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PROBABILISTIC GLOBAL WELL-POSEDNESS FOR A VISCOUS NONLINEAR WAVE EQUATION MODELING FLUID-STRUCTURE INTERACTION

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ABSTRACT. We prove probabilistic well-posedness for a 2D viscous nonlinear wave equation modeling fluid-structure interaction between a 3D incompressible, viscous Stokes flow and nonlinear elastodynamics of a 2D stretched membrane. The focus is on (rough) data, often arising in real-life problems, for which it is known that the deterministic problem is ill-posed. We show that random perturbations of such data give rise almost surely to the existence of a unique solution. More specifically, we prove almost sure global well-posedness for a viscous nonlinear wave equation with the subcritical initial data in the Sobolev space $\mathcal{H}^s(\mathbb{R}^2)$, $s > -\frac{1}{5}$, which are randomly perturbed using Wiener randomization. This result shows "robustness" of nonlinear FSI problems/models, and provides confidence that even for the "rough data" (data in $\mathcal{H}^s$, $s > -\frac{1}{5}$) random perturbations of such data (due to e.g., randomness in real-life data, numerical discretization, etc.) will almost surely provide a unique solution which depends continuously on the data in the $\mathcal{H}^s$ topology.

1. Background

This paper is motivated by a study of fluid-structure (FSI) interaction and the impact of *rough data* and *random perturbations* of such data on the solution of a nonlinear fluid-structure interaction problem, where the nonlinearity may come, e.g., from a nonlinear forcing of the structure. The main motivation derives from real-life applications that often exhibit such data and nonlinear forcing, e.g., coronary arteries contracting and expanding on the surface of a moving heart, where the forcing comes from the surrounding heart tissue.

In particular, we are interested in the flow of a viscous incompressible fluid modeled by the Stokes equations in a channel that is bounded by a two-dimensional membrane, modeled by a nonlinear wave equation. See Fig. 1. A competition between dissipative effects coming from the fluid viscosity, dispersion and nonlinear effects coming from the structure model are of particular interest, especially for the (rough) initial data for which one can show that the Sobolev $\mathcal{H}^s$ norm gets inflated very quickly after a small time $t_\epsilon > 0$ (the $\mathcal{H}^s$ norm gets greater than $1/\epsilon$), even for the initial data with a small $\mathcal{H}^s$ norm (less than ϵ). For example, due to the nonlinearity in the problem, "small" oscillations in the initial data get amplified quickly, giving rise to an ill-posed problem in Hadamard sense. This is often the case for the initial data in $\mathcal{H}^s$ with the Sobolev exponent s below a critical exponent s_{crit} (rough initial data), where s_{crit} is "given" by the "natural" scaling of the problem. In this paper we show that for a range of Sobolev exponents *below the critical exponent*, $s_{\min} < s < s_{\text{crit}}$ where $s_{\min} < 0$, random perturbations of such initial data via a Wiener randomization will

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still give rise to a globally well-posed problem almost surely. This result shows "robustness" of nonlinear FSI problems/models, and provides confidence that even for the "rough data" (data in $\mathcal{H}^s$, $s \in (s_{\min}, s_{\text{crit}})$), random perturbations due to a combination of factors (e.g., real-life data, numerical discretization, etc.) will almost surely provide a solution which depends continuously on the data in the $\mathcal{H}^s$ topology.

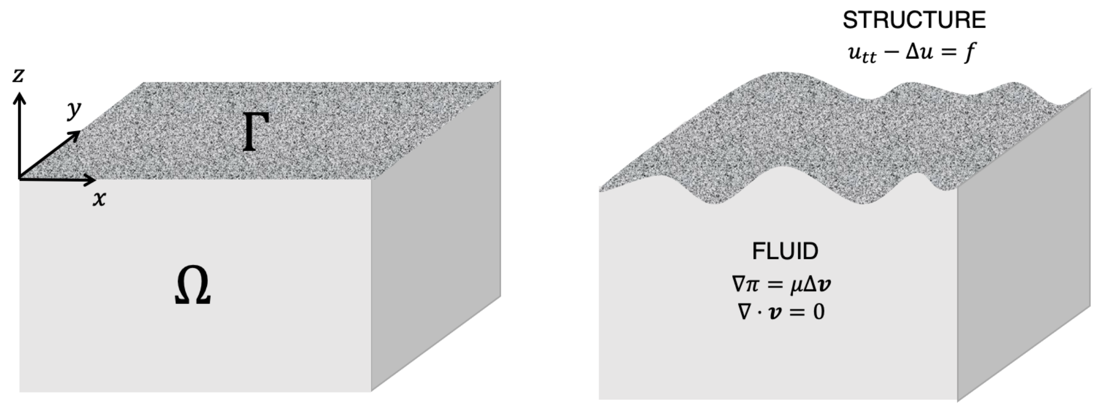

FIGURE 1. A sketch of the reference configurations for the structure and fluid (left), and the fluid-structure interaction system with nonzero vertical displacement of the structure (right).

More precisely, we study a prototype equation capturing dispersive, dissipative, and nonlinear (forcing) effects in fluid-structure interaction problems between the flow of an incompressible, viscous fluid modeled by the 3D Stokes equations, and the elastodynamics of a 2D elastic membrane modeled by a nonlinear wave equation. The prototype equation is the following *viscous nonlinear wave equation* (vNLW) defined on $\mathbb{R}^2$, given by a 2D nonlinear wave equation (NLW) augmented by the viscoelastic effects modeled by the *fractional Laplacian operator* (Dirichlet-to-Neumann operator) $D = |\nabla| = \sqrt{-\Delta}$ applied to the time derivative $\partial_t u$, where u denotes vertical membrane displacement:

$$\begin{cases} \partial_t^2 u - \Delta u + 2\mu D \partial_t u + |u|^{p-1} u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \qquad (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+, \tag{1.1}$$

Here $\mu > 0$ is a constant denoting fluid viscosity, and $\mathbb{R}_+ = [0, \infty)$.

Background on the viscous nonlinear wave equation. The viscous nonlinear wave equation (1.1) was derived in [38] by coupling the elastodynamics of a 2D elastic, prestressed membrane whose reference configuration is given by the infinite plane

$$\Gamma = \{(x, y, 0) \in \mathbb{R}^3\},$$

with the flow of an incompressible, viscous Newtonian fluid residing in the lower half space, which we will denote by

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : z < 0\}.$$

See Figure 1. The membrane and the fluid are *linearly coupled*, namely, the fluid domain remains fixed over time. The structure is assumed to only experience displacement in the

z direction, which is denoted by u, where u satisfies the following wave equation

$$\partial_t^2 u - \Delta u = f, \qquad (x, y, t) \in \mathbb{R}^2 \times \mathbb{R}_+. \tag{1.2}$$

Here f is the external loading force on the elastic membrane, which can be nonlinear, as we specify later.

The membrane interacts with the flow of an incompressible, viscous Newtonian fluid, defined on the domain Ω, which is fixed in time due to the assumption of linear coupling. In order to isolate the dynamical effect of the fluid viscosity on the structure, we model the fluid velocity $\boldsymbol{v} = (v_1, v_2, v_3)$ and pressure π by the stationary Stokes equations

$$\begin{cases} \nabla \pi = \mu \Delta \boldsymbol{v} \\ \nabla \cdot \boldsymbol{v} = 0 \end{cases} \qquad \text{on } \Omega, \tag{1.3}$$

where the constant, $\mu > 0$, denotes the fluid viscosity. See Figure 1.

The fluid and structure are coupled via a two-way coupling, specified by the following two coupling conditions:

(1) The *kinematic coupling condition*, which in our problem is a no-slip condition (the trace of the fluid velocity at the interface Γ is equal to the structure velocity):

$$\boldsymbol{v} = (\partial_t u)\boldsymbol{e_z}, \qquad \text{on } \Gamma, \tag{1.4}$$

where $\boldsymbol{e_z} = (0, 0, 1)$; and

(2) The *dynamic coupling condition*, which states that the elastodynamics of the membrane is driven by the jump across Γ between the normal component of the normal Cauchy fluid stress $\boldsymbol{\sigma}$ and the external forcing F_{ext}:

$$f = -\boldsymbol{\sigma} \boldsymbol{e_z} \cdot \boldsymbol{e_z}|_\Gamma + F_{\text{ext}},$$

where

$$\boldsymbol{\sigma} = -\pi \mathbf{Id} + 2\mu \boldsymbol{D}(\boldsymbol{v}). \tag{1.5}$$

Thus, the structure equation with the dynamic coupling condition reads

$$\partial_t^2 u - \Delta u = -\boldsymbol{\sigma} \boldsymbol{e_z} \cdot \boldsymbol{e_z}|_\Gamma + F_{\text{ext}}. \tag{1.6}$$

For completeness, we summarize the derivation here. We start by noting that the fluid load is given *entirely by the pressure*, due to the particular geometry of this model. Specifically,

$$-\boldsymbol{\sigma} \boldsymbol{e_z} \cdot \boldsymbol{e_z}|_\Gamma = \left(\pi - 2\mu \frac{\partial v_3}{\partial z} \right)\Big|_\Gamma = \pi|_\Gamma,$$

which follows from $\frac{\partial v_3}{\partial z}|_\Gamma = 0$ by the incompressibility condition, the kinematic coupling condition (1.4), and the fact that $v_1 = v_2 = 0$ on Γ. Hence, we obtain

$$\partial_t^2 u - \Delta u = \pi|_\Gamma + F_{\text{ext}}. \tag{1.7}$$

One can then "solve" the stationary Stokes equations (1.3) with the boundary condition (1.4) on Γ for π using a Fourier transform argument, to obtain the final result:

$$\pi|_\Gamma = -2\mu D \partial_t u. \tag{1.8}$$

We will only sketch the main steps of this derivation in the following, and refer readers to the full derivation in [38].

From (1.7), we see that the goal is to express $\pi|_\Gamma$ in terms of the structure displacement u and its derivatives. We use the fact that π and $\boldsymbol{v}$ satisfy the stationary Stokes equations (1.3) with a boundary condition provided by the kinematic coupling condition (1.4). We impose a boundary condition at infinity that $\boldsymbol{v}$ is bounded and π decreases to zero at infinity. From the stationary Stokes equations (1.3), one concludes that π is a harmonic function in Ω with a normal derivative along Γ given by

$$\frac{\partial \pi}{\partial z}\bigg|_\Gamma = \left(\mu \Delta_{x,y} v_3 + \mu \frac{\partial^2 v_3}{\partial z^2}\right)\bigg|_\Gamma. \tag{1.9}$$

Hence, we can find $\pi|_\Gamma$ in (1.8) by inverting the Dirichlet-Neumann operator on the lower half space Ω.

