Locally common graphs

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Abstract

Goodman proved that the sum of the number of triangles in a graph on n nodes and its complement is at least $n^3/24$; in other words, this sum is minimized, asymptotically, by a random graph with edge density $1/2$. Erdős conjectured that a similar inequality will hold for K_4 in place of K_3, but this was disproved by Thomason. But an analogous statement does hold for some other graphs, which are called *common graphs*. A characterization of common graphs seems, however, out of reach.

Franek and Rödl proved that K_4 is common in a weaker, local sense. Using the language of graph limits, we study two versions of locally common graphs. We sharpen a result of Jagger, Štovíček and Thomason by showing that no graph containing K_4 can be locally common, but prove that all such graphs are weakly locally common. We also show that not all connected graphs are weakly locally common.

1 Introduction

Let $\mathsf{inj}(F, G)$ denote the number of embeddings of the graph F in the graph G. The following inequality was proved by Goodman [6]:

$$\mathsf{inj}(K_3, G) + \mathsf{inj}(K_3, \overline{G}) \geq \frac{1}{4}|V(G)|^3, \tag{1}$$

where equality holds asymptotically if G is a random graph with edge density $1/2$. Erdős conjectured that a similar inequality will hold for K_4 in place of K_3, but this was disproved by Thomason [17] (see also Thomason [18] for a more "conceptual" proof). More generally, one can ask which graphs F satisfy

$$\mathsf{inj}(F, G) + \mathsf{inj}(F, \overline{G}) \geq \left(1 + o(1)\right) 2^{1 - |E(F)|} |V(G)|^{|V(F)|} \tag{2}$$

for every graph G, where the $o(1)$ refers to $|V(G)| \to \infty$. Such graphs F are called *common graphs*. So the triangle is common, but K_4 is not. (Throughout the paper, we are going to assume that the graphs are simple and, unless stressed otherwise, have no isolated nodes.)

Many classes of bipartite graphs are common, and it is conjectured that they all are. Among non-bipartite graphs, very few are known to be common. Franek and Rödl [5] proved that deleting an edge from K_4 we get a common graph. More recently Hatami, Hladky, Král, Norine and Razborov [7] proved that the 5-wheel is common, thus providing the first common graph with chromatic number 4. In the opposite direction, Jagger, Štovíček and Thomason [8] proved that no graph containing K_4 is common.

It will be more convenient to count homomorphisms instead of embeddings or copies of F. Let $\mathsf{hom}(F, G)$ denote the number of homomorphisms from F into G. We are interested in the case when $|V(G)| \to \infty$, when $\mathsf{inj}(F, G) = \mathsf{hom}(F, G) + O(|V(G)|^{|V(F)|-1})$, and so we could replace inj by hom in the definition of common graphs (2). It will be even better to consider the normalized version $t(F, G) = \mathsf{hom}(F, G)/|V(G)|^k$, which can be interpreted as the probability that a random map $\phi \colon V(F) \to V(G)$ preserves adjacency. With this notation, common graphs are those graphs F for which

$$t(F, G) + t(F, \overline{G}) \geq \left(1 + o(1)\right) 2^{1 - |E(F)|}$$

for simple graphs G with $|V(G)| \to \infty$.

Sidorenko [15] studied various "convexity" properties of graphs, one of which is closely related to common graphs. Let us say that a graph F has the *Sidorenko property*, if for every graph G,

$$t(F, G) \geq t(K_2, G)^{|E(F)|}.$$

It is easy to see that non-bipartite graphs do not have this property, and Sidorenko conjectured that all bipartite graphs do. A closely related conjecture, in a different language, was formulated earlier by Simonovits [16]. For us, the significance of this work is that *the Sidorenko property implies that the graph is common.* So the Sidorenko–Simonovits conjecture would imply that all bipartite graphs are common. Sidorenko's conjecture has been proved for several rather broad classes of bipartite graphs [9, 4]; for a description of these classes, we refer to these publications.

