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MINIMAL POLYNOMIALS OF THE IMAGES OF THE UNIPOTENT ELEMENTS OF NON-PRIME ORDER IN THE IRREDUCIBLE REPRESENTATIONS OF AN ALGEBRAIC GROUP OF TYPE F_4

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Abstract. The minimal polynomials of the images of the unipotent elements of non-prime order in the irreducible representations of an algebraic group of type F_4 in characteristics 3 and 7 are found. This completes the solution of the minimal polynomial problem for unipotent elements in the irreducible representations of such a group in an odd characteristic.

Keywords: an algebraic group of type F_4, unipotent elements, irreducible representations

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МИНИМАЛЬНЫЕ МНОГОЧЛЕНЫ ОБРАЗОВ УНИПОТЕНТНЫХ ЭЛЕМЕНТОВ НЕПРОСТОГО ПОРЯДКА В НЕПРИВОДИМЫХ ПРЕДСТАВЛЕНИЯХ АЛГЕБРАИЧЕСКОЙ ГРУППЫ ТИПА F_4

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Аннотация. Найдены минимальные многочлены образов унипотентных элементов непростого порядка в неприводимых представлениях алгебраической группы типа F_4 в характеристиках 3 и 7. Это завершает решение задачи о минимальных многочленах унипотентных элементов в неприводимых представлениях такой группы в нечетной характеристике.

Ключевые слова: алгебраическая группа типа F_4, унипотентные элементы, неприводимые представления

Introduction. The investigation of the minimal polynomial problem for the images of unipotent elements in irreducible representations of the simple algebraic groups is continued. In this paper the problem is solved for unipotent elements of non-prime order and irreducible representations of an algebraic group of type F_4 in characteristics 3 and 7. Earlier this problem has been solved for unipotent elements of prime order and all simple algebraic groups [1], for unipotent elements of non-prime order and irreducible representations of the classical algebraic groups in an odd characteristic [2], and for such elements in exceptional groups in certain characteristics [3]. In [3] the following cases are settled: the groups of types E_6 in characteristic at least 5, the groups of type E_7 in characteristics 5, 7, and 17, the groups of type E_8 in characteristics 7 and 29, the groups of type F_4 in characteristics 5 and 11, and the groups of type G_2 in all characteristics. As the groups of type F_4 have unipotent elements of non-prime order only in characteristics at most 11, now for these groups the minimal polynomial problem is completely solved in all odd characteristics.

The minimal polynomials of the images of individual elements in representations yield important invariants of these representations useful for solving problems on recognizing representations and linear groups by the presence of particular matrices. Results on these polynomials proven for irreducible representations of algebraic groups can be immediately transferred to absolutely irreducible representations of finite Chevalley groups in the defining characteristic what increases the range of their potential

applications. Therefore such results can be regarded as a contribution to the programme of extending the fundamental results of Hall and Higman [4] on the minimal polynomials of p-elements in finite irreducible p-solvable linear groups in characteristic p to groups that are not p-solvable. In [2] one can find a short discussion of some results on the minimal polynomial problem for irreducible representations of finite groups close to simple.

The main part. In what follows $\mathbb{C}$ is the complex field, K is an algebraically closed field of an odd characteristic p, $\mathbb{Z}$ and $\mathbb{Z}^+$ are the sets of integers and nonnegative integers, respectively, $G = F_4(K)$, $G_{\mathbb{C}} = F_4(\mathbb{C})$, ω_i, $1 \leq i \leq 4$, are the fundamental weights of G, $\omega(\varphi)$ is the highest weight of a representation φ. For an element x and a representation ρ of some algebraic group, the symbol $d_\rho(x)$ denotes the degree of the minimal polynomial of $\rho(x)$; $|x|$ is the order of x; $\langle \mu, \alpha \rangle$ is the value of a weight μ on a root α (the canonical pairing in the sense of [5, Section 1]). If φ is an irreducible representation of G, then $\varphi_{\mathbb{C}}$ is the irreducible representation of $G_{\mathbb{C}}$ with highest weight $\omega(\varphi)$. There exists a canonical bijection f from the set of unipotent conjugacy classes of G onto the analogical set for $G_{\mathbb{C}}$ determined with the help of the distinguished parabolic subgroups in the Levi subgroups of G (see, for instance, comments in the Introduction of [6]). In what follows if $x \in G$ is a unipotent element from a class C, then $x_{\mathbb{C}} \in f(C) \subset G_{\mathbb{C}}$. Recall that an irreducible representation of a semisimple algebraic group over K is p-restricted if all coefficients of its highest weight are less than p.

