Double copy of the multipole expansion

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The classical double copy relates solutions of biadjoint, gauge, and gravity theories. The ultimate origin and scope of this correspondence remains mysterious, such that it is important to build a clear physical intuition of how the double copy operates. To this end, we consider the multipole expansion of exact classical solutions. Using a recently developed twistor translation of the classical double copy, we use well-established techniques to show that the multipole moments of arbitrary vacuum type-D gravity fields are exactly related to their counterparts in gauge and biadjoint scalar theories by the single and zeroth copies. We cross-check our results using previously obtained results for the Kerr metric and also provide new results for the "square root" of the Kerr-Taub-NUT solution.

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I. INTRODUCTION

There is mounting evidence that our various theories of fundamental physics are more closely connected than previously thought. In this paper, we focus on a particular correspondence—the *classical double copy* that relates solutions of the field equations in (non-)Abelian gauge theories and gravity, as well as in a novel scalar theory with two different types of color charge (*biadjoint scalar field theory*). Inspired by the original double copy for scattering amplitudes in the corresponding quantum field theories [1,2] (which itself has a string theoretic origin [3]), the first classical double copy to appear was the *Kerr-Schild double copy* of Ref. [4] (see Refs. [5–18] for further developments, and Refs. [19,20] for related earlier work in a higher-spin context). An alternative exact double copy procedure is the *Weyl double copy* of Ref. [21] (see also Refs. [14,22–25]). This uses the spinorial rather than tensorial formalism of general relativity and includes the Kerr-Schild double copy as a special case. To date, it constitutes the most general exact statement of the classical double copy, although other formalisms also offer useful alternative insights [26–37].

As well as practical applications of this correspondence (see, e.g., Ref. [38]), there are also important conceptual issues to address, including understanding the ultimate origin of the classical double copy itself. To this end, Refs. [39,40] showed how one can derive both the form and scope of the Weyl double copy using well-established ideas from twistor theory [41–43] (see Refs. [44–46] for pedagogical reviews of this subject), as well as showing that the Weyl copy is more general than previously thought. Recently, Ref. [47] used the ideas of Refs. [48–50] to show that the Bern-Carrasco-Johansson (BCJ) double copy for scattering amplitudes, the twistor double copy of Refs. [39,40], and the Weyl double copy of Ref. [21] are precisely equivalent for type-D solutions, being related by well-defined integral transforms. This provides a firm basis for the validity of the exact classical double copy where it applies and allows us to use whichever form of the double copy happens to be convenient for a particular purpose.

As well as applying new mathematical techniques, another useful method for extending our understanding of the double copy is to take known physical or mathematical properties in biadjoint scalar, gauge, and gravity theories and to see how they match up (or otherwise). Recent examples include properties of solutions at strong coupling [51–54], symmetries [35,55–58], and geometric/topological information [8,59,60]. Even more simply, the original Kerr-Schild double copy of Ref. [4] told us that mass and energy map to charge in the gauge theory, which nicely mirrors the replacement of kinematic by color information in scattering amplitudes [1,2]. Arguably, however, a detailed physical understanding of how the double copy operates is still in its infancy. Further development of our physical intuition ought to proceed in tandem with

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mathematical developments, as it is very often the case that physical and/or geometric reasoning allows us to make conceptual breakthroughs that allow for technical progress. With this in mind, it becomes important to fully scrutinize exact cases of the double copy however special or restricted a given class of solutions may happen to be.

In this paper, we extend these ideas by considering the well-known *multipole expansion* of classical solutions. For any given solution, one may define a series of higher-rank *multipole tensors*, which completely characterize the spatial and temporal distributions of charge (energy/momentum) in the gauge (gravity) theory, respectively. The lowest-order results in this expansion constitute the total charge and mass mentioned above, and one may then ask whether higher multipole moments are strict double copies of each other. It turns out that this can be easily addressed in the twistor double copy of Refs. [39,40], which relies on the well-known Penrose transform connecting spacetime (spinor) fields with certain functions in an abstract twistor space. As has been argued by Curtis [61], the multipole tensors associated with a given spacetime field can be replaced by a description in terms of higher-rank twistors, which are straightforwardly defined from the twistor-space functions describing the fields. We combine this with the twistor-space double copy of Refs. [39,40] and thus obtain an explicit statement that the multipole expansion double copies for arbitrary type-D vacuum solutions. Our results provide a useful physical insight into how the double copy operates and may well be relevant for thinking about further applications. Note that it must be possible to arrive at similar results using the standard BCJ or Weyl double copies. However, given that these are both equivalent to the twistor double copy for the Petrov type-D solutions described by the Weyl double copy, we can choose whichever formalism we like, and it is Ref. [61] that suggests that the twistor double copy is the most convenient.

The structure of our paper is as follows. In Sec. II, we briefly review the twistor double copy of Refs. [39,40]. In Sec. III, we define more precisely the multipole expansion, guided by the arguments of Ref. [61]. In Sec. III B, we argue that multipole moments in different theories double copy for arbitrary instances of the Weyl double copy of Ref. [21]. Finally, we discuss our results and conclude in Sec. IV.

