k\}$ such that for each $i = 1, \ldots, k$ one has $C'_{\pi(i)} \sim C_i$ or $C'_{\pi(i)} \sim C_i^{-1}$. Now apply Proposition 5.10 to conclude that $C_1, \ldots, C_k$ can be moved to $C'_1, \ldots, C'_k$ by Reidemeister moves of type III and homotopic shifts (possibly after permuting indices). ∎

Chapter 6

Grid minors of graphs on the torus

We show that any graph G embedded on the torus with face-width $r \geq 5$ contains the toroidal $\lfloor \frac{2}{3}r \rfloor$-grid as a minor. (The *face-width* of G is the minimum value of $|C \cap G|$ where C ranges over all homotopically nontrivial closed curves on the torus. The *toroidal k-grid* is the product $C_k \times C_k$ of two copies of a k-circuit C_k.) For each fixed $r \geq 5$, the value $\lfloor \frac{2}{3}r \rfloor$ is largest possible.

This applies to a theorem of Robertson and Seymour showing, for each graph H embedded on any compact surface S, the existence of a number ρ_H such that every graph G embedded on S with face-width at least ρ_H contains H as a minor. Our result implies that for $H = C_k \times C_k$ embedded on the torus, $\rho_H := \lceil \frac{3}{2}k \rceil$ is the smallest possible value.

Our proof is based on deriving a result in the geometry of numbers stating that for any symmetric convex body K in $\mathbb{R}^2$ one has $\lambda_2(K) \cdot \lambda_1(K^*) \leq \frac{3}{2}$, and that this bound is smallest possible. (Here $\lambda_i(K)$ denotes the minimum value of λ such that $\lambda \cdot K$ contains i linearly independent integer vectors. K^* is the polar convex body.)

6.1 Introduction

For any graph G embedded on a surface S, the *face-width* (or *representativity*) $r(G)$ of G is the minimum of $|C \cap G|$, where C ranges over all homotopically nontrivial closed curves on S. Robertson and Seymour [24] showed:

> for each graph H embedded on a compact surface S there exists an integer ρ_H so that each graph G embedded on S with $r(G) \geq \rho_H$ contains H as a minor. (6.1)

In this chapter we determine the smallest value of ρ_H for a certain class of graphs H embedded on the torus, viz. the toroidal grids. For each $k \geq 3$, the *toroidal k-grid* is the product $C_k \times C_k$ of two k-circuits C_k. (By definition, $C_k \times C_k$ has vertices (i, j)

for $0 \le i,j \le k-1$, where (i,j) and (i',j') are adjacent if either $i = i'$ and $j = j' \pm 1 \pmod k$ or $j = j'$ and $i = i' \pm 1 \pmod k$.)

Clearly, each toroidal k-grid can be embedded on the torus. In fact, if $k \ge 4$, there is a unique embedding, up to homeomorphisms (of the torus and of the grid). (If $k \ge 5$, this follows easily from the fact that each face of the embedded graph should be a quadrangle, moreover, for $k \ge 5$ it is also a special case of a more general result of Robertson and Vitray [25]). For $k = 4$ this takes some elaboration.) For $k = 3$ the embedding is not unique as is shown by the two different embeddings of $C_3 \times C_3$ in Figure 6.1.

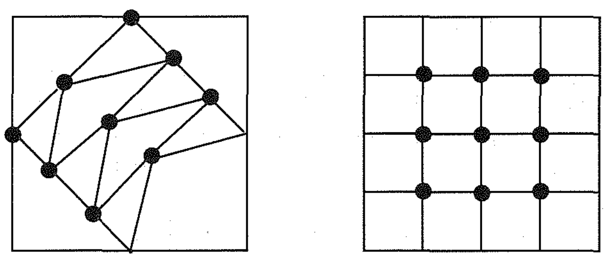


Figure 6.1: Different embeddings of $C_3 \times C_3$.

We show:

Theorem 6.1 *For the toroidal k-grid* $H = C_k \times C_k$ *embedded on the torus,* $\rho_H := \lceil \frac{3}{2}k \rceil$ *is the smallest integer value one can take for* ρ_H *in* (6.1).

We derive this from:

Theorem 6.2 *Any graph G embedded on the torus contains the toroidal* $\lfloor \frac{2}{3} r(G) \rfloor$*-grid as a minor (if* $r(G) \ge 5$*). For each integer* $r \ge 3$ *there exists a graph G embedded on the torus with* $r(G) = r$ *and not containing the toroidal* $\lfloor \frac{2}{3} r \rfloor + 1$*-grid as a minor.*

Proof of the implication Theorem 6.2 $\Rightarrow$ Theorem 6.1. Choose $k \ge 3$. Let G be a graph with $r(G) \ge \lceil \frac{3}{2}k \rceil$. Since $k = \lfloor \frac{2}{3} \lceil \frac{3}{2}k \rceil \rfloor \le \lfloor \frac{2}{3} r(G) \rfloor$, Theorem 6.2 implies that G contains the toroidal k-grid as a minor.

Let $r := \lceil \frac{3}{2}k \rceil - 1$. By Theorem 6.2 there exists a graph G on the torus with $r(G) = r$ and not containing the toroidal $\lfloor \frac{2}{3} r \rfloor + 1$-grid as a minor. Since $k = \lfloor \frac{2}{3} r \rfloor + 1$, Theorem 6.1 follows. ∎

The result derived in this chapter is related to some earlier results on vertex-disjoint circuits in graphs embedded on the torus, which we will describe here. The following theorem is concerned with the existence of vertex-disjoint circuits of a given homotopy type in a graph G embedded on the torus. Theorem 6.3 is a special case of a theorem proved by A. Schrijver in [28] for general compact surfaces. A necessary condition for the existence of k pairwise vertex-disjoint circuits of some prescribed free homotopy type in a given graph is that each closed curve D on the torus should intersect the graph 'often enough'. This is a 'cut condition' which turns out to be sufficient. To be more precise, define for any graph G embedded on the torus T and any closed curve D on T:

$$\text{cr}(G, D) = \text{ number of intersections of } G \text{ and } D. \tag{6.2}$$

Moreover, define for two closed curve C and D on T:

$$\text{cr}(C, D) = \text{ number of intersections of } C \text{ and } D; \tag{6.3}$$

$$\text{mincr}(C, D) = \min\{\text{cr}(\widetilde{C}, \widetilde{D}) | \widetilde{C} \sim C, \widetilde{D} \sim D\}. \tag{6.4}$$

(We count multiplicities.) Here $C \sim C'$ means that C is a closed curve freely homotopic to closed curve C'.