It is clear that the minimal polynomial of the image of a unipotent element in a rational representation of an algebraic group has the form $(t-1)^d$ and hence is completely determined by its degree. It is well known that the maximal order of a unipotent element in G is equal to 27 for $p = 3$ and to 49 for $p = 7$; if $p = 3$, only regular unipotent elements have order 27, other unipotent elements have smaller orders and are conjugate to elements from proper subsystem subgroups of G whose simple components are classical groups; for $p = 7$, the group G has two conjugacy classes of elements of order 49: regular unipotent elements and the class containing regular unipotent elements of a subsystem subgroup of type B_4 (see, for instance, [6]).

In what follows φ is a nontrivial irreducible representation of G with highest weight ω and M is a module affording φ.

T h e o r e m 1. *Let $p = 3$, $x \in G$, and $|x| = 9$. Then $d_\varphi(x) = 9$ or one of the following holds:*

1) $\omega = 3^j\omega_4$, *x is conjugate to a regular unipotent element from a subsystem subgroup with a simple component of type C_2, and $d_\varphi(x) = 5$;*

2) $\omega = 3^j\omega_4$, *x is conjugate to a regular unipotent element from a subsystem subgroup of type B_3, and $d_\varphi(x) = 7$;*

3) $\omega = 3^j\omega_1$, *x is such as in Item 1), and $d_\varphi(x) = 7$;*

4) $\omega = 3^j\omega_3$, *x is such as in Item 2), and $d_\varphi(x) = 8$ (here j is a nonnegative integer).*

There are 3 conjugacy classes in G that satisfy the assumptions of Item 1) of Theorem 1. Regular unipotent elements from subsystem subgroups of types C_2, $C_2 \times C_1$, and $C_2 \times C_1 \times C_1$ are their representatives.

T h e o r e m 2. *Let $p = 3$ and $x \in G$ be a regular unipotent element. Then $d_\varphi(x) = 27$ for $\omega(\varphi) \notin \{3^j\omega_1, 3^j\omega_4\}$, $d_\varphi(x) = 19$ for $\omega(\varphi) = 3^j\omega_1$, and $d_\varphi(x) = 15$ for $\omega(\varphi) = 3^j\omega_4$ (here j is a nonnegative integer).*

T h e o r e m 3. *Let $p = 7$, $x \in G$ be a regular unipotent element, and $z \in G$ be such element of a subsystem subgroup of type B_4. Assume that φ is p-restricted. Then*

$$d_\varphi(g) = \min\{d_{\varphi\mathbb{C}}(g_{\mathbb{C}}), 7d_\varphi(g^7), 49\}$$

for $g = x$ or z. In particular, $d_\varphi(z) = d_{\varphi\mathbb{C}}(z_{\mathbb{C}})$ for

$$\omega \in \{\omega_i, 1 \leq i \leq 4, 2\omega_1, 3\omega_1, \omega_1 + \omega_j, 2 \leq j \leq 4, 2\omega_1 + \omega_3, 2\omega_1 + \omega_4, \omega_1 + \omega_3 + \omega_4\},$$

$$d_\varphi(z) = 7 d_\varphi(z^7) \text{ for}$$

$$\omega \in \{a_4\omega_4, 2 \leq a_4 \leq 5, \omega_i + \omega_j, 1 < i < j \leq 4, \omega_k + 2\omega_4, 1 \leq k \leq 3, 2\omega_3, 2\omega_3 + \omega_4, \omega_1 + 3\omega_4, \omega_3 + 3\omega_4\},$$

$$d_\varphi(x) = d_{\varphi\mathbb{C}}(x_{\mathbb{C}}) \text{ for } \omega \in \{\omega_i, 1 \leq i \leq 4, 2\omega_1, 2\omega_4, \omega_1 + \omega_4, \omega_3 + \omega_4\},$$

and $d_\varphi(g) = 49$ otherwise for $g = x$ or z.