II. THE TWISTOR SPACE DOUBLE COPY

In this section, we review salient details of the Weyl double copy, together with its twistor space incarnation, referring the reader to Rfs. [39,40] for full details. First, we recall that massless free spacetime fields can be represented by multi-index spinors $\phi_{AB\ldots C}$ ($\bar{\phi}_{A'B'\ldots C'}$), representing the antiself-dual (self-dual) parts of the field respectively. Index values run from 0 to 1 and may be raised, lowered, and/or contracted using the Levi-Civita symbols ϵ^{AB}, etc. There are $2n$ indices for a spin-n field, and the resulting quantities then satisfy a special case of the general massless free-field equation,

$$\nabla^{AA'}\bar{\phi}_{A'\ldots C'} = 0, \qquad \nabla^{AA'}\phi_{AB\ldots C} = 0, \tag{1}$$

where $\nabla^{AA'}$ is the appropriate translation of the spacetime covariant derivative. These fields can be reinterpreted in twistor space $\mathbb{T}$, corresponding to solutions of the *twistor equation*,

$$\nabla_{A'}^{(A}\Omega^{B)} = 0 \Rightarrow \Omega^A = \omega^A - ix^{AA'}\pi_{A'}. \tag{2}$$

In the second equality, we have written the general solution in Minkowski space, in terms of constant spinors which may be grouped together to make a 4-component *twistor*,

$$Z^\alpha = (\omega^A, \pi_{A'}). \tag{3}$$

A nonlocal map between spacetime and twistor space is established by requiring that the field in Eq. (2) vanishes, such that the twistor components satisfy the incidence relation,

$$\omega^A = ix^{AA'}\pi_{A'}. \tag{4}$$

This is invariant under rescalings $Z^\alpha \to \lambda Z^\alpha$, $\lambda \in \mathbb{C}$, such that we need only consider *projective twistor space* $\mathbb{PT}$. A point in spacetime corresponds to a Riemann sphere in $\mathbb{PT}$, also referred to as a *(complex) line*. An important result known as the *Penrose transform* expresses massless free spacetime fields satisfying Eq. (1) via the contour integrals,

$$\bar{\phi}_{A'B'\ldots C'}(x) = \frac{1}{2\pi i}\oint_\Gamma d\pi_{E'}d\pi^{E'}\pi_{A'}\pi_{B'}\ldots\pi_{C'}[\rho_x f(Z^\alpha)], \tag{5}$$

where $\rho_x$from which we may straightforwardly write restricts all twistors to obey the incidence relation corresponding to spacetime point x, and the contour Γ lies on the appropriate Riemann sphere. The combined integrand and measure must be invariant under rescalings $Z^\alpha \to \lambda Z^\alpha$, which fixes the (holomorphic) function $f(Z^\alpha)$ to have homogeneity $-(2n+2)$ for a spin-n field. The above remarks imply that twistor functions of homogeneity -2, -4, and -6 correspond to spacetime gravity fields in scalar, gauge, and gravity theories, respectively. Denoting the corresponding twistor functions by the subscripts $\{\text{scal}, \text{EM}, \text{grav}\}$, Refs. [39,40] argued that one may define a gravity twistor function via[1]

[1]We have here skimmed over the fact that the twistor functions used throughout are not unique and are instead representatives of cohomology classes. The product of Eq. (6) is then interpreted to apply only to particular chosen representatives, and we return to this point in what follows.

$$f_{\rm grav}(Z^\alpha) = \frac{f^{(1)}_{\rm EM}(Z^\alpha) f^{(2)}_{\rm EM}(Z^\alpha)}{f_{\rm scal}(Z^\alpha)}, \tag{6}$$

leading to the spacetime *Weyl double copy* formula,

$$\phi_{A'B'C'D'}(x) = \frac{\phi^{(1)}_{(A'B'}(x)\phi^{(2)}_{C'D')}(x)}{\phi(x)}, \tag{7}$$

first presented in Ref. [21]. Here, ϕ is a biadjoint scalar field, $\phi^{(i)}_{A'B'}$ an electromagnetic spinor, and $\phi_{A'B'C'D}$ a Weyl spinor. The above discussion applies to the case of primed spinor fields in spacetime. For unprimed fields, one may consider the conjugate of Eq. (2), whose solutions are associated with *dual twistors* W_α. The notion of the Penrose transform can be straightforwardly adapted from Eq. (5),

$$\phi_{AB\ldots C}(x) = \frac{1}{2\pi i}\oint_\Gamma d\lambda_E d\lambda^E \lambda_A\lambda_B\ldots\lambda_C[\rho_x f(W_\alpha)], \tag{8}$$

and the twistor double copy of Eq. (6) similarly generalizes. We work with dual twistors by default in what follows, in order to match conventions with Ref. [61].

