Conditions for minimally tough graphs

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Abstract

Katona, Soltész, and Varga showed that no induced subgraph can be excluded from the class of minimally tough graphs. In this paper, we consider the opposite question, namely which induced subgraphs, if any, must necessarily be present in each minimally t-tough graph. Katona and Varga showed that for any rational number $t \in (1/2, 1]$, every minimally t-tough graph contains a hole. We complement this result by showing that for any rational number $t > 1$, every minimally t-tough graph must contain either a hole or an induced subgraph isomorphic to the k-sun for some integer $k \geq 3$. We also show that for any rational number $t > 1/2$, every minimally t-tough graph must contain either an induced 4-cycle, an induced 5-cycle, or two independent edges as an induced subgraph.

Keywords: toughness, minimally tough graph, chordal graph, strongly chordal graph, split graph, moplex

1 Introduction

In 1973, Chvátal defined a (finite, simple, and undirected) graph G to be t-tough, where t is a positive real number, if the removal of any vertex set S leaves at most $|S|/t$ components provided the removal of S disconnects the graph, and all graphs are considered 0-tough. The toughness of G is the largest real number t such that G is t-tough, whereby the toughness of complete graphs is defined as infinity [7]. As informally described by Chvátal himself, toughness "measures in a simple way how tightly various pieces of a graph hold together". Toughness aimed at generalizing the notion of Hamiltonicity since Hamiltonian graphs are 1-tough, and Chvátal conjectured that there exists a real number t_0 such that every t_0-tough graph is Hamiltonian [7]. His stronger conjecture was that every more than 3/2-tough graph is Hamiltonian – but this was disproved by Thomassen [2]. Thereafter it was conjectured (based on [9]) that every 2-tough graph is Hamiltonian – but this was also disproved, this time by Bauer et al. in 2000 [1]. However, the general conjecture remains open until this day.

A concept closely related to Chvátal's conjecture is that of minimally t-tough graphs, which Broersma et al. defined as graphs whose toughness is t but the deletion of any edge decreases their toughness [6]. This notion has been studied in a number of subsequent works (see, e.g., [14–16, 19]). In particular, Katona et al. showed in [15] that for every positive rational number t, any graph is an induced subgraph of some minimally t-tough graph.

In this paper, we continue the study of minimally t-tough graphs. First, we provide a necessary and sufficient condition for a graph G to be minimally tough. We then use this condition to derive that for every $t > 1$, every minimally t-tough, noncomplete graph must contain a hole or an induced subgraph isomorphic to the k-sun for some $k \geq 3$. This result complements the one of Katona et al. stating that for all rational number $t \in (1/2, 1]$, every minimally t-tough graph contains a hole [16]. In addition, this result shows that there are no minimally t-tough, strongly chordal graphs for any rational number $t > 1/2$. Furthermore, our approach also implies that for any rational number $t > 1$, if a minimally t-tough graph contains a universal vertex, then it contains a hole. Finally, we show that there are no minimally t-tough, split graphs for any rational number $t > 1/2$.

2 Preliminaries

In this section, we present some necessary definitions and claims. Let $\omega(G)$ denote the *number of components*[1], $\kappa(G)$ the *connectivity* and $\delta(G)$ the *minimum degree* of a graph G. For a vertex v and for a set of vertices W in

[1]Using $\omega(G)$ to denote the number of components might be confusing; most of the literature on toughness, however, uses this notation.

a graph G, the *degree* of v is denoted by $d_G(v)$, the *open neighborhood* and the *closed neighborhood* of v and those of W are denoted by $N_G(v)$, $N_G[v]$, $N_G(W)$, and $N_G[W]$, respectively. In all cases, the subscript G is dismissed whenever it does not cause confusion.

Definition 2.1. *Let t be a real number. A graph G is called* t-tough *if $|S| \geq t \cdot \omega(G-S)$ holds for any vertex set $S \subseteq V(G)$ that disconnects the graph (i.e., for any $S \subseteq V(G)$ with $\omega(G-S) > 1$). The* toughness *of G, denoted by $\tau(G)$, is the largest t for which G is t-tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$.*

A graph G is said to be minimally t-tough *if $\tau(G) = t$ and $\tau(G-e) < t$ for all $e \in E(G)$. A graph is called minimally tough if it is minimally t-tough for some real number t.*

Note that a graph is disconnected if and only if its toughness is 0.

It is not difficult to see that the toughness of any connected, noncomplete graph is a positive rational number, thus there exist no minimally tough graphs with nonpositive or with irrational toughness. Also note that every complete graph on at least two vertices is minimally ∞-tough.

The following claim easily follows from the definition of minimally tough graphs.

Claim 2.2 ([14])**.** *Let t be a positive rational number and G a minimally t-tough graph. For every edge $e \in G$,*

1. *the edge e is a bridge in G, or*
2. *there exists a vertex set $S = S(e) \subseteq V(G)$ with*

$$\omega(G-S) \leq \frac{|S|}{t} \quad \text{and} \quad \omega\big((G-e)-S\big) > \frac{|S|}{t},$$

and the edge e is a bridge in $G-S$.