According to [6, Tables A and D], for $p = 7$, the element z^7 is a long root element and x^7 is a product of commuting long and short root elements. Let $\omega = \sum_{i=1}^{4} a_i \omega_i$. Applying [1, Theorem 1.1, Proposition 1.3, and Table 4], one can show that

$$d_{\varphi_{\mathbb{C}}}(z_{\mathbb{C}}) = 14a_1 + 26a_2 + 18a_3 + 10a_4 + 1, \quad d_{\varphi_{\mathbb{C}}}(x_{\mathbb{C}}) = 22a_1 + 42a_2 + 30a_3 + 16a_4 + 1,$$

$$d_{\varphi}(z^7) = \min\{7, 1 + 2a_1 + 3a_2 + 2a_3 + a_4\}, \quad d_{\varphi}(x^7) = \min\{7, 1 + 3a_1 + 6a_2 + 4a_3 + 2a_4\}.$$

By the Steinberg tensor product theorem [7, Theorem 1.1], if ρ is an irreducible representation of a semisimple algebraic group over K, then $\rho \cong \otimes_{k=0}^{j} \rho_k \circ Fr^k$ where all ρ_k are p-restricted and Fr is the Frobenius morphism determined by raising the elements of K to the power p. Set $\omega'(\rho) = \sum_{k=0}^{j} \omega(\rho_k)$. The weight $\omega'(\rho)$ is uniquely determined. We call an irreducible representation ρ of a simple algebraic group Γ over K p-large if $\langle \varphi'(\rho), \beta \rangle \geq p$ for a maximal root β of Γ.

As for other simple algebraic groups and characteristics, when the minimal polynomial problem is solved for p-restricted representations, we can apply the Steinberg tensor product theorem and the formulas for a tensor product of unipotent Jordan blocks from [8] to pass to arbitrary irreducible representations. In particular, if $\varphi = \varphi_1 \otimes \varphi_2$ and $d_{\varphi_1}(x) + d_{\varphi_2}(x) > |x|$ for a unipotent element x, then

$$d_{\varphi}(x) = |x|. \tag{1}$$

Theorems 1, 2, 3, the results of [3] on the minimal polynomials of the images of unipotent elements in irreducible representations of $F_4(K)$ in characteristics 5 and 11, and those of [1] on analogous polynomials for elements of order p imply

T h e o r e m 4. *For a p-large representation φ of the group $F_4(K)$ in an odd characteristic p, the degree $d_{\varphi}(x) = |x|$ for each unipotent element x.*

We need some more notation. In what follows Γ is a simply connected simple algebraic group over K, $\Lambda(\Gamma)$, $\Lambda^+(\Gamma)$, $R(\Gamma)$, and $R^+(\Gamma)$, respectively, are the sets of weights, dominant weights, roots, and positive roots of Γ, r is the rank of Γ, $\Pi(\Gamma) = \{\alpha_1, ..., \alpha_r\}$ is a basis in $R(\Gamma)$; $\mathcal{X}_{\beta}$ and $x_{\beta}(t)$ are the root subgroup and the root element of Γ associated with a root β and an element t of the field; $\overline{cl(x)}$ is the Zariski closure of the conjugacy class containing an element x; $\langle H_1, ..., H_k \rangle \subset \Gamma$ is the subgroup generated by subgroups $H_1, ..., H_k$, $\Gamma(\beta_1, ..., \beta_k)$ is the subgroup of Γ generated by the root subgroups $\mathcal{X}_{\pm\beta_1}, ..., \mathcal{X}_{\pm\beta_k}$. Set $\Gamma(i_1, ..., i_k) = \Gamma(\alpha_{i_1}, ..., \alpha_{i_k})$ and $\Gamma(\beta, i_1, ..., i_k) = \Gamma(\beta, \alpha_{i_1}, ..., \alpha_{i_k})$. If $\beta = \sum_{i=1}^{r} b_i \alpha_i \in R^+(\Gamma)$ with $b_i \in \mathbb{Z}^+$, set $h(\beta) = \sum_{i=1}^{r} b_i$. We use the notation $\Gamma_{\mathbb{C}}$, ω_i, and $\omega(\varphi)$ in the same manner as for G.