Franek and Rödl [5] proved that K_4 is common in a "local" sense: the original conjecture of Erdős said that the number of K_4's in a graph and in its complement is minimized asymptotically by a random graph, and Franek and Rödl showed that this is true at least for graphs coming from a random graph by a small perturbation. A more natural formulation of this result was given in [11], using notions of graph limit theory (see below).

Somewhat surprisingly, it turns out that whether or not a graph is "locally" common depends on the topology we consider on graph limits. This leads to (at least) two different versions of this notion: "locally common" and "weakly locally common".

More recently Lovász [10] proved a "local" version of Sidorenko's conjecture, and characterized those graphs satisfying the weak local Sidorenko property [11]. If a graph is [locally, weakly locally] Sidorenko, then it is [locally, weakly locally] common, and so these (partial) results about the Sidorenko property have implications about common graphs. In particular, all bipartite graphs are locally common.

The goal of this paper is to show that every graph containing K_4 is locally common in the weakest sense, but not in a stronger sense. We give a rather general sufficient condition for a graph to be weakly locally common, and show that not all connected graphs are weakly locally common.

2 Preliminaries

2.1 Graph limits

We need some definition from the theory of graph limits; see [11] for more detail. A *kernel* is a symmetric bounded measurable function $W : [0,1]^2 \to \mathbb{R}$. (Instead of $[0,1]$ we could use any other standard probability space here, and we shall do so if it is more convenient.) A *graphon* is a kernel with values in $[0,1]$. We denote the set of kernels by $\mathcal{W}$, the set of graphons by $\mathcal{W}_0$, and the set of kernels with values in $[-1,1]$ by $\mathcal{W}_1$.

The significance of graphons is that they provide limit objects for convergent graph sequences. We call a sequence $(G_1, G_2, \dots)$ of (finite) simple graphs *convergent*, if the numerical sequence $t(F, G_n)$ is convergent for every simple graph F [1]. It was proved in [12] that for every convergent graph sequence there is graphon W such that

$$t(F, G_n) \to t(F, W) \qquad (n \to \infty),$$

where

$$t(F, W) = \int_{[0,1]^{V(F)}} \prod_{ij \in E(F)} W(x_i, x_j) \prod_{i \in V} dx_i. \tag{3}$$

Conversely, every graphon represents the limit of a convergent graph sequence.

These results make it possible to formulate our problems in a remainder-term-free form. A simple graph F is common if and only if

$$t(F, W) + t(F, 1 - W) \geq 2^{1 - |E(F)|} = 2t\left(F, \frac{1}{2}\right) \tag{4}$$

for every graphon W (where $1/2$ means the identically-$1/2$ graphon). We can multiply by $2^{|E(F)|}$, and write $W = (1 + U)/2$ (where $U \in \mathcal{W}_1$) to get the inequality

$$t(F, 1 + U) + t(F, 1 - U) \geq 2.$$

We call a simple graph F *locally common for perturbation* $\varepsilon > 0$, if $t(F, 1 + U) + t(F, 1 - U) \geq 2$ for every $U \in \mathcal{W}_1$ with $\|U\|_\infty \leq \varepsilon$. We say that F is *locally common*, if there is an $\varepsilon > 0$ such that F is locally common for perturbation ε.

A related notion is that the graph F is *weakly locally common*[1]: this means that for every $U \in \mathcal{W}_1$ there is an $\varepsilon_U > 0$ such that $t(F, 1 + \varepsilon U) + t(F, 1 - \varepsilon U) \geq 2$ for all $0 \leq \varepsilon \leq \varepsilon_U$.

It is clear that every common graph is locally common, and every locally common graph is weakly locally common. In the other direction, there are weakly locally common graphs which are not locally common, but it is still open whether there are locally common graphs which are not common.