Theorem 6.3 (A. Schrijver) *Let G be an undirected graph embedded on the torus T, and let C be a simple closed curve on T. Then G contains k pairwise disjoint circuits each freely homotopic to C if and only if*

$$\text{cr}(G, D) \geq k \cdot \text{mincr}(C, D)$$

for each closed curve D on T.

This theorem was extended to directed graphs by Seymour [33] and Ding, Schrijver and Seymour [5].

Represent the torus as the product $S^1 \times S^1$ of two copies of the unit circle S^1 in the complex plane. For $(m, n)^T \in \mathbb{Z}^2$, let $C_{m,n} : S^1 \longrightarrow S^1 \times S^1$ be the closed curve on the torus given by:

$$C_{m,n}(z) := (z^m, z^n) \tag{6.5}$$

for $z \in S^1$.

Now, as is well-known (cf. [35]) the $C_{m,n}$ form a system of representatives for the free homotopy classes of closed curves on the torus. Moreover,

$$\text{mincr}(C_{m,n}, C_{m',n'}) = |mn' - m'n|. \tag{6.6}$$

Let G be a graph embedded on the torus. Define for each $(m,n)^T \in \mathbb{Z}^2$ the value $\varphi_G(m,n)$ as the minimum number of intersections of C' and G (counting multiplicities) where C' ranges over all closed curves homotopic to $C_{m,n}$. That is, $\varphi_G : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ is defined by:

$$\varphi_G(m,n) := \min_{C \sim C_{m,n}} \operatorname{cr}(C,G), \tag{6.7}$$

So $r(G)$ is equal to the minimum value of $\varphi_G(m,n)$ over all vectors $(m,n)^T \neq (0,0)^T$ in $\mathbb{Z}^2$. It is not difficult to show that the function φ_G satisfies:

$$\varphi_G(km,kn) = |k| \cdot \varphi_G(m,n) \tag{6.8}$$

and

$$\varphi_G(m+m', n+n') \leq \varphi_G(m,n) + \varphi_G(m',n'). \tag{6.9}$$

Inequality (6.9) follows from the fact that if $C \sim C_{m,n}$ and $C' \sim C_{m',n'}$, and (m,n) and (m',n') are linearly independent, then C and C' have a crossing. We can concatenate C and C' at this crossing so as to obtain a closed curve $C'' \sim C_{m+m',n+n'}$ with $\operatorname{cr}(G,C'') = \operatorname{cr}(G,C) + \operatorname{cr}(G,C')$.

Let $P(G)$ be the following set in $\mathbb{R}^2$:

$$P(G) := \{(x,y)^T \in \mathbb{R}^2 | mx + ny \leq \varphi_G(m,n) \text{ for all } (m,n)^T \in \mathbb{Z}^2\}. \tag{6.10}$$

A *polytope* is the convex hull of a finite set of vectors. A polytope P is called an *integer* polytope if each vertex of P is an integer vector. The following theorem follows directly from the 'cutting plane theorem' of Chvátal [4]. It is an extension of a result of Hoffman [12] for polytopes.

Theorem 6.4 *Let C be a nonempty compact convex set in $\mathbb{R}^n$. Then C is an integer polytope if and only if $\max\{c^Tx | x \in C\}$ is integer for each integer vector $c \in \mathbb{R}^n$.*

It is standard convexity theory (cf. [29]) that:

$$\varphi_G(m,n) = \max\{mx + ny | (x,y)^T \in P(G)\}. \tag{6.11}$$

So by Theorem 6.4 it follows that $P(G)$ is an integer polytope in $\mathbb{R}^2$. Moreover $P(G)$ is closed and convex and symmetric (i.e., $P(G) = -P(G)$). Define the *height* $\operatorname{height}(K)$ of a polygon K by:

$$\operatorname{height}(K) := \min_{(m,n)^T \in \mathbb{Z}^2, (m,n)^T \neq (0,0)^T} \max\{mx + ny | (x,y)^T \in K\}. \tag{6.12}$$

So we have:

$$r(G) = \text{height}(P(G)). \tag{6.13}$$

Now Theorem 6.3 expressed in terms of the associated polygon P_G is:

Theorem 6.5 *Let G be an undirected graph embedded on the torus. Then G contains k pairwise disjoint nontrivial circuits if and only if $\frac{1}{k}P(G)$ contains a nonzero integer vector.*

Proof. In order to understand the relation between symmetric integer polygons in $\mathbb{R}^2$ and graphs embedded on the torus it is instructive to show the proof of this result.

By Theorem 6.3 it follows that G contains k pairwise disjoint circuits each freely homotopic to $C_{m',n'}$ if and only if for each closed curve D on T one has: $\text{cr}(G, D) \geq k \cdot \text{mincr}(C_{m',n'}, D)$. As each D is freely homotopic to $C_{m,n}$ for some $(m,n)^T \in \mathbb{Z}^2$, this statement is by (6.6) and (6.7) equivalent to:

$$\text{for each } (m,n)^T \in \mathbb{Z}^2 \text{ one has: } \phi_G(m,n) \geq k \cdot |mn' - nm'|.$$

This is the case if and only if $k \cdot (n', -m')^T \in P(G)$. ∎

A. Schrijver showed in [29] the following result.

Theorem 6.6 (A. Schrijver) *Let $r \geq 1$. Then for each symmetric integer polygon K of height r, the polygon $\lfloor \frac{3}{4} r \rfloor^{-1} K$ contains a nonzero integer vector.*