In the first case, we define $S = S(e) = \emptyset$.

The following relation between the toughness of a graph and its connectivity can be proved directly from the definition of toughness.

Claim 2.3 (Chvátal [7])**.** *For every noncomplete graph G, we have $\tau(G) \leq \kappa(G)/2$.*

Now we give the definitions and theorems needed for this paper. For two vertices $u, v \in V(G)$, a set $S \subseteq V(G) \setminus \{u, v\}$ is called a *u-v separator* if u and

v belong to different components of $G - S$. A u-v separator is called *minimal* if none of its proper subsets is a u-v separator. A set $S \subseteq V(G)$ is called a *(minimal) separator* if it is a (minimal) u-v separator for some $u, v \in V(G)$. Note that a minimal separator can be a proper subset of another minimal separator but for different pairs of vertices.

To give a characterization for minimal separators, we need the following definition.

Definition 2.4. *For a set $S \subset V(G)$, a component C of $G - S$ is said to be* S-full *if every vertex in S has a neighbor in C.*

Claim 2.5 (Golumbic, Exercise 10 of Chapter 4 in [13])**.** *In a graph G, a set $S \subseteq V(G)$ is a minimal separator if and only if the graph $G - S$ has at least two S-full components.*

In this work, we mainly focus on chordal, strongly chordal, and split graphs. A hole in a graph is an induced cycle of length at least 4. A graph is *chordal* if it is hole-free, i.e., if it does not contain an induced cycle of length at least 4. A graph is *strongly chordal* if it is chordal and every even cycle of length at least 6 has an odd chord, i.e., a chord whose endpoints are an odd distance apart in the cycle. A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set. It is not difficult to see that every split graph is chordal, but not necessarily strongly chordal.

A vertex or a set of vertices is called *simplicial* if its closed neighborhood is a clique. A classical theorem of Dirac states that every chordal graph has a simplicial vertex; moreover, every noncomplete, chordal graph contains at least two nonadjacent simplicial vertices [8].

Definition 2.6. *A vertex $u \in N[v]$ is a* maximum neighbor *of a vertex v if $N[w] \subseteq N[u]$ holds for all $w \in N[v]$.*

Note that a vertex can be its own maximum neighbor.

Building upon the notion of maximum neighbor, we introduce the following definition.

Definition 2.7. *An edge uu' is a* maximum neighboring edge *of a vertex v if $u, u' \in N(v)$ and $N[w] \subseteq N[u] \cup N[u']$ holds for all $w \in N[v]$.*

Note that if a vertex v has a maximum neighbor different from v itself, then this maximum neighbor and any other neighbor of v (if existing) form a maximum neighboring edge of v.

Definition 2.8. *A vertex s is called a* simple vertex *if its closed neighborhood can be linearly ordered by inclusion, i.e., for any $x, y \in N[s]$, if x precedes y, then $N[x] \subseteq N[y]$.*

Observe that every simple vertex is simplicial, and they have a maximum neighbor.

Definition 2.9. *Given an integer $k \geq 3$, a k-*sun *S_k is a graph whose vertex set can be partitioned into two sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ such that A is a clique and B is an independent set in S_k, and for all $i, j \in \{1, \ldots, k\}$, the vertices a_i and b_j are adjacent if and only if $i = j$ or $i \equiv j+1 \pmod{k}$ (see Figure 1).*

A graph is said to be sun-free *if it does not contain an induced k-sun for any integer $k \geq 3$.*

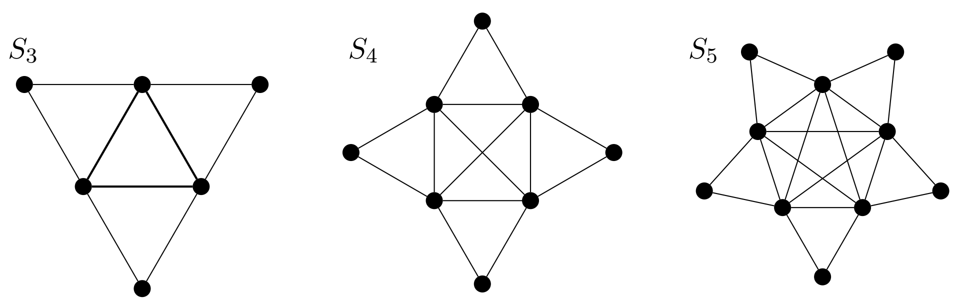

Figure 1: The 3-sun, the 4-sun, and the 5-sun.

Later, we make use of the following characterizations of strongly chordal graphs.

Theorem 2.10 (Farber [10])**.** *For any graph G, the following conditions are equivalent.*

1. *The graph G is strongly chordal.*
2. *The graph G is chordal and sun-free.*
3. *Each induced subgraph of G has a simple vertex.*

The following theorem gives a characterization of split graphs.