Bipartite graphs are locally common, but not known to be common. As cited above, Thomason [17] proved that the graph K_4 is not common, while Franek and Rödl [5] proved (in a different language) that K_4 is weakly locally common. It will follow from our results that K_4 is not locally common. Jagger, Štovíček and Thomason [8] proved that no graph containing K_4 as a subgraph is common. We are going to prove that a graph containing K_4 is always weakly locally common, but never locally common.

Similarly to common graphs, we can define "local" and "weakly local" versions of other extremal problems. We say that a simple graph F has the *local Sidorenko property for perturbation* ε, if $t(F, 1 + U) \geq 1$ for every $U \in \mathcal{W}_1$ with $\int U = 0$ and $\|U\|_\infty \leq \varepsilon$. It was proved in [10] that every bipartite graph F is locally Sidorenko for perturbation $\varepsilon = 1/(4|E(F)|)$.

We call a simple graph F *weakly locally Sidorenko*, if for every $U \in \mathcal{W}_1$ with $\int U = 0$ there is an $\varepsilon_U > 0$ such that $t(F, 1 + \varepsilon U) \geq 1$ for every $0 \leq \varepsilon \leq \varepsilon_U$. The weak local Sidorenko property is even easier to treat, as noted in [11], Section 16.5.3: *A simple graph has the weak local Sidorenko property if and only if it is a forest or its girth is even.*

These results immediately imply some facts about locally common graphs: every bipartite graph F is locally common for perturbation $1/(4|E(F)|)$, and every graph with even girth is weakly locally common. We are going to prove a more general sufficient condition for being weakly locally common.

2.2 Subgraph densities

We call a graph *mirror-symmetric*, if it is obtained by the following construction: we take a graph G, select a set S of mutually nonadjacent nodes in it, and glue together two copies of G along S.

The following simple fact has been noted in [3]:

Lemma 2.1. *If F is mirror-symmetric, then $t(F, U) \geq 0$ for every kernel U.*

(It is conjectured in [3] that this property characterizes mirror-symmetric graphs.)

Proof. No matter how we fix the variables in the definition of $t(F, U)$ corresponding to nodes in S, integrating the rest gives a square, which is nonnegative. $\qquad\square$

We say that a kernel U is *balanced*, if $\int_0^1 U(x, y)\, dy = 0$ for almost all $x \in [0, 1]$. Analogously, an edge-weighted graph is *balanced*, if for every node v, the sum of weights of edges incident with v is 0.

Lemma 2.2. *A kernel U is balanced if and only if $t(P_3, U) = 0$). If U is a balanced kernel, and F has a node of degree 1, then $t(F, U) = 0$.*

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[1] In [11], only this version was defined and called "locally common".

Proof. Lemma 2.1 implies that $t(P_3, U) \geq 0$ for every kernel U. The case of equality easily follows from the proof of the inequality.

If $\deg_F(u) = 1$, and v is its neighbor, then fixing x_i for $i \neq u$, integrating with respect to u gives 0, by the definition of being balanced. $\qquad\square$

Let $\mathsf{sub}(H, F)$ denote the number of subgraphs of F without isolated nodes isomorphic to H. The densities in the "perturbed" graphons can be expanded:

$$t(F, 1 + U) = \sum_{F' \subseteq F} t(F', U) = \sum_H \mathsf{sub}(H, F) t(H, U). \tag{5}$$

Hence

$$t(F, 1 + U) + t(F, 1 - U) = 2 \sum_{H: \ |E(H)| \ \text{even}} \mathsf{sub}(H, F) t(H, U) = 2 + 2p(F, U), \tag{6}$$

where

$$p(F, U) = \sum_{H: \ 0 < |E(H)| \ \text{even}} \mathsf{sub}(H, F) t(H, U). \tag{7}$$

Using this notation, we get the following rephrasing of the definitions of different versions of the common property.