III. MULTIPOLES AND THE DOUBLE COPY

The idea of multipoles is familiar from Newtonian physics in three-dimensional Euclidean space. A stationary electrostatic or Newtonian potential ϕ in a sourceless region satisfies Laplace's equation $\nabla^2\phi = 0$, and may be expanded as[2]

$$\phi = \frac{M}{r} + \frac{M_i x^i}{r^3} + \frac{M_{ij}x^i x^j}{r^5} + \ldots, \tag{9}$$

where $r = (x^i x^i)^{\frac{1}{2}}$, and the *multipole tensors* $\{M_{ij\ldots k}\}$ are constant tensors defined in terms of derivatives of the potential, evaluated at the origin $\mathcal{O}$. Upon shifting to a different point, the multipole moments change in a way that involves only lower-order multipoles. The extension of these ideas to general relativity has been discussed in Refs. [62–64], for general asymptotically flat spacetimes. We do not need the full complication of the latter, given that we are concerned with solutions of the massless free-field equation of Eq. (1) in Minkowski space. Given a constant unit timelike vector t^a, one may then consider the 3-space orthogonal to this, with induced metric

$$h_{ab} = \eta_{ab} - t_a t_b. \tag{10}$$

Reference [62] then showed that an appropriate generalization of the multipole tensors appearing in eq. (9) is provided by symmetric, trace-free tensor fields $Q^{a_1\ldots a_n}(x)$ satisfying

$$t_{a_1}Q^{a_1\ldots a_n} = 0,$$
$$\nabla^m Q^{a_1\ldots a_n} = \frac{n(2n-1)}{3}h^{m(a_1}Q^{a_2\ldots a_n)} - \frac{n(n-1)}{3}Q^{m(a_3\ldots a_n}h^{a_1 a_2)}, \tag{11}$$

where the nth such quantity is referred to as the 2^n-*multipole tensor*, and the second condition requires that the derivatives of multipole tensors depend only upon lower multipoles. This is the analogue of the "shifting the origin" property mentioned for Newtonian multipoles above and ensures that the set of tensors $\{Q^{a_1\ldots a_n}\}$ corresponds to the same solution of the field equation.

In the spinorial formalism, each spacetime index in Eq. (11) will become a pair of spinor indices. Contracting with the timelike vector appearing there, one may define the symmetric spinor field,

$$\omega^{A'_1\ldots A'_{2n}} = (6i)^n Q^{A'_1\ldots A'_n B_1\ldots B_n} t^{A'_{n+1}}_{B_1}\ldots t^{A'_{2n}}_{B_n}, \tag{12}$$

which turns out to solve a higher-rank generalization of the twistor equation of Eq. (2),

$$\nabla^{(L'}_{L}\omega^{A'_1\ldots A'_{2n})} = 0. \tag{13}$$

Thus, we can associate the spinors of Eq. (12) with multi-index *multipole twistors* $\{Q_{\alpha_1\ldots\alpha_{2n}}\}$. To see how this works in practice, consider a given physical spin-n field $\Psi_{A_1A_2\ldots A_{2n}}$. Then, one may define higher-spin fields iteratively by taking derivatives and contracting with the timelike vector appearing in Eq. (12),

$$\Psi^{(n)}_{A_1\ldots A_{2n}} = t_{A'A_1}\nabla^{A'}_{A_2}[\Psi^{(n-1)}_{A_3\ldots A_{2n}}]. \tag{14}$$

These constitute a spinorial analogue of the multiple derivatives appearing in the Newtonian formalism, whereby higher multipole moments contain more derivatives of the original potential. For a spin-1 field, one may write an explicit twistor space integral for the total conserved charge producing the field [65],

$$Q = -\frac{i}{4\pi^2}\oint f(W_\alpha)d^4W,$$
$$d^4W = \frac{1}{4!}\epsilon^{\alpha\beta\gamma\delta}dW_\alpha dW_\beta dW_\gamma dW_\delta, \tag{15}$$

where an appropriate contour must be chosen and where $f(W_\alpha)$ is the twistor function corresponding to the spacetime field. Given a higher-spin field as in Eq. (14), we can form multiple spin-1 fields by contracting with solutions of the twistor equation[3] $\{\alpha^{A_1\ldots A_{2n}}\}$,

[2]Throughout the paper, we use lower-case latin, upper-case latin, and greek indices for spacetime tensors, spacetime spinors, and twistors, respectively. Note, however, that the indices in Eq. (9) run only over spatial components, i.e., from 1 to 3.

[3]That the fields of Eq. (16) indeed satisfy the massless free-field equation of Eq. (1) follows from Eq. (13).

$$\Phi^{(n)}_{AB} = -i^n \alpha^{A_1\ldots A_{2n}}[\Psi^{(n+1)}_{ABA_1\ldots A_{2n}}]. \tag{16}$$

Each of these fields will have a conserved charge according to Eq. (15), and we may collect together all such charges in the twistor-covariant form,

$$q(A^{\alpha_1\ldots\alpha_{2n}}) = \frac{i^{n+1}}{4\pi^2}\oint W_{\alpha_1}\ldots W_{\alpha_{2n}} A^{\alpha_1\ldots\alpha_{2n}} f_{n+1}(W_\alpha) d^4W, \tag{17}$$

for symmetric twistors $\{A^{\alpha_1\cdots\alpha_{2n}}\}$, where $f_{n+1}(W_\alpha)$ is the twistor function corresponding to the spacetime higher-spin field ${}_{(n+1)}\Psi_{ABA_1\ldots A_{2n}}$ and multipole index n. Equation (17) defines a set of quantities dual to the $\{A^{\alpha_1\cdots\alpha_{2n}}\}$,

$$Q_{\alpha_1\ldots\alpha_{2n}} = \frac{i^{n+1}}{4\pi^2}\oint W_{\alpha_1}\ldots W_{\alpha_{2n}} f_{n+1}(W_\alpha) d^4W, \tag{18}$$

which are the multipole twistors we have been seeking. Note that the iterative structure of the higher-spin fields in Eq. (14) implies that the twistor functions $\{f_{n+1}\}$ in Eq. (18) can also be constructed iteratively, and there are various ways that this can be written. A fully invariant condition is [61]

$$f_{n+1} = i(R_\alpha W_\beta I^{\alpha\beta})^{-1} R_\gamma P^\gamma_\delta \frac{\partial f_n}{\partial W_\delta}, \tag{19}$$

where we have introduced the so-called *infinity twistors* for Minkowski spacetime,

$$I_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{A'B'} \end{pmatrix}, \qquad I^{\alpha\beta} = \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & 0 \end{pmatrix}. \tag{20}$$