Our main goal is to express v_3 in terms of u and its derivatives, as this will give the Neumann boundary condition for the harmonic function π in (1.9). By taking the Laplacian of the first equation in (1.9) and recalling that π is harmonic, we see that v_3 satisfies the biharmonic equation:

$$\Delta^2 v_3 = 0 \tag{1.10}$$

with boundary conditions given by the kinematic boundary condition (see (1.4)):

$$v_3|_\Gamma = \partial_t u \tag{1.11}$$

Furthermore, there is a boundary condition at infinity that v_3 must be bounded in Ω.

By taking the Fourier transform of (1.10) in the x and y variables but not the z variable, we can establish that

$$\widehat{v}_3(\xi, z) = \widehat{\partial_t u}(\xi) e^{|\xi| z} - |\xi| \widehat{\partial_t u}(\xi) z e^{|\xi| z}, \tag{1.12}$$

where ξ denotes the frequency variable corresponding to the x and y variables. For more details, see the explicit calculation in [38]. Then, by taking the Fourier transform of (1.9) in the x and y variables and using (1.11) and (1.12), we obtain

$$\frac{\partial \widehat{\pi}}{\partial z}\bigg|_\Gamma = -2\mu |\xi|^2 \widehat{\partial_t u}(\xi). \tag{1.13}$$

By taking the inverse Fourier transform, this gives the Neumann boundary condition for the harmonic function π in terms of (derivatives) of u. Recall that the Dirichlet-Neumann operator for the lower half plane with the vanishing boundary condition at infinity is given by $D = \sqrt{-\Delta}$; see [13]. By inverting this operator, we see that the Neumann-Dirichlet operator with the same boundary condition at infinity is given by the Riesz potential $D^{-1} = (-\Delta)^{-\frac{1}{2}}$ with a Fourier multiplier $|\xi|^{-1}$. Therefore, by applying the Neumann-Dirichlet operator to (the inverse Fourier transform of) (1.13), we obtain the desired result in (1.8).

As a model for nonlinear restoring external forcing effects, we consider a defocusing power nonlinearity of the form

$$F_{\text{ext}}(u) = -|u|^{p-1} u, \tag{1.14}$$

for positive integers $p > 1$. Such a power-type nonlinearity has been studied extensively for dispersive equations such as the nonlinear Schrödinger equations and the nonlinear wave equations; see, for example, [69]. Combining (1.7), (1.8), and (1.14) gives the final form of the viscous nonlinear wave equation, as stated in (1.1), in dimension $d = 2$. Although

$d = 2$ corresponds to the scenario described in this fluid-structure interaction model, the equation (1.1) can be stated in full generality for arbitrary dimension d.

Critical exponent and ill-posedness. Let us now turn to analytical aspects of the viscous NLW (1.1). When $\mu \geq 1$, this equation is purely parabolic, where the general solution to the homogeneous linear equation

$$\partial_t^2 u - \Delta u + 2\mu D\partial_t u = 0$$

with initial data $(u, \partial_t u)|_{t=0} = (u_0, u_1)$, is given by

$$u(t) = e^{-\mu|\nabla| + \sqrt{(\mu^2-1)|\nabla|^2}} f_1 + e^{-\mu|\nabla| - \sqrt{(\mu^2-1)|\nabla|^2}} f_2.$$

Noting that $-\mu|\xi| + \sqrt{(\mu^2-1)|\xi|^2} \sim \mu^{-1}|\xi|$ in this case ($\mu \geq 1$), the solution theory can be studied by simply using the Schauder estimate for the Poisson kernel (see Lemma 2.5 below). We will not pursue this direction in this paper. Instead, our main interest in this paper is to study the combined effect of the dissipative-dispersive mechanism, appearing in (1.1). As such, we will restrict our attention to $0 < \mu < 1$. Without loss of generality, we set $\mu = \frac{1}{2}$ as in [38] and focus on the following version of vNLW:

$$\begin{cases} \partial_t^2 u - \Delta u + D\partial_t u + |u|^{p-1}u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases} \tag{1.15}$$

As in the case of the usual NLW:

$$\partial_t^2 u - \Delta u + |u|^{p-1}u = 0, \tag{1.16}$$

the viscous NLW in (1.15) enjoys the following scaling symmetry. If $u(x,t)$ is a solution to (1.15), then $u^\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda t)$ is also a solution to (1.15) for any $\lambda > 0$. This induces the critical Sobolev regularity s_{crit} on $\mathbb{R}^d$ given by

$$s_{\text{crit}} = \frac{d}{2} - \frac{2}{p-1}$$

such that the homogeneous Sobolev norm on $\mathbb{R}^2$ remains invariant under this scaling symmetry. This scaling heuristics provides a common conjecture that an evolution equation is well-posed in H^s for $s > s_{\text{crit}}$, while it is ill-posed for $s < s_{\text{crit}}$. Indeed, for many dispersive PDEs, ill-posedness below a scaling critical regularity is known. In particular, the following form of strong ill-posedness, known as *norm inflation*, is established for many dispersive PDEs, including NLW; see [19, 11, 14, 37, 56, 63, 18, 64, 70, 60, 28]. Norm inflation in the case of the wave equation on $\mathbb{R}^d$ states the following: given any $\varepsilon > 0$, there exist a solution u to (1.16) and $t_\varepsilon \in (0, \varepsilon)$ such that

$$\|(u, \partial_t u)(0)\|_{\mathcal{H}^s} < \varepsilon \qquad \text{but} \qquad \|(u, \partial_t u)(t_\varepsilon)\|_{\mathcal{H}^s} > \varepsilon^{-1},$$

where

$$\mathcal{H}^s(\mathbb{R}^d) = H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d).$$

In [38], Kuan and Čanić studied this issue for vNLW (1.15). Due to the presence of the viscous term in (1.15), which induces some smoothing property, one may expect to have a different ill-posedness result but this was shown not to be the case. More precisely, Kuan and Čanić proved norm inflation for vNLW (1.15) in $\mathcal{H}^s(\mathbb{R}^d)$ for $0 < s < s_{\text{crit}}$ (for any odd integer $p \geq 3$) as in the case of the usual NLW. Moreover, they showed that the viscous contribution has the potential to slow down the speed of the norm inflation. See [38] for details. It is of interest to see if norm inflation in negative Sobolev spaces for the usual NLW [19, 60, 28] carries over to the viscous NLW. See [26].

This norm inflation for vNLW (1.15) shows that the equation is ill-posed in $\mathcal{H}^s(\mathbb{R}^d)$ for $0 < s < s_{\text{crit}}$, showing that there is no hope in studying well-posedness in this low regularity space in a deterministic manner.

However, we can go beyond the limit of deterministic analysis and consider our Cauchy problem with randomized initial data. The area of nonlinear dispersive equations with randomized initial data has become rather active in recent years [9, 11, 23, 49, 12, 5, 4, 65, 57, 6]. See also a survey paper [7] in this direction.

In fact, in [38] Kuan and Čanić considered the Cauchy problem (1.15) with $p = 5$ and dimension $d = 2$, and proved almost sure local well-posedness for *randomized initial data* in $\mathcal{H}^s(\mathbb{R}^2)$ for $s > -\frac{1}{6}$. In this manuscript we extend this result in two important directions:

(1) We prove *global* rather than local well-posedness in the probabilistic sense for randomized initial data in $\mathcal{H}^s(\mathbb{R}^2)$ $s > s_{\min}$ where $s_{\min} = -1/5$; and
(2) We extend the interval for the exponents s from $s_{\min} = -1/6$ to $s_{\min} = -1/5$, where the threshold $s_{\min} = -1/5$ seems to be sharp, see Remark 3.6(i).

Since the randomized initial data considered in this work are given in terms of Wiener randomization, we provide a brief description of Wiener randomization next.

Wiener randomization. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\operatorname{supp} \psi \subset [-1, 1]^d$, $\psi(-\xi) = \overline{\psi(\xi)}$, and

$$\sum_{n \in \mathbb{Z}^d} \psi(\xi - n) \equiv 1 \quad \text{for all } \xi \in \mathbb{R}^d.$$

Then, any function f on $\mathbb{R}^d$ can be written as

$$f = \sum_{n \in \mathbb{Z}^d} \psi(D - n) f, \tag{1.17}$$

where $\psi(D - n)$ denotes the Fourier multiplier operator with symbol $\psi(\,\cdot\, - n)$. Hence, $\psi(D-n)f$ localizes f in the frequency space around the frequency $n \in \mathbb{Z}^d$ over a unit scale. We recall a particular example of Bernstein's inequality:

$$\|\psi(D - n) f\|_{L^q(\mathbb{R}^d)} \lesssim \|\psi(D - n) f\|_{L^p(\mathbb{R}^d)} \tag{1.18}$$

for any $1 \le p \le q \le \infty$. This classical inequality follows from the localization in the frequency space due to the compact support of ψ, and Young's convolution inequality (see, for example, Lemma 2.1 in [49]).

We now introduce a randomization adapted to the uniform decomposition (1.17). For $j = 0, 1$, let $\{g_{n,j}\}_{n \in \mathbb{Z}^d}$ be a sequence of mean-zero complex-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that

$$g_{-n,j} = \overline{g_{n,j}} \tag{1.19}$$

for all $n \in \mathbb{Z}^d$, $j = 0, 1$. In particular, $g_{0,j}$ is real-valued. Moreover, we assume that $\{g_{0,j}, \operatorname{Re} g_{n,j}, \operatorname{Im} g_{n,j}\}_{n \in \mathcal{I}, j = 0,1}$ are independent, where the index set $\mathcal{I}$ is defined by

$$\mathcal{I} = \bigcup_{k=0}^{d-1} \mathbb{Z}^k \times \mathbb{Z}_+ \times \{0\}^{d-k-1}.$$

Note that $\mathbb{Z}^d = \mathcal{I} \cup (-\mathcal{I}) \cup \{0\}$. Then, given a pair (u_0, u_1) of functions on $\mathbb{R}^d$, we define the *Wiener randomization* (u_0^ω, u_1^ω) of (u_0, u_1) by

$$(u_0^\omega, u_1^\omega) = \bigg(\sum_{n \in \mathbb{Z}^d} g_{n,0}(\omega)\psi(D-n)u_0, \sum_{n \in \mathbb{Z}^d} g_{n,1}(\omega)\psi(D-n)u_1 \bigg). \tag{1.20}$$

See [73, 49, 5, 4]. We emphasize that thanks to (1.19), this randomization has the desirable property that if u_0 and u_1 are real-valued, then their randomizations u_0^ω and u_1^ω defined in (1.20) are also real-valued.

We make the following assumption on the probability distributions $\mu_{n,j}$ for $g_{n,j}$; there exists $c > 0$ such that

$$\int e^{\gamma \cdot x} d\mu_{n,j}(x) \le e^{c|\gamma|^2}, \quad j = 0, 1, \tag{1.21}$$

for all $n \in \mathbb{Z}^d$, (i) all $\gamma \in \mathbb{R}$ when $n = 0$, and (ii) all $\gamma \in \mathbb{R}^2$ when $n \in \mathbb{Z}^d \setminus \{0\}$. Note that (1.21) is satisfied by standard complex-valued Gaussian random variables, standard Bernoulli random variables, and any random variables with compactly supported distributions.