Proposition 2.3. (a) *A graph F is common if and only if $p(F, U) \geq 0$ for all $U \in \mathcal{W}_1$.*

(b) *A graph F is locally common if and only if there is a number $\varepsilon > 0$ such that $p(F, \varepsilon U) \geq 0$ for all $U \in \mathcal{W}_1$.*

(c) *A graph F is weakly locally common if and only if for every $U \in \mathcal{W}_1$ there is a number $\gamma_U > 0$ such that $p(F, \varepsilon U) \geq 0$ for all $0 < \varepsilon \leq \gamma_U$.*

Defining

$$c_r(F, U) = \sum_{H: \ |E(H)| = r} \mathsf{sub}(H, F) t(H, U) \quad (r = 0, 1, \dots), \tag{8}$$

we can express $p(F, \varepsilon U)$ as a polynomial in ε:

$$p(F, \varepsilon U) = \sum_{r=1}^{\lfloor |E(F)|/2 \rfloor} \varepsilon^{2r} c_{2r}(F, U). \tag{9}$$

Using this expansion, assertion (c) in Proposition 2.3 can be rephrased as follows: *A graph F is weakly locally common if and only if for every $U \in \mathcal{W}_1$, either $c_2(F, U) = c_4(F, U) = \cdots = 0$, or the first nonzero number in the sequence $c_2(F, U), c_4(F, U), \dots$ is positive.*

For a short proof of the result of Franek and Rödl [5] that the graph obtained from K_4 by deleting an edge is common, using this language, see [11], Section 16.5.4.

3   Locally common graphs

Our goal is to prove the following strengthening of the result of Jagger, Štovíček and Thomason [8], asserting that graphs containing K_4 are never common.

Theorem 3.1. *No graph containing K_4 is locally common.*

Proof. We start with some general consequences of the expansion formulas in the previous section. Let us introduce two operations on kernels: for a kernel U and $0 < \delta \leq 1$, define a kernel $U_\delta \in \mathcal{W}_1$ by

$$U_\delta(x, y) = \begin{cases} U(x/\delta, y/\delta), & \text{if } x, y \leq \delta, \\ 0, & \text{otherwise.} \end{cases}$$

For a kernel U and positive integer m, we define the "tensor power" kernel $U^{\otimes m} : [0,1]^m \times [0,1]^m \to [-1,1]$ by

$$U^{\otimes m}\big((x_1, \ldots, x_m), (y_1, \ldots, y_m)\big) = U(x_1, y_1) \cdots U(x_m, y_m).$$

It is straightforward that if $U \in \mathcal{W}_1$ is balanced, then so are U_δ and $U^{\otimes m}$. Furthermore, $t(F, U_\delta) = \delta^{|V(F)|} t(F, U)$ and $t(F, U^{\otimes m}) = t(F, U)^m$. (We will use an odd m in this construction, so that the sign of $t(F, U)$ is preserved.)

Substituting these expressions, we get the expansion

$$p(F, \varepsilon(U^{\otimes m})_\delta) = \sum_{H:\ 0<|E(H)|\ \text{even}} \mathsf{sub}(H, F)\, \varepsilon^{|E(H)|}\, \delta^{|V(H)|} t(H, U)^m$$

$$= \sum_{q=2}^{|V(F)|} \delta^q \sum_{\substack{H:\ |E(H)|\ \text{even} \\ |V(H)|=q}} \mathsf{sub}(H, F)\varepsilon^{|E(H)|} t(H, U)^m. \qquad (10)$$

Suppose that F is locally common for perturbation ε. Then $p(F, \varepsilon U) \geq 0$ for every kernel $U \in \mathcal{W}_1$, including every kernel of the form $(U^{\otimes m})_\delta$. The parameter ε is fixed, but we can play with the parameters δ and m.