Theorem 2.11 (Földes and Hammer [11])**.** *A graph is a split graph if and only if it is $(C_4, C_5, 2K_2)$-free, i.e., if it does not contain a cycle of length 4, a cycle of length 5, and a pair of independent edges as induced subgraphs.*

The following definition can be considered as a generalization of the concept of simplicial vertices.

Definition 2.12. *In a graph G, a set of vertices $M \subseteq V(G)$ is a* module *if each vertex $v \in V(G) \setminus M$ is either adjacent to every vertex in M or not adjacent to any vertex in M. A* clique module *is a module that is a clique. A* moplex *in a graph G is an inclusion-maximal clique module whose open neighborhood is either empty or is a minimal separator in G. A vertex that belongs to a moplex is said to be* moplicial.

By the following characterization of chordal graphs, it is not difficult to see that in a chordal graph every moplicial vertex is simplicial.

Theorem 2.13 (Dirac [8])**.** *A graph G is chordal if and only if every minimal separator in G forms a clique.*

The previously mentioned result of Dirac on the existence of simplicial vertices in chordal graphs [8] was strengthened by Berry and Bordat in [3] using the concept of moplexes, which, in the case of chordal graphs, can consist of simplicial vertices only.

Theorem 2.14 (Berry and Bordat [3])**.** *Every noncomplete graph contains at least two moplexes.*

The following lemma provides a link between simple vertices and moplexes.

Lemma 2.15. *If s is a simple vertex of a graph G, then s is moplicial.*

Proof. Let M denote the set of those vertices in G whose closed neighborhood is the same as that of s. Clearly, M is an inclusion-maximal clique module. If G is complete, then $M = V(G)$ and $N(M) = \emptyset$, thus M is a moplex. So assume that G is noncomplete. In this case, we claim that $N(M)$ is a minimal separator in G. By Claim 2.5, we need to show that $G - N(M)$ has at least two $N(M)$-full components.

Since M is a clique module, it forms an $N(M)$-full component in $G - N(M)$. Let v be a vertex in $N(M)$ with smallest degree. Since s is simple and $v \notin M$,

$$N[s] \subsetneq N[v] \subseteq N[w]$$

holds for any $w \in N(M)$. Let $u \in N[v] \setminus N[s]$, and let H denote the connected component of u in $G - N(M)$. Since $u \in N[v] \subseteq N[w]$ holds for any $w \in N(M)$, we can conclude that H is also an $N(M)$-full component in $G - N(M)$.

Therefore, M is a moplex, thus s is moplicial. □

Chordal graphs have several equivalent definitions; in this paper, we also rely on the one using the concept of clique trees.

Definition 2.16. *Let G be a graph and let $\mathcal{Q}_G$ denote the set of its maximal cliques. A* clique tree *of G is a tree with vertex set $\mathcal{Q}_G$ such that for every pair of distinct maximal cliques $Q, Q' \in \mathcal{Q}_G$, the set $Q \cap Q'$ is contained in every clique on the path connecting Q and Q' in the tree.*

Theorem 2.17 (Gavril [12], Blair and Payton [5])**.** *A graph is chordal if and only if it has a clique tree.*

With the help of clique trees, minimal separators in chordal graphs can be characterized as follows.

Lemma 2.18 (Blair and Payton [5])**.** *Let G be a connected, chordal graph and T a clique tree of G. A set of vertices $S \subseteq V(G)$ is a minimal separator in G if and only if S is the intersection of two maximal cliques corresponding to two adjacent nodes of T.*

The following lemma shows an interesting connection between moplexes in a chordal graph and its clique trees.

Lemma 2.19 (Berry and Bordat [4])**.** *For any moplex M in a connected, chordal graph G, there exists a clique tree T of G such that $N[M]$ is a maximal clique corresponding to a leaf of T.*

3 A characterization of non-minimally tough graphs

The following theorem characterizes graphs that are not minimally tough.

Theorem 3.1. *Let G be a connected, noncomplete graph and let $t = \tau(G)$. Then G is not minimally t-tough if and only if G contains an edge $e = uv$ such that the following conditions are met.*

1. *There exist at least $2t + 1$ pairwise internally vertex-disjoint u-v paths in G (including uv).*
2. *Every separator S in G that is also a u-v separator in $G - e$ satisfies*

$$|S| \geq t \cdot \big(\omega(G - S) + 1\big) .$$

Proof. First, assume that G is not minimally t-tough. Then, there exists an edge $e = uv$ in G such that $\tau(G - e) = t$. By Theorem 2.3, we have

$\kappa(G-e) \geq 2\tau(G-e) = 2t$, so by Menger's Theorem, there exist at least $2t$ pairwise internally vertex-disjoint paths from u to v in $G-e$, which proves Condition 1. Consider now a separator S in G that is also a u-v separator in $G-e$. Then e is a bridge in $G-S$, which implies $\omega\big((G-e)-S\big) = \omega(G-S)+1$. Since S is a separator in $G-e$ and $\tau(G-e) = t$, we obtain

$$|S| \geq t \cdot \omega\big((G-e)-S\big) = t \cdot \big(\omega(G-S)+1\big)\,,$$

which proves Condition 2.