Combining Theorem 6.6 with Theorem 6.5 and assertion (6.13) it follows (cf. [29]) that:

Any graph G embedded on the torus contains at least $\lfloor \frac{3}{4} r(G) \rfloor$ pairwise disjoint nontrivial circuits. (6.14)

The following extension of Theorem 6.5 is shown by A. Schrijver in [26]:

Theorem 6.7 (A. Schrijver) *Let $k \geq 3$; a graph G embedded on the torus contains a toroidal k-grid as a minor, if and only if $\frac{1}{k}P(G)$ contains two linearly independent integer vectors.*

Although Theorem 6.5 guarantees the existence of k pairwise disjoint circuits homotopic to a nontrivial curve C and k pairwise disjoint circuits homotopic to another nontrivial curve D so that C and D are not homotopic, the proof of the above result is nontrivial. The reason for this is that it is not easy to show that both sets of circuits can be appropriately contracted to give a toroidal k-grid.

Assertions (6.13) and Theorem 6.7 imply that to prove Theorem 6.2, it suffices to show:

Theorem 6.8 *Let $r \geq 3$. Then for each symmetric integer polygon K of height r, the polygon $\lfloor \frac{2}{3}r \rfloor^{-1}K$ contains two linearly independent integer vectors. Here $\lfloor \frac{2}{3}r \rfloor$ cannot be replaced by any larger integer.*

6.2 Geometry of numbers

The geometry of numbers is concerned with the study of symmetric convex bodies in $\mathbb{R}^n$. A *body* is a compact full-dimensional subset. One of the most well-known results in the geometry of numbers is the fundamental theorem of Minkowski which states that if the volume $\mathrm{vol}(K)$ of a symmetric convex body K exceeds 2^n then K contains a nonzero integer vector. Minkowski also defined the *successive minima* $\lambda_i(K)$ ($i = 1, \ldots, n$) of a convex body K:

$$\lambda_i(K) := \min\{\lambda | \lambda \cdot K \text{ contains } i \text{ linearly independent integer vectors}\}. \tag{6.15}$$

Note that the successive minima of K are invariant under unimodular transformations (linear transformations of $\mathbb{R}^n$ fixing $\mathbb{Z}^n$) of K. Let K^* denote the *polar* convex body:

$$K^* := \{y \in \mathbb{R}^2 | x^T y \leq 1 \text{ for all } x \in K\}. \tag{6.16}$$

Combining several results in the geometry of numbers one can show the following inequality:

$$1 \leq \lambda_i(K)\lambda_{n+1-i}(K^*) \leq n! \tag{6.17}$$

We show the following theorem improving this inequality for $n = 2$:

Theorem 6.9 *For each symmetric convex body K in $\mathbb{R}^2$ one has $\lambda_2(K) \cdot \lambda_1(K^*) \leq \frac{3}{2}$. The bound $\frac{3}{2}$ is smallest possible.*

Proof. (i) We may assume K is a polygon. We may also assume that each edge of K contains an integer vector in its relative interior, otherwise we can shift the edge until it contains an integer vector in its relative interior or until it 'disappears'. We show that if K contains no nonzero integer vectors in its interior then $\frac{3}{2}K^*$ contains two linearly independent integer vectors. Consider the following two cases:

I. If K has 6 edges or more, let $v_1, \ldots, v_{2k}$ be the vertices of K in cyclic order and let z_i be an integer vector in the relative interior of the edge connecting v_{i-1} and v_i ($i = 1, \ldots, 2k$ taking indices $\pmod{2k}$). As no two vectors z_i and $z_{i'}$ are equal mod 2 for $i, i' = 1, \ldots, k$ (otherwise $\frac{1}{2}(z_i + z_{i'})$ would be an integer vector in the interior of K), we may even assume $k = 3$. By Minkowski's theorem the volume of K is at most 4. Hence there exists an $i \in \{1, \ldots, 2k\}$ such that the volume $\text{vol}(Q_i)$ of the quadrangle $Q_i = 0, z_i, v_i, z_{i+1}$ is at most $\frac{4}{6}$. By the fact that the triangle $0, z_i, z_{i+1}$, which has volume $\frac{1}{2}$, is strictly contained in Q_i we find that $\text{vol}(Q_i) > \frac{1}{2}$. The symmetry around the origin of K ensures that $\text{vol}(Q_{i-1}) + \text{vol}(Q_i) + \text{vol}(Q_{i+1}) \leq 2$. As $\text{vol}(Q_i) > \frac{1}{2}$, at least one of $\text{vol}(Q_{i-1})$ or $\text{vol}(Q_{i+1})$ is strictly less than $\frac{3}{4}$. Assume $\text{vol}(Q_{i+1}) < \frac{3}{4}$. Let c_1 be the vector with $c_1^T z_i = c_1^T z_{i+1} = 1$. As the triangle $0, z_i, z_{i+1}$ has volume $\frac{1}{2}$, c_1 is an integer vector. Now:

$$\max\{c_1^T x | x \in K\} = c_1^T v_i = 2\,\text{vol}(Q_i) \leq 2 \cdot \frac{4}{6} = \frac{4}{3}.$$

Let c_2 be the vector with $c_2^T z_{i+1} = c_2^T z_{i+2} = 1$. Also c_2 is an integer vector and $\{c_1, c_2\}$ is a linearly independent set of vectors. Moreover:

$$\max\{c_2^T x | x \in K\} = c_2^T v_{i+1} = 2\,\text{vol}(Q_{i+1}) < 2 \cdot \frac{3}{4} = \frac{3}{2}.$$

So both c_1 and c_2 are elements of $\frac{3}{2}K^*$.