R_α is an arbitrary twistor, and we have introduced the projector [61]

$$\lambda P^\alpha_\beta = I^{\alpha\gamma} Q_{\gamma\beta}, \tag{21}$$

where λ is the relevant mass or charge parameter for a given theory. One thus has $\lambda = m$ in gravity, where m is the total mass of the system. In gauge or biadjoint theory, it will be the total charge of the system that is creating the field, which we denote by q and y, respectively.

A. The double copy of the multipole expansion

The multipole twistors introduced above allow us to address the double copy of the multipole expansion in a particularly compact and elegant way. Consider twistor functions corresponding to a biadjoint scalar, electromagnetic, and gravity solutions, respectively, which we label by $f_X(W_\alpha)$, $X \in \{\text{scal}, \text{EM}, \text{grav}\}$. From each of these, one may define a set of higher-spin twistor functions according to the iterative procedure of Eq. (19), denoted here by $f_X^{(n+1)}(W_\alpha)$. By Eq. (18), this immediately leads to a set of multipole twistors for each original spacetime field. This construction is shown in Table I, where each column contains twistor functions of the same homogeneity, leading to fields in position space with the same spin. We show the multipole twistors that arise from these in Table II. Note that the biadjoint scalar twistor function f_{scal} does not produce a physical multipole, but it is needed to generate all the $f_{\text{scal}}^{(n+1)}$ that do result in the multipole expansion. Furthermore, note that a gravitational monopole contribution is not directly obtained from the corresponding twistor functions. As the total mass, however, it is obtainable from the angular momentum twistor $Q^{\text{grav}}_{\alpha_1\alpha_2}$. For the $n = 0$ case, one finds integrals expressing the total charge generated by the biadjoint or EM field. From $n = 1$ upwards, there are multipole twistors in all three theories. For a given set of functions $\{f_X\}$ related by the twistor-space double copy, we can then associate each column of Table II with a classical double copy triple, as shown in Fig. 1.

The physical interpretation of the identifications in Fig. 1 is straightforward. Consider, for example, the 2-multipole tensors $Q^X_{\alpha\beta}$. This represents the angular momentum in gravity [42], whereas in electromagnetism, it is the charge dipole tensor, as expected given that the single copy turns mass into charge. Likewise, for the higher multipoles, the single copy replaces the relevant spatiotemporal distribution of mass/momentum with that of charge, with a further replacement to "biadjoint charge" in the zeroth copy.

It is one thing to formally identify the multipole twistors in different theories, as we have done in Fig. 1. It is quite

TABLE I. Twistor functions resulting in the multipole expansion for three different theories. Here, f_X is a twistor function corresponding to the physical spacetime field in theory X, and $f_X^{(n+1)}$ is a higher-spin field generated from this by the iterative procedure of Eq. (19). The function f_{scal} does not produce a physical multipole, but it generates the set of higher multipole twistor functions $f_{\text{scal}}^{(n+1)}$.

	Multipole index n			
Theory	0	1	2	3
Biadjoint scalar	$f^{(1)}_{\text{scal}}$	$f^{(2)}_{\text{scal}}$	$f^{(3)}_{\text{scal}}$	$f^{(4)}_{\text{scal}}$
Gauge	f_{EM}	$f^{(2)}_{\text{EM}}$	$f^{(3)}_{\text{EM}}$	$f^{(4)}_{\text{EM}}$
Gravity		f_{grav}	$f^{(3)}_{\text{grav}}$	$f^{(4)}_{\text{grav}}$

TABLE II. Multipole twistors arising from the twistor functions of Table I.

	Multipole index n			
Theory	0	1	2	3
Biadjoint scalar	Q^{scal}	$Q^{\text{scal}}_{\alpha_1\alpha_2}$	$Q^{\text{scal}}_{\alpha_1\alpha_2\alpha_3\alpha_4}$	$Q^{\text{scal}}_{\alpha_1\ldots\alpha_6}$
Gauge	Q^{EM}	$Q^{\text{EM}}_{\alpha_1\alpha_2}$	$Q^{\text{EM}}_{\alpha_1\alpha_2\alpha_3\alpha_4}$	$Q^{\text{EM}}_{\alpha_1\ldots\alpha_6}$
Gravity		$Q^{\text{grav}}_{\alpha_1\alpha_2}$	$Q^{\text{grav}}_{\alpha_1\alpha_2\alpha_3\alpha_4}$	$Q^{\text{grav}}_{\alpha_1\ldots\alpha_6}$