It is easy to see that, if $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^d)$ for some $s \in \mathbb{R}$, then the Wiener randomization (u_0^ω, u_1^ω) is almost surely in $\mathcal{H}^s(\mathbb{R}^d)$. Note that, under some non-degeneracy condition on the random variables $\{g_{n,j}\}$, there is almost surely no gain from randomization in terms of differentiability (see, for example, Lemma B.1 in [11]). Instead, the main feature of the Wiener randomization (1.20) is that (u_0^ω, u_1^ω) behaves better in terms of integrability. More precisely, if $u_j \in L^2(\mathbb{R}^d)$, $j = 0, 1$, then the randomized function u_j^ω is almost surely in $L^p(\mathbb{R}^d)$ for any finite $p \ge 2$. See [5].

Using the Wiener randomization of the initial data, we prove the main results of this paper, which are the local and global almost sure existence of a unique solution for the quintic vNWL in $\mathbb{R}^2$. More precisely, we have the following main results.

Main results. Fix $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^2)$ for some $s \in \mathbb{R}$ and let (u_0^ω, u_1^ω) denote the Wiener randomization of (u_0, u_1) defined in (1.20). Consider the following defocusing quintic vNLW on $\mathbb{R}^2$ with the random initial data:

$$\begin{cases} \partial_t^2 u - \Delta u + D\partial_t u + u^5 = 0 \\ (u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega) \end{cases} \qquad (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+. \tag{1.22}$$

Theorem 1.1. *Let $s > -\frac{1}{5}$. Then, the quintic vNLW (1.22) is almost surely* **locally** *well-posed with respect to the Wiener randomization (u_0^ω, u_1^ω) as initial data. More precisely, there exist $C, c, \gamma > 0$ and $0 < T_0 \ll 1$ such that for each $0 < T \le T_0$, there exists a set $\Omega_T \subset \Omega$ with the following properties:*

(i) $P(\Omega_T^c) < C \exp\bigg(- \dfrac{c}{T^\gamma \|(u_0, u_1)\|_{\mathcal{H}^s}^2} \bigg)$,

(ii) *For each $\omega \in \Omega_T$, there exists a (unique) local-in-time solution u to (1.22) with $(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)$ in the class*

$$V(t)(u_0^\omega, u_1^\omega) + C([0,T]; H^{s_0}(\mathbb{R}^2)) \cap L^{5+\delta}([0,T]; L^{10}(\mathbb{R}^2))$$

for some $s_0 = s_0(s) > \frac{3}{5}$, sufficiently close to $\frac{3}{5}$, and small $\delta > 0$ such that $s_0 \ge 1 - \frac{1}{5+\delta} - \frac{2}{10}$. Here, $V(t)$ denotes the linear propagator for the viscous wave equation defined in (2.2).

Remark 1.2. Let k_0 be the smallest integer such that $k_0 \geq T_0^{-1}$. Then, by setting

$$\Sigma = \bigcup_{k=k_0}^{\infty} \Omega_{k^{-1}},$$

we have:

(1) $P(\Sigma) = 1$, and
(2) for each $\omega \in \Sigma$, there exist a (unique) local-in-time solution u to (1.22) with $(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)$ on the time interval $[0, T_\omega]$ for some $T_\omega > 0$. More specifically, for $\omega \in \Omega_{k^{-1}}$, the random local existence time T_ω is given by $T_\omega = k^{-1}$.

The proof of Theorem 1.1 is based on the first order expansion [9, 11, 23, 5, 4, 38]:

$$u = z + v, \tag{1.23}$$

where $z = z^\omega$ denotes the random *linear* solution given by

$$z(t) = V(t)(u_0^\omega, u_1^\omega). \tag{1.24}$$

Then, (1.22) can be rewritten as

$$\begin{cases} \partial_t^2 v - \Delta v + D\partial_t v + (v+z)^5 = 0 \\ (v, \partial_t v)|_{t=0} = (0, 0) \end{cases} \tag{1.25}$$

and we study the fixed point problem (1.25) for v. In contrast with [38], where the proof of almost sure local well-posedness was based on the Strichartz estimate for the viscous wave equation with the diagonal Strichartz space $L^6([0,T]; L^6(\mathbb{R}^2))$, we prove Theorem 1.1, using the Schauder estimate for the Poisson kernel (Lemma 2.5) and work in a non-diagonal space $L^{5+\delta}([0,T]; L^{10}(\mathbb{R}^2))$. This was important to obtain higher regularity of $\vec{v}$ ($\mathcal{H}^1$ regularity), which allowed us to show boundedness of the energy, not otherwise attainable using Strichartz estimates. See Section 3 for details.

Once local almost sure well-posedness is established we obtain the following global almost sure well-posedness result:

Theorem 1.3. *Let $s > -\frac{1}{5}$. Then, the defocusing quintic vNLW (1.22) is almost surely globally well-posed with respect to the Wiener randomization (u_0^ω, u_1^ω) as initial data. More precisely, there exists a set $\Sigma \subset \Omega$ with $P(\Sigma) = 1$ such that, for each $\omega \in \Sigma$, there exists a (unique) global-in-time solution u to (1.22) with $(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)$ in the class*

$$V(t)(u_0^\omega, u_1^\omega) + C(\mathbb{R}_+; H^{s_0}(\mathbb{R}^2))$$

for some $s_0 > \frac{3}{5}$.

Here, the uniqueness holds in the following sense. Given any $t_0 \in \mathbb{R}_+$, there exists a random time interval $I(t_0, \omega) \ni t_0$ such that the solution $u = u^\omega$ constructed in Theorem 1.3 is unique in

$$V(t)(u_0^\omega, u_1^\omega) + C(I(t_0, \omega); H^{s_0}(\mathbb{R}^2)) \cap L^{5+\delta}(I(t_0, \omega); L^{10}(\mathbb{R}^2)),$$

where $s_0 > \frac{3}{5}$ and $\delta > 0$ are as in Theorem 1.1.

The main idea of the proof of Theorem 1.3 is based on a probabilistic energy estimate, see e.g., [11, 61]. With $\vec{v} = (v, \partial_t v)$, a smooth solution $\vec{v}$ to the defocusing vNLW (1.25) (with $z \equiv 0$ and general initial data) satisfies monotonicity of the energy (for the usual NLW):

$$E(\vec{v}) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} (\partial_t v)^2 dx + \frac{1}{6}\int_{\mathbb{R}^2} v^6 dx. \tag{1.26}$$

Indeed a simple integration by parts with (1.25) (with $z \equiv 0$) shows

$$\partial_t E(\vec{v}) = -\|\partial_t v\|^2_{\dot{H}^{\frac{1}{2}}} \leq 0.$$

For our problem, we proceed with the first order expansion (1.23) and thus the residual term $v = u - z$ only satisfies the perturbed vNLW (1.25). As such, the monotonicity of the energy $E(\vec{v})$ no longer holds. Nonetheless, by using the time integration by parts trick introduced by Oh and Pocovnicu [61], we establish a Gronwall type estimate for $E(\vec{v})$ to prove almost sure global well-posedness.

One important point to note is that as it is written, the local theory (Theorem 1.1) does not provide a sufficient regularity (i.e. $\mathcal{H}^1(\mathbb{R}^2)$) for $\vec{v}$ to guarantee finiteness of the energy $E(\vec{v})$. By using the Schauder estimate (Lemma 2.5), however, we can show that the residual term $\vec{v}(t)$ is smoother and indeed lies in $\mathcal{H}^1(\mathbb{R}^2)$ for strictly positive times. It is at this step that the dissipative nature of the equation plays an important role in this globalization argument. See Subsection 4.1 for details.

We conclude this introduction by a few remarks.

First, we expect that probabilistic *continuous dependence*, a notion introduced by Burq and Tzvetkov in [12], see also [65], can be extended from the range $s > -\frac{1}{6}$, proved by Kuan and Čanić in [38], to the entire range $s > -\frac{1}{5}$. We omit details here.

Secondly, we note that it is also possible to establish almost sure global well-posedness with respect to the Wiener randomization for the defocusing vNLW (1.15) on $\mathbb{R}^2$ with a *general defocusing nonlinearity* $|u|^{p-1}u$ for $p < 5$, provided that $s > -\frac{1}{p}$. For $p \leq 3$, a straightforward Gronwall type argument by Burq and Tzvetkov [12] applies. See also [65]. For $3 < p < 5$, one can adapt the argument in Sun and Xia [68] which interpolates the $p = 3$ case [12] and the $p = 5$ case [61] in the context of the usual NLW. See Remark 4.2 (ii) for a discussion on the $p > 5$ case.

Finally, we remark that the derivation discussed above but with a *random external forcing* F_{ext} in (1.7), leads to a stochastic version of vNLW. In [39], Kuan and Čanić studied the following stochastic vNLW on $\mathbb{R}^2$ with a multiplicative space-time white noise forcing:

$$\partial_t^2 u - \Delta u + 2\mu D \partial_t u = F(u)\xi, \tag{1.27}$$

where ξ denotes a space-time white noise on $\mathbb{R}^2 \times \mathbb{R}_+$. Under a suitable assumption on F, they proved global well-posedness of (1.27). We also note that well-posedness of stochastic vNLW with a (singular) additive noise on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ was recently considered in [48, 47].

The results of this work shed a new light on this active and important research area by showing the first global well-posedness result for a prototype fluid-structure interaction problem with randomly perturbed rough initial data in the case when the corresponding deterministic problem is ill-posed.

We begin by presenting the estimates that will be used to obtain the local and global almost sure well-posedness results.

2. Basic estimates

In this section, we go over the deterministic and probabilistic linear estimates that will be the basis of the proofs of the main results. For this purpose, we introduce the following **notation**:

- We write $A \lesssim B$ to denote an estimate of the form $A \le CB$ for some $C > 0$. Similarly, we write $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when we have $A \le cB$ for small $c > 0$.
- We define the operators D and $\langle \nabla \rangle$ by setting
$$D = |\nabla| = \sqrt{-\Delta} \qquad \text{and} \qquad \langle \nabla \rangle = \sqrt{1 - \Delta}, \tag{2.1}$$
viewed as Fourier multiplier operators with multipliers $|\xi|$ and $\langle \xi \rangle$, respectively.