Conversely, let $e = uv$ be an edge in G for which Conditions 1 and 2 hold. We claim that $\tau(G-e) = t$, which implies that G is not minimally t-tough. Let S be an arbitrary separator in $G-e$. We need to show that $|S| \geq t \cdot \omega\big((G-e)-S\big)$ holds.

Assume first that S is not a separator in G. Then $\omega(G-S) = 1$ and e is a bridge $G-S$, which implies

$$\omega\big((G-e)-S\big) = \omega(G-S)+1 = 2\,.$$

By Condition 1, we have $|S| \geq 2t$. Therefore,

$$|S| \geq 2t \geq t \cdot \omega\big((G-e)-S\big)\,.$$

Assume now that S is a separator in G, as well as a u-v separator in $G-e$. Then $\omega\big((G-e)-S\big) = \omega(G-S)+1$ and by Condition 2, we have

$$|S| \geq t \cdot \big(\omega(G-S)+1\big) = t \cdot \omega\big((G-e)-S\big)\,.$$

Finally, assume that S is a separator in G, but not a u-v separator in $G-e$. Then $\omega\big((G-e)-S\big) = \omega(G-S)$ and using the fact that G is t-tough, we obtain

$$|S| \geq t \cdot \omega(G-S) = t \cdot \omega\big((G-e)-S\big)\,.$$

This completes the proof. □

As a side remark, let us note that Theorem 3.1 can be strengthened further by the use of the following lemma.

Lemma 3.2. *Let G be a connected, noncomplete graph, let $t = \tau(G)$, and let $e = uv$ be an edge of G. Then the following two conditions are equivalent.*

1. *Every separator S in G that is also a u-v separator in $G-e$ satisfies*

$$|S| \geq t \cdot \big(\omega(G-S)+1\big)\,.$$

2. *Every separator S in G that is also a u-v separator in $G - e$ and whose every vertex has neighbors in at least two components of $(G - e) - S$ satisfies*
$$|S| \geq t \cdot \big(\omega(G - S) + 1\big)\,.$$

Proof. Condition 1 trivially implies Condition 2.

To prove the converse, assume that Condition 2 holds. We show that every separator S in G that is also a u-v separator in $G - e$ satisfies $|S| \geq t \cdot \big(\omega(G - S) + 1\big)$ by induction on $k = k(S)$ defined as the number of vertices in S that have neighbors in at most one component of $(G - e) - S$.

The base case corresponds to $k = 0$, where the desired inequality holds by Condition 2. Let S be an arbitrary separator in G that is also a u-v separator in $G - e$ with $k = k(S) > 0$, and assume that $|S'| \geq t \cdot \big(\omega(G - S') + 1\big)$ holds for any separator S' in G that is also a u-v separator in $G - e$ with $k(S') < k$. Let $w \in S$ be a vertex that has neighbors in at most one component of $(G - e) - S$ and let $S' = S \setminus \{w\}$. It is not difficult to see that S' is a separator in G and also a u-v separator in $G - e$, and $\omega\big((G - e) - S'\big) = \omega\big((G - e) - S\big)$, and $k(S') < k(S)$. Thus by the induction hypothesis,

$$\begin{aligned}|S| > |S'| \geq t \cdot \big(\omega(G - S') + 1\big) = t \cdot \omega\big((G - e) - S'\big) \\ = t \cdot \omega\big((G - e) - S\big) = t \cdot \big(\omega(G - S) + 1\big)\,,\end{aligned}$$

which completes the proof. □

Using Theorem 3.1, we now derive the following sufficient condition for a graph not to be minimally tough.

Lemma 3.3. *Let t be a positive rational number and let G be a graph containing two adjacent vertices u and v such that u and v have at least $2t$ common neighbors, at least t of which have all their neighbors in $N(u) \cup N(v)$. Then G is not minimally t-tough.*

Proof. We can assume that $\tau(G) = t$, otherwise G is clearly not minimally t-tough. We verify that the graph G and its edge uv satisfy Conditions 1 and 2 of Theorem 3.1. Clearly, Condition 1 follows from the assumption that u and v are adjacent and have at least $2t$ common neighbors. For Condition 2, let S be an arbitrary separator in G that is also a u-v separator in $G - e$. Let
$$T = \big\{w \in N(u) \cap N(v) \;\big|\; N(w) \subseteq N(u) \cup N(v)\big\}\,.$$
By the assumption of the lemma, $|T| \geq t$ holds. Furthermore, $T \subseteq S$ since S is a u-v separator in $G - e$. Consider the vertex set $S' = S \setminus T$. The definition

of T implies that S' is a separator in G with $\omega(G-S') = \omega(G-S)$. Since G is t-tough, we obtain

$$|S| = |S'| + |T| \geq t \cdot \omega(G-S') + t = t \cdot \big(\omega(G-S') + 1\big) = t \cdot \big(\omega(G-S) + 1\big)\,.$$

Therefore, Condition 2 holds as well, and we conclude that G is not minimally t-tough. □

4 Implications for chordal, strongly chordal, and split graphs

Using Lemma 3.3, we now prove the following result, which appears already in [16].