Letting $\delta \to 0$, we get that the first nonzero term in the outer sum must be positive. There is only one term with $q \leq 3$, namely $H = P_3$, and by Lemma 2.2, $t(P_3, U) > 0$ unless U is balanced. So let us assume that U is balanced. Then Lemma 2.2 implies that only those terms are nonzero where all degrees in H are at least 2. There are only two such graphs with $q = 4$, namely $H = C_4$ and $H = K_4$. Thus (simplifying by $\delta^4 \varepsilon^2$) we get a necessary condition for being locally common for perturbation ε:

$$\mathsf{sub}(C_4, F)t(C_4, U)^m + \varepsilon^2 \mathsf{sub}(K_4, F)t(K_4, U)^m \geq 0. \qquad (11)$$

for every balanced kernel $U \in \mathcal{W}_1$. Here $t(C_4, U) > 0$, so the condition is trivially satisfied if F contains no K_4. Our goal is to prove the converse.

Letting $m \to \infty$, this implies that

$$t(C_4, U) \geq \begin{cases} -t(K_4, U), & \text{if } \mathsf{sub}(K_4, F) > 0, \\ 0, & \text{otherwise.} \end{cases} \qquad (12)$$

This strange conclusion, which is independent of ε and almost independent of F, says the following: either $t(C_4, U) + t(K_4, U) \geq 0$ for every balanced $U \in \mathcal{W}_1$, or no locally common graph contains K_4. We show that the second alternative occurs, by constructing a kernel U violating the first inequality. The construction is carried out in several steps.
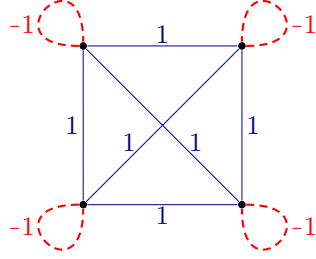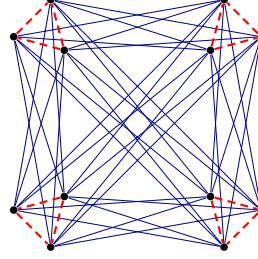
Figure 1/a: G_1

Figure 1/b: $G_2 = K_3 \times G_1$

$$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$
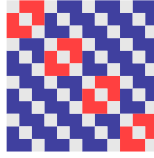

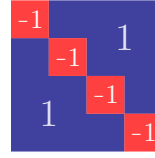Figure 2/a: G_1 Figure 2/b: G_2 Figure 2/c: $\lim_{n \to \infty} K_n \times G_1$

In Figures 1/a and 1/b, solid blue lines indicate edges with weight 1, red dashed lines indicate edges with weight -1. Figures 2/a and 2/b show their adjacency matrices, blue, grey and red represent 1, 0, -1, respectively. In Figures 1/b and 2/b, we used $n = 3$, namely, $G_2 = K_3 \times G_1$. In the language of graph limits, Figure 2/c shows the graphon of $K_n \times G_1$ in the limit $n \to \infty$.

Claim 1. *There exists a looped-simple graph G_1 with edgeweights ± 1 such that $t(C_4, G_1) + t(K_4, G_1) = -1/4$.*

Let G_1 obtained from K_4 by adding a loop with weight -1 at every node. Then $t(C_4, G_1) + t(K_4, G_1) = -1/4$ by direct calculation. Note that $t(C_4, G_1) \geq 0$, so $t(K_4, G_1) < 0$.

Claim 2. *There exists an arbitrarily large simple graph G_2 (without loops) with edgeweights ± 1 such that $t(C_4, G_2) + t(K_4, G_2) \leq -1/5$.*

Indeed, consider any looped-simple graph G with the properties of Claim 1, and take its categorical product $G_2 = K_n \times G$, where K_n is a large complete graph (without loops). Then G_2 has no loops, and

$$t(C_4, G_2) + t(K_4, G_2) = t(C_4, K_n)t(C_4, G_1) + t(K_4, K_n)t(K_4, G_1)$$

$$\to t(C_4, G_1) + t(K_4, G_1) = -\frac{1}{4} \qquad (n \to \infty).$$

So $t(C_4, G_2) + t(K_4, G_2) \leq -1/5$ if n is large enough.