2.1. Linear operators and the relevant linear estimates.

By writing (1.22) in the Duhamel formulation, we have
$$u(t) = V(t)(u_0^\omega, u_1^\omega) - \int_0^t W(t - t')u^5(t')dt',$$
where the linear propagator $V(t)$ is defined by
$$V(t)(u_0, u_1) = e^{-\frac{D}{2}t}\bigg(\cos\big(\tfrac{\sqrt{3}}{2}Dt\big) + \frac{1}{\sqrt{3}}\sin\big(\tfrac{\sqrt{3}}{2}Dt\big)\bigg)u_0 + e^{-\frac{D}{2}t}\frac{\sin\big(\frac{\sqrt{3}}{2}Dt\big)}{\frac{\sqrt{3}}{2}D}u_1, \tag{2.2}$$
and $W(t)$ is defined by
$$W(t) = e^{-\frac{D}{2}t}\frac{\sin\big(\frac{\sqrt{3}}{2}Dt\big)}{\frac{\sqrt{3}}{2}D}. \tag{2.3}$$
By letting
$$P(t) = e^{-\frac{D}{2}t} \tag{2.4}$$
denote the Poisson kernel (with a parameter $\frac{t}{2}$) and
$$S(t) = \frac{\sin\big(\frac{\sqrt{3}}{2}Dt\big)}{\frac{\sqrt{3}}{2}D},$$
we have
$$W(t) = P(t) \circ S(t).$$
By defining $U(t)$ by
$$U(t)(u_0, u_1) = \Big(\cos\big(\tfrac{\sqrt{3}}{2}Dt\big) + \frac{1}{\sqrt{3}}\sin\big(\tfrac{\sqrt{3}}{2}Dt\big)\Big)u_0 + \frac{\sin\big(\frac{\sqrt{3}}{2}Dt\big)}{\frac{\sqrt{3}}{2}D}u_1,$$
we have
$$V(t) = P(t) \circ U(t). \tag{2.5}$$

We first recall the Strichartz estimates for the homogeneous linear viscous wave equation (Theorem 3.2 in [38]). Given $\sigma > 0$, we say that a pair (q, r) is σ-admissible if $2 \le q, r \le \infty$ with $(q, r, \sigma) \neq (2, \infty, 1)$ and

$$\frac{2}{q} + \frac{2\sigma}{r} \le \sigma. \tag{2.6}$$

Lemma 2.1. *Given $\sigma > 0$, let (q, r) be a σ-admissible pair with $r < \infty$. Then, a solution u to the homogeneous linear wave equation on $\mathbb{R}^d$:*

$$\begin{cases} \partial_t^2 u - \Delta u + D\partial_t u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}$$

satisfies

$$\|(u, \partial_t u)\|_{L^\infty(\mathbb{R}_+; \mathcal{H}^s_x(\mathbb{R}^d))} + \|u\|_{L^q(\mathbb{R}_+; L^r_x(\mathbb{R}^d))} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{R}^d)}, \tag{2.7}$$

provided that the following scaling condition holds:

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s. \tag{2.8}$$

Remark 2.2. In view of the scaling condition (2.8), if a pair (q, r) satisfies (2.8) for some $s \ge 0$, then it is σ-admissible with $\sigma = d$.

Remark 2.3. We remark that the bounding constant in the estimate (2.7) depends only on $\sigma > 0$. See [38] for details.

Remark 2.4. In the usual Strichartz estimates for the homogeneous wave equation, one must impose an additional restriction on s that $0 \le s \le 1$. This is not present in the corresponding estimate for the homogeneous viscous wave equation in Lemma 2.1. Although a restriction on s is not explicitly stated in Lemma 2.1, s does have a limited range of possible values, due to the constraints imposed by the fact that $\sigma > 0$, $2 \le q, r \le \infty$, with $(q, r, \sigma) \neq (2, \infty, 1)$ in (2.6), and the scaling condition (2.8). The exponent s can take values in the range $-\frac{1}{2} < s \le \frac{d}{2}$ depending on the choice of parameters. One attains the lower end of the range by taking $q = 2$ and taking r arbitrarily close to 2, while one attains the upper endpoint of the range by taking $q, r = \infty$ for $s = \frac{d}{2}$.

Next, we state a Schauder-type estimate for the Poisson kernel $P(t)$, which allows us to exploit the dissipative nature of the dynamics.

Lemma 2.5. *Let $1 \le p \le q \le \infty$ and $\alpha \ge 0$. Then, we have*

$$\|D^\alpha P(t) f\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\alpha - d(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)} \tag{2.9}$$

for any $t > 0$.

Proof. Let $K_t(x)$ denote the kernel for $P(t)$, whose Fourier transform is given by $\widehat{K}_t(\xi) = e^{-\frac{|\xi|}{2}t}$. Then, we have

$$K_t(x) = t^{-d} K_1(t^{-1}x), \tag{2.10}$$

where $K_1(x)$ satisfies

$$K_1(x) = \frac{c_1}{(c_2 + |x|^2)^{\frac{d+1}{2}}}$$

for some $c_1, c_2 > 0$. In particular, we have $K_1 \in L^r(\mathbb{R}^d)$ for any $1 \le r \le \infty$.

We first consider the case $\alpha = 0$. For $1 \le r \le \infty$ with $\frac{1}{r} = \frac{1}{q} - \frac{1}{p} + 1$, it follows from (2.10) that

$$\|K_t\|_{L^r} = t^{-d(1-\frac{1}{r})}\|K_1\|_{L^r} = C_r t^{-d(\frac{1}{p}-\frac{1}{q})}. \tag{2.11}$$

Then, (2.9) follows from Young's inequality and (2.11).

Next, we consider the case $\alpha > 0$. Noting that $D^\alpha P(t) f = (D^\alpha K_t) * f$, we need to study the scaling property of $D^\alpha K_t$. On the Fourier side, we have

$$\widehat{D^\alpha K_t}(\xi) = |\xi|^\alpha e^{-\frac{|\xi|}{2}t} = t^{-\alpha}(t|\xi|)^\alpha e^{-\frac{|\xi|t}{2}} = t^{-\alpha}\widehat{D^\alpha K_1}(t\xi).$$

Namely, we have

$$D^\alpha K_t(x) = t^{-d-\alpha}(D^\alpha K_1)(t^{-1}x). \tag{2.12}$$

Then, proceeding as before, the bound (2.9) follows from Young's inequality and (2.12). □

2.2. **Probabilistic estimates.** In this subsection, we establish certain probabilistic Strichartz estimates. See also Lemma 5.3 in [38].

We first recall the following probabilistic estimate. See [11] for the proof.

Lemma 2.6. *Given $j = 0, 1$, let $\{g_{n,j}\}_{n\in\mathbb{Z}^d}$ be a sequence of mean-zero complex-valued, random variables, satisfying* (1.21)*, as in Subsection 1. Then, there exists $C > 0$ such that*

$$\Big\| \sum_{n\in\mathbb{Z}^d} g_{n,j}(\omega) c_n \Big\|_{L^p(\Omega)} \le C\sqrt{p}\|c_n\|_{\ell^2_n(\mathbb{Z}^d)}$$

for any $j = 0, 1$, any finite $p \ge 2$, and any sequence $\{c_n\} \in \ell^2(\mathbb{Z}^d)$.

We now establish the first probabilistic Strichartz estimate.

Proposition 2.7. *Given $(u_0, u_1) \in \mathcal{H}^0(\mathbb{R}^d)$, let (u_0^ω, u_1^ω) be its Wiener randomization defined in* (1.20)*, satisfying* (1.21)*. Then, given any $2 \le q, r < \infty$ and $\alpha \ge 0$, satisfying $q\alpha < 1$, there exist $C, c > 0$ such that*

$$P\Big(\|D^\alpha V(t)(u_0^\omega, u_1^\omega)\|_{L^q([0,T];L^r_x)} > \lambda\Big) \le C\exp\bigg(-c\frac{\lambda^2}{T^{\frac{2}{q}-2\alpha}\|(u_0,u_1)\|^2_{\mathcal{H}^0}}\bigg) \tag{2.13}$$

for any $T > 0$ and $\lambda > 0$.

Remark 2.8. (i) From (2.13), we conclude that

$$P\Big(\|D^\alpha V(t)(u_0^\omega, u_1^\omega)\| \le \lambda\Big) \longrightarrow 1,$$

as $\lambda \to \infty$ for fixed $T > 0$, or as $T \searrow 0$ for fixed $\lambda > 0$.

(ii) Let $\alpha_0 \ge 0$ and $q\alpha_0 < 1$. Then, by applying Proposition 2.7 with $\alpha = 0$ and $\alpha = \alpha_0$, we have

$$P\Big(\|\langle\nabla\rangle^{\alpha_0} V(t)(u_0^\omega, u_1^\omega)\|_{L^q([0,T];L^r_x)} > \lambda\Big) \le C\exp\bigg(-c\frac{\lambda^2}{T^{\frac{2}{q}-2\alpha_0}\|(u_0,u_1)\|^2_{\mathcal{H}^0}}\bigg) \tag{2.14}$$

for any $0 < T \le 1$ and $\lambda > 0$, where $\langle\nabla\rangle = \sqrt{1-\Delta}$ is as in (2.1). We also have

$$P\Big(\|\langle\nabla\rangle^{\alpha_0} V(t)(u_0^\omega, u_1^\omega)\|_{L^q([0,T];L^r_x)} > \lambda\Big) \le C\exp\left(-c\frac{\lambda^2}{T^{\frac{2}{q}}\|(u_0,u_1)\|^2_{\mathcal{H}^0}}\right) \tag{2.15}$$

for any $T \ge 1$ and $\lambda > 0$.

See also Lemma 5.3 in [38], where the case $q = r = 6$ was treated. The proof of Proposition 2.7 follows the usual proofs of the probabilistic Strichartz estimates via Minkowski's integral inequality [11, 23, 5] but also utilizes the Schauder estimate (Lemma 2.5).