Theorem 4.1. *For any rational number $t \in (1/2, 1]$, there exists no minimally t-tough, chordal graph.*

Proof. Let $t \in (1/2, 1]$ be a rational number and let G be a chordal graph with $\tau(G) = t$. Now we prove that G is not minimally t-tough.

Since $\tau(G) = t$, the graph G is connected and noncomplete. Since G is a chordal graph, it has a clique tree T, and since G is noncomplete, T has at least two nodes.

Let Q be a maximal clique in G corresponding to a leaf of T, and let Q' be the maximal clique of G corresponding to the unique neighbor of this leaf in T. By Lemma 2.18, $S = Q \cap Q'$ is a minimal separator in G. By Claim 2.3, $|S| \geq 2t > 1$, i.e., $|S| \geq 2$, thus S contains two adjacent vertices u and v.

Since Q and Q' are distinct maximal cliques in G, there exist a vertex $x \in Q \setminus Q'$ and a vertex $y \in Q' \setminus Q$. Clearly, the vertices x and y are common neighbors of u and v, thus u and v have at least $2 \geq 2t$ common neighbors. Now, we show that at least $1 \geq t$ of the common neighbors of u and v, namely x, have all their neighbors in $N(u) \cup N(v)$. For this, observe that the set of vertices of T that correspond to maximal cliques containing any fixed vertex in G induces a subtree of T. Since $x \in Q \setminus Q'$ and Q corresponds to a leaf in T, this implies that Q is the only maximal clique containing x in G. Therefore $N(x) = V(Q) \subseteq N(u) \cup N(v)$.

So by Lemma 3.3, we infer that G is not minimally t-tough. □

Theorem 4.1 can be equivalently stated as follows.

Corollary 4.2. *For any rational number $t \in (1/2, 1]$, every minimally t-tough graph contains a hole.*

By Theorem 2.14, every graph contains moplicial vertices, and in the following theorem, we study these vertices in minimally t-tough, chordal graphs with $t > 1/2$. For the definition of maximum neighboring edges, recall Definition 2.7.

Theorem 4.3. *Let $t > 1/2$ be a rational number and let G be a minimally t-tough, chordal graph. Then none of the moplicial vertices of G have a maximum neighbor or a maximum neighboring edge.*

Proof. Suppose to the contrary that G is a minimally t-tough, chordal graph containing a moplicial vertex s with a maximum neighbor u or with a maximum neighboring edge uv.

Since $t \neq 0$ and $t \neq \infty$, the graph G is connected and noncomplete. By Theorem 4.1, $t > 1$ holds.

As we observed earlier, since G is chordal and s is moplicial, Theorem 2.13 implies that s is simplicial. Note that since G is connected and noncomplete, s cannot be its own maximum neighbor. Also note that by Claim 2.3, s has at least two neighbors. Therefore, if s has a maximum neighbor u, then for any other neighbor v of s (which exists), uv is a maximum neighboring edge.

Since s is moplicial, there exists a moplex X containing s. Since s is simplicial, $N[s] = N[X]$ is the unique maximal clique containing s. Let $Q = N[X]$. Then by Lemma 2.19, Q corresponds to a leaf of some clique tree $\mathcal{T}$ of G; let Q' denote the maximal clique of G that corresponds to the only neighbor of this leaf in $\mathcal{T}$. By Lemma 2.18, $Q \cap Q'$ is a minimal separator in G, so by Claim 2.3, $|Q \cap Q'| \geq 2t$. Since G is noncomplete, $Q \neq Q'$, thus $|Q| \geq |Q \cap Q'| + 1 \geq 2t + 1$.

Now consider the vertices of $Q \setminus \{u, v\}$. Clearly, they are common neighbors of u and v, and since $Q = N[s]$ and uv is a maximum neighboring edge of s, the vertices of $Q \setminus \{u, v\}$ have all their neighbors in $N[u] \cup N[v]$. Clearly,

$$\big|Q \setminus \{u, v\}\big| = |Q| - 2 \geq (2t + 1) - 2 = 2t - 1 > t\,,$$

where the last inequality is valid since $t \geq 1$. Therefore, Lemma 3.3 implies that G is not minimally t-tough, which is a contradiction. □

Theorem 4.3 has several interesting consequences.

Corollary 4.4. *For any rational number $t > 1/2$, there exists no minimally t-tough, strongly chordal graph.*

Proof. Let $t > 1/2$ be a rational number and let G be a strongly chordal graph. By Theorem 2.10, G contains a simple vertex s, and by Lemma 2.15, s is moplicial. Since every simple vertex has a maximum neighbor, Theorem 4.3 implies that G is not minimally t-tough. □

Thus by Theorem 2.10, we can conclude the following.