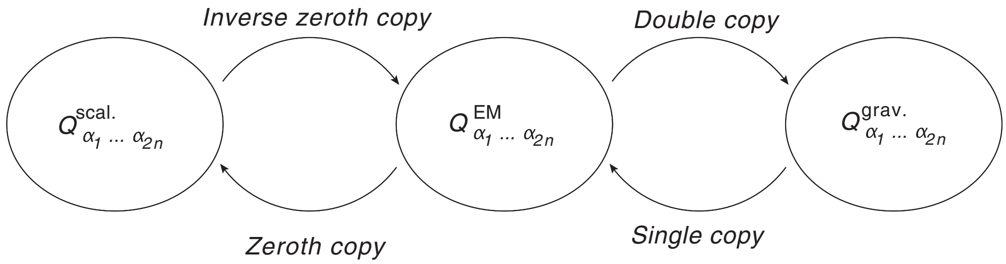

FIG. 1. Double copy structure of the multipole twistors appearing in a single column in Table II.

another thing to say that the multipole twistors in the different theories *are the same*, up to simple mass and charge replacements. Remarkably, this strong statement indeed turns out to be true for fields related by the original type-D Weyl double copy of Ref. [21], as we discuss in the following section.

B. Multipole moments of type-D solutions

As discussed in Ref. [66] and reviewed in Refs. [39,40], all vacuum type-D solutions arise from twistor functions of the form

$$f_{\text{grav}} = (A^{\alpha\beta} W_\alpha W_\beta)^{-3}, \tag{22}$$

where $A^{\alpha\beta}$ is a constant twistor that can be taken to be symmetric. We see that Eq. (22) has homogeneity -6 under rescalings of W_α, as required for a gravity solution. Furthermore, it has two poles in twistor space, which give rise, after performing the Penrose transform of Eq. (8) to position space, to the two two-fold degenerate principal spinors of the Weyl spinor that characterize it as being of type D. It turns out that the twistor $A^{\alpha\beta}$ can be straightforwardly related to the 2-multipole twistor for this field. Substituting Eq. (22) into Eq. (18) for $n = 1$, one may carry out the integral using a special case of

$$\oint W_{\alpha_1} \dots W_{\alpha_{2n}} (W_\alpha W_\beta A^{\alpha\beta})^{-(n+2)} d^4 W = \frac{\pi^3 i}{\Delta} \frac{(2n)!}{2^{2n-1}(n+1)!n!} B_{(\alpha_1\alpha_2} \dots B_{\alpha_{2n-1}\alpha_{2n})}, \tag{23}$$

where $B_{\alpha\beta}$ is the inverse of $A^{\alpha\beta}$ and Δ the determinant of the latter. One finds

$$Q_{\alpha\beta} = \frac{\pi}{8i\Delta} B_{\alpha\beta}, \qquad Q^{\alpha\beta} = \frac{8i\Delta}{\pi} A^{\alpha\beta}. \tag{24}$$

Given the general type-D gravity twistor function of Eq. (22), one may also identify the single and zeroth copies, giving rise to gauge and biadjoint scalar fields in spacetime, respectively. As explained in Refs. [39,40], these are

$$f_{\text{scal}} = \mathcal{N}_0 (A^{\alpha\beta} W_\alpha W_\beta)^{-1}, \qquad f_{\text{EM}} = \mathcal{N}_1 (A^{\alpha\beta} W_\alpha W_\beta)^{-2}. \tag{25}$$

We have here included arbitrary constant normalization factors in the scalar and electromagnetic functions, which are in any case not fixed in the Weyl double copy of Ref. [21]. Physically, one may absorb such constants by redefining the total amount of charge in a particular solution, but we fix them shortly. Let us now construct and compare the multipole twistors from these solutions. For each field, we may construct higher-spin twistor functions using the procedure of Eq. (19). Starting with the gravity function from Eq. (22), one finds

$$f^{(3)}_{\text{grav}} = -\frac{3i}{\lambda} (R_\alpha W_\beta I^{\alpha\beta})^{-1} (A^{\rho\lambda} W_\rho W_\lambda)^{-4} R_\gamma I^{\gamma\tau} Q_{\tau\delta} A^{\delta\sigma} W_\sigma, \tag{26}$$

where we have used Eq. (21). We may now use Eq. (24), which yields

$$f^{(3)}_{\text{grav}} = 3\left(-\frac{\pi}{4\Delta m}\right) (A^{\rho\lambda} W_\rho W_\lambda)^{-4}, \tag{27}$$

such that iterating this procedure leads to the formula

$$f^{(n)}_{\text{grav}} = \left(-\frac{\pi}{4\Delta m}\right)^{n-2} \frac{n!}{2} (W_\alpha W_\beta A^{\alpha\beta})^{-(n+1)}, \tag{28}$$

as quoted in Ref. [61]. Note that placing $n = 2$ in this formula reproduces the original gravity twistor function $f_{\text{grav}}(W_\alpha)$ itself. We may find the multipole twistors of Eq. (18) using Eq. (23), (24), yielding

$$Q^{\text{grav}}_{\alpha_1 \dots \alpha_{2n}} = \frac{1}{2} \frac{1}{(2m)^{n-1}} \frac{(2n)!}{n!} Q^{\text{grav}}_{(\alpha_1\alpha_2} \dots Q^{\text{grav}}_{\alpha_{2n-1}\alpha_{2n})}, \tag{29}$$