Proof. From (2.5) and Lemma 2.5 followed by Minkowski's integral inequality, we have

$$\begin{aligned}\Big\|\|D^\alpha V(t)(u_0^\omega, u_1^\omega)\|_{L^q_t([0,T];L^r_x)}\Big\|_{L^p(\Omega)} &\lesssim \Big\|\big\|t^{-\alpha}\|U(t)(u_0^\omega,u_1^\omega)\|_{L^r_x}\big\|_{L^q_t([0,T])}\Big\|_{L^p(\Omega)}\\ &\le \Big\|\big\|t^{-\alpha}\|U(t)(u_0^\omega,u_1^\omega)\|_{L^p(\Omega)}\big\|_{L^r_x}\Big\|_{L^q_t([0,T])}\end{aligned} \tag{2.16}$$

for any finite $p \ge \max(q,r)$. By Lemma 2.6, Minkowski's integral inequality, Bernstein's unit-scale inequality (1.18), and the boundedness of $U(t)$ from $\mathcal{H}^0(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$, we obtain

$$\begin{aligned}(2.16) &\lesssim \sqrt{p}\,\Big\|t^{-\alpha}\big\|\|\psi(D-n)U(t)(u_0,u_1)\|_{\ell^2_n}\big\|_{L^r_x}\Big\|_{L^q_t([0,T])}\\ &\le \sqrt{p}\,\Big\|t^{-\alpha}\big\|\|\psi(D-n)U(t)(u_0,u_1)\|_{L^r_x}\big\|_{\ell^2_n}\Big\|_{L^q_t([0,T])}\\ &\lesssim \sqrt{p}\,\Big\|t^{-\alpha}\|U(t)(u_0,u_1)\|_{L^2_x}\Big\|_{L^q_t([0,T])}\\ &\lesssim \sqrt{p}\,T^{\frac{1}{q}-\alpha}\|(u_0,u_1)\|_{\mathcal{H}^0},\end{aligned} \tag{2.17}$$

where we used $q\alpha < 1$ in the last step. Then, the tail estimate (2.13) follows from (2.17) and Chebyshev's inequality. See the proof of Lemma 3 in [5].[1] □

In establishing almost sure global well-posedness, we need to introduce several additional linear operators. Define $\widetilde{V}(t)$ by

$$\begin{aligned}\widetilde{V}(t)(u_0,u_1) &= \langle\nabla\rangle^{-1}\partial_t V(t)\\ &= -\frac{2\sqrt{3}}{3}\frac{D}{\langle\nabla\rangle}e^{-\frac{D}{2}t}\sin\big(\tfrac{\sqrt{3}}{2}Dt\big)u_0\\ &\quad + e^{-\frac{D}{2}t}\bigg(-\frac{1}{2}\frac{D}{\langle\nabla\rangle}\frac{\sin\big(\frac{\sqrt{3}}{2}Dt\big)}{\frac{\sqrt{3}}{2}D} + \frac{\cos\big(\frac{\sqrt{3}}{2}Dt\big)}{\langle\nabla\rangle}\bigg)u_1.\end{aligned} \tag{2.18}$$

Then, defining $\widetilde{U}(t)$ by

$$\begin{aligned}\widetilde{U}(t)(u_0,u_1) &= -\frac{2\sqrt{3}}{3}\frac{D}{\langle\nabla\rangle}\sin\big(\tfrac{\sqrt{3}}{2}Dt\big)u_0\\ &\quad + \bigg(-\frac{1}{2}\frac{D}{\langle\nabla\rangle}\frac{\sin\big(\frac{\sqrt{3}}{2}Dt\big)}{\frac{\sqrt{3}}{2}D} + \frac{\cos\big(\frac{\sqrt{3}}{2}Dt\big)}{\langle\nabla\rangle}\bigg)u_1,\end{aligned}$$

[1] Lemma 2.2 in the arXiv version.

we have

$$\widetilde{V}(t) = P(t) \circ \widetilde{U}(t). \tag{2.19}$$

Next, we state a probabilistic estimate involving the L^∞_t-norm, which plays an important role in establishing an energy bound for almost sure global well-posedness. The proof is based on an adaptation of the proof of Proposition 3.3 in [61] combined with the Schauder estimate (Lemma 2.5).

Proposition 2.9. *Given a pair (u_0, u_1) of real-valued functions defined on $\mathbb{R}^2$, let (u_0^ω, u_1^ω) be its Wiener randomization defined in (1.20), satisfying (1.21). Fix $T \gg 1 \geq T_0 > 0$ and let $V^*(t) = V(t)$ or $\widetilde{V}(t)$ defined in (2.2) and (2.18), respectively. Then, given any $2 \leq r \leq \infty$, $\alpha \geq 0$, and $\varepsilon_0 > 0$, there exist $C, c > 0$ such that*

$$P\Big(\|D^\alpha V^*(t)(u_0^\omega, u_1^\omega)\|_{L^\infty([T_0,T];L^r_x)} > \lambda\Big) \leq CT \exp\Bigg(-c\frac{\lambda^2}{T^2 T_0^{-2\alpha}\|(u_0,u_1)\|^2_{\mathcal{H}^{\varepsilon_0}}}\Bigg)$$

for any $\lambda > 0$.

Proof. Let $U^*(t) = P(-t) \circ V^*(t)$. Then, from Lemma 2.5 with (2.5) or (2.19), we have

$$\begin{aligned}
\|D^\alpha V^*(t)(u_0^\omega, u_1^\omega)\|_{L^\infty_t([T_0,T];L^r_x)} &\lesssim \big\| t^{-\alpha}\|U^*(t)(u_0^\omega, u_1^\omega)\|_{L^r_x}\big\|_{L^\infty_t([T_0,T])} \\
&\leq T_0^{-\alpha}\|U^*(t)(u_0^\omega, u_1^\omega)\|_{L^\infty_t([T_0,T];L^r_x)}
\end{aligned}$$

As in the proof of Proposition 3.3 in [61], the rest follows from Lemma 3.4 in [61], which established similar L^∞_t-bounds for the half-wave operators $e^{\pm itD}$. □

Remark 2.10. It is also possible to prove Proposition 2.9, using the Garsia-Rodemich-Rumsey inequality ([29, Theorem A.1]). See, for example, Lemma 2.3 in [34] in the context of the stochastic nonlinear wave equation.

3. Local well-posedness

In this section, we present the proof of Theorem 1.1. Instead of (1.25) with the zero initial data, we study (1.25) with general (deterministic) initial data (v_0, v_1):

$$\begin{cases} \partial_t^2 v - \Delta v + D\partial_t v + (v+z)^5 = 0 \\ (v, \partial_t v)|_{t=0} = (v_0, v_1). \end{cases} \tag{3.1}$$

We recall from (1.24) and (2.2) that $z = V(t)(u_0^\omega, u_1^\omega)$ is the random linear solution with the randomized initial data (u_0^ω, u_1^ω) which is the result of the Wiener randomization (1.20) performed on the given deterministic initial data $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^2)$.

Theorem 3.1. *Let $s > -\frac{1}{5}$. Fix $(v_0, v_1) \in \mathcal{H}^{s_0}(\mathbb{R}^2)$ for some $s_0 = s_0(s) > \frac{3}{5}$ sufficiently close to $\frac{3}{5}$. Then, there exist $C, c, \gamma > 0$ and $0 < T_0 \ll 1$ such that for each $0 < T \leq T_0$, there exists a set $\Omega_T \subset \Omega$ with the following properties:*

(i) *The following probability bound holds:*

$$P(\Omega_T^c) < C\exp\bigg(-\frac{c}{T^\gamma\|(u_0,u_1)\|^2_{\mathcal{H}^s}}\bigg). \tag{3.2}$$

(ii) *For each* $\omega \in \Omega_T$*, there exists a (unique) solution* $(v, \partial_t v)$ *to* (3.1) *with* $(v, \partial_t v)|_{t=0} = (v_0, v_1)$ *in the class*

$$(v, \partial_t v) \in C([0,T]; \mathcal{H}^{s_0}(\mathbb{R}^2)) \qquad \textit{and} \qquad v \in L^{5+\delta}([0,T]; L^{10}(\mathbb{R}^2)) \tag{3.3}$$

for small $\delta > 0$ *such that* $s_0 \geq 1 - \frac{1}{5+\delta} - \frac{2}{10}$.

In Subsection 3.1, we first state several linear estimates. We then present the proof of Theorem 3.1 in Subsection 3.2.

3.1. **Linear estimates.** In this subsection, we establish several nonhomogeneous linear estimates, which are slightly different from those in Theorem 3.3 in [38].

Lemma 3.2. *Let* $W(t)$ *be as in* (2.3)*. Then, given sufficiently small* $\delta > 0$*, we have*

$$\bigg\| \int_0^t W(t-t')F(t')dt' \bigg\|_{L_t^{5+\delta}([0,T];L_x^{10}(\mathbb{R}^2))} \lesssim \|F\|_{L^1([0,T];L_x^2(\mathbb{R}^2))} \tag{3.4}$$

for any $0 < T \leq 1$.

Proof. Let $\mathbf{P}_{\lesssim 1}$ be a smooth[2] projection onto spatial frequencies $\{|\xi| \leq 1\}$ and set $\mathbf{P}_{\gg 1} = \mathbf{Id} - \mathbf{P}_{\lesssim 1}$. In the following, we separately estimate the contributions from $\mathbf{P}_{\lesssim 1}F$ and $\mathbf{P}_{\gg 1}F$.

Let us first estimate the low frequency contribution. By Minkowski's integral inequality and Bernstein's unit-scale inequality (1.18) with $\sin x \leq x$ for $x \geq 0$, we have

$$\begin{aligned} &\bigg\| \int_0^t W(t-t')\mathbf{P}_{\lesssim 1}F(t')dt' \bigg\|_{L_t^{5+\delta}([0,T];L_x^{10})} \\ &\quad \leq \bigg\| \int_0^t \|\mathbf{1}_{[0,t]}(t')W(t-t')\mathbf{P}_{\lesssim 1}F(t')\|_{L_x^{10}}dt' \bigg\|_{L_t^{5+\delta}([0,T])} \\ &\quad \leq \bigg\| \int_0^t (t-t')\|\mathbf{1}_{[0,t]}(t')\mathbf{P}_{\lesssim 1}F(t')\|_{L_x^2}dt' \bigg\|_{L_t^{5+\delta}([0,T])} \\ &\quad \lesssim T^\theta \|F\|_{L^1([0,T];L_x^2)} \end{aligned} \tag{3.5}$$

for some $\theta > 0$.

Next, we estimate the high frequency contribution. Note that the pair $(5+\delta, 10)$ is σ-admissible for $\sigma \geq \frac{1}{2}$ in the sense of (2.6). Let

$$s_0 = 1 - \frac{1}{5+\delta} - \frac{2}{10} = \frac{3}{5} + \delta_0 \tag{3.6}$$

for some small $\delta_0 = \delta_0(\delta) > 0$. Then, by Minkowski's integral inequality and the homogeneous Strichartz estimate (Lemma 2.1), we have

$$\begin{aligned} &\bigg\| \int_0^t W(t-t')\mathbf{P}_{\gg 1}F(t')dt' \bigg\|_{L_t^{5+\delta}([0,T];L_x^{10})} \\ &\quad \leq \int_0^T \|\mathbf{1}_{[0,t]}(t')W(t-t')\mathbf{P}_{\gg 1}F(t')\|_{L_t^{5+\delta}([0,T];L_x^{10})}dt' \\ &\quad \lesssim \int_0^T \|\mathbf{P}_{\gg 1}F(t')\|_{H_x^{s_0-1}}dt' \\ &\quad \lesssim \|F\|_{L^1([0,T];L_x^2)}. \end{aligned} \tag{3.7}$$

[2]Namely, given by a smooth Fourier multiplier.

The desired bound (3.4) then follows from (3.5) and (3.7). □

Lemma 3.3. *Let $W(t)$ be as in* (2.3)*. Then, given $0 \le s \le 1$, we have*

$$\bigg\| \int_0^t W(t-t')F(t')dt' \bigg\|_{C([0,T];H_x^s(\mathbb{R}^2))} \lesssim \|F\|_{L^1([0,T];L_x^2(\mathbb{R}^2))}, \tag{3.8}$$

$$\bigg\| \partial_t \int_0^t W(t-t')F(t')dt' \bigg\|_{C([0,T];H_x^{s-1}(\mathbb{R}^2))} \lesssim \|F\|_{L^1([0,T];L_x^2(\mathbb{R}^2))}, \tag{3.9}$$

for any $0 < T \le 1$.