Claim 3. *There exists a simple graph G_3 with balanced edgeweights ± 1 such that $t(C_4, G_3) + t(K_4, G_3) < 0$.*

Let G_2 be a graph in Claim 2, and let $V(G_2) = [r]$. Note that r can be arbitrarily large.
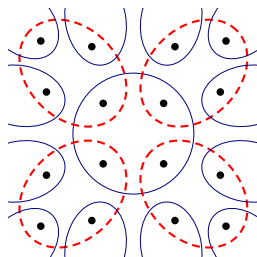
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Figure 3: The local structure of the hypergraph H for $r = 4$. The hyperedges A_i and B_j are shown by blue solid ellipses and red dashed ellipses, respectively.

There is an r-uniform r-partite hypergraph H with two families of edges $\{A_1, \ldots, A_N\}$ and $\{B_1, \ldots, B_N\}$ such that the sets A_i as well as the sets B_i form a partition of $V(H)$, and H has girth at least 5. (The dual hypergraph of an r-regular bipartite graph with large girth has these properties.) Let $V_1, \ldots, V_r$ be the partition classes of H. We glue a copy of G_2 on every A_i and every B_i (node u of G_2 is glued onto the node of A_i in V_u). In the sets A_i, we keep the original weighting of the edges; in the sets B_i, we multiply them by -1.

It is clear that the weighted graph G_3 constructed this way is balanced. Furthermore, every homomorphism $K_4 \to G_3$ maps K_4 into one of the A_i or into one of the B_i, and hence $\mathsf{hom}(K_4, G_3) = 2N\,\mathsf{hom}(K_4, G_2)$. This is not quite true for C_4 in place of K_4, but the difference is small: it counts those homomorphisms $C_4 \to G_3$ for which two opposite nodes of C_4 are mapped onto the same node v of G_3, and the other two nodes are mapped into different copies of G_2 containing v. Hence

$$\mathsf{hom}(C_4, G_3) - 2N\,\mathsf{hom}(C_4, G_2) \leq 2r^3 N,$$

and for $r > 5$,

$$
\begin{aligned}
t(C_4, G_3) + t(K_4, G_3) &= \frac{1}{r^4 N^4}\Big(\mathsf{hom}(C_4, G_3) + \mathsf{hom}(K_4, G_3)\Big) \\
&\leq \frac{1}{r^4 N^3}\Big(2\,\mathsf{hom}(C_4, G_2) + 2\,\mathsf{hom}(K_4, G_2) + 2r^3\Big) \\
&= \frac{2}{N^3}\Big(t(C_4, G_2) + t(K_4, G_2) + \frac{1}{r}\Big) < 0.
\end{aligned}
$$

This proves Claim 3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

4 Weakly locally common graphs

We have seen that every forest and every graph with even girth is weakly locally common. We prove more in the next Theorem. Let $g_{\text{even}}(F)$ denote the length of the shortest even cycle of F, where $g_{\text{even}} = \infty$ if F has no even cycle.

Theorem 4.1. *If F is not weakly locally common, then F has two (odd) cycles with at most one node in common, of lengths g_1 and g_2, such that either $g_1 < g_2$ and $g_1 + g_2 \leq g_{\text{even}}(F)$ or $g_1 = g_2$ and $g_1 + g_2 < g_{\text{even}}(F)$.*
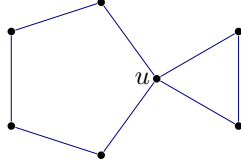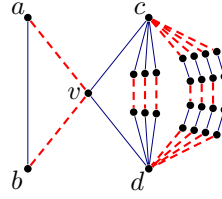
Figure 4/a: The graph F.   Figure 4/b: The graph G with $k = 3$.

In particular, if the length of the shortest even cycle in F is at most twice of the length of the shortest odd cycle in F, or F has no odd cycle, then F is weakly locally common.