II. If K has 4 edges, we may assume by applying unimodular transformations that the vectors in the relative interior of the edges are $\pm(1,0)$ and $\pm(0,1)$. Now K is defined by the following inequalities:

$$\begin{aligned} |\alpha x_1 + x_2| &\leq 1 \\ |x_1 - \beta x_2| &\leq 1 \\ |\alpha| \leq 1,\ |\beta| &\leq 1. \end{aligned}$$

The polygon K^* is:

$$K^* = \text{convex hull}\{\begin{pmatrix} \alpha \\ 1 \end{pmatrix}, \begin{pmatrix} -\alpha \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\beta \end{pmatrix}, \begin{pmatrix} -1 \\ \beta \end{pmatrix}\}. \tag{6.18}$$

We obtain:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{1+\alpha\beta}\begin{pmatrix} \alpha \\ 1 \end{pmatrix} + \frac{\alpha}{1+\alpha\beta}\begin{pmatrix} -1 \\ \beta \end{pmatrix} \in \frac{1+\alpha}{1+\alpha\beta}K^* \tag{6.19}$$

and

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\beta}{1+\alpha\beta}\begin{pmatrix} \alpha \\ 1 \end{pmatrix} + \frac{1}{1+\alpha\beta}\begin{pmatrix} 1 \\ -\beta \end{pmatrix} \in \frac{1+\beta}{1+\alpha\beta}K^* \tag{6.20}$$

and

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1-\beta}{1+\alpha\beta}\begin{pmatrix} -\alpha \\ -1 \end{pmatrix} + \frac{1+\alpha}{1+\alpha\beta}\begin{pmatrix} 1 \\ -\beta \end{pmatrix} \in \frac{2+\alpha-\beta}{1+\alpha\beta}K^*. \tag{6.21}$$

In view of the symmetry, we may assume that

$$\alpha \geq 0 \text{ and } \beta \geq \alpha. \tag{6.22}$$

Using the fact that for each $x \in \mathbb{R}$ one has $1 + x < \frac{3}{2}(1 + x^2)$, we obtain: $\frac{1+\alpha}{1+\alpha\beta} \leq \frac{1+\alpha}{1+\alpha^2} < \frac{3}{2}$. Hence

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \frac{3}{2} \cdot K^*. \tag{6.23}$$

If $\beta \leq \frac{1}{3}$ then $\frac{1+\beta}{1+\alpha\beta} \leq \frac{1+1/3}{1} < \frac{3}{2}$, so if $\beta \leq \frac{1}{3}$ then

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \frac{3}{2} \cdot K^*. \tag{6.24}$$

Finally, if $\beta > \frac{1}{3}$ then $3\alpha\beta \geq \alpha$ so

$$\frac{1+\beta}{1+\alpha\beta} + \frac{2+\alpha-\beta}{1+\alpha\beta} = \frac{3+\alpha}{1+\alpha\beta} \leq \frac{3+3\alpha\beta}{1+\alpha\beta} \leq 3, \tag{6.25}$$

implying that one of the vectors $(1,0)^T$ and $(1,-1)^T$ belongs to $\frac{3}{2}K^*$. So $\frac{3}{2}K^*$ contains 2 linearly independent integer vectors.

(ii) The body K defined as:

$$K = \{(x_1, x_2) \in \mathbf{R}^2 | -1 \leq x_1 - \frac{1}{2}x_2 \leq 1; -1 \leq x_2 \leq 1\}$$

with dual

$$K^* = \{(x_1, x_2) \in \mathbf{R}^2 | -1 \leq 1\frac{1}{2}x_1 + x_2 \leq 1; -1 \leq -\frac{1}{2}x_1 + x_2 \leq 1\}$$

is (upon unimodular transformations) the unique body K with $\lambda_1(K)\lambda_2(K^*) = 1\frac{1}{2}$. ∎

6.3 Proof of Theorem 6.2

We present a proof of Theorem 6.2. To see the first part of the theorem note that height$(K) = \lambda_1(K^*)$. So for any convex symmetric integer polygon K of height r we have that:

$$\lambda_2(K) \leq \frac{3}{2r} \leq \frac{1}{\lfloor \frac{2r}{3} \rfloor}, \tag{6.26}$$

which implies that the polygon $\lfloor \frac{2}{3}r \rfloor^{-1}K$ contains two linearly independent integer vectors.

To see that we cannot replace $\lfloor \frac{2}{3}r \rfloor$ by any larger integer, choose integers $r \geq 3$ and $k > 2r/3$. Let Q be the convex hull of $\pm(r,0)^T$ and $\pm(-\lfloor r/2 \rfloor, r)$. We show that if $(x,y)^T \in k^{-1}Q$ and $(x,y)^T \in \mathbb{Z}^2$ then $y = 0$. This implies that $k^{-1}Q$ does not contain two linearly independent integer vectors.

Suppose to the contrary that $y \neq 0$. We may assume that $y \geq 1$. Since $(x,y)^T \in k^{-1}Q$, there exists λ and μ such that

$$k\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} r \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\lfloor r/2 \rfloor \\ r \end{pmatrix}, \tag{6.27}$$

and such that $|\lambda| + |\mu| \leq 1$. Now $y = \mu r/k < 3\mu/2 \leq 3/2$, and hence $y = 1$ and $\mu = k/r$. Using the facts that $\lambda r = kx + \mu \lfloor \frac{r}{2} \rfloor$ (by (6.27)), $|x + \frac{1}{r}\lfloor \frac{r}{2} \rfloor| \geq \frac{1}{r}\lfloor \frac{r}{2} \rfloor$ (as x is an integer and as $\frac{1}{r}\lfloor \frac{r}{2} \rfloor \leq \frac{1}{2}$), $\lfloor \frac{r}{2} \rfloor \geq \frac{r}{2} - \frac{1}{2}$ and $k \geq \frac{2}{3}r + \frac{1}{3}$, we obtain

$$\begin{aligned} r = k + (1-\mu)r \geq k + |\lambda r| = k + |kx + \mu \lfloor \tfrac{r}{2} \rfloor| = k + k|x + \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor| \\ \geq k + k\tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor \geq k(1 + \tfrac{1}{r}(\tfrac{r}{2} - \tfrac{1}{2})) \geq (\tfrac{2}{3}r + \tfrac{1}{3})(\tfrac{3}{2} - \tfrac{1}{2r}) = r + \tfrac{1}{6} - \tfrac{1}{6r} > r, \end{aligned} \tag{6.28}$$

a contradiction. ∎

6.4 r-minimal integer polygons

In fact, we can show slightly stronger bounds for $\lambda_1(K)$ and $\lambda_2(K)$ if we know that K is an integer polygon with height at least r. To that aim we need the classification of *r-minimal* symmetric integer polygons, given in [26].