Proof. The first estimate (3.8) follows from Minkowski's integral inequality with (2.3). As for the second estimate (3.9), we first note from (2.3) that

$$\partial_t \int_0^t W(t-t')F(t')dt' = \int_0^t \partial_t W(t-t')F(t')dt',$$

where

$$\partial_t W(t) = e^{-\frac{D}{2}t}\bigg(\cos\big(\tfrac{\sqrt{3}}{2}Dt\big) - \frac{1}{\sqrt{3}}\sin\big(\tfrac{\sqrt{3}}{2}Dt\big)\bigg).$$

Then, the second estimate (3.9) follows from Minkowski's integral inequality and the boundedness of $\partial_t W(t)$ on $H^{s-1}(\mathbb{R}^2)$. □

3.2. **Local well-posedness.** We now present the proof of Theorem 3.1.

Proof of Theorem 3.1. Fix $s > -\frac{1}{5}$ and $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^2)$. Then, there exists small $\delta > 0$ such that

$$s > -\frac{1}{5+\delta}, \tag{3.10}$$

and we fix this choice of $\delta > 0$ for the remainder of the proof.

Fix $C_0 > 0$ and define the event $\Omega_T = \Omega_T(C_0)$ by setting

$$\Omega_T = \big\{\omega \in \Omega : ||z||_{L^{5+\delta}([0,T];L_x^{10})} \le C_0\big\}. \tag{3.11}$$

Then, from the probabilistic Strichartz estimate (Proposition 2.7) (see also (2.14)) with (1.24), (1.20), and (3.10) (which guarantees $\alpha_0 q < 1$ in invoking (2.14) with $\alpha_0 = -s$ and $q = 5+\delta$), we have

$$P(\Omega_T^c) \le C \exp\Bigg(- c\frac{C_0^2}{T^{\frac{2}{q}+2s}\|(u_0,u_1)\|_{\mathcal{H}^s}^2}\Bigg) \tag{3.12}$$

for any $0 < T \le 1$. We remark that the choice of $C_0 > 0$ does not matter, and that the specific value of $C_0 > 0$ affects only the size of $T_0 \ll 1$ and the constants in the estimate (3.2).

By writing (3.1) in the Duhamel formulation, we have

$$v(t) = \Gamma_{(v_0,v_1),z}(v)(t) := V(t)(v_0,v_1) - \int_0^t W(t-t')(v+z)^5(t')dt'.$$

For simplicity, we set $\Gamma = \Gamma_{(v_0,v_1),z}$. Let $\vec{\Gamma}(v) = (\Gamma(v), \partial_t\Gamma(v))$. Let $s_0 = s_0(\delta) = \frac{3}{5} + \delta_0$ as in (3.6). Then, given $T > 0$, define the solution space $Z(T)$ by setting

$$Z(T) = X(T) \times Y(T),$$

where $X(T)$ and $Y(T)$ are defined by

$$\begin{aligned} X(T) &= C([0,T]; H^{s_0}(\mathbb{R}^2)) \cap L^{5+\delta}([0,T]; L^{10}(\mathbb{R}^2)) \\ Y(T) &= C([0,T]; H^{s_0-1}(\mathbb{R}^2)). \end{aligned}$$

In order to prove Theorem 3.1, we show that there exists small $0 < T_0 \ll 1$ such that $\vec{\Gamma} : (v, \partial_t v) \mapsto (\Gamma(v), \partial_t \Gamma(v))$ is a strict contraction on an appropriate closed ball in $Z(T)$ for any $0 < T \le T_0$ and for any $\omega \in \Omega_T$, where Ω_T is as in (3.11). The probability estimate (3.2) on Ω_T^c follows from (3.12).

Fix arbitrary $\omega \in \Omega_T$ for $0 < T \le T_0$, where T_0 is to be determined later. Recall $\vec{\Gamma}(v) = (\Gamma(v), \partial_t \Gamma(v))$. Note that the ordered pair $(5+\delta, 10)$ is σ-admissible for $\sigma \ge \frac{1}{2}$ in the sense of (2.6) and furthermore, it satisfies the scaling condition (2.8) with s_0 as defined in (3.6). Then, by Lemmas 2.1, 3.2, and 3.3 with (3.11), we have

$$\begin{aligned} \|\vec{\Gamma}(v)\|_{Z(T)} &\lesssim \|(v_0, v_1)\|_{\mathcal{H}^{s_0}} + \|(v+z)^5\|_{L^1([0,T];L^2_x)} \\ &\lesssim \|(v_0, v_1)\|_{\mathcal{H}^{s_0}} + T^\theta \Big(\|v\|_{L^{5+\delta}([0,T];L^{10}_x)} + \|z\|_{L^{5+\delta}([0,T];L^{10}_x)} \Big) \\ &\lesssim \|(v_0, v_1)\|_{\mathcal{H}^{s_0}} + T^\theta \Big(\|\vec{v}\|_{Z(T)} + C_0 \Big) \end{aligned}$$

for some $\theta > 0$, where $\vec{v} = (v, \partial_t v)$.

A similar computation yields the following difference estimate:

$$\begin{aligned} \|\vec{\Gamma}(v) - \vec{\Gamma}(w)\|_{Z(T)} &\lesssim \|(v+z)^5 - (w+z)^5\|_{L^1([0,T];L^2_x)} \\ &\lesssim T^\theta \Big(\|v\|^4_{L^{5+\delta}([0,T];L^{10}_x)} + \|w\|^4_{L^{5+\delta}([0,T];L^{10}_x)} \\ &\qquad + \|z\|^4_{L^{5+\delta}([0,T];L^{10}_x)} \Big) \|v - w\|_{L^{5+\delta}([0,T];L^{10}_x)} \\ &\lesssim T^\theta \Big(\|\vec{v}\|^4_{Z(T)} + \|\vec{w}\|^4_{Z(T)} + C_0^4 \Big) \|\vec{v} - \vec{w}\|_{Z(T)}. \end{aligned}$$

Hence by choosing $T_0 > 0$ sufficiently small, depending on the initial choice of $C_0 > 0$ and $||(v_0, v_1)||_{\mathcal{H}^{s_0}}$, we see that $\vec{\Gamma} = \vec{\Gamma}_{(v_0,v_1),z}$ is a strict contraction on the ball in $Z(T)$ of radius $\sim 1 + \|(v_0, v_1)\|_{\mathcal{H}^{s_0}}$, whenever $\omega \in \Omega_T$ and $0 < T \le T_0$. This proves almost sure local well-posedness of (3.1) (and (1.22)) for $s > -\frac{1}{5}$. This concludes the proof of Theorem 3.1 (and hence of Theorem 1.1). $\square$

Let us conclude this section by stating some corollaries and remarks. Given $N \in \mathbb{N}$, let $\mathbf{P}_{\le N}$ denote a smooth projection onto the (spatial) frequencies $\{|\xi| \le N\}$. Then, consider the following perturbed vNLW:

$$\begin{cases} \partial_t^2 v_N - \Delta v_N + D\partial_t v_N + (v_N + z_N)^5 = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (\mathbf{P}_{\le N} v_0, \mathbf{P}_{\le N} v_1), \end{cases} \tag{3.13}$$

where z_N denotes the truncated random linear solution defined by

$$z_N(t) = V(t)(\mathbf{P}_{\le N} u_0^\omega, \mathbf{P}_{\le N} u_1^\omega).$$

Then, a slight modification of the proof of Theorem 3.1 yields the following approximation result.

Corollary 3.4. *Let $s > -\frac{1}{5}$ and $s_0 > \frac{3}{5}$ be as in Theorem 3.1. Fix $(v_0, v_1) \in \mathcal{H}^{s_0}(\mathbb{R}^2)$. Let Ω_T be as in Theorem 3.1. Furthermore, for each $\omega \in \Omega_T$, let $(v, \partial_t v)$ be the solution to* (3.1) *on $[0, T]$ with $(v, \partial_t v)|_{t=0} = (v_0, v_1)$ constructed in Theorem 3.1. By possibly shrinking the local existence time T by a constant factor (while keeping the definition* (3.11) *of Ω_T unchanged), for each $\omega \in \Omega_T$, the solution $(v_N, \partial_t v_N)$ to* (3.13) *converges to $(v, \partial_t v)$ in the class* (3.3) *as $N \to \infty$.*

Next, consider the following perturbed vNLW:

$$\begin{cases} \partial_t^2 v - \Delta v + D\partial_t v + (v + f)^5 = 0 \\ (v, \partial_t v)|_{t=t_0} = (v_0, v_1), \end{cases} \tag{3.14}$$

where f is a given deterministic function. As a corollary to the proof of Theorem 3.1, we have the following local well-posedness result of (3.14).

Corollary 3.5. *Let $s > -\frac{1}{5}$, $s_0 > \frac{3}{5}$, and small $\delta > 0$ be as in Theorem 3.1. Fix $(v_0, v_1) \in \mathcal{H}^{s_0}(\mathbb{R}^2)$ and fix $t_0 \in \mathbb{R}_+$. Suppose that*

$$f \in L^{5+\delta}([t_0, t_0 + 1]; L^{10}(\mathbb{R}^2)).$$

Then, there exists $T = T\big(\|(v_0, v_1)\|_{\mathcal{H}^{s_0}}, \|f\|_{L^{5+\delta}([t_0,t_0+T];L^{10}_x)}\big) > 0$ and a (unique) solution $(v, \partial_t v)$ to (3.14) *on the time interval $[t_0, t_0 + T]$ with $(v, \partial_t v)|_{t=t_0} = (v_0, v_1)$ in the class*

$$(v, \partial_t v) \in C([t_0, t_0 + T]; \mathcal{H}^{s_0}(\mathbb{R}^2)) \qquad \text{and} \qquad v \in L^{5+\delta}([t_0, t_0 + T]; L^{10}(\mathbb{R}^2)).$$

Remark 3.6. (i) In terms of the current approach based on the first order expansion (1.23), the threshold $s = -\frac{1}{5}$ seems to be sharp. Since we need to measure the quintic power in L^1 in time, this forces us to measure the random linear solution essentially in L^5 in time. In view of Proposition 2.7, local-in-time integrability of t^s in L^5 requires $s > -\frac{1}{5}$. It is worthwhile to note that the regularity restriction $s > -\frac{1}{5}$ comes only from the temporal integrability and does not have anything to do with the spatial integrability.

With a pth power nonlinearity $|u|^{p-1}u$, $p > 1$, (in place of the quintic power u^5), a similar argument shows almost sure local well-posedness of (1.22) for $s > -\frac{1}{p}$, which is essentially sharp (in terms of the first order expansion). For $p \notin 2\mathbb{N} + 1$, the nonlinearity is not algebraic and thus we need to proceed as in [57], where probabilistic well-posedness of the nonlinear Schrödinger equations with non-algebraic nonlinearities was studied. See [47] for details. See also Remark 4.2.