Corollary 4.5. *For any rational number $t > 1/2$, every minimally t-tough graph contains a hole or an induced k-sun for some integer $k \geq 3$.*

Furthermore, since every interval graph is strongly chordal [17], Corollary 4.4 directly implies the following.

Corollary 4.6. *For any rational number $t > 1/2$, there exists no minimally t-tough, interval graph.*

Three independent vertices in a graph form an *asteroidal triple* if every two of them are connected by a path avoiding the neighborhood of the third. Since the interval graphs are exactly those chordal graphs which are also asteroidal triple-free [17], Corollary 4.6 can equivalently stated as follows: for any rational number $t > 1$, every minimally t-tough graph contains either a hole or an asteroidal triple.

A graph is a *co-comparability graph* if its complement admits a transitive orientation (i.e., if (u, v) and (v, w) are arcs of the resulting directed graph, then (u, w) must be also an arc of it). It is natural to ask whether the result of Corollary 4.6 generalizes to the larger class of co-comparability graphs, but this is not the case. For an integer $k \geq 2$, let G be the graph consisting of two disjoint cliques of size k and a perfect matching between them. Then G is a co-bipartite graph and thus a co-comparability graph. Furthermore, G is claw-free and $\kappa(G) = k$ since we need to remove at least one vertex from each edge of the perfect matching to disconnect G and G is k-regular. Thus by [18], $\tau(G) = \kappa(G)/2 = k/2$. Since G is k-regular, Claim 2.3 implies $\tau(G - e) \leq \kappa(G - e)/2 \leq (k - 1)/2$. Therefore, G is a minimally $k/2$-tough, co-comparability graph.

We now move to another consequence of Theorem 4.3, one related to graphs with a *universal vertex*, that is, a vertex adjacent to all other vertices. Observe that there exist minimally t-tough graphs with $t > 1$ that have universal vertices: it is not difficult to see that wheels on $n \geq 5$ vertices (for some examples, see Figure 2) are minimally t_n-tough graphs having a universal vertex, where $t_n = (n + 1)/(n - 1)$ for odd n and $t_n = n/(n - 2)$ for even n.

In fact, it turns out that every minimally t-tough graph with $t > 1$ containing a universal vertex must also contain an induced subgraph isomorphic to a wheel. A graph G is said to be *wheel-free* if it does not contain any induced subgraph isomorphic to a wheel W_n for some $n \geq 5$. In this

Figure 2: The wheels on 5 and 6 vertices.

terminology, the above statement reads as follows: for any rational number $t > 1$, there exists no minimally t-tough, wheel-free graph with a universal vertex. Since a wheel-free graph can only have a universal vertex if it is chordal, it suffices to prove the analogous statement for chordal graphs.

Corollary 4.7. *For any rational number $t > 1$, there exists no minimally t-tough, chordal graph with a universal vertex.*

Proof. Let $t > 1$ be a rational number and let G be a chordal graph with a universal vertex. We need to show that G is not minimally t-tough. We may assume that G is noncomplete, otherwise it is clearly not minimally t-tough. Let U be the set of universal vertices in G and let $G' = G - U$. Since G is noncomplete, G' has vertices and G' is also a noncomplete.

Thus, by Theorem 2.14, G' contains a moplex M. Now we show that M is also a moplex in G. Since M is an inclusion-wise maximal clique module in G' and U is adjacent to every vertex in G, we infer that M is also an inclusion-wise clique module in G. Furthermore, the fact that $N_{G'}(M)$ is a minimal separator in G' implies that $N_G(M) = N_{G'}(M) \cup U$ is a minimal separator in G. Thus M is indeed a moplex in G.

Let $s \in M$ and $u \in U$ be arbitrary. Clearly, s is moplicial and u is a maximum neighbor of s, so by Theorem 4.3, G is not minimally t-tough. $\square$

The following claim characterizes minimally t-tough graphs with $t \leq 1$ having a universal vertex.

Theorem 4.8. *Let $t \leq 1$ be a rational number. If G is a minimally t-tough graph with a universal vertex, then $G \cong K_{1,b}$ and $t = 1/b$ for some integer $b \geq 2$.*

Proof. Let $t \leq 1$ be a rational number, G a minimally t-tough graph, and $u \in V(G)$ a universal vertex of G. Let $v \in V(G)$, $v \neq u$ be an arbitrary vertex. Since u is universal, $uv \in E(G)$. Let $S = S(uv) \subseteq V(G)$ be a vertex set guaranteed by Claim 2.2. Since uv is a bridge in $G-S$, clearly $u \notin S$. Since u is universal in G, the graph $(G-uv)-S$ has exactly two components: the one containing only v and the other containing all the remaining vertices. Therefore, $N(v) \setminus \{u\} \subseteq S$. Then by Claim 2.2,

$$\frac{\big|N(v)\big|-1}{t} = \frac{\big|N(v)\setminus\{u\}\big|}{t} \le \frac{|S|}{t} < \omega\big((G-e)-S\big) = 2\,,$$

that is,

$$\big|N(v)\big| < 2t+1\,.$$

Note that the above arguments and inequalities also hold in the case when uv is a bridge in G, in which case $S=\emptyset$.