Call a symmetric integer polygon K *r-minimal*, if height$(K) \geq r$ while height$(K') < r$ for each symmetric integer polygon $K' \neq K$ contained in K.

A. Schrijver showed in [26] that each r-minimal polygon is a quadrangle or a hexagon, where quadrangles arise as follows. Choose integer values $0 \leq \alpha < r$ and $0 \leq \beta < r$. Let $Q_{\alpha,\beta}$ be the convex hull of the points $\pm(r,\alpha)^T, \pm(-\beta,r)^T$. Then $Q_{\alpha,\beta}$ is r-minimal, and all symmetric r-minimal integer polygons that are quadrangles arise in this way, up to unimodular transformations.

The hexagons arise as follows. Choose integer values $0 < \alpha < r$, $0 < \beta < r$ and $0 < \gamma < r$. Let $H_{\alpha,\beta,\gamma}$ be the convex hull of the points $\pm(r,\alpha)^T, \pm(r-\beta,r)^T, \pm(-\gamma,r-\gamma)^T$. Again, $H_{\alpha,\beta,\gamma}$ is r-minimal, and all symmetric r-minimal integer polygons that are hexagons arise in this way, up to unimodular transformations.

Theorem 6.10 *Let Q be an r-minimal quadrangle then $\lambda_1(Q) < \frac{1+\sqrt{2}}{2r}$ and*

$$\lambda_2(Q) \leq \frac{3r+1}{2r^2+r+1} \text{ if } r \text{ is odd}$$

$$\lambda_2(Q) \leq \frac{3}{2r} \text{ if } r \text{ is even}$$

both bounds for $\lambda_2(Q)$ are smallest possible for fixed r.

Proof. According to the classification of r-minimal polygons we may assume that Q is of the form:

$$Q = Q_{\alpha,\beta} = \text{convex hull of } \pm\begin{bmatrix} r \\ \alpha \end{bmatrix} \text{ and } \pm\begin{bmatrix} -\beta \\ r \end{bmatrix}$$

with $\alpha, \beta \in \{0, \cdots, r-1\}$.
The inequalities determining $Q_{\alpha,\beta}$ are:

$$\begin{array}{c} -1 \leq \frac{r-\alpha}{r^2+\alpha\beta}x_1 + \frac{r+\beta}{r^2+\alpha\beta}x_2 \leq 1 \\ -1 \leq \frac{r+\alpha}{r^2+\alpha\beta}x_1 + \frac{\beta-r}{r^2+\alpha\beta}x_2 \leq 1. \end{array} \tag{6.29}$$

There is a unimodular transformation $U : \mathbb{R}^2 \to \mathbb{R}^2$ so that $UQ_{\alpha',\beta'} = Q_{\alpha,\beta}$ if and only if $\{\alpha',\beta'\} = \{\alpha,\beta\}$ or $\{\alpha,\beta\} = \{0,\gamma\}$ and $\{\alpha',\beta'\} = \{0, r-\gamma\}$ for some $\gamma \in \{1, \cdots, r-1\}$. For each vector (x_1,x_2) let the norm $||(x_1,x_2)||$ be the minimal positive value of λ for which $(x_1,x_2)^T \in \lambda \cdot Q_{\alpha,\beta}$. Suppose r is odd. Without loss of generality we are in one of the following two cases:

- **Case I** $\alpha = 0, \beta \in \{0, \cdots, \frac{r-1}{2}\}$.
  The norm $||(x_1,x_2)||$ can be calculated from (6.29). It follows that the two shortest integer vectors are: $(1,0)^T$ and $(0,1)^T$ where $(0,1)^T$ is the longer of the two. We have:
  $$||(0,1)|| = \frac{r+\beta}{r^2} \leq \frac{3r-1}{2r^2},$$
  showing that $\frac{3r-1}{2r^2}Q_{0,\beta}$ contains two linear independent integer vectors.

- **Case II** $\alpha, \beta \in \{1, \cdots, r-1\}$.
We may assume that $\alpha \leq \beta$. The three shortest integer vectors are:

$$(1,0)^T \text{ with norm } \|(1,0)\| = \frac{r+\alpha}{r^2+\alpha\beta},$$

$$(0,1)^T \text{ with norm } \|(0,1)\| = \frac{r+\beta}{r^2+\alpha\beta},$$

$$(1,-1)^T \text{ with norm } \|(1,-1)\| = \frac{2r+\alpha-\beta}{r^2+\alpha\beta}$$

and their opposites. One has: $\|(1,0)\| \leq \|(0,1)\|$ and $\|(1,0)\| \leq \|(1,-1)\|$. Suppose $2\beta \geq r+\alpha$ then $\|(1,-1)\| \leq \|(0,1)\|$ and we have:

$$\|(1,-1)\| = \frac{4r+2\alpha-2\beta}{2r^2+2\alpha\beta} \leq \frac{3r+\alpha}{2r^2+\alpha r+\alpha^2} \leq \frac{3r+1}{2r^2+r+1}.$$

If $2\beta \leq r+\alpha$ then $\|(0,1)\| \leq \|(1,-1)\|$ and:

$$\|(0,1)\| = \frac{2r+2\beta}{2r^2+2\alpha\beta} \leq \frac{3r+\alpha}{2r^2+\alpha r+\alpha^2} \leq \frac{3r+1}{2r^2+r+1}.$$

As $\frac{3r+1}{2r^2+r+1} \geq \frac{3r-1}{2r^2}$, we find that for every odd r and every $\alpha, \beta \in \{0, \cdots r-1\}$ the quadrangle $\frac{3r+1}{2r^2+r+1} Q_{\alpha,\beta}$ contains two linear independent integer vectors. No better bound for $\lambda_2(Q)$ is possible since: $\lambda \cdot Q_{1,\frac{r+1}{2}}$ does not contain two linear independent integer vectors for $\lambda < \frac{3r+1}{2r^2+r+1}$.