Proof. Suppose that F is not weakly locally common. Then the sequence $c_2(F, U)$, $c_4(F, U), \ldots$ has a nonzero term, and its first nonzero term, say $c_{2p}(F, U)$, is negative. We know that $c_2(F, U) \geq 0$ by Lemma 2.1, so $p > 1$. Hence $c_2(F, U) = 0$, which implies that U is balanced. In this case, $t(H, U) = 0$ for every graph H having a node of degree 1 by Lemma 2.2.

Let r be the smallest positive integer for which F has a subgraph H with $2r$ edges and $t(H, U) < 0$ (the inequality $c_{2p}(F, U) < 0$ implies that such a subgraph exists and $r \leq p$). We know by the above that $2 \leq r$. Lemma 2.2 implies that all degrees in H are at least 2. We have $t(F', U) = 0$ for every subgraph F' of F with $|E(F')| < 2r$ and $|E(F')|$ even. Since U is not almost everywhere zero, Lemma 2.1(a) implies that $g_{\text{even}} \geq 2r$ and since H is not an even cycle, it cannot contain an even cycle.

It is a well-known elementary exercise that every block (2-connected component) of such a graph is an odd cycle. So H contains two odd cycles C and C' of lengths g_1 and g_2 intersecting in at most one node. Therefore, $g_1 + g_2 \leq |E(H)| = 2r \leq g_{\text{even}}$.

To complete the proof, we have to exclude the case $g_1 = g_2 = r$. In this case $H = C \cup C'$, and H is mirror-symmetric, which implies by Lemma 2.1 that $t(H, U) \geq 0$. □

Corollary 4.2. *Every graph containing C_4 or C_6 is weakly locally common.*

Proposition 4.3. *There exist connected graphs that are not weakly locally common.*

Proof. Let F consist of a triangle and a pentagon, attached to each other at one node u. (Figure 4/a.) We construct a balanced edge-weighted graph G with edgeweights ± 1 such that $t(F, G) < 0$. (Figure 4/b.) We start with a 4-star with center node v and endnodes a, b, c, d. Let k be a large positive integer. We connect a and b by an edge; we attach k openly disjoint paths $Q_1, \ldots, Q_k$ of length 3 and $k+1$ further openly disjoint paths $R_1, \ldots, R_{k+1}$ of length 5 connecting c and d. We weight the following edges with -1: the edges va and vb; the middle edge of every path Q_i; and every second edge of each path R_i, starting at the end. The remaining edges are weighted with 1. It is clear that the weighting is balanced.

We claim that

$$t(F, G) < 0. \tag{13}$$

The normalization is irrelevant, so it suffices to show that $\mathsf{hom}(F, G) < 0$. Let $\phi : V(F) \to V(G)$ be a homomorphism. The triangle in F must be mapped onto the triangle in G. If the pentagon in F is mapped into the subgraph $G[S]$ induced by $S = \{v, a, b, c, d\}$, then the contribution of ϕ is positive, but the number of these

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maps is independent of k (52, in fact). If the image of the pentagon contains a node outside S, then it must contain one of the paths Q_i, and then u must be mapped onto v. The contribution from such a map is -1, and the number of such maps is $4k$. Thus $\mathsf{hom}(F, G) = 52 - 4k$, which is negative if $k > 13$. This proves (13).

The condition that G is balanced implies that $t(F', W_G) = 0$ if F' has a node with degree 1. The only subgraph of F with an even number of edges and with all degrees at least 2 is F itself, and hence $c_2(F, W_G) = c_4(F, W_G) = c_6(F, W_G) = 0$ but $c_8(F, W_G) = t(F, W_G) < 0$. Thus F is not weakly locally common. $\qquad\square$

5   Open problems

In the definition of locally common graphs, we can consider various norms on the space $\mathcal{W}$ instead of the L_∞ norm. Can the results above be extended to other norms? An important candidate is the *cut norm*, defined by

$$\|W\|_\square = \sup_{S,T \subseteq [0,1]} \left| \int_{S \times T} W(x,y) \, dx \, dy \right|,$$

playing an important role in the theory of graph limits. It was proved in [10] that every bipartite graph is locally Sidorenko with respect to the cut norm. Since the cut norm is continuous with respect to every "reasonable" norm on $\mathcal{W}$ (for an exact formulation of this fact see [11], Theorem 14.10), it follows that every bipartite graph is locally Sidorenko in every "reasonable" norm on $\mathcal{W}$.