So by Claim 2.3, $\lceil 2t\rceil \le \kappa(G) \le d(v) < 2t+1$ holds for any vertex $v \ne u$, moreover, since $d(v)$ is integer, $d(v) = \lceil 2t\rceil$. This means, that if $t \le 1/2$, then each vertex, except possibly for u, is of degree 1. Thus G is a star, that is, $G \cong K_{1,b}$ for some integer $b \ge 0$, and since the toughness of K_1 and K_2 is infinity, $G \cong K_{1,b}$ and $t = 1/b$ for some integer $b \ge 2$. If $1/2 < t \le 1$, then then each vertex, except for u, is of degree 2, thus G can be obtained from a star on at least 3 vertices by adding a matching covering every leaf of the star. However, no such graph is minimally t-tough with $1/2 < t \le 1$: if G has exactly 3 vertices, then $G \cong K_3$, whose toughness is infinity, and if G contains more than 3 vertices, then v is a cut-vertex in G, thus the toughness of G is at most $1/2$. ☐

Finally, we study minimally tough, split graphs. Our next result, Theorem 4.10, can also be found in [16], but with a significantly different proof.

Claim 4.9 ([16])**.** *Let t be a positive rational number and G a minimally t-tough, split graph partitioned into a clique Q and an independent set I. Let $e = uv$ be an edge between two vertices of Q and $S = S(e) \subseteq V(G)$ a vertex set guaranteed by Claim 2.2. Then*

$$S = \big(Q \setminus \{u,v\}\big) \cup \{w \in I \mid uw, vw \in E(G)\}\,.$$

Notice that for any integer $b \ge 2$, the star $K_{1,b}$ is a minimally t-tough, split graph for $t = 1/b \le 1/2$. On the other hand, there are no such graphs for $t > 1/2$.

Theorem 4.10. *For any rational number $t > 1/2$, there exists no minimally t-tough, split graph.*

Proof. Since every split graph is chordal, we can assume by Theorem 4.1 that $t > 1$. So let $t > 1$ be an arbitrary rational number and suppose to the contrary that there exists a minimally t-tough, split graph G. Then G is connected and noncomplete, otherwise its toughness would be either 0 or infinity, and not t. Since G is a split graph, its vertex set can be partitioned into a clique Q and an independent set I. We can assume that in this partition, Q is inclusion-wise maximal, i.e., none of the vertices of I is adjacent to every vertex of Q. Since G is noncomplete, $I \neq \emptyset$. By Claim 2.3, every vertex of G has degree at least $\kappa(G) \geq \lceil 2t \rceil \geq 3$ (since $t > 1$), in particular, so do the vertices of I (and, as we saw, $I \neq \emptyset$), thus $|Q| \geq 3$. Furthermore, $|I| \geq 2$ since $I \neq \emptyset$ and if $|I| = 1$ held, then the neighbors of the only vertex in I (and since each vertex is of degree at least 3, the only vertex in I has indeed neighbors) would be universal, contradicting Corollary 4.7.

Now we show that each vertex of I forms a moplex. Let $v \in I$ be an arbitrary vertex. Obviously, $\{v\}$ is a clique module, and now we show that it is an inclusion-wise maximal one. So suppose to the contrary that there exists a clique module M such that $v \in M$ and $M \neq \{v\}$. Since M is a clique and none of the vertices of I is adjacent to every vertex of Q, clearly $M \setminus \{v\} \subseteq N(v) \subsetneqq Q$ holds. Consider any vertex $w \in M \setminus \{v\}$ and any vertex $u \in Q \setminus N(v)$. Then $uv \notin E(G)$, and since $M \setminus \{v\} \subsetneqq Q$ and Q is a clique, $uw \in E(G)$. Since M is a clique module with $v, w \in M$ and $u \notin M$, this is a contradiction. So $\{v\}$ is indeed an inclusion-wise maximal clique module. Clearly, $N(v)$ is a minimal separator. Therefore, $\{v\}$ is a moplex. Thus each vertex of I is moplicial.

For any $v \in Q$, let $I_v = N(v) \cap I$. Now we show that $|I_u \cup I_v| \leq 3$ and $|I_u \setminus I_v| \leq 1$ hold for any $u, v \in Q$, $u \neq v$. Let $u, v \in Q$ be arbitrary vertices. Since Q is a clique, $uv \in E(G)$. Let $S = S(uv) \subseteq V(G)$ be a vertex set guaranteed by Claim 2.2. By Claim 4.9,

$$S = \big(Q \setminus \{u, v\}\big) \cup \{w \in I \mid uw, vw \in E(G)\}\,.$$

Then by Claim 2.2,

$$\begin{aligned}\frac{|Q| - 2 + |I_u \cap I_v|}{t} = \frac{|S|}{t} &< \omega\big((G - uv) - S\big) = \omega(G - S) + 1 \\ &= \big(|I| - |I_u \cup I_v| + 1\big) + 1 = |I| - |I_u \cup I_v| + 2\,.\end{aligned}$$