where the angular momentum twistor takes the explicit form

$$Q^{\text{grav}}_{\alpha\beta} = \begin{pmatrix} 0 & m t_A^{B'} \\ m t_B^{A'} & 2i\mu^{A'B'} \end{pmatrix}, \tag{30}$$

and $\mu^{A'B'}$ is the dipole spinor at the origin. Given that $\mu^{A'B'}$ can be shown to be proportional to the linear momentum spinor, we conclude that $Q^{\rm grav}_{\alpha\beta}$ is linear in the mass m. It is then useful to scale this out by writing

$$Q^{\rm grav}_{\alpha\beta} = m\bar{Q}^{\rm grav}_{\alpha\beta}. \tag{31}$$

The multipole twistors then become

$$Q^{\rm grav}_{\alpha_1\ldots\alpha_{2n}} = \frac{m}{2^n}\frac{(2n)!}{n!}\bar{Q}^{\rm grav}_{(\alpha_1\alpha_2}\ldots\bar{Q}^{\rm grav}_{\alpha_{2n-1}\alpha_{2n})}, \tag{32}$$

which makes clear that they are well defined as $m\to 0$, with a simple linear mass dependence. In principle, one may convert these multipole twistors back into multipole tensors. For the Kerr solution, a set of scalar *multipole moments* has been defined in the literature [64]. Let $\tilde{z}^a$ be a vector aligned with the axis of rotation of the black hole and Λ be the point at infinity after conformal compactification of the spacetime. Then, the multipole moments are given by

$$Q_n = \frac{1}{n!}Q_{a_1\ldots a_n}\tilde{z}^{a_1}\ldots\tilde{z}^{a_n}|_\Lambda, \tag{33}$$

where the notation on the right-hand side denotes that this be evaluated at Λ itself. As stated in Ref. [61], the multipole twistors of Eq. (29) do indeed reproduce the known multipole moments of the Kerr solution, first found in Ref. [64].

We may carry out the above procedure for the biadjoint scalar and gauge theories twistor functions of Eq. (25), and the resulting higher-spin twistor functions are given by

$$f^{(n)}_{\rm scal} = \mathcal{N}_0\left(-\frac{\pi}{4\Delta y}\right)^n\frac{n!}{2}(W_\alpha W_\beta A^{\alpha\beta})^{-(n+1)},$$
$$f^{(n)}_{\rm EM} = \mathcal{N}_1\left(-\frac{\pi}{4\Delta q}\right)^{n-1}\frac{n!}{2}(W_\alpha W_\beta A^{\alpha\beta})^{-(n+1)}, \tag{34}$$

where we have replaced the mass m in the gravity solution with the charge q in gauge theory and biadjoint charge y in the scalar theory. These functions reproduce the original fields for $n=0$ and $n=1$, respectively. We may choose to fix the arbitrary normalization constants $\mathcal{N}_i$ by requiring that the 2-multipole (dipole) tensor in each theory is simply related by replacing

$$m\to q\to y, \tag{35}$$

in going from gravity to gauge theory to biadjoint theory. This determines

$$\mathcal{N}_0 = \left(-\frac{\pi}{4\Delta y}\right)^{-2}, \qquad \mathcal{N}_1 = \left(-\frac{\pi}{4\Delta q}\right)^{-1}, \tag{36}$$

after which comparison of Eq. (34) with Eq. (28) shows that all higher-spin twistor functions agree across all three theories, so that one may simply replace the multipole twistors of Eq. (29) with the gauge and biadjoint scalar counterparts,

$$Q^{\rm scal}_{\alpha_1\ldots\alpha_{2n}} = \frac{y}{2^n}\frac{(2n)!}{n!}\bar{Q}^{\rm scal}_{(\alpha_1\alpha_2}\ldots\bar{Q}^{\rm scal}_{\alpha_{2n-1}\alpha_{2n})},$$
$$Q^{\rm EM}_{\alpha_1\ldots\alpha_{2n}} = \frac{q}{2^n}\frac{(2n)!}{n!}\bar{Q}^{\rm EM}_{(\alpha_1\alpha_2}\ldots\bar{Q}^{\rm EM}_{\alpha_{2n-1}\alpha_{2n})}, \tag{37}$$

where the bar notation on the right-hand side denotes that we have scaled the relevant coupling out of the two-index twistors, similarly to Eq. (31). A direct consequence of Eq. (37) is that the multipole moments of the gauge and biadjoint scalar fields corresponding to a given gravity field from Eq. (22) precisely match, after making the necessary mass-to-charge replacements. Our twistor analysis has applied for an arbitrary quadratic form in Eq. (22) which, as explained in Refs. [39,40], is a general statement for any (vacuum type-D) spacetime entering the original Weyl double copy of Ref. [21]. In particular, this must apply to the Kerr solution, whose scalar multipole moments are given by [64]

$$Q^{\rm Kerr}_n = i^{n-1}ma^n, \tag{38}$$

where a is the ring radius of the Kerr black hole. Our arguments in this paper then immediately imply that the multipole moments of the $\sqrt{\rm Kerr}$ solution, and its zeroth copy, are given by

$$Q^{\sqrt{\rm Kerr}}_n = i^{n-1}qa^n, \qquad Q^{\rm scal}_n = i^{n-1}ya^n, \tag{39}$$