Throughout the text dim V is the dimension of a subspace V. If $\lambda \in \Lambda^+(\Gamma)$, then $M(\lambda)$ and $V(\lambda)$ are the irreducible module and the Weyl module of Γ with highest weight λ; $\omega(m)$ is the weight of a weight vector m from some module. If H is a subgroup of Γ, then $M \mid H$ is the restriction of a Γ-module M to H. We assume that the weights and the roots of Γ are considered with respect to a fixed maximal torus T. If $T \cap H$ is a maximal torus in H, then $\omega \mid H$ is the restriction of a weight ω to $T \cap H$. In this case for a weight vector m from some Γ-module, we set $\omega_H(m) = \omega(m) \mid H$. If M is an irreducible Γ-module, then $v \in M$ is a nonzero highest weight vector. For $\Gamma = A_1(K)$, the set $\Lambda(\Gamma)$ is canonically identified with $\mathbb{Z}$: $a\omega_1 \rightarrow a$.

The following facts are used intensively in the proofs of the main results.

P r o p o s i t i o n 1 [2, a part of Proposition 2.5]. *Let M be a Γ-module, $x \in \Gamma$ be a unipotent element, and $|x| = p^{s+1} > p$.*

(a) *Assume that $l \leq s$ and $z = x^{p^l}$. Then $p^l(d_M(z) - 1) < d_M(z) \leq p^l d_M(z)$.*

(b) *Let $y = x^{p^s}$, $d_M(y) = a + 1$, $M_y = (y - 1)^a M$, and $d_{M_y}(x) = b$. Then $b \leq p^s$, $d_M(x) = ap^s + b$, and* dim $(x - 1)^{ap^s + b - 1} M =$ dim $(x - 1)^{b-1} M_y$.

L e m m a 1 [1, Lemma 2.20]. *Let Γ be a semisimple algebraic group, $x, y \in \Gamma$ be unipotent, and $y \in \overline{cl(x)}$. Then $d_{\varphi}(y) \leq d_{\varphi}(x)$ for each representation φ of Γ.*

L e m m a 2 [2, Lemma 2.42]. *Let $I \subset \Pi(\Gamma)$ be a proper subset and M be a Γ-module. Denote by Σ_I the set of integer linear combinations of the simple roots from I. Set $R_I = R^+(\Gamma) \cap \Sigma_I$ and $R' = R^+(\Gamma) \setminus R_I$. Let $m = m_1 + ... + m_k \in M$ and m_j, $1 \leq j \leq k$, be the weight components of m.*

If $k > 1$, assume that $\omega(m_i) - \omega(m_j) \in \Sigma_t$ for $1 \le i < j \le k$. Suppose that $x' \in \langle \mathcal{X}_\alpha \mid \alpha \in R' \rangle$, $x_I \in \langle \mathcal{X}_\alpha \mid \alpha \in R_1 \rangle$, $x = x'x_I$, and $(x_I - 1)^a m \ne 0$. Then $(x - 1)^a m \ne 0$. In particular, $d_N(x) \ge a + 1$ for any x-invariant subspace $N \subset M$ that contains m.

L e m m a 3. [3, Lemma 7] *Let M be a Γ-module, N be a composition factor of M, and $x \in \Gamma$ be a unipotent element. Then $d_M(x) \ge d_N(x)$.*

L e m m a 4. *Let $x \in U^+$ and $k = p^l$. Then $x^k \in \langle \mathcal{X}_\alpha \mid h(\alpha) \ge k \rangle$.*

To prove Lemma 4, it suffices to recall that for a weight vector m from an arbitrary Γ-module the vector $(x - 1)m = \Sigma m_i$ where m_i are weight vectors with $\omega(m_i) > \omega(m)$, and $(x - 1)^k m = (x^k - 1)m$.

L e m m a 5. *Let $\Gamma = A_1(K)$, $z = x_1(t) \in \Gamma$, $t \ne 0$, and N be an indecomposable Γ-module with highest weight $ap + p - 1$ generated by a highest weight vector v. Then $v \in (z - 1)^{p-1} N$.*

In the following lemma the symbol F_p denotes the field of p elements.