(ii) It would be of interest to investigate if higher order expansions, such as those in [6, 62], give any improvement over Theorem 3.1 on almost sure local well-posedness. One may also adapt the paracontrolled approach used for the stochastic NLW [33, 58, 10, 59] to study vNLW with random initial data.

4. Global well-posedness

In this section, we prove almost sure global well-posedness of (1.22). As noted in [23, 4], it suffices to prove the following "almost" almost sure global well-posedness result.

Proposition 4.1. *Let $s > -\frac{1}{5}$. Given $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^2)$, let (u_0^ω, u_1^ω) be its Wiener randomization defined in* (1.20)*, satisfying* (1.21)*. Then, given any $T, \varepsilon > 0$, there exists a set $\Omega_{T,\varepsilon} \subset \Omega$ such that*

(i) $P(\Omega_{T,\varepsilon}^c) < \varepsilon$,

(ii) *For each* $\omega \in \Omega_{T,\varepsilon}$, *there exists a (unique) solution* u *to* (1.22) *on* $[0,T]$ *with* $(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)$.

It is easy to see from the Borel-Cantelli lemma that almost sure global well-posedness (Theorem 1.3) follows once we prove "almost" almost sure global well-posedness stated in Proposition 4.1 above. See [23, 4]. Hence, the remaining part of this section is devoted to the proof of Proposition 4.1.

Fix $T \gg 1$. In order to extend our local-in-time result for the initial value problem (1.25) to a result on $[0,T]$ for arbitrary $T > 0$, we consider (3.14) with $f = z = V(t)(u_0^\omega, u_1^\omega)$, given explicitly by

$$\begin{cases} \partial_t^2 v - \Delta v + D\partial_t v + (v+z)^5 = 0 \\ (v, \partial_t v)|_{t=t_0} = (v_0, v_1), \end{cases} \tag{4.1}$$

where $t_0 \in \mathbb{R}^+$. In view of Corollary 3.5 and almost sure boundedness of the $L^{5+\delta}([0,T]; L_x^{10}(\mathbb{R}^2))$-norm of the random linear solution $z(t) = V(t)(u_0^\omega, u_1^\omega)$ thanks the probabilistic Strichartz estimate (Proposition 2.7), it suffices to control the $\mathcal{H}^{s_0}$-norm of the remainder term $\vec{v} = (v, \partial_t v)$, where $s_0 = \frac{3}{5} + \delta_0$ as in (3.6) and the remainder term v satisfies the initial value problem (1.25). This will allow us to extend the local-in-time result in Theorem 3.1 to $[0,T]$ by iteratively applying Corollary 3.5. In the next subsection, we first show a gain of regularity such that $(v(t), \partial_t v(t))$ indeed belongs to $\mathcal{H}^1(\mathbb{R}^2)$ as soon as $t > 0$. Then, the problem is reduced to controlling the growth of the energy $E(\vec{v})$ in (1.26), associated with the standard nonlinear wave equation, since the energy $E(\vec{v})$ controls the $\mathcal{H}^1$-norm (and hence $\mathcal{H}^{s_0}$-norm) of the remainder term $(v, \partial_t v)$, as needed to establish the result. See Subsection 4.2.

4.1. **Gain of regularity.** Consider the initial value problem (4.1). Fix $s > -\frac{1}{5}$ and $s_0 = \frac{3}{5} + \delta_0$ with small $\delta_0 > 0$ as in (the proof of) Theorem 3.1. Let (u_0^ω, u_1^ω) be the Wiener randomization of a given deterministic pair $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^2)$ and fix $(v_0, v_1) \in \mathcal{H}^{s_0}(\mathbb{R}^2)$.

Let $T \gg 1$ and let $z(t) = V(t)(u_0^\omega, u_1^\omega)$ be the random linear solution. Then, it follows from the probabilistic Strichartz estimate (Proposition 2.7) (see also (2.15)) that there exists an almost surely finite random constant $C_\omega = C_\omega(T) > 0$ such that

$$\|z\|_{L^{5+\delta}([0,T]; L_x^{10})} \le C_\omega. \tag{4.2}$$

Fix a good $\omega \in \Omega$ such that C_ω in (4.2) is finite. Then, from Corollary 3.5, we see that there exist $\tau_\omega > 0$ and a unique solution $\vec{v} = (v, \partial_t v)$ to (4.1) on the time interval $[t_0, t_0 + \tau_\omega]$ with $(v, \partial_t v)|_{t=t_0} = (v_0, v_1)$ in the class

$$(v, \partial_t v) \in C([t_0, t_0 + \tau_\omega]; \mathcal{H}^{s_0}(\mathbb{R}^2)) \qquad \text{and} \qquad v \in L^{5+\delta}([t_0, t_0 + \tau_\omega]; L^{10}(\mathbb{R}^2)).$$

We show that the solution $\vec{v} = (v, \partial_t v)$ to (4.1) in fact belongs to $C((t_0, t_0 + \tau_\omega]; \mathcal{H}^1(\mathbb{R}^2))$, thanks to the smoothing due to the Poisson kernel $P(t)$ in (2.4). Fix $t > t_0$. By (2.5) and Lemma 2.5, we have

$$\|V(t - t_0)(v_0, v_1)\|_{\mathcal{H}^1} \lesssim (t - t_0)^{-1+s_0} \|(v_0, v_1)\|_{\mathcal{H}^{s_0}}. \tag{4.3}$$

Then, from (4.3), Lemma 3.3 with $s = 1$, and (4.2), we have, for any $t_0 < t \le t_0 + \tau_\omega$,

$$\begin{aligned}\|\vec{v}(t)\|_{\mathcal{H}^1} &\lesssim (t - t_0)^{-1+s_0}\|(v_0, v_1)\|_{\mathcal{H}^{s_0}} + \|(v+z)^5\|_{L^1([t_0,t_0+\tau_\omega];L^2_x)}\\ &\lesssim (t - t_0)^{-1+s_0}\|(v_0, v_1)\|_{\mathcal{H}^{s_0}} + \tau_\omega^\theta\Big(\|v\|_{L^{5+\delta}([t_0,t_0+\tau_\omega];L^{10}_x)} + C_\omega\Big)\\ &< \infty.\end{aligned}$$

This proves the gain of regularity for $\vec{v} = (v, \partial_t v)$.[3] In the following, our main goal is to control the $\mathcal{H}^1$-norm of $\vec{v}(t)$ on $[0, T]$ for any given $T \gg 1$.

4.2. **Energy bound.** Fix $\varepsilon > 0$. Then, it follows from Theorem 3.1 that there exists Ω_{T_0} with sufficiently small $T_0 = T_0(\varepsilon) > 0$ such that

$$P(\Omega_{T_0}^c) < \frac{\varepsilon}{2} \tag{4.4}$$

and, for each $\omega \in \Omega_{T_0}$, the local well-posedness of (1.22) holds on $[0, T_0]$.

Fix a large target time $T \gg 1$. In the following, by excluding further a set of small probability, we construct the solution $\vec{v} = (v, \partial_t v)$ on the time interval $[T_0, T]$ and hence on $[0, T]$. Our goal is to control the growth of the $\mathcal{H}^1$-norm of $\vec{v}(t)$ on $[T_0, T]$. To do this, we closely follow the procedure for a similar energy bound for a defocusing quintic nonlinear wave equation in [61] in the remainder of this section. Since the same argument in the proof of Proposition 4.1 in [61] applies to establishing the energy bound in the current context, we only summarize the main steps below and refer the reader to [61] for details.

Step 1: *Reduction to an energy bound.* In order to control the $\mathcal{H}^1$ norm of $vectv(t)$ on $[T_0, T]$, it suffices to control the $\dot{\mathcal{H}}^1$ norm on $[T_0, T]$, where $\dot{\mathcal{H}}^1(\mathbb{R}^2)$-norm, where $\dot{\mathcal{H}}^1(\mathbb{R}^2) := \dot{H}^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. This is due to the fundamental theorem of calculus:

$$\|v(t)\|_{L^2_x} = \bigg\| \int_0^t \partial_t v(t')dt' \bigg\|_{L^2_x} \le T\|\partial_t v\|_{L^\infty_T L^2_x},$$

for $0 < t \le T$. The $\dot{\mathcal{H}}^1$ norm of $\vec{v}$ is further controlled by the energy $E(\vec{v})$ in (1.26). *Hence, it suffices to control the energy $E(\vec{v})$ on $[T_0, T]$.*

Step 2: *Statement of desired energy growth inequality on $[T_0, T]$.* We will derive an energy inequality to estimate $E(\vec{v})(t) - E(\vec{v})(T_0)$ for $t \in [T_0, T]$. We remark that if we were considering the case of the cubic nonlinearity (rather than a quintic nonlinearity), we can follow the Gronwall argument by Burq and Tzvetkov [12]. In the current quintic case, however, this argument fails. To overcome this difficulty, we employ the integration-by-parts trick introduced by Pocovnicu and the second author [61] in studying almost sure global well-posedness of the energy-critical defocusing quintic NLW on $\mathbb{R}^3$.

Let $z(t) = V(t)(u_0^\omega, u_1^\omega)$ be the random linear solution defined in (1.24). With $\widetilde{V}(t)$ as in (2.18), define $\widetilde{z}$ by

$$\widetilde{z}(t) = \langle\nabla\rangle^{-1}\partial_t z(t) = \widetilde{V}(t)(u_0^\omega, u_1^\omega). \tag{4.5}$$

[3]Here, we did not show the continuity in time of $\vec{v}$ in $\mathcal{H}^1(\mathbb{R}^2)$ but this can be done by a standard argument, which we omit.