Next, suppose that r is even. Consider the body $\frac{3}{2r} \cdot Q_{0,\frac{r}{2}}$:

$$\{(x_1, x_2) | -1 \leq \frac{2}{3}x_1 + x_2 \leq 1; \ -1 \leq \frac{2}{3}x_1 - \frac{1}{3}x_2 \leq 1\}.$$

This body contains the integer vectors $(1,0)^T; (0,1)^T$ and $(1,-1)^T$ and their opposites but $\lambda \cdot Q_{0,\frac{r}{2}}$ does not contain two linear independent integer vectors for $\lambda < \frac{3}{2r}$.

Finally, we show the bound for $\lambda_1(Q)$. Assuming $\alpha \leq \beta$ as in Case II, we obtain:

$$\|(1,0)\| = \frac{r+\alpha}{r^2+\alpha\beta} \leq \frac{r+\alpha}{r^2+\alpha^2} < \frac{1+\sqrt{2}}{2r}. \tag{6.30}$$

The last inequality follows from the fact that $(1+x) \leq \frac{1+\sqrt{2}}{2}(1+x^2)$ with equality only if $x = \sqrt{2}-1$. ∎

Theorem 6.11 *Let H be an r-minimal hexagon then*

$$\lambda_1(Q) \leq \frac{4r}{3r^2+1} \text{ and } \lambda_2(Q) \leq \frac{3}{2r+1} \text{ if } r \text{ is odd}$$

$$\lambda_1(Q) \leq \frac{4}{3r} \text{ and } \lambda_2(Q) \leq \frac{3r-2}{2r^2-r} \text{ if } r \text{ is even}$$

and all bounds are smallest possible.

Proof. According to the classification of r-mininal polygons we may assume that H is of the form:

$$H = H_{\alpha,\beta,\gamma} = \text{ convex hull of } \pm \begin{bmatrix} r \\ \alpha \end{bmatrix} \pm \begin{bmatrix} r-\beta \\ r \end{bmatrix} \pm \begin{bmatrix} -\gamma \\ r-\gamma \end{bmatrix}$$

with $\alpha, \beta, \gamma \in \{1, \cdots, r-1\}$.
The inequalities determining $H_{\alpha,\beta,\gamma}$ are:

$$\begin{array}{l} -1 \leq \frac{r-\alpha}{r^2-\alpha r+\beta\alpha} x_1 + \frac{\beta}{r^2-\alpha r+\beta\alpha} x_2 \leq 1 \\ -1 \leq \frac{\gamma}{r^2-\beta r+\gamma\beta} x_1 + \frac{\beta-\gamma-r}{r^2-\beta r+\gamma\beta} x_2 \leq 1 \\ -1 \leq \frac{\gamma-r-\alpha}{r^2-\gamma r+\alpha\gamma} x_1 + \frac{r-\gamma}{r^2-\gamma r+\alpha\gamma} x_2 \leq 1. \end{array} \tag{6.31}$$

For each vector (x_1, x_2) let the norm $||(x_1, x_2)||$ be the minimum positive value of λ such that $(x_1, x_2)^T \in \lambda \cdot H_{\alpha,\beta,\gamma}$. The norm $||(x_1, x_2)||$ can be calculated from (6.31). It follows that the three shortest integer vectors are:

$$(1,0)^T \text{ with norm: } ||(1,0)|| = \frac{r+\alpha-\gamma}{r^2-\gamma r+\alpha\gamma},$$

$$(0,1)^T \text{ with norm: } ||(0,1)|| = \frac{r+\gamma-\beta}{r^2-\beta r+\gamma\beta},$$

$$(1,1)^T \text{ with norm: } ||(1,1)|| = \frac{r+\beta-\alpha}{r^2-\alpha r+\beta\alpha},$$

and their opposites. By symmetry we may assume that $\alpha \leq \gamma$. Then

$$||(1,0)|| = \frac{r+\alpha-\gamma}{r^2-\gamma r+\alpha\gamma} \leq \frac{r}{r^2-\gamma r+\gamma^2} = \frac{r}{(\gamma-\frac{r}{2})^2+\frac{3}{4}r^2}. \tag{6.32}$$

Inequality (6.32) immediately implies the bounds for $\lambda_1(Q)$ given in the theorem. (If r is odd, then $|\gamma - \frac{r}{2}| \geq 1$.) Moreover, for each r the bound for $\lambda_1(Q)$ is smallest possible. To see this, consider the body $H := H_{\lfloor\frac{r}{2}\rfloor,\lfloor\frac{r}{2}\rfloor,\lfloor\frac{r}{2}\rfloor}$, with $\lambda_1(H) = \frac{4}{3r}$ if r is even and $\lambda_1(H) = \frac{4r}{3r^2+1}$ if r is odd.

In order to obtain the bounds for $\lambda_2(Q)$ note that there is a unimodular transformation $U : \mathbb{R}^2 \to \mathbb{R}^2$ so that $UH_{\alpha',\beta',\gamma'} = H_{\alpha,\beta,\gamma}$ if and only if (α',β',γ') is a cyclic permutation of (α,β,γ) or of $(r-\gamma, r-\beta, r-\alpha)$. From this it follows that we may assume without loss of generality that $\alpha,\beta \in \{1,\cdots,\lfloor\frac{r}{2}\rfloor\}$ and $\gamma \in \{1,\cdots,r-1\}$. As

$$\max_{1\leq x_1\leq r-1, 1\leq x_2\leq\lfloor\frac{r}{2}\rfloor} \frac{r+x_2-x_1}{r^2-x_1r+x_2x_1} = \frac{r+\lfloor\frac{r}{2}\rfloor-1}{r^2-r+\lfloor\frac{r}{2}\rfloor}, \tag{6.33}$$

we obtain that $\|(1,0)\|$ and $\|(1,1)\|$ are less than or equal to:

$$\frac{r+\lfloor\frac{r}{2}\rfloor-1}{r^2-r+\lfloor\frac{r}{2}\rfloor}$$

as required. The body:

$$H_{1,\lfloor\frac{r}{2}\rfloor,r-1}$$

shows that this bound cannot be improved because for odd r:

$$\|(1,0)\| = \frac{2}{2r-1},$$

$$\|(0,1)\| = \frac{3r-1}{2r^2-r+1} \geq \frac{3}{2r+1} = \|(1,1)\|$$

and for even r:

$$\|(1,0)\| = \frac{2}{2r-1},$$

$$\|(0,1)\| = \|(1,1)\| = \frac{3r-2}{2r^2-r},$$

showing that for $\lambda < \frac{r+\lfloor\frac{r}{2}\rfloor-1}{r^2-r+\lfloor\frac{r}{2}\rfloor}$ the hexagon $\lambda \cdot H_{1,\lfloor\frac{r}{2}\rfloor,r-1}$ does not contain two linear independent integer vectors. ∎

Combining Theorem 6.10 and Theorem 6.11 we obtain:

Let Q be an integer polygon with height$(Q) \geq r$. Then

$$\begin{array}{llllll} \lambda_1(Q) \leq & \frac{4r}{3r^2+1} & \text{and} & \lambda_2(Q) \leq & \frac{3r+1}{2r^2+r+1} & \text{if } r \text{ is odd} \\ \lambda_1(Q) \leq & \frac{4}{3r} & \text{and} & \lambda_2(Q) \leq & \frac{3}{2r} & \text{if } r \text{ is even} \end{array} \tag{6.34}$$

and all bounds are smallest possible.

Bibliography

[1] R. Baer, Kurventypen auf Flächen, *J. Reine Angew. Math.* **156** (1927), 231-246.

[2] S.S. Cairns, *Introductory Topology*, The Ronald Press Company, New York, 1961.

[3] J.W.S. Cassels, *An introduction to the geometry of numbers*, Springer-Verlag, Berlin, 1959.

[4] V. Chvátal, Edmonds polytopes and a hierarchy of combinatorial problems, *Discrete Math.* 4 (1973), 305-337.

[5] G. Ding, A. Schrijver, and P.D Seymour, Disjoint cycles in directed graphs on the torus and the Klein bottle, *J. Combin. Theory Ser. B*, **58** (1993), 40-45.

[6] A. Frank and A. Schrijver, Edge-disjoint circuits in graphs on the torus, *J. Combin. Theory Ser. B* **55** (1992), 9-17.

[7] P.J. Giblin, *Graphs, surfaces and homology*, Chapman and Hall, London, 1977.

[8] M. de Graaf and A. Schrijver, *Decompositions of graphs on surfaces and a homotopic circulation theorem*, preprint, 1994.

[9] M. de Graaf and A. Schrijver, Grid minors of graphs on the torus, *J. Combin. Theory. Ser. B*, to appear.

[10] M. de Graaf and A. Schrijver, *Homotopy and crossings of systems of curves on a compact surface*, preprint, 1993.

[11] M. de Graaf and A. Schrijver, *Making curve systems minimally crossing by Reidemeister moves*, preprint, 1994.

[12] A.J. Hoffman, A generalization of max flow-min cut, *Math. Programming.* **6** (1974), 352-359.

[13] L.H. Kauffman, *On knots*, Annals of Mathematics Studies (number 115), Princeton University Press, Princeton, 1987.

[14] P. Koebe, Riemannsche Mannigfaltigkeiten und nichteuklidische Raumformen (Vierte Mitteilung), *S. B. Deutsch. Akad. Wiss. Berlin Kl. Math. Phys.* (1929), 414-457.

[15] S. Lins, A minimax theorem on circuits in projective graphs, *J. Combin. Theory Ser. B* **30** (1981), 253-262.

[16] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.* **10** (1927), 96-115.

[17] H. Poincaré, Analysis situs, *Journal de l'École Polytechnique* **t.1**, 1-121 (1895). (Reprinted in: *Oeuvres de Henri Poincaré*, Tome VI, Gauthiers-Villars, Paris, 1953, 193-288.)

[18] H. Poincaré, Complément a l'Analysis situs, *Rendiconti del Circolo Matematico di Palermo* **t.13**, 285-343 (1899). (Reprinted in: *Oeuvres de Henri Poincaré*, Tome VI, Gauthiers-Villars, Paris, 1953, 290-337.)

[19] S. Randby, to appear.

[20] K. Reidemeister, Elementäre Begründung der Knotentheorie, *Abh. math. Semin. Hamburg. Univ.* **Bd. 5** (1926), 24-32.

[21] B. L. Reinhart, Algorithms for Jordan curves on compact surfaces, *Ann. Math.* **75** (1962) (2), 209-222.

[22] G. Ringel, Teilungen der Ebene durch Geraden oder topologische Geraden, *Math. Z.* **64** (1955), 79-102.

[23] N. Robertson and P.D. Seymour, Graph Minors. I - XVI, (to be) published in *J. Combin. Theory Ser. B.*

[24] N. Robertson and P.D. Seymour, Graph Minors. VII. Disjoint paths on a surface. *J. Combin. Theory, Ser. B* **45** (1988), 212-254.

[25] N. Robertson and R. Vitray, Representativity of surface embeddings, in: B. Korte, L. Lovász, H.J. Prömel, A. Schrijver (Eds.), *Paths, flows, and VLSI-layout*, Algorithms and Combinatorics **9**, Springer-Verlag, New York, 1991, 293-328.

[26] A. Schrijver, Classification of minimal graphs of given face-width on the torus, *J. Combin. Theory Ser. B*, to appear.

[27] A. Schrijver, Decompositions of graphs on surfaces and a homotopic circulation theorem, *J. Combin. Theory Ser. B* **51** (1991), 161-210.

[28] A. Schrijver, Disjoint circuits of prescribed homotopies in a graph on a compact surface, *J. Combin. Theory. Ser. B* **51** (1991), 127-159.