L e m m a 6 [2, Lemma 2.38]. *Let $\beta_1, ..., \beta_k \in \Lambda^+(\Gamma)$, $t_1, ..., t_k \in \mathbb{Z}$, and $\bar{t}_j$ be the image of t_j under the natural homomorphism $\mathbb{Z} \to \mathrm{F}_p$. Let $x = \prod_{j=1}^k x_{\beta_j}(\bar{t}_j) \in \Gamma$, $x_{\mathbb{C}} = \prod_{j=1}^k x_{\beta_j}(t_j) \in \Gamma_{\mathbb{C}}$, and φ be an irreducible representation of Γ. Then $d_\varphi(x) \le d_{\varphi\mathbb{C}}(x_{\mathbb{C}})$.*

T h e o r e m 5 [2, Theorem 1.1]. *For a p-large representation ρ of a simple algebraic group Γ of a classical type over K and a unipotent element $x \in \Gamma$, the degree of the minimal polynomial of $\rho(x)$ is equal to the order of x.*

P r o p o s i t i o n 2 [3, a part of Proposition 5]. *Let $H \subset G$ be a subsystem subgroup of type C_3 or B_4 and φ be a p-large irreducible representation of G. Then the restriction $\varphi \mid H$ has a p-large composition factor.*

On the proofs of the main results. The size of this article does not permit us to include the proofs even for the main results, so we present the schemes of these proofs. It is clear that $d_\varphi(u) \le |u|$ for all unipotent elements u and representations φ. Set $l = 9$ if $p = 3$ and u is a regular unipotent element, and $l = p$ otherwise. Put $d'_\varphi(u) = \min\{|u|, d_{\varphi\mathbb{C}}(u_{\mathbb{C}}), ld_\varphi(u^l)\}$. Then Proposition 1 and Lemma 6 imply that $d_\varphi(u) \le d'_\varphi(u)$. Hence we are done if we can show that $d_\varphi(u) \ge d'_\varphi(u)$. The latter inequality holds in the majority of cases, but there are some exceptions for $p = 3$ (see Theorem 2). The proofs of Theorem 1 and Theorem 3 for elements of a subsystem subgroup of type B_4 are based on quite close arguments. We analyze the restrictions of the representations being considered to proper subsystem subgroups and apply the results on the minimal polynomials of the images of unipotent elements in irreducible representations of the classical groups [2, Theorems 1.7 and 1.10] and on such polynomials for elements of order p [1, Theorem 1.1, Proposition 1.3, Algorithm 1.4, and Table 4], Theorem 5, Proposition 2, and Lemmas 3 and 1.

One easily observes that $G(\alpha_2 + 2\alpha_3 + 2\alpha_4, 1, 2, 3) \cong B_4(K)$. We use this group when analyzing the restrictions of representations to a subsystem subgroup of type B_4.

Now assume that $x \in G$ is a regular unipotent element. Set $y = x^9$ for $p = 3$ and $y = x^7$ for $p = 7$. Define M_y such as in Proposition 1. Assume that φ is p-restricted. First suppose that $p = 3$. Now we state two lemmas that play an important role in the proof of Theorem 2. Let α be the maximal root of G. It is well known that $\alpha = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$.

L e m m a 7. *Let $x \in U^+$. Then $y \in \mathcal{X}_\alpha$.*

On the proof of Lemma 7. Set $R_9 = \{\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha\}$. One easily concludes that R_9 is the set of roots β of G with $h(\beta) \ge 9$. By Lemma 4, $y \in \langle \mathcal{X}_\beta \mid \beta \in R_9 \rangle$. Using conjugation by elements of U^+, one can assume that $y = x_\delta(t)$ where $\delta \in R_9$. Analyzing the action of the elements x and y on the irreducible module N with highest weight ω_4, we deduce that $\delta = \alpha$, otherwise we would get that $d_N(x) > 15$ and obtain a contradiction due to [6, Table 3].

L e m m a 8. *Let $\Gamma = C_3(K)$, $u \in \Gamma$ be a regular unipotent element, and ρ be an irreducible representation of Γ. Assume that $d_\rho(u) < 9$. Then $\omega(\rho) \in \{0, 3^j\omega_1, 3^j\omega_3\}$ $(j \in \mathbb{Z}^+)$.*

The proof of Lemma 8 follows from [2, Table VII] and [8, Lemma 6.14 and Theorem 6.4].