Then, given $0 < T_0 < T$, we set $A(T_0, T)$ as

$$\begin{aligned} A(T_0, T) = 1 &+ \|z\|^2_{L^\infty([T_0,T];L^\infty_x)} + \|z\|^{10}_{L^{10}([T_0,T];L^{10}_x)} + \|z\|^6_{L^\infty([T_0,T];L^6_x)} \\ &+ \|\widetilde{z}\|^6_{L^6([T_0,T];L^6_x)} + \big\|\langle\nabla\rangle^{s_1}\widetilde{z}\big\|_{L^\infty([T_0,T];L^\infty_x)}, \end{aligned} \tag{4.6}$$

where $s_1 > \frac{1}{2}$ is sufficiently close to $\frac{1}{2}$ (to be chosen later). Since we can control $A(T_0, T)$ with high probability by the probabilistic Strichartz estimate in Proposition 2.9, our goal is to obtain an energy inequality of the following form and apply Gronwall's inequality:

$$E(\vec{v})(t) \lesssim E(\vec{v})(T_0) + A(T_0, T) + A(T_0, T)\int_{T_0}^t E(\vec{v})(t')dt'. \tag{4.7}$$

Step 3: *Calculation of* $\frac{d}{dt}E(\vec{v})$. By using the equation (1.25), we have

$$\frac{d}{dt}E(\vec{v})(t) = -\int_{\mathbb{R}^2}(D^{\frac{1}{2}}\partial_t v)^2 dx - \int_{\mathbb{R}^2}\partial_t v\big((v+z)^5 - v^5\big)dx. \tag{4.8}$$

By using the fact that $-\int_{\mathbb{R}^2}(D^{\frac{1}{2}}\partial_t v)^2 dx \le 0$ and proceeding as in [61], we deduce that

$$\begin{aligned} E(\vec{v})(t) - E(\vec{v})(T_0) &\le -\int_{T_0}^t\int_{\mathbb{R}^2} z(t')\partial_t(v(t')^5)dt'dx - \int_{T_0}^t\int_{\mathbb{R}^2}\partial_t v(t')\mathcal{N}(z,v)(t')dxdt' \\ &=: \mathrm{I}(t) + \mathrm{II}(t) \end{aligned} \tag{4.9}$$

for any $t \in [T_0, T]$, where $\mathcal{N}(z, v)$ denotes the lower order terms in v:

$$\mathcal{N}(z,v) = 10z^2v^3 + 10z^3v^2 + 5z^4v + z^5.$$

Step 4: *Estimate of* $\mathrm{II}(t)$. We have reduced the goal of showing the energy bound estimate (4.7) to estimating the quantities $\mathrm{I}(t)$ and $\mathrm{II}(t)$. Using the same arguments in [61], we have that

$$\begin{aligned} |\mathrm{II}(t)| &\lesssim \big(1 + \|z\|^2_{L^\infty([T_0,T];L^\infty_x)}\big)\int_{T_0}^t E(\vec{v})(t')dt' + \|z\|^{10}_{L^{10}([T_0,T];L^{10}_x)} \\ &\le A(T_0,T)\int_{T_0}^t E(\vec{v})(t')dt' + A(T_0,T). \end{aligned} \tag{4.10}$$

Step 5: *Estimate of* $\mathrm{I}(t)$. The estimate of $\mathrm{I}(t)$ is more involved. We proceed by using the integration-by-parts trick in [61] and integrating by parts in time, to obtain two quantities $\mathrm{I}_1(t)$ and $\mathrm{I}_2(t)$:

$$\mathrm{I}(t) = -\int_{\mathbb{R}^2} z(t')v(t')^5 dx\Big|_{T_0}^t + \int_{\mathbb{R}^2}\int_{T_0}^t \partial_t z(t')v(t')^5 dt'dx =: \mathrm{I}_1(t')\Big|_{T_0}^t + \mathrm{I}_2(t). \tag{4.11}$$

As for the first term I_1, we use Young's inequality with exponents 6 and 6/5 to obtain

$$\begin{aligned} |\mathrm{I}_1(t) - \mathrm{I}_1(T_0)| &\lesssim \varepsilon_0^{-6}\|z(T_0)\|^6_{L^6_x} + \varepsilon_0^{\frac{6}{5}}\|v(T_0)\|^6_{L^6_x} + \varepsilon_0^{-6}\|z(t)\|^6_{L^6_x} + \varepsilon_0^{\frac{6}{5}}\|v(t)\|^6_{L^6_x} \\ &\lesssim \varepsilon_0^{-6}\|z\|^6_{L^\infty([T_0,T];L^6_x)} + \varepsilon_0^{\frac{6}{5}}E(\vec{v})(T_0) + \varepsilon_0^{\frac{6}{5}}E(\vec{v})(t) \end{aligned} \tag{4.12}$$

for some small constant $\varepsilon_0 > 0$ (to be chosen later).

Next, we consider the second term I_2 in (4.11). While we closely follow the argument in [61], we summarize the procedure here for readers' convenience. From (4.5), we have

$$\mathrm{I}_2(t) = \int_{T_0}^t\int_{\mathbb{R}^2}\langle\nabla\rangle\widetilde{z}(t')\cdot v(t')^5 dxdt'. \tag{4.13}$$

Given dyadic $M \geq 1$, let $\mathbf{Q}_M$ denote the *nonhomogeneous* Littlewood-Paley projector onto the (spatial) frequencies $\{|\xi| \sim M\}$. This means that $\mathbf{Q}_1$ is a smooth projector onto the (spatial) frequencies $\{|\xi| \lesssim 1\}$ and by convention, $\mathbf{Q}_{2^{-1}} = 0$. Then, define $\mathcal{I}(t)$ by

$$\mathcal{I}(t) \coloneqq \int_{\mathbb{R}^2} \langle \nabla \rangle \widetilde{z}(t) \cdot v(t)^5 dx$$

and note that by using a Littlewood-Paley frequency decomposition, we have that

$$\mathcal{I}(t) \sim \sum_{\substack{M \geq 1 \\ \text{dyadic}}} \mathcal{I}^M(t) \tag{4.14}$$

for dyadic $M = 2^k$ with k a nonnegative integer, and

$$\mathcal{I}^M(t) := \sum_{k=-1}^{1} M \int_{\mathbb{R}^2} \mathbf{Q}_{2^k M} \widetilde{z}(t) \mathbf{Q}_M\big(v(t)^5\big) dx.$$

We also set

$$\mathcal{I}^{M \geq 2}(t) = \mathcal{I}(t) - \mathcal{I}^1(t).$$

• **Case 1:** $M = 1$. In this case, we can bound the contribution to $|\,\mathrm{I}_2(t)|$ by Young's inequality as

$$\begin{aligned} \bigg| \int_{T_0}^t \mathcal{I}^1(t')dt' \bigg| &\lesssim \|\widetilde{z}\|^6_{L^6([T_0,T];L^6_x)} + \int_{T_0}^t \|v(t')\|^6_{L^6_x} dt' \\ &\lesssim A(T_0,T) + \int_{T_0}^t E(\vec{v})(t')dt'. \end{aligned} \tag{4.15}$$

• **Case 2:** $M \geq 2$. We can follow the full details given in [61] and apply the Littlewood-Paley decomposition on each factor of v in the quintic term v^5, in order to obtain the following estimate:

$$|\mathcal{I}^{M \geq 2}(t)| \lesssim \big\| \langle \nabla \rangle^{s_2 - \theta} \widetilde{z}(t) \big\|_{L^\infty_x} E(\vec{v}), \tag{4.16}$$

provided that $2(1 - s_2 + \theta+) \leq 1$. Hence, by setting $s_1 = s_2 - \theta$, it follows from (4.6) and (4.16) that

$$\bigg| \int_{T_0}^t \mathcal{I}^{M \geq 2}(t')dt' \bigg| \lesssim A(T_0,T) \int_{T_0}^t E(\vec{v})(t')dt', \tag{4.17}$$

provided that $2(1 - s_2 + \theta+) \leq 1$, namely $s_2 \geq \frac{1}{2} + \theta+$, which is satisfied by choosing $s_2 > \frac{1}{2}$ and $\theta > 0$ sufficiently small. This determines the choice of $s_1 = s_2 - \theta$ in (4.6).

Therefore, from (4.13), (4.14), (4.15), and (4.17), we obtain

$$\begin{aligned} |\,\mathrm{I}_2(t)| &\lesssim \|\widetilde{z}\|^6_{L^6([T_0,T];L^6_x)} + \Big(1 + \big\| \langle \nabla \rangle^{s_1} \widetilde{z} \big\|_{L^\infty([T_0,T];L^\infty_x)}\Big) \int_{T_0}^t E(\vec{v})(t')dt' \\ &\lesssim A(T_0,T) + A(T_0,T) \int_{T_0}^t E(\vec{v})(t')dt'. \end{aligned} \tag{4.18}$$

Step 6: *Final estimate and Gronwall inequality.* Putting (4.9), (4.10), (4.11), (4.12), and (4.18) together and choosing sufficiently small $\varepsilon_0 > 0$ in (4.12), we obtain

$$E(\vec{v})(t) \lesssim E(\vec{v})(T_0) + A(T_0,T) + A(T_0,T) \int_{T_0}^t E(\vec{v})(t')dt',$$

for any $t \in [T_0, T]$. Therefore, from Gronwall's inequality, we conclude that

$$E(\vec{v})(t) \lesssim C\big(T_0, T, E(\vec{v})(T_0), A(T_0, T)\big) \tag{4.19}$$

for any $t \in [T_0, T]$.

Remark 4.2. (i) In order to justify the formal computation in this subsection, we need to proceed with the smooth solution $(v_N, \partial_t v_N)$ associated with the frequency truncated random initial data (for example, to guarantee finiteness of the term $-\int_{\mathbb{R}^2}(D^{\frac{1}{2}}\partial_t v)^2 dx$ in (4.8)) and then take $N \to \infty$, using the approximation argument (Corollary 3.4). This argument, however, is standard and thus we omit details. See, for example, [61].

(ii) In this section, we followed the argument in [61] to obtain an energy bound in the quintic case. In this argument, the first term after the first inequality in (4.10) provides the restriction $p \le 5$ on the degree of the nonlinearity $|u|^{p-1}u$. For $p > 5$, we will need to apply the integration by parts trick to lower order terms as well. See for example [44] in the context of the standard NLW. In a recent preprint [47], Liu extended Theorems 1.1 and 1.3 to the super-quintic case ($p > 5$) and proved almost sure global well-posedness of the defocusing vNLW (1.1) in $\mathcal{H}^s(\mathbb{R}^2)$ for $s > -\frac{1}{p}$.

4.3. **Proof of Proposition 4.1.** Fix a target time $T \gg 1$ and small $\varepsilon > 0$. Then, let T_0 be as in (4.4). With $A(T_0, T)$ as in (4.6), set

$$A_\lambda = \big\{\omega \in \Omega : A(T_0, T) < \lambda\big\}$$

for $\lambda > 0$. From Proposition 2.9, there exists $\lambda_0 \gg 1$ such that

$$P(A^c_{\lambda_0}) < \frac{\varepsilon}{2}. \tag{4.20}$$

Now, set $\Omega_{T,\varepsilon} = \Omega_{T_0} \cap A_{\lambda_0}$. Then, from (4.4) and (4.20), we have $P(\Omega^c_{T,\varepsilon}) < \varepsilon$.

Let $\omega \in \Omega_{T,\varepsilon}$. From (4.6) and Hölder's inequality, we see that $A(T_0, T)$ controls the $L^{5+\delta}_t([T_0, T]; L^{10}_x)$-norm of z:

$$\|z\|_{L^{5+\delta}([T_0,T];L^{10}_x)} \lesssim T^\theta \lambda_0^{\frac{1}{10}}$$

for some $\theta > 0$, where $\delta > 0$ is as in (the proof of) Theorem 3.1. Then, together with the energy bound (4.19) and the discussion in Section 4.1, we can iteratively apply Corollary 3.5 (see also the discussion right after Proposition 4.1) and construct a solution $u = z + v$ to (1.22) on $[0, T]$ with $(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)$ for each $\omega \in \Omega_{T,\varepsilon}$. This proves Proposition 4.1 and hence almost sure global well-posedness (Theorem 1.3).

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