respectively. That is, they are simply obtained by replacing the mass in the gravitational case with the relevant coupling parameter, as described above. There is a novel cross-check of the gauge theory results that one may perform. Although the electromagnetic multipole moments of the $\sqrt{\rm Kerr}$ solution have not been previously calculated in the literature, one may instead consider a charged Kerr black hole, otherwise known as a Kerr-Newman black hole [67,68]. This is a solution of the Einstein-Maxwell equations, and as such consists of a metric plus a gauge field. The $\sqrt{\rm Kerr}$ solution can be obtained by setting the mass of the solution to zero, leaving a gauge field living in Minkowski space, which is known to correspond to the single copy of the gravity solution. The combined gravitational and electromagnetic multipole moments of the Kerr-Newman solution have been calculated in Ref. [69]. The gravity moments agree with the pure Kerr solution, and the electromagnetic ones indeed match Eq. (39), thus verifying our results.

Let us now return to a technical issue that we glossed over above. As discussed above, the twistor double copy of Refs. [39,40] relies upon forming products of "functions" in twistor space. However, the Penrose transform of Eq. (5) is invariant under redefining $f(Z^\alpha)$ according to

$$f(Z^\alpha) \to f(Z^\alpha) + h_N(Z^\alpha) + h_S(Z^\alpha), \tag{40}$$

where $h_{N,S}$ have poles only on one side of the integration contour Γ (i.e., respectively only in the northern or southern hemispheres of the Riemann sphere corresponding to the spacetime point x). The set of functions related by such transformations forms a *Čech) cohomology class*, such that any given $f(Z^\alpha)$ constitutes a representative member of the class. As discussed in detail in Ref. [40], the nonlinear product of Eq. (6) that is needed to obtain the Weyl double copy in spacetime does not allow one to first perform redefinitions according to Eq. (40). It therefore seems that special representatives are needed to make the double copy product structure manifest, leaving a conceptual puzzle as how such representatives must be chosen. A number of discussions of this issue have recently appeared in the literature, and the first of these was in Ref. [70], which considered purely radiative spacetimes, i.e., those that can be completely prescribed using data defined at future null infinity. In such cases, there exists a natural way to pick out a special cohomology representative in each theory entering the double copy [71], thus making the twistor double copy unambiguous. A second discussion can be found in Ref. [72], which showed how the twistor double copy of Refs. [39,40], based on Čech cohomology, can be systematically translated into the different viewpoint of *Dolbeault cohomology*. The quantities entering the Penrose transform then become differential forms, and Ref. [72] showed that in the Euclidean signature the Weyl double copy in spacetime implies a natural product structure in twistor space, provided *harmonic* differential forms are chosen to represent each spacetime field. How this procedure relates to the Čech representatives used in Refs. [39,40]—not to mention the radiative double copy of ref. [70]—remains unclear.

Much closer to our present study, however, is the more recent work of Ref. [48], which constructs certain classical spacetimes from momentum-space scattering amplitudes using a two-step procedure: (i) gravitational amplitudes in momentum space are transformed into twistor space and (ii) the resulting quantities from step (i) are Penrose transformed to give spacetime fields, where further analytic continuation may be required to fix a desired signature. The relevant Penrose transform is in the Čech cohomology language, and step (i) thus picks out a particular Čech cohomology representative for the spacetime of interest. As shown in Ref. [47], the Čech representatives picked out by this procedure are precisely those entering the original twistor double copy of Refs. [39,40] and given here by Eq. (22). Thus, the known exact classical double copies in momentum, twistor, and position space amount to the same thing. Locality in all three spaces can also be shown to arise from the precise nature of three-point scattering amplitudes in momentum space [47]. This provides a response to those critics of position-space double copies, who may regard them as somehow speculative and/or coincidental: they are in fact rigorously derivable from the BCJ double copy for scattering amplitudes, but such that *exact* position-space double copies may not be available for arbitrary solutions. For general nonlinear solutions, one may still form classical double copies, but one must typically proceed order-by-order in the coupling, as has been emphasised, for example, in Refs. [73–77].

A simple consequence of the above discussion is that our multipole double copy indeed applies to all vacuum type-D solutions that are described by the Weyl double copy [21], where (as noted in Refs. [39,40]) different choices for the constant twistor $A^{\alpha\beta}$ in Eq. (22) map out the space of such solutions (see also Ref. [66]). As a further example, let us consider the *Kerr-Taub-NUT solution*, in which a Kerr black hole is dressed by a NUT charge N, which gives rise to a rotational character of the gravitational field that survives at infinity. As argued only very recently [78], it is possible to derive multipole moments for such a space, even though strictly speaking the spacetime is not asymptotically flat.[4] Here, we may easily obtain the multipole moments of Kerr-Taub-NUT as follows. First, we may note that the three-point amplitudes corresponding to the Kerr-Taub-NUT solution in position space can be obtained from the amplitudes $\mathcal{M}_\pm$ for the Kerr solution by the simple procedure [50]

$$\mathcal{M}_\pm \to e^{\pm i\theta}\mathcal{M}_\pm, \tag{41}$$

where subscripts denote the helicity of the emitted graviton. The results of Refs. [47,48] linking momentum-space amplitudes with twistor-space functions then immediately imply that the twistor function of Eq. (22) (for the Kerr case) is simply multiplied by a similar factor $e^{-i\theta}$ to obtain the Kerr-Taub-NUT result.[5] This has the effect of replacing the mass in our above results as follows:

$$m \to m e^{-i\theta} \equiv M - iN, \tag{42}$$

where we have defined the parameters,

$$M = m\cos\theta, \qquad N = m\sin\theta. \tag{43}$$

These play the role of the mass and NUT charge in the Kerr-Taub-NUT solution [50], and it follows from our above analysis that the scalar multipole moments of the Kerr-Taub-NUT solution will be simply given by