Similarly to common graphs and Sidorenko graphs, we can define "local" and "weakly local" versions of other extremal properties, but little is known in this direction.

Are there any non-common graphs that are locally common in the cut norm or the L_∞ norm? Is there a graph that is locally common with respect to the L_∞ norm, but not with respect to the cut norm? Can weakly locally common graphs be characterized similarly as weakly locally Sidorenko graphs?

References

[1] C. Borgs, J. Chayes, L. Lovász, V.T. Sós and K. Vesztergombi: Counting graph homomorphisms, in: *Topics in Discrete Mathematics* (ed. M. Klazar, J. Kratochvil, M. Loebl, J. Matoušek, R. Thomas, P. Valtr), Springer (2006), 315–371.

[2] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós and K. Vesztergombi: Convergent Graph Sequences I: Subgraph frequencies, metric properties, and testing, *Advances in Math.* **219** (2008), 1801–1851.

[3] O.A. Camarena, E. Csóka, T. Hubai, G. Lippner and L. Lovász: Positive graphs, *European Journal of Combinatorics* **52**, Part B (2016), 290–301.

[4] D. Conlon and J. Lee: Sidorenko's conjecture for blow-ups, https://arxiv.org/abs/1809.01259

[5] F. Franek and V. Rödl: Ramsey Problem on Multiplicities of Complete Subgraphs in Nearly Quasirandom Graphs, *Graphs and Combinatorics* **8** (1992), 299–308.

[6] A.W. Goodman: On sets of aquaintences and strangers at any party, *Amer. Math. Monthly* **66** (1959) 778–783.

[7] H. Hatami, J. Hladky, D. Kral, S. Norine and A. Razborov: Non-three-colorable common graphs exist, http://arxiv.org/abs/1105.0307

[8] C. Jagger, P. Štovíček and A. Thomason: Multiplicities of subgraphs, *Combinatorica* **16** (1996), 123–141.

[9] J.L.X. Li and B. Szegedy: On the logarithimic calculus and Sidorenko's conjecture,
http://arxiv.org/abs/math/1107.1153

[10] L. Lovász: Subgraph densities in signed graphons and the local Sidorenko conjecture, *Electr. J. Combin.* **18** (2011), P127 (21pp).

[11] L. Lovász: *Large networks and graph limits*, Amer. Math. Soc., Providence, R.I. (2012).

[12] L. Lovász and B. Szegedy: Limits of dense graph sequences, *J. Combin. Theory B* **96** (2006), 933–957.

[13] A.F. Sidorenko: Inequalities for functionals generated by bipartite graphs (Russian) *Diskret. Mat.* **3** (1991), 50–65; translation in *Discrete Math. Appl.* **2** (1992), 489–504.

[14] A.F. Sidorenko: A correlation inequality for bipartite graphs, *Graphs and Combin.* **9** (1993), 201–204.

[15] A.F. Sidorenko: Randomness friendly graphs, *Random Struc. Alg.* **8** (1996), 229–241.

[16] M. Simonovits: Extremal graph problems, degenerate extremal problems, and supersaturated graphs, in: *Progress in Graph Theory*, NY Academy Press (1984), 419–437.

[17] A. Thomason: A disproof of a conjecture of Erdős in Ramsey theory, *J. London Math. Soc.* **39** (1898), 246–255.

[18] A. Thomason: Graph Products and Monochromatic Multiplicities, *Combinatorica* **17** (1997), 125–134.