One easily observes that the group $H = G(2, 3, 4) \subset C_G(y)$ and that $H \cong C_3(K)$. Put $I = \{\alpha_2, \alpha_3, \alpha_4\}$, $x' = x_1(1)$, and $x_I = x_2(1)x_3(1)x_4(1)$. Take $x = x'x_I$. Applying Lemmas 8 and 3, we can conclude that $d_{M_y}(x_I) = 9$ if φ is p-restricted and $\omega(\varphi) \notin \{\omega_1, \omega_4\}$. Now Theorem 2 follows from Lemma 2, Proposition 1, [6, Tables 3 and 4], and Formula (1).

Now let $p = 7$. Lemma 1 implies that for proving Theorem 3 for the element x, it suffices to consider only irreducible representations φ with $d_\varphi(z) < 49$ for a regular unipotent element z of a subsystem subgroup of type B_4, in particular, Proposition 2 and Theorem 5 allow us to exclude p-large representations.

Now we state three results that are used for computing the minimal polynomial of $\varphi(x)$ in the remaining cases.

P r o p o s i t i o n 3. *Put* $R_0 = R^+(G) \setminus \{\alpha_1, \alpha_3, \alpha_4, \alpha_3 + \alpha_4\}$, $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$, *and* $\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$. *One can assume that* $x = x_0 x_1(t_1)u$, *where* $x_0 \in \langle \mathcal{X}_\alpha \mid \alpha \in R_0 \rangle$, $u = x_3(t_3)x_4(t_4)x_{\alpha_3 + \alpha_4}(t)$, $y = x_\beta(t_\beta)x_\gamma(t_\gamma)$, *and* $t_1 t_3 t_4 t_\beta t_\gamma \neq 0$.

On the proof of Proposition 3. Taking into account the commutation relations in G, one easily observes that a regular unipotent element lying in U^+ can be represented in such form as in the assertion of the proposition. By [6, Table D], y is a product of commuting nontrivial long and short root elements. Set

$$R_7 = \{\beta, \gamma, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\}.$$

One easily checks that R_7 coincides with the set of roots δ with $h(\delta) \geq 7$. By Lemma 4, $y \in \langle \mathcal{X}_\delta \mid \delta \in R_7 \rangle$ if $x \in U^+$. Taking these facts into account and applying the conjugation in U^+, we obtain the required equality for y.

L e m m a 9. *The group* $C_G(y)$ *contains a subgroup* $A \cong A_1(K)$ *such that* $u \in A \subset G(3, 4)$ *and* A *contains a nontrivial element from* $\langle \mathcal{X}_{-3}, \mathcal{X}_{-4} \rangle$; $G(1) \subset C_G(y)$.

L e m m a 10. *Let* $\Gamma = A_2(K)$, $g \in U^+(\Gamma)$ *be a regular unipotent element, N be an irreducible p-restricted Γ-module with highest weight* $\omega = a_1\omega_1 + a_2\omega_2$, *and* $w \in N$ *be a nonzero lowest weight vector. Assume that the Weyl module* $V(\omega)$ *is irreducible. Put* $l = \min\{2(a_1 + a_2), 6\}$. *Then the vector* $(z - 1)^l w$ *has a nontrivial weight component of weight* $\omega(w) + b_1\alpha_1 + b_2\alpha_2$ *with* $b_1 + b_2 = l$.

To prove Theorem 3 for x, we show that in all cases under consideration $d_{M_y}(x_1(t_1)u) \geq d'_\varphi(x) - 7(d_M(y) - 1)$ and then apply Lemma 2 and Proposition 1. To estimate the parameter $d_{M_y}(x_1(t_1)u)$, we have to deal not only with composition factors of the restriction $M \mid G(1)A$, but with sections (quotient modules of submodules) of this restriction as well. Here Lemma 10 is used.

Conclusion. Theorems 1 and 2 can be used for computing the minimal polynomials of the images of unipotent elements in irreducible representations of the group $E_6(K)$ for $p = 3$. This would complete the solution of the minimal polynomial problem for the latter group in an odd characteristic.

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