Since G is t-tough and Q is a separator (by $|I| \geq 2$),

$$|I| = \omega(G - Q) \leq \frac{|Q|}{t}\,.$$

Thus,

$$
\begin{aligned}
|I| \le \frac{|Q|}{t} &= \frac{|Q|-2+|I_u \cap I_v|}{t} + \frac{2-|I_u \cap I_v|}{t} \\
&< |I| - |I_u \cup I_v| + 2 + \frac{2-|I_u \cap I_v|}{t},
\end{aligned}
$$

that is,

$$
|I_u \cup I_v| < 2 + \frac{2-|I_u \cap I_v|}{t} \le 2 + \frac{2}{t} < 4,
$$

where the last inequality is valid since $t > 1$. Therefore, $|I_u \cup I_v| \le 3$.

By Corollary 4.7, no vertex of Q is adjacent to all vertices of I, thus $Q \setminus \{v\}$ is a separator, and since G is t-tough,

$$
|I| - |I_v| + 1 = \omega\Big(G - \big(Q \setminus \{v\}\big)\Big) \le \frac{\big|Q \setminus \{v\}\big|}{t} = \frac{|Q|-1}{t}
$$

holds. Thus,

$$
\begin{aligned}
|I| - |I_v| + 1 \le \frac{|Q|-1}{t} &= \frac{|Q|-2+|I_u \cap I_v|}{t} + \frac{1-|I_u \cap I_v|}{t} \\
&< |I| - |I_u \cup I_v| + 2 + \frac{1-|I_u \cap I_v|}{t},
\end{aligned}
$$

that is,

$$
|I_u \setminus I_v| \le 1 + \frac{1-|I_u \cap I_v|}{t} \le 1 + \frac{1}{t} < 2,
$$

where the last inequality is valid since $t > 1$. Therefore, $|I_u \setminus I_v| \le 1$, and similarly, $|I_v \setminus I_u| \le 1$.

By the inequalities $|I_u \cup I_v| \le 3$ for any $u, v \in Q$, clearly $|I_u| \le 3$ holds for any $u \in Q$, and it is not difficult to see that if $|I_u| = 3$ for some $u \in Q$, then $|I| = 3$ and thus u is a universal vertex, contradicting Corollary 4.7. Thus, $|I_u| \le 2$ for every $u \in Q$.

If $|I_u| = 1$ for all $u \in Q$, then any vertex in Q is a maximum neighbor of its only neighbor in I, which contradicts Theorem 4.3 since the vertices in I are moplicial.

Thus, there exists a vertex $u \in Q$ with $|I_u| = 2$. Let $I_u = \{a, b\}$. Let $A = N(a)$ and $B = N(b)$. Obviously, $u \in A \cap B$, thus $\{a, b\} \subseteq N(A) \cap I$ and $\{a, b\} \subseteq N(B) \cap I$. In addition, $N(A) \cap I \ne \{a, b\}$, otherwise u would be a maximum neighbor of a, contradicting Theorem 4.3. Similarly, $N(B) \cap I \ne \{a, b\}$. Let $c \in \big(N(A) \cap I\big) \setminus \{a, b\}$ and let $v \in A \setminus \{u\}$ be a neighbor of c (such a vertex v exists since c is of degree at least 3). Now, $N(B) \cap I \ne \{a, b, c\}$, otherwise for any $v' \in N(B) \setminus \{u\}$ which is a neighbor of c, the edge uv' would be a maximum neighboring edge of b, contradicting Theorem 4.3. Let $d \in \big(N(B) \cap I\big) \setminus \{a, b, c\}$, and let $w \in B \setminus \{u, v\}$ be a neighbor of d (such a vertex w exists since d is of degree at least 3). Now $a, c \in I_v$ and $b, d \in I_w$, where a, b, c, d are four distinct vertices, contradicting $|I_v \cup I_w| \leq 3$. □

By Theorem 2.11, the above result, i.e., Theorem 4.10, can be equivalently stated as follows.

Corollary 4.11. *For any rational number $t > 1/2$, every minimally t-tough graph contains either an induced 4-cycle or an induced 5-cycle or a pair of independent edges as an induced subgraph.*

In conclusion, we pose the following.

Conjecture 4.12. *For any rational number $t > 1/2$, there exists no minimally t-tough, chordal graph.*

If true, the conjecture would provide a common generalization of Theorems 4.1 and 4.10 and Corollaries 4.4 and 4.7.

Acknowledgments

The research by Gyula Y. Katona is supported by the BME-Artificial Intelligence FIKP grant of EMMI (BME FIKP-MI/SC), and the research project K-132696 by the Ministry of Innovation and Technology of Hungary from the National Research, Development and Innovation Fund. This work is supported in part by the Slovenian Research Agency (I0-0035, research program P1-0285, research projects J1-3001, J1-3002, J1-3003, J1-4008, J1-9110, N1-0102, and N1-0160, and a Young Researchers Grant).

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