[4] All that is needed to apply the multipole formalism of Refs. [62–64] is the presence of an asymptotically flat spacelike hypersurface orthogonal to the time direction. See Ref. [78] for a full discussion.

[5] In choosing the dual twistor space double copy, we have picked out a particular helicity of the graviton. The other helicity is obtained from the (nondual) twistor space double copy.

$$Q_n^{\mathrm{KTN}} = i^{n-1}(M - iN)a^n. \tag{44}$$

This matches the recent computation of Ref. [78] and is itself a significant check that the original twistor multipole construction of Ref. [61] indeed corresponds to the Geroch-Hansen multipole formalism of Refs. [62–64]. The analogue of Eq. (35) for gauge and biadjoint theories is

$$q \to qe^{-i\theta} \equiv Q - i\tilde{Q}, \qquad y \to ye^{-i\theta} \equiv Y - i\tilde{Y}, \tag{45}$$

from which we may straightforwardly write the multipole moments,

$$Q_n^{\sqrt{\mathrm{KTN}}} = i^{n-1}(Q - i\tilde{Q})a^n, \qquad Q_n^{\mathrm{scal}} = i^{n-1}(Y - i\tilde{Y})a^n, \tag{46}$$

In the gauge theory, for example, these will be the multipole moments associated with a stationary spinning electromagnetic dyon at the origin, where Q and $\tilde{Q}$ can be interpreted as the electric and magnetic charges, respectively. Again, it is the case that the multipole moments in gauge or biadjoint theory are essentially identical to their gravitational counterparts, up to the replacements of the relevant charges/couplings, as follows directly from the twistor formalism.

IV. DISSCUSSION

In this paper, we have considered whether the multipole expansions of fields in biadjoint scalar, gauge, and gravity theories can be related by the classical double copy. By combining a twistor formulation of the multipole expansion [61] with a recently developed twistor language for the classical double copy [39,40], we have shown that the multipole moments for arbitrary type-D vacuum solutions indeed match up in different theories, subject to appropriate mass/charge replacements.[6]

Our results provide a nice illustration of the efficiency of the twistor double copy, but are of interest in their own right. It is often the case that a single copy of a given gravity solution can be found, but not easily interpreted. A canonical case of this is the single copy of the Kerr black hole, first formally identified in Ref. [4] and denoted as $\sqrt{\mathrm{Kerr}}$ in subsequent literature (see, e.g., Refs. [80–82]). It is known that this solution occurs by replacing the source for the Kerr black hole (a rotating disk of mass) with a similar gauge theory source (a rotating disk of charge). However, the nature of the sources is subtly different in the two theories [4], such that it is not clear what impact this has on the fields themselves. Multipole moments, however, allow us to fully characterize the structure of fields in a gauge-invariant way. Thus, the fact that the multipole moments for the Kerr and $\sqrt{\mathrm{Kerr}}$ solutions are essentially identical tells us a great deal of information about how to physically interpret the single copy, by recycling our intuition gathered from the Kerr black hole. Furthermore, the fact that our results apply for any type-D vacuum solution makes this a rather powerful statement, that may well help in interpreting and extending the double copy in future.

Our focus on type-D solutions is due to the fact that these linearize the field equations in biadjoint, gauge, and gravity theories. The twistor double copy—which relies on the standard Penrose transform restricted to the linearized level of each respective theory—is then an exact statement. Whether or not the twistor double copy can be extended to genuine nonlinear solutions is an open question, but it is worth noting that nonlinear twistor constructions exist in both gauge and gravity theories if one restricts to the (anti) self-dual sector [83,84]. A related question is whether our conclusions about multipole moments extend to solutions of the arbitrary Petrov type. Although Refs. [39,40] showed that the twistor double copy could be used to generate Weyl double copies for solutions of the arbitrary Petrov type, these were restricted to the linearized level only due to the limitations of the Penrose transform. Whether or not the multipole moments of genuine nontype-D solutions can be double copied depends on whether twistor methods can be extended to these nonlinear solutions. Progress in this area may also come from the recent connection between scattering amplitudes and the twistor double copy uncovered in Ref. [48], which we discuss above.

A particularly nice aspect of our results is that the multipole expansion in biadjoint theory also matches that in the gauge and gravity theories for the wide class of solutions we have considered. This adds a powerful weight to the observations made in Refs. [39,40], namely, that the twistor double copy allows us to understand the *inverse zeroth copy* from biadjoint scalar theory to gauge theory. That is, we have seen directly that the multipoles of vacuum type-D gravity solutions and their single copies are essentially inherited directly from a much simpler scalar theory. It is interesting to ponder what other physical quantities can be phrased in such an appealing manner.

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