[29] A. Schrijver, Graphs on the torus and geometry of numbers, *J. Combin. Theory Ser. B.*, **58** (1993), 147-158.

[30] A. Schrijver, Homotopy and crossings of systems of curves on a surface, *Linear Algebra Appl.* **114/115** (1989), 157-167.

[31] A. Schrijver, On the uniqueness of kernels, *J. Combin. Theory Ser. B* **55** (1992), 146-160.

[32] A. Schrijver, *Theory of linear and integer programming*, Wiley, Chichester, 1986.

[33] P.D. Seymour, Directed circuits on the torus, *Combinatorica* **11** (1991), 261-273.

[34] E. Steinitz, Polyeder und Raumeinteilungen, in: W.F. Meyer and H. Mohrmann (Eds.), *Encyclopädie der mathematischen Wissenschaften, mit Einschluss ihrer Anwendungen*, Dritter Band (Geometrie), Erster Teil, Zweite Hälfte, IIAB12, Teubner, Leipzig, 1922, 1 - 139.

[35] J. Stillwell, *Classical topology and combinatorial group theory*, Springer-Verlag, New York, 1980.

[36] J. Stillwell, *Geometry of Surfaces*, Springer-Verlag, New York, 1992.

[37] P.G. Tait, On links, *Proc. Roy. Soc. Edinburgh* **9** (1875-1878), 321-332.

[38] K. Wagner, Zum Basisproblem für Graphenmengen, insbesondere für die nicht in die projektive Ebene einbettbaren Graphen, *J. Combin. Theory* **2** (1967), 168-185.

Grafen en krommen op oppervlakken

— samenvatting —

Dit proefschrift is gebaseerd op onderzoek naar circuits in grafen en naar collecties van gesloten krommen op een oppervlak. Het blijkt dat resultaten aangaande collecties van gesloten krommen eigenschappen impliceren voor circuits in grafen op oppervlakken, en omgekeerd.

Hoofdstuk 1 geeft een overzicht van de belangrijkste resultaten van dit proefschrift, legt het verband tussen een stelling uit dit proefschrift met een stelling van H. Poincaré en geeft enkele toepassingen van dit type onderzoek.

Hoofdstuk 2 richt zich op collecties van gesloten krommen. We beschouwen collecties van gesloten krommen op een oppervlak met de eigenschap dat geen enkel drietal krommen een punt gemeenschappelijk heeft. We tonen aan dat dergelijke collecties 'minimaal kruisend' gemaakt kunnen worden door middel van operaties die verwant zijn met de door K. Reidemeister geïntroduceerde operaties in de studie van knopen. Van belang hierbij is dat in elke stap noch het aantal doorsnijdingen van een kromme met zichzelf, noch het aantal doorsnijdingen tussen twee krommen toeneemt. Een verzameling van krommen heet hier *minimaal kruisend* als het niet mogelijk is door middel van continue deformaties van de krommen het aantal doorsnijdingen van een kromme met zichzelf of het aantal doorsnijdingen van een tweetal krommen te verminderen.

In Hoofdstuk 3 wordt een stelling afgeleid over de decompositie van grafen die zijn ingebed op een compact oppervlak en waarvan ieder punt een even graad heeft. Hierbij maken we gebruik van de resultaten uit Hoofdstuk 2. Voor het projectieve vlak was deze stelling reeds verkregen door S. Lins en voor compacte oriënteerbare oppervlakken door A. Schrijver. Voor deze gevallen geeft onze methode een eenvoudiger bewijs, dat tevens geldt voor alle compacte oppervlakken.

De decompositiestelling impliceert een stelling aangaande circulaties in een graaf. Deze 'homotope circulatiestelling' geeft noodzakelijke en voldoende voorwaarden voor het bestaan van circulaties in een graaf op een compact oppervlak, waarbij iedere circulatie vrij homotoop is met een gegeven gesloten kromme en iedere kant capaciteit 1 heeft.

In Hoofdstuk 4 wordt de homotope circulatiestelling gebruikt voor het bestuderen van collecties van gesloten krommen. We beschouwen hier twee collecties en geven

noodzakelijke en voldoende voorwaarden voor het gelijk zijn van beide verzamelingen, eventueel op de doorlooprichting van de krommen en homotopie na. Deze stelling is verwant met een stelling van H. Poincaré die de homologie-klasse van een collectie gesloten krommen karakteriseert.

In Hoofdstuk 5 wordt uit de equivalentie van collecties van gesloten krommen een stelling afgeleid over de uniciteit van 'kernen'. Een *kern* is een graaf ingebed op een compact oppervlak waarbij elke verwijdering of contractie van een kant resulteert in een graaf waarvoor er een gesloten kromme bestaat die minder doorsnijdingen heeft met deze graaf dan met de oorspronkelijke. We beschouwen twee kernen G en G' ingebed op een compact oppervlak zodanig dat voor elke homotopieklasse beide grafen hetzelfde minimale aantal doorsnijdingen met krommen uit deze homotopieklasse hebben. We tonen aan dat G' uit G verkregen kan worden door herhaaldelijke toepassing van een drietal operaties. (Hierbij wordt aangenomen dat ieder facet van G topologisch gezien een open cirkelschijf is.) Voor het projectieve vlak was deze stelling reeds bewezen door S. Randby, voor oriënteerbare oppervlakken door A. Schrijver.

Hoofdstuk 6 bestudeert grafen ingebed op de torus en heeft betrekking op een stelling van N. Robertson en P.D. Seymour. Deze stelling zegt dat voor iedere graaf H ingebed op een compact oppervlak S er een getal ρ_H bestaat, zodat iedere graaf G die is ingebed op S en die met elke niet-nulhomotope gesloten kromme tenminste ρ_H doorsnijdingen heeft, de graaf H als minor bevat. We laten zien dat indien G is ingebed op de torus en H gelijk is aan het product van twee circuits op k punten, de waarde $\lceil \frac{3}{2}k \rceil$ de kleinst mogelijke waarde voor ρ_H is. Dit wordt bewezen met behulp van de meetkunde der getallen.

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